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Occupation densities for certain processes related to subfractional Brownian motion

Ibrahima Mendy[a] **and Ibrahim Dakaou**[b]

[a]*Université de Ziguinchor*
[b]*Université de Maradi*

Abstract. In this paper, we establish the existence of a square integrable occupation density for two classes of stochastic processes. First, we consider a Gaussian process with an absolutely continuous random drift, and second we handle the case of a (Skorohod) integral with respect to subfractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. The proof of these results uses a general criterion for the existence of a square integrable local time, which is based on the techniques of Malliavin calculus.

1 Introduction

Local times for semimartingale have been widely studied. See, for example, the monograph Revuz and Yor (1994) and the references therein. On the other hand, local times of Gaussian processes have also been the object of a rich probabilistic literature; see, for example, the book of Marcus and Rosen (2006). A general criterion for existence of a local time for a wide class of anticipating processes, which are not semimartingale or Gaussian processes, was established by Imkeller and Nualart (1994). The proof of this result combines the technique of Malliavin calculus with the criterion given by Geman and Horowitz (1980). This criterion was applied in Imkeller and Nualart (1994) to the Brownian motion with an anticipating drift, and to indefinite Skorohod integral processes. The same criterion was applied in Es-Sebaiy et al. (2010) to the fractional Brownian motion with an anticipating drift, and to indefinite Skorohod integral with respect to fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. The aim of this paper is to extend the result of Es-Sebaiy et al. (2010) to the case of subfractional Brownian motion. First, we consider a subfractional Brownian motion $S^H = \{S_t^H, t \in [0, 1]\}$ with an absolutely continuous random drift

$$X_t = S_t^H + \int_0^t u_s \, ds,$$

where u is a stochastic process measurable with respect to the σ-field generated by S^H. Under reasonable regularity hypotheses imposed to the process u,

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we prove the existence of a square integrable occupation density with respect to the Lebesque measure for the process X. Our second example is represented by indefinite divergence (Skorohod) integral $X = \{X_t, t \in [0, 1]\}$ with respect to the subfractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, that is,

$$X_t = \int_0^t u_s \delta S_s^H.$$

We provide integrability conditions on the integrand u and its iterated derivatives in the sense of Malliavin calculus in order to deduce the existence of a square integrable occupation densities for X.

The paper is organized as follows. In Section 2, we give some preliminaries on subfractional Brownian motion and on Malliavin calculus with respect to Gaussian process. In Section 3, we prove the existence of the occupation densities for perturbed subfractional Brownian motion process, and in Section 4 we treat the case of indefinite divergence integral processes with respect to the subfractional Brownian motion.

2 Preliminaries

2.1 Subfractional Brownian motion

The subfractional Brownian motion $S^H = \{S_t^H, t \in [0, 1]\}$ with parameter $H \in (0, 1)$ is a centered Gaussian process with covariance function

$$C_H(s,t) = \mathbb{E}[S_t^H S_s^H] = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |t-s|^{2H}], \qquad s, t \geq 0.$$

If $H = \frac{1}{2}$, the process S^H is a standard Brownian motion. The increments of S^H satisfy

$$\begin{aligned} &[(2-2^{2H-1}) \wedge 1]|t-s|^{2H} \\ &\quad \leq \mathbb{E}(S_t^H - S_s^H)^2 \\ &\quad \leq [(2-2^{2H-1}) \vee 1]|t-s|^{2H} \qquad \text{if } H \in (0,1). \end{aligned} \tag{2.1}$$

For the most information about subfractional Brownian motion, see Bojdecki, Gorostiza and Talarczyk (2004); Dzhaparidze and van Zanten (2004).

2.2 Malliavin calculus

Let $\{B_t, t \in [0, 1]\}$ be a centered Gaussian process with covariance

$$R(t,s) = \mathbb{E}(B_t B_s),$$

defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By $\mathcal{H}_B$, we denote the canonical Hilbert space associated to B defined as the closure of the linear space generated by the indicator functions $\{\mathbf{1}_{[0,t]}, t \in [0,1]\}$ with respect to the inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}_B} = R(t,s), \qquad s,t \in [0,1].$$

The mapping $\mathbf{1}_{[0,t]} \to B_t$ can be extended to an isometry between $\mathcal{H}_B$ and the first Gauusian chaos generated by B. We denote by $B(\varphi)$ the image of an element $\varphi \in \mathcal{H}_B$ by this isometry.

We will first introduce some elements of the Malliavin calculus associated with B. We refer to Nualart (2006) for detailed account these notions. For a smooth random variable $F = f(B(\varphi_1), \dots, B(\varphi_n))$, for $\varphi_i \in \mathcal{H}_B$ and $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), the derivative of F with respect to B is defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \big(B(\varphi_1), \dots, B(\varphi_n)\big)\varphi_j.$$

For any integer $k \geq 1$ and any real number $p \geq 1$, we denote by $\mathbb{D}^{k,p}$ the Sobolev space defined as the closure of the space of smooth random variables with respect to the norm

$$\|F\|_{k,p}^p = \mathbb{E}\big(|F|^p\big) + \sum_{j=1}^k \|D^j F\|_{L^p(\Omega, \mathcal{H}_B^{\otimes j})}^p.$$

Similarly, for a given Hilbert space V, we can define Sobolev spaces of V-valued random variables $\mathbb{D}^{k,p}(V)$.

Consider the adjoint δ of D in L^2. Its domain is the class of elements $u \in L^2(\Omega, \mathcal{H}_B)$ such that

$$\mathbb{E}\big(\langle DF, u\rangle_{\mathcal{H}_B}\big) \leq C\|F\|_2,$$

for any $F \in \mathbb{D}^{1,2}$, and $\delta(u)$ is the element of $L^2(\Omega)$ given by

$$\mathbb{E}\big(\delta(u)F\big) = \mathbb{E}\big(\langle DF, u\rangle_{\mathcal{H}_B}\big)$$

for any $F \in \mathbb{D}^{1,2}$. We will make use of the notation $\delta(u) = \int_0^1 u_s \delta B_s$. It is well known that $\mathbb{D}^{1,2}(\mathcal{H}_B)$ is included in the domain of δ. Note that $\mathbb{E}(\delta(u)) = 0$ and the variance of $\delta(u)$ is given by

$$\mathbb{E}\big(\delta(u)^2\big) = \mathbb{E}\big(\|u\|_{\mathcal{H}_B}^2\big) + \mathbb{E}\big(\langle Du, (Du)^*\rangle_{\mathcal{H}_B \otimes \mathcal{H}_B}\big), \tag{2.2}$$

if $u \in \mathbb{D}^{1,2}(\mathcal{H}_B)$, where $(Du)^*$ is the adjoint of Du in the Hilbert space $\mathcal{H}_B \otimes \mathcal{H}_B$. We have Meyer's inequality

$$\mathbb{E}\big(|\delta(u)|^p\big) \leq C_p\big(\mathbb{E}\big(\|u\|_{\mathcal{H}_B}^p\big) + \mathbb{E}\big(\|Du\|_{\mathcal{H}_B \otimes \mathcal{H}_B}^p\big)\big), \tag{2.3}$$

for $p \geq 1$. We will make use of the property

$$F\delta(u) = \delta(Fu) + \langle DF, u \rangle_{\mathcal{H}_B}, \tag{2.4}$$

if $F \in \mathbb{D}^{1,2}$ and $u \in \mathrm{Dom}(\delta)$ such that $Fu \in \mathrm{Dom}(\delta)$. We also need the commutativity relationship between D and δ

$$D\delta(u) = u + \int_0^1 Du_s \delta B_s, \tag{2.5}$$

if $u \in \mathbb{D}^{1,2}(\mathcal{H}_B)$ and the process $\{Du_s, s \in [0,1]\}$ belong to the domain of δ. Throughout this paper, we will denote by C a generic constant that may be different from line to line.

For a measurable function $x : [0,1] \to \mathbb{R}$, we define the occupation measure

$$\mu(x)(C) = \int_0^1 \mathbf{1}_C(x_s)\, ds,$$

where C is a Borel subset of $\mathbb{R}$ and we will say that x has one occupation density with respect to the Lebesque measure λ if the measure μ is absolutely continuous with respect to λ. The occupation density of the function x will be the derivative $\frac{d\mu}{d\lambda}$. For a continuous process $\{X_t, t \in [0,1]\}$, we will say that X has a occupation density on $[0,1]$ if for almost all $\omega \in \Omega$, $X(\omega)$ has an occupation density on $[0,1]$. We will use the following criterium for the existence of occupation densities (see Es-Sebaiy et al., 2010; Imkeller and Nualart, 1994). Set $T = \{(s,t) \in [0,1]^2 : s < t\}$.

Theorem 2.1. *Let $\{X_t, t \in [0,1]\}$ be a continuous stochastic process such that $X_t \in \mathbb{D}^{2,2}$ for every $t \in [0,1]$. Suppose that there exists a sequence of random variables $\{F_n, n \geq 1\}$ with $\bigcup_n \{F_n \neq 0\} = \Omega$ a.s. and $F_n \in \mathbb{D}^{1,1}$ for every $n \geq 1$, two sequences $\alpha_n > 0, \delta_n > 0$, a measurable bounded function $\gamma : [0,1] \to \mathbb{R}$, and a constant $\theta > 0$, such that:*

(a) *For every $n \geq 1$, $|t-s| \leq \delta_n$, and on $\{F_n \neq 0\}$ we have*

$$\left\langle \gamma D(X_t - X_s), \mathbf{1}_{[s,t]} \right\rangle_{\mathcal{H}_B} > \alpha_n |t-s|^{\theta}, \qquad \text{a.s.} \tag{2.6}$$

(b) *For every $n \geq 1$,*

$$\int_T \mathbb{E}\big(\langle \gamma DF_n, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}_B}\big)|t-s|^{-\theta}\, dt\, ds < \infty. \tag{2.7}$$

(c) *For every $n \geq 1$,*

$$\int_T \mathbb{E}\big(\big|F_n \langle \gamma^{\otimes 2} DD(X_t - X_s), \mathbf{1}_{[s,t]}^{\otimes 2} \rangle_{\mathcal{H}_B^{\otimes 2}}\big|\big)|t-s|^{-2\theta}\, dt\, ds < \infty. \tag{2.8}$$

Then the process $\{X_t, t \in [0,1]\}$ admits a square integrable occupation density on $[0,1]$.

3 Occupation density for subfractional Brownian motion with random drift

We study in this part the existence of the occupation density for subfractional Brownian motion perturbed by a absolute continuous random drift. For the reader's convenience, we recall the following.

Theorem 3.1 (See Es-Sebaiy et al., 2010, Theorem 2, page 137). *Let* $\{B_t, t \in [0,1]\}$ *be a centered Gaussian process satisfying*

$$C_1(t-s)^{2\rho} \leq \mathbb{E}\big(|B_t - B_s|^2\big) \leq C_2(t-s)^{2\rho},$$

for some $\rho \in (0,1)$ *with* C_1, C_2 *two positive constants not depending on* t, s.

Consider the process $\{X_t, t \in [0,1]\}$ *given by*

$$X_t = B_t + \int_0^t u_s\, ds,$$

and suppose that the process u *satisfies the following conditions*:

(i) $u \in \mathbb{D}^{2,2}(L^2([0,1]))$.
(ii) $\mathbb{E}((\int_0^1 \|D^2 u_t\|^p_{\mathcal{H}\otimes\mathcal{H}}\, dt)^{\lambda/p}) < \infty$, *for some* $\lambda > 1$, $p > \frac{1}{1-\rho}$.

Then the process X *has a square integrable occupation density on the interval* $[0,1]$.

The main result of this section is the following.

Theorem 3.2. *Let* $\{S_t^H, t \in [0,1]\}$ *be a subfractional Brownian motion with parameter* $H \in (0,1)$. *Consider the process* $\{X_t, t \in [0,1]\}$ *given by*

$$X_t = S_t^H + \int_0^t u_s\, ds,$$

and suppose that the process u *satisfies the following conditions*:

1. $u \in \mathbb{D}^{2,2}(L^2([0,1]))$.
2. $\mathbb{E}((\int_0^1 \|D^2 u_t\|^p_{\mathcal{H}_{S^H}\otimes\mathcal{H}_{S^H}}\, dt)^{q/p}) < \infty$, *for some* $q > 1$, $p > \frac{1}{1-H}$.

Then the process X *has a square integrable occupation density on the interval* $[0,1]$.

Proof. Since S^H is Gaussian process then by using (2.1) and Theorem 3.1 we obtain the result. □

4 Occupation density for Skorohod integrals with respect to the subfractional Brownian motion

We study here the existence of occupation densities for indefinite divergence integrals with respect to subfractional Brownian motion. Consider a process of the form $X_t = \int_0^t u_s \delta S_s^H$, $t \in [0,1]$, where S^H is subfractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, and u is an element of $\mathbb{D}^{1,2}(L^2([0,1])) \subset \mathrm{Dom}(\delta)$.

We know that the covariance of subfractional Brownian motion can be written as

$$\mathbb{E}(S_t^H S_s^H) = \int_0^t \int_0^s \phi_H(u,v)\,du\,dv = C_H(s,t), \tag{4.1}$$

where $\phi_H(u,v) = H(2H-1)[|u-v|^{2H-2} - (u+v)^{2H-2}]$.

Formulae (4.1) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{S^H}} = \alpha_H \int_0^1 \int_0^1 [|u-v|^{2H-2} - (u+v)^{2H-2}]\varphi_u \psi_v\,du\,dv \tag{4.2}$$

for any pair step functions φ and ψ on $[0,1]$ and where $\alpha_H = H(2H-1)$. Consider the kernel

$$n_H(t,s) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-1/2)} s^{3/2-H} \left(\int_s^t (x^2 - s^2)^{H-3/2}\,dx \right) 1_{(0,t)}(s). \tag{4.3}$$

By Dzhaparidze and van Zanten (2004, Theorem 3.2, page 45), we have

$$C_H(s,t) = c_H^2 \int_0^{s\wedge t} n_H(t,u) n_H(s,u)\,du, \tag{4.4}$$

where

$$c_H^2 = \frac{\Gamma(1+2H)\sin \pi H}{\pi}.$$

Property (4.4) implies that $C_H(s,t)$ is nonnegative definite. Consider the linear operator n_H^* from $\mathcal{E}$ (set of step functions on $[0,1]$) to $L^2([0,1])$ defined by

$$n_H^*(\varphi)(s) = c_H \int_s^1 \varphi_r \frac{\partial n_H}{\partial r}(r,s)\,dr.$$

Using (4.2) and (4.4), we have

$$\begin{aligned}
&\langle n_H^* \varphi, n_H^* \psi \rangle_{L^2([0,1])} \\
&\quad = c_H^2 \int_0^1 \left(\int_s^1 \frac{\partial n_H}{\partial r}(r,s)\,dr \varphi_r \right) \left(\int_s^1 \varphi_r \frac{\partial n_H}{\partial u}(u,s)\,du \psi_u \right) ds \\
&\quad = c_H^2 \int_0^1 \int_0^1 \left(\int_0^{r\wedge u} \frac{\partial n_H}{\partial r}(r,s) \frac{\partial n_H}{\partial u}(u,s)\,ds \right) \varphi_r \psi_u\,dr\,du
\end{aligned}$$

$$= c_H^2 \int_0^1 \int_0^1 \frac{\partial^2 C_H}{\partial r \, \partial u}(r, u) \varphi_r \psi_u \, dr \, du$$
$$= \alpha_H \int_0^1 \int_0^1 \left[|u - r|^{2H-2} - (u + r)^{2H-2} \right] \varphi_r \psi_u \, dr \, du$$
$$= \langle \varphi, \psi \rangle_{\mathcal{H}_{S^H}}.$$

As a consequence, the operator n_H^* provides an isometry between the Hilbert space $\mathcal{H}_{S^H}$ and $L^2([0, 1])$. Hence, the process W defined by

$$W_t = S^H\big((n_H^*)^{-1} (\mathbf{1}_{[0,t]}) \big)$$

is a Wiener process, and the process S^H has an integral representation of the form

$$S_t^H = c_H \int_0^t n_H(t, s) \, dW_s,$$

because $(n_H^*)(\mathbf{1}_{[0,t]})(s) = c_H n_H(t, s)$. By Dzhaparidze and van Zanten (2004), we have

$$W_t = \int_0^t \psi_H(t, s) \, dS_s^H,$$

where

$$\begin{aligned} \psi_H(t, s) = {} & \frac{s^{H-1/2}}{\Gamma(3/2 - H)} \\ & \times \Big[t^{H-3/2} \big(t^2 - s^2 \big)^{1/2-H} \\ & \quad - (H - 3/2) \int_s^t \big(x^2 - s^2 \big)^{1/2-H} x^{H-3/2} \, dx \Big] 1_{(0,t)}(s). \end{aligned} \tag{4.5}$$

We can find a linear space of functions contained in $\mathcal{H}_{S^H}$ in the following way. Let $|\mathcal{H}_{S^H}|$ be the linear space of measurable functions on $[0, 1]$ such that

$$\|\varphi\|_{|\mathcal{H}_{S^H}|}^2 = c_H^2 \int_0^1 \left(\int_s^1 |\varphi_r| \frac{\partial n_H}{\partial r}(r, s) \, dr \right)^2 ds < \infty. \tag{4.6}$$

From the above computations, it is easy to check that

$$\|\varphi\|_{|\mathcal{H}_{S^H}|}^2 = \alpha_H \int_0^1 \int_0^1 |\varphi_r| |\varphi_u| \phi_H(r, u) \, dr \, du. \tag{4.7}$$

It is not difficult to show that $|\mathcal{H}_{S^H}|$ is a Banach space with the norm $\| \cdot \|_{|\mathcal{H}_{S^H}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}_{S^H}|$. We will obtain the following lemma.

Lemma 4.1. *The canonical Hilbert space $\mathcal{H}_{S^H}$ associated to S^H satisfies*:

$$L^2([0, 1]) \subset L^{1/H}([0, 1]) \subset \mathcal{H}_{S^H}, \tag{4.8}$$

where $H > \frac{1}{2}$.

Proof. For any $H > \frac{1}{2}$, we have $\phi_H(t,s) \le \alpha_H |u-r|^{2H-2}$. Then, for any measurable function φ on $[0,1]$ we have

$$\|\varphi\|_{|\mathcal{H}_{S^H}|} \le \|\varphi\|_{|\mathcal{H}_{B^H}|}. \tag{4.9}$$

Using (4.9) and the fact that $L^2([0,1]) \subset L^{1/H}([0,1]) \subset |\mathcal{H}_{B^H}|$ (see Nualart, 2006), we obtain

$$L^2([0,1]) \subset L^{1/H}([0,1]) \subset |\mathcal{H}_{B^H}| \subset |\mathcal{H}_{S^H}|.$$

It remains to show that $|\mathcal{H}_{S^H}| \subset \mathcal{H}_{S^H}$. For any measurable function φ on $[0,1]$, we have

$$\begin{aligned}\|\varphi\|_{\mathcal{H}_{S^H}} &= \alpha_H \int_0^1 \int_0^1 \varphi_r \varphi_u \phi_H(r,u)\, dr\, du \\ &\le \alpha_H \int_0^1 \int_0^1 |\varphi_r||\varphi_u| \phi_H(r,u)\, dr\, du = \|\varphi\|_{|\mathcal{H}_{S^H}|}.\end{aligned}$$

Hence, $|\mathcal{H}_{S^H}| \subset \mathcal{H}_{S^H}$. □

For any $0 \le s < t \le 1$, and $u \in [0,1]$ we set

$$f_{s,t}(u) := \int_s^t \phi_H(u,v)\, dv. \tag{4.10}$$

The following is the main result of this section.

Theorem 4.2. *Consider the stochastic process $X_t = \int_0^t u_s \delta S_s^H$ where the integrand u satisfy the following conditions for some $q > \frac{2H}{1-H}$ and $p > 1$ such that $\frac{1}{p} + 2 < H(p+1)$:*

(H_1) $u \in \mathbb{D}^{3,2}(L^2([0,1]))$;
(H_2) $\int_0^1 \int_0^1 [\mathbb{E}(|D_t u_s|^p) + \mathbb{E}(\||D_t D u_s|\|_{\mathcal{H}_{S^H}}^p) + \mathbb{E}(\||D_t D D u_s|\|_{\mathcal{H}_{S^H} \otimes \mathcal{H}_{S^H}}^p)]\, ds\, dt < \infty$;
(H_3) $\int_0^1 \mathbb{E}(|u_t|^{-p(q+1)/(p-1)})\, dt < \infty$.

Then the process $\{X_t, t \in [0,1]\}$ admits a square integrable occupation density on $[0,1]$.

Proof. We are going to show conditions (a), (b) and (c) of Theorem 2.1.

Before, we note that by Es-Sebaiy et al. (2010), there exists a function $\gamma : [0,1] \to \{-1,1\}$ such that $\gamma_t u_t = |u_t|$.

Proof of condition (a): Fix $0 \le s < t \le 1$. From (2.5), we obtain

$$D(X_t - X_s) = u \mathbf{1}_{[s,t]} + \int_s^t D u_r \delta S_r^H,$$

and we can write

$$\langle \gamma D(X_t - X_s), \mathbf{1}_{[s,t]}\rangle_{\mathcal{H}_{SH}} = \langle |u|\mathbf{1}_{[s,t]}, \mathbf{1}_{[s,t]}\rangle_{\mathcal{H}_{SH}} + \left\langle \gamma \int_s^t Du_r \delta S_r^H, \mathbf{1}_{[s,t]} \right\rangle_{\mathcal{H}_{SH}}. \tag{4.11}$$

We first study the term

$$\langle |u|\mathbf{1}_{[s,t]}, \mathbf{1}_{[s,t]}\rangle_{\mathcal{H}_{SH}} = \int_s^t \int_s^t |u_\alpha| \phi_H(\alpha,\beta)\, d\alpha\, d\beta = \int_s^t |u_\alpha| f_{s,t}(\alpha)\, d\alpha.$$

For any $q > 1$, we have (see Bojdecki, Gorostiza and Talarczyk, 2004)

$$\begin{aligned}\mathbb{E}(S_t^H - S_s^H)^2 &= \int_s^t f_{s,t}(\alpha)\, d\alpha \\ &= \int_s^t \big(|u_\alpha| f_{s,t}(\alpha)\big)^{q/(q+1)} \big(|u_\alpha| f_{s,t}(\alpha)\big)^{-q/(q+1)} f_{s,t}(\alpha)\, d\alpha,\end{aligned}$$

and using Hölder's inequality with order $\frac{q+1}{q}$ and $q+1$, we obtain

$$\begin{aligned}&\mathbb{E}(S_t^H - S_s^H)^2 \\ &\quad \le \left(\int_s^t |u_\alpha| f_{s,t}(\alpha)\, d\alpha\right)^{q/(q+1)} \left(\int_s^t |u_\alpha|^{-q} f_{s,t}(\alpha)\, d\alpha\right)^{1/(q+1)}.\end{aligned} \tag{4.12}$$

Hence,

$$\begin{aligned}f_{s,t}(\alpha) &\le H(2H-1) \int_0^1 \big[|\alpha - \beta|^{2H-2} + (\alpha+\beta)^{2H-2}\big]\, d\beta \\ &= H\big[(1-\alpha)^{2H-1} + (1+\alpha)^{2H-1}\big] \le 2H,\end{aligned} \tag{4.13}$$

using (2.1), (4.12) and (4.13), we get

$$\int_s^t |u_\alpha| f_{s,t}(\alpha)\, d\alpha \ge C|t-s|^{2H(q+1)/q} Z_q^{-1/q}, \tag{4.14}$$

where $Z_q = \int_0^1 |u_\alpha|^{-q}\, d\alpha$.

On the other hand, for the second summand in the right-hand side of (4.11) we can write, using Hölder's inequality

$$\begin{aligned}\left|\left\langle \gamma \int_s^t Du_r \delta S_r^H, \mathbf{1}_{[s,t]}\right\rangle_{\mathcal{H}_{SH}}\right| &\le \int_0^1 \left|\int_s^t D_\alpha u_r \delta S_r^H\right| f_{s,t}(\alpha)\, d\alpha \\ &\le \left(\int_0^1 f_{s,t}(\alpha)^{p/(p-1)}\, d\alpha\right)^{(p-1)/p} \\ &\quad \times \left(\int_0^1 \left|\int_s^t D_\alpha u_r \delta S_r^H\right|^p d\alpha\right)^{1/p}.\end{aligned} \tag{4.15}$$

We can write

$$\begin{aligned}\Big(\int_0^1 f_{s,t}(\alpha)^{p/(p-1)}\,d\alpha\Big)^{(p-1)/p}&\\ &=\alpha_H\Big\|\int_s^t[|\cdot-\beta|^{2H-2}-|\cdot+\beta|^{2H-2}]\,d\beta\Big\|_{L^{p/(p-1)}}\\ &\le\alpha_H\Big\|\int_s^t|\cdot-\beta|^{2H-2}\,d\beta\Big\|_{L^{p/(p-1)}}\\ &\le\alpha_H\|\mathbf{1}_{[s,t]}*|\cdot|^{2H-2}\mathbf{1}_{[-1,1]}\|_{L^{p/(p-1)}(\mathbb{R})}.\end{aligned}\tag{4.16}$$

Young's inequality with exponents a and b in $(1,\infty)$ such that $\frac{1}{a}+\frac{1}{b}=2-\frac{1}{p}$ yields

$$\begin{aligned}&\|\mathbf{1}_{[s,t]}*|\cdot|^{2H-2}\mathbf{1}_{[-1,1]}\|_{L^{p/(p-1)}(\mathbb{R})}\\ &\quad\le\|\mathbf{1}_{[s,t]}\|_{L^a(\mathbb{R})}\||\cdot|^{2H-2}\mathbf{1}_{[-1,1]}\|_{L^b(\mathbb{R})}.\end{aligned}\tag{4.17}$$

Choosing $b<\frac{1}{2-2H}$ and letting $\eta=\frac{1}{a}<2H-\frac{1}{p}$ we obtain from (4.15)–(4.17)

$$\begin{aligned}&\Big|\Big\langle\gamma\int_s^t Du_r\delta S_r^H,\mathbf{1}_{[s,t]}\Big\rangle_{\mathcal{H}_{S^H}}\Big|\\ &\quad\le C|t-s|^\eta\Big(\int_0^1\Big|\int_s^t D_\alpha u_r\delta S_r^H\Big|^p\,d\alpha\Big)^{1/p}.\end{aligned}\tag{4.18}$$

Now we will apply Garsia–Rodemich–Ramsey's lemma (see Garsia, Rodemich and Rumsey, 1970/1971) with $\phi(x)=x^p$, $p(x)=x^{(m+2)/p}$ and to the continuous function $u_s=\int_0^s D_\alpha u_r\delta S_r^H$ and (Alòs and Nualart, 2003, Theorem 5), we get

$$\Big|\int_s^t D_\alpha u_r\delta S_r^H\Big|^p\le C|t-s|^m\int_0^1\int_0^1\frac{|\int_x^y D_\alpha u_r\delta S_r^H|^p}{|x-y|^{m+2}}\,dx\,dy.$$

As a consequence

$$\Big(\int_0^1\Big|\int_s^t D_\alpha u_r\delta S_r^H\Big|^p\,d\alpha\Big)^{1/p}\le C|t-s|^{m/p}Y_{m,p}^{1/p},\tag{4.19}$$

where

$$Y_{m,p}=\int_0^1\int_0^1\int_0^1\frac{|\int_x^y D_\alpha u_r\delta S_r^H|^p}{|x-y|^{m+2}}\,dx\,dy\,d\alpha.$$

Substituting (4.19) into (4.18) yields

$$\Big|\Big\langle\gamma\int_s^t Du_r\delta S_r^H,\mathbf{1}_{[s,t]}\Big\rangle_{\mathcal{H}_{S^H}}\Big|\le C|t-s|^{\eta+m/p}Y_{m,p}^{1/p},\tag{4.20}$$

and from (4.14) and (4.20) into (4.11), we get

$$\begin{aligned}\langle \gamma D(X_t - X_s), \mathbf{1}_{[s,t]}\rangle_{\mathcal{H}_{SH}} &\geq |t-s|^{2H(q+1)/q} Z_q^{-1/q} - C|t-s|^{\eta+m/p} Y_{m,p}^{1/p}\\ &= |t-s|^{2H(q+1)/q}\big(Z_q^{-1/q} - C|t-s|^{\delta} Y_{m,p}^{1/p}\big),\end{aligned}$$

where $\delta = \eta + \frac{m}{p} - 2H - \frac{2H}{q}$. With a right choice of η the exponent δ is positive, provide that $m + \frac{1}{p} - 2H - \frac{2H}{q} > 0$, because $\eta < 2H - \frac{1}{p}$. Taking account that $\frac{2H}{q} < 1 - H$, it suffices that

$$m > \frac{1}{q} + 1 - H. \tag{4.21}$$

We construct now the sequence $\{F_n, n \geq 1\}$. Fix a natural number $n \geq 2$, and choose a function $\varphi_n(x)$, which is infinitely differentiable with compact support, such that $\varphi_n(x) = 1$ if $|x| \leq n-1$, and $\varphi_n(x) = 0$ if $|x| \geq n$. Set $F_n = \varphi_n(G)$, where $G = Z_q + Y_{m,p}$. Then clearly the sequences α_n and δ_n required in Theorem 2.1 can be constructed on the set $\{F_n \neq 0\}$, with $\theta = 2H + \frac{2H}{q}$.

It only remains to show that the random variables F_n are in the space $\mathbb{D}^{1,1}$. For this we have to show that the random variables $\|DZ_q\|_{\mathcal{H}_{SH}}$ and $\|DY_{m,p}\|_{\mathcal{H}_{SH}}$ are integrable on the set $\{G \leq n\}$. First notice that, as in the proof of Proposition 4.1 of Imkeller and Nualart (1994), we can show that $\mathbb{E}(\|DZ_q\|_{\mathcal{H}_{SH}}) < \infty$. This follows from the integrability conditions (H_3) and

$$\int_0^1 \mathbb{E}(\|Du_t\|_{\mathcal{H}_{SH}}^p)\, dt < \infty, \tag{4.22}$$

which holds of (H_2), the continuous embedding of $L^{1/H}([0,1])$ into $\mathcal{H}_{SH}$ (see Lemma 4.1), and the fact that $pH \geq 1$. On the other hand, we can write

$$DY_{m,p} = p\int_0^1\int_0^1\int_0^1 |\xi_{x,y,\alpha}|^{p-1}\operatorname{sign}(\xi_{x,y,\alpha}) D\xi_{x,y,\alpha}|x-y|^{-m-2}\, dx\, dy\, d\alpha,$$

where $\xi_{x,y,\alpha} = \int_x^y D_\alpha u_r \delta S_r^H$. Thus

$$\begin{aligned}&\|DY_{m,p}\|_{\mathcal{H}_{SH}}\\ &\quad\leq p\int_0^1\int_0^1\int_0^1 |\xi_{x,y,\alpha}|^{p-1}\|D\xi_{x,y,\alpha}\|_{\mathcal{H}_{SH}}|x-y|^{-m-2}\, dx\, dy\, d\alpha\\ &\quad\leq p(Y_{m,p})^{(p-1)/p}\left(\int_0^1\int_0^1\int_0^1 \|D\xi_{x,y,\alpha}\|_{\mathcal{H}_{SH}}^p|x-y|^{-m-2}\, dx\, dy\, d\alpha\right)^{1/p}.\end{aligned}$$

Now, to show that $\mathbf{1}_{(G\leq n)}\|DY_{m,p}\|_{\mathcal{H}_{SH}}$ belong to $L^1(\Omega)$, it suffices to show that the random variable

$$Y = \int_0^1\int_0^1\int_0^1 \|D\xi_{x,y,\alpha}\|_{\mathcal{H}_{SH}}^p|x-y|^{-m-2}\, dx\, dy\, d\alpha$$

has a finite expectation. Since, for any $0 \le y < x \le 1$

$$D\xi_{x,y,\alpha} = \mathbf{1}_{[y,x]} D_\alpha u + \int_y^x D D_\alpha u_s \delta S_s^H,$$

we have

$$\begin{aligned} Y &\le C\Big(\int_0^1 \int_0^1 \int_0^1 \|\mathbf{1}_{[y,x]} D_\alpha u\|_{\mathcal{H}_{S^H}}^p |x-y|^{-m-2}\, dx\, dy\, d\alpha \\ &\quad + \int_0^1 \int_0^1 \int_0^1 \Big\| \int_y^x D D_\alpha u_s \delta S_s^H \Big\|_{\mathcal{H}_{S^H}}^p |x-y|^{-m-2}\, dx\, dy\, d\alpha\Big) \\ &= C(Y_1 + Y_2). \end{aligned}$$

From the continuous embedding of $L^{1/H}([0,1])$ into $\mathcal{H}_{S^H}$, we obtain

$$\begin{aligned} Y_1 &\le C \int_0^1 \int_0^1 \int_0^1 \|\mathbf{1}_{[y,x]} D_\alpha u\|_{L^{1/H}([0,1])}^p |x-y|^{-m-2}\, dx\, dy\, d\alpha \\ &\le C|x-y|^{pH-1} \int_0^1 \int_0^1 \int_0^1 \int_y^x |D_\alpha u_r|^p |x-y|^{-m-2}\, dx\, dy\, d\alpha. \end{aligned}$$

Hence, $\mathbb{E}(Y_1) < \infty$, by Fubini's theorem (Imkeller and Nualart, 1994, Proposition 3.1) and condition (H_2), provided

$$m < pH - 1. \tag{4.23}$$

On the other hand, using the estimate (2.3), and again the continuous embedding of $L^{1/H}([0,1])$ into $\mathcal{H}_{S^H}$ yields

$$\begin{aligned} &\mathbb{E}\Big(\Big\| \int_y^x D D_\alpha u_s \delta S_s^H \Big\|_{\mathcal{H}_{S^H}}^p\Big) \\ &\quad \le C\mathbb{E}\big(\|D_\alpha D u_\cdot \mathbf{1}_{[y,x]}(\cdot)\|_{\mathcal{H}_{S^H}^{\otimes 2}}^p + \|D_\alpha D D u_\cdot \mathbf{1}_{[y,x]}(\cdot)\|_{\mathcal{H}_{S^H}^{\otimes 3}}^p\big) \\ &\quad \le C\mathbb{E}\big(\||D_\alpha D u_\cdot| \mathbf{1}_{[y,x]}(\cdot)\|_{L^{1/H}([0,1],\mathcal{H}_{S^H})}^p \\ &\qquad + \||D_\alpha D D u_\cdot| \mathbf{1}_{[y,x]}(\cdot)\|_{L^{1/H}([0,1],\mathcal{H}_{S^H}^{\otimes 2})}^p\big) \\ &\quad \le C|x-y|^{pH-1}\Big(\int_y^x \mathbb{E}\big(\||D_\alpha D u_r|\|_{\mathcal{H}_{S^H}}^p\big)\, dr \\ &\qquad + \int_y^x \mathbb{E}\||D_\alpha D D u_r|\|_{\mathcal{H}_{S^H}^{\otimes 2}}^p\Big). \end{aligned}$$

As before we obtain $\mathbb{E}(Y_2) < \infty$ by Fubini's theorem and condition (H_2), provided (4.23) holds. Notice that condition $\frac{1}{p} + 2 < H(p+1)$ implies that we can choose an m such that (4.21) and (4.23) hold.

Proof of condition (b): Define $A_n = \{G \le n\}$. Then condition (b) in Theorem 2.1 follows from

$$\int_T \mathbb{E}(\langle \gamma DF_n, \mathbf{1}_{[s,t]}\rangle_{\mathcal{H}_{SH}})|t-s|^{-\theta}\, dt\, ds$$
$$\le C\int_T \mathbb{E}(\mathbf{1}_{A_n}\langle \gamma DG, \mathbf{1}_{[s,s]}\rangle_{\mathcal{H}_{SH}})|t-s|^{-\theta}\, dt\, ds$$
$$\le C\mathbb{E}(\mathbf{1}_{A_n}\|DG\|_{\mathcal{H}_{SH}})\int_T |t-s|^{H-\theta}\, dt\, ds,$$

since $C\mathbb{E}(\mathbf{1}_{A_n}\|DG\|_{\mathcal{H}_{SH}}) < \infty$ and $\theta - H = H + \frac{2H}{q} < 1$.

Proof of condition (c): We have

$$D_\alpha D_\beta(X_t - X_s) = \mathbf{1}_{[s,t]}(\beta)D_\alpha u_\beta + \mathbf{1}_{[s,t]}(\alpha)D_\beta u_\alpha + \int_s^t D_\alpha D_\beta u_r \delta S_r^H.$$

Hence,

$$\begin{aligned}
&\langle \gamma^{\otimes 2} DD(X_t - X_s), \mathbf{1}_{[s,t]}^{\otimes 2}\rangle_{\mathcal{H}_{SH}^{\otimes 2}}\\
&\quad = \langle \gamma^{\otimes 2}\mathbf{1}_{[s,t]}(\beta)D_\alpha u_\beta, \mathbf{1}_{[s,t]}^{\otimes 2}\rangle_{\mathcal{H}_{SH}^{\otimes 2}}\\
&\qquad + \langle \gamma^{\otimes 2}\mathbf{1}_{[s,t]}(\alpha)D_\beta u_\alpha, \mathbf{1}_{[s,t]}^{\otimes 2}\rangle_{\mathcal{H}_{SH}^{\otimes 2}}\\
&\qquad + \left\langle \gamma^{\otimes 2}\int_s^t D_\alpha D_\beta u_r \delta S_r^H, \mathbf{1}_{[s,t]}^{\otimes 2}\right\rangle_{\mathcal{H}_{SH}^{\otimes 2}}\\
&\quad = J_1(s,t) + J_2(s,t) + J_3(s,t).
\end{aligned}$$

For $i = 1, 2, 3$, we set $A_i = \mathbb{E}(F_n \int_T |t-s|^{-2\theta} J_i(s,t)\, dt\, ds)$. Let us compute first

$$\begin{aligned}
A_1 &\le C\int_T\int_T |t-s|^{2H-2\theta}\mathbb{E}(\||D_\alpha u_\beta|\mathbf{1}_{[s,t]}(\beta)\|_{\mathcal{H}_{SH}^{\otimes 2}})\, ds\, dt\\
&\le C\int_T\int_T |t-s|^{2H-2\theta}\mathbb{E}(\||D_\alpha u_\beta \mathbf{1}_{[s,t]}(\beta)|\|_{\mathcal{H}_{BH}^{\otimes 2}})\, ds\, dt\\
&= C\int_T\int_T |t-s|^{2H-2\theta}\left(\int_s^t\int_s^t \varphi(\beta,y)\, d\beta\, dy\right)^{1/2} ds\, dt,
\end{aligned}$$

where

$$\varphi(\beta,y) = \int_0^1\int_0^1 \mathbb{E}(|D_\alpha u_\beta||D_x u_y|\phi'(\alpha,x)\phi'(\beta,y))\, d\alpha\, dx,$$

with $\phi'(\beta,y) = \alpha_H|y-\beta|^{2H-2}$. By Fubini's theorem $A_1 < \infty$, because $2H - 2\theta > -2$, which is equivalent to $q > H$, and

$$\int_s^t\int_s^t \varphi(\beta,y)\, d\beta\, dy \le \mathbb{E}(\||Du|\|_{\mathcal{H}_{BH}^{\otimes 2}}^2)$$

and this is finite because of the inclusion of $L^2([0,1])$ in $\mathcal{H}_{B^H}$ (see Nualart, 2006). In the same way, we can show that $A_2 < \infty$. Finally,

$$A_3 = \mathbb{E}\left(F_n \int_T |t-s|^{-2\theta} \left| \left\langle \gamma^{\otimes 2} \int_s^t D_\alpha D_\beta u_r \delta S_r^H, \mathbf{1}_{[s,t]}^{\otimes 2} \right\rangle_{\mathcal{H}_{S^H}^{\otimes 2}} \right| dt\, ds \right)$$

$$\leq \int_T |t-s|^{2H-2\theta} \mathbb{E}\left(\left\| \int_s^t D_\alpha D_\beta u_r \delta S_r^H \right\|_{\mathcal{H}_{S^H}^{\otimes 2}} \right) dt\, ds,$$

and we conclude as before by using for example the bound (2.3) for the norm of the Skorohod integral and the condition (H_2). $\square$

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UFR Sciences et Technologies
Département de Mathématiques
Université de Ziguinchor
BP 523 Ziguinchor
Sénégal
E-mail: imendy@univ-zig.sn

Département de Mathématiques
Faculté des Sciences et Techniques
Université de Maradi
BP 465 Maradi
Niger
E-mail: idakaou@yahoo.fr