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Objective Bayesian Inference for a Generalized Marginal Random Effects Model

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Abstract. An objective Bayesian inference is proposed for the generalized marginal random effects model $p(\mathbf{x}|\mu, \sigma_\lambda) = f((\mathbf{x} - \mu\mathbf{1})^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}(\mathbf{x} - \mu\mathbf{1}))/\sqrt{\det(\mathbf{V} + \sigma_\lambda^2\mathbf{I})}$. The matrix $\mathbf{V}$ is assumed to be known, and the goal is to infer μ given the observations $\mathbf{x} = (x_1, \ldots, x_n)^T$, while σ_λ is a nuisance parameter. In metrology this model has been applied for the adjustment of inconsistent data $x_1, \ldots, x_n$, where the matrix $\mathbf{V}$ contains the uncertainties quoted for $x_1, \ldots, x_n$.

We show that the reference prior for grouping $\{\mu, \sigma_\lambda\}$ is given by $\pi(\mu, \sigma_\lambda) \propto \sqrt{\mathbf{F}_{22}}$, where $\mathbf{F}_{22}$ denotes the lower right element of the Fisher information matrix $\mathbf{F}$. We give an explicit expression for the reference prior, and we also prove propriety of the resulting posterior as well as the existence of mean and variance of the marginal posterior for μ. Under the additional assumption of normality, we relate the resulting reference analysis to that known for the conventional balanced random effects model in the asymptotic case when the number of repeated within-class observations for that model tends to infinity.

We investigate the frequentist properties of the proposed inference for the generalized marginal random effects model through simulations, and we also study its robustness when the underlying distributional assumptions are violated. Finally, we apply the model to the adjustment of current measurements of the Planck constant.

Keywords: objective Bayesian inference, reference prior, random effects model.

1 Introduction

We consider the model

$$p(\mathbf{x}|\mu, \sigma_\lambda) = \frac{1}{\sqrt{\det(\mathbf{V} + \sigma_\lambda^2\mathbf{I})}} f\left((\mathbf{x} - \mu\mathbf{1})^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}(\mathbf{x} - \mu\mathbf{1})\right), \tag{1}$$

where $\mathbf{1}$ is a vector of ones, and $\mathbf{I}$ denotes the identity matrix of an appropriate order. The goal is to infer μ given observations $(x_1, \ldots, x_n)^T = \mathbf{x}$. The $n \times n$ symmetric positive definite matrix $\mathbf{V}$ is assumed to be known, while σ_λ denotes a nuisance parameter. In the multivariate normal case, (1) is the marginal model of the random effects model

$$\mathbf{X} = \mu\mathbf{1} + \boldsymbol{\lambda} + \boldsymbol{\varepsilon} \quad \text{with} \quad \boldsymbol{\lambda} \sim N(\mathbf{0}, \sigma_\lambda^2\mathbf{I}) \quad \text{and} \quad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \mathbf{V}), \tag{2}$$

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where $\boldsymbol{\lambda}$ and $\boldsymbol{\varepsilon}$ are independent. We will therefore call (1) a generalized marginal random effects model. For the particular normal case, we will refer to the model as the normal marginal random effects model. We note that $\mathbf{X}|\mu, \sigma_\lambda \sim E_n(\mu\mathbf{1}, \mathbf{V} + \sigma_\lambda^2\mathbf{I}, f)$ (n-dimensional elliptically contoured distribution with location vector $\mu\mathbf{1}$, dispersion matrix $(\mathbf{V} + \sigma_\lambda^2\mathbf{I})$, and density generator $f(\cdot)$, cf. Gupta et al. (2013)) under (1), and model (2) is obtained as a special case when setting $f(u) = \exp(-u/2)/(2\pi)^{n/2}$. Following the definition of elliptically contoured distributions (cf. Definition 1 in Gómez et al. (2003)), the function $f(\cdot)$ should be a non-negative Lebesgue measurable function on $[0, \infty)$ such that

$$\int_0^\infty t^{n-1} f(t^2) dt < \infty$$

holds.

Model (1) is relevant in metrology for the adjustment of inconsistent data. For example, under the additional assumption of normality, the model has been proposed for the determination of a reference value required in the analysis of interlaboratory comparisons (see, e.g., Kacker (2004), Toman and Possolo (2009)), or for the determination of a fundamental constant (cf. Toman et al. (2012)). The matrix $\mathbf{V}$ contains the uncertainty assessments about $\mathbf{X}$ made by the corresponding laboratories, and the simple model $\mathbf{X} \sim N(\mu\mathbf{1}, \widetilde{\mathbf{V}})$ is applied with $\widetilde{\mathbf{V}} = \mathbf{V} + \sigma_\lambda^2\mathbf{I}$. The additional term $\sigma_\lambda^2\mathbf{I}$ accounts for a possible underrating of quoted uncertainties. However, the normality assumption is rather stringent and may not be adequate which motivates our distributional generalization. We also refer to Rukhin and Possolo (2011) and Possolo (2013) who considered model (2) with the normal distribution being replaced by a Laplace distribution and a t-distribution, respectively.

The elements in the matrix $\mathbf{V}$ may actually be viewed as further parameters of model (1) that ought to be included in a Bayesian inference. However, such a model is no longer identifiable (from a single multivariate observation $\mathbf{x}$), and a Bayesian inference based on non-informative priors would not be possible then. Similar to applications in metrology, model (1) with known $\mathbf{V}$ may be seen as a simple model for the inference of μ based on a single observation $\mathbf{x}$ and a possibly underrated covariance matrix $\mathbf{V}$. Formally, the results given in this paper are conditional on $\mathbf{V}$.

The one way random effects model (2) is a standard model in statistics that has long been researched from both classical statistics (see, e.g., Cochran (1937, 1954), Yates and Cochran (1938), Rao (1997), Searle et al. (2006)) and Bayesian statistics (cf., for example, Hill (1965), Tiao and Tan (1965), Datta and Gosh (1995), Browne and Draper (2006), Gelman (2006)). However, in the form (2), i.e., with known residual variance and without repeated within-class observations, the random effects model has hardly been treated in the statistical literature. One exception is Rukhin and Possolo (2011) who considered this (type of) model, albeit with the Gaussian distributions replaced by Laplace distributions. The one way random effects model is usually considered in combination with repeated within-class observations, for instance in the form

$$X_{ij} = \mu + \lambda_i + \epsilon_{ij}, \quad \text{where} \quad \lambda_i \sim N(0, \sigma_\lambda^2), \quad \text{and} \quad \epsilon_{ij} \sim N(0, \sigma^2) \tag{3}$$

for $j = 1, \ldots, n_i$ and $i = 1, \ldots, n$. Furthermore, the variance σ^2 enters as a further unknown. Software is widely available for the treatment of this model, e.g., in R Development Core Team (2008), for both classical or Bayesian inferences. Also the Berger & Bernardo reference prior has already been derived both for the balanced case (Berger and Bernardo, 1992b) and the unbalanced case (Ye, 1990).

We derive the Berger & Bernardo reference prior (cf. Berger and Bernardo (1992a)) for model (1) (with known $\mathbf{V}$) based on the grouping $\{\mu, \sigma_\lambda\}$. First we show that the Fisher information matrix $\mathbf{F}$ does not depend on μ, and that hence the sought reference prior is given by $\pi(\mu, \sigma_\lambda) \propto \sqrt{\mathbf{F}_{22}}$. We then provide the reference prior in explicit form, and we show propriety of the resulting posterior as well as the existence of mean and variance of the marginal posterior for μ. We will establish a relationship between the corresponding (marginal) reference posterior in the balanced case and the reference posterior obtained for model (1) under the additional assumption of normality. The inferential properties of the resulting posterior are investigated by simulations for several density generators and a particular scenario, and we will report coverage probabilities and mean lengths of 95% credible intervals. In addition, we study the robustness of the inference when distributional assumptions are violated.

The paper is organized as follows. In Section 2, we derive the reference prior for the generalized random effects model (1) and examine properties of the corresponding posterior. We investigate the frequentist properties of the resulting inference in terms of simulations for different distributions and a particular scenario in Section 3. In Section 4, we finally consider as an example the adjustment of measurement results for the Planck constant, and we compare our results to those published in the physical literature (cf. Mohr et al. (2012)). Section 5 presents concluding remarks and possibilities of future research. For ease of notation we will subsequently suppress the dependence of the results on $\mathbf{V}$. Furthermore, the range of integrals is assumed to be $\mathbb{R}^n$ unless indicated otherwise.

2 Reference prior and reference posterior

We start by deriving an explicit expression for the Fisher information matrix for model (1). The information matrix does not depend on μ but only on the density generator $f(\cdot)$, and hence the Berger & Bernardo reference prior for grouping $\{\mu, \sigma_\lambda\}$ follows immediately. We then prove propriety of the resulting posterior, and also the existence of mean and variance of the marginal posterior for μ. Finally, we present a relation between the marginal reference posterior for μ for the normal marginal random effects model and the reference posterior known for the balanced random effects model (2).

Lemma 1. *The Fisher information matrix for model (1) is given by*

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}_{11} & 0 \\ 0 & \mathbf{F}_{22} \end{pmatrix} \tag{4}$$

where

$$\mathbf{F}_{11} = 4 \cdot \mathbf{1}^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1/2} E\left(\mathbf{Z}\mathbf{Z}^T \left(\frac{f'\left(\mathbf{Z}^T\mathbf{Z}\right)}{f\left(\mathbf{Z}^T\mathbf{Z}\right)}\right)^2\right) (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1/2} \mathbf{1},$$

$$\mathbf{F}_{22} = 4\sigma_\lambda^2 \mathrm{tr}\left((\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-2}\right) E\left((Z_1^4 - Z_1^2 Z_2^2)\left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right) +$$

$$\sigma_\lambda^2 \big(\mathrm{tr}\left((\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1}\right)\big)^2 \left(1 + 4E\left(Z_1^2 Z_2^2 \left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right) + 4E\left(Z_1^2 \frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)\right), \quad (5)$$

with $\mathbf{Z} = (Z_1, \ldots, Z_n)^T \sim E_n(\mathbf{0}, \mathbf{I}, f)$.

Proof. Under model (1) the log-likelihood is given by

$$L(\mu, \sigma_\lambda; \mathbf{x}) = -\frac{1}{2} \log(\det(\mathbf{V} + \sigma_\lambda^2 \mathbf{I})) + \log\left(f\left((\mathbf{x} - \mu\mathbf{1})^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1})\right)\right)$$

from which we obtain

$$\frac{\partial L(\mu, \sigma_\lambda; \mathbf{x})}{\partial \mu} = -2 \frac{\mathbf{1}^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1}) f'\left((\mathbf{x} - \mu\mathbf{1})^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1})\right)}{f\left((\mathbf{x} - \mu\mathbf{1})^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1})\right)}$$

and

$$\begin{aligned} \frac{\partial L(\mu, \sigma_\lambda; \mathbf{x})}{\partial \sigma_\lambda} &= -\sigma_\lambda \mathrm{tr}\left((\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1}\right) \\ -\quad & 2\sigma_\lambda (\mathbf{x} - \mu\mathbf{1})^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-2} (\mathbf{x} - \mu\mathbf{1}) \frac{f'\left((\mathbf{x} - \mu\mathbf{1})^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1})\right)}{f\left((\mathbf{x} - \mu\mathbf{1})^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1})\right)}. \end{aligned}$$

Now, it holds that

$$\begin{aligned} & E\left(\frac{\partial L(\mu, \sigma_\lambda; \mathbf{x})}{\partial \mu}\right)^2 = \frac{4}{\sqrt{\det(\mathbf{V} + \sigma_\lambda^2 \mathbf{I})}} \\ \times \quad & \int f\left((\mathbf{x} - \mu\mathbf{1})^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1})\right) \\ \times \quad & \left(\frac{\mathbf{1}^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1}) f'\left((\mathbf{x} - \mu\mathbf{1})^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1})\right)}{f\left((\mathbf{x} - \mu\mathbf{1})^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1} (\mathbf{x} - \mu\mathbf{1})\right)}\right)^2 d\mathbf{x}. \end{aligned}$$

Making the transformation $\mathbf{x} = \mu\mathbf{1} + (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{1/2} \mathbf{y}$ with Jacobian $\sqrt{\det(\mathbf{V} + \sigma_\lambda^2 \mathbf{I})}$, we obtain

$$\mathbf{F}_{11} \quad = \quad E\left(\frac{\partial L(\mu, \sigma_\lambda; \mathbf{x})}{\partial \mu}\right)^2 = 4 \int \frac{\left(\mathbf{1}^T (\mathbf{V} + \sigma_\lambda^2 \mathbf{I})^{-1/2} \mathbf{y} f'\left(\mathbf{y}^T \mathbf{y}\right)\right)^2}{f\left(\mathbf{y}^T \mathbf{y}\right)} d\mathbf{y}$$

$$
\begin{aligned}
&= 4 \cdot \mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1/2} \left(\int \mathbf{y}\mathbf{y}^T \frac{\left(f'\left(\mathbf{y}^T\mathbf{y}\right)\right)^2}{f\left(\mathbf{y}^T\mathbf{y}\right)} \mathrm{d}\mathbf{y} \right) (\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1/2}\mathbf{1} \\
&= 4 \cdot \mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1/2} E\left(\mathbf{Z}\mathbf{Z}^T \left(\frac{f'\left(\mathbf{Z}^T\mathbf{Z}\right)}{f\left(\mathbf{Z}^T\mathbf{Z}\right)} \right)^2 \right) (\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1/2}\mathbf{1},
\end{aligned}
$$

where $\mathbf{Z} \sim E_n(\mathbf{0}, \mathbf{I}, f)$.

Similarly, using the transformation $\mathbf{x} = \mu\mathbf{1} + (\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{1/2}\mathbf{y}$, the relation

$$
\begin{aligned}
\mathbf{F}_{21} &= E\left(\frac{\partial L(\mu, \sigma_\lambda; \mathbf{x})}{\partial \mu} \right) \left(\frac{\partial L(\mu, \sigma_\lambda; \mathbf{x})}{\partial \sigma_\lambda} \right) \\
&= 2\sigma_\lambda \mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1/2} \int h(\mathbf{y})\mathrm{d}\mathbf{y},
\end{aligned}
$$

follows, where

$$
h(\mathbf{y}) = \mathbf{y} f'\left(\mathbf{y}^T\mathbf{y}\right) \left(\mathrm{tr}\left((\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\right) + 2\frac{\mathbf{y}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{y} f'\left(\mathbf{y}^T\mathbf{y}\right)}{f\left(\mathbf{y}^T\mathbf{y}\right)} \right).
$$

Since $h(-\mathbf{y}) = -h(\mathbf{y})$, we get $\mathbf{F}_{21} = 0$.

Finally, using the same transformation, we obtain

$$
\begin{aligned}
\mathbf{F}_{22} &= E\left(\frac{\partial L(\mu, \sigma_\lambda; \mathbf{x})}{\partial \sigma_\lambda} \right)^2 \\
&= \sigma_\lambda^2 \left(\mathrm{tr}\left((\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\right)\right)^2 + 4\sigma_\lambda^2 \int \frac{\left(\mathbf{y}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{y} f'\left(\mathbf{y}^T\mathbf{y}\right)\right)^2}{f\left(\mathbf{y}^T\mathbf{y}\right)} \mathrm{d}\mathbf{y} \\
&+ 4\sigma_\lambda^2 \mathrm{tr}\left((\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\right) \int \mathbf{y}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{y} f'\left(\mathbf{y}^T\mathbf{y}\right) \mathrm{d}\mathbf{y}.
\end{aligned}
$$

Decomposing $\mathbf{V}$ as $\mathbf{V} = \mathbf{H}\mathbf{D}\mathbf{H}^T$, where $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_n)$ is the diagonal matrix of eigenvalues and $\mathbf{H}$ the corresponding orthogonal matrix of eigenvectors, leads to

$$
\int \mathbf{y}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{y} f'\left(\mathbf{y}^T\mathbf{y}\right) \mathrm{d}\mathbf{y} = \sum_{i=1}^{n} \left\{ (d_i + \sigma_\lambda^2)^{-1} \int w_i^2 f'\left(\mathbf{w}^T\mathbf{w}\right) \mathrm{d}\mathbf{w} \right\},
$$

where the last equality is obtained by using the transformation $\mathbf{w} = \mathbf{H}^T\mathbf{y}$. Since

$$
\int w_i^2 f'\left(\mathbf{w}^T\mathbf{w}\right) \mathrm{d}\mathbf{w} = E\left(Z_i^2 \frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})} \right)
$$

does not depend on i or on σ_λ, we get

$$
\int \mathbf{y}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{y} f'\left(\mathbf{y}^T\mathbf{y}\right) \mathrm{d}\mathbf{y} = \mathrm{tr}\left((\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\right) E\left(Z_1^2 \frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})} \right).
$$

Similarly, for the first integral we obtain

$$\begin{aligned}
&\int \frac{\left(\mathbf{y}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{y}f'\left(\mathbf{y}^T\mathbf{y}\right)\right)^2}{f\left(\mathbf{y}^T\mathbf{y}\right)}\mathrm{d}\mathbf{y} \\
= \;& \sum_{i=1}^n\sum_{j=1}^n (d_i+\sigma_\lambda^2)^{-1}(d_j+\sigma_\lambda^2)^{-1}\int w_i^2 w_j^2 \frac{f'\left(\mathbf{w}^T\mathbf{w}\right)^2}{f\left(\mathbf{w}^T\mathbf{w}\right)}\mathrm{d}\mathbf{w} \\
= \;& \sum_{i=1}^n (d_i+\sigma_\lambda^2)^{-2} E\left(Z_1^4\left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right) \\
+ \;& \sum_{i=1}^n\sum_{j=1,j\neq i}^n (d_i+\sigma_\lambda^2)^{-1}(d_j+\sigma_\lambda^2)^{-1} E\left(Z_1^2 Z_2^2\left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right) \\
= \;& \operatorname{tr}\left((\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-2}\right) E\left((Z_1^4 - Z_1^2 Z_2^2)\left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right) \\
+ \;& \left(\operatorname{tr}\left((\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\right)\right)^2 E\left(Z_1^2 Z_2^2\left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right).
\end{aligned}$$

Putting the results for both integrals together completes the proof of the lemma. □

The results of Lemma 1 show that the Fisher information matrix is finite if

$$E\left(Z_1 Z_2\left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right) < \infty, \quad E\left(Z_1^2\left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right) < \infty, \quad E\left(Z_1^2\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right) < \infty,$$
$$E\left(Z_1^4\left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right) < \infty, \quad \text{and} \quad E\left(Z_1^2 Z_2^2\left(\frac{f'(\mathbf{Z}^T\mathbf{Z})}{f(\mathbf{Z}^T\mathbf{Z})}\right)^2\right) < \infty. \tag{6}$$

The conditions in (6) depend only on the density generator $f(\cdot)$, i.e., on the type of the elliptically contoured distribution. Consequently, throughout the paper, we assume that the density generator is chosen such that the expectations in (6) are finite.

The reference prior for the generalized marginal random effects model is generally improper and needs to be determined as the limit of proper priors restricted to a sequence of compact subsets for μ and σ_λ. Since the Fisher information matrix (4) does not depend on μ, and by using a sequence of nested compact subsets of the form $\Omega_\mu^l \times \Omega_{\sigma_\lambda}^l, l = 1, 2, \ldots$, where $\Omega_\mu^1 \subset \Omega_\mu^2 \subset \cdots$ with $\bigcup \Omega_\mu^l = (-\infty, \infty)$, and $\Omega_{\sigma_\lambda}^1 \subset \Omega_{\sigma_\lambda}^2 \subset \cdots$ with $\bigcup \Omega_{\sigma_\lambda}^l = (0, \infty)$, we immediately obtain from the Corollary to Proposition 5.29 in Bernardo and Smith (2000) the Berger & Bernardo reference prior $\pi(\mu, \sigma_\lambda)$ for the generalized marginal random effects model (1) and grouping $\{\mu, \sigma_\lambda\}$ (i.e., with σ_λ as the nuisance parameter) as

$$\pi(\mu, \sigma_\lambda) \propto \sqrt{\mathbf{F}_{22}}, \tag{7}$$

where $\mathbf{F}_{22}$ is given by (5).

Next we show that the conditional reference posterior for μ belongs to the family of elliptically contoured distributions.

Proposition 1. *The conditional reference posterior $\pi(\mu|\sigma_\lambda, \mathbf{x})$ for the generalized marginal random effects model (1) and grouping $\{\mu, \sigma_\lambda\}$ (i.e., with σ_λ as the nuisance parameter) is given by*

$$\pi(\mu|\sigma_\lambda, \mathbf{x}) \quad \propto \quad f_{\sigma_\lambda,\mathbf{x}}\left(\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}\left(\mu - \frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}\right)^2\right),$$

where

$$f_{\sigma_\lambda,\mathbf{x}}(u) = f\left(\mathbf{x}^T\mathbf{R}(\sigma_\lambda)\mathbf{x} + u\right) \qquad u \geq 0, \tag{8}$$

with

$$\mathbf{R}(\sigma_\lambda) = (\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1} - \frac{(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}. \tag{9}$$

Proof. The joint posterior for μ and σ_λ under the generalized marginal random effects model (1) is given by

$$\pi(\mu, \sigma_\lambda|\mathbf{x}) \propto \pi(\mu, \sigma_\lambda)\frac{f\left((\mathbf{x}-\mu\mathbf{1})^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}(\mathbf{x}-\mu\mathbf{1})\right)}{\sqrt{\det(\mathbf{V}+\sigma_\lambda^2\mathbf{I})}}.$$

In using (9) we get

$$\begin{aligned} & (\mathbf{x}-\mu\mathbf{1})^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}(\mathbf{x}-\mu\mathbf{1}) \\ = \quad & \mathbf{x}^T\mathbf{R}(\sigma_\lambda)\mathbf{x} + \mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}\left(\mu - \frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}\right)^2. \end{aligned}$$

Hence,

$$\begin{aligned} & \pi(\mu, \sigma_\lambda|\mathbf{x}) \propto \pi(\mu, \sigma_\lambda) \\ \times \quad & \frac{f\left(\mathbf{x}^T\mathbf{R}(\sigma_\lambda)\mathbf{x} + \mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}\left(\mu - \frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}\right)^2\right)}{\sqrt{\det(\mathbf{V}+\sigma_\lambda^2\mathbf{I})}} \\ = \quad & \frac{\pi(\mu, \sigma_\lambda)}{\sqrt{\det(\mathbf{V}+\sigma_\lambda^2\mathbf{I})}} f_{\sigma_\lambda,\mathbf{x}}\left(\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}\left(\mu - \frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}\right)^2\right), \end{aligned}$$

where $f_{\sigma_\lambda,\mathbf{x}}(\cdot)$ is given in (8). Noting that the reference prior $\pi(\mu, \sigma_\lambda)$ does not depend on μ completes the proof of the proposition. □

Proposition 2. *The marginal posterior $\pi(\sigma_\lambda|\mathbf{x})$ obtained for the reference prior (7) is given by*

$$\pi(\sigma_\lambda|\mathbf{x}) \propto C(\sigma_\lambda)\sqrt{\frac{\mathbf{F}_{22}}{\det(\mathbf{V}+\sigma_\lambda^2\mathbf{I})\,(\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1})}}, \tag{10}$$

where $\mathbf{F}_{22}$ is given in (5) and

$$C(\sigma_\lambda) \quad = \quad \int_{-\infty}^{\infty} f_{\sigma_\lambda,\mathbf{x}}\left(u^2\right) du = \int_{-\infty}^{\infty} f\left(\mathbf{x}^T\mathbf{R}(\sigma_\lambda)\mathbf{x} + u^2\right) du. \tag{11}$$

Proof. This result follows directly from (7), the fact that $\mathbf{F}_{22}$ does not depend on μ, and the last equality in the proof of Proposition 1. $\square$

The result of Proposition 1 shows that the conditional posterior mean for μ given σ_λ belongs to the family of elliptically contoured distributions. Using the properties of the elliptically contoured distributions, we get

$$E\left(\mu|\sigma_\lambda, \mathbf{x}\right) = \frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}$$

and

$$\mathrm{Var}\left(\mu|\sigma_\lambda, \mathbf{x}\right) = \frac{1}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}} E\left(W^2|\sigma_\lambda, \mathbf{x}\right),$$

where $W|\sigma_\lambda, \mathbf{x} \sim E_1(0, 1, cf_{\sigma_\lambda, \mathbf{x}})$, if both quantities exist. As a result, mean and variance of the marginal posterior $\pi(\mu|\mathbf{x})$ can be calculated by the following one-dimensional integrals

$$E\left(\mu|\mathbf{x}\right) = E\left(E\left(\mu|\sigma_\lambda, \mathbf{x}\right)\right) \tag{12}$$

and

$$\mathrm{Var}\left(\mu|\mathbf{x}\right) = E\left(\mathrm{Var}\left(\mu|\sigma_\lambda, \mathbf{x}\right)\right) + \mathrm{Var}\left(E\left(\mu|\sigma_\lambda, \mathbf{x}\right)\right), \tag{13}$$

where the expectation in (12), and the expectation and the variance in (13), are calculated with respect to the marginal posterior $\pi(\sigma_\lambda|\mathbf{x})$ given in (10).

In Theorem 1, we provide conditions on n which ensure propriety of the posterior and also the existence of (12) and (13).

Theorem 1. *The posterior $\pi(\mu, \sigma_\lambda|\mathbf{x})$ obtained for the reference prior from (7) is proper if $n \geq 2$, and for the according marginal posterior $\pi(\mu|\mathbf{x})$ mean or variance exist if $n \geq 3$ or $n \geq 4$, respectively.*

Proof. The application of the joint posterior from the proof of Proposition 1 leads to

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty \pi(\mu, \sigma_\lambda|\mathbf{x}) \mathrm{d}\mu \mathrm{d}\sigma_\lambda \\ \propto \quad &\int_0^\infty \frac{\sqrt{\mathbf{F}_{22}}}{\sqrt{\det(\mathbf{V}+\sigma_\lambda^2\mathbf{I})}\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} C(\sigma_\lambda) \mathrm{d}\sigma_\lambda, \end{aligned}$$

where $C(\sigma_\lambda)$ is given in (11). First, we note that no singularity is present at $\sigma_\lambda = 0$ and that (cf., Lemma 9 in Gómez et al. (2003))

$$C(\sigma_\lambda = 0) \leq \int_0^\infty f(u) \mathrm{d}u < \infty.$$

At infinity we get

$$\frac{\sqrt{\mathbf{F}_{22}}}{\sqrt{\det(\mathbf{V}+\sigma_\lambda^2\mathbf{I})}\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} \approx \sigma_\lambda^{-n}$$

and

$$\lim_{\sigma_\lambda\to\infty} C(\sigma_\lambda) = \int_{-\infty}^{\infty} \lim_{\sigma_\lambda\to\infty} f\left(\mathbf{x}^T\mathbf{R}(\sigma_\lambda)\mathbf{x}+u^2\right)\mathrm{d}u = \int_{-\infty}^{\infty} f\left(u^2\right)\mathrm{d}u < \infty.$$

Hence,

$$\pi(\sigma_\lambda|\mathbf{x}) = O(\sigma_\lambda^{-n}) \tag{14}$$

as $\sigma_\lambda \to \infty$, and the posterior is proper if and only if $n \geq 2$.

For the mean of the marginal posterior $\pi(\mu|\mathbf{x})$, we first note that (cf. Proposition 1)

$$\mu|\sigma_\lambda,\mathbf{x} \sim E_1\left(\frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}, \frac{1}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}, cf_{\sigma_\lambda,\mathbf{x}}\right),$$

and thus

$$\mu|\sigma_\lambda,\mathbf{x} \overset{d}{=} \frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}} + \frac{1}{\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} W|\sigma_\lambda,\mathbf{x},$$

where $W|\sigma_\lambda,\mathbf{x} \sim E_1\left(0,1,cf_{\sigma_\lambda,\mathbf{x}}\right)$ and the symbol $\overset{d}{=}$ denotes equality in distribution. Hence,

$$\begin{aligned} & E(\mu|\mathbf{x}) = E\left(E(\mu|\sigma_\lambda,\mathbf{x})\right) \\ = \quad & E\left(\frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}\right) && (15) \\ + \quad & E\left(\frac{1}{\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} E(W|\sigma_\lambda,\mathbf{x})\right). && (16) \end{aligned}$$

Consequently, the mean of the marginal posterior $\pi(\mu|\mathbf{x})$ exists if and only if the expectations (15) and (16) exist. Because no singularity is present at zero, $f_{\sigma_\lambda,\mathbf{x}}(\cdot) \to f(\cdot)$ as $\sigma_\lambda \to \infty$ (see the proof for propriety), and since also

$$\frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}} \approx \frac{\mathbf{1}^T\mathbf{x}}{n}$$

holds as $\sigma_\lambda \to \infty$, the expectation in (15) exists if and only if $n \geq 2$. Furthermore, as

$$\begin{aligned} & E\left(\frac{1}{\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} E(W|\sigma_\lambda,\mathbf{x})\right) \\ = \quad & \int_0^\infty \frac{1}{\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} \pi(\sigma_\lambda|\mathbf{x}) \int_{-\infty}^{\infty} w f_{\sigma_\lambda,\mathbf{x}}(w^2)\mathrm{d}w\mathrm{d}\sigma_\lambda \\ \to \quad & \mathcal{I}_1 + \int_{\sigma_\lambda^{(\infty)}}^\infty \frac{1}{\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} \pi(\sigma_\lambda|\mathbf{x}) \int_{-\infty}^{\infty} w f(w^2)\mathrm{d}w\mathrm{d}\sigma_\lambda \\ = \quad & \mathcal{I}_1 + \left(\int_{-\infty}^{\infty} w f(w^2)\mathrm{d}w\right) \int_{\sigma_\lambda^{(\infty)}}^\infty \frac{1}{\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} \pi(\sigma_\lambda|\mathbf{x},)\mathrm{d}\sigma_\lambda, \end{aligned}$$

for significantly large $\sigma_\lambda^{(\infty)}$ with $\mathcal{I}_1 = \int_0^{\sigma_\lambda^{(\infty)}} \frac{\pi(\sigma_\lambda|\mathbf{x})}{\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} \int_{-\infty}^{\infty} w f_{\sigma_\lambda,\mathbf{x}}(w^2) \mathrm{d}w \mathrm{d}\sigma_\lambda < \infty$, and because

$$\frac{1}{\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} \approx \frac{\sigma_\lambda}{\sqrt{n}}$$

holds for $\sigma_\lambda \to \infty$ and $\pi(\sigma_\lambda|\mathbf{x}) = O(\sigma_\lambda^{-n})$ asymptotically (cf. (14)), we get that the expectation in (16) exists if and only if $n \geq 3$, i.e., $n \geq 3$ ensures that the mean of the marginal posterior $\pi(\mu|\mathbf{x})$ exists.

Finally, using the expression for the variance of the marginal posterior $\pi(\mu|\mathbf{x})$ and performing the same analysis, we conclude that the variance exists if

$$E\left(\frac{1}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}} E(W^2|\sigma_\lambda,\mathbf{x})\right) < \infty,$$

which is true if and only if $n \geq 4$. □

Subsequently, we provide some further explicit results under the additional assumption of normality, i.e., we assume $f(u) = \exp(-u/2)/(2\pi)^{n/2}$.

Theorem 2. *The Berger & Bernardo reference prior $\pi(\mu, \sigma_\lambda)$ for the normal random effects model (i.e., model (1) with $f(u) = \exp(-u/2)/(2\pi)^{n/2}$) and grouping $\{\mu, \sigma_\lambda\}$ (i.e., with σ_λ as the nuisance parameter) is given by*

$$\pi(\mu, \sigma_\lambda) \propto \sqrt{\sigma_\lambda^2 \cdot \operatorname{tr}\left((\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-2}\right)}.$$

Proof. For the normal marginal random effects model with $f(u) = \exp(-u/2)/(2\pi)^{n/2}$ we get $f'(u) = -f(u)/2$, $E(\mathbf{Z}\mathbf{Z}^T) = \mathbf{I}$, $E(Z_1^4) = 3$, $E(Z_1^2 Z_2^2) = E(Z_1^2)E(Z_2^2) = 1$. Application of Lemma 1 yields

$$\mathbf{F} = \begin{pmatrix} \mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1} & 0 \\ 0 & 2\sigma_\lambda^2 \operatorname{tr}((\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-2}) \end{pmatrix},$$

and using (7) then completes the proof. □

We note that this prior has already been used (but not derived) in Toman et al. (2012). From Proposition 1 we immediately get that the conditional reference posterior for μ is normally distributed

$$\mu|\sigma_\lambda,\mathbf{x} \sim N\left(\frac{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}, \frac{1}{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}\right), \tag{17}$$

and from Proposition 2 (together with Theorem 2) we obtain the marginal posterior $\pi(\sigma_\lambda|\mathbf{x})$ as

$$\pi(\sigma_\lambda|\mathbf{x}) \propto \frac{\sqrt{\sigma_\lambda^2 \cdot \operatorname{tr}\left((\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-2}\right)}}{\sqrt{\det(\mathbf{V}+\sigma_\lambda^2\mathbf{I})}\sqrt{\mathbf{1}^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}} \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{R}(\sigma_\lambda)\mathbf{x}\right), \tag{18}$$

where $\mathbf{R}(\sigma_\lambda)$ is defined in Proposition 1.

We note that for reasons of stability the term $\mathbf{x}^T\mathbf{R}(\sigma_\lambda)\mathbf{x}$ in (18) ought to be evaluated in numerical calculations as

$$\mathbf{x}^T\mathbf{R}(\sigma_\lambda)\mathbf{x} = \min_{\mu} \tilde{\chi}^2(\mu) = \tilde{\chi}^2(\breve{\mu})$$

where

$$\tilde{\chi}^2(\mu) = (\mathbf{x} - \mu\mathbf{1})^T \left(\mathbf{V} + \sigma_\lambda^2\mathbf{I}\right)^{-1} (\mathbf{x} - \mu\mathbf{1})$$

and

$$\breve{\mu} = \frac{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}.$$

Unfortunately, no closed expression is available for the posterior $\pi(\mu|\mathbf{x})$, and numerical means have to be applied. Markov chain Monte Carlo methods (cf. Robert and Casella (2004)) may be used or, since only two parameters are involved in our problem, numerical integration (see, e.g., Evans and Swartz (2000)). The results reported in Sections 3 and 4 were obtained by the latter approach.

The explicit formula for the marginal reference posterior $\pi(\sigma_\lambda|\mathbf{x})$ given in (18), together with the conditional posterior $\pi(\mu|\sigma_\lambda, \mathbf{x})$ from (17), can be utilized in the numerical calculation of marginal posterior mean and standard deviation,

$$E\left(\mu|\mathbf{x}\right) = \int_0^\infty \frac{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}} \pi(\sigma_\lambda|\mathbf{x})\mathrm{d}\sigma_\lambda,$$

and

$$\mathrm{Var}\left(\mu|\mathbf{x}\right) =$$
$$\int_0^\infty \left\{ \frac{1}{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}} + \left(E\mu|\mathbf{x} - \frac{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}} \right)^2 \right\} \pi(\sigma_\lambda|\mathbf{x})\mathrm{d}\sigma_\lambda.$$

A shortest 95% credible interval can be obtained by minimizing over $\beta \in (0, 0.05)$ the length of the interval

$$[a_{0.05-\beta}, a_{1-\beta}],$$

where a_γ is the solution of

$$\gamma = \int_0^\infty \Phi\left(a_\gamma; \frac{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{x}}{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}, \frac{1}{\mathbf{1}^T(\mathbf{V} + \sigma_\lambda^2\mathbf{I})^{-1}\mathbf{1}}\right) \pi(\sigma_\lambda|\mathbf{x})\mathrm{d}\sigma_\lambda. \tag{19}$$

In (19), the symbol $\Phi(y; a, b^2)$ denotes the distribution function of the normal distribution with mean a and variance b^2 at y.

We finally note that the objective Bayesian inference obtained for the normal marginal random effects model is related to that obtained for the customary random effects model (3) when using the corresponding reference prior given in Berger and Bernardo (1992b) according to

Theorem 3. *Consider the balanced random effects model M_{brem} (3) with $n_1 = n_2 = \cdots = n_n =: n_0$ and the normal marginal random effects model M_{nmrem} (i.e., model (1) with $f(u) = \exp(-u/2)/(2\pi)^{n/2}$) where $\mathbf{V} = \frac{\sigma_0^2}{n_0}\mathbf{I}$ with known σ_0. Then asymptotically as $n_0 \to \infty$ the posterior $\pi(\mu, \sigma_\lambda | \mathbf{x}, M_{nmrem})$ obtained for the reference prior from Theorem 2 coincides with the marginal posterior $\pi(\mu, \sigma_\lambda | \mathbf{x}, M_{brem})$ obtained for the reference prior $\pi(\mu, \sigma_\lambda, \sigma | M_{brem}) \propto \sigma_\lambda \sigma^{-1}(n_0\sigma_\lambda^2 + \sigma^2)^{-1}$ (derived in Berger and Bernardo (1992b) for the grouping $\{\mu, (\sigma_\lambda, \sigma)\}$) and model M_{brem} (3), provided that the underlying variance σ^2 in M_{brem} (3) equals σ_0^2.*

Proof. Using $\mathbf{x}_i = (x_{i1}, \ldots, x_{in_0})^T$ the likelihood under model M_{brem} in (3) is

$$l(\mu, \sigma_\lambda, \sigma; \mathbf{x}_1, \ldots, \mathbf{x}_n, M_{brem}) \propto \prod_{i=1}^{n} \frac{\exp\left[-\frac{1}{2}(\mathbf{x}_i - \mu\mathbf{1})^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2 \mathbf{1}\mathbf{1}^T\right)^{-1} (\mathbf{x}_i - \mu\mathbf{1})\right]}{\sqrt{\det\left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)}}.$$

From

$$\begin{aligned}
(\mathbf{x}_i - \mu\mathbf{1})^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} (\mathbf{x}_i - \mu\mathbf{1}) \quad &= \\
(\mu - \hat{\mu})^2 \left(\mathbf{1}^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} \mathbf{1}\right) \quad &+ \quad \mathbf{x}_i^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} \mathbf{x}_i \\
&- \quad \hat{\mu}^2 \left(\mathbf{1}^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} \mathbf{1}\right),
\end{aligned}$$

where

$$\hat{\mu} = \frac{\mathbf{1}^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} \mathbf{x}_i}{\mathbf{1}^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} \mathbf{1}},$$

together with

$$\left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} = \sigma^{-2}\mathbf{I} - \frac{\sigma_\lambda^2/\sigma^2}{(n_0\sigma_\lambda^2 + \sigma^2)}\mathbf{1}\mathbf{1}^T,$$

we immediately observe that

$$\hat{\mu} = \frac{1}{n_0}\mathbf{1}^T\mathbf{x}_i = \overline{\mathbf{x}}_i$$

holds, as well as

$$\mathbf{1}^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} \mathbf{1} = \left(\sigma^2/n_0 + \sigma_\lambda^2\right)^{-1}.$$

From

$$\mathbf{x}_i^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} \mathbf{x}_i - \hat{\mu}^2 \left(\mathbf{1}^T \left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right)^{-1} \mathbf{1}\right) = \frac{1}{\sigma^2} \sum_{j=1}^{n_0} (x_{ij} - \overline{\mathbf{x}}_i)^2,$$

and

$$\det\left(\sigma^2\mathbf{I} + \sigma_\lambda^2\mathbf{1}\mathbf{1}^T\right) = \left(\sigma^2\right)^{n_0} \left(\sigma^2/n_0 + \sigma_\lambda^2\right) \left(\sigma^2/n_0\right)^{-1},$$

we thus get

$$
\begin{aligned}
l(\mu,\sigma_\lambda,\sigma;\mathbf{x}_1,\ldots,\mathbf{x}_n,M_{brem}) &\propto \prod_{i=1}^{n}\left\{\frac{\exp\left(-\frac{1}{2}\frac{(\mu-\overline{\mathbf{x}}_i)^2}{(\sigma^2/n_0+\sigma_\lambda^2)}\right)}{\sqrt{\sigma^2/n_0+\sigma_\lambda^2}}\right.\\
&\times \left.\frac{\exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^{n_0}\left(x_{ij}-\overline{\mathbf{x}}_i\right)^2\right)}{\sigma^{(n_0-1)}}\right\}\\
&\propto \prod_{i=1}^{n}\left\{\frac{\exp\left(-\frac{1}{2}\frac{(\mu-\overline{\mathbf{x}}_i)^2}{(\sigma^2/n_0+\sigma_\lambda^2)}\right)}{\sqrt{\sigma^2/n_0+\sigma_\lambda^2}}\right\}\times\frac{\exp\left(-\frac{n_0}{2\sigma^2}u^2\right)}{(\sigma/\sqrt{n_0})^{n(n_0-1)}},
\end{aligned}
$$

with $u^2=\frac{1}{n_0}\sum_{i=1}^{n}\sum_{j=1}^{n_0}\left(x_{ij}-\overline{\mathbf{x}}_i\right)^2$.

Let $\tilde{\sigma}=\sigma/\sqrt{n_0}$. The posterior for $\mu,\sigma_\lambda,\tilde{\sigma}$ is given by

$$
\begin{aligned}
&p(\mu,\sigma_\lambda,\tilde{\sigma}|\mathbf{x}_1,\ldots,\mathbf{x}_n,M_{brem})\\
\propto\ &\prod_{i=1}^{n}\left\{\frac{\exp\left(-\frac{1}{2}\frac{(\mu-\overline{\mathbf{x}}_i)^2}{(\tilde{\sigma}^2+\sigma_\lambda^2)}\right)}{\sqrt{\tilde{\sigma}^2+\sigma_\lambda^2}}\right\}\times\frac{\exp\left(-\frac{1}{2\tilde{\sigma}^2}u^2\right)}{\tilde{\sigma}^{n(n_0-1)}}\sigma_\lambda\tilde{\sigma}^{-1}(\sigma_\lambda^2+\tilde{\sigma}^2)^{-1}\\
=\ &\prod_{i=1}^{n}\left\{\frac{\exp\left(-\frac{1}{2}\frac{(\mu-\overline{\mathbf{x}}_i)^2}{(\tilde{\sigma}^2+\sigma_\lambda^2)}\right)}{\sqrt{\tilde{\sigma}^2+\sigma_\lambda^2}}\right\}\times\frac{\sigma_\lambda}{\sigma_\lambda^2+\tilde{\sigma}^2}\frac{\exp\left(-\frac{1}{2\tilde{\sigma}^2}u^2\right)}{\tilde{\sigma}^{n(n_0-1)+1}},
\end{aligned}
$$

which leads to

$$
p(\mu,\sigma_\lambda|\mathbf{x}_1,\ldots,\mathbf{x}_n,M_{brem}) \propto \int_0^\infty\prod_{i=1}^{n}\left\{\frac{\exp\left(-\frac{1}{2}\frac{(\mu-\overline{\mathbf{x}}_i)^2}{(\tilde{\sigma}^2+\sigma_\lambda^2)}\right)}{\sqrt{\tilde{\sigma}^2+\sigma_\lambda^2}}\right\}\times\frac{\sigma_\lambda}{\sigma_\lambda^2+\tilde{\sigma}^2}\delta_{n_0}(\tilde{\sigma})\mathrm{d}\tilde{\sigma},
$$

where

$$
\delta_{n_0}(\tilde{\sigma})\propto\frac{\exp\left(-\frac{1}{2\tilde{\sigma}^2}u^2\right)}{\tilde{\sigma}^{n(n_0-1)+1}}
$$

is a sequence of distributions whose variance tends to zero and which are asymptotically concentrated at $\tilde{\sigma}=u/\sqrt{nn_0}$. Since u^2 is consistent for $n\sigma_0^2$, it follows that asymptotically the distributions $\delta_{n_0}(\tilde{\sigma})$ have support only at $\sigma_0/\sqrt{n_0}$, and thus

$$
\begin{aligned}
&\lim_{n_0\to\infty}p(\mu,\sigma_\lambda|\mathbf{x}_1,\ldots,\mathbf{x}_n,M_{brem})\\
\propto\ &\lim_{n_0\to\infty}\int_0^\infty\prod_{i=1}^{n}\left\{\frac{\exp\left(-\frac{1}{2}\frac{(\mu-\overline{\mathbf{x}}_i)^2}{(\tilde{\sigma}^2+\sigma_\lambda^2)}\right)}{\sqrt{\tilde{\sigma}^2+\sigma_\lambda^2}}\right\}\times\frac{\sigma_\lambda}{\sigma_\lambda^2+\tilde{\sigma}^2}\delta_{n_0}(\tilde{\sigma})\mathrm{d}\tilde{\sigma}\\
=\ &\sigma_\lambda(v^2+\sigma_\lambda^2)^{-1}\prod_{i=1}^{n}\frac{\exp\left(-\frac{(\mu-\overline{\mathbf{x}}_i)^2}{v^2+\sigma_\lambda^2}\right)}{\sqrt{v^2+\sigma_\lambda^2}},
\end{aligned}
\tag{20}
$$

where $v^2=\sigma_0^2/n_0$.

But (20) is just the same as the reference posterior for model (2) when using the reference prior from Theorem 2. □

We note that Theorem 3 holds when the uncertainty with a single observation in the marginal random effects model decreases with increasing n_0. The same applies to the uncertainty associated with the mean $\overline{X}_i = \sum_{j=1}^{n_0} X_{ij}/n_0$ of the repeated observations in the balanced random effects model (2). As $n_0 \to \infty$ the unknown variance σ^2 becomes known, and so the means $\overline{X}_i = \sum_j X_{ij}/n_0$ follow the normal marginal random effects model. Theorem 3 ensures that this is reflected by the corresponding references analyses.

3 Simulation study

The frequentist properties of the reference posterior for the generalized marginal random effects model (1) are investigated in terms of simulations for several density generators $f(\cdot)$ and a particular scenario. We also explore the robustness of results when the assumption about the density generator is violated. We focus on coverage probabilities and mean lengths of shortest 95% credible intervals. The settings chosen for the simulations are motivated by applications in metrology.

Without loss of generality, $\mu = 0$ and $\sigma_\lambda = 1$ were used throughout. Two different values of n were considered, namely $n = 11$ and $n = 22$, and the matrix $\mathbf{V}$ was taken to be of autoregressive structure, i.e., $\mathbf{V} = \mathbf{U}^{1/2}\boldsymbol{\Omega}\mathbf{U}^{1/2}$ with $\mathbf{U} = \mathrm{diag}(u_1^2, \ldots, u_n^2)$ and $\boldsymbol{\Omega} = (\rho^{|i-j|})_{i,j=1,\ldots,n}$. In order to capture different situations, the u_i were chosen differently for each single simulated data set. Specifically, the u_i were drawn randomly from a uniform distribution on the interval $[0.01, 0.5]$. Several values of ρ were used, namely $\{-0.9, -0.6, -0.3, 0, 0.3, 0.6, 0.9\}$, and the following three density generators considered:

(i) Normal marginal random effects model (1) with $f(u) \propto \exp(-u/2)$;

(ii) Rescaled t_3 marginal random effects model (1) with[1] $f(u) \propto (1+u/(d-2))^{-(n+d)/2}$;

(iii) Laplace marginal random effects model (1) with (cf. Eltoft et al. (2006))

$$\begin{aligned} f(u) &= \frac{1}{(2\pi)^{n/2}} \int_0^\infty z^{-n/2} \exp\left(-\frac{u}{2z} - z\right) \mathrm{d}z \\ &\propto u^{-n/4+1/2} K_{n/2-1}(\sqrt{2u}), \end{aligned}$$

where $K_\alpha(x)$ denotes the modified Bessel function of the second kind (see, e.g., Andrews et al. (2000)).

[1] Since the covariance matrix of a random vector which has a t-distribution with d degrees of freedom is equal to $\frac{d}{d-2}(\mathbf{V} + \sigma_\lambda^2\mathbf{I})$, we adjust the samples from t-distribution by the factor $\sqrt{\frac{d-2}{d}}$ in order to ensure that the covariance matrices are the same in all scenarios. For Scenarios (i) and (iii), the covariance matrix is equal to the dispersion matrix $(\mathbf{V} + \sigma_\lambda^2\mathbf{I})$.

True/Fitted		Normal		t_3		Laplace	
		Coverage	Length	Coverage	Length	Coverage	Length
Normal	$\rho = -0.9$	0.947	1.323 ± 0.005	0.948	1.327 ± 0.005	0.953	1.309 ± 0.004
	$\rho = -0.6$	0.950	1.337 ± 0.004	0.951	1.337 ± 0.004	0.950	1.315 ± 0.005
	$\rho = -0.3$	0.952	1.369 ± 0.004	0.950	1.363 ± 0.005	0.951	1.325 ± 0.005
	$\rho = 0.0$	0.950	1.373 ± 0.004	0.947	1.366 ± 0.005	0.951	1.339 ± 0.004
	$\rho = 0.3$	0.948	1.354 ± 0.004	0.948	1.350 ± 0.004	0.952	1.364 ± 0.004
	$\rho = 0.6$	0.954	1.457 ± 0.004	0.952	1.450 ± 0.004	0.954	1.414 ± 0.004
	$\rho = 0.9$	0.956	1.479 ± 0.004	0.951	1.471 ± 0.004	0.957	1.534 ± 0.004
		Coverage	Length	Coverage	Length	Coverage	Length
t_3	$\rho = -0.9$	0.948	1.057 ± 0.012	0.948	1.055 ± 0.012	0.942	1.043 ± 0.011
	$\rho = -0.6$	0.944	1.070 ± 0.012	0.945	1.070 ± 0.012	0.940	1.044 ± 0.011
	$\rho = -0.3$	0.952	1.068 ± 0.012	0.954	1.065 ± 0.012	0.941	1.053 ± 0.011
	$\rho = 0.0$	0.945	1.070 ± 0.010	0.945	1.065 ± 0.010	0.941	1.069 ± 0.011
	$\rho = 0.3$	0.948	1.106 ± 0.011	0.947	1.093 ± 0.011	0.946	1.100 ± 0.011
	$\rho = 0.6$	0.953	1.244 ± 0.012	0.952	1.213 ± 0.012	0.952	1.156 ± 0.011
	$\rho = 0.9$	0.956	1.248 ± 0.010	0.953	1.216 ± 0.011	0.957	1.274 ± 0.011
		Coverage	Length	Coverage	Length	Coverage	Length
Laplace	$\rho = -0.9$	0.952	1.195 ± 0.010	0.952	1.195 ± 0.010	0.939	1.167 ± 0.010
	$\rho = -0.6$	0.948	1.195 ± 0.010	0.946	1.188 ± 0.010	0.934	1.171 ± 0.010
	$\rho = -0.3$	0.953	1.203 ± 0.010	0.952	1.198 ± 0.010	0.933	1.179 ± 0.010
	$\rho = 0.0$	0.947	1.220 ± 0.010	0.944	1.121 ± 0.010	0.936	1.194 ± 0.010
	$\rho = 0.3$	0.955	1.268 ± 0.010	0.951	1.248 ± 0.010	0.940	1.221 ± 0.010
	$\rho = 0.6$	0.956	1.230 ± 0.010	0.953	1.221 ± 0.010	0.945	1.270 ± 0.010
	$\rho = 0.9$	0.957	1.342 ± 0.009	0.954	1.322 ± 0.009	0.946	1.380 ± 0.009

Table 1: Coverage probabilities and mean lengths of 95% (shortest) credible intervals for $n = 11$. Each row refers to a particular scenario and has been analyzed in turn by assuming all three density generators. An upper bound on the standard error of the reported coverages is 0.007.

Scenario (i) presents the most famous elliptical model which is used in many applications in metrology. In contrast, both models from scenarios (ii) and (iii) correspond to heavy-tailed elliptically contoured distributions.

For all three scenarios coverage probabilities and mean lengths of credible intervals were determined. Note that since the u_i and hence the matrix $\mathbf{V}$ are varied for each simulated data set the reported coverage probabilities are average coverage probabilities where the average refers to different situations. Each single data set was analyzed using (in turn) the density generator of all three scenarios. In this way, the robustness of the analyses w.r.t. the distributional assumption is investigated. For each scenario (and each chosen correlation ρ) 5.000 data sets were drawn and analyzed. Tables 1 and 2 contain the corresponding results. For each reported mean length we state the standard deviation of the corresponding 5.000 samples divided by $\sqrt{5.000}$; for the coverage probability we give an upper bound of the standard deviation of the estimates.

For all considered scenarios and choices of ρ coverage probabilities are close to 95%, and similar mean lengths of 95% credible intervals are observed when the density generator is changed in the analysis. Hence, the results are robust against a mis-specification of the underlying distribution. Interestingly, coverage probabilities for credible intervals obtained under the assumption of the Laplace distribution are slightly smaller than those calculated under the normal distribution and the t-distribution when the true model is Scenario (iii) and $n = 11$. This difference becomes negligible if n increases (see Table 2). Credible intervals calculated in the case of positive correlations are larger than those obtained for negative correlations. For $n = 22$ (Table 2) results are similar to those for $n = 11$ (Table 1). A difference is observed in the mean lengths of 95% credible intervals which are for $n = 11$ larger by a factor of about $\sqrt{2}$, which is expected since the sample size is doubled.

True/Fitted		Normal		t_3		Laplace	
		Coverage	Length	Coverage	Length	Coverage	Length
Normal	$\rho = -0.9$	0.951	0.884 ± 0.002	0.951	0.887 ± 0.002	0.944	0.880 ± 0.002
	$\rho = -0.6$	0.948	0.900 ± 0.002	0.947	0.901 ± 0.002	0.945	0.885 ± 0.002
	$\rho = -0.3$	0.954	0.895 ± 0.002	0.953	0.895 ± 0.002	0.945	0.893 ± 0.002
	$\rho = 0.0$	0.950	0.914 ± 0.002	0.949	0.912 ± 0.002	0.946	0.904 ± 0.002
	$\rho = 0.3$	0.950	0.927 ± 0.002	0.949	0.924 ± 0.002	0.945	0.924 ± 0.002
	$\rho = 0.6$	0.955	0.965 ± 0.002	0.952	0.959 ± 0.002	0.947	0.966 ± 0.002
	$\rho = 0.9$	0.951	1.145 ± 0.002	0.941	1.120 ± 0.002	0.947	1.102 ± 0.002
		Coverage	Length	Coverage	Length	Coverage	Length
t_3	$\rho = -0.9$	0.944	0.710 ± 0.009	0.944	0.712 ± 0.009	0.939	0.700 ± 0.009
	$\rho = -0.6$	0.945	0.703 ± 0.007	0.946	0.705 ± 0.007	0.938	0.704 ± 0.010
	$\rho = -0.3$	0.948	0.723 ± 0.007	0.947	0.721 ± 0.007	0.939	0.710 ± 0.009
	$\rho = 0.0$	0.947	0.739 ± 0.008	0.945	0.734 ± 0.008	0.940	0.721 ± 0.009
	$\rho = 0.3$	0.955	0.761 ± 0.010	0.951	0.752 ± 0.010	0.945	0.744 ± 0.009
	$\rho = 0.6$	0.954	0.763 ± 0.007	0.951	0.749 ± 0.007	0.951	0.788 ± 0.009
	$\rho = 0.9$	0.965	1.018 ± 0.007	0.957	0.966 ± 0.007	0.952	0.908 ± 0.009
		Coverage	Length	Coverage	Length	Coverage	Length
Laplace	$\rho = -0.9$	0.949	0.779 ± 0.006	0.949	0.780 ± 0.006	0.945	0.766 ± 0.006
	$\rho = -0.6$	0.950	0.798 ± 0.006	0.950	0.798 ± 0.006	0.944	0.771 ± 0.006
	$\rho = -0.3$	0.951	0.801 ± 0.006	0.950	0.798 ± 0.006	0.940	0.777 ± 0.006
	$\rho = 0.0$	0.949	0.804 ± 0.006	0.949	0.800 ± 0.006	0.942	0.789 ± 0.006
	$\rho = 0.3$	0.948	0.819 ± 0.006	0.948	0.813 ± 0.006	0.948	0.809 ± 0.006
	$\rho = 0.6$	0.959	0.841 ± 0.006	0.958	0.830 ± 0.006	0.951	0.851 ± 0.006
	$\rho = 0.9$	0.948	1.011 ± 0.006	0.948	0.980 ± 0.006	0.949	0.968 ± 0.006

Table 2: Coverage probabilities and mean lengths of 95% (shortest) credible intervals for $n = 22$. Each row refers to a particular scenario and has been analyzed in turn by assuming all three density generators. An upper bound on the standard error of the reported coverages is 0.007.

The fact that the reference posterior is not much affected by the density generator (chosen for the analysis) is similar to the situation met for a general location–scale model such as $p(\mathbf{x}|\mu, \sigma_B) = f((\mathbf{x} - \mu\mathbf{1})^T(\sigma_B^2\mathbf{V})^{-1}(\mathbf{x} - \mu\mathbf{1}))/\sqrt{\det(\sigma_B^2\mathbf{V})}$. Fernández and Steel (1999) have derived the reference prior for such models, and Arellano-Valle et al. (2006) as well as Osiewalski and Steel (1993) have given results which show that the posterior for such a general location–scale model does not depend on the density generator $f(\cdot)$.[2]

4 Adjustment of measurements for the Planck constant

As an application we consider the adjustment of measurements for the Planck constant. Table 3 and Figure 1 show the measurement results from Table XXVI in Mohr et al. (2012). The data are estimates of the Planck constant and the goal is to derive an improved estimate by combining these data. An accurate estimate of the Planck constant is required in order to re-define the kilogram (cf. Mills et al. (2006)), and therefore the reliability of the uncertainty quoted for a combined estimate is important. In Mohr et al. (2012), the data have already been analyzed and we show the according results in Table 4.

The data in Table 3 appear to be inconsistent w.r.t. quoted uncertainties,[3] cf. Toman et al. (2012). We applied the generalized marginal random effects model (2) to model the data and to infer μ. As density generator we used a normal distribution, a t-distribution

[2]We are grateful to a referee for pointing out this fact.

[3]The matrix $\mathbf{V}$ has been taken as follows: the diagonal elements are the squared quoted standard uncertainties, accompanied with four non-zero off-diagonal elements $V_{ij} = \rho_{ij}u_iu_j$ with correlations ρ_{ij} that were reported (and accounted for) in Mohr et al. (2012), see Table 3.

Identification	$h/(Js)$	Relative standard uncertainty
NPL-79	$6.6260730 \times 10^{-34}$	1.0×10^{-6}
NIST-80	$6.6260657 \times 10^{-34}$	1.3×10^{-6}
NMI-89	$6.6260684 \times 10^{-34}$	5.4×10^{-7}
NPL-90	$6.6260682 \times 10^{-34}$	2.0×10^{-7}
PTB-91	$6.6260670 \times 10^{-34}$	6.3×10^{-7}
NIM-95	6.626071×10^{-34}	1.6×10^{-6}
NIST-98	$6.62606891 \times 10^{-34}$	8.7×10^{-8}
NIST-07	$6.62606891 \times 10^{-34}$	3.6×10^{-8}
METAS-11	$6.6260691 \times 10^{-34}$	2.9×10^{-7}
IAC-11	$6.62607009 \times 10^{-34}$	3.0×10^{-8}
NPL-12	$6.6260712 \times 10^{-34}$	2.0×10^{-7}

Table 3: Values for the Planck constant from Table XXVI in Mohr et al. (2012) sorted according to the time of measurement. In Table XXI of Mohr et al. (2012), also the correlations, $r(\text{NIST-98}, \text{NIST-07}) = 0.14$ and $r(\text{NPL-90}, \text{NPL-12}) = 0.003$, were given.

Method	$\hat{\mu} \times 10^{34}$ Js	$u(\hat{\mu})/\hat{\mu} \times 10^8$	$(\text{CI} - \hat{\mu})/\hat{\mu} \times 10^8$
GMREM (normal distribution)	6.6260694	7.2	[−14.9,13.8]
GMREM (t-distribution)	6.6260693	6.6	[−13.7,12.8]
GMREM (Laplace distribution)	6.6260693	6.6	[−13.5,13.6]
Codata2010	6.6260696	4.4	–

Table 4: Estimates $\hat{\mu}$ for the Planck constant obtained from the data from Table 3, together with relative posterior standard deviations and 'relative' 95% credible intervals obtained for the generalized marginal random effects model (GMREM) using a normal distribution, a (rescaled) t-distribution with 3 degrees of freedom, and a Laplace distribution. In addition, the CODATA 2010 results are given, where the CODATA 2010 estimate has been rounded in accordance with its uncertainty.

with 3 degrees of freedom, and a Laplace distribution. The results[4] are given in Table 4 and illustrated in Figure 1. Figure 2 shows the marginal posteriors for μ obtained for the three density generators.

The estimates obtained for the generalized marginal random effects models and the three density generators are similar, and they are consistent with the result given in Mohr et al. (2012). However, the standard uncertainties obtained for the generalized marginal random effects models are significantly larger than the standard uncertainty quoted for the Codata 2010 result, with the results obtained for the normal marginal random effects model being most conservative. Since the reliability in the uncertainty quoted for the Planck constant is important, we would rather recommend the results obtained by the normal marginal random effects model than those published in the physical literature.

[4]The data of Table 3 including one further result have already been analyzed in Toman et al. (2012) using the normal marginal random effects model with a diagonal matrix $\mathbf{V} = \text{diag}(u_1^2, \ldots, u_n^2)$.

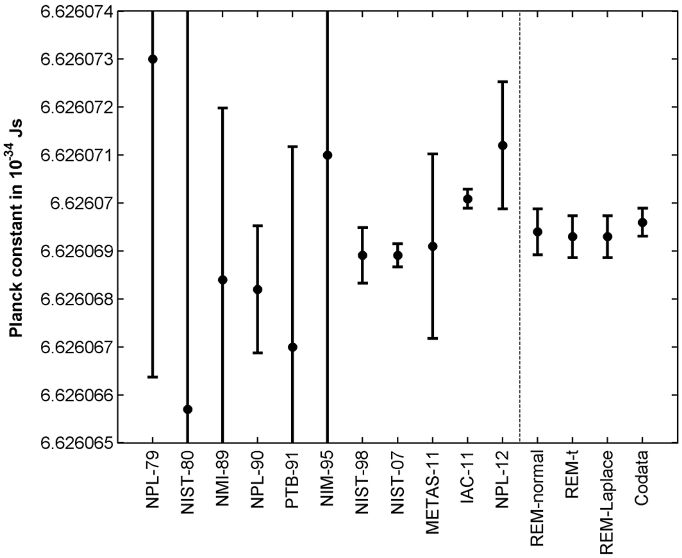

Figure 1: The data for the Planck constant from Table 3 together with estimates obtained by the generalized marginal random effects model (with three different density generators) as well as the adjusted value given in Mohr et al. (2012). Error bars indicate standard uncertainties.

5 Discussion

We have introduced the generalized marginal random effects model $p(\mathbf{x}|\mu,\sigma_\lambda) = f((\mathbf{x}-\mu\mathbf{1})^T(\mathbf{V}+\sigma_\lambda^2\mathbf{I})^{-1}(\mathbf{x}-\mu\mathbf{1}))/\sqrt{\det(\mathbf{V}+\sigma_\lambda^2\mathbf{I})}$ for the purpose of adjusting measurement results that are inconsistent with respect to the uncertainties quoted for them. When the density generator $f(\cdot)$ is a normal distribution, the model corresponds to a marginal random effects model, but this is not true in general. We considered an objective Bayesian inference for this model and derived the Berger & Bernardo reference prior for grouping $\{\mu,\sigma_\lambda\}$. The corresponding reference posterior has sound theoretical properties, i.e., the posterior, as well as first and second moments, of the marginal posterior for μ exist under mild assumptions, and the resulting inference also showed good frequentist behavior in a simulation study.

The results of the objective Bayesian inference appear to be insensitive with respect to the assumed underlying distribution. This motivates to use the assumption of a normal distribution from the start for which the model is equivalent to a random effects model. The fact that the reference posterior seems to be insensitive with respect to the density generator $f(\cdot)$ is similar to the situation of the general location–scale model where the reference posterior for the location parameter is a t-distribution independently from the type of the underlying distribution used (cf. Osiewalski and Steel (1993), Fernández and Steel (1999), Arellano-Valle et al. (2006)).

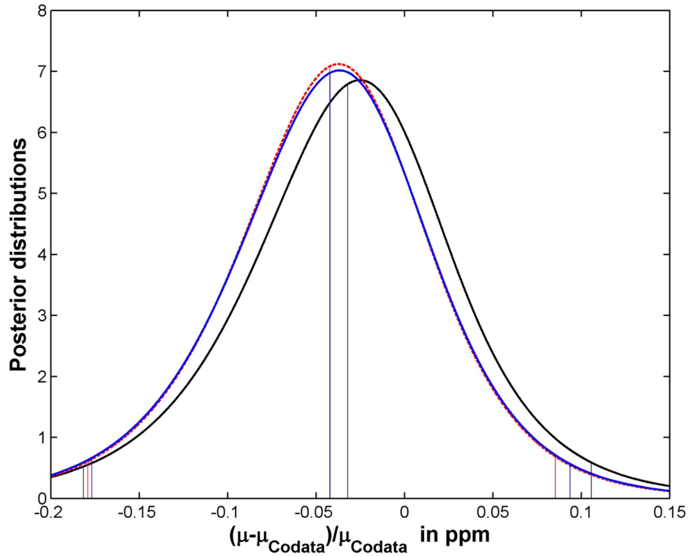

Figure 2: Posterior distributions for μ, together with means and credible intervals, obtained for the Planck data from Table 3 on the basis of the generalized marginal random effects model using as the density generators a normal distribution (black line), a t-distribution with 3 degrees of freedom (red line), and a Laplace distribution (blue line).

Future research may generalize the results of this paper by relaxing the assumption that the matrix $\mathbf{V}$ is known exactly. Instead, one may treat this matrix as further unknowns in connection with an informative prior centered around the uncertainties quoted for the measurement results.

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