

## ON THE BLOCK MAXIMA METHOD IN EXTREME VALUE THEORY: PWM ESTIMATORS<sup>1</sup>

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In extreme value theory, there are two fundamental approaches, both widely used: the block maxima (BM) method and the peaks-over-threshold (POT) method. Whereas much theoretical research has gone into the POT method, the BM method has not been studied thoroughly. The present paper aims at providing conditions under which the BM method can be justified. We also provide a theoretical comparative study of the methods, which is in general consistent with the vast literature on comparing the methods all based on simulated data and fully parametric models. The results indicate that the BM method is a rather efficient method under usual practical conditions.

In this paper, we restrict attention to the i.i.d. case and focus on the probability weighted moment (PWM) estimators of Hosking, Wallis and Wood [*Technometrics* (1985) **27** 251–261].

**1. Introduction.** The *block maxima* (BM) approach in extreme value theory (EVT), consists of dividing the observation period into nonoverlapping periods of equal size and restricts attention to the maximum observation in each period [see, e.g., Gumbel (1958)]. The new observations thus created follow—under domain of attraction conditions, cf. (2) below—approximately an extreme value distribution,  $G_\gamma$  for some real  $\gamma$ . Parametric statistical methods for the extreme value distributions are then applied to those observations.

In the *peaks-over-threshold* (POT) approach in EVT, one selects those of the initial observations that exceed a certain high threshold. The probability distribution of those selected observations is approximately a generalized Pareto distribution Pickands (1975).

In the case of the POT method, exact conditions under which the statistical method is justified can be described by a second-order term [see, e.g., Drees (1998) and de Haan and Ferreira (2006), Section 2.3]. In the case of block maxima, usually it is taken for granted that the maxima follow very well an extreme value distribution. In this paper, we take this misspecification into account by quantifying it in terms of a second-order expansion; cf. Condition 2.1 below. Since  $G_\gamma$  is not the exact distribution for those observations, a bias may appear.

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The POT method picks up all “relevant” high observations. The BM method on the one hand misses some of these high observations and, on the other hand, might retain some lower observations. Hence the POT seems to make better use of the available information.

There are practical reasons for using the BM method:

- The only available information may be block maxima, for example, yearly maxima with long historical records or long range simulated data sets [Kharin et al. \(2007\)](#).
- The BM method may be preferable when the observations are not exactly independent and identically distributed (i.i.d.). For example, there may be a seasonal periodicity in case of yearly maxima or, there may be short range dependence that plays a role within blocks but not between blocks; cf. for example, [Katz, Parlange and Naveau \(2002\)](#) and [Madsen, Rasmussen and Rosbjerg \(1997\)](#) for further discussion.
- The BM method may be easier to apply since the block periods appear naturally in many situations [[Naveau et al. \(2009\)](#), [van den Brink, Können and Opsteegh \(2005\)](#), [de Valk \(1993\)](#)]. On the other hand, the POT method allows for greater flexibility in many cases since it might be difficult to change the block size in practice.

When working with BM, there are two sets of estimators that are widely used: the maximum likelihood (ML) estimators [e.g., [Prescott and Walden \(1980\)](#)] and the probability weighted moment (PWM) estimators [Hosking, Wallis and Wood \(1985\)](#). Recently, [Dombry \(2013\)](#) has proved consistency of the former. The present paper concentrates on the latter. Our work has given rise to the paper [Bücher and Segers \(2014\)](#) on the multivariate case.

The PWM estimators under the  $G_\gamma$  model are very popular, for example, in applications to hydrologic and climatologic extremes, because of their computational simplicity, good performance for small sample sizes and robustness even for location and scale parameters [[Diebolt et al. \(2008\)](#), [Katz, Parlange and Naveau \(2002\)](#), [Caires \(2009\)](#), [Hosking \(1990\)](#)].

The relative merits of POT and BM have been discussed in several papers, all based on simulated data: [Cunnane \(1973\)](#) states that for  $\gamma = 0$  and ML estimators, the POT estimate for a high quantile is better only if the number of exceedances is larger than 1.65 times the number of blocks; [Wang \(1991\)](#) writes that POT is as efficient as BM model for high quantiles, based on PWM estimators; [Madsen, Pearson and Rosbjerg \(1997\)](#) and [Madsen, Rasmussen and Rosbjerg \(1997\)](#) write that POT is preferable for  $\gamma > 0$ , whereas for  $\gamma < 0$ , BM is more efficient, again with the number of exceedances larger than the number of blocks; [Martins and Stedinger \(2001\)](#) state that the gains (when using historical data) with the BM model are in the range of the gains with the POT model, based on ML estimators; [Caires \(2009\)](#) in a vast simulation study writes that with POT samples having an average of two or more observations per block, the estimates are more accurate

than the corresponding BM estimates, and with more than 200 years of data the accuracies of the two approaches are similar and rather good, based on several estimators including the PWM and ML estimators.

From all these studies, some even with mixed views, the following two features seem dominant. First, POT is more efficient than BM in many circumstances, though needing, on average, a number of exceedances larger than the number of blocks. Secondly, POT and BM often have comparable performances, for example, for large sample sizes.

Our theoretical comparison shows that BM is rather efficient. The asymptotic variances of both extreme value index and quantile estimators are always lower for BM than for POT. Moreover, the approximate minimal mean square error is also lower for BM under usual circumstances. The optimal number of exceedances is generally higher than the optimal number of blocks.

The paper is organized as follows. In Section 2, we state exact conditions to justify the BM method, along with the asymptotic normality result for the PWM estimators including high quantile estimators. In Section 3, we provide a theoretical comparison between the two methods, BM and POT. The analysis is based on a uniform expansion of the relevant quantile process given in Section 2.1. This expansion also provides a basis for analysing alternative estimators besides the PWM estimator. Proofs are postponed to Section 4.

Throughout the paper, we assume that the observations are i.i.d. In future work, we shall extend the results to the non-i.i.d. case and to the maximum likelihood estimator.

**2. The estimators and their properties.** Let  $\tilde{X}_1, \tilde{X}_2, \dots$  be i.i.d. random variables with distribution function  $F$ . Define for  $m = 1, 2, \dots$  and  $i = 1, 2, \dots, k$  the block maxima

$$(1) \quad X_i = \max_{(i-1)m < j \leq im} \tilde{X}_j.$$

Hence, the  $m \times k$  observations are divided into  $k$  blocks of size  $m$ . Write  $n = m \times k$ , the total number of observations. We study the model for large  $k$  and  $m$ , hence we shall assume that  $n \rightarrow \infty$ ; in order to obtain meaningful limit results, we have to require that both  $m = m_n \rightarrow \infty$  and  $k = k_n \rightarrow \infty$ , as  $n \rightarrow \infty$ .

The main assumption is that  $F$  is in the domain of attraction of some extreme value distribution

$$G_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma}), \quad \gamma \in \mathbb{R}, 1 + \gamma x > 0,$$

that is, for appropriately chosen  $a_m > 0$  and  $b_m$  and all  $x$

$$(2) \quad \begin{aligned} \lim_{m \rightarrow \infty} P\left(\frac{X_i - b_m}{a_m} \leq x\right) &= \lim_{m \rightarrow \infty} F^m(a_m x + b_m) \\ &= G_\gamma(x), \quad i = 1, 2, \dots, k. \end{aligned}$$

This can be written as

$$\lim_{m \rightarrow \infty} \frac{1}{m - \log F(a_mx + b_m)} = (1 + \gamma x)^{1/\gamma},$$

which is equivalent to the convergence of the inverse functions:

$$\lim_{m \rightarrow \infty} \frac{V(mx) - b_m}{a_m} = \frac{x^\gamma - 1}{\gamma}, \quad x > 0,$$

with  $V = (-1/\log F)^\leftarrow$ . Hence,  $b_m$  can be chosen to be  $V(m)$ . This is the first-order condition. For our analysis, we also need a second-order expansion as follows.

CONDITION 2.1 (Second-order condition). Suppose that for some positive function  $a$  and some positive or negative function  $A$  with  $\lim_{t \rightarrow \infty} A(t) = 0$ ,

$$\lim_{t \rightarrow \infty} \frac{(V(tx) - V(t))/a(t) - (x^\gamma - 1)/\gamma}{A(t)} = \int_1^x s^{\gamma-1} \int_1^s u^{\rho-1} du ds = H_{\gamma,\rho}(x),$$

for all  $x > 0$  [see, e.g., de Haan and Ferreira (2006), Theorem B.3.1]. Note that the function  $|A|$  is regularly varying with index  $\rho \leq 0$ .

Let  $X_{1,k}, \dots, X_{k,k}$  be the order statistics of the block maxima  $X_1, \dots, X_k$ . The statistics  $\beta_0 = k^{-1} \sum_{i=1}^k X_{i,k}$  and

$$(3) \quad \beta_r = \frac{1}{k} \sum_{i=1}^k \frac{(i-1) \cdots (i-r)}{(k-1) \cdots (k-r)} X_{i,k}, \quad r = 1, 2, 3, \dots, k > r,$$

are unbiased estimators of  $EX_1 F^{rm}(X_1)$  [Landwehr, Matalas and Wallis (1979)]. The PWM estimators for  $\gamma$ , as well as the location  $b_m$  and scale  $a_m = a([m])$ , are simple functionals of  $\beta_0, \beta_1$  and  $\beta_2$ . The estimator  $\hat{\gamma}_{k,m}$  for  $\gamma$  is defined as the solution of the equation

$$\frac{3\hat{\gamma}_{k,m} - 1}{2\hat{\gamma}_{k,m} - 1} = \frac{3\beta_2 - \beta_0}{2\beta_1 - \beta_0},$$

$$(4) \quad \hat{a}_{k,m} = \frac{\hat{\gamma}_{k,m}}{2\hat{\gamma}_{k,m} - 1} \frac{2\beta_1 - \beta_0}{\Gamma(1 - \hat{\gamma}_{k,m})} \quad \text{and} \quad \hat{b}_{k,m} = \beta_0 + \hat{a}_{k,m} \frac{1 - \Gamma(1 - \hat{\gamma}_{k,m})}{\hat{\gamma}_{k,m}},$$

where  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, x > 0$  [Hosking, Wallis and Wood (1985)]. The rationale behind the estimator of  $\gamma$  becomes clear when checking the statement of Theorem 2.3 below.

2.1. *Asymptotic normality.* The following theorem is the basis for analysing estimators in the BM approach. Let  $\lceil u \rceil$  represent the smallest integer larger than or equal to  $u$ .

**THEOREM 2.1.** *Assume that  $F$  is in the domain of attraction of an extreme value distribution  $G_\gamma$  and that Condition 2.1 holds. Let  $m = m_n \rightarrow \infty$  and  $k = k_n \rightarrow \infty$  as  $n \rightarrow \infty$ , in such a way that  $\sqrt{k}A(m) \rightarrow \lambda \in \mathbb{R}$ . Let  $0 < \varepsilon < 1/2$  and  $\{X_{i,k}\}_{i=1}^k$  be the order statistics of the block maxima  $X_1, X_2, \dots, X_k$ . Then, with  $\{E_k\}_{k \geq 1}$  an appropriate sequence of Brownian bridges,*

$$\begin{aligned} & \sqrt{k} \left( \frac{X_{\lceil ks \rceil, k} - b_m}{a_0(m)} - \frac{(-\log s)^{-\gamma} - 1}{\gamma} \right) \\ &= \frac{E_k(s)}{s(-\log s)^{1+\gamma}} + \sqrt{k}A_0(m)H_{\gamma, \rho} \left( \frac{1}{-\log s} \right) \\ & \quad + (s^{-1/2-\varepsilon}(1-s)^{-1/2-\gamma-\rho-\varepsilon})o_P(1), \end{aligned}$$

as  $n \rightarrow \infty$ , where the  $o_P(1)$  term is uniform for  $1/(k+1) \leq s \leq k/(k+1)$ . The functions  $a_0(m)$  and  $A_0(m)$  are chosen as in Lemma 4.2 below.

**THEOREM 2.2.** *Assume the conditions of Theorem 2.1 with  $\gamma < 1/2$ . Then*

$$\begin{aligned} & \sqrt{k} \left( \frac{(r+1)\beta_r - b_m}{a_m} - D_r(\gamma) \right) \\ & \rightarrow^d (r+1) \int_0^1 s^{r-1} (-\log s)^{-1-\gamma} E(s) ds + \lambda I_r(\gamma, \rho) =: Q_r, \end{aligned}$$

as  $n \rightarrow \infty$ , jointly for  $r = 0, 1, 2, 3, \dots$ , where  $\rightarrow^d$  means convergence in distribution,  $E$  is Brownian bridge,

$$D_r(\xi) = \frac{(r+1)^\xi \Gamma(1-\xi) - 1}{\xi}, \quad \xi < 1$$

$[D_r(0) = \log(r+1) - \Gamma'(1)$  as defined by continuity], and

$$I_r(\gamma, \rho) = \begin{cases} \frac{1}{\rho} (D_r(\gamma + \rho) - D_r(\gamma)), & \rho \neq 0, \\ D'_r(\gamma) = \frac{(r+1)^\gamma}{\gamma} (-\Gamma'(1-\gamma) + \log(r+1)\Gamma(1-\gamma) - (r+1)^{-\gamma} D_r(\gamma)), & \gamma \neq 0, \rho = 0, \\ D'_r(0) = \frac{1}{2} (\log^2(r+1) + \Gamma''(1) - 2\log(r+1)\Gamma'(1)), & \gamma = 0, \rho = 0. \end{cases}$$

Note that  $\Gamma'(1-\gamma) = \int_0^\infty u^{-\gamma} e^{-u} (\log u) du$  and  $\Gamma''(1) = 1.97811$ .

REMARK 2.1. The condition  $\sqrt{k}A(m) \rightarrow \lambda \in \mathbb{R}$  means that the growth of  $k_n$ , the number of blocks, is restricted with respect to the growth of  $m_n$ , the size of a block, as  $n \rightarrow \infty$ . In particular this condition implies that  $(\log k)/m \rightarrow 0$ , as  $n \rightarrow \infty$ .

THEOREM 2.3. Under the conditions of Theorem 2.2, as  $n \rightarrow \infty$ ,

$$\begin{aligned} \sqrt{k}(\hat{\gamma}_{k,m} - \gamma) &\rightarrow^d \frac{1}{\Gamma(1-\gamma)} \left( \frac{\log 3}{1-3^{-\gamma}} - \frac{\log 2}{1-2^{-\gamma}} \right)^{-1} \\ &\quad \times \left\{ \frac{\gamma}{3^\gamma - 1} (Q_2 - Q_0) - \frac{\gamma}{2^\gamma - 1} (Q_1 - Q_0) \right\} \\ &=: \Delta, \\ \sqrt{k} \left( \frac{\hat{a}_{k,m}}{a_m} - 1 \right) &\rightarrow^d \frac{\gamma}{(2^\gamma - 1)\Gamma(1-\gamma)} (Q_1 - Q_0) \\ &\quad + \Delta \left\{ \frac{\log 2}{\gamma} \left( \frac{-\gamma}{1-2^{-\gamma}} + \frac{1}{\log 2} \right) + \frac{\Gamma'(1-\gamma)}{\Gamma(1-\gamma)} \right\} \\ &=: \Lambda, \\ \sqrt{k} \frac{\hat{b}_{k,m} - b_m}{a_m} &\rightarrow^d Q_0 + \frac{\gamma\Gamma'(1-\gamma) - 1 + \Gamma(1-\gamma)}{\gamma^2} \Delta + \frac{1 - \Gamma(1-\gamma)}{\gamma} \Lambda \\ &=: \Xi; \end{aligned}$$

where for  $\gamma = 0$  the formulas should read as (defined by continuity):

$$\begin{aligned} \sqrt{k}\hat{\gamma}_{k,m} &\rightarrow^d \left( \frac{\log 3}{2} - \frac{\log 2}{2} \right)^{-1} \left( \frac{1}{\log 3} (Q_2 - Q_0) - \frac{1}{\log 2} (Q_1 - Q_0) \right), \\ \sqrt{k} \left( \frac{\hat{a}_{k,m}}{a_m} - 1 \right) &\rightarrow^d \frac{1}{\log 2} (Q_1 - Q_0) + \Delta \left( \frac{\log 2}{2} + \Gamma'(1) \right), \\ \sqrt{k} \frac{\hat{b}_{k,m} - b_m}{a_m} &\rightarrow^d Q_0 - \Gamma''(1)\Delta + \Gamma'(1)\Lambda. \end{aligned}$$

REMARK 2.2. A slight modification of  $\hat{\gamma}_{k,m}$  produces the explicit estimator

$$(5) \quad \hat{\gamma}_{k,m}^* = \frac{1}{\log 2} \log \left( \frac{4\beta_3 - \beta_0}{2\beta_1 - \beta_0} - 1 \right),$$

which is the solution of  $(4\hat{\gamma}_{k,m}^* - 1)(2\hat{\gamma}_{k,m}^* - 1)^{-1} = (4\beta_3 - \beta_0)(2\beta_1 - \beta_0)^{-1}$ . The conditions of Theorem 2.2 imply

$$\begin{aligned} \sqrt{k}(\hat{\gamma}_{k,m}^* - \gamma) &\rightarrow^d \frac{1}{\Gamma(1-\gamma)} \left( \frac{\log 4}{1-4^{-\gamma}} - \frac{\log 2}{1-2^{-\gamma}} \right)^{-1} \\ &\quad \times \left\{ \frac{\gamma}{4^\gamma - 1} (Q_3 - Q_0) - \frac{\gamma}{2^\gamma - 1} (Q_1 - Q_0) \right\}. \end{aligned}$$

2.2. *High quantile estimation.* Our estimator for  $x_n = F^{\leftarrow}(1 - p_n) = V(1/(-\log(1 - p_n)))$ , with  $p_n$  small, is

$$\hat{x}_{k,m} = \hat{b}_{k,m} + \hat{a}_{k,m} \frac{(mp_n)^{-\hat{\gamma}_{k,m}} - 1}{\hat{\gamma}_{k,m}}.$$

**THEOREM 2.4.** *Assume the conditions of Theorem 2.2 with  $\rho$  negative, or zero with  $\gamma$  negative. Moreover, assume that the probabilities  $p_n$  satisfy*

$$\lim_{n \rightarrow \infty} mp_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log(mp_n)}{\sqrt{k}} = 0$$

[in case  $\rho < 0$  the latter can be simplified to  $\lim_{n \rightarrow \infty} (\log p_n)/\sqrt{k} = 0$ ]. Then

$$\sqrt{k} \frac{(\hat{x}_{k,m} - x_n)}{a_m q_\gamma(1/(mp_n))} \rightarrow^d \Delta + (\gamma_-)^2 \Xi - \gamma_- \Lambda - \lambda \frac{\gamma_-}{\gamma_- + \rho}$$

as  $n \rightarrow \infty$ , where  $\gamma_- = \min(0, \gamma)$  and  $q_\gamma(t) = \int_1^t s^{\gamma-1} \log s \, ds$ .

**3. Theoretical comparison between BM and POT methods.** In this section, we develop a theoretical comparison between the BM and POT methods, by comparing the two PWM estimators for the two methods [Hosking and Wallis (1987) and Hosking, Wallis and Wood (1985), resp., for POT and BM].

First, we introduce the PWM-POT estimators for  $\gamma$  and  $a(n/k)$ , where  $k$  is the number of selected order statistics,  $\{\tilde{X}_{n-i,n}\}_{i=0}^{k-1}$ , from the original sample  $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$ . The statistics

$$P_n = \frac{1}{k} \sum_{i=0}^{k-1} \tilde{X}_{n-i,n} - \tilde{X}_{n-k,n} \quad \text{and} \quad Q_n = \frac{1}{k} \sum_{i=0}^{k-1} \frac{i}{k} (\tilde{X}_{n-i,n} - \tilde{X}_{n-k,n})$$

are estimators for  $a(n/k)(1 - \gamma)^{-1}$  and  $a(n/k)(2(2 - \gamma)^{-1})$ , respectively. Consequently, the PWM estimators are

$$\hat{\gamma}_{k,n} = 1 - \left( \frac{P_n}{2Q_n} - 1 \right)^{-1} \quad \text{and} \quad \hat{a}(n/k) = P_n \left( \frac{P_n}{2Q_n} - 1 \right)^{-1}.$$

The quantile estimator is

$$\hat{x}_{k,n} = \tilde{X}_{n-k,n} + \hat{a}(n/k) \frac{(k/(np_n))^{\hat{\gamma}_{k,n}} - 1}{\hat{\gamma}_{k,n}}.$$

Asymptotic normality under conditions equivalent to the ones in Theorems 2.3 and 2.4 holds [see, e.g., Cai, de Haan and Zhou (2013)], if  $\rho \in [-1, 0]$  with a caveat for  $\rho = -1$  [for certain cases the functions  $A$  in the corresponding second-order conditions may not be the same asymptotically resulting in different values of  $\lambda$  in the limiting distributions; cf. Drees, de Haan and Li (2003)].

For BM,  $k$  is defined as the number of blocks and, for POT,  $k$  is defined as the number of selected top order statistics. Hence, in both cases  $k$  means the number of selected observations. For the theoretical comparison, we confine ourselves to the range  $\rho \in [-1, 0]$  and  $\gamma \in [-1, 1/2)$ , a usual range in many applications.

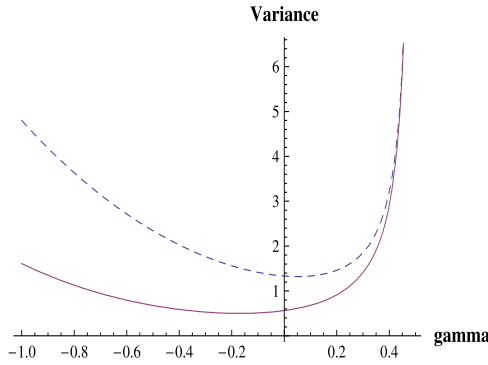


FIG. 1. Asymptotic variances of  $\gamma$  PWM estimators with dashed line for POT.

*Extreme value index estimators.*

- First, we compare asymptotic variance and bias for a common value of  $k$ :  
 The asymptotic variances of the two  $\gamma$  estimators are shown in Figure 1: the curve from BM is always below the other one, meaning lower values for the asymptotic variance for all values of  $\gamma$ . The asymptotic biases are compared in Figure 2, through the ratio “bias BM/bias POT”. Recall that the bias depends on both first- and second-order parameters  $\gamma$  and  $\rho$ . Contrary to what is observed for the variance, the bias of BM is always larger but for  $\rho = 0$  they are the same regardless the value of  $\gamma$ , equal to 1 [or  $\lambda$  if one takes into account the asymptotic contribution of  $\sqrt{k}A(n/k)$  to the biases].
- Next, we compare asymptotic mean square errors for the “optimal choice” of  $k$  (i.e., that value that makes the limiting mean square error of  $\hat{\gamma} - \gamma$  minimal), which is different in the two cases:

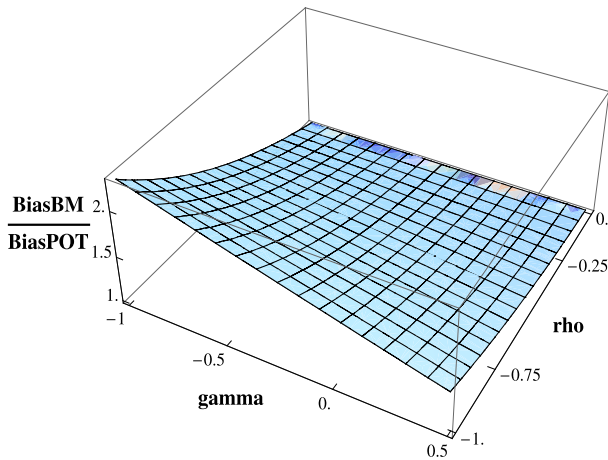


FIG. 2. Ratio of asymptotic bias of  $\gamma$  PWM estimators.



An asymptotic expression of the “asymptotic minimal mean square error” (MINMSE in the sequel) is obtained in the following way. Suppose  $\rho < 0$ . First we find for each estimator the optimal  $k$  in the sense of minimizing the approximate asymptotic mean square error. Denote by  $\sigma_i^2 = \sigma_i^2(\gamma)$  and  $B_i^2 = B_i^2(\gamma, \rho)$  ( $i = 1, 2$ ; “1” refers to PWM-BM and “2” refers to PWM-POT) the asymptotic variance and squared bias of the estimators. Under Condition 2.1, we can write  $A^2(t) = \int_t^\infty s(u) du$  with  $s(\cdot)$  decreasing and  $2\rho - 1$  regularly varying. The limiting mean square error is, approximately,

$$(6) \quad \inf_k \left( \frac{\sigma_i^2}{k} + A^2(n/k) B_i^2 \right)$$

or, writing  $r$  for  $n/k$ ,  $\inf_r ((r/n)\sigma_i^2 + B_i^2 \int_r^\infty s(u) du)$ . Setting the derivative equal to zero and using properties of regularly varying functions one finds for the optimal choice of  $r$ ,  $r_0^{(i)} \sim (1/s)^\leftarrow(n) (B_i^2/\sigma_i^2)^{1/(1-2\rho)}$  and, in terms of  $k$ ,

$$k_0^{(i)} \sim \frac{n}{(1/s)^\leftarrow(n)} \left( \frac{\sigma_i^2}{B_i^2} \right)^{1/(1-2\rho)}.$$

Note that the optimal  $k_0^{(i)}$  is different but of the same order for both methods. Next, inserting  $k_0^{(i)}$  in (6), after some manipulation we get the following asymptotic expression for MINMSE,

$$\frac{1 - 2\rho}{-2\rho} \frac{(1/s)^\leftarrow(n)}{n} (B_i^2)^{1/(1-2\rho)} (\sigma_i^2)^{-2\rho/(1-2\rho)}.$$

It follows that MINMSE(BM)/MINMSE(POT) is, approximately,

$$\left( \frac{B_1^2(\gamma, \rho)}{B_2^2(\gamma, \rho)} \right)^{1/(1-2\rho)} \left( \frac{\sigma_1^2(\gamma)}{\sigma_2^2(\gamma)} \right)^{-2\rho/(1-2\rho)},$$

which does not depend on  $n$ , just on  $\gamma$  and  $\rho$ .

The contour plot of “MINMSE(BM)/MINMSE(POT)” is represented in Figure 3. It can be seen that the BM has lower MINMSE for a large range of  $(\gamma, \rho)$  combinations. Note that this range includes  $\gamma$  negative and  $\gamma$  positive close to zero which seem to be common values in many practical situations, for example, in hydrologic and climatologic extremes. Only for  $\gamma > 0.2$  approximately, MINMSE for POT can be lower depending on  $\rho$ .

Finally, comparing the optimal sample sizes (cf. Figure 4 with contour plot of the ratio of the optimal values of  $k$ ), one sees that POT requires systematically larger optimal sample size even when the approximate MINMSE is smaller for POT than BM.

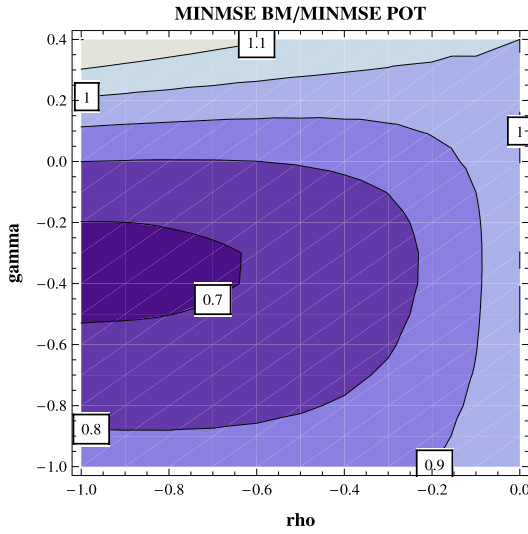


FIG. 3. Contour plot for the ratio of asymptotic minimal mean square error of  $\gamma$  PWM estimators.

*Quantile estimators.* We repeat the previous analysis for the quantile estimators:

- The asymptotic variances of the two estimators are compared in Figure 5: again the curve from BM is always below the other one meaning lower values for the asymptotic variance for all values of  $\gamma$ . In Figures 6 and 7, the asymptotic bias

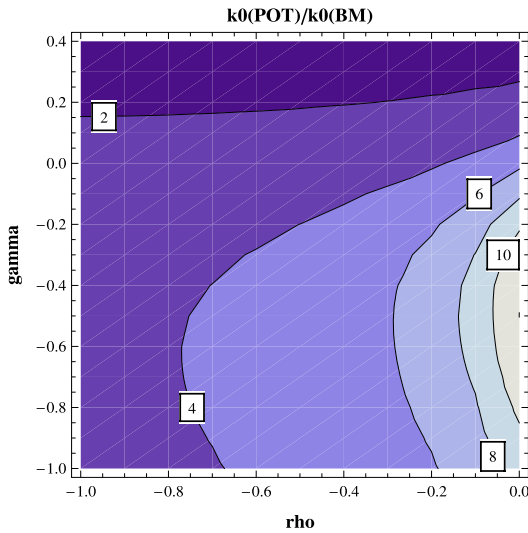


FIG. 4. Contour plot for the ratio of the optimal values of  $k$ .

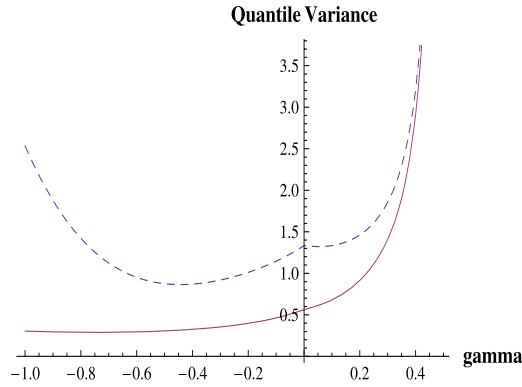


FIG. 5. Asymptotic variances of quantile PWM estimators with dashed line for POT.

is represented for each case separately. Note that for  $\gamma$  negative, the bias for BM approaches zero when  $\rho \uparrow 0$  whereas in the POT case it escapes to  $-\infty$ .

- The contour plot for the ratio “MINMSE(BM)/MINMSE(POT)” is represented in Figure 8. Again the BM method has lower MINMSE for a large range of  $(\gamma, \rho)$  combinations. The “irregularity” around  $\gamma \approx -0.2$  is due to a change of sign in the bias in the POT case. Finally, Figure 9 gives the contour plot for the ratio of the optimal values of  $k$ , which is smaller than one when  $\gamma$  is small and  $\rho$  is closer to zero.

*In conclusion*, for both the extreme value index and quantile PWM estimators, the ones from the BM method have always lower asymptotic variances. Moreover, at an optimal level the BM gives lower MINMSE, thus being more efficient, under many practical situations. This is in agreement with some of Sofia Caires’ (2009)

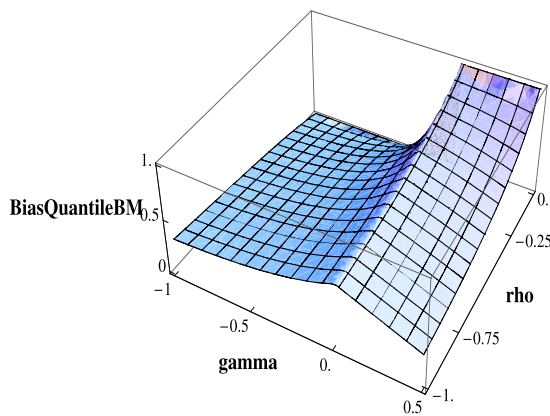


FIG. 6. Asymptotic bias of quantile PWM estimator under BM method.

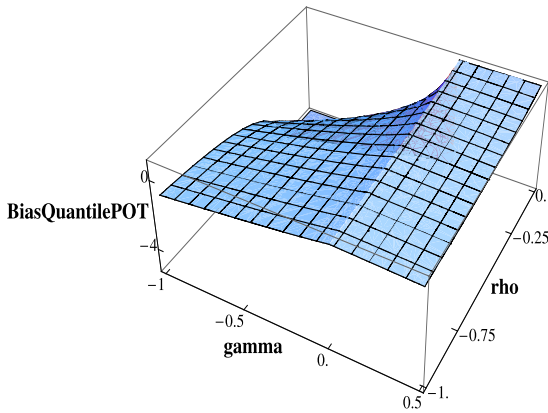


FIG. 7. Asymptotic bias of quantile PWM estimator under POT method.

conclusions, for example, that for equal sample sizes or with more than 200 years of data the uncertainty or the error of the estimates are lower for BM than for POT.

**4. Proofs.** Throughout this section,  $Z$  represents a unit Fréchet random variable, that is, one with distribution function  $F(x) = e^{-1/x}$ ,  $x > 0$ , and  $\{Z_{i,k}\}_{i=1}^k$  are the order statistics from the associated i.i.d. sample of size  $k$ ,  $Z_1, \dots, Z_k$ . Similarly,  $\{X_{i,k}\}_{i=1}^k$  represents the order statistics of the block maxima  $X_1, \dots, X_k$  from (1) and,  $X_{\lceil u \rceil, k} = X_{r,k}$  for  $r - 1 < u \leq r$ ,  $r = 1, \dots, k$ . Recall the function  $V$

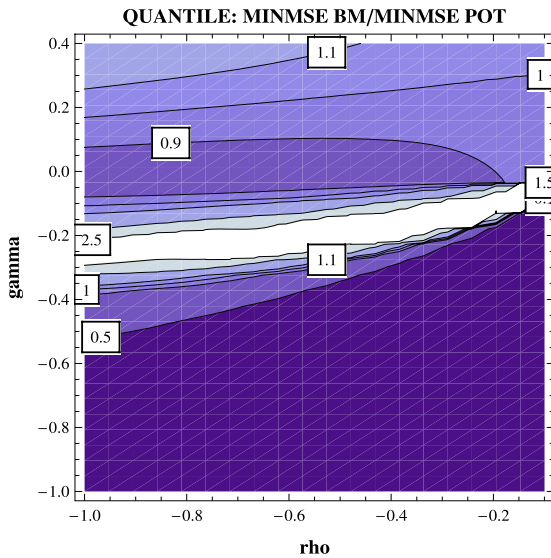


FIG. 8. Contour plot for the ratio of asymptotic minimal mean square error of quantile PWM estimators.

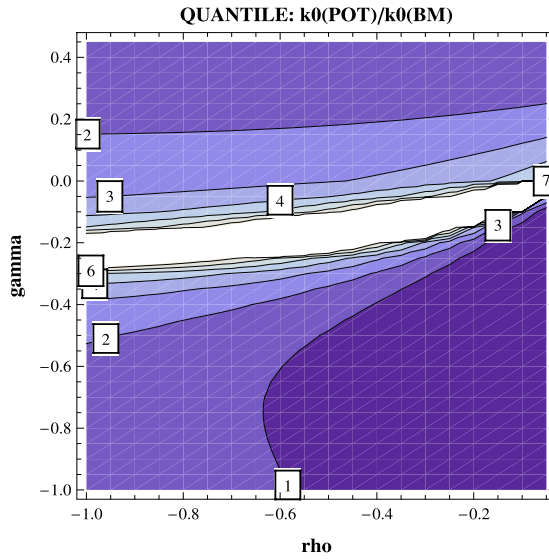


FIG. 9. Contour plot for the ratio of the optimal values of  $k$  in quantile case.

from Section 2. The following representation will be useful:

$$(7) \quad X =^d V(mZ).$$

We start by formulating a number of auxiliary results.

LEMMA 4.1. 1. As  $k \rightarrow \infty$ ,

$$(\log k)Z_{1,k} \rightarrow^P 1.$$

2. [Csörgő and Horváth (1993), page 381] Let  $0 < \nu < 1/2$ . With  $\{E_k\}_{k \geq 1}$ , an appropriate sequence of Brownian bridges,

$$\sup_{1/(k+1) \leq s \leq k/(k+1)} \frac{s(-\log s)}{(s(1-s))^\nu} \left| \sqrt{k}((-\log s)Z_{\lceil ks \rceil, k} - 1) - \frac{E_k(s)}{s(-\log s)} \right| = o_P(1),$$

as  $k \rightarrow \infty$  ( $\lceil u \rceil$  represents the smallest integer larger or equal to  $u$ ).

3. Similarly, with  $0 < \nu < 1/2$  for an appropriate sequence  $\{E_k\}_{k \geq 1}$  of Brownian bridges and  $\xi \in \mathbb{R}$ ,

$$\begin{aligned} & \sup_{1/(k+1) \leq s \leq k/(k+1)} (s(1-s))^{-\nu} \\ & \times \left| \sqrt{ks}(-\log s)^{1+\xi} \left( \frac{Z_{\lceil ks \rceil, k}^\xi - 1}{\xi} - \frac{(-\log s)^{-\xi} - 1}{\xi} \right) - E_k(s) \right| = o_P(1), \end{aligned}$$

as  $k \rightarrow \infty$ .

The following is an easily obtained variant of Theorem B.3.10 of de Haan and Ferreira (2006).

LEMMA 4.2. *Under Condition 2.1, there are functions  $A_0(t) \sim A(t)$  and  $a_0(t) = a(t)(1 + o(A(t)))$ , as  $t \rightarrow \infty$ , such that for all  $\varepsilon, \delta > 0$  there exists  $t_0 = t_0(\varepsilon, \delta)$  such that for  $t, tx > t_0$ ,*

$$(8) \quad \left| \frac{(V(tx) - V(t))/a_0(t) - (x^\gamma - 1)/\gamma}{A_0(t)} - H_{\gamma, \rho}(x) \right| \leq \varepsilon \max(x^{\gamma+\rho+\delta}, x^{\gamma+\rho-\delta}).$$

Moreover,

$$(9) \quad \left| \frac{a_0(tx)/a_0(t) - x^\gamma}{A_0(t)} - x^\gamma(x^\rho - 1)/\rho \right| \leq \varepsilon \max(x^{\gamma+\rho+\delta}, x^{\gamma+\rho-\delta})$$

and

$$\left| \frac{A_0(tx)}{A_0(t)} - x^\rho \right| \leq \varepsilon \max(x^{\rho+\delta}, x^{\rho-\delta}).$$

Note that

$$H_{\gamma, \rho}(x) = \begin{cases} \frac{1}{\rho} \left( \frac{x^{\gamma+\rho} - 1}{\gamma + \rho} - \frac{x^\gamma - 1}{\gamma} \right), & \rho \neq 0 \neq \gamma, \\ \frac{1}{\gamma} \left( x^\gamma \log x - \frac{x^\gamma - 1}{\gamma} \right), & \rho = 0 \neq \gamma, \\ \frac{1}{\rho} \left( \frac{x^\rho - 1}{\rho} - \log x \right), & \rho \neq 0 = \gamma, \\ \frac{1}{2} (\log x)^2, & \rho = 0 = \gamma. \end{cases}$$

PROOF OF THEOREM 2.1. By representation (7),

$$\begin{aligned} & \frac{X_{\lceil ks \rceil, k} - b_m}{a_0(m)} - \frac{(-\log s)^{-\gamma} - 1}{\gamma} \\ &= {}^d \left( \frac{V(mZ_{\lceil ks \rceil, k}) - b_m}{a_0(m)} - \frac{V(m/-\log s) - b_m}{a_0(m)} \right) \\ & \quad + \left( \frac{V(m/-\log s) - b_m}{a_0(m)} - \frac{(-\log s)^{-\gamma} - 1}{\gamma} \right) \\ &= \text{I (random part)} + \text{II (bias part)}. \end{aligned}$$

We start with part I,

$$\begin{aligned} I &= \left\{ (-\log s)^{-\gamma} \frac{V((-\log s)Z_{\lceil ks \rceil, k} m / -\log s) - V(m / -\log s)}{a_0(m / -\log s)} \right\} \\ &\quad \times \left\{ \frac{a_0(m / -\log s)}{a_0(m)} (-\log s)^\gamma \right\} \\ &= I.1 \times I.2. \end{aligned}$$

According to (9) of Lemma 4.2, for each  $\varepsilon, \delta > 0$  there exists  $t_0$  such that the factor I.2 is bounded (above and below) by

$$1 + A_0(m) \left\{ \frac{(-\log s)^{-\rho} - 1}{\rho} \pm \varepsilon \max((-\log s)^{-\rho+\delta}, (-\log s)^{-\rho-\delta}) \right\}$$

provided  $m \geq t_0$  and  $s \geq e^{-m/t_0}$ . According to (8) of Lemma 4.2, for factor I.1 we have the bounds

$$\begin{aligned} &(-\log s)^{-\gamma} \frac{((-\log s)Z_{\lceil ks \rceil, k})^\gamma - 1}{\gamma} + A_0\left(\frac{m}{-\log s}\right) (-\log s)^{-\gamma} \\ &\quad \times \left\{ H_{\gamma, \rho}((-\log s)Z_{\lceil ks \rceil, k}) \pm \varepsilon \max((-\log s)Z_{\lceil ks \rceil, k})^{\gamma+\rho+\delta}, \right. \\ &\quad \left. ((-\log s)Z_{\lceil ks \rceil, k})^{\gamma+\rho-\delta} \right\} \\ &= I.1a + I.1b \end{aligned}$$

provided  $s \geq e^{-m/t_0}$  and  $m / \log k \geq t_0$  [the latter inequality eventually holds true since  $\sqrt{k}A_0(m)$  is bounded]. Note that  $m / \log k \geq t_0$  implies  $mZ_{1, k} \geq 2t_0$  which implies (Lemma 4.1)  $mZ_{\lceil ks \rceil, k} \geq 2t_0$  for all  $s$ .

For term I.1a, we use Lemma 4.1.3:

$$\frac{(Z_{\lceil ks \rceil, k})^\gamma - 1}{\gamma} - \frac{(-\log s)^{-\gamma} - 1}{\gamma}$$

is bounded (above and below) by

$$\frac{1}{\sqrt{k}} \frac{E_k(s)}{s(-\log s)^{1+\gamma}} \pm \frac{\varepsilon}{\sqrt{k}} \frac{(s(1-s))^\nu}{s(-\log s)^{1+\gamma}},$$

for some  $\varepsilon > 0$ ,  $0 < \nu < 1/2$  and all  $s \in [1/(k+1), k/(k+1)]$ .

Next, we turn to term I.1b. By Lemma 4.2,  $(-\log s)^{-\gamma} A_0(\frac{m}{-\log s})$  is bounded (above and below) by

$$A_0(m) \{ (-\log s)^{-\gamma-\rho} \pm \varepsilon \max((-\log s)^{-\gamma-\rho+\delta}, (-\log s)^{-\gamma-\rho-\delta}) \}$$

provided  $s > e^{-m/t_0}$  and  $m/\log k > t_0$ . Furthermore for  $\rho \neq 0 \neq \gamma$  and  $s \in [1/(k+1), k/(k+1)]$ , by Lemma 4.1.3,

$$\begin{aligned} &H_{\gamma,\rho}((-\log s)Z_{\lceil ks \rceil,k}) \\ &= \frac{1}{\rho} \left\{ \frac{((-\log s)Z_{\lceil ks \rceil,k})^{\gamma+\rho} - 1}{\gamma + \rho} - \frac{((-\log s)Z_{\lceil ks \rceil,k})^\gamma - 1}{\gamma} \right\} \\ &= \frac{1}{\rho} \left\{ (-\log s)^{\gamma+\rho} \left[ \frac{Z_{\lceil ks \rceil,k}^{\gamma+\rho} - 1}{\gamma + \rho} - \frac{(-\log s)^{-\gamma-\rho} - 1}{\gamma + \rho} \right] \right. \\ &\quad \left. - (-\log s)^\gamma \left[ \frac{Z_{\lceil ks \rceil,k}^\gamma - 1}{\gamma} - \frac{(-\log s)^{-\gamma} - 1}{\gamma} \right] \right\} \end{aligned}$$

is bounded by

$$\begin{aligned} &\frac{1}{\rho} \left\{ (-\log s)^{\gamma+\rho} \left[ \frac{1}{\sqrt{k}} \frac{E_k(s)}{s(-\log s)^{1+\gamma+\rho}} \pm \frac{\varepsilon}{\sqrt{k}} \frac{(s(1-s))^{\nu}}{s(-\log s)^{1+\gamma+\rho}} \right] \right. \\ &\quad \left. - (-\log s)^\gamma \left[ \frac{1}{\sqrt{k}} \frac{E_k(s)}{s(-\log s)^{1+\gamma}} \mp \frac{\varepsilon}{\sqrt{k}} \frac{(s(1-s))^{\nu}}{s(-\log s)^{1+\gamma}} \right] \right\} \\ &= \pm \frac{2\varepsilon}{\rho\sqrt{k}} \frac{(s(1-s))^{\nu}}{s(-\log s)}, \end{aligned}$$

and similarly for cases other than  $\rho \neq 0 \neq \gamma$ . The remaining part of I.1b, namely  $\pm \varepsilon \max((-\log s)Z_{\lceil ks \rceil,k}^{\gamma+\rho+\delta}, (-\log s)Z_{\lceil ks \rceil,k}^{\gamma+\rho-\delta})$ , is similar.

Part II, by the inequalities of Lemma 4.2, is bounded by

$$A_0(m) \left\{ H_{\gamma,\rho} \left( \frac{1}{-\log s} \right) \pm \varepsilon \max((-\log s)^{-\gamma-\rho+\delta}, (-\log s)^{-\gamma-\rho-\delta}) \right\}$$

hence it contributes  $\sqrt{k}A_0(m)H_{\gamma,\rho}(\frac{1}{-\log s})$  to the result.

Collecting all the terms, one finds the result.  $\square$

PROOF OF THEOREM 2.2. Let, for  $r = 0, 1, 2, 3, \dots$ ,

$$J_k^{(r)}(s) = \frac{(\lceil ks \rceil - 1) \cdots (\lceil ks \rceil - r)}{(k-1) \cdots (k-r)}, \quad s \in [0, 1].$$

Note that  $J_k^{(r)}(s) \rightarrow s^r$ , as  $k \rightarrow \infty$ , uniformly in  $s \in [0, 1]$ , and

$$\begin{aligned} \frac{1}{k} \sum_{i=1}^k \frac{(i-1) \cdots (i-r)}{(k-1) \cdots (k-r)} &= \int_0^1 J_k^{(r)}(s) ds = \frac{1}{r+1} \\ &= \int_0^1 s^r ds. \end{aligned}$$



Then

$$\begin{aligned}
 & \sqrt{k} \left( \frac{(r+1)\beta_r - b_m}{a_m} - \frac{(r+1)^\gamma \Gamma(1-\gamma) - 1}{\gamma} \right) \\
 &= \sqrt{k} \left( \frac{(r+1) \int_0^1 X_{\lceil ks \rceil, k} J_k^{(r)}(s) ds - b_m}{a_m} \right. \\
 &\quad \left. - (r+1) \int_0^1 \frac{(-\log s)^{-\gamma} - 1}{\gamma} s^r ds \right) \\
 &= \sqrt{k}(r+1) \int_0^1 \left( \frac{X_{\lceil ks \rceil, k} - b_m}{a_m} - \frac{(-\log s)^{-\gamma} - 1}{\gamma} \right) J_k^{(r)}(s) ds \\
 &\quad - \sqrt{k}(r+1) \int_0^1 \frac{(-\log s)^{-\gamma} - 1}{\gamma} (s^r - J_k^{(r)}(s)) ds \\
 &= \sqrt{k}(r+1) \int_0^{1/(k+1)} \left( \frac{X_{\lceil ks \rceil, k} - b_m}{a_m} - \frac{(-\log s)^{-\gamma} - 1}{\gamma} \right) J_k^{(r)}(s) ds \\
 &\quad + \sqrt{k}(r+1) \int_{1/(k+1)}^{k/(k+1)} \left( \frac{X_{\lceil ks \rceil, k} - b_m}{a_m} - \frac{(-\log s)^{-\gamma} - 1}{\gamma} \right) J_k^{(r)}(s) ds \\
 &\quad + \sqrt{k}(r+1) \int_{k/(k+1)}^1 \left( \frac{X_{\lceil ks \rceil, k} - b_m}{a_m} - \frac{(-\log s)^{-\gamma} - 1}{\gamma} \right) J_k^{(r)}(s) ds \\
 &\quad - \sqrt{k}(r+1) \int_0^1 \frac{(-\log s)^{-\gamma} - 1}{\gamma} (s^r - J_k^{(r)}(s)) ds \\
 &= \text{I.1} + \text{I.2} + \text{I.3} + \text{I.4}.
 \end{aligned}$$

For I.4: since  $(s^r - J_k^{(r)}(s)) = O(1/k)$  uniformly in  $s$ ,  $\text{I.4} = O(1/\sqrt{k})$ .

For I.1, note that

$$(10) \quad \int_0^{1/(k+1)} \sqrt{k} \frac{X_{\lceil ks \rceil, k} - b_m}{a_m} s^r ds = o_P(1).$$

This follows since, the left-hand side of (10) equals, in distribution,

$$\frac{\sqrt{k}}{(k+1)^{r+1}} \frac{V(mZ_{1,k}) - V(m)}{a_m}$$

which, by Lemmas 4.1.1, 4.2 and the fact that  $m/\log k \rightarrow \infty$ , is bounded (below and above) by

$$\frac{\sqrt{k}}{(k+1)^{r+1}} \left\{ \frac{Z_{1,k}^\gamma - 1}{\gamma} + A(m)H_{\gamma, \rho}(Z_{1,k}) \pm \varepsilon A(m) \max(Z_{1,k}^{\gamma+\rho+\delta}, Z_{1,k}^{\gamma+\rho-\delta}) \right\}.$$

This is easily seen to converge to zero in probability, since  $Z_{1,k}^\xi/\sqrt{k} = \{(\log k)Z_{1,k}\}^\xi \log^{-\xi} k/\sqrt{k} \rightarrow^P 0$  for all real  $\xi$  and  $\sqrt{k}A(m) \rightarrow \lambda$ . Hence,  $\text{I.1} = o_P(1)$ .

Next, we show that

$$(11) \quad \int_{k/(k+1)}^1 \sqrt{k} \frac{X_{\lceil ks \rceil, k} - b_m}{a_m} J_k^{(r)}(s) ds = o_P(1).$$

The left-hand side equals, in distribution, since  $J_k^{(r)}(s) \equiv 1$  for  $s \in (k(k+1)^{-1}, 1)$ ,

$$\left(1 - \frac{k}{k+1}\right) \sqrt{k} \frac{V(mZ_{k,k}) - V(m)}{a_m}.$$

Lemma 4.1 yields

$$\frac{V(mZ_{k,k}) - V(m)}{a_m} = \frac{Z_{k,k}^\gamma - 1}{\gamma} + A(m) \{H_{\gamma, \rho}(Z_{k,k}) \pm \varepsilon Z_{k,k}^{\gamma+\rho+\delta}\},$$

which is (since  $Z_{k,k}^\gamma/k^\gamma$  converges to a positive random variable) of the order  $O_P(k^\gamma)$ . Hence, the integral is of order  $(k+1)^{-1} \sqrt{k} k^\gamma$  which tends to zero since  $\gamma < 1/2$ .

Finally, I.2 has the same asymptotic behaviour as

$$(r+1) \int_{1/(k+1)}^{k/(k+1)} \sqrt{k} \left( \frac{X_{\lceil ks \rceil, k} - b_m}{a_m} - \frac{(-\log s)^{-\gamma} - 1}{\gamma} \right) s^r ds,$$

which, by Theorem 2.1 tends to

$$(r+1) \int_0^1 s^{r-1} (-\log s)^{-1-\gamma} E(s) ds + \lambda(r+1) \int_0^1 H_{\gamma, \rho} \left( \frac{1}{-\log s} \right) s^r ds.$$

For the evaluation of the latter integral note that for  $\xi < 1$ ,

$$(r+1) \int_0^1 s^r (-\log s)^{-\xi} ds = (r+1)^{\xi-1} \int_0^\infty v^{-\xi} e^{-v} dv = (r+1)^{\xi-1} \Gamma(1-\xi).$$

Moreover, note that

$$(r+1) \int_0^1 s^r \frac{(-\log s)^{-\xi} - 1}{\xi} ds = \frac{(r+1)^\xi \Gamma(1-\xi) - 1}{\xi}, \quad \xi < 1$$

$[D_r(0) = \log(r+1) - \Gamma'(1)$  as defined by continuity], and  $(r+1) \times \int_0^1 H_{\gamma, \rho} \left( \frac{1}{-\log s} \right) s^r ds = I_r(\gamma, \rho)$ .  $\square$

PROOF OF THEOREM 2.3. From Theorem 2.2, we obtain

$$\begin{aligned} \sqrt{k} \left( \frac{2\beta_1 - \beta_0}{a_m} - \frac{2^\gamma - 1}{\gamma} \Gamma(1-\gamma) \right) &\rightarrow^d Q_1 - Q_0, \\ \sqrt{k} \left( \frac{3\beta_2 - \beta_0}{a_m} - \frac{3^\gamma - 1}{\gamma} \Gamma(1-\gamma) \right) &\rightarrow^d Q_2 - Q_0 \end{aligned}$$

hence, by Cramér’s delta method,

$$\begin{aligned} & \sqrt{k} \left( \frac{3^{\hat{\gamma}_{k,m}} - 1}{2^{\hat{\gamma}_{k,m}} - 1} - \frac{3^\gamma - 1}{2^\gamma - 1} \right) \\ &= \sqrt{k} \left( \frac{3\beta_2 - \beta_0}{2\beta_1 - \beta_0} - \frac{3^\gamma - 1}{2^\gamma - 1} \right) \\ &\rightarrow^d \frac{1}{\Gamma(1 - \gamma)} \frac{3^\gamma - 1}{2^\gamma - 1} \left( \frac{\gamma}{3^\gamma - 1} (Q_2 - Q_0) - \frac{\gamma}{2^\gamma - 1} (Q_1 - Q_0) \right). \end{aligned}$$

It follows that  $\hat{\gamma}_{k,m} \rightarrow^P \gamma$ , and hence

$$\sqrt{k} \left( \frac{r^{\hat{\gamma}_{k,m}} - 1}{r^\gamma - 1} - 1 \right) = \sqrt{k} \frac{r^{\hat{\gamma}_{k,m} - \gamma} - 1}{1 - r^{-\gamma}}$$

has the same limit distribution as

$$\sqrt{k} (\hat{\gamma}_{k,m} - \gamma) \frac{\log r}{1 - r^{-\gamma}}, \quad r = 2, 3.$$

It follows that

$$\begin{aligned} & \sqrt{k} \left( \frac{3^{\hat{\gamma}_{k,m}} - 1}{2^{\hat{\gamma}_{k,m}} - 1} - \frac{3^\gamma - 1}{2^\gamma - 1} \right) \\ &= \frac{3^\gamma - 1}{2^\gamma - 1} \left[ \sqrt{k} \left( \frac{3^{\hat{\gamma}_{k,m}} - 1}{3^\gamma - 1} - 1 \right) - \sqrt{k} \left( \frac{2^{\hat{\gamma}_{k,m}} - 1}{2^\gamma - 1} - 1 \right) \right] \end{aligned}$$

has the same limit distribution as

$$\frac{3^\gamma - 1}{2^\gamma - 1} \sqrt{k} (\hat{\gamma}_{k,m} - \gamma) \left( \frac{\log 3}{1 - 3^{-\gamma}} - \frac{\log 2}{1 - 2^{-\gamma}} \right)$$

and, consequently,

$$\begin{aligned} & \sqrt{k} (\hat{\gamma}_{k,m} - \gamma) \\ &\rightarrow^d \frac{1}{\Gamma(1 - \gamma)} \left( \frac{\log 3}{1 - 3^{-\gamma}} - \frac{\log 2}{1 - 2^{-\gamma}} \right)^{-1} \\ &\quad \times \left( \frac{\gamma}{3^\gamma - 1} (Q_2 - Q_0) - \frac{\gamma}{2^\gamma - 1} (Q_1 - Q_0) \right). \end{aligned}$$

For the asymptotic distribution of  $\hat{a}_{k,m}$  we write

$$\begin{aligned} & \sqrt{k} \left( \frac{\hat{a}_{k,m}}{a_m} - 1 \right) \frac{\hat{\gamma}_{k,m}}{(2^{\hat{\gamma}_{k,m}} - 1)\Gamma(1 - \hat{\gamma}_{k,m})} \\ &\quad \times \left\{ \sqrt{k} \left( \frac{2\beta_1 - \beta_0}{a_m} - \frac{2^\gamma - 1}{\gamma} \Gamma(1 - \gamma) \right) \right. \\ &\quad \left. + \sqrt{k} \left( \frac{2^\gamma - 1}{\gamma} \Gamma(1 - \gamma) - \frac{2^{\hat{\gamma}_{k,m}} - 1}{\hat{\gamma}_{k,m}} \Gamma(1 - \hat{\gamma}_{k,m}) \right) \right\}, \end{aligned}$$

and the statement follows, for example, by Cramér’s delta method.

For the asymptotic distribution of  $\hat{b}_{k,m}$ , we write

$$\begin{aligned} & \sqrt{k} \left( \frac{\hat{b}_{k,m} - b_m}{a_m} \right) \\ &= \sqrt{k} \left( \frac{\beta_0 - b_m}{a_m} - \frac{\Gamma(1 - \gamma) - 1}{\gamma} \right) \\ & \quad - \frac{\hat{a}_{k,m}}{a_m} \sqrt{k} \left( \frac{\Gamma(1 - \hat{\gamma}_{k,m}) - 1}{\hat{\gamma}_{k,m}} - \frac{\Gamma(1 - \gamma) - 1}{\gamma} \right) \\ & \quad + \frac{\Gamma(1 - \gamma) - 1}{\gamma} \sqrt{k} \left( \frac{\hat{a}_{k,m}}{a_m} - 1 \right) \end{aligned}$$

and the statement follows, for example, by Cramér’s delta method.  $\square$

PROOF OF THEOREM 2.4. The proof follows the line of the comparable result for the POT method [see, e.g., de Haan and Ferreira (2006), Chapter 4.3]. Let  $c_n = 1/(mp_n)$ . Then

$$\begin{aligned} & \frac{\sqrt{k}(\hat{x}_{k,m} - x_n)}{a_m q_\gamma(c_n)} \\ &= \frac{\sqrt{k}}{a_m q_\gamma(c_n)} \left( \hat{b}_{k,m} + \hat{a}_{k,m} \frac{c_n^{\hat{\gamma}_{k,m}} - 1}{\hat{\gamma}_{k,m}} - V \left( \frac{1}{-\log(1 - p_n)} \right) \right) \\ &= \frac{\sqrt{k}}{q_\gamma(c_n)} \frac{\hat{b}_{k,m} - b_m}{a_m} + \frac{\hat{a}_{k,m}}{a_m} \frac{\sqrt{k}}{q_\gamma(c_n)} \left( \frac{c_n^{\hat{\gamma}_{k,m}} - 1}{\hat{\gamma}_{k,m}} - \frac{c_n^\gamma - 1}{\gamma} \right) \\ & \quad - \frac{\sqrt{k}}{q_\gamma(c_n)} \left( \frac{V(m/(-m \log(1 - p_n))) - V(m)}{a_m} - \frac{c_n^\gamma - 1}{\gamma} \right) \\ & \quad + \frac{c_n^\gamma - 1}{\gamma q_\gamma(c_n)} \sqrt{k} \left( \frac{\hat{a}_{k,m}}{a_m} - 1 \right). \end{aligned}$$

Similarly, as on pages 135–137 of de Haan and Ferreira (2006), this converges in distribution to

$$\Delta + (\gamma_-)^2 \Xi - \gamma_- \Lambda - \lambda \frac{\gamma_-}{\gamma_- + \rho}. \quad \square$$

### APPENDIX: ASYMPTOTIC VARIANCES AND BIASES OF THE PWM ESTIMATORS

The following provides a basis for an algorithm to calculate the asymptotic variances/covariances and biases of the PWM estimators.

Let  $Q_r = (r + 1) \int_0^1 s^{r-1} (-\log s)^{-1-\gamma} E(s) ds + \lambda I_r(\gamma, \rho)$ ,  $r = 0, 1, 2$ , as defined in Theorem 2.2. For  $r, m = 0, 1, 2$ ,

$$\begin{aligned}
 & \text{Cov}(Q_r, Q_m) \\
 &= (r + 1)(m + 1) \\
 & \quad \times \int_0^1 \int_0^1 s^{r-1} u^{m-1} (-\log s)^{-1-\gamma} (-\log u)^{-1-\gamma} EB(s)B(u) ds du \\
 (12) \quad &= (r + 1)(m + 1) \int_0^1 u^{m-1} (1 - u) (-\log u)^{-1-\gamma} \int_0^u s^r (-\log s)^{-1-\gamma} ds du \\
 & \quad + (r + 1)(m + 1) \\
 & \quad \times \int_0^1 s^{r-1} (1 - s) (-\log s)^{-1-\gamma} \int_0^s u^m (-\log u)^{-1-\gamma} du ds
 \end{aligned}$$

using the fact that  $EB(s)B(u) = \min(s, u) - su = s(1 - u)$  for  $0 < s < u$ . These integrals can be evaluated numerically (we have used Mathematica software).

From Theorems 2.3 and 2.4, after some calculations,

$$(13) \quad \sqrt{k}(\hat{\gamma}_{k,m} - \gamma) \rightarrow^d C_\gamma(k_{\gamma,0}Q_0 + k_{\gamma,1}Q_1 + k_{\gamma,2}Q_2),$$

$$(14) \quad \sqrt{k} \left( \frac{\hat{a}_{k,m}}{a_m} - 1 \right) \rightarrow^d k_{a,0}Q_0 + k_{a,1}Q_1 + k_{a,2}Q_2,$$

$$(15) \quad \sqrt{k} \frac{\hat{b}_{k,m} - b_m}{a_m} \rightarrow^d k_{b,0}Q_0 + k_{b,1}Q_1 + k_{b,2}Q_2,$$

$$(16) \quad \sqrt{k} \frac{(\hat{x}_{k,m} - x_n)}{a_m q_\gamma(c_n)} \rightarrow^d k_{x,0}Q_0 + k_{x,1}Q_1 + k_{x,2}Q_2,$$

where, for  $\gamma \neq 0$ ,

$$C_\gamma = \frac{1}{\Gamma(1 - \gamma)} \left( \frac{\log 3}{1 - 3^{-\gamma}} - \frac{\log 2}{1 - 2^{-\gamma}} \right)^{-1},$$

$$k_{\gamma,0} = \frac{\gamma(3^\gamma - 2^\gamma)}{(3^\gamma - 1)(2^\gamma - 1)}, \quad k_{\gamma,1} = \frac{-\gamma}{2^\gamma - 1}, \quad k_{\gamma,2} = \frac{\gamma}{3^\gamma - 1};$$

$$C_a = \frac{\log 2}{\gamma} \left( \frac{1}{\log 2} - \frac{\gamma}{1 - 2^{-\gamma}} \right) + \frac{\Gamma'(1 - \gamma)}{\Gamma(1 - \gamma)},$$

$$k_{a,0} = C_\gamma k_{\gamma,0} C_a - \frac{\gamma}{(2^\gamma - 1)\Gamma(1 - \gamma)},$$

$$k_{a,1} = C_\gamma k_{\gamma,1} C_a + \frac{\gamma}{(2^\gamma - 1)\Gamma(1 - \gamma)},$$

$$k_{a,2} = C_\gamma k_{\gamma,2} C_a - \frac{\gamma}{(2^\gamma - 1)\Gamma(1 - \gamma)};$$

$$C_b = \frac{\gamma\Gamma'(1-\gamma) - 1 + \Gamma(1-\gamma)}{\gamma^2},$$

$$k_{b,0} = 1 + C_\gamma k_{\gamma,0} C_b + k_{\gamma,0} \frac{1 - \Gamma(1-\gamma)}{\gamma},$$

$$k_{b,1} = C_\gamma k_{\gamma,1} C_b + k_{\gamma,1} \frac{1 - \Gamma(1-\gamma)}{\gamma},$$

$$k_{b,2} = C_\gamma k_{\gamma,2} C_b + k_{\gamma,2} \frac{1 - \Gamma(1-\gamma)}{\gamma};$$

$$k_{x,0} = C_\gamma k_{\gamma,0} + (\gamma_-)^2 k_{b,0} - \gamma_- k_{a,0},$$

$$k_{x,1} = C_\gamma k_{\gamma,1} + (\gamma_-)^2 k_{b,1} - \gamma_- k_{a,1}, \quad k_{x,2} = C_\gamma k_{\gamma,2} + (\gamma_-)^2 k_{b,2} - \gamma_- k_{a,2}$$

and, for  $\gamma = 0$ ,

$$C_\gamma = 2(\log(3/2))^{-1}, \quad k_{\gamma,0} = (\log 2)^{-1} - (\log 3)^{-1},$$

$$k_{\gamma,1} = -(\log 2)^{-1}, \quad k_{\gamma,2} = (\log 3)^{-1};$$

$$C_a = 2^{-1} \log 2 + \Gamma'(1), \quad C_b = -\Gamma''(1)$$

and the rest follow similarly by continuity. Then the asymptotic variances, covariances and biases follow by combining (12) with (13)–(16) in the obvious way.

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