

NOISE-STABILITY AND CENTRAL LIMIT THEOREMS FOR EFFECTIVE RESISTANCE OF RANDOM ELECTRIC NETWORKS¹

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We investigate the (generalized) Walsh decomposition of point-to-point effective resistances on countable random electric networks with i.i.d. resistances. We show that it is concentrated on low levels, and thus point-to-point effective resistances are uniformly stable to noise. For graphs that satisfy some homogeneity property, we show in addition that it is concentrated on sets of small diameter. As a consequence, we compute the right order of the variance and prove a central limit theorem for the effective resistance through the discrete torus of side length n in \mathbb{Z}^d , when n goes to infinity.

1. Introduction. Consider a piece of conductive material whose resistivity possesses some microscopic disorder. One way to account for this disorder is to suppose that the material is an electric network made of tiny random resistances. Once this model is assumed, one typically wants to understand the behaviour of the macroscopic resistivity of the material. To make the picture more accurate, imagine that each edge of the lattice \mathbb{Z}^d is equipped with a resistance $r(e)$ belonging to some interval $[1, \Lambda]$ (we shall not prescribe any resistance unit). Suppose in addition that all resistances are random, independent and identically distributed. Our macroscopic piece of material is now the box $\mathcal{B}_n = \{0, \dots, n\}^d$, two sides of which we distinguish: $A_n = \{x \in \mathcal{B}_n \text{ s.t. } x_1 = 0\}$ and $Z_n = \{x \in \mathcal{B}_n \text{ s.t. } x_1 = n\}$. The *effective resistance* of the box \mathcal{B}_n is then defined as

$$\mathcal{R}_n = \inf_{\theta} \sum_{e \in E_n} r(e) \theta^2(e),$$

where the sum is over the set E_n of edges inside \mathcal{B}_n and the infimum is taken over all unit flows on E_n from A_n to Z_n (all precise definitions are postponed until Section 2). In the literature, the *effective conductivity* is more often the main character. It is simply the reciprocal value of the effective resistance and can also be defined as

$$(1) \quad \mathcal{C}_n = \inf_v \sum_{e \in E_n} c(e) (dv(e))^2,$$

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where the infimum is over all functions v on \mathcal{B}_n having value 0 on A_n and 1 on Z_n , $c(e) = 1/r(e)$ is the conductance of edge e , and $dv(e) := v(e_-) - v(e_+)$ is the difference of v along edge e . The unique minimizer in the definition of \mathcal{C}_n is the function that is 0 on A_n and 1 on Z_n and is discrete harmonic on $\mathcal{B}_n \setminus \{A_n, Z_n\}$. It is worth mentioning that the setting above is also relevant to describe the pressure field of a fluid through a weakly porous medium when the circulation of the fluid can be modelled with Poiseuille's law. The central problem is now to understand the asymptotic behaviour of \mathcal{R}_n (or, equivalently, of \mathcal{C}_n) as n goes to infinity.

A first step in this direction was accomplished in the setting of stochastic homogenization theory (cf. [19], Chapter 7). It is shown in [21], Section 3, that a law of large numbers holds (see also [22, 29] and [6] for related results). Namely, there is some positive constant μ such that

$$\frac{1}{n^{d-2}} \mathcal{C}_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mu \quad \text{that is} \quad n^{d-2} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \frac{1}{\mu}.$$

To understand the scaling, notice that the function $v_{\text{hom}} : x \mapsto x_1/n$ gives an upper bound of order n^{d-2} on the value of \mathcal{C}_n , and a flow θ satisfying $\theta(e) = \frac{1}{(n+1)^{d-1}} dv_{\text{hom}}$ gives an upper bound of order n^{2-d} on the value of \mathcal{R}_n .

A second step is to understand the fluctuations of \mathcal{C}_n and \mathcal{R}_n . If the optimal function in the definition of \mathcal{C}_n were v_{hom} , then \mathcal{C}_n would merely be a sum of $\Theta(n^d)$ i.i.d. random variables, each of variance $\Theta(n^{-4})$. The variance of \mathcal{C}_n would thus be of order $\Theta(n^{d-4})$, and that of \mathcal{R}_n of order $\Theta(n^{4-3d})$. A lower bound of this order was given by Wehr (cf. [31]) under some technical assumptions (see also Section 3.2 below). More recently, an upper bound of the same order was obtained by [16] for a different, but closely related quantity. We shall present in more details the work of Gloria and Otto at the end of this introduction.

The main purpose of the present paper is to derive the right order of the variances of \mathcal{C}_n and \mathcal{R}_n and in addition to make a step further in the understanding of their fluctuations by deriving Gaussian central limit theorems for these quantities. However, for technical reasons we shall only be able to do this in a translation invariant setting, namely for the effective resistance through the torus; cf. Theorem 5.2. This is the main result of the article. Our approach to obtain this result is however quite general and not restricted to graphs like \mathbb{Z}^d . Indeed, we shall study in Section 3 *the generalized Walsh decomposition* of point-to-point effective resistance on general infinite networks. This decomposition, sometimes called the Efron–Stein decomposition, is an extension of the Fourier–Walsh decomposition on the discrete cube and is related to a notion of noise sensitivity introduced in [2]. From now on, we shall drop the term “generalized” for simplicity. The Walsh decomposition of a square integrable function f of the resistances reads

$$f = \sum_S f_S,$$

where the sum runs over all finite subsets of the set of edges, f_S is a function of $(r(e))_{e \in S}$ for any S and f_S is orthogonal to $f_{S'}$ whenever $S \neq S'$. This decomposition has two immediate interesting features. First, the variance of f may be expressed as

$$\text{Var}(f) = \sum_{S \neq \emptyset} \|f_S\|_2^2.$$

Second, if f^ε is obtained from f by resampling independently each input $r(e)$ with probability ε , the correlation between f and its ε -noised version f^ε equals (see Proposition 3.3 for further details)

$$\text{Corr}(f, f^\varepsilon) = \frac{\sum_{S \neq \emptyset} (1 - \varepsilon)^{|S|} \|f_S\|_2^2}{\sum_{S \neq \emptyset} \|f_S\|_2^2}.$$

Thus, if f is nonconstant one may associate a ‘‘spectral probability measure’’ Q_f to f on the set of nonempty finite subsets of the set of edges:

$$Q_f(S) = \frac{\|f_S\|_2^2}{\text{Var}(f)},$$

and we see that understanding the distribution of $|S|$ under Q_f allows to control the noise-sensitivity of f . Our first result, Theorem 3.5, shows that the second moment of $|S|$ under Q_f when f is a point-to-point effective resistance, is bounded above by a constant depending only on Λ . This implies, loosely speaking, that the Walsh decomposition of the effective resistance is always concentrated on low levels. More precisely, consider the set of distributions of $|S|$ under Q_f when f runs over all possible point-to-point resistances on graphs equipped with independent resistances in $[1, \Lambda]$ with Λ fixed. Then our first result implies that this set of probability measures is tight. It implies also that effective resistances are always uniformly stable to noise in the sense of [2] (cf. Corollary 3.6) and that the Efron–Stein inequality is always sharp for estimating the variance of the effective resistance; cf. Corollary 3.7. Then we shall improve this result on a class of graphs which possess some homogeneity property. These graphs that we shall qualify as having *homogeneous currents* contain all quasi-transitive graphs; cf. Corollary 3.11. On those graphs we shall show that the Walsh decomposition is, loosely speaking, concentrated on sets of small diameter. This is the key to obtain a central limit theorem since the sets of resistances with bounded diameter exhibit only finite range dependence; cf. Section 4.

In Section 5, we shall adapt this general approach to the effective resistance through the discrete torus, deriving the optimal variance estimate and the Gaussian central limit theorem already mentioned.

We end this introduction by giving more details on the work [16], and comparing our results to theirs. Their work is close to the homogenization theory framework. Consider the discrete elliptic differential operator $d^*(cd(\cdot))$ corresponding

to random, translation invariant and ergodic conductances $c = (c(e))_{e \in \mathbb{E}^d}$ on \mathbb{Z}^d . Precise definitions of d^* and d are given in Section 2.1, but let us just mention that it gives, for a function v on \mathbb{Z}^d :

$$d^*(cdv)(x) = \sum_{y \sim x} c(x, y)(v(x) - v(y)).$$

Then, using the words of [16], homogenization theory (namely [22]) shows that there exists a constant matrix A such that the solution operator of $\nabla^*(A\nabla(\cdot))$ describes the large scale behaviour of the solutions operator of $d^*(cd(\cdot))$. Furthermore, A can be characterized by the so called corrector: for any ξ in \mathbb{R}^d , there exists a unique function ϕ_ξ on \mathbb{Z}^d (which is a function also of the conductances) such that $\nabla\phi_\xi$ is stationary, $\phi_\xi(0) = 0$, $\mathbb{E}(\nabla\phi_\xi) = 0$ and such that $g_\xi : x \mapsto \xi \cdot x + \phi_\xi(x)$ is discrete harmonic for $d^*(cd(\cdot))$ on \mathbb{Z}^d . Then A is characterized by

$$\xi \cdot A\xi = \mathbb{E}[\nabla g_\xi \cdot A\nabla g_\xi].$$

When the conductances are i.i.d., A equals μ times the identity matrix, and the constant μ is the same as in the law of large numbers of C_n stated above. We shall fix $\xi = (1, 0, \dots, 0)$ in the sequel. When one is interested in computing μ , Gloria and Otto remark that the preceding characterization is not computationally tractable. Thus, one has to find a way to efficiently estimate μ . The quantity C_n/n^{d-2} is therefore a reasonable estimator for μ . Putting aside for a moment the problem of controlling its bias, this is where the knowledge of its variance, and even of a central limit theorem, may be useful. Unfortunately, C_n lacks stationarity. This is a handicap for error analysis, as Gloria and Otto noticed in [16] for a quantity very similar to C_n . Next, they introduce a stationary approximation of the voltage, namely ϕ_T solving

$$\frac{1}{T}\phi_T + d^*(c(\xi + d\phi_T)) = 0 \quad \text{in } \mathbb{Z}^d.$$

Let η_L be an averaging cutoff function with support in $(0, n)^d$ (and some extra regularity condition). When T is large with respect to n , they show that the quantity

$$\mathcal{A}_n := \sum_{e \in \mathbb{E}^d} c(e)(\xi + d\phi_T)(e)^2 \eta_L(e),$$

is a good proxy for $n^{2-d}C_n$ and, furthermore, they show that the variance of \mathcal{A}_n is of order at most n^{-d} , with some extra polylogarithmic factor in T for $d = 2$. This order coincides with the variance order conjectured above for $n^{2-d}C_n$.

What we shall obtain in Theorem 5.2 is an optimal variance estimate and a central limit theorem for the effective conductance on the discrete torus of length n when n goes to infinity. To compare our results to those obtained by Gloria and Otto, we shall say that the precise quantity that we analyse is practically computable and stationary. Furthermore, in some sense, the discrete tori converge to \mathbb{Z}^d

better than the discrete cubes since they avoid boundary effects. Thus, the effective conductance on the torus may be a better estimator of μ . Notice, however, that the convergence of the normalized effective conductance to μ is known (cf. [9] and [28]) but not the rate of convergence. It would be interesting to investigate this rate, for instance, in the spirit of [17]. Second, our method works the same way whether $d = 2$ or not, and this is an advantage over Gloria and Otto's result, which makes a distinction between the two. Finally, the fact that we obtain a central limit theorem is really a step forward compared to [16] which only obtains variance estimates. On the other hand, Gloria and Otto obtain other interesting results, that we do not get by our method, notably concerning the integrability of the corrector itself (Proposition 2.1 in [16]).

After this paper was submitted for publication, we learned the existence of two preprints which address essentially the same question. Nolen [27] defines a continuous version of the effective conductance on the torus (but with discrete randomness) and shows a Gaussian approximation. He uses essentially two arguments: a second-order Poincaré inequality due to Chatterjee [11], and the results of Gloria and Otto on the boundedness in L^p of the corrector. The drawbacks of this approach are twofold. First, the bound obtained by Nolen in dimension 2 is suboptimal, because in dimension 2, integrability results of Gloria and Otto are weaker. Then the use of Chatterjee's inequality forces the elliptic conductances to have a special form of distribution (notably, it must be absolutely continuous with respect to the Lebesgue measure). In return, Nolen obtains a bound on the variation distance between the normalized effective conductance and the standard Gaussian distribution, which is of course a stronger conclusion than ours. The other preprint is by Biskup, Salvi and Wolff [4]. It shows a central limit theorem for the effective conductance on the grid with linear boundary condition. One serious limitation of their approach is that they require a small ellipticity contrast (i.e., Λ close enough to 1 in our setting). On the other hand, this paper has the advantage of giving an asymptotic equivalent of the variance of the effective conductance.

2. Preliminaries.

2.1. *Effective resistance and minimal current.* An excellent reference for background on electric networks is the book [23], Chapters 2 and 9 and we shall try to stick to its notation.

In the sequel, $G = (V, E)$ will be a *countable, locally finite, oriented, symmetric and connected graph*. Symmetric means that E is a symmetric subset of V^2 , that is, each edge of G occurs with both orientations in E , countable means here that both V and E are at most countable and locally finite means that every vertex has finite degree. When $e \in E$, we let e_- denote the tail of e and e_+ its head, we denote by $-e := (e_+, e_-)$ the edge e with reversed direction and let $E_{1/2}$ be a subset of E such that for every edge e , exactly one of e and $-e$ belongs to $E_{1/2}$.

For every collection $r \in (0, \infty)^{E_{1/2}}$, one may define the *electric network* (G, r) : it must be understood as a resistive network, where each edge e is a resistor with resistance $r(e)$. We shall sometimes use the notation $c(e)$ to denote the *conductance* of edge e , that is, $c(e) = 1/r(e)$. We define the *co-boundary operator* d from \mathbb{R}^V to \mathbb{R}^E by

$$dv(e) = v(e_-) - v(e_+),$$

and the *boundary operator* d^* from \mathbb{R}^E to \mathbb{R}^V by

$$d^*\theta(x) = \sum_{e_- = x} \theta(e).$$

Notice that dv plays the role of a gradient and $d^*\theta$ the role of a divergence.

For a fixed collection r , we define $\ell^2_-(E, r)$ as the Hilbert space of *antisymmetric functions on the edges that have bounded energy*:

$$\ell^2_-(E, r) = \{\theta \in \mathbb{R}^E \text{ s.t. } \mathcal{E}_r(\theta) < \infty \text{ and } \forall e \in E, \theta(e) = -\theta(-e)\},$$

where

$$\mathcal{E}_r(\theta) := \sum_{e \in E_{1/2}} r(e)\theta^2(e),$$

endowed with the scalar product:

$$(\theta, \theta')_r = \sum_{e \in E_{1/2}} r(e)\theta(e)\theta'(e).$$

We shall denote by $\|\theta\|_r := (\theta, \theta)_r^{1/2}$ the norm associated to this scalar product. Thus, $\mathcal{E}_r(\theta)$ is the square of the norm in $\ell^2_-(E, r)$ of θ and it is called the *energy* of θ . In the main part of the present paper (from Section 3.3 on), we shall be interested in *elliptic networks*, that is, networks (G, r) for which there is a finite constant $\Lambda \geq 1$ such that $r \in [1, \Lambda]^E$. In the whole article, $C(\Lambda)$ [resp., $C(\Lambda, G)$] will denote a constant, depending only on Λ (resp., on Λ and G), that may vary from time to time. Of course, all the sets $\ell^2_-(E, r)$ for $r \in [1, \Lambda]^E$ are the same, and we shall define this common set as $\ell^2_-(E)$:

$$(2) \quad \ell^2_-(E) := \left\{ \theta \in \mathbb{R}^E \text{ s.t. } \sum_{e \in E_{1/2}} \theta^2(e) < \infty \text{ and } \forall e \in E, \theta(e) = -\theta(-e) \right\}.$$

Let I be a nonnegative real number, and u and v two distinct vertices of G . A member θ of $\ell^2_-(E, r)$ is called a *flow of intensity I from u to v* if:

$$(3) \quad d^*\theta = I(\mathbf{1}_u - \mathbf{1}_v).$$

This means θ satisfies the node law on the network, except at u where a net flow of value I enters the network, and at v where a net flow of value I leaves the network. When θ is a flow from u to v , we say that θ is a unit flow if its intensity is 1. Among flows, some are particularly important: the *currents*, which satisfy Kirchhoff's cycle law as stated precisely in the following definition.

DEFINITION 2.1. For any $e \in E$, let $\chi_e = \mathbf{1}_{\{e\}} - \mathbf{1}_{\{-e\}}$ denote the unit flow along e . A current $i \in \ell_-^2(E, r)$ from u to v is a flow from u to v which satisfies Kirchhoff's cycle law: if e_1, \dots, e_n is an oriented cycle in G , then

$$\left(\sum_{k=1}^n \chi^{e_k}, i \right)_r = 0.$$

Currents are the flows which derive from a potential: if i is a current, there exists a function v on V such that $r(e)i(e) = dv(e)$ for any edge $e \in E$.

We may now define the effective resistance between two points u and v on the network (G, r) as the minimal energy of a unit flow between u and v :

$$\mathcal{R}^{u,v}(r) := \inf \left\{ \sum_{e \in E_{1/2}} r(e)\theta^2(e) \text{ s.t. } \theta \in \ell_-(E, r) \text{ is a unit flow from } u \text{ to } v \right\}.$$

Since one minimizes a Hilbert norm on a nonempty closed convex set (nonempty because it contains the flows induced by the paths from u to v), the infimum above is attained by a unique flow (cf. Proposition 9.2 of [23]). It turns out that this flow has the additional property of being a current. It is called the minimal unit current from u to v and we shall denote it by $i_r^{u,v}$. The term *minimal* stems from the fact that it minimizes the energy among all unit currents from u to v .

It is important to notice that currents from u to v of prescribed intensity may or may not be unique depending on the particular network; see Chapter 9 in [23]. On finite networks, however, it is well known that currents are unique (see, e.g., Chapter 2 in [23]). A useful fact about minimal currents is that they are limits of currents on finite graphs. Let us be more precise. Let $(G_n)_{n \geq 0}$ be a sequence of finite subgraphs of G that exhausts G , that is, such that $G_n \subset G_{n+1}$ and such that $G = \bigcup_{n \geq 0} G_n$. Suppose that u and v belong to G_0 and denote by G_n^W the “wired” network obtained from G by identifying all vertices outside G_n as a single vertex. Notice that one may identify the edges of G_n and G_n^W as subsets of E . Let $i_{r,n}^W$ be the (unique) current from u to v on G_n^W and see it as an element of $\ell_-(E, r)$ by putting zero flow on edges not in G_n^W . Then $i_{r,n}^W$ converges in $\ell_-(E, r)$ (and thus pointwise) as n goes to infinity (see Proposition 9.2 in [23]).

We finish this section with a useful lemma: the absolute value of a minimal unit current is at most one on any edge. This is intuitively clear since it must carry a unit mass from u to v and also minimize the energy.

LEMMA 2.2. For any distinct vertices u and v on a network (G, r) , and any edge e ,

$$|i_r^{u,v}(e)| \leq 1.$$

PROOF. Suppose first that G is finite. Let e be any edge of G and suppose, without loss of generality, that $i_r(e) > 0$. Let f denote the voltage associated to i_r with value zero at v (see Chapter 2 in [23]). It satisfies, for any edge e' :

$$df(e') = i_r(e'),$$

notably f is discrete harmonic on $V \setminus \{u, v\}$, that is,

$$\forall x \in V \setminus \{u, v\}, \quad \sum_{y \sim x} c(x, y)(f(x) - f(y)) = 0,$$

and $f(u) > 0$ (see, e.g., equation (2.3) in [23]). Furthermore, it satisfies the maximum principle on $V \setminus \{u, v\}$ (see Section 2.1 in [23]): for any $W \subset V \setminus \{u, v\}$ let ∂W be the set of vertices which are adjacent to a vertex in W . Then the maximum and the minimum of f on $\overline{W} = W \cup \partial W$ are attained on ∂W . Now, consider the set

$$A = \{x \in G \text{ s.t. } f(x) > f(e_+)\}.$$

A is a connected set of vertices containing u and e_- , and not containing v nor e_+ . Indeed, A clearly contains u and e_- . Furthermore, if A had a connected component W not containing u , then from the maximum principle, the maximum of f on \overline{W} would be obtained at some $x \in \partial W$, showing that there is $y \in W$ such that $x \sim y$ and $f(x) \geq f(y)$, but then x would be in A , contradicting the fact that W is a connected component of A .

Let Π be the set of edges with the tail in A and the head in A^c . Thus, e belongs to Π . Because of the node law,

$$\sum_{x \in A} d^* i_r(x) = d^* i_r(u) = 1,$$

and on the other hand,

$$\begin{aligned} \sum_{x \in A} d^* i_r(x) &= \sum_{x \in A} \sum_{e' \in E} i_r(e') \mathbf{1}_{e'_- = x} \\ &= \sum_{e' \in E} i_r(e') \sum_{x \in A} \mathbf{1}_{e'_- = x} \\ &= \sum_{e' \in E} i_r(e') \mathbf{1}_{e'_- \in A} \\ &= \sum_{e' \in \Pi} i_r(e'), \end{aligned}$$

since i_r is antisymmetric, but of course, $i_r(e') \geq 0$ for any e' in A . Thus,

$$0 < i_r(e) \leq \sum_{e' \in \Pi} i_r(e') = 1.$$

This shows the result on finite graphs. It implies the general result since the minimal unit current between u and v is the pointwise limit of a sequence of minimal unit currents between u and v on finite graphs. \square

2.2. *Partial derivatives of the effective resistance and the minimal current.* The functions $r \mapsto i_r^{u,v}$ and $r \mapsto \mathcal{R}^{u,v}(r)$ are smooth functions, as the next lemma shows. In the sequel, $\partial_e f$ denotes the partial derivative with respect to $r(e)$ of a function f on $(0, \infty)^{E_{1/2}}$ and $\partial_{e,e'}^2 f$ denotes $\partial_e \partial_{e'} f$.

LEMMA 2.3. *The functions $r \mapsto i_r^{u,v}(e)$, for any edge e , and $r \mapsto \mathcal{R}^{u,v}(r)$ admit partial derivatives of all orders. In addition, for any distinct vertices u, v and edges e, e' :*

- (i) $\forall e' \neq e, \partial_{e'} i_r^{u,v}(e) = \frac{i_r^{u,v}(e')}{r(e')} i_r^{e'}(e) = \frac{i_r^{u,v}(e')}{r(e)} i_r^e(e')$.
- (ii) $\forall e, \partial_e i_r^{u,v}(e) = \frac{i_r^{u,v}(e)}{r(e)} (i_r^e(e) - 1)$.
- (iii) $\forall e, \partial_e \mathcal{R}^{u,v}(r) = (i_r^{u,v}(e))^2$.

PROOF. Let us first suppose that G is finite. Then it is well known that $i_r^{u,v}(e)$ and $\mathcal{R}^{u,v}(r)$ are rational functions of r with no positive pole. See, for instance, [7], Theorem 2, page 46. The idea goes back to Kirchhoff (see [20] for an english translation of the original paper). The fact that $\partial_e \mathcal{R}^{u,v}(r) = (i_r^{u,v}(e))^2$ is also known; cf., for instance, [23], Exercise 2.69. One easy way to see it is as follows. Let r' be a collection of resistance differing from r only on edge e . Then, using the minimality of $i_r^{u,v}$,

$$\mathcal{E}_{r'}(i_{r'}^{u,v}) - \mathcal{E}_r(i_{r'}^{u,v}) \leq (\mathcal{R}^{u,v}(r') - \mathcal{R}^{u,v}(r)) \leq \mathcal{E}_{r'}(i_{r'}^{u,v}) - \mathcal{E}_r(i_{r'}^{u,v}),$$

and thus

$$(4) \quad (r'(e) - r(e))(i_{r'}^{u,v}(e))^2 \leq (\mathcal{R}^{u,v}(r') - \mathcal{R}^{u,v}(r)) \leq (r'(e) - r(e))(i_r^{u,v}(e))^2.$$

Letting $r'(e)$ go to $r(e)$ shows that $\partial_e \mathcal{R}^{u,v}(r) = (i_r^{u,v}(e))^2$.

To compute the partial derivatives of $r \mapsto i_r^{u,v}(e)$, let us differentiate the flow condition (3) and Kirchhoff's cycle law of Definition 2.1 with respect to $r(e')$. We obtain

$$\forall x \in V, \quad d^*[\partial_{e'} i_r^{u,v}](x) = 0,$$

and for every cycle γ on G ,

$$\sum_{e \in \gamma} [r(e) \partial_{e'} i_r^{u,v}(e) + i_r^{u,v}(e') \chi_{e'}(e)] = 0.$$

Thus, if one defines

$$j(e) = \partial_{e'} i_r^{u,v}(e) + \frac{i_r^{u,v}(e')}{r(e')} \chi_{e'}(e),$$

we get

$$\forall x \notin \{e'_-, e'_+\}, \quad d^* j(x) = 0,$$

and, for every cycle γ on G ,

$$\sum_{e \in \gamma} r(e)j(e) = 0.$$

Thus, j is a current from e_- to e_+ or from e_+ to e_- , depending on the sign of $d^*j(e'_-)$. Its intensity is deduced from

$$d^*j(e'_-) = d^*[\partial_{e'}i_r^{u,v}](e_-) + \frac{i_r^{u,v}(e')}{r(e')} = \frac{i_r^{u,v}(e')}{r(e')}.$$

Thus, from the unicity of currents on finite graphs, one gets

$$j = \frac{i_r^{u,v}(e')}{r(e')}i_r^{e'}.$$

Consequently,

$$\forall e' \neq e, \quad \partial_{e'}i_r^{u,v}(e) = \frac{i_r^{u,v}(e')}{r(e')}i_r^{e'}(e)$$

and

$$\forall e', \quad \partial_{e'}i_r^{u,v}(e') = \frac{i_r^{u,v}(e')}{r(e')}j(e') = \frac{i_r^{u,v}(e')}{r(e')}(i_r^{e'}(e') - 1).$$

Finally, for any $e \neq e'$,

$$\partial_{e',e}^2 \mathcal{R}^{u,v}(r) = \partial_{e'}(i_r^{u,v}(e))^2 = 2i_r^{u,v}(e)\frac{i_r^{u,v}(e')}{r(e')}i_r^{e'}(e).$$

But since $\partial_{e',e}^2 \mathcal{R}^{u,v}(r) = \partial_{e,e'}^2 \mathcal{R}^{u,v}(r)$, we obtain

$$i_r^{u,v}(e)\frac{i_r^{u,v}(e')}{r(e')}i_r^{e'}(e) = i_r^{u,v}(e')\frac{i_r^{u,v}(e)}{r(e)}i_r^e(e').$$

Now, take $(u, v) = e$ and $(u, v) = e'$ and notice that $i_r^e(e)$ and $i_r^{e'}(e')$ are always different from zero. We obtain

$$\frac{i_r^e(e')i_r^{e'}(e)}{r(e')} = \frac{(i_r^e(e'))^2}{r(e)}$$

and

$$\frac{i_r^e(e')i_r^{e'}(e)}{r(e)} = \frac{(i_r^{e'}(e))^2}{r(e')}.$$

Thus, one deduces that $i_r^e(e') = 0$ if and only if $i_r^{e'}(e) = 0$ and in any case,

$$(5) \quad \frac{i_r^e(e')}{r(e)} = \frac{i_r^{e'}(e)}{r(e')}.$$

This last relation is called the *reciprocity law*. See [23], Chapter 2 for another proof. This concludes the proof of the lemma on finite graphs.

Now, let \mathbf{G} be infinite, r belong to $(0, \infty)^{\mathbf{E}_{1/2}}$ and e, e' in $\mathbf{E}_{1/2}$. As explained in Section 2.1, for any u and v , $i_r^{u,v}$ is the limit, in $\ell^2_-(\mathbf{E}, r)$ of a sequence $i_{r,n}^W$ of unit currents from u to v on “wired” finite graphs G_n^W . Notably, $i_r^{u,v}(e)$ is the pointwise limit of $i_{r,n}^W(e)$. From the formulas of the derivatives on finite graphs and Lemma 2.2, one sees that $r(e') \mapsto i_{r,n}^W(e)$ form an equi-continuous family of functions on any compact interval I of $(0, \infty)$. It follows from Arzela–Ascoli’s theorem that the convergence of $i_{r,n}^{u,v}(e)$ to $i_r^{u,v}(e)$ is uniform when $r(e')$ runs over I . Notably, this implies the continuity of $r(e') \mapsto i_r^{u,v}(e)$ for any e, e' . Then, from the formulas (i) and (ii) of the derivative $\partial_{e'} i_r^{u,v}(e)$ one sees that the derivative itself converges uniformly when $r(e')$ runs over I . Then $r(e') \mapsto i_r^{u,v}(e)$ is differentiable on $(0, \infty)$ and its derivative is the limit of the derivatives $\partial_{e'} i_{r,n}^{u,v}(e)$. This shows the formulas for $\partial_{e'} i_r^{u,v}(e)$. Formula (iii) is then a consequence of (4), since $r(e) \mapsto i_r^{u,v}(e)$ is continuous. \square

REMARK 1. A similar formula, relating the partial derivative of the voltage drop through e with respect to the conductance $c(e')$ to the voltage induced through e' by a voltage source between e_- and e_+ was used in [26], Proposition 1, in [16], Lemma 2.4 and in [4], Proposition 2.5.

The formula satisfied by the partial derivatives of $r \mapsto i_r^{u,v}$ in Lemma 2.3 allows us to control $i_r^{u,v}$ after a finite number of modifications of the individual resistances.

LEMMA 2.4. For any subset $S \subset \mathbf{E}_{1/2}$, define $r^{S \leftarrow r'}$ by:

$$r^{S \leftarrow r'}(e) = \begin{cases} r'(e), & \text{if } e \in S, \\ r(e), & \text{else.} \end{cases}$$

Then:

(i) For any $e \in \mathbf{E}_{1/2}$, if $r'(e) \leq r(e)$,

$$|i_r^{u,v}(e)| \leq |i_{r^{S \leftarrow r'}}^{u,v}(e)| \leq \frac{r(e)}{r'(e)} |i_r^{u,v}(e)|.$$

(ii) Let $g(x, y) = \max\{\frac{x}{y}, \frac{y}{x}\}$ for x and y in $(0, +\infty)$. For any nonempty, finite subset $S \subset \mathbf{E}_{1/2}$, any edge $e \in \mathbf{E}_{1/2}$, and any distinct vertices u and v , if $e \notin S$,

$$|i_{r^{S \leftarrow r'}}^{u,v}(e)| \leq |i_r^{u,v}(e)| + \left(\sum_{e' \in S} |i_{r'}^{u,v}(e')| \right) \prod_{e' \in S} g(r(e'), r'(e')),$$

and if $e \in S$,

$$|i_{r^{S \leftarrow r'}}^{u,v}(e)| \leq \left(\sum_{e' \in S} |i_{r'}^{u,v}(e')| \right) \prod_{e' \in S} g(r(e'), r'(e')).$$

PROOF. First, let us prove (i). Let $e \in E_{1/2}$ and consider $\{r(e'), e' \neq e\}$ fixed in $(0, \infty)^{E_{1/2}}$. To simplify notation, for $x > 0$, define

$$f(x) := i_{r e \leftarrow x}^{u,v}(e)$$

and

$$g(x) := i_{r e \leftarrow x}^e(e).$$

Then one gets from Lemma 2.3,

$$f'(x) = \frac{f(x)}{x}(g(x) - 1).$$

This is a homogeneous differential equation of order 1 on $(0, \infty)$ which implies that f is of constant sign: either it is zero on $(0, \infty)$, or it is positive on $(0, \infty)$, or it is negative on $(0, \infty)$. Suppose that it is not identically zero and orient e so that f is positive. Notice that $g(x) \in]0, 1]$ for every $x > 0$. Then f' is negative, which shows that for any $x_0 \leq x_1$,

$$f(x_1) \leq f(x_0).$$

But also,

$$(-\ln f)'(x) \leq \frac{1}{x}$$

and thus

$$f(x_0) \leq \frac{x_1}{x_0} f(x_1).$$

This shows (i), and notably implies the following:

$$(6) \quad |i_{r e \leftarrow r'}(e)| \leq |i_r(e)| \max \left\{ 1, \frac{r(e)}{r'(e)} \right\}.$$

Now, let e' and e be distinct edges in $E_{1/2}$. Using Lemma 2.3, and dropping the superscript u, v ,

$$\frac{\partial i_{r e' \leftarrow x}(e)}{\partial x} = \frac{i_{r e' \leftarrow x}(e')}{x} i_{r e' \leftarrow x}^{e'}(e).$$

Recall from Lemma 2.2 that $|i_{r e' \leftarrow x}^{e'}(e)|$ is not larger than one, and from inequality (6),

$$|i_{r e' \leftarrow x}(e')| \leq |i_r(e')| \max \left\{ 1, \frac{r(e')}{x} \right\}.$$

Thus,

$$\begin{aligned} |i_{r,e' \leftarrow r'}(e) - i_r(e)| &\leq |i_r(e')| \left| \int_{r(e)}^{r(e')} \frac{1}{x} \max\left\{1, \frac{r(e')}{x}\right\} dx \right| \\ &\leq |i_r(e')| \max\{r(e'), r'(e')\} \left| \frac{1}{r(e')} - \frac{1}{r'(e')} \right| \\ &= |i_r(e')|(g(r(e'), r'(e')) - 1). \end{aligned}$$

Thus,

$$(7) \quad |i_{r,e' \leftarrow r'}(e)| \leq |i_r(e)| + |i_r(e')|(g(r(e'), r'(e')) - 1).$$

Notice that we have now established (ii) for sets S of size 1. Now, let us prove the first part of (ii) by induction on the size of the set S . Let $e \notin S$. Using inequality (7),

$$\begin{aligned} |i_{r,S \leftarrow r'}(e)| &= |i_{(r,S \setminus \{e'\} \leftarrow r')e' \leftarrow r'}(e)| \\ &\leq |i_{r,S \setminus \{e'\} \leftarrow r'}(e)| + |i_{r,S \setminus \{e'\} \leftarrow r'}(e')|(g(r(e'), r'(e')) - 1). \end{aligned}$$

From the induction hypothesis,

$$|i_{r,S \setminus \{e'\} \leftarrow r'}(e)| \leq |i_r(e)| + \left(\sum_{e'' \in S \setminus \{e'\}} |i_r(e'')| \right) \prod_{e'' \in S \setminus \{e'\}} g(r(e''), r'(e''))$$

and

$$|i_{r,S \setminus \{e'\} \leftarrow r'}(e')| \leq |i_r(e')| + \left(\sum_{e'' \in S \setminus \{e'\}} |i_r(e'')| \right) \prod_{e'' \in S \setminus \{e'\}} g(r(e''), r'(e'')).$$

Gathering terms, and noting that g is not smaller than 1 allows to complete the induction step. Finally, the second part of (ii) is a consequence of the first part and inequality (7). Indeed, if $e \in S$,

$$\begin{aligned} |i_{r,S \leftarrow r'}(e)| &= |i_{(r,S \setminus \{e\} \leftarrow r')e \leftarrow r'}(e)| \\ &\leq |i_{r,S \setminus \{e\} \leftarrow r'}(e)|g(r(e), r'(e)) \\ &\leq \left(|i_r(e)| + \sum_{e' \in S \setminus \{e\}} |i_r(e')| \prod_{e' \in S \setminus \{e\}} g(r(e'), r'(e')) \right) g(r(e), r'(e)) \\ &\leq \sum_{e' \in S} |i_r(e')| \prod_{e' \in S} g(r(e'), r'(e')). \end{aligned} \quad \square$$

2.3. *The random setting.* For any $e \in E_{1/2}$, we let μ_e denote some probability measure on $(0, \infty)$. The collection of resistances r will be supposed to be random with distribution $\mathbb{P} := \bigotimes_{e \in E_{1/2}} \mu_e$. Furthermore, in the sequel, r' will usually denote an independent copy of r .

We shall always suppose that the resistances are square integrable. Recall that the network is said to be *elliptic* if there is a constant $\Lambda > 1$ such that $r \in [1, \Lambda]^{E_{1/2}}$. This will be a crucial assumption from Section 3.3 on. Finally, we will use the notation:

$$\forall e \in E, \quad m_p(e) = \mathbb{E}[|r(e) - \mathbb{E}(r(e))|^p]^{1/p}.$$

3. The Walsh decomposition.

3.1. *Definition and basic properties.* For any $e \in E_{1/2}$ let Δ_e be the following operator on $L^2(\mathbb{R}^{E_{1/2}}, \mathbb{P})$:

$$\Delta_e f(r) = f(r) - \int f(r) d\mu_e(r(e)).$$

From now on, $S \subset E_{1/2}$ will always mean that S is a *finite* subset of $E_{1/2}$. For $S \subset E_{1/2}$, we shall denote by r_S the collection $(r(e))_{e \in S}$ of random variables (which is empty if S is empty). Let f be in $L^2(\mathbb{R}^{E_{1/2}}, \mathbb{P})$ and notice that

$$\mathbb{E}[f(r)|r_S] = \int f(r) \bigotimes_{e \in S^c} d\mu_e(r(e)).$$

Then, for any $S \subset E_{1/2}$ we define

$$f_S(r) = \sum_{T \subset S} (-1)^{|S \setminus T|} \mathbb{E}[f(r)|r_T].$$

Notice that $f_\emptyset = \mathbb{E}(f)$. It is easy to see that an alternative definition is

$$f_S(r) = \mathbb{E}\left[\left(\prod_{e \in S} \Delta_e\right) f(r) \middle| r_S\right],$$

with the usual convention that when S is empty, the product of operators over S is the identity. Then $(f_S)_{S \subset E_{1/2}}$ is an orthogonal decomposition of f known as the Efron–Stein or the (generalized) Walsh decomposition (cf. [8, 15, 18] and [25], e.g.). The basic properties of this decomposition are gathered in the following proposition, where infinite sums in $L^2(\mathbb{R}^{E_{1/2}}, \mathbb{P})$ are understood as follows: $\sum_S f_S$ is the limit in $L^2(\mathbb{R}^{E_{1/2}}, \mathbb{P})$ of the net $S \mapsto \sum_{T \subset S} f_T$, defined on the set of finite subsets of $E_{1/2}$ with inclusion as partial order (in other words, this corresponds to unconditional summability in L^2).

PROPOSITION 3.1. For any f and g in $L^2(\mathbb{R}^{E_{1/2}}, \mathbb{P})$,

$$f = \sum_S f_S,$$

$$\mathbb{E}(fg) = \sum_S \mathbb{E}(f_S g_S),$$

and thus

$$S \neq S' \Rightarrow \mathbb{E}(f_S g_{S'}) = 0.$$

Furthermore, for any $e \in E_{1/2}$,

$$\Delta_e f = \sum_{S \ni e} f_S.$$

As a consequence, for any integer $k \geq 1$,

$$\sum_{e_1, \dots, e_k \in E_{1/2}} \left\| \left(\prod_{i=1}^k \Delta_{e_i} \right) f \right\|^2 = \sum_S |S|^k \|f_S\|_2^2.$$

PROOF. Let us first suppose that $E_{1/2}$ is finite. Then Proposition 3.1 is well known (cf. [8]) but we shall quickly recall the proof for the sake of completeness.

For a subset S any subset of $E_{1/2}$, let L_S be the operator on $L^2(\mathbb{R}^{E_{1/2}})$ defined by

$$(8) \quad L_S f(r) = \int f(r) \prod_{e \in S} d\mu_e(r(e)).$$

Let 1 denote the identity operator. Notice that $L_{\{e\}}$ and $L_{\{e'\}}$ commute for any e and e' . Since $\Delta_{e'} = 1 - L_{\{e\}}$, $\Delta_{e'}$ and $L_{\{e\}}$ commute and

$$1 = \prod_{e \in E_{1/2}} (\Delta_{\{e\}} + L_{\{e\}}) = \sum_{S \subset E_{1/2}} L_{S^c} \prod_{e \in S} \Delta_{\{e\}}.$$

Since

$$f_S = \left(\prod_{e \in S} \Delta_{\{e\}} \right) f,$$

this shows that $f_\emptyset = \mathbb{E}(f)$ and $f = \sum_{S \subset E_{1/2}} f_S$. Now, remark that for any edge e ,

$$L_{\{e\}} \Delta_e = 0.$$

Thus, for any S and any $e \in S$,

$$L_{\{e\}} f_S = 0.$$

This implies that $\Delta_e f = \sum_{S \ni e} f_S$. Now, if $S \neq S'$, suppose, for instance, that there is some $e \in S \setminus S'$:

$$\mathbb{E}[f_S g_{S'}] = L_{E_{1/2}}(f_S g_{S'}) = L_{E_{1/2}} L_{\{e\}}(f_S g_{S'}) = L_{E_{1/2}}(g_{S'} L_{\{e\}}(f_S)) = 0.$$

This implies

$$\mathbb{E}(fg) = \sum_S \mathbb{E}(f_S g_S).$$

Finally,

$$\begin{aligned} \sum_{e_1, \dots, e_k \in E_{1/2}} \left\| \left(\prod_{i=1}^k \Delta_{e_i} \right) f \right\|^2 &= \sum_{e_1, \dots, e_k \in E_{1/2}} \left\| \sum_{S \supset \{e_1, \dots, e_k\}} f_S \right\|^2 \\ &= \sum_{e_1, \dots, e_k \in E_{1/2}} \sum_{S \supset \{e_1, \dots, e_k\}} \|f_S\|^2 \\ &= \sum_{S \subset E_{1/2}} \sum_{\{e_1, \dots, e_k\} \subset S} \|f_S\|^2 \\ &= \sum_{S \subset E_{1/2}} |S|^k \|f_S\|^2. \end{aligned}$$

Now, let us suppose that $E_{1/2}$ is countable, and take some exhaustion $(E_n)_{n \geq 0}$ of the edges: E_n is finite for any n , $E_n \subset E_{n+1}$ and $E_{1/2} = \bigcup_n E_n$. Denote by f_n the conditional expectation of f with respect to r_{E_n} . We have

$$f_n = L_{E_n^c} f$$

and thus,

$$(9) \quad (f_n)_S = f_S \mathbf{1}_{S \subset E_n},$$

which implies

$$f_n = \sum_{S \subset E_n} f_S \quad \text{and} \quad \|f_n\|_2^2 = \sum_{S \subset E_n} \|f_S\|_2^2.$$

Since $(f_n)_{n \in \mathbb{N}}$ converges to f in L^2 , we know that $\|f_n\|_2^2$ converges to $\|f\|_2^2$. It is then standard to see that $\sum_S f_S$ forms a Cauchy net and that its limit is the same as the limit of f_n , that is f . It shows notably that $\mathbb{E}(f^2) = \sum_S \mathbb{E}(f_S^2)$, from which one derives $\mathbb{E}(fg) = \sum_S \mathbb{E}(f_S g_S)$. All the other properties can then easily be derived by standard limit arguments. \square

A trivial consequence of Proposition 3.1 is the Efron–Stein inequality (cf. [15]).

COROLLARY 3.2 (Efron–Stein’s inequality). *For any f in $L^2(\mathbb{R}^{E_{1/2}}, \mathbb{P})$*

$$\text{Var}(f) \leq \sum_{e \in E_{1/2}} \|\Delta_e f\|^2.$$

PROOF. Since the Efron–Stein decomposition is orthogonal and $f_\emptyset = \mathbb{E}(f)$,

$$\begin{aligned} \text{Var}(f) &= \mathbb{E}(f^2) - \mathbb{E}(f)^2 \\ &= \sum_S \mathbb{E}(f_S^2) - f_\emptyset^2 \\ &= \sum_{S \neq \emptyset} \mathbb{E}(f_S^2). \end{aligned}$$

On the other hand, since $\Delta_e f = \sum_{S \ni e} f_S$,

$$\begin{aligned} \sum_{e \in E_{1/2}} \|\Delta_e f\|^2 &= \sum_{e \in E_{1/2}} \sum_{S \ni e} \mathbb{E}(f_S^2) \\ &= \sum_S \sum_{e \in S} \mathbb{E}(f_S^2) \\ &= \sum_S |S| \mathbb{E}(f_S^2) \\ &\geq \sum_{S \neq \emptyset} \mathbb{E}(f_S^2). \end{aligned}$$

□

It is also clear that the Efron–Stein inequality is an equality if and only if $f = \sum_{|S| \leq 1} f_S$. This means that f is a constant plus a sum of independent random variables. One sees also that if the variance of f is concentrated on functions f_S such that S is small, then the Efron–Stein inequality is sharp up to a multiplicative constant.

There are a number of models of statistical physics flavour where the Efron–Stein inequality is not sharp. Chatterjee (cf. [10]) calls this phenomenon “super-concentration.” This holds, for instance, for the first passage percolation time between two distant points on \mathbb{Z}^d when $d \geq 2$ (cf. [1, 2]), which may be defined in our setting as

$$(10) \quad T_r(u, v) := \inf_{\gamma: u \rightarrow v} \sum_{e \in \gamma} r(e),$$

where the infimum is over all paths from u to v . Since super-concentration implies that some part of the variance of f is concentrated on large sets, one sees that, informally, it is related to high complexity (or high nonlinearity) of the function f .

It is also related to some noise-sensitivity of the function. Indeed, there is a close link between the Walsh decomposition and a notion of noise introduced by [2]. Let r and r' be two independent random variables with the same distribution $\mathbb{P} := \otimes_{e \in E_{1/2}} \mu_e$. Let $\varepsilon \in]0, 1[$. One constructs a noisy version r^ε of r by replacing with probability ε , at random and independently for any edge e , the variable $r(e)$ by its independent copy $r'(e)$.

PROPOSITION 3.3. *For any $f \in L^2(\mathbb{R}^{E_{1/2}}, \mathbb{P})$,*

$$\mathbb{E}[f(r^\varepsilon)|r] = \sum_S (1 - \varepsilon)^{|S|} f_S(r),$$

and thus,

$$\text{Cov}(f(r^\varepsilon), f(r)) = \sum_{S \neq \emptyset} (1 - \varepsilon)^{|S|} \|f_S\|_2^2.$$

PROOF. Let r and r' be two independent copies of law \mathbb{P} . Let S_ε be the (possibly infinite) random subset of \mathbf{E} drawn at random as follows: $(\mathbf{1}_{e \in S_\varepsilon})_{e \in \mathbf{E}_{1/2}}$ are i.i.d. with distribution Bernoulli of parameter $\varepsilon \in [0, 1]$, independent of (r, r') . Now, we define the following linear operators from $L^2(\mathbb{R}^{\mathbf{E}_{1/2}})$ to $L^2(\mathbb{R}^{\mathbf{E}_{1/2}} \times \mathbb{R}^{\mathbf{E}_{1/2}})$:

$$L'_{\{e\}} f(r) = f(r^{e \leftarrow r'}).$$

Then the noisy version of $f(r)$ may be written as

$$f(r^\varepsilon) = \prod_{e \in S_\varepsilon} L'_{\{e\}} f(r).$$

Notice that

$$\begin{cases} \forall e \neq e', & L'_{\{e\}} L_{\{e'\}} = L_{\{e'\}} L'_{\{e\}}, \\ \forall e, & L'_{\{e\}} L_{\{e\}} = L_{\{e\}}, \\ \forall e, & L_{\{e\}} L'_{\{e\}} = L'_{\{e\}}. \end{cases}$$

Thus,

$$L'_{\{e\}} \Delta_e = L'_{\{e\}} - L_{\{e\}}.$$

Whence

$$\begin{aligned} f(r^\varepsilon) &= \sum_{S \subset \mathbf{E}_{1/2}} \prod_{e \in S_\varepsilon} L'_{\{e\}} f_S(r) \\ &= \sum_{S \subset \mathbf{E}_{1/2}} \prod_{e \in S_\varepsilon} L'_{\{e\}} \prod_{e \in S^c} L_{\{e\}} \prod_{e \in S} \Delta_e f(r) \\ &= \sum_{S \subset \mathbf{E}_{1/2}} \prod_{e \in S^c} L_{\{e\}} \prod_{e \in S \cap S_\varepsilon} (L'_{\{e\}} - L_{\{e\}}) \prod_{e \in S \setminus S_\varepsilon} \Delta_e f(r). \end{aligned}$$

Now, denote by $L'_{\{e\}}$ the operator on $L^2(\mathbb{R}^{\mathbf{E}_{1/2}} \times \mathbb{R}^{\mathbf{E}_{1/2}})$ which integrates $r'(e)$. Notice that

$$L'_{\{e\}}(L'_{\{e\}} - L_{\{e\}}) = 0 \quad \text{and} \quad L'_{\{e\}} \Delta_e = \Delta_e.$$

Then

$$\begin{aligned} \mathbb{E}[f(r^\varepsilon) | r, S_\varepsilon] &= \prod_{e \in \mathbf{E}_{1/2}} L'_{\{e\}}((r, r') \mapsto f(r^\varepsilon)) \\ &= \sum_{S \subset \mathbf{E}_{1/2}} \prod_{e \in S^c} L_{\{e\}} \prod_{e \in S \setminus S_\varepsilon} \Delta_e f(r) \mathbf{1}_{S \cap S_\varepsilon = \emptyset} \\ &= \sum_{S \subset \mathbf{E}_{1/2}} \prod_{e \in S^c} L_{\{e\}} \prod_{e \in S} \Delta_e f(r) \mathbf{1}_{S \cap S_\varepsilon = \emptyset}. \end{aligned}$$

Thus,

$$\mathbb{E}[f(r^\varepsilon)|r] = \sum_{S \subset E_{1/2}} f_S(r) \mathbb{P}(S \cap S_\varepsilon = \emptyset) = \sum_{S \subset E_{1/2}} (1 - \varepsilon)^{|S|} f_S(r). \quad \square$$

To see another, closely related, interpretation of sensitivity to noise, called chaos, see [10]. Contrarily to what happens to the first passage percolation times, simulations suggest that the minimal current is extremely immune to noise. This tends to suggest that the Efron–Stein inequality could always be sharp in this context, and this is what we shall prove in the following sections. We shall in fact prove much more in the context of *current-homogeneous graphs*, namely that the Walsh decomposition is concentrated not only on sets of small size, but already on sets of small diameter.

Finally, to end the parallel between first passage percolation and effective resistance, note that there is a means to interpolate between those two quantities. Indeed, let us define, for $p \in [1, 2]$:

$$\mathcal{R}_r^{u,v}(p) = \inf_{\theta: u \rightarrow v} \sum_{e \in E_{1/2}} r(e) |\theta(e)|^p,$$

where the infimum is over all unit flows from u to v , and recall the definition (10) of $T_r(u, v)$, the minimum passage time from u to v . Then $\mathcal{R}_r^{u,v}(1) = T_r(u, v)$ and $\mathcal{R}_r^{u,v}(2) = \mathcal{R}^{u,v}(r)$. The quantity $\mathcal{R}_r^{u,v}(p)$ is called the p -resistance between u and v . Since the distribution of the Walsh decomposition is dramatically different when p equals 1 or 2, it would be interesting to investigate the evolution of the Walsh decomposition of $\mathcal{R}_r^{u,v}(p)$ when p varies continuously from 2 to 1.

3.2. *Concentration of the Walsh decomposition on low levels.* First, let us study the bound given by the Efron–Stein inequality.

LEMMA 3.4. *For any $e \in E_{1/2}$,*

$$\alpha_-(e) \mathbb{E}[r^2(e)(i_r^{u,v}(e))^4] \leq \|\Delta_e \mathcal{R}^{u,v}\|_2^2 \leq \alpha_+(e) \mathbb{E}[r^2(e)(i_r^{u,v}(e))^4],$$

where

$$\alpha_-(e) = \frac{\mathbb{E}[(r(e) - r'(e))_+^2 \min\{1, 1/(r^4(e))\}]}{\mathbb{E}[\max\{r^2(e), 1/(r^2(e))\}]}$$

and

$$\alpha_+(e) = \frac{\mathbb{E}[(r'(e) - r(e))_+^2 \max\{1, 1/(r^4(e))\}]}{\mathbb{E}[\min\{r^2(e), 1/(r^2(e))\}]}.$$

PROOF. Let r and r' be two independent random variables with the same distribution $\mathbb{P} := \otimes_{e \in E_{1/2}} \mu_e$. Inequality (4) gives

$$\begin{aligned} \|\Delta_e \mathcal{R}^{u,v}\|_2^2 &= \mathbb{E}[(\mathcal{R}^{u,v}(r) - \mathcal{R}^{u,v}(r^{e \leftarrow r'}))_+^2] \\ &\geq \mathbb{E}[(r(e) - r'(e))_+^2 i_r^{u,v}(e)^4]. \end{aligned}$$

Now, Lemma 2.4 allows to decouple positive functions of $r(e)$ and powers of $|i_r(e)|$. Let F and G be nonnegative functions on $(0, +\infty)$ and p be a positive real number. Then

$$\begin{aligned} (11) \quad &\mathbb{E}[F(r(e)) |i_r^{u,v}(e)|^p] \\ &\leq \mathbb{E}[G(r(e)) |i_r^{u,v}(e)|^p] \frac{\mathbb{E}[F(r(e)) \max\{1, 1/(r^p(e))\}]}{\mathbb{E}[G(r(e)) \min\{1, 1/(r^p(e))\}]} \end{aligned}$$

Indeed, using Lemma 2.4, for any r ,

$$|i_r^{u,v}(e)| \leq \max\left\{1, \frac{1}{r(e)}\right\} |i_{r^{e \leftarrow 1}}^{u,v}(e)|.$$

Thus,

$$\begin{aligned} \mathbb{E}[F(r(e)) |i_r^{u,v}(e)|^p] &\leq \mathbb{E}\left[F(r(e)) \max\left\{1, \frac{1}{r^p(e)}\right\} |i_{r^{e \leftarrow 1}}^{u,v}(e)|^p\right] \\ &= \mathbb{E}\left[F(r(e)) \max\left\{1, \frac{1}{r^p(e)}\right\}\right] \mathbb{E}[|i_{r^{e \leftarrow 1}}^{u,v}(e)|^p], \end{aligned}$$

since $r(e)$ and $i_{r^{e \leftarrow 1}}^{u,v}(e)$ are independent. Similarly,

$$\begin{aligned} \mathbb{E}[G(r(e)) |i_r^{u,v}(e)|^p] &\geq \mathbb{E}\left[G(r(e)) \min\left\{1, \frac{1}{r^p(e)}\right\} |i_{r^{e \leftarrow 1}}^{u,v}(e)|^p\right] \\ &= \mathbb{E}\left[G(r(e)) \min\left\{1, \frac{1}{r^p(e)}\right\}\right] \mathbb{E}[|i_{r^{e \leftarrow 1}}^{u,v}(e)|^p]. \end{aligned}$$

Thus,

$$\|\Delta_e \mathcal{R}^{u,v}\|_2^2 \geq \frac{\mathbb{E}[(r(e) - r'(e))_+^2 \min\{1, 1/(r^4(e))\}]}{\mathbb{E}[\max\{r^2(e), 1/(r^2(e))\}]} \mathbb{E}[r^2(e) i_{r^{e \leftarrow 1}}^{u,v}(e)^4].$$

On the other hand,

$$\begin{aligned} \|\Delta_e \mathcal{R}^{u,v}\|_2^2 &= \mathbb{E}[(\mathcal{R}^{u,v}(r^{e \leftarrow r'}) - \mathcal{R}^{u,v}(r))_+^2] \\ &\leq \mathbb{E}[(r'(e) - r(e))_+^2 i_r^{u,v}(e)^4] \\ &\leq \frac{\mathbb{E}[(r'(e) - r(e))_+^2 \max\{1, 1/(r^4(e))\}]}{\mathbb{E}[\min\{r^2(e), 1/(r^2(e))\}]} \mathbb{E}[r^2(e) i_r^{u,v}(e)^4]. \quad \square \end{aligned}$$

The following theorem shows that the Walsh decompositions of point-to-point effective resistances are uniformly concentrated (in terms of the L^2 -norm) on sets of small size.

THEOREM 3.5. *There is a universal constant $C \in (0, +\infty)$ such that if one defines in $[0, +\infty]$:*

$$K(\mu) = C \sup_e (\mathbb{E}[r^8(e)] + \mathbb{E}[r^{-8}(e)])^6 \sup_e \frac{\mathbb{E}[(r(e) - r'(e))^2(1/(r^6(e)) + r^6(e))]}{\mathbb{E}[(r(e) - r'(e))^2_+ \min\{1, 1/(r^4(e))\}]},$$

then for any graph G and any pair of vertices (u, v) ,

$$\sum_S |S|^2 \|\mathcal{R}_S^{u,v}\|_2^2 \leq K(\mu) \sum_{S \neq \emptyset} \|\mathcal{R}_S^{u,v}\|_2^2.$$

Consequently, for any $k \geq 1$, any graph G and any pair of vertices (u, v)

$$\sum_{|S| \geq k} \|\mathcal{R}_S^{u,v}\|_2^2 \leq \frac{K(\mu)}{k^2} \sum_{S \neq \emptyset} \|\mathcal{R}_S^{u,v}\|_2^2.$$

PROOF. We fix u and v and drop the superscript u, v . Proposition 3.1 implies

$$\sum_S |S|^2 \|\mathcal{R}_S\|_2^2 = \sum_{e, e'} \|\Delta_e \Delta_{e'} \mathcal{R}\|_2^2.$$

The first step is to prove the following:

$$(12) \quad \sum_{e \neq e'} \|\Delta_e \Delta_{e'} \mathcal{R}\|_2^2 \leq D(\mu) \sum_e \|\Delta_e \mathcal{R}^{u,v}\|_2^2.$$

Suppose for the moment that (12) is true. Then

$$\begin{aligned} \sum_S |S|^2 \cdot \|\mathcal{R}_S^{u,v}\|_2^2 &= \sum_{e, e'} \|\Delta_e \Delta_{e'} \mathcal{R}^{u,v}\|_2^2 \\ &= \sum_{e \neq e'} \|\Delta_e \Delta_{e'} \mathcal{R}^{u,v}\|_2^2 + \sum_e \|\Delta_e \mathcal{R}^{u,v}\|_2^2 \\ (13) \quad &\leq (D(\mu) + 1) \sum_e \|\Delta_e \mathcal{R}^{u,v}\|_2^2 \\ &= (D(\mu) + 1) \sum_S |S| \|\mathcal{R}_S^{u,v}\|_2^2. \end{aligned}$$

Then, for any $k \in \mathbb{N}^*$,

$$\begin{aligned} \sum_{|S| \geq k} |S| \|\mathcal{R}_S^{u,v}\|_2^2 &\leq \frac{1}{k} \sum_S |S|^2 \|\mathcal{R}_S^{u,v}\|_2^2 \\ &\leq \frac{(D(\mu) + 1)}{k} \sum_S |S| \|\mathcal{R}_S^{u,v}\|_2^2. \end{aligned}$$

Thus, for every $k \geq 2(D(\mu) + 1)$,

$$\sum_S |S| \|\mathcal{R}_S^{u,v}\|_2^2 \leq k \sum_{0 < |S| < k} \|\mathcal{R}_S^{u,v}\|_2^2 + \frac{1}{2} \sum_S |S| \|\mathcal{R}_S^{u,v}\|_2^2,$$

whence, for every $k \geq 2(D(\mu) + 1)$,

$$\sum_S |S| \|\mathcal{R}_S^{u,v}\|_2^2 \leq 2k \sum_{0 < |S| < k} \|\mathcal{R}_S^{u,v}\|_2^2.$$

Notably,

$$\sum_S |S| \|\mathcal{R}_S^{u,v}\|_2^2 \leq 2\lceil 2(D(\mu) + 1) \rceil \sum_{S \neq \emptyset} \|\mathcal{R}_S^{u,v}\|_2^2.$$

Plugging this into inequality (13) ends the proof of the first inequality of the theorem with

$$K(\mu) = 2\lceil 2(D(\mu) + 1) \rceil.$$

It remains to prove (12). Let us present first the main idea in the elliptic setting. We know from Lemma 2.3 that

$$(14) \quad \partial_{e,e'}^2 \mathcal{R}(r) = \partial_e [i_r(e')]^2 = 2i_r(e')i_r(e) \frac{i_r^e(e')}{r(e)}.$$

Approximating $\Delta_e \Delta_{e'} \mathcal{R}^{u,v}$ by $\partial_{e,e'}^2 \mathcal{R}(r)$, one gets

$$\begin{aligned} \sum_{e \neq e'} \|\Delta_e \Delta_{e'} \mathcal{R}^{u,v}\|_2^2 &\lesssim \sum_{e \neq e'} \mathbb{E}[i_r(e')^2 i_r(e)^2 (i_r^e(e'))^2] \\ &\lesssim \sum_{e \neq e'} \mathbb{E}[(i_r(e')^4 + i_r(e)^4) (i_r^e(e'))^2]. \end{aligned}$$

The reciprocity law (5) gives that $i_r^e(e')$ and $i_r^{e'}(e)$ are of the same order:

$$\begin{aligned} \sum_{e \neq e'} \|\Delta_e \Delta_{e'} \mathcal{R}^{u,v}\|_2^2 &\lesssim \sum_{e \neq e'} \mathbb{E}[i_r(e)^4 (i_r^e(e'))^2] + \sum_{e \neq e'} \mathbb{E}[i_r(e')^4 (i_r^{e'}(e))^2] \\ &= 2 \sum_{e \neq e'} \mathbb{E}[i_r(e)^4 (i_r^e(e'))^2] \\ &\lesssim \sum_e \mathbb{E}\left[i_r(e)^4 \sum_{e'} r(e') (i_r^e(e'))^2 \right], \end{aligned}$$

but $\sum_{e'} r(e') (i_r^e(e'))^2$ is the effective resistance from e_- to e_+ , which is of order at most 1 [in fact at most $r(e)$]. Thus,

$$\begin{aligned} \sum_{e \neq e'} \|\Delta_e \Delta_{e'} \mathcal{R}^{u,v}\|_2^2 &\lesssim \sum_e \mathbb{E}[i_r(e)^4] \\ &\lesssim \sum_e \|\Delta_e \mathcal{R}^{u,v}\|_2^2, \end{aligned}$$

from Lemma 3.4.

Now, let us enter the details of the proof of (12) in the general case. Let r and r' be two independent random variables with the same distribution $\mathbb{P} := \otimes_{e \in E_{1/2}} \mu_e$. Remark that for $e \neq e'$:

$$\begin{aligned} \Delta_e \Delta_{e'} \mathcal{R}(r) &= \mathbb{E}[\mathcal{R}(r) - \mathcal{R}(r^{e' \leftarrow r'}) - \mathcal{R}(r^{e \leftarrow r'}) + \mathcal{R}(r^{(e,e') \leftarrow r'}) | r] \\ &= \mathbb{E}\left[\int_{r'(e)}^{r(e)} \int_{r'(e')}^{r(e')} \partial_{e,e'}^2 \mathcal{R}(r^{(e,e') \leftarrow (x,y)}) dx dy \middle| r\right]. \end{aligned}$$

Thus,

$$\begin{aligned} \|\Delta_e \Delta_{e'} \mathcal{R}\|_2^2 &\leq \mathbb{E}\left[(r(e) - r'(e))^2 (r(e') - r'(e'))^2 \sup_{\substack{x \in [r(e), r'(e)] \\ y \in [r'(e'), r'(e')]} (\partial_{e,e'}^2 \mathcal{R}(r^{(e,e') \leftarrow (x,y)}))^2\right], \end{aligned}$$

where we make the abuse of notation of writing $[a, b]$ for $[\min\{a, b\}, \max\{a, b\}]$. Lemma 2.4 shows that for x in $[r(e), r'(e)]$ and y in $[r'(e'), r'(e')]$,

$$|i_{r^{(e,e') \leftarrow (x,y)}}(e)| \leq [|i_{r^{(e,e') \leftarrow 1}}(e)| + |i_{r^{(e,e') \leftarrow 1}}(e')|] g(1, x) g(1, y),$$

and the same bound holds for $|i_{r^{(e,e') \leftarrow (x,y)}}(e)|$. Furthermore, using Lemma 2.3 and the reciprocity law (5),

$$\begin{aligned} \frac{1}{x} |i_{r^{(e,e') \leftarrow (x,y)}}^e(e')| &\leq \frac{1}{x} |i_{r^{(e,e') \leftarrow (x,1)}}^e(e')| \max\left\{1, \frac{1}{y}\right\} \\ &= |i_{r^{(e,e') \leftarrow (x,1)}}^{e'}(e)| \max\left\{1, \frac{1}{y}\right\} \\ &\leq |i_{r^{(e,e') \leftarrow 1}}^{e'}(e)| \max\left\{1, \frac{1}{x}\right\} \max\left\{1, \frac{1}{y}\right\} \\ &= |i_{r^{(e,e') \leftarrow 1}}^e(e')| \max\left\{1, \frac{1}{x}\right\} \max\left\{1, \frac{1}{y}\right\}. \end{aligned}$$

Thus, using $(a + b)^4 \leq 8(a^4 + b^4)$,

$$\|\Delta_e \Delta_{e'} \mathcal{R}\|_2^2 \leq 2A(e) A(e') \mathbb{E}[|i_{r^{(e,e') \leftarrow 1}}(e)|^4 + |i_{r^{(e,e') \leftarrow 1}}(e')|^4] |i_{r^{(e,e') \leftarrow 1}}^e(e')|^2,$$

where

$$A(e) = 4\mathbb{E}\left[(r(e) - r'(e))^2 \left(\frac{1}{r^6(e)} + r^6(e)\right)\right].$$

Then we use again the same decoupling device based on Lemma 2.4, in the spirit of (11). We get

$$|i_{r^{(e,e') \leftarrow 1}}^e(e')| \leq \frac{|i_r^e(e')|}{r(e)} \max\{1, r(e)\} \max\{1, r(e')\}$$

and

$$|i_{r(e,e')\leftarrow 1}(e)| \leq [|i_r(e)| + |i_r(e')|]g(r(e), 1)g(r(e'), 1).$$

Thus,

$$\begin{aligned} & \mathbb{E}[(i_{r(e,e')\leftarrow 1}(e)^4 + i_{r(e,e')\leftarrow 1}(e')^4)i_{r(e,e')\leftarrow 1}^e(e')^2] \\ & \leq B(e)B(e')\mathbb{E}\left[(i_r^4(e) + i_r^4(e'))\frac{i_r^e(e')^2}{r(e)^2}r(e)r(e')\right], \end{aligned}$$

where

$$B(e) = \frac{4}{\mathbb{E} \min\{1/(r^7(e)), r^3(e)\}}.$$

Whence

$$(15) \quad \|\Delta_e \Delta_{e'} \mathcal{R}\|_2^2 \leq 2A(e)A(e')B(e)B(e')\mathbb{E}\left[(i_r^4(e) + i_r^4(e'))\frac{i_r^e(e')^2}{r(e)^2}r(e)r(e')\right].$$

Now notice that for e fixed,

$$\sum_{e'} i_r^e(e')^2 r(e') = \mathcal{R}^e(r) \leq r(e).$$

We obtain

$$\begin{aligned} \sum_{e \neq e'} \|\Delta_e \Delta_{e'} \mathcal{R}\|_2^2 & \leq 2 \sup_{e'} \{A(e')B(e')\} \sum_e A(e)B(e)\mathbb{E}\left[\frac{i_r^4(e)}{r^2(e)}\right] \\ & \leq 2 \sup_e \{A^2(e)B^2(e)\} \sum_e A(e)B(e)C(e)\mathbb{E}[r^2(e)i_r^4(e)], \end{aligned}$$

where $C(e)$ is given by (11):

$$C(e) = \frac{\mathbb{E}[1/(r^2(e)) \max\{1, 1/(r^4(e))\}]}{\mathbb{E}[\min\{r^2(e), 1/(r^2(e))\}]}.$$

Thus, using Lemma 3.4, where

$$D(\mu) = 2 \sup_e \{A^2(e)B^2(e)\} \sup_e \frac{A(e)B(e)C(e)}{\alpha_-(e)}.$$

Finally, elementary calculus shows that

$$D(\mu) \leq C \sup_e (\mathbb{E}[r^8(e)] + \mathbb{E}[r^{-8}(e)])^6 \sup_e \frac{\mathbb{E}[(r(e) - r'(e))^2(1/(r^6(e)) + r^6(e))]}{\mathbb{E}[(r(e) - r'(e))_+^2 \min\{1, 1/(r^4(e))\}]},$$

for some universal constant C . \square

REMARK 2. $K(\mu)$ is finite if the resistances are elliptic, or alternatively if

$$\sup_e \{ \mathbb{E}(r^8(e)) + \mathbb{E}(r^{-8}(e)) \} < \infty$$

and

$$\inf_e \mathbb{E} |r(e) - \mathbb{E}(r(e))| > 0.$$

Indeed, in this case, Cauchy–Schwarz inequality shows that

$$\inf_e \mathbb{E} \left[(r(e) - r'(e))_+^2 \min \left\{ 1, \frac{1}{r^4(e)} \right\} \right] < 0.$$

An easy consequence is that point-to-point effective resistances are uniformly stable to noise.

COROLLARY 3.6. *On any random network with independent uniformly elliptic resistances, or more generally with $K(\mu)$ finite, point-to-point effective resistances are uniformly stable to noise:*

$$\inf_{u,v \in V} \text{Corr}(\mathcal{R}^{u,v}(r), \mathcal{R}^{u,v}(r^\varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} 1.$$

PROOF. Let us fix u and v , and let f denote the function $r \mapsto \mathcal{R}^{u,v}(r)$. From Proposition 3.3,

$$\begin{aligned} \text{Cov}(f(r), f(r^\varepsilon)) &= \sum_{S \neq \emptyset} (1 - \varepsilon)^{|S|} \|f_S\|^2 \\ &= \text{Var}(f) - \sum_{S \neq \emptyset} (1 - (1 - \varepsilon)^{|S|}) \|f_S\|^2 \\ &\geq \text{Var}(f) - \log \frac{1}{1 - \varepsilon} \sum_{S \neq \emptyset} |S| \|f_S\|^2 \\ &\geq \text{Var}(f) \left(1 - K(\mu) \log \frac{1}{1 - \varepsilon} \right). \end{aligned}$$

This shows that $\text{Corr}(\mathcal{R}^{u,v}(r), \mathcal{R}^{u,v}(r^\varepsilon))$ tends uniformly (in u and v) to 1 when ε tends to zero. \square

A trivial consequence of Theorem 3.5 is that the Efron–Stein inequality, Corollary 3.2, is always tight for point-to-point effective resistances.

COROLLARY 3.7. *For any graph G and any pair of vertices (u, v) ,*

$$\text{Var}(\mathcal{R}^{u,v}) \leq \sum_{e \in E_{1/2}} \|\Delta_e \mathcal{R}^{u,v}\|_2^2 \leq K(\mu) \text{Var}(\mathcal{R}^{u,v}).$$

To finish this section, we shall recall the setting of the effective resistance through a box in \mathbb{Z}^d , which was described in the [Introduction](#). Let $\mathcal{B}_n = \{0, \dots, n\}^d$ be equipped with random resistances r on its set of edges E_n , and let us define $A_n = \{x \in \mathcal{B}_n \text{ s.t. } x_1 = 0\}$ and $Z_n = \{x \in \mathcal{B}_n \text{ s.t. } x_1 = n\}$. Then consider the graph with vertex set \mathcal{B}_n , where all the vertices of A_n are identified to a single vertex on one side, and all the vertices of Z_n are identified on the other side. The *effective resistance* through the box \mathcal{B}_n is defined as

$$\mathcal{R}_n = \mathcal{R}_r^{A_n, Z_n}.$$

In [31], it is shown that under some hypotheses on the distribution of the resistances,

$$(16) \quad \text{Var}(\mathcal{C}_n) \geq Cn^{d-4},$$

where $\mathcal{C}_n = 1/\mathcal{R}_n$, and C is some positive constant, depending on d and the common distribution of the conductances. Let us show how one can recover (16) in our setting. One could work directly on \mathcal{C}_n , but will we rather show that

$$(17) \quad \text{Var}(\mathcal{R}_n) \geq Cn^{4-3d},$$

and then translate this bound on $\text{Var}(\mathcal{C}_n)$. Let us suppose that

$$(18) \quad \sup_n \sup_{e \in E_n} \{\mathbb{E}[r(e)] + \mathbb{E}[r^{-1}(e)]\} < \infty.$$

Then one may see that $\mathbb{E}[\mathcal{R}_n] = \Theta(n^{2-d})$, and $\mathbb{E}[\mathcal{C}_n] = \Theta(n^{d-2})$ (cf. the argument in Section 5 of [3]).

Let us call i_r the unit current in the definition of $\mathcal{R}_r^{A_n, Z_n}$, and order the edges in E_n in some arbitrary fixed way. We have the following martingale representation:

$$\begin{aligned} \mathcal{R}_n - \mathbb{E}[\mathcal{R}_n] &= \sum_{e \in E_n} \mathbb{E}[\mathcal{R}_n(r)|(r_{e'})_{e' \geq e}] - \mathbb{E}[\mathcal{R}_n(r)|(r_{e'})_{e' > e}] \\ &= \sum_{e \in E_n} \mathbb{E}[\Delta_e \mathcal{R}_n(r)|(r_{e'})_{e' \geq e}]. \end{aligned}$$

Thus,

$$\begin{aligned} \text{Var}(\mathcal{R}_n) &= \sum_{e \in E_n} \mathbb{E}[\mathbb{E}[\Delta_e \mathcal{R}_n(r)|(r_{e'})_{e' \geq e}]^2] \\ &= \frac{1}{2} \sum_{e \in E_n} \mathbb{E}[\mathbb{E}[\mathcal{R}_n(r) - \mathcal{R}_n(r^{e \leftarrow r'})|(r_{e'})_{e' \geq e}, r'(e)]^2] \\ &= \frac{1}{2} \sum_{e \in E_n} \mathbb{E}[\mathbb{E}[\mathcal{R}_n(r) - \mathcal{R}_n(r^{e \leftarrow r'})_+|(r_{e'})_{e' \geq e}, r'(e)]^2] \\ &\quad + \frac{1}{2} \sum_{e \in E_n} \mathbb{E}[\mathbb{E}[\mathcal{R}_n(r) - \mathcal{R}_n(r^{e \leftarrow r'})_-|(r_{e'})_{e' \geq e}, r'(e)]^2], \end{aligned}$$

since the sign of $(\mathcal{R}_n(r) - \mathcal{R}_n(r^{e \leftarrow r'}))$ is the sign of $r(e) - r'(e)$. Using inequality (4),

$$\begin{aligned} \text{Var}(\mathcal{R}_n) &\geq \frac{1}{2} \sum_{e \in E_n} \mathbb{E}[(r(e) - r'(e))_+^2 \mathbb{E}[i_r^2(e) | r(e), r'(e)]^2] \\ &\quad + \frac{1}{2} \sum_{e \in E_n} \mathbb{E}[(r(e) - r'(e))_-^2 \mathbb{E}[i_{r^{e \leftarrow r'}}^2(e) | r(e), r'(e)]^2] \\ &= \sum_{e \in E_n} \mathbb{E}[(r(e) - r'(e))^2 \mathbb{E}[i_r^2(e) | r(e)]^2] \\ &\geq \sum_{e \in E_n} \mathbb{E} \left[(r(e) - r'(e))^2 \min \left\{ 1, \frac{1}{r^4(e)} \right\} \mathbb{E}[i_{r^{e \leftarrow 1}}^2(e) | r(e)]^2 \right] \\ &= \sum_{e \in E_n} \mathbb{E} \left[(r(e) - r'(e))^2 \min \left\{ 1, \frac{1}{r^4(e)} \right\} \mathbb{E}[i_{r^{e \leftarrow 1}}^2(e)]^2 \right] \\ &\geq \sum_{e \in E_n} \frac{\mathbb{E}[(r(e) - r'(e))^2 \min\{1, 1/(r^4(e))\}]}{\mathbb{E}[\max\{r(e), 1/(r(e))\}]^2} \mathbb{E}[r(e)i_r^2(e)]^2, \end{aligned}$$

where the inequalities follows from Lemma 2.4 as in the proof of (11). Define

$$\tilde{\alpha} := \inf_n \inf_{e \in E_n} \frac{\mathbb{E}[(r(e) - r'(e))^2 \min\{1, 1/(r^4(e))\}]}{\mathbb{E}[\max\{r(e), 1/(r(e))\}]^2}.$$

Then, using Jensen’s inequality,

$$\begin{aligned} \text{Var}(\mathcal{R}_n) &\geq \tilde{\alpha} \sum_{e \in E_n} \mathbb{E}[r(e)i_r^2(e)]^2 \\ &\geq \tilde{\alpha} \#(E_n) \left(\frac{1}{\#(E_n)} \sum_{e \in E_n} \mathbb{E}[r(e)i_r^2(e)] \right)^2 \\ &= \tilde{\alpha} \frac{1}{\#(E_n)} \mathbb{E}[\mathcal{R}_n]^2 \\ &\geq C \tilde{\alpha} n^{4-3d}. \end{aligned}$$

This shows inequality (17) under the condition $\tilde{\alpha} > 0$. Now, suppose that for some $p > 2$,

$$(19) \quad \sup_n \sup_{e \in E_n} \mathbb{E}[r(e)^p] < \infty,$$

and let us show how one may recover (16) from (17). Notice first that bounding \mathcal{R}_n by the energy of the deterministic flow which splits uniformly on A_n and goes straight to Z_n , there is a constant c_d such that

$$\mathcal{R}_n \leq c_d \frac{\sum_{e \in E_n} r(e)}{n^{2d-2}} =: S_n.$$

Using Rosenthal’s inequality (see [30], Theorem 3), there is a finite positive constant $k(p)$ such that for any $c > 1$ and $p > 2$,

$$\begin{aligned} \mathbb{P}(\mathcal{R}_n \geq c\mathbb{E}[S_n]) &\leq \mathbb{P}(S_n \geq c\mathbb{E}[S_n]) \\ &\leq \frac{\mathbb{E}[|S_n - \mathbb{E}(S_n)|^p]}{(c - 1)^p \mathbb{E}[S_n]^p} \\ &= \frac{\mathbb{E}[|\sum_{e \in E_n} (r(e) - \mathbb{E}[r(e)])|^p]}{(c - 1)^p \mathbb{E}[\sum_{e \in E_n} r(e)]^p} \\ &\leq \frac{k(p) \max\{\sum_{e \in E_n} \mathbb{E}[r(e)^p], (\sum_{e \in E_n} \mathbb{E}[r(e)^2])^{1/2}\}}{(c - 1)^p \mathbb{E}[\sum_{e \in E_n} r(e)]^p} \\ &\leq \frac{k'(p)n^{d(1-p)}}{(c - 1)^p}. \end{aligned}$$

Now,

$$\begin{aligned} \text{Var}(C_n) &\geq \mathbb{E}\left[\left(\frac{1}{\mathcal{R}_n} - \frac{1}{\mathbb{E}[\mathcal{R}_n]}\right)^2\right] \\ &= \mathbb{E}\left[\frac{(\mathcal{R}_n - \mathbb{E}[\mathcal{R}_n])^2}{\mathcal{R}_n^2 \mathbb{E}[\mathcal{R}_n]^2}\right] \\ &\geq \frac{\text{Var}(\mathcal{R}_n)}{c^2 \mathbb{E}[\mathcal{R}_n]^2 \mathbb{E}[S_n]^2} - \mathbb{E}\left[\frac{(\mathcal{R}_n - \mathbb{E}[\mathcal{R}_n])^2}{\mathcal{R}_n^2 \mathbb{E}[\mathcal{R}_n]^2} \mathbf{1}_{\mathcal{R}_n > c\mathbb{E}[S_n]}\right] \\ &\geq C\tilde{\alpha} \frac{n^{4-3d}}{c^2 n^{8-4d}} - \mathbb{E}\left[\frac{(\mathcal{R}_n - \mathbb{E}[\mathcal{R}_n])^2}{\mathcal{R}_n^2 \mathbb{E}[\mathcal{R}_n]^2} \mathbf{1}_{\mathcal{R}_n > c\mathbb{E}[S_n]}\right] \\ &\geq C\tilde{\alpha} \frac{n^{d-4}}{c^2} - \frac{\mathbb{P}(\mathcal{R}_n \geq c\mathbb{E}[S_n])}{\mathbb{E}[\mathcal{R}_n]^2} \\ &\geq C\tilde{\alpha} \frac{n^{d-4}}{c^2} - C'n^{2d-4} \frac{k'(p)n^d n^{d(1-p)}}{(c - 1)^p} \\ &\geq C'n^{d-4}, \end{aligned}$$

for n large enough, since $p > 2$. This shows inequality (16) when $\tilde{\alpha}$ is finite and (18) and (19) hold. To state a simple moment condition, for i.i.d. resistances with positive variance, one gets (16) under the condition that $\mathbb{E}[r^p(e)] + \mathbb{E}[c(e)] < \infty$ for some $p > 2$. Notice that the moments of order $p > 2$ are used only to go from the bound on $\text{Var}(\mathcal{R}_n)$ to a bound on $\text{Var}(C_n)$. Alternatively, one could work directly on C_n , thanks to the formula (1). One would need (this is not difficult) to establish the analogs of Lemma 2.3 and 2.4 for v_r , the minimizer in (1), instead of i_r . Then the line of proof which gave (17) would lead to (16). To state a simple moment condition, for i.i.d. conductances with positive variance, one would

obtain (16) under the condition that $\mathbb{E}[r(e)] + \mathbb{E}[c(e)^2] < \infty$. In any case, our conditions seem to be much weaker than the conditions in [31]. For instance, no power-law distribution satisfies Wehr’s assumption, and he requires absolute continuity w.r.t. the Lebesgue measure. We record the result of our calculations in the following lemma, for the neat case where the resistances are i.i.d.

LEMMA 3.8. *Suppose that the resistances are i.i.d., not constant and that $r(e)$ and $c(e)$ are integrable. Then*

$$\text{Var}(\mathcal{R}_n) \geq Cn^{4-3d}.$$

If furthermore $r(e)$ has a finite moment of order $p > 2$, then

$$\text{Var}(C_n) \geq C'n^{d-4}.$$

The positive constants C and C' depend only on the common distribution of the resistances.

Finally, let us emphasize the fact that we are unfortunately unable to show that

$$(20) \quad \mathbb{E} \left[\sum_{e \in E_n} r^2(e) i_{r,n}^4(e) \right] \leq Cn^{4-3d}.$$

Otherwise, we would obtain from Corollary 3.7 the correct order for the variance of \mathcal{R}_n . Notice that (20) does not necessarily hold when the resistances are not supposed to have identical distribution, even in the elliptic case. To understand why, let v_r be the discrete harmonic function on $\mathcal{B}_n \setminus (A_n \cup Z_n)$ with value 1 on A_n and 0 on Z_n . Discrete harmonic at x means that

$$d^*(c.dv_r)(x) = 0.$$

Then (20) is equivalent to

$$(21) \quad \mathbb{E} \left[\frac{1}{\#E_n} \sum_{e \in E_n} (ndv_r(e))^4 \right] \leq C'.$$

When n goes to infinity, one may compare v_r to a continuous analog. Let $(c_x)_{x \in \mathbb{R}^d}$ be a deterministic elliptic collection of conductance matrices. Let \tilde{v} be the function on $[0, 1]^d$ with value 1 on $\{x \text{ s.t. } x_1 = 0\}$, 0 on $\{x \text{ s.t. } x_1 = 1\}$ and satisfying on $(0, 1)^d$:

$$\text{div}(c.\nabla\tilde{v}) = 0.$$

The continuous analog of (21) is the fact that $\nabla\tilde{v}$ belongs to $L^4([0, 1]^d)$. However, it is well known that this may be false if the ellipticity constant Λ is not close enough to 1. On \mathbb{R}^2 , a counterexample is given in [14]; see the discussion after Proposition 1.1 therein.

3.3. *Further results for elliptic networks with homogeneous currents.* From now on, the networks will be elliptic: r belongs to $[1, \Lambda]^{E_{1/2}}$ for some $\Lambda \geq 1$. Recall that all sets $\ell^2_-(E, r)$ are the same for r in $[1, \Lambda]^{E_{1/2}}$, since the norms with seights r are all equivalent, the common set is denoted by $\ell^2_-(E)$; cf. (2) and we shall refer to the common norm topology of the sets $\ell^2_-(E, r)$ as the *strong topology*. We shall consider the *graph distance*, denoted by d , on V . Then, if e and e' are two edges in E , let $d(e, e')$ be the maximal distance between two endpoints, one of which is in e and the other in e' . For any edge e and any collection of resistances r in $[1, \Lambda]^{E_{1/2}}$, the flow i_r^e belongs to $\ell^2_-(E, r)$. Thus,

$$(22) \quad \sum_{e': d(e', e) \geq L} r(e')(i_r^e(e'))^2 \xrightarrow{L \rightarrow \infty} 0.$$

Below, we shall be interested in graphs where the above convergence holds *uniformly in e and r* . We shall say that such a graph has *homogeneous currents*.

DEFINITION 3.9. Let $G = (V, E)$ be a countable, oriented, symmetric and connected graph. Let $\Lambda \geq 1$ be a real number, and define

$$(23) \quad \alpha(G, L, \Lambda) = \sup \left\{ \sum_{e': d(e', e) \geq L} r(e')(i_r^e(e'))^2 \text{ s.t. } e \in E, r \in [1, \Lambda]^{E_{1/2}} \right\}.$$

The graph G is said to have Λ -homogeneous currents if

$$\alpha(G, L, \Lambda) \xrightarrow{L \rightarrow +\infty} 0.$$

It is natural to expect that for every $\Lambda \geq \Lambda'$ strictly larger than 1, G has Λ -homogeneous currents if it has Λ' -homogeneous currents [the other direction being trivial since $\alpha(G, L, \Lambda)$ is monotone in Λ]. However, we could not prove this.

The first fundamental observation, due to Mikaël de la Salle, is that for a fixed edge e , the convergence in (22) always hold uniformly in r , thanks to a compactness argument.

PROPOSITION 3.10. *Let u and v be two vertices of G , and suppose that $(G_L)_{L \geq 0}$ is a sequence of finite connected graphs that exhausts G and such that G_0 contains u and v . Then*

$$\sup \left\{ \sum_{e' \in G_L^c} r(e')(i_r^{u,v}(e'))^2 \text{ s.t. } r \in [1, \Lambda]^{E_{1/2}} \right\} \xrightarrow{L \rightarrow +\infty} 0.$$

PROOF (DUE TO MIKAËL DE LA SALLE). Let us fix u and v two vertices of G . We equip $[1, \Lambda]^{E_{1/2}}$ with the product topology. For a fixed $\theta \in \ell^2_-(E)$ and $\varepsilon > 0$, let F be a finite subset of edges such that $\sum_{e \in F^c} \theta^2(e) < \varepsilon$. Then

$$\left| \sum_e r(e)\theta^2(e) - \sum_e r'(e)\theta^2(e) \right| \leq \sum_{e \in F} |r(e) - r'(e)|\theta^2(e) + \Lambda\varepsilon,$$

and thus the function $r \mapsto \sum_e r(e)\theta^2(e)$ is continuous for the product topology. Then $r \mapsto \mathcal{R}^{u,v}(r)$ is an infimum of continuous functions on $[1, \Lambda]^{\mathbb{E}_{1/2}}$, and thus, it is upper semi-continuous on $[1, \Lambda]^{\mathbb{E}_{1/2}}$. Now, define

$$c := \lim_{L \rightarrow +\infty} \sup \left\{ \sum_{e' \in \mathbb{G}_L^c} r(e')(i_r^{u,v}(e'))^2 \text{ s.t. } r \in [1, \Lambda]^{\mathbb{E}} \right\},$$

which exists by the monotonicity in L of the right-hand side.

One may find a sequence $(r_L)_{L \geq 1}$ in $[1, \Lambda]^{\mathbb{E}_{1/2}}$ such that

$$c = \lim_{L \rightarrow +\infty} \sum_{e' \in \mathbb{G}_L^c} r_L(e')(i_{r_L}^{u,v}(e'))^2.$$

By Lemma 2.2, the sequence $(i_{r_L}^{u,v})_{L \geq 1}$ lies in $[-1, 1]^{\mathbb{E}}$ [and even in $\ell^2_-(\mathbb{E})$] which is compact for the product topology by Cantor’s diagonal argument. Also, the sequence $(r_L)_{L \geq 1}$ lies in the compact set $[1, \Lambda]^{\mathbb{E}_{1/2}}$. Thus, one may find an increasing sequence of integers $(L_k)_{k \geq 1}$ such that for any $e \in \mathbb{E}$, $(r_{L_k})_{k \geq 1}(e)$ converges to some value $r(e)$ in $[1, \Lambda]$ and $(i_{r_{L_k}}^{u,v}(e))_{k \geq 1}$ converges to some $\theta(e) \in [-1, 1]$. Then θ is a unit flow from u to v . From the upper-semi-continuity of $r \mapsto \mathcal{R}^{u,v}(r)$,

$$\begin{aligned} \mathcal{R}^{u,v}(r) &\geq \limsup_{k \rightarrow +\infty} \mathcal{R}^{u,v}(r_{L_k}) \\ &= \limsup_{k \rightarrow +\infty} \sum_{e'} r_{L_k}(e')(i_{r_{L_k}}^{u,v}(e'))^2 \\ &= \limsup_{k \rightarrow +\infty} \left(\sum_{e' \in \mathbb{G}_{L_k}} r_{L_k}(e')(i_{r_{L_k}}^{u,v}(e'))^2 + \sum_{e' \in \mathbb{G}_{L_k}^c} r_{L_k}(e')(i_{r_{L_k}}^{u,v}(e'))^2 \right) \\ &= \limsup_{k \rightarrow +\infty} \sum_{e' \in \mathbb{G}_{L_k}} r_{L_k}(e')(i_{r_{L_k}}^{u,v}(e'))^2 + c \\ &\geq \sup_{L'} \limsup_{k \rightarrow +\infty} \sum_{e' \in \mathbb{G}_{L'}} r_{L_k}(e')(i_{r_{L_k}}^{u,v}(e'))^2 + c \\ &= \sup_{L'} \sum_{e' \in \mathbb{G}_{L'}} r(e')(\theta(e'))^2 + c \\ &= \sum_{e'} r(e')(\theta(e'))^2 + c \\ &\geq \mathcal{R}^{u,v}(r) + c. \end{aligned}$$

This shows that $c = 0$ and proves the proposition. \square

Notice that in Proposition 3.10, the ellipticity hypothesis is used in a crucial way. This proposition will allow us to find our first graphs with homogeneous

currents: the *quasi-transitive* graphs. There is no universal definition, but in this article we shall say that a graph $G = (V, E)$ with automorphism group $\text{Aut}(G)$ is quasi-transitive if its set of edges E is composed of a finite number of distinct orbits under the natural action of $\text{Aut}(G)$ on E .

COROLLARY 3.11. *Let $G = (V, E)$ be a countable, oriented, symmetric and connected graph. Suppose that G is quasi-transitive. Then, for any $\Lambda \geq 1$, G has Λ -homogeneous currents.*

PROOF. The quasi-transitivity hypothesis implies that there exists a finite set of edges e_1, \dots, e_r such that

$$\alpha(G, L, \Lambda) = \max_{i=1, \dots, r} \sup \left\{ \sum_{e' : d(e', e_i) \geq L} r(e') (i_r^{e_i}(e'))^2 \text{ s.t. } r \in [1, \Lambda]^{E_{1/2}} \right\}.$$

But, from Proposition 3.10, for any i ,

$$\sup \left\{ \sum_{e' : d(e', e_i) \geq L} r(e') (i_r^{e_i}(e'))^2 \text{ s.t. } r \in [1, \Lambda]^{E_{1/2}} \right\} \xrightarrow{L \rightarrow +\infty} 0,$$

taking G_L to be the graph whose edges are all the edges of E at distance at most L from e_i and whose vertices are the endpoints of those edges. Thus, $\alpha(G, L, \Lambda)$ goes to zero as L goes to infinity, and G has Λ -homogeneous currents. \square

A consequence of Corollary 3.11 is that \mathbb{Z}^d has homogeneous currents. This can also be seen in a more robust way using the powerful tool of *elliptic Harnack inequality*. Indeed, let $B_L(e)$ be the vertices at distance at most L from e . Then $\sum_{d(e', e) \geq L} r(e') (i_r^e(e'))^2$ is upper-bounded by the oscillation on $B_L(e)^c$ of the voltage induced by the flow i_r^e . Since this voltage is a bounded function, harmonic on $\mathbb{Z}^d \setminus e$, one may then show, using the Harnack inequality of [13] as in [24], Section 6, that $\alpha(\mathbb{Z}^d, L, \Lambda)$ decays at least as quickly as a negative power of L . This argument can be carried out on any graph satisfying the conditions of [13] plus the additional condition that the annuli between $B_L(e)$ and $B_{4L}(e)$ are connected and may be covered by a bounded number of balls of radius L . For instance, this shows that any graph roughly isometric to \mathbb{Z}^d has homogeneous currents.

Now, let us give an (artificial) example of a graph which does not have homogeneous currents. A perfect binary tree of depth k is a rooted binary tree where every leaf is at depth k and all other vertices have two children. For any $k \in \mathbb{N}^*$, let T_k be two copies of a perfect binary trees of depth k glued at the leaves. The result has two roots. Now, to construct our graph (see Figure 1), we start from \mathbb{N}^* , with the usual notion of graph on it, and for any $k \in \mathbb{N}^*$, we do the following construction. We add T_k by glueing one of its roots on the vertex $x_{2k} := 2k$ of \mathbb{N} and call the remaining root x'_{2k} . Then we add a copy T'_k of T_k by glueing one of its root on the vertex $x_{2k+1} := 2k + 1$ of \mathbb{N} and call the remaining root x'_{2k+1} .

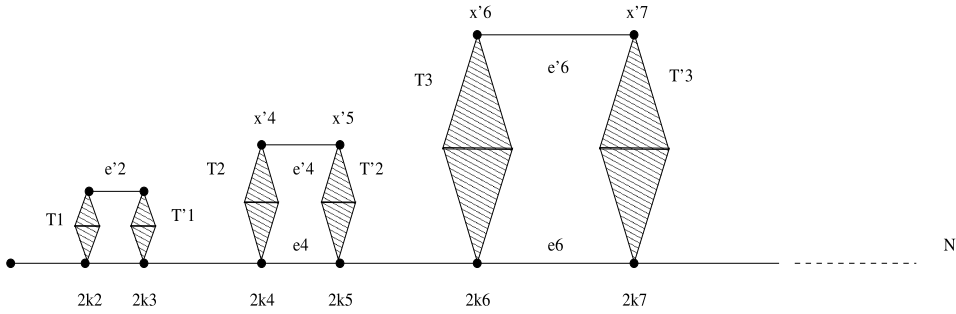


FIG. 1. An example of a graph without homogeneous currents.

Then we join x'_{2k} and x'_{2k+1} by an edge. We denote by G the resulting graph, let $e_{2k} := (x_{2k}, x_{2k+1})$, $e'_{2k} := (x'_{2k}, x'_{2k+1})$ and equip this graph with unit resistances. Now, the resistance between the root and the leafs of a complete binary tree of any depth is at most $1/2$. Thus, $\mathcal{R}_{T_k}^{x_{2k}, x'_{2k}}$, the resistance between x_{2k} and x'_{2k} in the graph T_k is at most 1. Similarly, $\mathcal{R}_{T'_k}^{x'_{2k+1}, x_{2k+1}}$ is at most 1. Since resistances in series add, one sees that

$$i_r^{e_{2k}}(e'_{2k}) = \frac{r(e_{2k})}{\mathcal{R}_{T_k}^{x_{2k}, x'_{2k}} + r(e'_{2k}) + \mathcal{R}_{T'_k}^{x'_{2k+1}, x_{2k+1}}} \geq \frac{1}{3}.$$

On the other hand, the graph-theoretical distance between e_{2k} and e'_{2k} is $2k + 1$. Thus, for any $\Lambda \geq 1$, and $L \geq 1$,

$$\alpha(G, L, \Lambda) \geq \sum_{e': d(e', e_{2L}) \geq L} r(e)(i_r^{e_{2L}}(e'))^2 \geq (i_r^{e_{2L}}(e'_{2L}))^2 \geq \frac{1}{9}.$$

Whence G does not have homogeneous currents.

Finally, the reasoning in the proof of Proposition 3.10 allows also to obtain some regularity of the functions $r \mapsto i_r^{u,v}$ and $r \mapsto \mathcal{R}^{u,v}(r)$ when $[1, \Lambda]^{E_{1/2}}$ is equipped with the product topology (it will not be used in the sequel).

PROPOSITION 3.12. *Let u and v be two vertices of G . If $[1, \Lambda]^{E_{1/2}}$ is equipped with the product topology and $\ell_-^2(E)$ with the strong topology, then the following maps are continuous:*

$$\left\{ \begin{array}{l} [1, \Lambda]^{E_{1/2}} \rightarrow \ell_-^2(E), \\ r \mapsto i_r^{u,v}, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} [1, \Lambda]^{E_{1/2}} \rightarrow \mathbb{R}, \\ r \mapsto \mathcal{R}^{u,v}(r). \end{array} \right.$$

PROOF. Take any r in $[1, \Lambda]^E$ and any sequence (r_L) converging to r . As in the proof of Proposition 3.10, one may extract a sequence $(r_{L_k})_{k \geq 1}$ such that

$(i_{r_{L_k}}^{u,v})_{k \geq 1}$ converges pointwise to some θ which is a unit flow from u to v . Then

$$\begin{aligned} \mathcal{R}^{u,v}(r) &\geq \limsup_{k \rightarrow +\infty} \mathcal{R}^{u,v}(r_{L_k}) \geq \liminf_{k \rightarrow +\infty} \mathcal{R}^{u,v}(r_{L_k}) \\ &\geq \liminf_{k \rightarrow +\infty} \sum_{e' \in G_{L_k}} r_{L_k}(e') (i_{r_{L_k}}^{u,v}(e'))^2 \\ &\geq \sum_{e'} r(e') (\theta(e'))^2, \end{aligned}$$

by Fatou’s lemma. Since $i_r^{u,v}$ is the unique minimizer of $\theta \mapsto \sum_{e'} r(e')\theta(e')$ over the unit flows from u to v , this shows at once that $\theta = i_r^{u,v}$ and that $\mathcal{R}^{u,v}(r_{L_k})$ converges to $\mathcal{R}^{u,v}(r)$ as k goes to infinity. Since this is true for any subsequence $(r_{L_k})_k$ such that $(i_{r_{L_k}}^{u,v})_k$ converges pointwise, we deduce that $i_{r_L}^{u,v}$ converges pointwise to $i_r^{u,v}$ and that $\mathcal{R}^{u,v}(r_L)$ converges to $\mathcal{R}^{u,v}(r)$. Notably, $r \mapsto \mathcal{R}^{u,v}(r)$ is continuous for the product topology.

Now, recall that i_{r_L} is a current, thus there exists a function v such that $r(e)i_{r_L}(e) = dv(e)$ for any e . Notice also that $d^*(i_{r_L}^{u,v} - i_r^{u,v}) = 0$. This implies that i_{r_L} is orthogonal in $\ell^2_-(E, r)$ to $i_{r_L}^{u,v} - i_r^{u,v}$. Indeed, suppose first that the network is finite. Then

$$\begin{aligned} \sum_e r_L(e) i_{r_L}^{u,v}(e) (i_{r_L}^{u,v}(e) - i_r^{u,v}(e)) &= \sum_e dv(e) (i_{r_L}^{u,v}(e) - i_r^{u,v}(e)) \\ &= \sum_{x \in V} v(x) d^*(i_{r_L}^{u,v} - i_r^{u,v})(x) \\ &= 0. \end{aligned}$$

This continues to hold when G is not finite since $i_{r_L}^{u,v}$ and $i_r^{u,v}$ are limits in $\ell^2_-(E)$ (for the strong topology) of wired currents on finite graphs. Thus,

$$\|i_{r_L}^{u,v} - i_r^{u,v}\|_{r_L}^2 = \sum_{e' \in E_{1/2}} r_L(e') (i_{r_L}^{u,v}(e'))^2 - \mathcal{R}^{u,v}(r_L).$$

From the dominated convergence theorem, since r_L converges pointwise to r ,

$$\lim_{L \rightarrow \infty} \sum_{e' \in E_{1/2}} r_L(e') (i_{r_L}^{u,v})^2 = \mathcal{R}^{u,v}(r).$$

Thus,

$$\lim_{L \rightarrow \infty} \|i_{r_L}^{u,v} - i_r^{u,v}\|_{r_L}^2 = 0,$$

which shows that $r \mapsto i_r^{u,v}$ is continuous. \square

3.3.1. *Concentration on sets of small diameter.* A small variation on the proof of Theorem 3.5 allows to obtain the following result, which shows that on graphs with homogeneous currents, the Walsh decomposition is concentrated on sets of small diameter.

THEOREM 3.13. *For any graph G , any $\Lambda \geq 1$ and any $L \geq 1$,*

$$\sum_{\text{diam}(S) \geq L} \|\mathcal{R}_S^{u,v}\|_2^2 \leq C(\Lambda)\alpha(G, L, \Lambda) \sum_{S \neq \emptyset} \|\mathcal{R}_S^{u,v}\|_2^2.$$

PROOF. Let L be a positive integer. From inequality (15), one gets

$$\begin{aligned} & \sum_{(e,e') : d(e,e') \geq L} \|\Delta_e \Delta_{e'} \mathcal{R}^{u,v}\|_2^2 \\ & \leq C(\Lambda) \mathbb{E} \left[\sum_{d(e,e') \geq L} A(e)A(e')(i_r^4(e) + i_r^4(e')) \frac{i_r^e(e')^2}{r(e)^2} r(e)r(e') \right] \\ & = 2C(\Lambda) \mathbb{E} \left[\sum_{d(e,e') \geq L} A(e)A(e')i_r^4(e) \frac{i_r^e(e')^2}{r(e)^2} r(e)r(e') \right] \\ & \leq 2C(\Lambda) \sup_{e'} A(e') \mathbb{E} \left[\sum_e \frac{1}{r(e)} A(e)i_r^4(e) \sum_{\substack{e' \text{ s.t.} \\ d(e,e') \geq L}} i_r^e(e')^2 r(e') \right] \\ & \leq 2C(\Lambda) \sup_{e'} A(e') \alpha(G, L, \Lambda) \mathbb{E} \left[\sum_e A(e)r^2(e)i_r^4(e) \right] \\ & \leq C'(\Lambda)\alpha(G, L, \Lambda) \sum_{S \neq \emptyset} \|\mathcal{R}_S^{u,v}\|_2^2. \end{aligned}$$

Then, Theorem 3.5 leads to

$$\sum_{d(e,e') \geq L} \|\Delta_e \Delta_{e'} \mathcal{R}^{u,v}\|_2^2 \leq C(\Lambda)\alpha(G, L, \Lambda) \sum_{S \neq \emptyset} \|\mathcal{R}_S^{u,v}\|_2^2.$$

Finally, notice that

$$\begin{aligned} \sum_{d(e,e') \geq L} \|\Delta_e \Delta_{e'} \mathcal{R}^{u,v}\|_2^2 &= \sum_{d(e,e') \geq L} \sum_{S \supset \{e,e'\}} \|\mathcal{R}_S^{u,v}\|_2^2 \\ &= \sum_S \sum_{\substack{e,e' \in S \\ d(e,e') \geq L}} \|\mathcal{R}_S^{u,v}\|_2^2 \\ &\geq \sum_{\text{diam}(S) \geq L} \|\mathcal{R}_S^{u,v}\|_2^2. \end{aligned}$$

□

3.3.2. *The first level carries a significant weight.* A corollary of Theorem 3.13 is that on graphs with homogeneous currents, the first level of the Walsh decomposition carries a significant part of the L^2 -norm of the centered point-to-point resistances.

COROLLARY 3.14. *Suppose that G has Λ -homogeneous currents and degree bounded by δ . Then there is a constant $C(\Lambda, \delta, G)$ such that*

$$\sum_e \|\mathcal{R}_{\{e\}}^{u,v}\|_2^2 \leq \text{Var}(\mathcal{R}^{u,v}) = \sum_{S \neq \emptyset} \|\mathcal{R}_S^{u,v}\|_2^2 \leq C(\Lambda, \delta, G) \sum_e \|\mathcal{R}_{\{e\}}^{u,v}\|_2^2.$$

PROOF. Let us fix u and v two vertices of G , and let us drop the superscript u, v to lighten the notation. Since G has Λ -homogeneous currents, one may find L large enough (depending on G and Λ) so that

$$\sum_{S \neq \emptyset} \|\mathcal{R}_S\|_2^2 \leq 2 \sum_{\substack{\text{diam}(S) \leq L \\ S \neq \emptyset}} \|\mathcal{R}_S\|_2^2.$$

For L a positive integer, using Lemma 3.15 below,

$$\begin{aligned} & \sum_{\substack{\text{diam}(S) \leq L \\ S \neq \emptyset}} \|\mathcal{R}_S\|_2^2 \\ & \leq C(\Lambda, L) \sum_{\text{diam}(S) \leq L} \sum_{e \in S} \|\mathcal{R}_{\{e\}}\|_2^2 \\ & = C(\Lambda, L) \sum_{e \in E} \|\mathcal{R}_{\{e\}}\|_2^2 |\{S \text{ s.t. } e \in S \text{ and } \text{diam}(S) \leq L\}| \\ & \leq C'(\Lambda, L, \delta) \sum_{e \in E} \|\mathcal{R}_{\{e\}}\|_2^2. \end{aligned} \quad \square$$

In the proof above, we used the following lemma, which allows to control $\|\mathcal{R}_S^{u,v}\|_p$ for small sets S . It will be used again in the central limit approximation of Theorem 4.1.

LEMMA 3.15. *For any $p \geq 1$,*

$$C_1(\Lambda) m_p(e) \|(i_r^{u,v}(e))^2\|_1 \leq \|\mathcal{R}_{\{e\}}^{u,v}\|_p \leq C_2(\Lambda) m_p(e) \|(i_r^{u,v}(e))^2\|_1.$$

Furthermore, for any S such that $0 < |S| \leq L$,

$$\|\mathcal{R}_S^{u,v}\|_p \leq C(\Lambda)^L \sum_{e \in S} \|\mathcal{R}_{\{e\}}^{u,v}\|_p.$$

PROOF. Let r and r' be two independent random variables with the same distribution $\mathbb{P} := \otimes_{e \in E_{1/2}} \mu_e$. For any function F in $L^2(\mathbb{R}^{E_{1/2}})$ and any $p \geq 1$,

$$\begin{aligned} \|F_{\{e\}}\|_p^p &= \mathbb{E}[|\mathbb{E}(\Delta_e F | r(e))|^p] \\ &= \mathbb{E}[|\mathbb{E}(F(r^{e \leftarrow r'}) - F(r) | r'(e))|^p] \\ &\leq \mathbb{E}[|\mathbb{E}(F(r^{e \leftarrow r'}) - F(r) | r(e), r'(e))|^p] \\ &= 2\mathbb{E}[\mathbb{E}[(F(r^{e \leftarrow r'}) - F(r))\mathbf{1}_{r'(e) > r(e)} | r(e), r'(e)]^p]. \end{aligned}$$

Recall from (4) that

$$\begin{aligned} 0 &\leq (r'(e) - r(e))_+ i_{r^{e \leftarrow r'}}^2(e) \leq (\mathcal{R}(r^{e \leftarrow r'}) - \mathcal{R}(r))\mathbf{1}_{r'(e) > r(e)} \\ &\leq (r'(e) - r(e))_+ i_r^2(e). \end{aligned}$$

Lemma 2.4 allows then to decouple $(r'(e) - r(e))_+$ and $i_r^2(e)$ [or $(r'(e) - r(e))_+$ and $i_{r^{e \leftarrow r'}}^2(e)$]:

$$\begin{aligned} \|\mathcal{R}_{\{e\}}\|_p^p &\leq 2\mathbb{E}[(r'(e) - r(e))_+^p \mathbb{E}[i_r^2(e) | r(e), r'(e)]^p] \\ &\leq C(\Lambda)^p \mathbb{E}[(r'(e) - r(e))_+^p \mathbb{E}[i_{r^{e \leftarrow 1}}^2(e) | r(e), r'(e)]^p] \\ &= C(\Lambda)^p \mathbb{E}[(r'(e) - r(e))_+^p \mathbb{E}[i_{r^{e \leftarrow 1}}^2(e)]^p] \\ &\leq C'(\Lambda)^p m_p(e)^p \mathbb{E}[i_r^2(e)]^p. \end{aligned}$$

On the other hand, for any function F in $L^2(\mathbb{R}^{E_{1/2}})$ and any $p \geq 1$

$$\begin{aligned} &|\mathbb{E}[F(r^{e \leftarrow r'}) - F(r) | r(e), r'(e)]|^p \\ &\leq (|\mathbb{E}[F(r^{e \leftarrow r'}) - \mathbb{E}(F) | r(e), r'(e)]| + |\mathbb{E}[F(r) - \mathbb{E}(F) | r(e), r'(e)]|)^p \\ &\leq 2^{p-1} (|\mathbb{E}[F(r^{e \leftarrow r'}) - \mathbb{E}(F) | r(e), r'(e)]|^p \\ &\quad + |\mathbb{E}[F(r) - \mathbb{E}(F) | r(e), r'(e)]|^p), \end{aligned}$$

and thus:

$$\mathbb{E}[|\mathbb{E}[F(r^{e \leftarrow r'}) - F(r) | r(e), r'(e)]|^p] \leq 2^p \mathbb{E}[|\mathbb{E}[F(r^{e \leftarrow r'}) - \mathbb{E}(F) | r(e), r'(e)]|^p].$$

Furthermore,

$$F_{\{e\}} = \mathbb{E}[F(r^{e \leftarrow r'}) - \mathbb{E}(F) | r(e), r'(e)]$$

and

$$\begin{aligned} &\mathbb{E}[|\mathbb{E}[F(r^{e \leftarrow r'}) - F(r) | r(e), r'(e)]|^p] \\ &= 2\mathbb{E}[|\mathbb{E}[(F(r^{e \leftarrow r'}) - F(r))\mathbf{1}_{r(e) > r'(e)} | r(e), r'(e)]|^p]. \end{aligned}$$

Thus,

$$\|F_{\{e\}}\|_p^p \geq \frac{1}{2^{p-1}} \mathbb{E}[\mathbb{E}[|(F(r^{e \leftarrow r'}) - F(r))\mathbf{1}_{r(e) > r'(e)}| | r(e), r'(e) |]^p].$$

Now, as for the upper bound, one uses (4) and Lemma 2.4:

$$\begin{aligned} \|\mathcal{R}_{\{e\}}\|_p^p &\geq \frac{1}{2^{p-1}} \mathbb{E}[\mathbb{E}[(r'(e) - r(e))_+ i_{r^{e \leftarrow r'}}^2 | r(e), r'(e) |]^p] \\ &\geq C(\Lambda)^p \mathbb{E}[\mathbb{E}[(r'(e) - r(e))_+ i_{r^{e \leftarrow 1}}^2 | r(e), r'(e) |]^p] \\ &= C(\Lambda)^p \mathbb{E}[(r'(e) - r(e))_+^p] \mathbb{E}[i_{r^{e \leftarrow 1}}^2 | r(e) |]^p \\ &\geq m_p(e)^p C'(\Lambda)^p \mathbb{E}[i_r^2 | r(e) |]^p. \end{aligned}$$

This proves the first part of the lemma. Now, let S be such that $0 < |S| \leq L$ and let $e \in S$. For a subset S of \mathbf{E} , recall the definition of L_S in (8). Notice first that for any function F in $L^2(\mathbb{R}^{\mathbf{E}_{1/2}})$,

$$\begin{aligned} F_S(r) &= \mathbb{E}\left[\left(\prod_{e' \in S \setminus \{e\}} (1 - L_{\{e'\}})\right) \Delta_e F(r) \middle| r_S\right] \\ &= \mathbb{E}\left[\left(\sum_{I \subset S \setminus \{e\}} (-1)^{|I|} L_I\right) \Delta_e F(r) \middle| r_S\right] \\ &= \mathbb{E}\left[\sum_{I \subset S \setminus \{e\}} (-1)^{|I|} L_I (\Delta_e F)(r) \middle| r_S\right]. \end{aligned}$$

Thus, using Jensen’s inequality,

$$\|F_S\|_p^p \leq 2^{p|S \setminus \{e\}|} \mathbb{E}[\mathbb{E}[|\Delta_e F(r)| | r_S]^p].$$

Since this is true for any $e \in S$, we have

$$(24) \quad \|F_S\|_p^p \leq \min_{e \in S} 2^{p|S \setminus \{e\}|} \mathbb{E}[\mathbb{E}[|\Delta_e F(r)| | r_S]^p].$$

Now,

$$\mathbb{E}[|\Delta_e F(r)| | r_S]^p \leq \mathbb{E}[|F(r) - F(r^{e \leftarrow r'})| | r_S]^p.$$

Using (4) and Lemma 2.4,

$$\begin{aligned} \mathbb{E}[|\mathcal{R}(r) - \mathcal{R}(r^{e \leftarrow r'})| | r_S] &\leq C(\Lambda) \mathbb{E}[|r(e) - r'(e)| i_{r^{e \leftarrow 1}}^2 | r_S] \\ &= C(\Lambda) \mathbb{E}[|r(e) - r'(e)| | r_S] \mathbb{E}[i_{r^{e \leftarrow 1}}^2 | r_S]. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}[\mathbb{E}[|\Delta_e \mathcal{R}(r)| | r_S]^p] &\leq C(\Lambda)^p m_p(e)^p \mathbb{E}[\mathbb{E}[i_r^2 | r_S]^p] \\ &\leq C(\Lambda)^{pL} m_p(e)^p \left(\sum_{e' \in S} \mathbb{E}[i_r^2 | r_S]\right)^p, \end{aligned}$$

where the last inequality follows from the second part of Lemma 2.4. Gathering this inequality and (24),

$$\begin{aligned} \|\mathcal{R}_S\|_p &\leq (2C(\Lambda))^L \min_{e \in S} m_p(e) \sum_{e' \in S} \mathbb{E}[i_r^2(e')] \\ &\leq (2C(\Lambda))^L \sum_{e' \in S} m_p(e') \mathbb{E}[i_r^2(e')] \\ &\leq C'(\Lambda)^L \sum_{e' \in S} \|\mathcal{R}_{\{e'\}}^{u,v}\|_p, \end{aligned}$$

using the first part of the lemma. \square

4. Central limit theorem. Even if one takes two vertices u and v far apart, there is not necessarily a Gaussian central limit theorem for the effective resistance between them, since the influence of an edge near u , for instance, may well represent a positive fraction of the total variance of the resistance. However, let us define the *influence of an edge e* on the effective resistance between u and v by

$$I^{u,v}(e) := \|\mathcal{R}_{\{e\}}^{u,v}\|_2^2.$$

Then, if the maximal influence of an edge is small with respect to the variance and if the graph has homogeneous currents and bounded degree, one may obtain a Gaussian approximation. The following theorem shows a result in this direction, whereas another instance of this phenomenon will be described on a sequence of finite graphs, the discrete tori, in Section 5.

THEOREM 4.1. *Let \mathbb{G} be a graph with homogeneous currents and bounded degree, equipped with elliptic resistances in $[1, \Lambda]$. For vertices u and v such that $\text{Var}(\mathcal{R}^{u,v}) > 0$, define*

$$\beta(u, v) = \frac{\sup_{e \in E} I^{u,v}(e)}{\text{Var}(\mathcal{R}^{u,v})}$$

and

$$\overline{\mathcal{R}}^{u,v} := [\mathcal{R}^{u,v} - \mathbb{E}(\mathcal{R}^{u,v})] / \sqrt{\text{Var}(\mathcal{R}^{u,v})}.$$

Let Φ be the standard Gaussian distribution function and let $F^{u,v}$ be the distribution function of $\overline{\mathcal{R}}^{u,v}$. There is a function $f : \mathbb{R}^+ \rightarrow [0, 1]$, depending on \mathbb{G} and Λ only, such that

$$f(x) \xrightarrow{x \rightarrow 0} 0$$

and

$$\|F^{u,v} - \Phi\|_\infty \leq f(\beta(u, v)).$$

PROOF. Let us fix the vertices u and v . For every integer L , one defines $J = J(L)$ as

$$J(L) = \{S \subset E \text{ s.t. } \text{diam}(S) \leq L \text{ and } S \neq \emptyset\}.$$

Let W_L be the random variable:

$$W_L = \frac{\sum_{S \in J(L)} \mathcal{R}_S^{u,v}}{\sqrt{\sum_{S \in J(L)} \|\mathcal{R}_S^{u,v}\|_2^2}}.$$

Since G has homogeneous currents, we know that for L large enough, W_L will be close (in L^2 -norm) to $\overline{\mathcal{R}}^{u,v}$. On the other hand, since $\mathcal{R}_S^{u,v}$ depends only on $(r(e))_{e \in S}$, we know that for every fixed L , W_L is a sum of random variables with only local dependence. Thus, one may use the work of [12] to control the distance to normality.

To be more precise, let F_L be the distribution function of W_L . For S in $J(L)$, define, using the notation of [12],

$$\begin{aligned} A_S &= \{S_1 \in J(L) \text{ s.t. } S \cap S_1 \neq \emptyset\}, \\ B_S &= \{S_2 \in J(L) \text{ s.t. } \exists S_1 \in A_S, S_1 \cap S_2 \neq \emptyset\}, \\ C_S &= \{S_3 \in J(L) \text{ s.t. } \exists S_2 \in B_S, S_2 \cap S_3 \neq \emptyset\}. \end{aligned}$$

Finally, let $N(C_S) = \{S' \in J(L) \text{ s.t. } C_S \cap C_{S'} \neq \emptyset\}$ and

$$\kappa = \max_S \{|N(C_S)|, |\{S' \text{ s.t. } S \in C_{S'}\}|\}.$$

It is clear that

$$C_S \subset \{S_3 \in J(L) \text{ s.t. } \exists e \in S, e' \in S_3, d(e, e') \leq 2L\}.$$

Thus,

$$N(C_S) \subset \{S' \in J(L) \text{ s.t. } \exists e \in S, e' \in S', d(e, e') \leq 6L\},$$

which shows that

$$|N(C_S)| \leq |S| \delta^{6L} 2^{\delta^L}$$

and

$$|\{S' \text{ s.t. } S \in C_{S'}\}| \leq |S| \delta^{2L} 2^{\delta^L}.$$

Thus, one may use Theorem 2.4 of [12] with $p = 3$ and $\kappa \leq 2^{8\delta^L}$ where δ is a bound on the degrees in G . We obtain

$$\|F_L - \Phi\|_\infty \leq \kappa \frac{\sum_{S \in J(L)} \|\mathcal{R}_S^{u,v}\|_3^3}{(\sum_{S \in J(L)} \|\mathcal{R}_S^{u,v}\|_2^2)^{3/2}}.$$

Using Corollary 3.14 and Lemma 3.15,

$$\begin{aligned} \|F_L - \Phi\|_\infty &\leq C_1(L, \Lambda, \mathbf{G}) \frac{\sum_{S \in J(L)} (\sum_{e \in S} \|\mathcal{R}_{\{e\}}^{u,v}\|_3)^3}{(\sum_{e \in E} \|\mathcal{R}_{\{e\}}^{u,v}\|_2^2)^{3/2}} \\ &\leq C_1(L, \Lambda, \mathbf{G}) \frac{\sum_{S \in J(L)} |S|^{2/3} \sum_{e \in S} \|\mathcal{R}_{\{e\}}^{u,v}\|_3^3}{(\sum_{e \in E} \|\mathcal{R}_{\{e\}}^{u,v}\|_2^2)^{3/2}} \\ &\leq C'_1(L, \Lambda, \mathbf{G}) \frac{\sum_{e \in E} \|\mathcal{R}_{\{e\}}^{u,v}\|_2^3}{(\sum_{e \in E} \|\mathcal{R}_{\{e\}}^{u,v}\|_2^2)^{3/2}}, \end{aligned}$$

one gets

$$\|F_L - \Phi\|_\infty \leq C(L) \sqrt{\beta(u, v)},$$

where $C(L) = C(\Lambda, \mathbf{G}, L)$ is a positive nondecreasing function of L .

On the other hand, Theorem 3.13 ensures that

$$\left\| \mathcal{R}^{u,v} - \mathbb{E}(\mathcal{R}^{u,v}) - \sum_{S \in J(L)} \mathcal{R}_S^{u,v} \right\|^2 \leq \varepsilon(L) \text{Var}(\mathcal{R}^{u,v}),$$

where $\varepsilon(L) = \varepsilon(\Lambda, \mathbf{G}, L)$ is a positive nonincreasing function of L which goes to zero as L goes to infinity. This implies

$$\|\overline{\mathcal{R}}^{u,v} - W_L\|^2 \leq 4\varepsilon(L).$$

Notice that Φ is 1-Lipschitz (in fact, $1/\sqrt{2\pi}$ -Lipschitz), so for any $\eta > 0$ and any $t \in \mathbb{R}$

$$\begin{aligned} F^{u,v}(t) - \Phi(t) &\leq F^{u,v}(t) - \Phi(t + \eta) + \eta \\ &\leq F^{u,v}(t) - F_L(t + \eta) + C(L) \sqrt{\beta(u, v)} + \eta \\ &\leq \mathbb{P}(W_L - \overline{\mathcal{R}}^{u,v} > \eta) + C(L) \sqrt{\beta(u, v)} + \eta \\ &\leq \frac{\|\overline{\mathcal{R}}^{u,v} - W_L\|_2^2}{\eta^2} + C(L) \sqrt{\beta(u, v)} + \eta \\ &\leq \frac{4\varepsilon(L)}{\eta^2} + C(L) \sqrt{\beta(u, v)} + \eta. \end{aligned}$$

Symmetrically, one gets, for any $\eta > 0$ and any $t \in \mathbb{R}$

$$|F^{u,v}(t) - \Phi(t)| \leq \frac{4\varepsilon(L)}{\eta^2} + C(L) \sqrt{\beta(u, v)} + \eta.$$

Optimizing in η gives

$$\|F^{u,v} - \Phi\|_\infty \leq 3\varepsilon^{1/3}(L) + C(L) \sqrt{\beta(u, v)}.$$

It remains to optimize in L . Let $L_0(x) = \sup\{L \in \mathbb{N} \text{ s.t. } 3\varepsilon^{1/3}(L) \geq \sqrt{x}C(L)\}$. Then

$$\|F^{u,v} - \Phi\|_\infty \leq 6\varepsilon^{1/3}(L_0(\beta(u, v))).$$

Since $L_0(x)$ goes to infinity as x goes to zero, $f(x) := 6\varepsilon^{1/3}(L_0(x))$ answers the theorem. \square

It is in general difficult to apply this result because it is difficult to bound $\beta(u, v)$. However, notice that the influence of an edge is always bounded. Thus, on a bounded graph with homogeneous currents, if the variance of $\mathcal{R}^{u,v}$ goes to infinity as u and v move apart, one gets a central limit theorem. Notice that the last point is equivalent to showing that $\mathbb{E}[\sum_e (i_r^{u,v}(e))^4]$ goes to infinity. It may be shown, for instance, that this is true on some wedges of \mathbb{Z}^2 , using the idea of Nash–Williams inequality. For instance, let $h(x) = x^\alpha$ with $\alpha \leq 1/3$, let $V = \{(x, y) \in \mathbb{Z}^2 \text{ s.t. } |y| \leq h(|x|)\}$ and let G be the subgraph of \mathbb{Z}^2 induced by V . Then one may derive a central limit theorem for $\mathcal{R}^{0,v}$ on G when the distance $d(0, v)$ goes to infinity. Since this example is not quasi-transitive, one has to use the Harnack inequality to prove that the graph has homogeneous currents.

Notice also that already on \mathbb{Z}^2 , the variance of the resistance is only of order 1 (cf. [26]), and thus one cannot expect a central limit theorem for point-to-point effective resistance when the resistances are i.i.d. (since the influence of the edges near the source and the sink is of order 1). In this respect, the interest of Theorem 4.1 is rather limited. However, one should rather think of it as a first step, with a clean statement, toward central limit theorems for resistances on sequences of finite graphs, as will be made clear in Section 5.

5. CLT for the effective resistance of the d -dimensional torus. In this section, we investigate when n becomes large the effective resistance of the torus \mathbb{T}_n^d equipped with nonconstant i.i.d. resistances from $[1, \Lambda]$. Here, \mathbb{T}_n^d is the graph (V_n^d, E_n^d) where $V_n^d = (\mathbb{Z}/n\mathbb{Z})^d$ and E_n^d is the set of oriented edges of the torus: two vertices x and y of V_n^d are joined by an edge from x to y if there is some $i \in \{1, \dots, n\}$ such that $x_i - y_i \in \{-1, 1\}$ and for all $j \neq i$, $x_j = y_j$. One chooses also exactly one edge of each orientation as follows:

$$E_{1/2}^n = \{(x, y) \in E_n^d \text{ s.t. } \exists i \in \{1, \dots, n\} y_i - x_i = 1\}.$$

Recall that $\ell_-^2(E_n^d, r)$ is the Hilbert space

$$\ell_-^2(E_n^d, r) = \{\theta \in \mathbb{R}^{E_n^d} \text{ s.t. } \mathcal{E}_r(\theta) < \infty \text{ and } \forall e \in E_n^d, \theta(e) = -\theta(-e)\},$$

where

$$\mathcal{E}_r(\theta) := \sum_{e \in E_{1/2}^n} r(e)\theta^2(e),$$

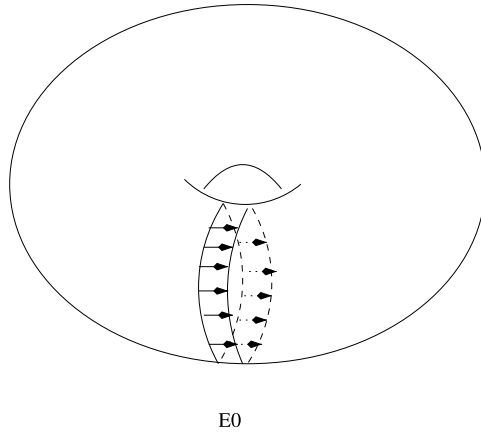


FIG. 2. The cut E_0 on a 2-dimensional torus.

endowed with the scalar product

$$(\theta, \theta')_r = \sum_{e \in E_{1/2}^n} r(e)\theta(e)\theta'(e).$$

Also, for resistances r in $[1, \Lambda]^{E_{1/2}^n}$, all the sets $\ell_-^2(E_n^d, r)$ are the same, and we denote this space by $\ell_-^2(E_n^d)$.

Since \mathbb{T}_n^d has no boundary, our first objective is to define the effective resistance in a natural and translation invariant way. First, we define a special *cut along direction 1* (see Figure 2):

$$E_0 := \{(x, y) \in E_{1/2}^n \text{ s.t. } x \sim y, x_1 = 0 \text{ and } y_1 = 1\},$$

and the flows which *cross the torus along direction 1, with intensity 1*:

$$\Theta_0^n := \left\{ \theta \in \ell_-^2(E_n^d) \text{ s.t. } d^*\theta = 0 \text{ and } \sum_{e \in E_0} \theta(e) = 1 \right\}.$$

Notice that the elements of Θ_0^n are sourceless flows and that the definition is independent of the choice of the cut (along direction 1). Indeed, if we define, for any i in $\{0, \dots, n - 1\}$,

$$E_i := \{(x, y) \in V_n^d \times V_n^d \text{ s.t. } x \sim y, x_1 = i \text{ and } y_1 = i + 1\},$$

then, for any i and any $\theta \in \ell_-^2(E_{1/2}^n)$, we have

$$(25) \quad \sum_{e \in E_i} \theta(e) - \sum_{e \in E_{i-1}} \theta(e) = \sum_{x \text{ s.t. } x_1=i} d^*\theta(x).$$

Thus, for any sourceless flow θ , $\sum_{e \in E_0} \theta(e)$ equals 1 if and only if $\sum_{e \in E_i} \theta(e) = 1$ for some i in $\{0, \dots, n - 1\}$. Thus, for any translation τ on the torus,

$$(26) \quad \theta \in \Theta_0^n \iff \theta \circ \tau \in \Theta_0^n.$$

DEFINITION 5.1. One defines the effective resistance of the torus as

$$(27) \quad \mathcal{R}_n := \inf_{\theta \in \Theta_0^n} \mathcal{E}_r(\theta).$$

Using the Nash–Williams inequality, it is easy to show that $\mathcal{R}_n = \Theta(1/n^{d-2})$. Using the ideas of the preceding sections, one may show that \mathcal{R}_n satisfies a central limit theorem. This is the main result of the present paper.

THEOREM 5.2. Suppose that $r(e)$, $e \in E_{1/2}^n$ are i.i.d. with positive variance and support in $[1, \Lambda]$ for some $\Lambda > 1$. Then

$$\text{Var}(\mathcal{R}_n) = \Theta(n^{4-3d}),$$

and the resistance satisfies a central limit theorem:

$$\frac{\mathcal{R}_n - \mathbb{E}(\mathcal{R}_n)}{\sqrt{\text{Var}(\mathcal{R}_n)}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1).$$

It is straightforward to show that a similar statement holds for what should be called the effective conductance of the torus \mathbb{T}_n^d , $C_n := 1/\mathcal{R}_n$ or the mean conductivity of the torus \mathbb{T}_n^d : $A_n = n^{2-d}C_n$. One obtains

$$\text{Var}(A_n) = \Theta(n^{-d})$$

and

$$\frac{A_n - \mathbb{E}(A_n)}{\sqrt{\text{Var}(A_n)}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1).$$

In [5], the definition of A_n is different, based on the discrete cube and not on the torus, however the behaviour should be the same. [5] obtains only suboptimal bounds for the variance of the mean conductivity. In [16], an optimal bound on the variance is obtained for a related quantity based on the corrector of homogenization theory. Again, the behaviour should be the same. In any case, the central limit theorem is new.

Our strategy to show Theorem 5.2 is simply to apply the ideas of Theorem 4.1 to this setting, where the infinite graph is traded against a growing sequence of finite graphs. First, notice that there is a unique minimal flow reaching the infimum in the definition of the resistance. It is characterized by an orthogonality criterion which is the analog of the Kirchhoff cycle law (see Definition 2.1) in this setting, so we shall call it a pseudo-Kirchhoff cycle law. Define the following tangent vector space to Θ_0^n :

$$\vec{\Theta} := \left\{ \theta \in \ell_-^2(E_n^d) \text{ s.t. } d^*\theta = 0 \text{ and } \sum_{e \in E_0} \theta(e) = 0 \right\}.$$

LEMMA 5.3. *The infimum in (27) is attained at a unique flow $i_{r,n}^{\text{per}}$ which is the unique flow in Θ_0^n that satisfies the following pseudo-Kirchhoff cycle law:*

$$\forall h \in \vec{\Theta}, \quad (h, i_{r,n}^{\text{per}})_r = 0.$$

PROOF. This is standard since $(\cdot, \cdot)_r$ is a scalar product. Let $\theta_0 \in \Theta_0^n$ be fixed. Notice that $\Theta_0^n = \theta_0 + \vec{\Theta}$. Define $i_{r,n}^{\text{per}}$ as the orthogonal projection of θ_0 on $\vec{\Theta}^\perp$ for $(\cdot, \cdot)_r$. It is the unique element i of $\ell^2_-(E_n^d)$ satisfying

$$\theta_0 - i \in \vec{\Theta} \quad \text{and} \quad \forall h \in \vec{\Theta}, \quad (h, i)_r = 0,$$

which is equivalent to

$$i \in \Theta_0^n \quad \text{and} \quad \forall h \in \vec{\Theta}, \quad (h, i)_r = 0.$$

This shows the last part of the lemma. Now, for any $\theta \in \Theta_0^n$, $\theta - i_{r,n}^{\text{per}}$ belongs to $\vec{\Theta}$, and thus

$$\begin{aligned} \mathcal{E}_r(\theta) &= \mathcal{E}_r(\theta - i_{r,n}^{\text{per}}) + 2(\theta - i_{r,n}^{\text{per}}, i_{r,n}^{\text{per}})_r + \mathcal{E}_r(i_{r,n}^{\text{per}}) \\ &= \mathcal{E}_r(\theta - i_{r,n}^{\text{per}}) + \mathcal{E}_r(i_{r,n}^{\text{per}}), \end{aligned}$$

which shows that the infimum of $\mathcal{E}_r(\theta)$ over $\theta \in \Theta_0^n$ is uniquely attained at $i_{r,n}^{\text{per}}$. □

Since the minimal flow is sourceless but satisfies the additional condition that the net flow through E_0 is 1, one needs to adapt the setting of the first sections in order to prove Theorem 5.2. For any $e \in E_{1/2}^n$, let

$$\Theta_e^n := \left\{ \text{unit flows } \theta \text{ from } e_- \text{ to } e_+ \text{ s.t. } \sum_{e' \in E_0} \theta(e') = \mathbf{1}_{e \in E_0} \right\}.$$

From (25), one sees that for any unit flow θ from e_- to e_+ , and any i ,

$$\sum_{e' \in E_i} \theta(e') = \sum_{e' \in E_0} \theta(e') + \mathbf{1}_{e \in E_i} - \mathbf{1}_{e \in E_0}.$$

Thus, one sees that for any translation τ on the torus, and any edge e ,

$$(28) \quad \theta \in \Theta_e^n \iff \theta \circ \tau \in \Theta_{\tau^{-1}(e)}^n.$$

Then, one may show as in the proof of Lemma 5.3 that $\theta \mapsto \mathcal{E}_r(\theta)$ has a unique minimizer on Θ_e^n , that we shall call $j_{r,n}^e$, and which is characterized by the same pseudo-Kirchhoff cycle law.

LEMMA 5.4. *The following infimum*

$$\inf_{\theta \in \Theta_e^n} \mathcal{E}_r(\theta)$$

is attained at a unique flow $j_{r,n}^e$, which is the orthogonal projection of χ_e on $\vec{\Theta}^\perp$ in $\ell^2_-(\mathbb{E}_n^d, r)$. It is the unique flow in Θ_e^n that satisfies the pseudo-Kirchhoff cycle law:

$$\forall h \in \vec{\Theta}, \quad (h, j_{r,n}^e)_r = 0.$$

PROOF. The proof is completely similar to the proof of Lemma 5.3, since $\vec{\Theta}$ is again the tangent vector space to Θ_e^n . \square

The role of $j_{r,n}^e$ will be similar to that of i_r^e in the first sections, as hinted by the following lemma.

LEMMA 5.5. *The functions $r \mapsto i_{r,n}^{\text{per}}(e)$, for any edge e , and $r \mapsto \mathcal{R}_n(r)$ admit partial derivatives of all orders. In addition, for any edges e, e' :*

- (i) $\forall e' \neq e, \partial_{e'} i_{r,n}^{\text{per}}(e) = \frac{i_{r,n}^{\text{per}}(e')}{r(e')} j_{r,n}^{e'}(e) = \frac{i_{r,n}^{\text{per}}(e')}{r(e')} j_{r,n}^e(e')$.
- (ii) $\forall e, \partial_e i_{r,n}^{\text{per}}(e) = \frac{i_{r,n}^{\text{per}}(e)}{r(e)} (j_{r,n}^e(e) - 1)$.
- (iii) $\forall e, \partial_e \mathcal{R}_n(r) = (i_{r,n}^{\text{per}}(e))^2$.

PROOF. The fact that $r \mapsto i_{r,n}^{\text{per}}(e)$ and $r \mapsto \mathcal{R}_n(r)$ admits partial derivatives of all order is analogous to the classical case; cf. the proof of Lemma 2.3.

Let us fix some edge $e' \in \mathbb{E}_{1/2}^n$. One may thus differentiate the node law and the pseudo-Kirchhoff cycle law of Lemma 5.3 with respect to $r(e')$ to obtain

$$\forall x \in \mathbb{V}_n^d, \quad d^*[\partial_{e'} i_{r,n}^{\text{per}}](x) = 0$$

and

$$\forall h \in \vec{\Theta}, \quad \sum_e h(e)r(e) \left(\partial_{e'} i_{r,n}^{\text{per}}(e) + \frac{i_{r,n}^{\text{per}}(e')}{r(e')} \chi_{e'}(e) \right) = 0.$$

Thus, if we define

$$\theta(e) = \partial_{e'} i_{r,n}^{\text{per}}(e) + \frac{i_{r,n}^{\text{per}}(e')}{r(e')} \chi_{e'}(e),$$

we see that

$$\forall x \notin e', \quad d^*\theta(x) = 0$$

and

$$\forall h \in \vec{\Theta}, \quad (h, \theta)_r = 0.$$

Furthermore,

$$d^*\theta(e'_-) = \frac{i_{r,n}^{\text{per}}(e')}{r(e')}$$

and

$$\sum_{e \in E_0} \theta(e) = \frac{i_{r,n}^{\text{per}}(e')}{r(e')} \mathbf{1}_{e' \in E_0}.$$

It follows from the characterization of $j_{r,n}^{e'}$ that

$$\theta = \frac{i_{r,n}^{\text{per}}(e')}{r(e')} j_{r,n}^{e'}.$$

This gives the proof of the first two equations. The proof of the last one is analogous to the classical case; cf. Lemma 2.3. \square

When n is large, we would like to compare $j_{r,n}^e$ to a flow on the whole lattice \mathbb{Z}^d . To do this, we shall couple the network $(\mathbb{Z}^d, \mathbb{E}^d)$ with all the tori by “unwrapping” each torus on \mathbb{Z}^d . *This construction will be used throughout the section.* Since our main objects are elements of Θ_0^n and Θ_e^n , we mainly need to identify the set of edges E_n^d , equipped with their resistances, as subsets of \mathbb{E}^d , and then be careful about what happens to the boundary operator through this identification.

First, fix e to be any edge such that e_- is the origin of \mathbb{Z}^d . Let $r \in [1, \Lambda]^{\mathbb{E}^d_{1/2}}$ be a fixed collection of resistances and define

$$V_n = \{-\lfloor n/2 \rfloor, \dots, \lfloor (n-1)/2 \rfloor\}^d,$$

where $\lfloor \cdot \rfloor$ is the integer part, and

$$E_{1/2}^n = \{(x, y) \in V_n \times \mathbb{Z}^d \text{ s.t. } \exists i \in \{1, \dots, n\} y_i - x_i = 1 \text{ and } \forall j \neq i, x_j = y_j\}$$

so that these sets are roughly centered around the origin. Now, we let G_n^{per} be the network with edge set $E_{1/2}^n$, resistances induced by r and set of vertices all the endpoints of the edges of $E_{1/2}^n$ with periodic condition, that is, vertices with identical coordinates modulo n are identified (this takes care of the boundary operator on the torus). Clearly, G_n^{per} is isomorphic to \mathbb{T}_n^d , and we shall thus use the notation $E_0, \Theta_e^n, \Theta_0^n$ and $j_{r,n}^e$ on G_n^{per} as well.

LEMMA 5.6. *Let e be any fixed edge such that e_- is the origin of \mathbb{Z}^d . Let i_r^e be the minimal current on \mathbb{Z}^d from e_- to e_+ , then $j_{r,n}^e$ converges to i_r^e in $\ell^2_-(\mathbb{E}^d, r)$.*

PROOF. We let G_n^F be the subgraph of \mathbb{Z}^d induced by the set of vertices V_n . Also, we define G_n^W be the graph obtained from \mathbb{Z}^d by identifying all the vertices outside $\{-\lfloor n/2 \rfloor + 1, \dots, \lfloor (n-1)/2 \rfloor\}^d$. See Figure 3. We equip these graphs with resistances given by r . Since these graphs have sets of edges which are still subsets of \mathbb{E}^d , there is no ambiguity about what resistance is assigned to which edge.

We need now to introduce some terminology from the theory of electrical networks. Let $G = (V, E)$ be a graph equipped with resistances $r = (r(e))_{e \in E}$ and

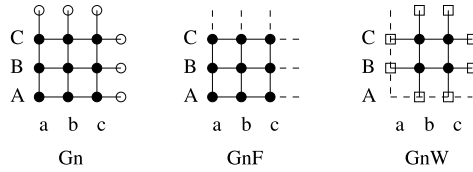


FIG. 3. The three graphs G_n^{per} , G_n^F and G_n^W when $n = 3$ and $d = 2$. Squares are identified as a single vertex, while empty circles are subject to periodic identification modulo n .

suppose that $(H_n)_{n \geq 0}$ is a sequence of finite subgraphs of G that exhausts G , as in Section 2.1. A *star* of a graph $G = (V, E)$ is a member of $\ell^2_-(E, r)$ of the form $\sum_{e_- = x} c(e) \chi_e$ for $x \in V$. Let \star (resp., \star_n) denote the closed subspace spanned by the stars in $\ell^2_-(E, r)$ (resp., by the stars of H_n^W). A *cycle* is a member of $\ell^2_-(E, r)$ of the form $\sum_{i=1}^n \chi^{e_k}$ with e_1, \dots, e_n an oriented cycle in G . Let \diamond (resp., \diamond_n) denote the closed subspace spanned by the cycles in $\ell^2_-(E, r)$ (resp., by the cycles of H_n). To understand the introduction of wired and free networks, notice that all stars of the wired network H_n^W are stars in G , except for the star at the “extra vertex” which represents all the vertices outside H_n , but this additional star is just the opposite of all the stars in H_n , thus \star_n is a subspace of \star . Furthermore, since $(H_n)_n$ exhausts G , each star of G is a member of \star_n for n large enough. This shows that $\star = \overline{\bigcup_n \star_n}$. Also, all the cycles of H_n are cycles of G , and thus \diamond_n is a subspace of \diamond , and each cycle of \diamond is in \diamond_n for n large enough. This shows that $\diamond = \overline{\bigcup_n \diamond_n}$.

For a closed subspace V , denote by P_V the projection on V in $\ell^2_-(E, r)$. Then, for any edge e in H_0 , $P_{\star_n} \chi_e$ is the unique current on H_n^W between e_- and e_+ and it converges in $\ell^2_-(E, r)$ to $i_r^e = P_{\star} \chi_e$, the minimal current on G from e_- to e_+ (this is a simple consequence of the fact that $\star = \overline{\bigcup_n \star_n}$; see Exercise 9.2 in [23], and also Propositions 9.1 and 9.2 therein). Also, $P_{\diamond_n^\perp} \chi_e$ converges to $i_r^{F,e} = P_{\diamond^\perp} \chi_e$, the *free current* from e_- to e_+ . In general, the free current $i_r^{F,e}$ is a current that may or may not be equal to i_r^e . However, on \mathbb{Z}^d (and on finite graphs of course), it is known that those currents coincide. One says in this case that “currents are unique.” Another way to express this is to say that $\star = \diamond^\perp$. An argument can be given as follows: this is trivial on \mathbb{Z} , because free and wired currents between u and v can be easily computed to be just 1 from u to v and 0 elsewhere, then the unicity of currents is preserved under cartesian product (Exercise 9.7 in [23]), this shows currents are unique on \mathbb{Z}^d with unit resistances. Finally, the unicity of currents is preserved under “rough isometries” (Theorem 9.9 in [23]), which include elliptic perturbations of the weights.

Now, we return to our setting and let \star_n (resp., \star_n^{per}) be the linear subspace of $\ell^2_-(E_{1/2}^n)$ spanned by the stars of G_n^W (resp., G_n^{per}). Let also \diamond_n (resp., \diamond_n^{per}) the linear subspace spanned by the cycles of G_n^F (resp., G_n^{per}). Since G_n^F is a strict subgraph of G_n^{per} ,

$$\diamond_n \subset \diamond_n^{\text{per}}.$$

Furthermore, any cycle in G_n^F must traverse E_0 in one direction the same number of times that it traverses it in the other direction (notice that this is not true on G_n^{per} , due to the periodic boundary conditions). Thus,

$$\diamond_n \subset \diamond_n^{\text{per}} \cap \left\{ \theta \in \ell^2_-(\mathbb{E}_{1/2}^n) \text{ s.t. } \sum_{e \in E_0} \theta(e) = 0 \right\}.$$

Also, note that the stars in G_n^W are generated by the stars at vertices in $\{-\lfloor n/2 \rfloor + 1, \dots, \lfloor (n-1)/2 \rfloor\}^d$, since the star at the “exterior vertex” equals the opposite of the sum of all the other stars. But all those stars are stars of G_n^{per} . Thus,

$$\star_n \subset \star_n^{\text{per}}.$$

Recall from Lemma 5.4 that $j_{r,n}^e$ is the orthogonal projection in $\ell^2_-(\mathbb{E}_n^d, r)$ of χ_e on $H_n = \overrightarrow{\Theta}^\perp$. It is easy to see that

$$\overrightarrow{\Theta} = \left\{ \theta \in \diamond_n^{\text{per}} \text{ s.t. } \sum_{e \in E_0} \theta(e) = 0 \right\}.$$

Indeed, on G_n^{per} , the condition $d^*\theta = 0$ may be written as $\theta \in (\star_n^{\text{per}})^\perp$, which is equivalent to $\theta \in \diamond_n^{\text{per}}$ since G_n^{per} is a finite graph. Notice that $j_{r,n}^e$ is a flow between e_- and e_+ on G_n^{per} but not necessarily on \mathbb{Z}^d . The inclusions above give

$$\star_n \subset H_n \subset \diamond_n^\perp.$$

Let \star (resp., \diamond) denote the closed linear subspace spanned by stars (resp., cycles) in $\ell^2_-(\mathbb{E}^d, r)$. Then, according to the elements of the theory of electrical networks stated above, $P_{\star_n} \chi_e$ (resp., $P_{\diamond_n^\perp} \chi_e$) converges to $i_r^e = P_{\star} \chi_e$, the minimal current on \mathbb{Z}^d from e_- to e_+ (resp., $i_r^{F,e} = P_{\diamond^\perp} \chi_e$, the free current on \mathbb{Z}^d from e_- to e_+). But on \mathbb{Z}^d , currents are unique. As a consequence, $(j_{r,n}^e)_{n \geq 1}$ converges also to i_r^e , the minimal current on \mathbb{Z}^d from e_- to e_+ . \square

In order to adapt the notion of graphs with homogeneous currents to this setting of sequences of finite graphs, define

$$\alpha(d, L, \Lambda) := \sup_{n \geq 1} \sup_{r \in [1, \Lambda]^{E_{1/2}}} \sup_{e \in \mathbb{T}_n^d} \sum_{d(e', e) \geq L} r(e') (j_{r,n}^e(e'))^2.$$

PROPOSITION 5.7. *The d -dimensional discrete tori have homogeneous currents in the sense that for any $\Lambda \geq 1$,*

$$\alpha(d, L, \Lambda) \xrightarrow{L \rightarrow +\infty} 0.$$

PROOF. We will adapt the proof of Proposition 3.10, using the convergence of $j_{r,n}^e$ given by Lemma 5.6. Let $V = \mathbb{Z}^d$ and $E = \mathbb{E}^d$.

Let e_1, \dots, e_d be the d edges going from 0 to some point of nonnegative coordinates. Thanks to the translation property (28), one has, for any translation τ on the torus:

$$j_{r \circ \tau^{-1}, n}^{\tau(e)} \circ \tau = j_{r,n}^e.$$

Thus,

$$\alpha(d, L, \Lambda) := \sup_{n \geq 1} \sup_{e \in \{e_1, \dots, e_d\}} \sup_{r \in [1, \Lambda]^{\mathbb{E}_{1/2}}} \sum_{d(e', e) \geq L} r(e') (j_{r,n}^e(e'))^2.$$

Let e be any fixed edge such that e_- is the origin, and define

$$c := \limsup_{L \rightarrow \infty} \sup_{n \geq 1} \sup_{r \in [1, \Lambda]^{\mathbb{E}_{1/2}}} \sum_{d(e', e) \geq L} r(e') (j_{r,n}^e(e'))^2.$$

It is thus enough to prove that $c = 0$. Notice that when n is fixed,

$$\limsup_{L \rightarrow \infty} \sup_{r \in [1, \Lambda]^{\mathbb{E}_{1/2}}} \sum_{d(e', e) \geq L} r(e') (j_{r,n}^e(e'))^2 = 0,$$

since the sequence in L is zero for L large enough. Thus, one may find a sequence $(r_L, n_L)_{L \geq 1}$ in $[1, \Lambda]^{\mathbb{E}_{1/2}} \times \mathbb{N}$ such that

$$c = \lim_{L \rightarrow \infty} \sum_{d(e', e) \geq L} r_L(e') (j_{r_L, n_L}^e(e'))^2,$$

and $n_L \xrightarrow{L \rightarrow \infty} +\infty$. The sequence j_{r_L, n_L}^e is bounded in $\ell_-^2(\mathbb{E}^d)$. Thus, by compactness of $[1, \Lambda]^{\mathbb{E}_{1/2}}$ one may extract a sequence from $(r_L, n_L)_{L \geq 1}$, that we shall still denote by $(r_L, n_L)_{L \geq 1}$ to lighten the notation, such that $(j_{r_L, n_L}^e(e'))_L$ converges $\theta(e')$ for any e' , and $(r_L(e'))_L$ converges to some $r(e') \in [1, \Lambda]$ for any e' . Notice that θ is a unit flow on the whole lattice \mathbb{Z}^d since for any L , $(j_{r_L, n_L}^e)_L$ is a unit flow on G_n^{per} . Using the minimality of i_r^e ,

$$\begin{aligned} \mathcal{E}_r(i_r^e) &\leq \mathcal{E}_r(\theta) \\ &= \sum_{e' \in \mathbb{E}_{1/2}} r(e') \theta^2(e') \\ &\leq \limsup_{L \rightarrow \infty} \left(\sum_{d(e', e) < L} r_L(e') (j_{r_L, n_L}^e(e'))^2 \right) \\ (29) \quad &= \limsup_{L \rightarrow \infty} \left(\mathcal{E}_{r_L}(j_{r_L, n_L}^e) - \sum_{d(e', e) \geq L} r_L(e') (j_{r_L, n_L}^e(e'))^2 \right) \\ &= \limsup_{L \rightarrow \infty} \mathcal{E}_{r_L}(j_{r_L, n_L}^e) - c \\ &\leq \limsup_{L \rightarrow \infty} \mathcal{E}_{r_L}(j_{r, n_L}^e) - c, \end{aligned}$$

where in the last inequality we used the minimality property of j_{r_L, n_L}^e . Now, using Minkowski’s inequality,

$$\begin{aligned} \mathcal{E}_{r_L}(j_{r, n_L}^e) &= \sum_{e' \in E_{1/2}} r_L(e')(j_{r, n_L}^e(e'))^2 \\ &\leq \left(\sqrt{\sum_{e' \in E_{1/2}} r_L(e')(i_r^e(e'))^2} + \sqrt{\Lambda \sum_{e' \in E_{1/2}} (j_{r, n_L}^e(e') - i_r^e(e'))^2} \right)^2. \end{aligned}$$

Since $(r_L)_{L \geq 1}$ converges simply to r and is bounded by Λ , the dominated convergence theorem gives

$$\sum_{e' \in E_{1/2}} r_L(e')(i_r^e(e'))^2 \xrightarrow{L \rightarrow \infty} \mathcal{E}_r(i_r^e),$$

Lemma 5.6 says that $(j_{r, n_L}^e)_{L \geq 1}$ converges to i_r^e in $\ell_-^2(E_{1/2})$. Thus,

$$\limsup_{L \rightarrow \infty} \mathcal{E}_{r_L}(j_{r, n_L}^e) \leq \mathcal{E}_r(i_r^e).$$

Plugging this into (29) shows that $c = 0$. \square

Now, one may complete the proof of Theorem 5.2.

PROOF OF THEOREM 5.2. With Lemma 5.5 at hand, it is easy to reproduce the proofs of Lemma 2.4, Corollary 3.14 and Theorem 3.13 with $\alpha(G, L, \Lambda)$ replaced by $\alpha(d, L, \Lambda)$. One obtains notably the existence of constants $C(\Lambda)$ and $C(\Lambda, d)$ such that for any n , and any $L \geq 1$,

$$\sum_{\text{diam}(S) \geq L} \|(\mathcal{R}_n)_S\|_2^2 \leq C(\Lambda)\alpha(G, L, \Lambda) \sum_{S \neq \emptyset} \|(\mathcal{R}_n)_S\|_2^2$$

and

$$\sum_e \|(\mathcal{R}_n)_{\{e\}}\|_2^2 \leq \text{Var}(\mathcal{R}_n) = \sum_{S \neq \emptyset} \|(\mathcal{R}_n)_S\|_2^2 \leq C(\Lambda, d) \sum_e \|(\mathcal{R}_n)_{\{e\}}\|_2^2.$$

Now, thanks to the translation invariance of the model given by (26) and the fact that the edge-resistances are i.i.d.,

$$\beta_n := \sup_n \sup_{e \in E_{1/2}^n} \frac{\|(\mathcal{R}_n)_{\{e\}}\|_2^2}{\text{Var}(\mathcal{R}_n)} = \Theta(1/n^d)$$

and

$$\sup_{e \in E_{1/2}^n} \mathbb{E}[(i_{r, n}^{\text{per}}(e))^2] = \Theta\left(\frac{1}{n^d} \mathbb{E}(\mathcal{R}_n)\right) = \Theta\left(\frac{1}{n^{2d-2}}\right).$$

This already shows that

$$\text{Var}(\mathcal{R}_n) = \Theta\left(\sup_{e \in E_{1/2}^n} \mathbb{E}[(i_{r,n}^{\text{per}}(e))^2]^2\right) = \Theta\left(\frac{1}{n^{3d-4}}\right).$$

Now, Proposition 5.7 allows to repeat the proof of Theorem 4.1. Let F_n be the distribution function of $\frac{\mathcal{R}_n - \mathbb{E}(\mathcal{R}_n)}{\sqrt{\text{Var}(\mathcal{R}_n)}}$. One obtains the existence of a function f having limit 0 at 0^+ and such that for any n ,

$$\|F_n - \Phi\|_\infty \leq f(\beta_n).$$

This completes the proof of the central limit theorem. \square

6. Perspectives.

We end this article with some questions left open.

First, it is not clear whether the notion of homogeneous currents is really useful to get a central limit theorem. For instance, in the counterexample of Figure 1, one sees that the currents i_r^e still spread most of their mass at very localized places, namely near e and near the edge e'_k which is in the same connected component as e . Thus, one could be able to adapt the proof of the central limit theorem in this special case. One may wonder whether the sole hypotheses of bounded degree and small influences are enough to get a central limit theorem. On the other hand, if the homogeneous currents hypothesis is proved really necessary, it would be important to understand which graphs satisfy it, and whether it is stable under perturbations like quasi-isometries.

Second, the most obvious question left open is the one raised in the Introduction, that is to determine the order of the variance and to show a central limit theorem for the resistance on the cube of side length n in \mathbb{Z}^d , and not only on the torus. More generally, consider a domain Ω of \mathbb{R}^d with two disjoint subsets of its boundary, A and Z . Let G_n be the graph induced by $\Omega \cap \frac{1}{n}\mathbb{Z}^d$ and let \mathcal{R}_n be the effective resistance between A and Z on G_n . Then we conjecture that a Gaussian central limit theorem holds for \mathcal{R}_n .

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