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GAUSSIAN-TYPE LOWER BOUNDS FOR THE DENSITY OF SOLUTIONS OF SDES DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

BY M. BESALÚ, A. KOHATSU-HIGA[1] AND S. TINDEL

Université de Lorraine, Ritsumeikan University and Japan Science and Technology Agency, and Université de Lorraine

In this paper we obtain Gaussian-type lower bounds for the density of solutions to stochastic differential equations (SDEs) driven by a fractional Brownian motion with Hurst parameter H. In the one-dimensional case with additive noise, our study encompasses all parameters $H \in (0, 1)$, while the multidimensional case is restricted to the case $H > 1/2$. We rely on a mix of pathwise methods for stochastic differential equations and stochastic analysis tools.

1. Introduction. Let $B = (B^1, \dots, B^d)$ be a d-dimensional fractional Brownian motion (fBm in the sequel) defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with Hurst parameter $H \in (0, 1)$. Recall that this means that B is a centered Gaussian process indexed in $[0, 1]$, whose coordinate processes are independent, and their covariance structure is defined by

$$R(t,s) := \mathbf{E}\big[B_s^j B_t^j\big] = \tfrac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big) \qquad \text{for } s,t \in [0,1] \text{ and } j = 1,\dots,d. \tag{1}$$

This implies that the variance of an increment is given by

$$\mathbf{E}\big[\big(B_t^j - B_s^j\big)^2\big] = |t-s|^{2H} \qquad \text{for } s,t \in [0,1]. \tag{2}$$

In particular, this process is γ-Hölder continuous a.s. for any $\gamma < H$ and is an H-self similar process. This converts fBm into a natural generalization of Brownian motion and explains the fact that it is used in applications [17, 26, 27].

We are concerned here with the following class of stochastic differential equations (SDEs) in $\mathbb{R}^m$ driven by B on the time interval $[0, 1]$:

$$X_t = a + \int_0^t V_0(X_s)\,ds + \sum_{i=1}^d \int_0^t V_i(X_s)\,dB_s^i, \tag{3}$$

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where $a \in \mathbb{R}^m$ is a generic initial condition, and $\{V_i; 0 \le i \le d\}$ is a collection of smooth and bounded vector fields of $\mathbb{R}^m$. Though equation (3) can be solved thanks to rough paths methods in the general case $H \in (1/4, 1)$, $d \ge 1$, we shall consider in the sequel three situations which can be handled without recurring to this kind of technique:

(1) The one-dimensional case with additive noise and $H \in (0, 1)$, which can be treated via simple ODE techniques.
(2) The one-dimensional situation, namely $m = d = 1$ with $H \in (1/2, 1)$, where the equation can be solved thanks to a Doss–Sussman-type methodology, as mentioned in [19].
(3) The case of a Hurst exponent $H \in (1/2, 1)$, for which Young integration methods are available; see, for example, [14, 24, 29].

Hence, we always understand the solution to equation (3) according to the three settings mentioned above. We shall see, however, that rough path-type arguments shall be involved in some of our proofs.

The process defined as the solution of (3) is obviously worth studying, and a natural step in this direction is to analyze the density of the random variable X_t for a fixed $t > 0$. In this respect, the following results are available in our cases of interest:

(1) For $m = d = 1$, the existence of density for $\mathcal{L}(X_t)$ is examined in [19].
(2) Whenever $H > 1/2$ and in a multidimensional setting, the existence of density is established in [25], while smoothness under elliptic assumptions is handled in [15].

Let us also mention that for multidimensional equation (3) and $H \in (1/4, 1/2)$, rough path techniques also enable the study of densities of the solution. We refer to [5, 6] for existence and [7] for smoothness results for $\mathcal{L}(X_t)$. However, the only Gaussian-type estimate for the density we are aware of, is the one contained in [3], which relies heavily on a skew-symmetric assumption for the vector fields $V_1, \dots, V_d$.

The current article is thus dedicated to give Gaussian-type lower bounds for the density of X_t. More specifically, we work under the following assumptions on the coefficients of equation (3):

HYPOTHESIS 1.1. The coefficients $V_0, \dots, V_d$ of equation (3) satisfy the following conditions:

(1) If $m = d = 1$, then $V_0, V_1 \in \mathcal{C}_b^3$, and we also assume $\lambda \le |V_1| \le \Lambda$.
(2) In the multidimensional case, the vector fields $V_0, \dots, V_d$ belong to the space $\mathcal{C}_b^\infty$ of smooth functions bounded together with all their higher order derivatives. Furthermore, if $V(x)$ denotes the matrix $(V_1(x), \dots, V_d(x)) \in \mathbb{R}^{m\times d}$ for all $x \in \mathbb{R}^m$, then we assume the following uniform elliptic condition:

$$\lambda \mathrm{Id}_m \le V(x)V^*(x) \le \Lambda \mathrm{Id}_m \qquad \text{for all } x \in \mathbb{R}^m, \tag{4}$$

where the inequalities are understood in the matrix sense and where λ and Λ are two given strictly positive constants which are independent of x.

With these hypotheses in hand, our main goal is to prove the following result:

THEOREM 1.2. *Consider equation* (3), *under the following three specific situations*:

(I) $m = d = 1$, $H \in (0, 1)$, $V_0 \in \mathcal{C}_b^1$ *and the noise is additive* (*i.e.*, V_1 *is a nonvanishing real constant*).

(II) $m = d = 1$, $H \in (1/2, 1)$ *and Hypothesis* 1.1(1) *is satisfied for* V_0, V_1.

(III) *Arbitrary* $m, d \in \mathbb{N}$, $H \in (1/2, 1)$ *and* $V_0, \dots, V_d$ *satisfy Hypothesis* 1.1(2).

Then the solution X_t *of equation* (3) *possesses a density* $p_t(x)$ *such that for every* $x \in \mathbb{R}^m$ *and* $t \in (0, 1]$, *we have*

$$p_t(x) \geq \frac{c_1}{t^{mH}} \exp\left(-\frac{c_2|x-a|^2}{t^{2H}}\right), \tag{5}$$

for some constants c_1, c_2 *only depending on* d, m *and* $V_0, \dots, V_d$.

As mentioned above, this is (to the best of our knowledge) the first Gaussian-type lower bound obtained for the density of the solution of the SDE driven by fBm in a general setting. It should also be mentioned that lower bound (5) can be complemented by a similar upper bound contained in [4].

Let us say a few words about the methodology we rely on in order to obtain our lower bound (5). Generally speaking it is based on Malliavin calculus tools, but the three results mentioned in Theorem 1.2 are proved in different ways:

(1) In the one-dimensional additive case, we invoke a recent formula for densities introduced in [20] which yields an easy way to estimate p_t in the case of additive stochastic equations. We thus include this study for didactical purposes, and also because we obtain (slightly nonoptimal) Gaussian upper and lower bounds with elegant methods. Observe that this technique proves to be useful (generally speaking) for equations with additive noise, as assessed in a SPDE context in [23].

(2) The one-dimensional case with multiplicative noise is based on the Doss–Sussmann transform and Girsanov-type arguments. It is rather easy to implement and yields results when the criterion of [20] cannot be applied.

(3) As far as the general case is concerned, it will be basically handled, thanks to the decomposition of random variables, using increments independent of Gaussian increments strategy introduced in [2, 16], which has also been invoked successfully, for example, in [9]. However, let us point out two important differences between the fBm and the diffusion case:

(i) In the case of SDE (3) without drift coefficient V_0, the first step of the method implemented (for a fixed $t \in (0,1]$) in [2, 16] amounts to introducing a partition $\{t_j; 0 \le j \le n\}$ such that $t_0 = 0$ and $t_n = t$, with n large enough, and then splitting X_t into small contributions of the form

$$X_{t_{j+1}} - X_{t_j} = \sum_{i=1}^{d} V_i(X_{t_j})\big[B^i_{t_{j+1}} - B^i_{t_j}\big] + \sum_{i=1}^{d} \int_{t_j}^{t_{j+1}} \big[V_i(X_s) - V_i(X_{t_j})\big]\, dB^i_s. \tag{6}$$

Then a main conditionally Gaussian contribution $V_i(X_{t_j})[B^i_{t_{j+1}} - B^i_{t_j}]$ is identified on the right-hand side of equation (6), while the other terms are a small remainder in the Malliavin calculus sense in comparison with the first. Roughly speaking, Gaussian lower bound (5) is then obtained by adding those main contributions and proving that the remainder does not significantly modify the estimate. However, let us highlight the fact that this general scheme does not fit to the fractional Brownian motion setting.

Indeed, due to the fBm dependence structure, the main contributions to the variance of X_t in the current situation come from the cross terms $\mathbf{E}[(B^i_{t_{j+1}} - B^i_{t_j})(B^i_{t_{k+1}} - B^i_{t_k})]$ for $j \ne k$. We have thus decided to express equation (3) as an anticipative Stratonovich-type equation with respect to the Wiener process induced by B. This is known to be an inefficient way to solve the original equation, but turns out to be very useful in order to analyze the law of X_t. We shall detail this strategy at Section 5.1.

(ii) In the case of an equation driven by usual Brownian motion, the Malliavin–Sobolev norms involved in the computations give deterministic contributions after conditioning, due to the independence of increments of the Wiener process. This is not true, however, in the fBm case, and we thus need to add a proper localization to the arguments in [2, 16].

The adaptation of the Brownian methodology to our fBm context is thus nontrivial. Note that we could also have tried to resort to the powerful global bounds given in [18] in order to get our Gaussian lower bounds. Unfortunately, the exponential moments conditions imposed in the latter reference are too restrictive to be applied to Malliavin derivatives of SDEs driven by fBm.

Our article is structured as follows: Section 2 is devoted to recall some useful facts on fractional Brownian motion and stochastic differential equations. We handle the one-dimensional case with additive noise at Section 3 and the one-dimensional case with multiplicative noise in Section 4 with different methodologies. Finally, the bulk of our article focuses on the general multidimensional case contained in Section 5. Some auxiliary results used in Section 5 dealing with stochastic derivatives are given in an Appendix.

NOTATION. Throughout this paper, unless otherwise specified, we use $|\cdot|$ for Euclidean norms and $\|\cdot\|_{L^p}$ for the $L^p(\Omega)$ norm with respect to the underlying probability measure $\mathbf{P}$. For a random variable X, $\mathcal{L}(X)$ denotes its law and for a σ-field $\mathcal{F}$, $X \in \mathcal{F}$ denotes the fact that X is $\mathcal{F}$-measurable.

Consider a finite-dimensional vector space V and a subset $U \subset \mathbb{R}^d$. The space of V-valued Hölder continuous functions defined on U, with k-derivatives which are γ-Hölder continuous with $\gamma \in (0,1)$, will be denoted by $\mathcal{C}^{k+\gamma}(U;V)$, or just $\mathcal{C}^{k+\gamma}$ when $U=[0,1]$. For a function $g \in \mathcal{C}^{\gamma}(V)$ and $0 \le s < t \le 1$, we shall consider the semi-norms

$$\|g\|_{s,t,\gamma} = \sup_{s\le u<v\le t} \frac{|g_v - g_u|_V}{|v-u|^{\gamma}}. \tag{7}$$

The semi-norm $\|g\|_{0,1,\gamma}$ will simply be denoted by $\|g\|_{\gamma}$. Similarly, for an open set U, $\mathcal{C}_b^1(U;V)$ denotes the space of bounded continuously differentiable functions with bounded first derivative. For $x, y \in \mathbb{R}^m$, we set $\mathbf{1}_{\{y\ge x\}} := \prod_{k=1}^m \mathbf{1}_{\{y_k \ge x_k\}}$. Vectors $x \in \mathbb{R}^m$ denote column vectors, their jth component is denoted by x^j and the transpose of x is denoted by x^*. The identity matrix of order $m \times m$ is denoted by Id_m.

Finally, let us mention that generic constants will be denoted by c, c_H, c_V, etc., independently of their actual value which may change from one line to the next. This rule will also apply for the constants M and M' which will appear as localization parameters, with the following additional convention: each time a localization constant appears, it increases its value by the addition of a fixed universal constant from the previous value. For a detailed explanation, see (16).

2. Stochastic calculus for fractional Brownian motion. This section is devoted to giving some of the basic elements of stochastic calculus with respect to B. For some fixed $H \in (0,1)$, we consider $(\Omega, \mathcal{F}, \mathbf{P})$ the canonical probability space associated with the fractional Brownian motion (in short fBm) with Hurst parameter H. That is, $\Omega = \mathcal{C}_0([0,1];\mathbb{R}^d)$ is the Banach space of continuous functions vanishing at 0 equipped with the supremum norm, $\mathcal{F}$ is the Borel sigma-algebra and $\mathbf{P}$ is the unique probability measure on Ω such that the canonical process $B = \{B_t = (B_t^1, \ldots, B_t^d), t \in [0,1]\}$ is a fBm with Hurst parameter H. In this context, let us recall that B is a d-dimensional centered Gaussian process, whose covariance structure is induced by equation (2).

2.1. *Malliavin calculus tools.* Gaussian techniques are obviously essential in the analysis of fBm driven differential equations like (3), and we proceed here to introduce some of them; see Chapter 5 in [21] for further details.

2.1.1. *Wiener space associated to fBm.* Let $\mathcal{E}$ be the space of $\mathbb{R}^d$-valued step functions on $[0,1]$, and $\mathcal{H}$ the closure of $\mathcal{E}$ under the distance defined by the scalar product

$$\big\langle (\mathbf{1}_{[0,t_1]}, \ldots, \mathbf{1}_{[0,t_d]}), (\mathbf{1}_{[0,s_1]}, \ldots, \mathbf{1}_{[0,s_d]}) \big\rangle_{\mathcal{H}} = \sum_{i=1}^d R(t_i, s_i).$$

The space $\mathcal{H}$ is isometric to the reproducing kernel Hilbert space associated to B.

Furthermore, if $(e_1, \dots, e_d)$ designates the canonical basis of $\mathbb{R}^d$, one constructs an isometry $K^*: \mathcal{H} \to L^2([0,1]; \mathbb{R}^d)$ such that $K^*(\mathbf{1}_{[0,t]} e_i) = \mathbf{1}_{[0,t]}\ K_H(t, \cdot) e_i$, where the kernel $K = K_H$ is given by

$$K(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2}\, du, \qquad H > \frac{1}{2},$$

$$K(t,s) = c_{H,1} \left(\frac{s}{t}\right)^{1/2-H} (t-s)^{H-1/2} + c_{H,2} s^{1/2-H} \int_s^t (u-s)^{H-1/2} u^{H-3/2}\, du, \qquad H < \frac{1}{2}, \tag{8}$$

for $0 \le s \le t$ and some explicit universal constants $c_H, c_{H,1}, c_{H,2}$. With a slight abuse of notation we will denote the associated integral operator by $Kf(x) = \int_0^x f(s) K(x,s)\, ds$. Note that we have that $R(s,t) = \int_0^{s\wedge t} K(t,r) K(s,r)\, dr$. Moreover, let us observe that K^* can be represented in the following form: for $H \in (1/2, 1)$, we have

$$[K^*\varphi]_t = \int_t^1 \varphi_r\, \partial_r K(r,t)\, dr$$

while for $H \in (0, 1/2)$ it holds that

$$[K^*\varphi]_t = K(1,t)\varphi_t + \int_t^1 (\varphi_r - \varphi_t)\, \partial_r K(r,t)\, dr.$$

When $H \in (1/2, 1)$ it can be shown that $L^{1/H}([0,1], \mathbb{R}^d) \subset \mathcal{H}$, and when $H \in (0, 1/2)$ one has $\mathcal{C}^\gamma \subset \mathcal{H} \subset L^2([0,1])$ for all $\gamma > \frac{1}{2} - H$. We shall also use the following representations of the inner product in $\mathcal{H}$:

(i) For $H \in (1/2, 1)$ and $\phi, \psi \in \mathcal{H}$, we have

$$\langle K^*\phi, K^*\psi \rangle_{L^2([0,1])} = \langle \phi, \psi \rangle_{\mathcal{H}} = c_H \int_0^1 \int_0^1 |s-t|^{2H-2} \langle \phi_s, \psi_t \rangle_{\mathbb{R}^d}\, ds\, dt. \tag{9}$$

(ii) For $H \in (0, 1/2)$, consider any family of partitions $\pi = (t_j)$ of $[0,1]$, and set $Q_{jk} = \sum_{i=1}^d \mathbf{E}[\Delta_j^i(B) \Delta_k^i(B)]$ with $\Delta_j^i(B) = B_{t_j}^i - B_{t_{j-1}}^i$. Then for $\phi, \psi \in \mathcal{H}$, we have

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \lim_{|\pi| \to 0} \sum_{j,k} \langle \phi_{t_{j-1}}, \psi_{t_{k-1}} \rangle_{\mathbb{R}^d} Q_{jk}. \tag{10}$$

Let us also recall that there exists a d-dimensional Wiener process W defined on $(\Omega, \mathcal{F}, \mathbf{P})$ such that B can be expressed as

$$B_t = \int_0^t K(t,r)\, dW_r, \qquad t \in [0,1]. \tag{11}$$

This formula will be referred to as Volterra's representation of fBm. Formula (11) has various important implications. For example, it is readily checked that $\mathcal{F}_t \equiv \sigma\{B_s; 0 \le s \le t\} = \sigma\{W_s; 0 \le s \le t\}$. This filtration will appear in the sequel.

2.1.2. *Malliavin calculus for* B. Isometry arguments allow us to define the Wiener integral $B(h) = \int_0^1 \langle h_s, dB_s \rangle$ for any element $h \in \mathcal{H}$, such that it satisfies $\mathbf{E}[B(h_1)\ B(h_2)] = \langle h_1, h_2 \rangle_{\mathcal{H}}$ for any $h_1, h_2 \in \mathcal{H}$. An $\mathcal{F}$-measurable real valued random variable F is then said to be cylindrical if it can be written, for a given $n \geq 1$, as

$$F = f(B(h^1), \ldots, B(h^n)) = f\left(\int_0^1 \langle h_s^1, dB_s \rangle, \ldots, \int_0^1 \langle h_s^n, dB_s \rangle\right),$$

where $h^i \in \mathcal{H}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ bounded function with bounded derivatives. The set of cylindrical random variables is denoted by $\mathcal{S}$.

The Malliavin derivative with respect to B is defined as follows: for $F \in \mathcal{S}$, the derivative of F is the $\mathbb{R}^d$ valued stochastic process $(\mathbf{D}_t F)_{0 \leq t \leq 1}$ given by

$$\mathbf{D}_t F = \sum_{i=1}^n h_t^i \frac{\partial f}{\partial x_i}(B(h^1), \ldots, B(h^n)).$$

More generally, we can introduce iterated derivatives. We will use the following notation, depending on the situation. For $F \in \mathcal{S}$, we set for $\mathbf{i} = (i_1, \ldots, i_k)$ and $\mathbf{t} = (t_1, \ldots, t_k)$

$$\mathbf{D}_{\mathbf{t}}^k = \mathbf{D}_{t_1, \ldots, t_k}^k F = \mathbf{D}_{t_1} \cdots \mathbf{D}_{t_k} F \quad \text{or} \quad \mathbf{D}_{\mathbf{t}}^{\mathbf{i}} F = \mathbf{D}_{t_1, \ldots, t_k}^{i_1, \ldots, i_k} F = \mathbf{D}_{t_1}^{i_1} \cdots \mathbf{D}_{t_k}^{i_k} F.$$

For any $p \geq 1$, it can be checked that the operator $\mathbf{D}^k$ is closable from $\mathcal{S}$ into $L^p(\Omega; \mathcal{H}^{\otimes k})$. We denote by $\mathbf{D}^{k,p}$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(\mathbf{E}[F^p] + \sum_{j=1}^k \mathbf{E}\left[\|\mathbf{D}^j F\|_{\mathcal{H}^{\otimes j}}^p\right]\right)^{1/p},$$

for $k \geq 0$ and $p \geq 1$. In particular, $\|F\|_{0,p} \equiv \|F\|_p = (\mathbf{E}[F^p])^{1/p}$. As it is usually the case in Malliavin calculus with respect to W, the spaces $\mathbf{D}^{k,p}(\mathcal{H})$ are also defined. The dual operator of $\mathbf{D}$ is denoted by $\boldsymbol{\delta}$, which corresponds to the Skorohod integral with respect to the fBm B on the interval [0, 1]. The space of smooth processes $\mathbf{L}^{k,p}(\mathcal{H})$ is induced by the following norm:

$$\|u\|_{\mathbf{L}^{k,p}(\mathcal{H})}^p = \mathbf{E}\left[\|u\|_{\mathcal{H}}^p\right] + \sum_{l=1}^k \mathbf{E}\left[\|\mathbf{D}^l u\|_{\mathcal{H}^{\otimes(l+1)}}^p\right].$$

Finally, the set of smooth integrands is defined as $\mathbf{D}^\infty(\mathcal{H}) = \bigcap_{k,p \geq 1} \mathbf{D}^{k,p}(\mathcal{H})$, and the Malliavin covariance matrix of F is denoted by $\boldsymbol{\Gamma}_F$.

As mentioned in the Introduction, our lower bound (5) will be obtained by considering equation (3) as an equation driven by the underlying Wiener process W defined in (11), meaning that we shall also use stochastic analysis estimates with respect to W. We refer to Chapter 1 in [21] for this classical setting, and just mention here a some notation: we denote by D the differentiation operator with respect to W and by δ the corresponding dual operator (Skorohod integral). The respective norms in the Sobolev spaces $D^{k,p}(L^2([0,1]))$ are denoted by $\|\cdot\|_{k,p}$ and the space of smooth integrands by $L^{k,p}$. The following simple relation between $\mathbf{D}$ and D is then shown in [21], Proposition 5.2.1:

PROPOSITION 2.1. *Let $D^{1,2}$ be the Malliavin–Sobolev space corresponding to the Wiener process W. Then $\mathbf{D}^{1,2}=(K^*)^{-1}D^{1,2}$, and for any $F\in D^{1,2}$ we have $\mathrm{D}F=K^*\mathbf{D}F$ whenever both members of the relation are well defined.*

In fact the above proposition says that the derivatives $\mathbf{D}$ and D are somewhat interchangeable. Indeed, using formula (5.14) in [21], which gives an explicit formula for $(K^*)^{-1}$, one obtains such a property. In particular, we will use that for $F\in\mathcal{F}_t$ with $F\in D^{k,p}$ and for $\mathbf{u}=(u_1,\dots,u_k)\in[0,1]^k$ and $\mathbf{r}=(r_1,\dots,r_n)$, we have

$$|\mathrm{D}^k_{\mathbf{u}}F|\le \operatorname*{ess\,sup}_{u_i\le r_i;i=1,\dots,k}|\mathbf{D}^k_{\mathbf{r}}F|K(t,u_1)\cdots K(t,u_k). \tag{12}$$

For the proof of (12) and other useful properties, see Appendix.

Some of our computations in Section 5 will rely on some conditional Malliavin calculus arguments, for which some definitions need to be recalled. First, for a given $t\in[0,1]$ and $F\in L^2(\Omega)$, we shorten notation and write

$$\mathbf{E}_t[F]:=\mathbf{E}[F|\mathcal{F}_t],$$

and also set $\mathbf{P}_t$ for the respective conditional probability and $\mathrm{Cov}_t(G)$ for the conditional covariance matrix of a Gaussian vector G. We shall only use conditional Malliavin calculus with respect to the underlying Wiener process W, for which we recall the following definitions: For a random variable F and $t\in[0,1]$, let $\|F\|_{k,p,t}$ and $\Gamma_{F,t}$ be the quantities defined (for $k\ge 0$, $p>0$) by

$$\begin{aligned}&\|F\|_{k,p,t}=\Biggl(\mathbf{E}_t[F^p]+\sum_{j=1}^k\mathbf{E}_t\bigl[\|D^jF\|^p_{(L^2_t)^{\otimes j}}\bigr]\Biggr)^{1/p}\quad\text{and}\\&\Gamma_{F,t}=\bigl(\langle F^i,F^j\rangle_{L^2_t}\bigr)_{1\le i,j\le d},\end{aligned}\tag{13}$$

where we have set $L^2_t\equiv L^2([t,1])$.

With this notation in hand, we give a conditional version of the integration by parts formula with respect to the Wiener process W, borrowed from [21], Proposition 2.1.4.

PROPOSITION 2.2. *Fix $n\ge 1$. Let $F,Z_s,G\in(D^\infty)^d$ be three random vectors where Z_s is $\mathcal{F}_s$-measurable and $(\det_{\Gamma_{F+Z_s}})^{-1}$ has finite moments of all orders. Let $g\in\mathcal{C}^\infty_p(\mathbb{R}^d)$. Then, for any multi-index $\alpha=(\alpha_1,\dots,\alpha_n)\in\{1,\dots,d\}^n$, there exists a r.v. $H^s_\alpha(F,G)\in\bigcap_{p\ge1}\bigcap_{m\ge0}D^{m,p}$ such that*

$$\mathbf{E}\bigl[(\partial_\alpha g)(F+Z_s)G|\mathcal{F}_s\bigr]=\mathbf{E}\bigl[g(F+Z_s)H^s_\alpha(F,G)|\mathcal{F}_s\bigr], \tag{14}$$

where $H^s_\alpha(F, G)$ is recursively defined by

$$H^s_{(i)}(F, G) = \sum_{j=1}^{d} \delta_s\big(G\big(\Gamma^{-1}_{F,s}\big)_{ij} DF^j\big),$$
$$H^s_\alpha(F, G) = H^s_{(\alpha_n)}\big(F, H^s_{(\alpha_1,\dots,\alpha_{n-1})}(F, G)\big).$$

Here δ_s denotes the Skorohod integral with respect to the Wiener process W on the interval $[s, 1]$. Furthermore, the following norm estimates with $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}$ hold true:

$$\big\|H^s_\alpha(F, G)\big\|_{p,s} \le c\big\|\det(\Gamma_{F,s})^{-1}\big\|^n_{2^{n-1}q_1,s}\|F\|^{2(dn+1)}_{n+2,2^n q_2,s}\|G\|_{n,q_3,s}.$$

We will also resort to a localized version of the above bounds. Namely, we introduce a family of functions $\Phi_{M,\epsilon} : \mathbb{R}_+ \to \mathbb{R}_+$ indexed by $M, \epsilon > 0$, which are regularizations of $\mathbf{1}_{\{x\le M\}}$. Specifically, we define a function $\phi_\epsilon = \epsilon^{-1}\phi : \mathbb{R} \to \mathbb{R}$ with

$$\phi(x) := c_\phi \exp\left(-\frac{1}{1 - x^2}\right)\mathbf{1}_{\{|x|<1\}},$$

where c_ϕ is a normalization constant chosen in order to have $\int_{\mathbb{R}} \phi(x)\,dx = 1$. Then we define

$$\Phi_{M,\epsilon}(z) := 1 - \int_{-\infty}^{z} \phi_\epsilon(x - M)\,dx. \tag{15}$$

It is then readily checked that $\Phi_{M,\epsilon}(z) = 0$ for $z > M + \epsilon$, $\Phi_{M,\epsilon}(z) = 1$ on $[0, M - \epsilon]$ and $\Phi_{M,\epsilon} \in C^\infty_b$. We will use the above localization function in two situations: one for $M \gg 1$, $\epsilon = 1$, and in this case we simplify the notation using $\Phi_M \equiv \Phi_{M,1}$. In a second case M will not be a large quantity and therefore we will have to choose ϵ accordingly.

Consider now $Z \in D^\infty$. Under the same conditions as for Proposition 2.2, we get a conditional integration by parts formula of form (14) localized by Z, with the following modification on the estimation of the norms of H^s_α:

$$\begin{aligned}&\big\|H^s_\alpha\big(F, G\Phi_M(Z)\big)\big\|_{p,s}\\ &\quad\le c\big\|\det(\Gamma_{F,s})^{-1}\Phi_{M'}(Z)\big\|^{k_3}_{p_3,s}\big\|F\Phi_{M'}(Z)\big\|^{k_4}_{k_2,p_2,s}\big\|G\Phi_{M'}(Z)\big\|_{k_1,p_1,s},\end{aligned} \tag{16}$$

for some appropriate positive integers $k_1, p_1, k_2, p_2, k_3, p_3, k_4$, and where we recall our convention on increasing constants $M' > M$. In fact, to obtain the above inequality is enough to notice that there exist constants M' and C which may depend on M and $k \in \mathbb{N}$ such that $\Phi_M(Z) \le C\Phi_{M'}(Z)^k$ and $|\partial^k_z\Phi_M(Z)| \le C\Phi_{M'}(Z)$. Notice that (16) is valid for localizations of the form $\Phi_{M,\epsilon}(Z)$ as well.

2.2. *Differential equations driven by fBm.* Recall that X is the solution of (3), and that our working assumptions are summarized in Hypothesis 1.1. We have distinguished 3 situations:

(1) The one-dimensional additive case, for which equation (3) can be reduced to an ordinary differential equation by considering the process $Z = X - B$.

(2) The one-dimensional multiplicative case, handled thanks to the Doss–Sussman transform; see, for example, [19].

(3) The multidimensional case with $H \in (1/2, 1)$, solved in a pathwise way by interpreting stochastic integrals as generalized Riemann–Stieljes-type integrals.

In this section we give a brief account on the known results in the last situation.

In the case $H \in (1/2, 1)$, (3) is solved thanks to a fixed point argument, after interpreting the stochastic integral in the (pathwise) Young sense; see, for example, [14]. Let us recall that Young's integral can be defined in the following way:

PROPOSITION 2.3. *Let $f \in \mathcal{C}^\gamma$, $g \in \mathcal{C}^\kappa$ with $\gamma + \kappa > 1$, and $0 \le s \le t \le 1$. Then the integral $\int_s^t g_\xi \, df_\xi$ is well defined as a Riemann–Stieltjes integral. Moreover, the following estimation is fulfilled*:

$$\left| \int_s^t g_\xi \, df_\xi \right| \le C \|f\|_\gamma \|g\|_\kappa |t-s|^\gamma,$$

where the constant C only depends on γ and κ.

With this definition in mind and under Hypothesis 1.1, we can solve (3) uniquely, in the Young sense. Specifically, it is proven in [24] that equation (3) driven by B admits a unique γ-Hölder continuous solution X, for any $\frac{1}{2} < \gamma < H$. Moreover, the following moments bounds are shown in [15]:

PROPOSITION 2.4. *Let $H \in (1/2, 1)$, and assume that $V_0, \dots, V_d$ satisfy Hypothesis 1.1. Then for $t \in [0, 1]$ and $\frac{1}{2} < \gamma < H$, we have*

$$\|X\|_{0,t,\infty} \le |a| + c_V \|B\|_{0,t,\gamma}^{1/\gamma}, \tag{17}$$

where we have set $\|X\|_{0,t,\infty} := \sup\{|X_s|; 0 \le s \le t\}$ and where we recall that $\|B\|_{0,t,\gamma}$ is defined by (7). Moreover $X_t \in \mathbf{D}^\infty$ and for $n \ge 1$, $\mathbf{i} = (i_1, \dots, i_n) \in \{1, \dots, d\}^n$ and $0 \le s < t \le 1$ the following bound holds true:

$$\sup_{s \le u, r_1, \dots, r_n \le t} \left| \mathbf{D}_\mathbf{r}^\mathbf{i} X_u \right| \le C_{V,n} \exp\left(c_{V,n} \|B\|_{s,t,\gamma}^{1/\gamma} \right). \tag{18}$$

We remark that $\mathbf{D}_{r_1,\dots,r_n}^{i_1,\dots,i_n} X_u$ is a continuous function except if $r_i = u$ for some i, where it is càdlàg, and therefore the above supremum is well defined.

Furthermore, a bound for γ-Hölder norms with $\frac{1}{2} < \gamma < H$ is provided in [12], equation (10.15), for X together with its Malliavin derivatives:

PROPOSITION 2.5. *Under the same assumptions as for Proposition 2.4, we have*

$$\|X\|_{s,t,\gamma} \le c_{1,V}\big(\|B\|_{s,t,\gamma} \vee \|B\|_{s,t,\gamma}^{1/\gamma}\big),$$
$$\|\mathbf{D}^{\mathbf{i}}_{\mathbf{r}} X_u\|_{s,t,\gamma} \le c_{2,V,n}\exp\big(c_{3,V,n}\|B\|_{s,t,\gamma}^{1/\gamma}\big).$$

REMARK 2.6. Assume $H > 1/2$ and the other hypothesis of Proposition 2.4 again. As mentioned, for example, in [8], Section 7, the Young-type integrals $\int_0^t V_i(X_s)\,dB_s^i$ in (3) coincide with the Russo–Vallois definition of integral and also with the Stratonovich integral of Malliavin calculus. We shall use these identifications later on, and they will be detailed in Section 5.2. For the time being, let us just stress the following fact: in order to harmonize notation, we shall often write $\int_0^t V_i(X_s) \circ dB_s^i$ for the Young integral (instead of $\int_0^t V_i(X_s)\,dB_s^i$), in order to recall that it can also be interpreted in the Stratonovich sense.

3. One-dimensional additive case. This section is devoted to prove our main Theorem 1.2 in the particular case $m = d = 1$ with additive noise. In this context, one can take advantage of the results obtained by Nourdin and Viens in [20] in order to derive Gaussian-type upper and lower bounds for p_t. Let us then first recall what those results are.

3.1. *General bounds on densities of one-dimensional random variables*. Recall that we denote the Malliavin–Sobolev spaces with respect to the fBm B by $\mathbf{D}^{k,p}$, and consider a real-valued centered random variable $F \in \mathbf{D}^{1,2}$. We define a function g on $\mathbb{R}$ by

$$g(z) := \mathbf{E}\big[\langle \mathbf{D}F, -\mathbf{D}\mathbf{L}^{-1}F\rangle_{\mathcal{H}} | F = z\big],$$

where the operator $\mathbf{L}$ is the Ornstein–Uhlenbeck operator associated to the fBm B (see [21] for further details), which can be defined using the chaos expansion by the formula $\mathbf{L} = -\sum_{n=0}^{\infty} n\mathbf{J}_n$. Based on the function g, the following simple criterion for Gaussian-type bounds has been obtained in [20]:

PROPOSITION 3.1. *Let $F \in \mathbf{D}^{1,2}$ with $\mathbf{E}[F] = 0$. If there exist $c_1, c_2 > 0$ such that*

$$c_1 \le g(F) \le c_2, \qquad \mathbf{P}\text{-}a.s., \tag{19}$$

then the law of F has a density ρ satisfying, for almost all $z \in \mathbb{R}$,

$$\frac{\mathbf{E}[|F|]}{2c_2}\exp\left(-\frac{z^2}{2c_1}\right) \le \rho(z) \le \frac{\mathbf{E}[|F|]}{2c_1}\exp\left(-\frac{z^2}{2c_2}\right).$$

Interestingly enough, Nourdin and Viens [20], Proposition 3.7, also give an alternative formula for $g(F)$ which is suitable for computational purposes. Indeed, if we write $\mathbf{D}F = \Phi_F(B)$ in the above Proposition, where $\Phi_F : \mathbb{R}^{\mathcal{H}} \to \mathcal{H}$ is a measurable mapping, then the following relation holds true:

$$g(F) = \int_0^\infty e^{-\theta} \mathbf{E}\big[\big\langle \Phi_F(B), \Phi_F\big(e^{-\theta} B + \sqrt{1-e^{-2\theta}} B'\big)\big\rangle_{\mathcal{H}} | F\big]\, d\theta, \tag{20}$$

where B' stands for an independent copy of B, and is such that B and B' are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbf{P} \times \mathbf{P}')$. Here we abuse the notation by letting $\mathbf{E}$ be the mathematical expectation with respect to $\mathbf{P} \times \mathbf{P}'$, while $\mathbf{E}'$ is the mathematical expectation with respect to $\mathbf{P}'$ only. One can thus recast relation (20) as

$$g(F) = \int_0^\infty \mathbf{E}\big[\mathbf{E}'\big[\langle \mathbf{D}F, \mathbf{D}F^\theta \rangle_{\mathcal{H}}\big] | F\big]\, d\theta, \tag{21}$$

where, for any random variable X defined in $(\Omega, \mathcal{F}, \mathbf{P})$, X^θ denotes the following shifted random variable in $\Omega \times \Omega'$:

$$X^\theta(\omega, \omega') = X\big(e^{-\theta}\omega + \sqrt{1-e^{-2\theta}}\omega'\big), \qquad \omega \in \Omega, \omega' \in \Omega'.$$

3.2. *Main result in the additive one-dimensional case.* Before stating our result let us point out that we assume throughout this subsection $V_1 \equiv \sigma$. That is, X is the solution of

$$X_t = x + \int_0^t V_0(X_s)\, ds + \sigma B_t, \qquad t \in [0, 1], \tag{22}$$

where $\sigma > 0$ is a strictly positive constant, V_0 satisfies $\|V_0'\|_\infty \le M$ for some constant $M > 0$ and B is a fBm with $H \in (0, 1)$. Under this setting, we are able to get the following bounds:

THEOREM 3.2. *Assume that V_0 satisfies that $\|V_0'\|_\infty \le M$, for some constant $M > 0$, $\sigma > 0$ and $H \in (0, 1)$. Then, for all $t \in (0, 1]$, X_t possesses a density p_t, and there exist some strictly positive constants $c_1 < c_3$ and $c_2 < c_4$ depending only on M and H such that for all $z \in \mathbb{R}$,*

$$\frac{c_1}{\sigma t^H} \exp\left(-\frac{(z-m)^2}{c_2 \sigma^2 t^{2H}}\right) \le p_t(z) \le \frac{c_3}{\sigma t^H} \exp\left(-\frac{(z-m)^2}{c_4 \sigma^2 t^{2H}}\right). \tag{23}$$

REMARK 3.3. The advantage of the Nourdin–Viens method of estimating densities is that upper and lower bounds are obtained with similar proofs. The drawback is the restriction to one-dimensional additive situations. Also notice that the exponents in equation (24) are optimal, meaning that our density bounds mimic the fBm case. See also Theorem 4.3 for the nonconstant diffusion case.

STRATEGY OF THE PROOF. We first notice that we can reduce our problem to prove that

$$\frac{\mathbf{E}[|X_t - m|]}{c_1\sigma^2 t^{2H}} \exp\left(-\frac{(z-m)^2}{c_2\sigma^2 t^{2H}}\right) \le p_t(z) \le \frac{\mathbf{E}[|X_t - m|]}{c_2\sigma^2 t^{2H}} \exp\left(-\frac{(z-m)^2}{c_1\sigma^2 t^{2H}}\right). \tag{24}$$

Indeed, one can check in our context that $\mathbf{E}[|X_t - m|] \asymp \sigma t^H$. This easy step is left to the reader for the sake of conciseness, and it naturally allows us to go from (24) to (23). Now in order to prove (24), we obviously rely heavily on Proposition 3.1. We thus define $F = X_t - \mathbf{E}[X_t]$, where X_t is the solution of (22). We get a centered random variable, and we shall prove that there exists two constants $0 < K_1 < K_2$ such that

$$K_1\sigma^2 t^{2H} \le g(F) \le K_2\sigma^2 t^{2H}. \tag{25}$$

Notice first that in the present case, it is easily seen that for any $t > 0$, we have $X_t \in \mathbf{D}^{1,2}$; this is a particular case of [25]. Furthermore, the Malliavin derivative of X_t satisfies the following equation for $r \le t$:

$$\mathbf{D}_r X_t = \int_r^t V_0'(X_s)\mathbf{D}_r X_s \, ds + \sigma.$$

This equation can be solved explicitly, and we obtain

$$\mathbf{D}_r X_t = \sigma e^{\int_r^t V_0'(X_s)\, ds}. \tag{26}$$

In particular, the bound

$$\sigma e^{-tM} \le \mathbf{D}_r X_t \le \sigma e^{tM} \tag{27}$$

holds true almost surely for $M = \|V_0'\|_\infty$.

Observe that we shall bound $g(F)$ thanks to relation (27). More specifically, we will show that for each $\theta \in \mathbb{R}_+$ we have (almost surely)

$$c_3 t^{2H}\sigma^2 \le \langle \mathbf{D}F, \mathbf{D}F^\theta \rangle_{\mathcal{H}} \le c_4 t^{2H}\sigma^2, \tag{28}$$

for two strictly positive constants $c_3 < c_4$. This deterministic bound easily yields (19) and thus (24). We now separate the cases $H \in (1/2, 1)$ and $H \in (0, 1/2)$ in order to get relation (28). Notice that the Brownian case, that is, $H = 1/2$, is well known, and it is thus omitted here for the sake of conciseness. □

3.3. *Case* $H > \frac{1}{2}$. Recall that we wish to prove (28) thanks to relation (27). Furthermore, owing to expression (9) for the inner product in $\mathcal{H}$, we can write $\langle \mathbf{D}F, \mathbf{D}F^\theta \rangle_{\mathcal{H}}$ as

$$\begin{aligned} \langle \mathbf{D}F, \mathbf{D}F^\theta \rangle_{\mathcal{H}} &= c_H \int_0^t \int_0^t \mathbf{D}_u X_t \mathbf{D}_v X_t^\theta |u-v|^{2H-2}\, du\, dv \\ &= c_H\sigma^2 \int_0^t \int_0^t e^{\int_u^t V_0'(X_s)\, ds} e^{\int_v^t V_0'(X_s^\theta)\, ds} |u-v|^{2H-2}\, du\, dv. \end{aligned} \tag{29}$$

Therefore the lower and upper bounds in (27) follow from plugging inequality (27) into relation (29).

3.4. *Case* $0 < H < \frac{1}{2}$. As in the case $H > \frac{1}{2}$, our aim is to prove (27). We thus go back to equation (21), and we observe that we can reduce the problem to the existence of two constants $0 < c_1 < c_2$ such that

$$c_1 t^{2H} \le \langle \mathbf{D}X_t, \mathbf{D}X_t^{\theta} \rangle_{\mathcal{H}} \le c_2 t^{2H}. \tag{30}$$

The proof of these inequalities will rely on the following quadratic programming lemma, which is a slight variation of [7], Lemma 6.2:

LEMMA 3.4. *Let $Q \in \mathbb{R}^n \otimes \mathbb{R}^n$ be a strictly positive symmetric matrix such that $\sum_{j=1}^n Q_{ij} \ge 0$ for all $i = 1, \dots, n$. For two positive constants a and b, consider the sets $\mathcal{A} = [a, \infty)^n$ and $\mathcal{B} = [b, \infty)^n$. Then*

$$\inf\{x^* Q \tilde{x}; \tilde{x} \in \mathcal{A}, x \in \mathcal{B}\} = ab \sum_{i,j=1}^n Q_{ij}.$$

PROOF. Set $\mathbf{a} = a\mathbf{1} \in \mathbb{R}^n$ and $\mathbf{b} = b\mathbf{1} \in \mathbb{R}^n$. The Lagrangian of our quadratic programming problem is a function $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ defined as

$$L(x, \tilde{x}, \lambda_1, \lambda_2) = x^* Q \tilde{x} - \lambda_1^*(x - \mathbf{b}) - \lambda_2^*(\tilde{x} - \mathbf{a}).$$

It is readily checked that $\nabla_x L(x, \tilde{x}, \lambda_1, \lambda_2) = Q\tilde{x} - \lambda_1$ and $\nabla_{\tilde{x}} L(x, \tilde{x}, \lambda_1, \lambda_2) = Qx - \lambda_2$, which vanishes for $x = Q^{-1}\lambda_2$ and $\tilde{x} = Q^{-1}\lambda_1$. Therefore,

$$\begin{aligned} \inf\{L(x, \tilde{x}, \lambda_1, \lambda_2); x, \tilde{x} \in \mathbb{R}^n\} &= L(Q^{-1}\lambda_2, Q^{-1}\lambda_1, \lambda_1, \lambda_2) \\ &= -\lambda_1^* Q^{-1} \lambda_2 + \lambda_1^* \mathbf{b} + \lambda_2^* \mathbf{a} =: G(\lambda_1, \lambda_2). \end{aligned}$$

We have thus obtained a dual problem of the form

$$\max\{G(\lambda_1, \lambda_2); \lambda_1, \lambda_2 \in \mathbb{R}^n_+\}. \tag{31}$$

Let us now solve Problem (31). We first maximize G without positivity constraints on λ_1 and λ_2: we get $\nabla_{\lambda_1} G(\lambda_1, \lambda_2) = -Q^{-1}\lambda_2 + \mathbf{b}$ and $\nabla_{\lambda_2} G(\lambda_1, \lambda_2) = -\lambda_1^* Q^{-1} + \mathbf{a}$, which vanishes for $\lambda_1^\circ = Q\mathbf{a}$ and $\lambda_2^\circ = Q\mathbf{b}$. Observe now that our assumption $\sum_{j=1}^n Q_{ij} \ge 0$ for all $i = 1, \dots, n$ implies $\lambda_1^\circ, \lambda_2^\circ \ge 0$, so that λ_1° and λ_2° are feasible for the dual problem. Hence

$$\max\{G(\lambda_1, \lambda_2); \lambda_1, \lambda_2 \in \mathbb{R}^n_+\} = G(\lambda_1^\circ, \lambda_2^\circ) = ab \sum_{i,j=1}^n Q_{ij},$$

which completes the proof. $\square$

Importantly enough, Lemma 3.4 can be applied in order to get a lower bound on $\mathcal{H}$ norms:

PROPOSITION 3.5. *Let B be a one-dimensional fBm on $[0, \tau]$, let $\mathcal{H} \equiv \mathcal{H}_\tau$ be the associated reproducing kernel Hilbert space and $f, \tilde{f} \in \mathcal{H}$ such that $f_u \ge b$ and $\tilde{f}_u \ge a$ for any $u \in [0, \tau]$. Then $\langle f, \tilde{f} \rangle_{\mathcal{H}} \ge ab\tau^{2H}$.*

PROOF. Recall that, owing to relation (10), we have $\langle f, \tilde{f}\rangle_{\mathcal{H}} = \lim_{|\pi|\to 0} I_\pi(f, \tilde{f})$, where π stands for a generic partition $\{0 = t_0 < \cdots < t_n = \tau\}$ and

$$I_\pi(f, \tilde{f}) = \sum_{i,j=1}^{n} f_{t_{i-1}} Q_{ij} \tilde{f}_{t_{j-1}} \qquad \text{with } Q_{ij} = \mathbf{E}\big[\Delta_i(B)\Delta_j(B)\big],$$

where we recall that $\Delta_i(B) = B_{t_i} - B_{t_{i-1}}$. We assume for the moment that Q satisfies the hypothesis of Lemma 3.4, and we get

$$I_\pi(f, \tilde{f}) \geq ab \sum_{i,j=1}^{n} Q_{ij} = ab \sum_{i,j=1}^{n} \mathbf{E}\big[\Delta_i(B)\Delta_j(B)\big] = ab\mathbf{E}\big[B_\tau^2\big] = ab\tau^{2H},$$

which is our claim.

Let us now prove that Q satisfies the hypothesis of Lemma 3.4. First, the strict positivity of Q stems from the local nondeterminism of B; see, for example, [28]. Indeed, for $u \in \mathbb{R}^n$ we have

$$u^* Q u = \operatorname{Var}\left(\sum_{j=0}^{n-1} u_j \Delta_j(B)\right) \geq c_n \sum_{j=1}^{n} u_j^2 |t_j - t_{j-1}|^{2H},$$

where the lower bound is the definition of local nondeterminism. Thus $u^* Q u > 0$ as long as $u \neq 0$.

Let us now check that for a fixed i we have $\sum_{j=1}^{n} Q_{ij} \geq 0$. To this end, write

$$\sum_{j=1}^{n} Q_{ij} = \mathbf{E}\big[\Delta_i(B) B_\tau\big] = \int_{t_i}^{t_{i+1}} \partial_u R(\tau, u)\, du.$$

Going back to expression (1), it is now easily seen that for $u < \tau$ we have

$$\partial_u R(\tau, u) = H\big(u^{2H-1} + (\tau - u)^{2H-1}\big) > 0,$$

which completes the proof. $\square$

We can now go back to the proof of relation (30), which is divided again into two steps:

Step 1: Lower bound. Thanks to relation (27), we have that $\sigma e^{-tM} \leq \mathbf{D}_r X_t$. Thus we just have to apply Proposition 3.5 to the Malliavin derivative in order to obtain

$$\langle \mathbf{D} X_t, \mathbf{D} X_t^\theta\rangle_{\mathcal{H}} \geq \sigma^2 t^{2H} e^{-2M}, \tag{32}$$

which is our desired lower bound.

Step 2: Upper bound. In order to obtain an upper bound for $g(F)$, we will use the representation of $\mathcal{H}$ through fractional derivatives. Indeed, apply first the Cauchy–Schwarz inequality in order to get

$$\langle \mathbf{D} X_t, \mathbf{D} X_t^\theta\rangle_{\mathcal{H}} \leq \|\mathbf{D} X_t\|_{\mathcal{H}} \|\mathbf{D} X_t^\theta\|_{\mathcal{H}}. \tag{33}$$

We then invoke Lemma A.1 to bound $\|\mathbf{D}X_t^\theta\|_{\mathcal{H}}$. This boils down to estimating

$$a = \sup_{r\in[0,t]} \left|\mathbf{D}_r X_t^\theta\right| \quad \text{and} \quad b = \sup_{r,v\in[0,t]} \frac{\mathbf{D}_r X_t^\theta - \mathbf{D}_v X_t^\theta}{(v-r)^\gamma},$$

with $1/2 - H < \gamma < 1/2$ and any $\theta \ge 0$.

Now starting from expression (26) and owing to the fact that V_0' is uniformly bounded by M, we trivially get $a \le \sigma e^M$. As far as b is concerned, we write

$$\left|\mathbf{D}_r X_t^\theta - \mathbf{D}_v X_t^\theta\right| \le \sigma e^{\int_v^t V_0'(X_s^\theta)\,ds} \left|1 - e^{\int_r^v V_0'(X_s^\theta)\,ds}\right| \le \sigma M e^{2M}(v-r).$$

We thus end up with the inequalities

$$a \le \sigma e^M \quad \text{and} \quad b \le \sigma M e^{2M} t^{1-\gamma}.$$

We now apply Lemma A.1 with constants a and b, and we obtain

$$\|\mathbf{D}X_t\|_{\mathcal{H}} \le c_H\left(\sigma e^M t^H + \sigma M e^{2M} t^{1+H}\right) \le 2c_H \sigma M e^{2M} t^H,$$

and hence

$$\left\langle \mathbf{D}X_t, \mathbf{D}X_t^\theta\right\rangle_{\mathcal{H}} \le 4c_H \sigma^2 M^2 e^{4M} t^{2H}.$$

Finally, putting together the last bound and (32), we get (25) in the case $H \in (0, 1/2)$, which completes the proof of Theorem 3.2.

4. One-dimensional nonvanishing diffusion coefficient case. We turn now to the case $m = d = 1$, $H \in (\frac{1}{2}, 1)$ for a nonconstant elliptic coefficient σ. Observe that this special case is treated in a separate section because (i) the Gaussian bound is obtained with weaker conditions on the coefficients than in the multidimensional case, and (ii) the proof is shorter due to specific one-dimensional techniques based on the Doss–Sussman transform and Girsanov's theorem. This is detailed below.

REMARK 4.1. The Doss–Sussman transform can be justified for any $H \in (0, 1)$ in our context. However, the computations related to Girsanov's transform become much more involved when $H < 1/2$, and this is why we restrict our analysis to $H > 1/2$ in the sequel.

4.1. *Doss–Sussmann transformation.* The idea of the method is to first consider a one-dimensional equation of Stratonovich-type without drift and then apply Girsanov's theorem for fBm in order to obtain a characterization of the density.

In order to carry out this strategy, we start by using an independent copy of $(\Omega, \mathcal{F}, \mathbf{P})$ called $(\Omega', \mathcal{F}', \mathbf{P}')$ supporting a fBm denoted by B'. On $(\Omega', \mathcal{F}', \mathbf{P}')$, let Y be the unique solution to

$$Y_t = a + \int_0^t V_1(Y_s) \circ dB_s', \tag{34}$$

where the integral is interpreted either in the Young or Stratonovich sense (as recalled in Remark 2.6), and where $V_1 \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$, $V_1 \neq 0$ and $H \in (\frac{1}{2}, 1)$. We also call W' the underlying Wiener process appearing in the Volterra-type representation (11) for B'. We now recall here some details from Doss and Sussmann's classical computations adapted to our fBm context.

Indeed, as in [19], let us recall that the solution of equation (34) can be expressed as $Y_t = F(B'_t, a)$, $t > 0$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is the flow associated to V_1,

$$\frac{\partial F}{\partial x}(x, y) = V_1\bigl(F(x, y)\bigr), \qquad F(0, y) = y. \tag{35}$$

We remark that if V_1 is bounded, then F satisfies $|F(x, y)| \leq c(1 + |x| + |y|)$.

Next we relate the solution X of equation (3) to the process Y defined by (34). This step is partially borrowed from [22], and we refer to that paper for further details. Indeed, thanks to a Girsanov-type transform, the following characterization of the law of the solution to (3) is shown for $m = d = 1$: For any bounded measurable function $U : \mathbb{R} \to \mathbb{R}$, one has

$$\mathbf{E}_{\mathbf{P}}\bigl[U(X_t)\bigr] = \mathbf{E}_{\mathbf{P}'}\bigl[U\bigl(F\bigl(B'_t, a\bigr)\bigr)\xi\bigr], \tag{36}$$

where $\xi \equiv \xi_t = \frac{d\mathbf{P}}{d\mathbf{P}'}$ is the random variable defined by

$$\xi = \exp\biggl(\int_0^t \biggl[\mathcal{M}_s\, dW'_s - \frac{1}{2}\mathcal{M}_s^2\, ds\biggr]\biggr), \tag{37}$$

where we have set $\mathcal{M} = K^{-1}(\int_0^\cdot V_0 V_1^{-1}(Y_u)\, du)$.

Notice that in definition (37), the operator K has been alluded to in Section 2.1.1. It should be observed that K, K^{-1} can also be defined, respectively, for $H \geq \frac{1}{2}$ and an appropriate function h, by (see details in [21], Chapter 5)

$$\begin{aligned} K(h)(s) &= I_{0+}^1\bigl(s^{H-1/2}\bigl(I_{0+}^{H-1/2}\bigl(s^{1/2-H}h\bigr)\bigr)\bigr)(s) \quad \text{and} \\ K^{-1}(h)(s) &= s^{H-1/2}\bigl(D_{0+}^{H-1/2}\bigl(s^{1/2-H}h'\bigr)\bigr)(s). \end{aligned}$$

We also recall that in the last equation, I_{0+}^α and D_{0+}^α denote the fractional integral and fractional derivative, whose expressions are

$$I_{0+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha-1} f(y)\, dy$$

and

$$D_{0+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\biggl(\frac{f(x)}{x^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x - y)^{\alpha+1}}\, dy\biggr).$$

It is easily seen from the expressions of K_H^{-1} and $D_{0+}^{H-1/2}$ that $K_H^{-1}h$ is an adapted transformation; see also expression (39) below. Hence the term ξ in (36) corresponds to the usual Girsanov correction term. Furthermore, notice that in order for (36) to be satisfied, it is required that $\int_0^{\cdot} V_0 V_1^{-1}(Y_u)\,du \in I_{0+}^{H+1/2}(L^2[0,1])$. This condition is satisfied due to the γ-Hölderianity of Y for any $\gamma < H$.

Actually one should prove that Novikov-type conditions are satisfied for ξ in order to apply Girsanov's transform and get relation (36). This is achieved in the following lemma:

LEMMA 4.2. *Let ξ be the random variable defined by* (37), *and assume that Hypothesis* 1.1(1) *is satisfied. Then*

$$\mathcal{M}_s \le c_V \beta_s \qquad \textit{with } \beta_s := s^{1/2-H} + \|B'\|_{H-1/2+\varepsilon}, \tag{38}$$

for any arbitrarily small $\varepsilon > 0$. Furthermore $\mathbf{E}_{\mathbf{P}'}[\xi] = 1$, which justifies the Girsanov identity (36). *That is, under $\mathbf{P}$, $B = B' + \int_0^{\cdot} V_0 V_1^{-1}(Y_u)\,du$ is a H-fBm.*

PROOF. According to the expression of K_H^{-1}, we have

$$\mathcal{M}_s = \frac{1}{\Gamma(H-1/2)}\left(\mathcal{M}_s^1 + \left(H - \tfrac{1}{2}\right)\mathcal{M}_s^2\right), \tag{39}$$

where we have set

$$\mathcal{M}_s^1 \equiv \frac{V_0 V_1^{-1}(Y_s)}{s^{H-1/2}},$$

$$\mathcal{M}_s^2 \equiv s^{H-1/2} \int_0^s \frac{s^{1/2-H} V_0 V_1^{-1}(Y_s) - u^{1/2-H} V_0 V_1^{-1}(Y_u)}{(s-u)^{H+1/2}}\,du.$$

The term $\mathcal{M}_s^1$ is easily bounded: we invoke the uniform ellipticity of V_1 and the regularity of V_0 and V_1, which yields $\mathcal{M}_s^1 \le c s^{-(H-1/2)}$. We now bound $\mathcal{M}_s^2$: let us decompose this term as $\mathcal{M}_s^2 = \mathcal{M}_s^{21} + \mathcal{M}_s^{22}$, with

$$\mathcal{M}_s^{21} = \int_0^s \frac{1-(s/u)^{H-1/2}}{(s-u)^{H+1/2}} V_0 V_1^{-1}(Y_u)\,du \quad \text{and}$$

$$\mathcal{M}_s^{22} = \int_0^s \frac{V_0 V_1^{-1}(Y_s) - V_0 V_1^{-1}(Y_u)}{(s-u)^{H+1/2}}\,du.$$

Then, resorting again to the fact that $V_0 V_1^{-1}$ is bounded and with the obvious change of variable $r = u/s$, we get

$$|\mathcal{M}_s^{21}| \le \frac{c_V}{s^{H-1/2}} \int_0^1 \frac{r^{H-1/2} - 1}{r^{H-1/2}(1-r)^{H+1/2}}\,dr \le \frac{c_{V,H}}{s^{H-1/2}}.$$

In order to handle the term $\mathcal{M}_s^{22}$, we start by writing

$$\mathcal{M}_s^{22} \le c_V \int_0^s \frac{|F(B'_s, a) - F(B'_u, a)|}{(s-u)^{H+1/2}}\,du,$$

and thanks to the Lipschitz properties of F plus elementary integral computations, we obtain

$$\mathcal{M}_s^{22} \le c_{V,H} \|B'\|_{H-1/2+\varepsilon}.$$

Therefore, summarizing our estimates on $\mathcal{M}^1$, $\mathcal{M}^{21}$ and $\mathcal{M}^{22}$, the proof of our claim (38) is now completed.

Now let us have a closer look at the process β: it is readily checked that $\|B'\|_\gamma$ admits quadratic exponential moments for any $\gamma < H$; see Theorem 3 in [22]. In particular, one can choose $\gamma = H - 1/2 + \varepsilon$ for ε small enough, and hence there exists $\lambda > 0$ such that the expected value $\mathbf{E}[\exp(\lambda \int_0^t \beta^2(s)\, ds)]$ is a finite quantity. Owing to a version of Novikov's condition stated in [11], Theorem 1.1, we deduce that $\mathbf{E}[\xi] = 1$. This completes the proof. $\square$

4.2. *Main result in the Doss–Sussman framework.* As in the additive case of Section 3, we are able to get both upper and lower Gaussian bounds in a one-dimensional context:

THEOREM 4.3. *Assume that $H \in (1/2, 1)$ and V_0, V_1 satisfy the assumptions of Hypothesis 1.1(1). Then there exist constants C_1 and C_2 such that for all $t \in (0, 1]$, the solution X_t to equation (3) possesses a density p_t satisfying for all $x \in \mathbb{R}$,*

$$\begin{aligned} \frac{1}{C_1\sqrt{2\pi t^{2H}}} \exp\left(-C_1 \frac{(x-a)^2}{2t^{2H}}\right) &\le p_t(x) \\ &\le \frac{1}{C_2\sqrt{2\pi t^{2H}}} \exp\left(-C_2 \frac{(x-a)^2}{2t^{2H}}\right). \end{aligned} \tag{40}$$

PROOF. In this proof one should separate 4 cases: (a) $\lambda \le V_1(z) \le \Lambda$ with subcases $x > a$ and $x \le a$ and (b) $-\Lambda \le V_1(z) \le -\lambda$ with subcases $x < -a$ and $x \ge -a$. These situations are treated thanks to the same kind of arguments, and we will thus assume in the proof that $x \ge a$ and $\lambda \le V_1(z) \le \Lambda$ for all $z \in \mathbb{R}$. We now divide our proof in two steps.

Step 1: Upper bound. We start from an equivalent of (36) for densities, which is justified by [15], Theorem 7, and a duality argument

$$p_t(x) = \mathbf{E}_{\mathbf{P}'}\big[\delta_x\big(F\big(B'_t, a\big)\big)\xi\big], \tag{41}$$

where ξ is the random variable defined in (37). We now integrate by parts in order to get

$$p_t(x) = \mathbf{E}_{\mathbf{P}'}\big[\mathbf{1}_{\{F(B'_t,a)\ge x\}} H\big(F\big(B'_t, a\big), \xi\big)\big],$$

with

$$H\big(F\big(B'_t,a\big),\xi\big)=\delta\bigg(\frac{\xi \mathrm{D}F(B'_t,a)}{\|\mathrm{D}F(B'_t,a)\|^2_{L^2([0,t])}}\bigg), \tag{42}$$

where D, δ, respectively, stand (with a slight abuse of notation) for the Malliavin derivative and divergence operator for the Brownian motion W' under $\mathbf{P}'$. Let us further simplify the expression for the random variable $H(F(B'_t,a),\xi)$: setting $K_t(u)\equiv K(t,u)\mathbf{1}_{[0,t]}(u)$, it is readily checked that we have

$$\mathrm{D}_u F\big(B'_t,a\big)=\partial_x F\big(B'_t,a\big)K_t(u) \quad \text{and} \quad \big\|\mathrm{D}F\big(B'_t,a\big)\big\|^2_{L^2([0,t])}=\big|\partial_x F\big(B'_t,a\big)\big|^2 t^{2H}.$$

Plugging this information into (42), and defining $Z:=\xi(\partial_x F(B'_t,a))^{-1}$, we end up with

$$H\big(F\big(B'_t,a\big),\xi\big)=\frac{\delta(ZK_t)}{t^{2H}}=K_1-K_2,$$

where

$$K_1=\frac{ZB'_t}{t^{2H}} \quad \text{and} \quad K_2=\frac{\langle \mathrm{D}Z,K_t\rangle_{L^2([0,t])}}{t^{2H}}.$$

We have thus obtained

$$p_t(x)=\mathbf{E}_{\mathbf{P}'}[\mathbf{1}_{\{F(B'_t,a)\ge x\}}K_1]-\mathbf{E}_{\mathbf{P}'}[\mathbf{1}_{\{F(B'_t,a)\ge x\}}K_2]=:p_t^1(x)-p_t^2(x), \tag{43}$$

and we shall upper bound these two terms separately.

The term $p_t^1(x)$ can be bounded as follows: for $q_1,q_2,q_3>1$ large enough and a parameter $1<q_4=1+\varepsilon$ with an arbitrarily small $\varepsilon>0$, we have

$$\begin{aligned} p_t^1(x)\le{}& \frac{\mathbf{E}_{\mathbf{P}'}^{1/q_1}[|B'_t|^{q_1}]}{t^{2H}}\mathbf{P}'^{1/q_2}\big(F\big(B'_t,a\big)\ge x\big) \\ &\times \mathbf{E}_{\mathbf{P}'}^{1/q_3}\big[\big|\partial_x F\big(B'_t,a\big)\big|^{-q_3}\big]\mathbf{E}_{\mathbf{P}'}^{1/q_4}\big[\xi^{q_4}\big]. \end{aligned} \tag{44}$$

We now bound the right-hand side of this inequality:

(i) We obviously have $\frac{\mathbf{E}_{\mathbf{P}'}^{1/q_1}[|B'_t|^{q_1}]}{t^{2H}}\le ct^{-H}$, since B' is a $\mathbf{P}'$-fBm.

(ii) Let us prove that there exist two positive constants c_1 and c_2 such that, for all $x\ge 0$,

$$\mathbf{P}'^{1/q_2}\big(F\big(B'_t,a\big)\ge x\big)\le c_1\exp\bigg(-\frac{c_2(x-a)^2}{t^{2H}}\bigg). \tag{45}$$

Indeed, for a fixed $a\in\mathbb{R}$, set $Q\equiv\mathbf{P}'(F(B'_t,a)\ge x)$, and decompose this term as $Q=Q_1+Q_2$ with

$$Q_1=\mathbf{P}'\big(F\big(B'_t,a\big)\ge x,B'_t\ge 0\big) \quad \text{and} \quad Q_2=\mathbf{P}'\big(F\big(B'_t,a\big)\ge x,B'_t<0\big).$$

Since we have assumed $x > a$ and $V_1 > \lambda > 0$, it is readily checked that $Q_2 = 0$. In the sequel we thus bound the term Q_1. Toward this aim, appealing to relation (35), we write

$$Q_1 = \mathbf{P}'\left(\int_0^{B_t'} V_1\big(F(z,a)\big)\,dz \ge x-a,\, B_t' \ge 0\right).$$

Next recall that we have assumed $\lambda \le V_1(z) \le \Lambda$ for all $z \in \mathbb{R}$. Hence we have $\int_0^\zeta V_1(F(z,a))\,dz \le \Lambda\zeta$ for all $\zeta \ge 0$, and thus

$$Q_1 \le \mathbf{P}'\big(\Lambda B_t' \ge x-a,\, B_t' \ge 0\big) = \mathbf{P}'\big(\Lambda B_t' \ge x-a\big) \le \exp\left(-\frac{(x-a)^2}{\Lambda^2 t^{2H}}\right),$$

which is consistent with relation (45). The proof is now completed by a similar analysis of the term Q_2.

(iii) Equation (35) and the nondegeneracy assumptions on V_1 show that $\partial_x F$ is bounded from below by a constant, so that we get the trivial bound

$$\mathbf{E}_{\mathbf{P}'}^{1/q_3}\big[\big|\partial_x F\big(B_t',a\big)\big|^{-q_3}\big] \le c.$$

(iv) Set $S = \int_0^t \mathcal{M}_s\,dW_s'$ and $D = \int_0^t \mathcal{M}_s^2\,ds$, where $\mathcal{M} \equiv K_H^{-1}(\int_0^\cdot V_0 \times V_1^{-1}(Y_u)\,du)$ as above, and where we recall that $q_4 = 1+\varepsilon$ with an arbitrarily small $\varepsilon > 0$. It is readily checked that

$$\xi^{q_4} = \exp\left(q_4 S - \frac{q_4}{2}D\right) = \exp\left(q_4 S - \frac{q_4^2}{2}D\right)\exp\left(\frac{q_\varepsilon}{2}D\right),$$

where $q_\varepsilon = q_4^2 - q_4 = \varepsilon(1+\varepsilon)$. Now observe that the term $\exp(q_4 S - \frac{q_4^2}{2}D)$ is a Girsanov change of measure which corresponds to a shift on B' of the form

$$\hat{B} = B' - q_4\int_0^\cdot V_0 V_1^{-1}(Y_u)\,du = B - (q_4 - 1)\int_0^\cdot V_0 V_1^{-1}(Y_u)\,du.$$

Calling $\hat{\mathbf{P}}'$ the probability under which $\hat{B}$ is a fBm, we get

$$\mathbf{E}_{\mathbf{P}'}[\xi^{q_4}] = \mathbf{E}_{\hat{\mathbf{P}}'}\left[\exp\left(\frac{q_\varepsilon}{2}D\right)\right]. \tag{46}$$

Now plug estimate (38) into (46). This yields

$$\begin{aligned} D &\le c_V\big(1 + \|B'\|_{H-1/2}^2\big) \\ &\le c_V\left(1 + \left\|\hat{B} + q_4\int_0^\cdot V_0 V_1^{-1}(Y_u)\,du\right\|_{H-1/2}^2\right) \\ &\le c_V\big(1 + \|\hat{B}\|_{H-1/2}^2\big). \end{aligned}$$

Going back to relation (46) and taking into account the fact that q_ε can be chosen arbitrarily small, we get $\mathbf{E}_{\mathbf{P}'}[\xi^{q_4}] < \infty$.

Gathering all the above estimates into (44), we have thus obtained that

$$p_t^1(x) \le \frac{c_1}{t^H} \exp\left(-\frac{c_2(x-a)^2}{t^{2H}}\right).$$

The upper bound for $p_t^2(x)$ [defined in (43)] is obtained along the same lines, and we spare the details to the reader. Let us just mention that more Malliavin derivatives of ξ and $F(B', a)$ are involved in the computations, and this is where we use both the nondegeneracy and smoothness assumptions on V. Then taking into account the estimates on $p_t^1(x)$ and $p_t^2(x)$ in (43), we end up with our global upper bound in (40).

Step 2: Lower bound. Our strategy to obtain the lower bound in (40) is based on the following decomposition:

$$p_t(x) = \mathbf{E}_{\mathbf{P}'}\left[\delta_x\left(F\left(B_t', a\right)\right)(\xi_t - \xi_{c_1 t})\right] + \mathbf{E}_{\mathbf{P}'}\left[\delta_x\left(F\left(B_t', a\right)\right)\xi_{c_1 t}\right] =: \rho_t^1 + \rho_t^2, \tag{47}$$

where c_1 is a constant to be determined later. Observe that the main term will be ρ_t^2, which means that we consider a two-point partition of the interval $[0, t]$, and we perform a one-step decomposition of X_t (or Y_t) on $[0, c_1 t]$ and $[c_1 t, t]$, as opposed to the general time interval partition in Section 5.

First, we start studying the main term ρ_t^2: Note that due to (11), we can apply Girsanov's theorem in order to get

$$\begin{aligned}
\rho_t^2 &= \mathbf{E}_{\mathbf{P}'}\left[\mathbf{E}_{\mathbf{P}'}\left[\delta_x\left(F\left(B_t', a\right)\right)|\mathcal{F}_{c_1 t}\right]\xi_{c_1 t}\right] \\
&= \mathbf{E}_{\mathbf{P}'}\left[\exp\left(-\frac{(F^{-1}(x,a) - \int_0^{c_1 t} K(t,s)\,dW_s')^2}{2\int_{c_1 t}^t K^2(t,s)\,ds}\right)\frac{\partial_x F^{-1}(x,a)}{\sqrt{2\pi \int_{c_1 t}^t K^2(t,s)\,ds}}\xi_{c_1 t}\right] \\
&= \mathbf{E}_{\mathbf{P}}[L_{c_1,t}],
\end{aligned}$$

where we have set

$$\begin{aligned}
L_{c_1,t} &:= \exp\left(-\frac{(F^{-1}(x,a) - \int_0^{c_1 t} K(t,s)\,dW_s + \int_0^{c_1 t} V_0 V_1^{-1}(X_s)\,ds)^2}{2\int_{c_1 t}^t K^2(t,s)\,ds}\right) \\
&\qquad \times \frac{\partial_x F^{-1}(x,a)}{\sqrt{2\pi \int_{c_1 t}^t K^2(t,s)\,ds}}.
\end{aligned}$$

In order to determine a lower bound for the above expression, we use the following information:

(i) We have $\partial_x F^{-1}(x,a) \ge [V_1(F(x,a))]^{-1} \ge \Lambda^{-1}$.

(ii) We apply the inequality $(m+a)^2 \ge \frac{1}{2}m^2 - 2a^2$ to $m \equiv F^{-1}(x,a) - \int_0^{c_1 t} K(t,s)\,dW_s$ and a defined by $a^2 \equiv (\int_0^{c_1 t} V_0 V_1^{-1}(X_s)\,ds)^2 \le c_V t^2$.

(iii) Gaussian convolution identities can be invoked in order to compose the quadratic exponential term defining $L_{c_1,t}$ with the expected value with respect to the Gaussian random variable $\int_0^{c_1 t} K(t,s)\,dW_s$.

(iv) The following trivial bound holds true: $\int_{c_1 t}^t K^2(t,s)\,ds \le \int_0^t K^2(t,s)\,ds = t^{2H}$. These ingredients easily entail that

$$\rho_t^2 \ge \frac{c}{\sqrt{2\pi\hat{\sigma}^2}} \exp\left(-\frac{F^{-1}(x,a)^2}{2\hat{\sigma}^2}\right),$$

for $\hat{\sigma}^2 = 2\int_{c_1 t}^t K^2(t,s)\,ds + \int_0^{c_1 t} K^2(t,s)\,ds$, and we observe that $\sigma^2 \le \hat{\sigma}^2 \le 2\sigma^2$.

Now we estimate the first term ρ_t^1 in (47) and prove that it is upper bounded by a quantity which is smaller than half of the lower bound we have just obtained. For this term we need to use again the integration by parts estimates carried out in (41). In order not to repeat arguments we just mention the main steps: we start by writing

$$\rho_t^1 = \mathbf{E}_{\mathbf{P}'}\left[\delta_x\left(F(B_t', a)\right)(\xi_t - \xi_{c_1 t})\right] = \mathbf{E}_{\mathbf{P}'}\left[\mathbf{1}_{\{F(B_t',a)\le x\}} H\left(F(B_t', a), \xi_t - \xi_{c_1 t}\right)\right],$$

and we decompose this expression into $p^1 - p^2$ like in (43), except for the fact that this time Z is replaced by $Z_t := ((\xi_t - \xi_{c_1 t})\,\partial_x F(B_t', a))^{-1}$.

We wish to take advantage of the fact that $\xi_t - \xi_{c_1 t}$ is a small quantity whenever c_1 is close to 1. For this, define the process $\mathcal{M}_{c_1 t,\cdot}$ as $\mathcal{M}_{c_1 t,s} = K_H^{-1}(\int_{c_1 t}^{\cdot} V_0 V_1^{-1}(Y_u)\,du)$, consider $\theta \in [0,1]$ and define

$$\xi_t(\theta) := \xi_{c_1 t} \exp\left(\theta \int_{c_1 t}^t \mathcal{M}_{c_1 t,s}\,dW_s' - \frac{\theta^2}{2}\int_{c_1 t}^t \mathcal{M}_{c_1 t,s}^2\,ds\right).$$

Then by the mean value theorem, we have

$$\xi_t - \xi_{c_1 t} = \int_0^1 d\theta \xi_t(\theta)\left(\int_{c_1 t}^t \mathcal{M}_s\,dW_s' - \theta\int_{c_1 t}^t \mathcal{M}_s^2\,ds\right).$$

Applying Fubini's theorem, one sees that the same estimates as in (44) appear again with the following exceptions: (i) The last term in the decomposition becomes $\mathbf{E}_{\mathbf{P}'}^{1/q_4}[(\xi_t(\theta))^{q_4}]$, which is handled in the same fashion as before. (ii) There is another term appearing in the decomposition, namely

$$\mathbf{E}_{\mathbf{P}'}^{1/q_5}\left[\left(\int_{c_1 t}^t \mathcal{M}_s\,dW_s' - \theta\int_{c_1 t}^t \mathcal{M}_s^2\,ds\right)^{q_5}\right].$$

Using (38) and the same estimates for stochastic integrals as in step 1, one obtains that the latter term is upper bounded by $c(1-c_1^{2-2H})t^{2-2H}$. Therefore taking c_1 sufficiently close to 1 one obtains that this upper bound is smaller than $1/2$ of the lower bound previously obtained. The proof is now complete. □

5. General lower bound. We now wish to obtain Gaussian-type lower bounds for the multi-dimensional case of equation (3). However, the computations in this section will be performed on the following simplified version for notational sake (adaptation of our calculations to the drift case are straightforward):

$$X_t = a + \sum_{i=1}^{d} \int_0^t V_i(X_s) \circ dB_s^i, \tag{48}$$

where $a \in \mathbb{R}^m$ is a generic initial condition, $V_i : \mathbb{R}^m \to \mathbb{R}^m$ $i = 1, \ldots, d$ is a collection of smooth and bounded vectors fields and $B^1, \ldots, B^d$ are d independent fBm's with $H \in (1/2, 1)$. Recall that our goal is then to prove relation (5) in this context. To this end, we shall assume that Hypothesis 1.1 [especially relation (4)] is satisfied for the remainder of the article. Observe that, as in Section 4, equation (48) is written in the Stratonovich sense. Relations between Stratonovich and Young integrals will be investigated in Section 5.2.

5.1. *Preliminary considerations.* Let us recall briefly the strategy used in [2, 16] in order to obtain Gaussian lower bounds for solutions of stochastic differential equations. The argument starts with some additional notation: Recall that the natural filtration of B, which is also the natural filtration of the underlying Wiener process W defined by (11), is denoted by $\mathcal{F}_t$. As we have introduced in Section 2.1, we write $\mathbf{E}_t$ for the conditional expectation with respect to $\mathcal{F}_t$. Under our working Hypothesis 1.1, let us also mention that the following result is available (see [4, 15] for further details):

PROPOSITION 5.1. *Under Hypothesis* 1.1, *there exists a unique solution to* (48). *Then for any* $t \in (0, 1]$, *the random variable* X_t *is nondegenerate in the sense of Definition* 2.1.1 *in* [21], *namely*: (i) $X_t \in \mathbf{D}^\infty$; (ii) *the Malliavin matrix* Γ_{X_t} *is almost surely invertible and satisfies* $\Gamma_{X_t}^{-1} \in \bigcap_{p \geq 1} L^p(\Omega)$. *In particular, the density of* X_t *admits the representation* $p_t(x) = \mathbf{E}[\delta_x(X_t)]$, *where* δ_x *stands for the Dirac measure at point* x.

With this preliminary result in hand, the quantity $\mathbf{E}[\delta_x(X_t)]$ will be analyzed by means of the succesive evaluation of conditional densities of an approximation sequence $\{F_j; 0 \leq j \leq n\}$ such that $X_t = F_n$. We thus consider $p_t(x) = \mathbf{E}[\delta_x(F_n)]$. The discretization procedure is based on a corresponding partition of the time interval as $\pi : 0 = t_0 < \cdots < t_n = t$, and the sequence of random variables F_j which satisfy the relation $F_j \in \mathcal{F}_{t_j}$.

Let us give some hints about the general strategy for the discretization: it is designed to take advantage of conditional Malliavin calculus, which allows one to capture the convolution property of Gaussian distributions. We shall thus assume for the moment a structure of the form

$$F_j = F_{j-1} + I_j + R_j, \tag{49}$$

where we recall that $F_{j-1} \in \mathcal{F}_{t_{j-1}}$. In formula (49), the term I_j will stand for a Gaussian random variable (conditionally to $\mathcal{F}_{t_{j-1}}$), and R_j refers to a small remainder term, whose contribution to the density of F_j can be neglected with respect to the one induced by I_j just like in the argument in (47). The local Gaussian bound (5) will be obtained from the density of the sum $\sum_{j=1}^{n} I_j$. The argument will finish by an application of the Chapman–Kolmogorov formula.

As suggested by equation (6) and setting $\Delta^i_{j+1}(B) := B^i_{t_{j+1}} - B^i_{t_j}$, a natural candidate consists of taking $F_j = X_{t_j}$, which yields

$$(50)\quad I_j = \sum_{i=1}^{d} V_i(X_{t_j})\Delta^i_{j+1}(B) \quad \text{and} \quad R_j = \sum_{i=1}^{d} \int_{t_{j-1}}^{t_j} [V_i(X_s) - V_i(X_{t_j})]\, dB^i_s.$$

However, this simple and natural guess is not suitable for the fBm case. Indeed, the analysis of the variances of I_j induced from decomposition (50) reveals that a significant amount is generated by the covariances between the increments $\Delta^i_j(B)$. Now, if we write

$$(51)\qquad t^{2H} = \mathbf{E}[(B^i_t)^2] = \mathbf{E}\left[\left(\sum_{j=1}^{n} \Delta^i_j(B)\right)^2\right] = \sum_{j,k=1}^{n} \mathbf{E}[\Delta^i_j(B)\Delta^i_k(B)],$$

we realize that the diagonal terms on the right-hand side expression only account for a term of the form $\sum_j |t_j - t_{j-1}|^{2H}$, which vanishes as the mesh of the partition goes to 0 when $H \in (1/2, 1)$. This means that our decomposition (50) will not be able to capture the correct amount of variance contained in X_t, and has to be modified.

There are at least two natural generalizations of the Euler-type method described above:

(1) Take into account the off-diagonal terms in (51), and perform a block type analysis.

(2) Express the equation as an equation driven by the Wiener process W defined by relation (11), and take advantage of the independence of the increments of W.

We have not been able to implement the strategy (1) above without cumbersome calculations, and we have thus chosen to follow the second approach. Toward this aim, we first recall how to define equation (48) as a Stratonovich equation with respect to W.

5.2. *Fractional equations as Stratonovich-type equations.* In order to handle equation (48) as an equation with respect to W, let us first introduce the following functional space:

DEFINITION 5.2. Let $|\mathcal{H}|$ be the space of measurable functions $\phi : [0,1] \to \mathbb{R}^d$ such that

$$\|\phi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^1 \left(\int_0^1 |\phi_r||\phi_u||r-u|^{2H-2}\, dr\right) du < +\infty.$$

Note that $|\mathcal{H}|$ endowed with the norm $\|\cdot\|_{|\mathcal{H}|}$ is a Banach space of functions, which is also a subspace of $\mathcal{H}$.

In the sequel we also consider random elements with values in $|\mathcal{H}|$. In particular, the norm of ϕ in $\mathbf{D}^{1,2}(|\mathcal{H}|)$ is given by

$$\|\phi\|_{\mathbf{D}^{1,2}(|\mathcal{H}|)} = \mathbf{E}\big[\|\phi\|^2_{|\mathcal{H}|}\big] + \mathbf{E}\big[\|\mathbf{D}\phi\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|}\big].$$

As mentioned before, the Young-type integrals we have handled so far can be identified with Stratonovich-type integrals with respect to B, and finally as anticipative Stratonovich-type integrals with respect to W. In order to state these results more formally, let us recall what we mean by Stratonovich integrals with respect to B:

DEFINITION 5.3. Let $u = \{u_t, t \in [0,1]\}$ be a $\mathbb{R}^d$-valued process defined on $(\Omega, \mathcal{F}, \mathbf{P})$, whose paths are supposed to be integrable. The Stratonovich (or symmetric, or Russo–Vallois) integral of u with respect to B is denoted by $\sum_{k=1}^d \int_0^1 u_s^k \circ dB_s^k$ and is defined as

$$\sum_{k=1}^d \int_0^1 u_s^k \circ dB_s^k = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \sum_{k=1}^d \int_0^1 u_s^k \big(B_{s+\varepsilon}^k - B_{s-\varepsilon}^k\big)\, ds,$$

whenever the limit exists. In the same way, the indefinite Stratonovich integral is defined as

$$\sum_{k=1}^d \int_0^t u_s^k \circ dB_s^k = \sum_{k=1}^d \int_0^1 \big(u_s^k \mathbf{1}_{[0,t]}(s)\big) \circ dB_s^k \qquad \text{for } t \in [0,1]. \tag{52}$$

The following result is borrowed from [1], Proposition 3 and [10], Proposition 4.2 and page 193 (we also refer to [1], Section 5, for considerations on the indefinite Stratonovich integral). It gives the link between Stratonovich and Young integrals with respect to B.

PROPOSITION 5.4. *Let* $u = \{u_t, t \in [0,1]\} \in \mathbf{D}^{1,2}(|\mathcal{H}|)$, *such that*

$$\int_0^1 \int_0^1 |\mathbf{D}_s u_t| |t-s|^{2H-2}\, ds\, dt < \infty. \tag{53}$$

Then:

(i) *The Stratonovich integral* $\sum_{k=1}^d \int_0^1 u_s^k \circ dB_s^k$ *in the sense of Definition* 5.3 *exists*, *and we also have*

$$\sum_{k=1}^d \int_0^1 u_s^k \circ dB_s^k = \boldsymbol{\delta}(u) + \alpha_H \sum_{k=1}^d \int_0^1 \int_0^1 \mathbf{D}_s^k u_t |t-s|^{2H-2}\, ds\, dt. \tag{54}$$

(ii) *Whenever* $u \in \mathcal{C}^\gamma$ *a.s. with* $\gamma > 1/2$ *and* $H \in (1/2, 1)$, *the Stratonovich integral* $\sum_{k=1}^d \int_0^1 u_s^k \circ dB_s^k$ *coincides with the Young integral* $\sum_{k=1}^d \int_0^1 u_s^k \, dB_s^k$.

REMARK 5.5. In the Brownian case (which corresponds to the limiting case $H \searrow 1/2$), one may wonder about the relation between our pathwise-type Stratonovich integral and the Stratonovich integral of a square integrable adapted process $u \in L_a^2$. The easiest way to carry out this comparison might be to start with relation (54). Indeed, on the right-hand side of this identity, the Skorohod integral $\boldsymbol{\delta}(u)$ coincides with Itô's integral as long as $u \in L_a^2$. As far as the terms $\alpha_H \int_0^1 \int_0^1 \mathbf{D}_s^k u_t |t-s|^{2H-2} \, ds \, dt$ is concerned, let us first mention that the measure $2\alpha_H |t-s|^{2H-2} \, ds \, dt$ converges to the Lebesgue measure on the diagonal $\{(s,t) \in [0,1]^2; s = t\}$ as $H \searrow 1/2$. We thus end up morally with a sum of terms of the form $\frac{1}{2} \int_0^1 \mathbf{D}_t^k u_t \, dt$. The identification of this term with the bracket $\frac{1}{2}\langle u, W\rangle_1$ is then standard and is detailed in [21], Remark 2, page 175.

The next Proposition allows us to interpret the stochastic integral appearing in (48) as a Stratonovich-type integral.

PROPOSITION 5.6. *Let* $X = \{X_t, t \in [0,1]\}$ *be the solution to* (48), *and assume Hypothesis* 1.1 *holds true. Then* $X \in \mathbf{D}^{1,2}(|\mathcal{H}|)$ *and satisfies the equation*

$$X_t = a + \sum_{k=1}^d \int_0^t V_k(X_u) \circ dB_u^k,$$

where the indefinite Stratonovich integral is defined by (52), *and can be decomposed as a Skorohod integral plus a trace term as in* (54).

PROOF. According to Propositions 2.4 and 5.4, we just have to prove that $X \in \mathbf{D}^{1,2}(|\mathcal{H}|)$ and satisfies relation (53). We first focus on proving the relation

$$\mathbf{E}\big[\|X\|_{|\mathcal{H}|}^2\big] + \mathbf{E}\big[\|\mathbf{D}X\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2\big] < \infty.$$

In order to see the first part of this inequality, invoke relation (17), and write

$$\begin{aligned}\mathbf{E}\big[\|X\|_{|\mathcal{H}|}^2\big] &= \alpha_H \int_0^1 \int_0^1 \mathbf{E}\big[|X_r||X_s|\big]|r-s|^{2H-2} \, dr \, ds \\ &\le c\mathbf{E}\big[\|X\|_\infty^2\big] \int_0^1 \int_0^1 |r-s|^{2H-2} \, dr \, ds < c_1.\end{aligned}$$

Along the same lines and owing to (18), it is also readily checked that $\mathbf{E}[\|\mathbf{D}X\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2] < \infty$ and that relation (53) holds true, which completes the proof. Note that due to Proposition 5.4(ii) and Proposition 2.5, we obtain the other assertions. $\square$

Finally, the following corollary is the key to the effective decomposition we shall use in order to get our Gaussian lower bound on p_t:

COROLLARY 5.7. *Let the same assumptions as for Proposition 5.6 hold true. For $0 \le s \le t \le 1$ and $\varphi \in |\mathcal{H}|$, we define*

$$K_t^*(\varphi)_s := \int_s^t \varphi_r \, \partial_r K(r,s) \, dr.$$

Then the process $K_t^*(V_k(X))_\cdot \in \mathrm{Dom}(\delta)$ and satisfies the equation

$$\begin{aligned} X_t &= a + \sum_{k=1}^d \int_0^t \big[K_t^*\big(V_k(X)\big)\big]_s \circ dW_s^k \\ &= a + \sum_{k=1}^d \int_0^t \left(\int_s^t \partial_u K(u,s) V_k(X_u) \, du \right) \circ dW_s^k, \end{aligned} \tag{55}$$

where the anticipative Stratonovich integrals with respect to W can be decomposed as a Skorohod integral plus a trace term as follows:

$$\begin{aligned} &\sum_{k=1}^d \int_0^t \big[K_t^*\big(V_k(X)\big)\big]_s \circ dW_s^k \\ &\qquad = \delta\big(K_t^*\big(V(X)\big)\big) + \sum_{k=1}^d \int_0^t D_s^k \big[K_t^*\big(V_k(X)\big)\big]_s \, ds. \end{aligned} \tag{56}$$

PROOF. For notational sake, we give some details of the proof for $n = d = 1$, the easy adaptation to the multidimensional case being omitted. We also set $V \equiv V_1$. According to Proposition 5.6 and relation (54), we have $X_t = a + S_t + c_H T_t$, with

$$S_t = \boldsymbol{\delta}\big(V(X)\mathbf{1}_{[0,t]}\big) \quad \text{and} \quad T_t = \int_0^1 \int_0^1 \mathbf{D}_r \big(V(X)\mathbf{1}_{[0,t]}\big)_s |r-s|^{2H-2} \, dr \, ds.$$

Then owing to [21], Proposition 5.2.2, we have $S_t = \delta(K^*(V(X)\mathbf{1}_{[0,t]}))$. In addition, a direct and easy computation shows that $K^*(V(X)\mathbf{1}_{[0,t]}) = K_t^*(V_k(X))\mathbf{1}_{[0,t]}$, so that we have obtained

$$S_t = \delta\big(K_t^*\big(V_k(X)\big)\big),$$

that is, the first term in (56).

Next, for a function $\varphi : [0,1]^2 \to \mathbb{R}$ set

$$\big[K^{*,\otimes 2}\varphi\big]_{r_1,r_2} = \int_{r_1}^1 \int_{r_2}^1 \partial_{s_1} K(s_1,r_1) \, \partial_{s_2} K(s_2,r_2) \varphi_{s_1 s_2} \, ds_1 \, ds_2.$$

Thanks to a slight extension of (9), we get

$$\begin{aligned} T_t &= \int_0^1 \big[K^{*,\otimes 2}\big(\mathbf{D}V(X)\mathbf{1}_{[0,t]}\big)\big]_{s,s} \, ds = \int_0^1 D_s \big[K^*\big(V(X)\mathbf{1}_{[0,t]}\big)\big]_s \, ds \\ &= \int_0^t D_s \big[K_t^*\big(V(X)\big)\big]_s \, ds, \end{aligned}$$

where the second relation is due to Proposition 2.1, and the third one stems from the fact that $K^*(V(X)\mathbf{1}_{[0,t]}) = K_t^*(V_k(X))\mathbf{1}_{[0,t]}$. Gathering the expressions we have obtained for the two terms S_t and T_t, the proof of our claim (56) is now complete. $\square$

5.3. *Discretization procedure*. We now proceed to the decomposition of $F_n := X_t$ as announced in (49), starting from the expression of F_j for $j = 0, \dots, n$. Indeed, according to expression (55), a natural approximation sequence for X_t based on a partition $0 = t_0 < \cdots < t_n = t$ of $[0, t]$ is the following:

$$F_i = F_{i-1} + I_i + R_i, \tag{57}$$

where, introducing the additional notation

$$\eta_i(u) := \inf(u, t_i) \quad \text{and} \quad g_{i,s}^k := \int_s^t \partial_u K(u, s) V_k(X_{\eta_i(u)})\, du, \tag{58}$$

we set (note that $g_{i-1,s}^k \in \mathcal{F}_{t_{i-1}}$)

$$\begin{aligned} F_{i-1} &:= \sum_{k=1}^d \int_0^{t_{i-1}} g_{i-1,s}^k \circ dW_s^k, \\ I_i &:= \sum_{k=1}^d \int_{t_{i-1}}^{t_i} g_{i-1,s}^k \circ dW_s^k = \sum_{k=1}^d V_k(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K(t, s)\, dW_s^k, \end{aligned} \tag{59}$$

where the last integral above is simply a Wiener integral with respect to W. We also introduce a family of random variables R_i defined by

$$R_i := \sum_{k=1}^d \int_{t_{i-1}}^{t_i} Q_s^k \circ dW_s^k, \tag{60}$$

where Q is the process defined by

$$Q_s^k := \int_s^t \partial_u K(u, s) \big[V_k(X_{\eta_i(u)}) - V_k(X_{t_{i-1}})\big]\, du. \tag{61}$$

Observe that if V is elliptic and bounded, it is clear from expression (59) that $\sum_i \mathrm{Cov}_{t_{i-1}}(I_i) \asymp t^{2H} Id_m$ up to a constant, independently of the particular values of the t_i's. We shall see, however, how to choose those values in Condition 5.10.

Finally we introduce some random variables $\Phi_M(\mathrm{N}_{\gamma,p}^i(B))$ for $i = 1, \dots, n$ which allow us to control the supremum norm of the solution of equation (48) and of their stochastic derivatives. This argument needs to be added in the methodology of [2, 16], and therefore we have to tailor the arguments therein to our situation. The localization random variables are based on the family of functionals $\mathrm{N}_{\gamma,p}^i(B)$ defined by

$$\mathrm{N}_{\gamma,p}^i(B) = \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} \frac{|B_v - B_u|^{2p}}{|v - u|^{2\gamma p + 2}}\, du\, dv,$$

which can be compared to Hölder-type norms and have the advantage that they can be differentiated with respect to B. In fact, we can see the aim of introducing this functional in the following proposition, which is direct consequence of the Garsia–Rodemich–Rumsey's lemma; see, for example, [13].

PROPOSITION 5.8. *Let $H > \frac{1}{2}$ and p such that $0 < \gamma < H - \frac{1}{2p}$. Then we have $\|B\|_{t_{i-1},t_i,\gamma} \le c_{\gamma,p}[\mathrm{N}^i_{\gamma,p}(B)]^{1/2p}$.*

The next step is to study the conditional densities of the approximation sequence F_i. To this end, one has to control various terms for which the localization technique of Malliavin Calculus turns out to be useful. Specifically, recall that we have introduced families of functions Φ_M, $\Phi_{M,\epsilon}$ given by expression (15). In the sequel we localize our expectations using functionals of the type $\Phi_M(\mathrm{N}^i_{\gamma,p}(B))$ and $\Phi_{c_i,\epsilon}(\sum_{j=1}^d \int_{t_{i-1}}^{t_i} |\mathrm{D}^j_r R_i|^2\,dr)$ for some constants c_i, ϵ of the form

$$c_i := \frac{\lambda}{4}\int_{t_{i-1}}^{t_i} K^2(t,s)\,ds > 0 \quad \text{and} \quad \epsilon_i := \frac{c_i}{2} > 0. \tag{62}$$

Furthermore, in order to ease notation, notice that we will simply write

$$\Phi_M \equiv \Phi_M\bigl(\mathrm{N}^i_{\gamma,p}(B)\bigr) \quad \text{and} \quad \Phi_{c_i,\epsilon_i} \equiv \Phi_{c_i,\epsilon_i}\left(\sum_{j=1}^d \int_{t_{i-1}}^{t_i} \bigl|\mathrm{D}^j_r R_i\bigr|^2\,dr\right). \tag{63}$$

With this additional notation in hand, we can proceed to the first step of our approximation scheme: since F_i is $\mathcal{F}_{t_{i-1}}$ conditionally nondegenerate and the localizations Φ_M and $\Phi_{c_i,\epsilon_i} \in D^\infty$, we can write

$$\mathbf{E}_{t_{i-1}}\bigl[\delta_x(F_i)\bigr] = \mathbf{E}_{t_{i-1}}\bigl[\delta_x(F_i)\Phi_M\Phi_{c_i,\epsilon_i}\bigr] + \mathbf{E}_{t_{i-1}}\bigl[\delta_x(F_i)(1-\Phi_M\Phi_{c_i,\epsilon_i})\bigr],$$

and due to the nonnegativity of the second term, we have

$$\mathbf{E}_{t_{i-1}}\bigl[\delta_x(F_i)\bigr] \ge \mathbf{E}_{t_{i-1}}\bigl[\delta_x(F_i)\Phi_M\Phi_{c_i,\epsilon_i}\bigr].$$

Recalling that $F_i = F_{i-1} + I_i + R_i$, we then obtain the following decomposition:

$$\mathbf{E}_{t_{i-1}}\bigl[\delta_x(F_i)\Phi_M\Phi_{c_i,\epsilon_i}\bigr] = J_{1,i} + J_{2,i} + J_{3,i}, \tag{64}$$

where

$$J_{1,i} = \mathbf{E}_{t_{i-1}}\bigl[\delta_x(F_{i-1}+I_i)\bigr], \qquad J_{2,i} = \mathbf{E}_{t_{i-1}}\bigl[\delta_x(F_{i-1}+I_i)(\Phi_M\Phi_{c_i,\epsilon_i}-1)\bigr] \tag{65}$$

and

$$J_{3,i} = \sum_{j=1}^m \mathbf{E}_{t_{i-1}}\left[\Phi_M\Phi_{c_i,\epsilon_i}\int_0^1 \partial_{x_j}\delta_x(F_{i-1}+I_i+\rho R_i)R_i^j\,d\rho\right]. \tag{66}$$

Our aim is now to prove that in this decomposition, $J_{1,i}$ should yield the main contribution, while $J_{2,i}$ is small because of the quantity $(\Phi_M\Phi_{c_i,\epsilon_i}-1)$ whenever M and n are large enough, and $J_{3,i}$ is small due to the presence of the difference between $X_{t_i} - X_{t_{i-1}}$ in R_i. We shall implement this strategy in the next subsections.

5.4. *Upper and lower bounds on $J_{1,i}$.* The main information which will be used about $J_{1,i}$ is the following:

PROPOSITION 5.9. *Let $J_{1,i}$ be defined by (65). Then under Hypothesis 1.1 we have*

$$J_{1,i} = \mathbf{E}_{t_{i-1}}[\delta_x(F_{i-1} + I_i)] = \frac{\exp(-(1/2)(x - F_{i-1})^* \Sigma_{i-1}^{-1}(x - F_{i-1}))}{(2\pi)^{m/2}|\Sigma_{i-1}|^{1/2}}, \tag{67}$$

where Σ_{i-1} is a deterministic (conditionally to $\mathcal{F}_{t_{i-1}}$) matrix such that

$$\lambda\left(\int_{t_{i-1}}^{t_i} K^2(t,u)\,du\right) Id_m \le \Sigma_{i-1} \le \Lambda\left(\int_{t_{i-1}}^{t_i} K^2(t,u)\,du\right) Id_m,$$

and where the two strictly positive constants λ, Λ satisfy (4).

PROOF. The fact that I_{i-1} is conditionally Gaussian is clear from expression (59), and this immediately yields our claim (67). Furthermore,

$$\begin{aligned}
\Sigma_{i-1} &:= \mathrm{Cov}_{t_{i-1}}(I_i) = \mathbf{E}_{t_{i-1}}[I_i I_i^*] \\
&= \mathbf{E}_{t_{i-1}}\Bigg[\left(\sum_{k=1}^d V_k(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K(t,u)\,dW_u^k\right) \\
&\qquad \times \left(\sum_{l=1}^d V_l^*(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K(t,u)\,dW_u^l\right)\Bigg] \\
&= \sum_{k=1}^d V_k(X_{t_{i-1}}) V_k^*(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K^2(t,u)\,du,
\end{aligned}$$

which completes the proof of our second claim, thanks to Hypothesis 1.1. □

The previous proposition induces a natural choice for the partition (t_i) in terms of the kernel K:

CONDITION 5.10. We choose the partition $0 = t_0 < \cdots < t_n = t$ of $[0,t]$ such that we have $\int_{t_{i-1}}^{t_i} K^2(t,u)\,du = \frac{t^{2H}}{n} =: \sigma_n^2$ for all $i = 1,\dots,n$.

With this choice in hand, let us note the following properties for further use:

LEMMA 5.11. *Let $t_0,\dots,t_n$ be the partition of $[0,t]$ defined by Condition 5.10. Then:*

(i) *The partition is constructed in a unique way.*
(ii) *We have $0 \le t_i - t_{i-1} \le c_H n^{-1/(2H)}$ for all $i = 1,\dots,n$.*
(iii) *The parameters c_i defined at (62) are all equal to $\frac{\lambda t^{2H}}{4n}$.*

PROOF. Our first claim stems from the fact that $\int_0^t K^2(t,u)\,du = t^{2H}$ and $v \mapsto \int_v^\tau K^2(t,u)\,du$ is a strictly decreasing function for all $0 \le v \le \tau \le t$.

In order to prove our item (ii), recall expression (8), from which we easily deduce the bound

$$K(t,s) \ge c_H (t-s)^{H-1/2}. \tag{68}$$

Consider now a fixed point $\tau \in (0,t]$ and $0 \le v \equiv v_\tau < \tau \le t$ such that $\int_v^\tau K^2(t,u)\,du = \frac{t^{2H}}{n}$. Thanks to bound (68) we have $v_\tau \ge w_\tau$ where $w_\tau \equiv w$ is defined by

$$c_H \int_w^\tau (t-u)^{2H-1}\,du = \frac{t^{2H}}{n} \quad \Longleftrightarrow \quad c_H\big[(t-w)^{2H} - (t-\tau)^{2H}\big] = \frac{t^{2H}}{n}.$$

In addition, since $2H > 1$, we have $(t-w)^{2H} - (t-\tau)^{2H} \ge (\tau - w)^{2H}$ for $w < \tau < t$, which means that $w_\tau \ge x_\tau$ where x_τ is defined by the equation $(\tau - x)^{2H} = \frac{c_H t^{2H}}{n}$. The latter equation can be solved explicitly as $x_\tau = \tau - \frac{c_H t}{n^{1/(2H)}}$, and summarizing our last considerations we end up with the relation

$$\tau - v_\tau \le \frac{c_H t}{n^{1/(2H)}},$$

which easily yields our assertion (ii). The proof of (iii) is straightforward. □

Now we state the following corollary to Proposition 5.9, whose immediate proof is left to the reader:

COROLLARY 5.12. *Let $J_{1,i}$ be defined by* (65). *Then under Hypothesis* 1.1 *and Condition* 5.10 *we have for $\sigma_n^2 = \frac{t^{2H}}{n}$*

$$J_{1,i} \ge \frac{1}{(2\pi)^{m/2}(\Lambda\sigma_n^2)^{m/2}} \exp\left(-\frac{|x - F_{i-1}|^2}{2\lambda\sigma_n^2}\right). \tag{69}$$

Summarizing the considerations of this section, we have obtained that the main contribution to $\mathbf{E}_{t_{i-1}}[\delta_x(F_i)]$, $J_{1,i}$, is of the order given by (69). Most of our work is now devoted to prove that the contributions of $J_{2,i}$ and $J_{3,i}$ are smaller than a fraction of (69) if M, n are conveniently chosen.

5.5. *Upper bounds for $J_{2,i}$.* We start the control of $J_{2,i}$ by stating a bound in terms of the localization we have chosen:

PROPOSITION 5.13. *Let $J_{2,i}$ be the quantity defined by* (65). *Then there exists positive constants $c_{\lambda,\Lambda}$, k_1, k_2 and p_1 independent of n such that*

$$|J_{2,i}| \le c_{\lambda,\Lambda}(\sigma_n^2)^{-k_2} L_{n,i}^{\gamma,p}(k_1,p_1)$$

$$\text{where } L_{n,i}^{\gamma,p}(k_1,p_1) \equiv \|1 - \Phi_M \Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}},$$

with $\sigma_n^2 = \frac{t^{2H}}{n}$, *and where we recall that the norms* $\|\cdot\|_{k,p,t}$ *have been introduced at equation* (13) *and the random variables* Φ_M, Φ_{c_i,ϵ_i} *at equation* (63).

PROOF. Our strategy hinges on the conditional integration by parts formula we have introduced in Proposition 2.2, which gives for some constants k_i, p_i, $i = 1, \dots, 4$,

$$
\begin{aligned}
|J_{2,i}| &= \left|\mathbf{E}_{t_{i-1}}\left[\mathbf{1}_{\{F_{i-1}+I_i>x\}} H^{t_{i-1}}_{(1,\dots,m)}(I_i, 1-\Phi_M\Phi_{c_i,\epsilon_i})\right]\right| \\
&\le c_{1,q}\left\|\det(\Gamma_{I_i,t_{i-1}})^{-1}\right\|^{k_3}_{p_3,t_{i-1}} \|I_i\|^{k_4}_{k_2,p_2,t_{i-1}} \|1-\Phi_M\Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}}.
\end{aligned}
\tag{70}
$$

Here, we have used that $\mathbf{1}_{\{F_{i-1}+I_i>x\}} \le 1$.

In order to bound the right-hand side of (70) we start by computing the Malliavin derivatives of I_i. Recall that due to (59), we have for $j = 1, \dots, d$, $\alpha > 1$ and $r, r_1, \dots, r_\alpha > t_{i-1}$ that

$$
\mathrm{D}^j_r I_i = V_j(X_{t_{i-1}}) K(t,r) \mathbf{1}_{[t_{i-1},t_i]}(r) \quad \text{and} \quad \mathrm{D}^\alpha_{r_1\dots r_\alpha} I_i = 0.
$$

As far as $\Gamma_{I_i,t_{i-1}}$ is concerned, it is a conditionally deterministic quantity such that for $i, j = 1, \dots, d$, we can write

$$
\begin{aligned}
\Gamma_{I_i,t_{i-1}} &= \sum_{j=1}^d \langle \mathrm{D}^j I_i, \mathrm{D}^j I_i^* \rangle_{L^2([t_{i-1},t_i])} \\
&= \sum_{j=1}^d V_j(X_{t_{i-1}}) V_j^*(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} K^2(t,s)\, ds = \sigma_n^2 V(X_{t_{i-1}}) V^*(X_{t_{i-1}}).
\end{aligned}
$$

Using the ellipticity condition of Hypothesis 1.1(2) for V, we thus obtain that

$$
0 \le \Gamma^{-1}_{I_i,t_{i-1}} \le \frac{1}{\lambda\sigma_n^2} Id_m.
$$

Therefore $\|I_i\|^{k_4}_{k_2,p_2,t_{i-1}} \le C(\sigma_n^2 \Lambda)^{k_4/2}$ and

$$
\left\|\det(\Gamma_{I_i,t_{i-1}})^{-1}\right\|^{k_3}_{p_3,t_{i-1}} \le \left(\frac{1}{\lambda\sigma_n^2}\right)^{mk_3}.
$$

Substituting these inequalities in (70), our proof is now finished. □

From the above Proposition 5.13, we see that in order to get a convenient bound for $J_{2,i}$ we need to study the random variable $\|1-\Phi_M\Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}}$. A suitable information for us will be the following bound:

PROPOSITION 5.14. *Assume Condition* 5.10 *and consider any* $\gamma \in (\frac{1}{2}, H)$ *and* $k_1, p_1 \ge 1$. *Let* $L^{\gamma,p}_{n,i}(k_1,p_1) = \|1-\Phi_M\Phi_{c_i,\epsilon_i}\|_{k_1,p_1,t_{i-1}}$ *be the random variable defined at Proposition* 5.13. *Then for any* $p \ge \frac{k_1}{2}$, $\gamma > 0$ [*recall that* $\Phi_M \equiv$

$\Phi_M(\mathrm{N}^i_{\gamma,p}(B))]$ *such that* $2p(H-\gamma)-2>k_1H$ *the following holds true*: *For any* $\eta>0$ *there exists* $c_{p,k_1,p_1,\gamma,H,M,\eta}>0$ *such that*

$$\mathbf{E}\big[L^{\gamma,p}_{n,i}(k_1,p_1)\big]\le c_{p,k_1,p_1,\gamma,H,M,\eta}n^{-\eta}. \tag{71}$$

PROOF. Let us first highlight what the parameters involved in the proof are: recall that c_i and ϵ_i were defined in (62). And although not explicitly written, Φ_M depends on γ and p. From now on, and through the proof we fix the values of γ, H, k_1, p_1, n and p satisfying the inequalities in the statement of the proposition.

As a preliminary step, we also observe that, due to the Hölder inequality, it is enough to find a proper bound for $\|1-\Phi_M\|_{k_1,p_1,t_{i-1}}$ and $\|\Phi_M(1-\Phi_{c_i,\epsilon_i})\|_{k_1,p_1,t_{i-1}}$ separately. We first handle the term $\|1-\Phi_M\|_{k_1,p_1,t_{i-1}}$.

Now we will obtain a general estimate to be used in the proof. By Chebyshev's inequality, for any $k_2\ge 1$ and $\frac{1}{2}<\gamma<H$,

$$\mathbf{E}\big[|1-\Phi_M|^2\big]\le\mathbf{P}\big(\mathrm{N}^i_{\gamma,p}(B)>M-1\big)\le\frac{\mathbf{E}[|\mathrm{N}^i_{\gamma,p}(B)|^{k_2}]}{(M-1)^{k_2}}. \tag{72}$$

We now find an upper bound for $\mathbf{E}[|\mathrm{N}^i_{\gamma,p}(B)|^{k_2}]$. A simple application of Jensen's inequality yields

$$\begin{aligned}\mathbf{E}\big[\big|\mathrm{N}^i_{\gamma,p}(B)\big|^{k_2}\big]&=\mathbf{E}\bigg[\bigg(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}\frac{|B_v-B_u|^{2p}}{|v-u|^{2p\gamma+2}}\,du\,dv\bigg)^{k_2}\bigg]\\&\le c|t_i-t_{i-1}|^{2(k_2-1)}\bigg(\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}\frac{\mathbf{E}[|B_v-B_u|^{2pk_2}]}{|v-u|^{(2p\gamma+2)k_2}}\,du\,dv\bigg)\\&\le c_{k_2,p,\gamma,H}|t_i-t_{i-1}|^{2k_2p(H-\gamma)}.\end{aligned} \tag{73}$$

We remark that all above integrals and expectations are finite due to the condition $2p(H-\gamma)-2>k_1H$. Furthermore, the quantity $|t_i-t_{i-1}|^{2k_2p(H-\gamma)}$ can be made as small as we wish by taking k_2, p and n large enough. We will play on these parameters later on.

Let us start the estimation for the high-order derivatives of $1-\Phi_M$. For this, we first notice that, for any $\mathbf{r}$ of length greater or equal to 1 and any $\mathbf{i}$, we have $\mathbf{D}^{\mathbf{i}}_{\mathbf{r}}(1-\Phi_M)=-\mathbf{D}^{\mathbf{i}}_{\mathbf{r}}\Phi_M$, so that we shall bound $\mathbf{D}^{\mathbf{i}}_{\mathbf{r}}\Phi_M$ in the sequel. Next we need to define the set of multi-indices $\mathcal{A}_n=\{(l_1,\dots,l_n);l_i\in\{0,\dots,n\},l_1+\cdots+l_n=n\}$. In fact, one can easily check that there exist (explicit) random variables $\mu^{\mathbf{i}}_{p,l,\gamma,H}(\mathbf{r})$, defined for $l\le n\le k_1$, $\mathbf{r}=(r_1,\dots,r_n)$ with $r_1\le\cdots\le r_n$ and $\mathbf{i}=(i_1,\dots,i_n)\in\{1,\dots,d\}^n$, such that the following inequality holds for a positive constant $C_{p,l,\gamma,H}(\mathbf{i},\mathbf{r})$:

$$\big|\mathbf{D}^{\mathbf{i}}_{\mathbf{r}}\Phi_M\big|\le\sum_{l=1}^{n}\big|\partial^l_z\Phi_M\big(\mathrm{N}^i_{\gamma,p}(B)\big)\big|\big|\mu^{\mathbf{i}}_{p,l,\gamma,H}(\mathbf{r})\big|, \tag{74}$$

and where the random variables $\mu^{\mathbf{i}}_{p,l,\gamma,H}(\mathbf{r})$ satisfy

$$|\mu^{\mathbf{i}}_{p,l,\gamma,H}(\mathbf{r})| \le C_{p,l,\gamma,H} \prod_{\mathbf{l}\in\mathcal{A}_l;\, j=1}^{l} \mu_{p,l_j,\gamma,H}$$

$$\text{with } \mu_{p,l,\gamma,H} = \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} \frac{|B_\xi - B_\eta|^{2p-l}}{|\xi-\eta|^{2\gamma p+2}}\, d\xi\, d\eta.$$

Note that all the integrals above are well defined due to the restrictions $2p \ge k_1$ and $2p(H-\gamma) - 2 > k_1 H$.

Next, we estimate the moments of $\mu^{\mathbf{i}}_{p,l,\gamma,H}(\mathbf{r})$ as follows. For any $\kappa \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathbf{E}\big[|\mu_{p,l,\gamma,H}|^\kappa\big] \\
&\quad \le C_{p,l,\gamma,H}(t_i - t_{i-1})^{2(\kappa-1)} \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} \frac{\mathbf{E}[|B_\xi - B_\eta|^{(2p-l)\kappa}]}{|\xi-\eta|^{(2\gamma p+2)\kappa}}\, d\xi\, d\eta \\
&\quad \le c_{p,l,\kappa,\gamma,H}|t_i - t_{i-1}|^{2p\kappa(H-\gamma)-l\kappa H}.
\end{aligned} \tag{75}$$

Therefore $\|\mu_{p,l,\gamma,H}\|_\kappa \le c_{p,l,\kappa,\gamma,H}|t_i - t_{i-1}|^{2p(H-\gamma)-lH}$. Note again that here, we have used the hypothesis $2p(H-\gamma) - 2 > k_1 H$.

Let us now turn to the estimation of $\mathbf{D}^n_{\mathbf{r}}\Phi_M$. Starting from relation (74), we get for $n \ge 1$,

$$\begin{aligned}
&\|\mathbf{D}^n_{\mathbf{r}}\Phi_M\|^2_{|\mathcal{H}([t_{i-1},t_i])|^{\otimes n}} \\
&\quad \le \sum_{l,m=1}^{n} \prod_{\mathbf{l}\in\mathcal{A}_l;\, j=1}^{l} \prod_{\mathbf{l}\in\mathcal{A}_m;\, k=1}^{m} |\mu_{p,l_j,\gamma,H}||\mu_{p,m_k,\gamma,H}| \\
&\qquad\qquad \times \big|\partial_z^l \Phi_M\big(\mathrm{N}^i_{\gamma,p}(B)\big)\big|\big|\partial_z^m \Phi_M\big(\mathrm{N}^i_{\gamma,p}(B)\big)\big| \\
&\qquad \times \int_{[t_{i-1},t_i]^{2n}} \prod_{i=1}^{n} |r_i - s_i|^{2(H-1)}\, dr_i\, ds_i.
\end{aligned}$$

Finally, plugging our previous inequalities (73) and (75) and resorting to Hölder's inequality with $\mathbf{q} = (q_1, \dots, q_{l+m+1})$ where $q_1^{-1} + \cdots + q_{l+m+1}^{-1} = 1$, we have for $k_1 \ge 1$,

$$\begin{aligned}
&\mathbf{E}\big[\|\mathbf{D}^{k_1}\Phi_M\|^{p_1}_{L^2([t_{i-1},t_i]^n)}\big] \\
&\quad \le c_{p,k_1,\kappa,\gamma,H}\|\Phi_M\|^{p_1}_{k_1,\infty}\mathbf{P}^{1/q_1}\big(\mathrm{N}^i_{\gamma,p}(B) > M-1\big) \\
&\qquad \times \sum_{l,m=1}^{k_1} \prod_{\mathbf{l}\in\mathcal{A}_l;\, j=1}^{l} \prod_{\mathbf{l}\in\mathcal{A}_m;\, k=1}^{m} \|\mu_{p,l_j,\gamma,H}\|^{p_1}_{q_{j+1}}\|\mu_{p,m_k,\gamma,H}\|^{p_1}_{q_{l+1+k}}
\end{aligned}$$

$$\times \left(\int_{[t_{i-1},t_i]^{2n}} \prod_{i=1}^{k_1} |r_i - s_i|^{2(H-1)} \, dr_i \, ds_i \right)^{p_1}$$
$$\leq c_{p,k_1,p_1,\mathbf{q},\gamma,H,k_2} \|\Phi_M\|_{n,\infty}^{p_1} |t_i - t_{i-1}|^{(k_2 q_1^{-1}+4)pp_1(H-\gamma)},$$

where we have set $\|\Phi_M\|_{n,\infty} := \sum_{l=0}^{n} \|\partial_z^l \Phi_M\|_\infty$. Therefore the result follows from (73) and the above inequality by noting that $|t_i - t_{i-1}| \leq c_H n^{-1/(2H)}$ and taking k_2 big enough. We remark that this result also gives that $\|\Phi_M\|_{k_1,p_1,t_{i-1}} \leq c_{p,k_1,p_1,\mathbf{q},\gamma,H}$.

The calculation for $\|\Phi_M(1-\Phi_{c_i,\epsilon_i})\|_{k_1,p_1,t_{i-1}}$ is similar, recalling that the norm of the Malliavin derivatives of Φ_M are bounded, and noting that instead of applying the operator $\mathbf{D}_{\mathbf{r}}^{k_1}$, it is better to use directly the derivative operator $\mathrm{D}_{\mathbf{r}}^{k_1}$ with Lemma A.2. We skip details for sake of conciseness. Observe, however, that in this case, the derivatives of $1-\Phi_{c_i,\epsilon_i}$ blow up as c_i, ε_i get small. Still, one remarks that the final proof is based on the fact that for any $k_6 > 0$, Chebyshev's inequality and the proof of Lemma A.4 (postponed to the Appendix) imply that

$$\mathbf{P}\left(\sum_{j=1}^{d} \int_{t_{i-1}}^{t_i} |\mathrm{D}_r^j R_i|^2 \, dr > \frac{\lambda}{8} \int_{t_{i-1}}^{t_i} K^2(t,s) \, ds \right)$$
$$\leq \left(\frac{\lambda}{8} \int_{t_{i-1}}^{t_i} K^2(t,s) \, ds \right)^{-k_6} \mathbf{E}\left[\left(\sum_{j=1}^{d} \int_{t_{i-1}}^{t_i} |\mathrm{D}_r^j R_i|^2 \, dr \right)^{k_6} \right]$$
$$\leq c(\lambda\sigma_n^2)^{-k_6} (t_i - t_{i-1})^{(2\gamma+1)k_6} \leq c n^{-(\gamma/H)k_6}.$$

Here we have used the result in Lemma 5.11(ii) and Condition 5.10. $\square$

5.6. *Upper bounds for $J_{3,i}$.* We now turn to the main technical issue in our computations, namely the bound on $J_{3,i}$. Our aim is thus to prove the following proposition:

PROPOSITION 5.15. *Let $J_{3,i}$ be the quantity defined by* (66). *Then there exist $c > 0$ and $k > 0$ such that for any $H - \frac{1}{2} < \gamma < H$,*

$$|J_{3,i}| \leq \frac{c_{M,V,m}(t_i - t_{i-1})^\gamma}{(\sigma_n^2)^{m/2}} \leq \frac{c_{M,V,m}}{n^{\gamma/2H}(\sigma_n^2)^{m/2}}. \tag{76}$$

PROOF. We start from expression (66) and normalize $I_i + \rho R_i$ in the following way: we just set $I_i + \rho R_i = \sigma_n \mathcal{U}_i$, where $\mathcal{U}_i := \sigma_n^{-1}(I_i + \rho R_i)$. We thus have

$$J_{3,i} = \sum_{j=1}^{m} \mathbf{E}_{t_{i-1}}\left[\Phi_M \Phi_{c_i,\epsilon_i} \int_0^1 \partial_{x_j} \delta_x (F_{i-1} + \sigma_n \mathcal{U}_i) R_i^j \, d\rho \right].$$

Along the same lines as in (70), the integration by parts formula (16) now yields

$$J_{3,i}=\sigma_n^{-(m+1)}\sum_{j=1}^{m}\int_0^1 \mathbf{E}_{t_{i-1}}\big[\mathbf{1}_{\{I_i+\rho R_i>x-F_{i-1}\}}H^{t_{i-1}}_{(j,1,\dots,m)}\big(\mathcal{U}_i,R_i^j\Phi_M\Phi_{c_i,\epsilon_i}\big)\big]\,d\rho.$$

Hence the following bound holds true (see [21], page 102):

$$|J_{3,i}|\le c_{1,q}\sigma_n^{-(m+1)}A_1\int_0^1 A_2(\rho)A_3(\rho)\,d\rho,$$

where the quantities A_1, $A_2(\rho)$, $A_3(\rho)$ are, respectively, defined by

$$A_1=\max_{j=1,\dots,m}\|R_i^j\Phi_{M'}\|_{k_1,p_1,t_{i-1}},\qquad A_2(\rho)=\|\det(\Gamma^{-1}_{\mathcal{U}_i,t_{i-1}})\Phi_{M'}\Phi_{c_i,\epsilon_i}\|^{k_3}_{p_3,t_{i-1}}$$

and

$$A_3(\rho)=\|\mathcal{U}_i\Phi_{M'}\|^{k_4}_{k_2,p_2,t_{i-1}},$$

and where we also recall that R_i^j is defined by (60). Then the first inequality in (76) follows from Lemmas A.4, A.5 and A.6 which have been postponed to the Appendix, and by choosing γ such that $H-\frac{1}{2}<\gamma$. In order to go from the first inequality in (76) to the second one, we simply apply Lemma 5.11. $\square$

5.7. *Lower bound.* Let us first summarize the considerations of the previous section: starting from decomposition (64) and applying Corollary 5.12, Propositions 5.13, 5.14 and 5.15 and the forthcoming relation (86), we have obtained the following facts: the inequality $\mathbf{E}_{t_{i-1}}[\delta_x(F_i)]\ge J_{1,i}+J_{2,i}+J_{3,i}$ holds true, and thus

$$\begin{aligned}\mathbf{E}_{t_{i-1}}\big[\delta_x(F_i)\big]\ge{}&\frac{1}{(2\pi)^{m/2}(\Lambda\sigma_n^2)^{m/2}}\exp\biggl(-\frac{|x-F_{i-1}|^2}{2\lambda\sigma_n^2}\biggr)\\&-c_{\lambda,\Lambda}\big(\sigma_n^2\big)^{-k_2}L^{\gamma,p}_{n,i}(k_1,p_1)-\frac{c_{M,V,m}}{n^{\gamma/2H}(\sigma_n^2)^{m/2}},\end{aligned}\tag{77}$$

with the additional information $\mathbf{E}[L^{\gamma,p}_{n,i}(k_1,p_1)]\le C_{M,\eta}n^{-\eta}$ for an arbitrarily large exponent η.

We are now ready to prove the main theorem of this article:

PROOF OF THEOREM 1.2. With equation (77) in hand, we shall follow the strategy designed in [2, 16]: Fix $x-a$ throughout the proof, and define the balls $B_i=B(y_i,c_1\sigma_n)$ for $i=1,\dots,n$ where $y_i=a+\frac{i}{n}(x-a)$. We also define below an additional sequence $\{x_i;i=1,\dots,n\}$, such that $x_i\in B_i$ and $x_n=x$. The constant c_1 will be fixed later on (see Figure 1).

We shall now proceed in a backward recursive way on the index i. For instance, in order to go from n to $n-1$, we resort to (77) in order to write

$$\mathbf{E}\big[\delta_x(F_n)\big]=\mathbf{E}\big[\mathbf{E}_{t_{n-1}}\big[\delta_x(F_n)\big]\big]\ge\frac{c_{V,m}}{\sigma_n^m}\mathbf{E}\biggl[\exp\biggl(-\frac{|x-F_{n-1}|^2}{2\lambda\sigma_n^2}\biggr)-c_{M,V,m}n^{-\kappa}\biggr],$$

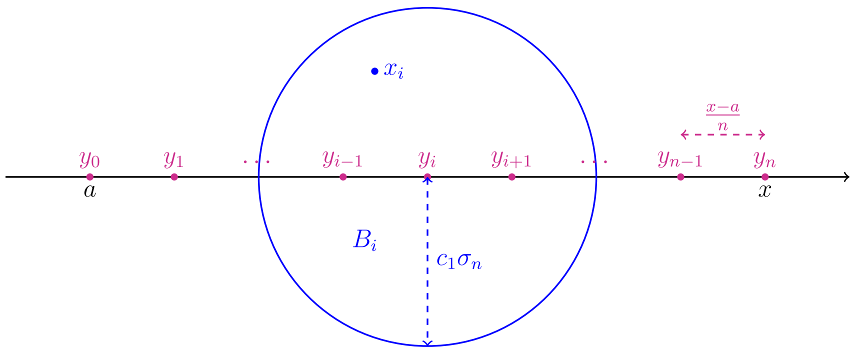

FIG. 1. *Space partition for the lower bound, with sequence* $y_1, \dots, y_n$ *and* x_i.

for a certain strictly positive constant κ. Hence

$$\begin{aligned}
&\mathbf{E}[\delta_x(F_n)] \\
&\quad \ge \frac{c_{V,m}}{\sigma_n^m} \int_{\mathbb{R}} \mathbf{E}\left[\left(\exp\left(-\frac{|x-F_{n-1}|^2}{2\lambda\sigma_n^2}\right) - c_{M,V,m} n^{-\kappa}\right)\delta_{x_{n-1}}(F_{n-1})\right] dx_{n-1} \\
&\quad \ge \frac{c_{V,m}}{\sigma_n^m} \int_{B_{n-1}} \mathbf{E}\left[\left(\exp\left(-\frac{|x-F_{n-1}|^2}{2\lambda\sigma_n^2}\right) - c_{M,V,m} n^{-\kappa}\right)\delta_{x_{n-1}}(F_{n-1})\right] dx_{n-1}.
\end{aligned}$$

We now observe the following: if we wish the term $\delta_{x_{n-1}}(F_{n-1})$ to give a nonnull contribution, the relations

$$x_{n-1} \in B(y_{n-1}, c_1\sigma_n), \qquad x - y_{n-1} = \frac{x-a}{n},$$
$$\sigma_n = \frac{t^H}{n^{1/2}}, \qquad |F_{n-1} - x_{n-1}| \le c_1\sigma_n$$

must be satisfied. Moreover, from these conditions, it is easily seen that $|x - F_{n-1}| \le 4c_1\sigma_n$ whenever $n \ge \frac{|x-a|^2}{c_1 t^{2H}}$. We thus define a constant $c_2 \ge \frac{1}{4c_1}$ such that

$$n = \frac{c_2|x-a|^2}{t^{2H}}. \tag{78}$$

Then if we take c_1 such that $\exp(-\frac{8c_1^2}{\lambda}) \ge \frac{1}{2}$ and n such that $c_{M,V,m} n^{-\kappa} \le 1/4$, we obtain

$$\mathbf{E}[\delta_x(F_n)] \ge \frac{c_{V,m}}{4\sigma_n^m} \int_{B_{n-1}} \mathbf{E}[\delta_{x_{n-1}}(F_{n-1})]\, dx_{n-1}.$$

These arguments can now be iterated backward from $i = n-1$ to 1, and the reader can easily check that the only additional required condition is the compatibility relation $y_{i+1} - y_i \le c_1\sigma_n$ (this will be verified below). Denoting by α_m the volume of a unit ball in $\mathbb{R}^m$ [viz. $\alpha_m = \pi^{m/2}/\Gamma(\frac{m}{2}+1)$], we end up with

$$\begin{aligned}
\mathbf{E}\big[\delta_x(F_n)\big] &\geq \left(\frac{c_{V,m}}{4\sigma_n^m}\right)^n \big|B(0, c_1\sigma_n)\big|^{n-1} \\
&= \left(\frac{c_{V,m}}{4}\right)^n \left(\frac{n^{1/2}}{t^H}\right)^{nm} \left(\frac{c_1 t^H}{n^{1/2}}\right)^{m(n-1)} \alpha_m^{n-1} \\
&= \left(\frac{c_{V,m}}{4}\right)^n (c_1^m \alpha_m)^{n-1} \left(\frac{n^{1/2}}{t^H}\right)^m \\
&= \frac{1}{\alpha_m (c_1 t^H)^m} \exp\left(n \ln\left(\frac{c_{V,m} c_1^m \alpha_m}{4}\right) + \frac{m}{2}\ln(n)\right).
\end{aligned} \tag{79}$$

Once here, we are reduced to tune our parameters according to the following constraints:

(i) Recalling (78), we have that if c_1 is taken small enough so that $\rho \equiv -\ln(c_{V,m} c_1^m \alpha_m/4) > 0$ and (as alluded to above) such that $\exp(-8c_1^2/\lambda) \geq \frac{1}{2}$ and $n\ln(\rho) + m\ln(n) \geq 0$ for all $n \in \mathbb{N}$, we get

$$\exp\left(n \ln\left(\frac{c_{V,m} c_1^m \alpha_m}{4}\right)\right) = \exp\left(-\frac{\rho c_2 \|x-a\|^2}{t^{2H}}\right).$$

We remark here that the values of c_1, c_2 and $c_{M,V,m}$ are fixed independently of n. It is now easily seen that our bound (79) is of the form (5).

(ii) We now choose the constant c_2 in (78) so that the compatibility relation $y_{i+1} - y_i \leq c_1\sigma_n$ is satisfied. Toward this aim, recall that

$$|y_{i+1} - y_i| = \frac{|x-a|}{n} = \frac{|x-a|}{n^{1/2}} \frac{1}{n^{1/2}},$$

and since $n = c_2 \frac{|x-a|^2}{t^{2H}}$, we get

$$|y_{i+1} - y_i| = \frac{|x-a|}{n^{1/2}} c_2^{-1/2} \frac{t^H}{|x-a|} = c_2^{-1/2}\sigma_n.$$

It is thus sufficient to take $c_2^{-1/2} \leq c_1 \wedge (2c_1^{1/2})$, which also satisfies that $n \geq \frac{|x-a|^2}{4c_1 t^{2H}}$. This completes our proof. $\square$

APPENDIX: SOME PROPERTIES OF STOCHASTIC DERIVATIVES

We start this technical section with a general bound on the space $\mathcal{H}$ related to fBm.

LEMMA A.1. *Let $H \in (0, 1/2)$, $t \in (0, 1]$ and consider the space $\mathcal{H}$ defined on $[0, t]$ as in Section 2.1. Let f be an element of $\mathcal{C}^\gamma([0,t])$ for $1/2 - H < \gamma < 1/2$, with $\|f\|_\infty \leq a$ and $\|f\|_{0,t,\gamma} \leq b$. Then*

$$\|f\|_{\mathcal{H}} \leq c_H\big(at^H + bt^{\gamma+H}\big).$$

PROOF. For a function g defined on $[0,t]$, recall that its fractional derivative is given by

$$D_{t^-}^{1/2-H} g_u = \frac{g_u}{(t-u)^{1/2-H}} + \int_u^t \frac{g_u - g_v}{(v-u)^{3/2-H}}\, dv. \tag{80}$$

Consider now $f \in \mathcal{C}^\gamma([0,t])$ satisfying the conditions above, and set $g_u = u^{-(1/2-H)} f_u$. According to [21], formula (5.31), we have

$$\|f\|_{\mathcal{H}}^2 \le c_H \int_0^t s^{1-2H} \left|D_{t^-}^{1/2-H} g_s\right|^2 ds. \tag{81}$$

We now proceed to estimate the right-hand side of relation (81).

Indeed, plugging definition (80) into (81), it is readily checked that

$$\|f\|_{\mathcal{H}}^2 \le c_H \left(\int_0^t A_s^2\, ds + \int_0^t B_s^2\, ds \right)$$

$$\text{with } A_s = \frac{f_s}{(t-s)^{1/2-H}},\ B_s = \int_s^t \frac{f_s - \psi_v f_v}{(v-s)^{3/2-H}}\, dv,$$

where we have set $\psi_v = (s/v)^{1/2-H}$. It is then easily seen that $\int_0^t A_s^2\, ds \le c_H a^2 t^{2H}$. In order to bound B, notice that the function ψ is well defined on $[s,t]$ and satisfies $\psi_s = 1$, $\psi_v \le 1$ and $|\psi_v'| \le v^{-1}$.

$$\begin{aligned} |f_s - \psi_v f_v| &\le |f_s - f_v|\psi_v + |f_s||1-\psi_v| \\ &\le b(v-s)^\gamma + a|1-\psi_v|^\gamma \le \left(b + \frac{a}{s^\gamma}\right)(v-s)^\gamma. \end{aligned}$$

Dividing this inequality by $(v-s)^{3/2-H}$, recalling that $\gamma \le 1/2$ and integrating over $[s,t]$, we get

$$|B_s| \le c_H \left(b + \frac{a}{s^\gamma}\right)(t-s)^{\gamma-(1/2-H)},$$

which entails that

$$\int_0^t B_s^2\, ds \le c_H \left(a^2 t^{2H} + b^2 t^{2(\gamma+H)}\right).$$

Gathering our bounds on $\int_0^t A_s^2\, ds$ and $\int_0^t B_s^2\, ds$, our proof is now complete. $\square$

Let us now state a bound on Malliavin derivatives.

PROOF OF RELATION (12). We focus on the first derivative case, the other ones being handled in a similar fashion. We will thus prove that

$$|\mathrm{D}_u F| \le \operatorname*{ess\,sup}_{u \le r} |\mathbf{D}_r F| K(t,u).$$

Indeed, according to Proposition 2.1, we have that for $F \in \mathcal{F}_t$,

$$|D_u F| = \left|\left[K_t^* \mathbf{D} F\right]_u\right| = \left|\int_u^t \mathbf{D}_r F \, \partial_r K(r,u) \, dr\right| \leq \operatorname*{ess\,sup}_{u \leq r \leq t} |\mathbf{D}_r F| K(t,u),$$

which is exactly our claim. $\square$

We now turn to the bounds on the process Q featuring in the definition of our remainders R_i [see decomposition (57) of X_t]:

LEMMA A.2. *Let X be the solution to (48), let η_i be the function defined by (58) and Q the process given by (61). If $r_1, s \in (t_{i-1}, t_i)$, then the following bounds hold true*:

$$|Q_s^k| \leq c_V K(t,s) |t_i - t_{i-1}|^\gamma Z_0^i, \tag{82}$$

$$|D_{r_1}^l Q_s^k| \leq c_V K(t,s) K(t,r_1) Z_1^i, \tag{83}$$

for $\mathcal{F}_1$-measurable random variables Z_0^i, Z_1^i defined by $Z_0^i = \|B\|_{t_{i-1},t,\gamma} \vee \|B\|_{t_{i-1},t,\gamma}^\gamma$ and

$$Z_1^i = \sup\{|\mathbf{D}_{r_1}^l (X_v - X_{t_{i-1}})|, t_{i-1} \leq r_1 \leq v \leq t_i\}, \tag{84}$$

admitting moments of all orders. In general, we can extend these results to Malliavin derivatives of arbitrary order $\ell \geq 1$ in the following way: for $r_1, s \in (t_{i-1}, t_i)$ and $r_2, \dots, r_\ell < t_i$, we have

$$|D_{r_1,\dots,r_\ell}^{j_1,\dots,j_\ell} Q_s^k| \leq c_V K(t,s) Z_\ell^i \prod_{j=1}^n K(t, r_j), \tag{85}$$

for $Z_\ell^i \equiv \sup\{|\mathbf{D}_{r_1,\dots,r_\ell}^{j_1,\dots,j_\ell}(X_v - X_{t_{i-1}})|, t_{i-1} \leq r_i \leq v \leq t_i, i = 1, \dots, n\}$, which is a $\mathcal{F}_1$-measurable random variable with moments of all orders.

PROOF. Bound (82) is an easy consequence of (61), Proposition 2.5 and the fact that $\partial_u K(u,s) \geq 0$. Moreover, observe that whenever $r_1 > t_{i-1}$, we have $D_{r_1} V_k(X_{t_{i-1}}) = 0$. Hence, using Proposition 2.1, we get

$$\begin{aligned}
|D_{r_1}^l Q_s^k| &= \left|\int_{s \vee r_1}^t \partial_u K(u,s) D_{r_1}^l V_k(X_{\eta_i(u)}) \, du\right| \\
&= \left|\int_{s \vee r_1}^t \partial_u K(u,s) \left[K_t^* \mathbf{D}_\cdot^l V_k(X_{\eta_i(u)})\right]_{r_1} du\right| \\
&= \left|\int_{s \vee r_1}^t \partial_u K(u,s) \left(\int_{r_1}^t \mathbf{D}_{r_2}^l V_k(X_{\eta_i(u)}) \, \partial_{r_2} K(r_2, r_1) \, dr_2\right) du\right|.
\end{aligned}$$

It is thus readily checked that

$$|D_{r_1}^l Q_s^k| \leq c_V Z_1^i \left|\int_{s \vee r_1}^t \partial_u K(u,s) K(t,r_1) \, du\right| \leq c_V Z_1^i K(t,s) K(t,r_1).$$

The general result (85) is now obtained by means of an induction argument and resorting to the same techniques as in the case of the first order derivative (namely $\ell = 1$). $\square$

REMARK A.3. Note that due to the definition (84) of Z_l^i and Proposition 2.5 which controls the derivatives of X using the Hölder norms of B, the random variables Z verify

$$|Z_j^i| \le C_V \exp\bigl(C_V \|B\|_{t_{i-1},t_i,\gamma}^{1/\gamma}\bigr),$$

for any $\gamma \in (\frac{1}{2}, H)$. Hence, applying Proposition 5.8 we obtain

$$|Z_j^i| \le C_V \exp\bigl(C_{V,\gamma}\bigl(\mathrm{N}_{\gamma,p}^i(B)\bigr)^{1/2\gamma p}\bigr),$$

for any p such that $0 < \gamma < H - \frac{1}{2p}$. This relation yields in particular that $Z_j^i \in \bigcap_{q\ge 1} L^q(\Omega)$. Furthermore, once we localize by the random variables Φ_M or $\Phi_{M'}$, we end up with

$$\max_{0\le l\le k}\bigl(Z_l^i \Phi_{M'}\bigr) \le c_{M,V,m} \qquad \text{with } c_{M,V,m} = c_{V,m} \exp\bigl(c_{V,m}\bigl(M'\bigr)^{1/2\gamma p}\bigr). \tag{86}$$

In the next proposition, we give norm estimates for the remainder terms R_i needed in the upper bound for $J_{3,i}$.

LEMMA A.4. *In the setting of Proposition 5.6 and Corollary 5.7, with definition (60) and (63), the following estimate is valid*:

$$\|R_i^j \Phi_{M'}\|_{k_1,p_1,t_{i-1}} \le c_{V,M}(t_i - t_{i-1})^\gamma \sigma_n. \tag{87}$$

PROOF. This result obviously involves the control of many derivative terms. For the sake of conciseness, we only sketch the bound for $\mathrm{D}R_i$. Now recall that

$$R_i = \sum_{k=1}^{d} \int_{t_{i-1}}^{t_i} Q_s^k \circ dW_s^k.$$

We now apply a small variant of [21], Proposition 1.3.8, to Stratonovich integrals, which states that for $r \in [t_{i-1}, t_i]$, we have

$$\mathrm{D}_r^j R_i = Q_r^j + \sum_{k=1}^{d} \int_{t_{i-1}}^{t_i} \mathrm{D}_r^j Q_s^k \circ dW_s^k. \tag{88}$$

Let us now evaluate the $L^2[t_{i-1}, t_i]$ norm of $\mathrm{D}_r^j R_i$. The main contribution for this norm comes from the term Q on the right-hand side of (88), for which we obtain, according to (82),

$$\begin{aligned}\int_{t_{i-1}}^{t_i} \bigl(Q_r^j\bigr)^2 dr &\le c_V |t_i - t_{i-1}|^{2\gamma} \bigl(Z_0^i\bigr)^2 \int_{t_{i-1}}^{t_i} K^2(t,r)\, dr \\ &= c_V \bigl(Z_0^i\bigr)^2 |t_i - t_{i-1}|^{2\gamma} \sigma_n^2,\end{aligned}$$

and thus

$$\mathbf{E}_{t_{i-1}}^{1/p_1}\big[\|Q\|_{L^2([t_{i-1},t_i])}^{p_1}\Phi_{M'}\big] \le c_V|t_i - t_{i-1}|^{\gamma}\sigma_n \mathbf{E}_{t_{i-1}}^{p_1}\big[(Z_0^i)^{p_1}\Phi_{M'}\big]$$
$$\le c_{V,M}|t_i - t_{i-1}|^{\gamma}\sigma_n,$$

which is consistent with our claim (87).

Let us give another example of term which has to be analyzed in order to bound the norm of $\mathrm{D}_r^j R_i$: the term A defined as

$$A := \mathbf{E}_{t_{i-1}}^{1/p_1}\left[\left(\int_{t_{i-1}}^{t_i} dr \int_{t_{i-1}}^{t_i} ds\big[\mathrm{D}_r^j Q_s^k\big]^2\right)^{p_1/2}\Phi_{M'}\right].$$

Along the same lines as above, using (82), we find

$$A \le c_{M,V}\int_{t_{i-1}}^{t_i} ds\, K^2(t,s)\int_{t_{i-1}}^{t_i} dr\, K^2(t,r) = c_{M,V}\sigma_n^4,$$

which is a remainder term with respect to (87). Notice that many other higher order terms have to be evaluated in order to complete the proof. We omit these cumbersome but routine developments for sake of conciseness. □

We now turn to the bound on $A_2(\rho)$:

LEMMA A.5. *Recall that $A_2(\rho)$ is defined as $A_2(\rho) = \|\det(\Gamma_{\mathcal{U}_i,t_{i-1}})^{-1}\Phi_{M'} \times \Phi_{c_i,\epsilon_i}\|_{p_3,t_{i-1}}^{k_3}$. Then this quantity is uniformly bounded in n, ρ and $\omega \in \Omega$.*

PROOF. Recall that $\mathcal{U}_i = \sigma_n^{-1}(I_i + \rho R_i)$, and remark that using Proposition 4 in [2], we have that

$$\det(\Gamma_{\mathcal{U}_i,t_{i-1}})^{-1}\Phi_{c_i,\epsilon_i} \le \sigma_n^{2m}\left(\frac{1}{2}\lambda\int_{t_{i-1}}^{t_i} K^2(t,s)\,ds - \sum_{j=1}^{d}\int_{t_{i-1}}^{t_i}\big|\mathrm{D}_r^j R_i\big|^2\,dr\right)^{-m}\Phi_{c_i,\epsilon_i}.$$

Moreover, we have localized $\sum_{j=1}^d \int_{t_{i-1}}^{t_i}|\mathrm{D}_r^j R_i|^2\,dr$ by Φ_{c_i,ϵ_i} with $c_i = \frac{\lambda\sigma_n^2}{8}$. Thus we end up with

$$\det(\Gamma_{\mathcal{U}_i,t_{i-1}})^{-1}\Phi_{c_i,\epsilon_i} \le \sigma_n^2\left(\frac{\lambda}{4}\int_{t_{i-1}}^{t_i} K^2(t,s)\,ds\right)^{-1},$$

from which the result follows. □

The estimates for $A_3(\rho)$ are obtained in a similar fashion. In fact, we have:

LEMMA A.6. *The same conclusion as in Lemma A.5 holds true for the quantity $A_3(\rho) = \|\mathcal{U}_i\Phi_{M'}\|_{k_2,p_2,t_{i-1}}^{k_4}$.*

PROOF. With respect to Lemma A.4, we only need to consider additionally the bound for

$$\|I_i \Phi_{M'}\|_{k_2,p_2,t_{i-1}} \leq c\|I_i\|_{k_2,p_3,t_{i-1}}\|\Phi_{M'}\|_{k_2,p_4,t_{i-1}}.$$

The above follows from Hölder's inequality. Therefore the result follows from straightforward calculations for I_i as in the proof of Proposition 5.13. □

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M. BESALÚ
INRIA NANCY GRAND EST
AND INSTITUT ÉLIE CARTAN LORRAINE
UNIVERSITÉ DE LORRAINE
B.P. 239
54506 VANDŒUVRE-LÈS-NANCY
FRANCE
E-MAIL: mireiabesalu@gmail.com

A. KOHATSU-HIGA
DEPARTMENT OF MATHEMATICAL SCIENCES
RITSUMEIKAN UNIVERSITY
AND JAPAN SCIENCE AND TECHNOLOGY AGENCY
1-1-1 NOJIHIGASHI
KUSATSU, SHIGA, 525-8577
JAPAN
E-MAIL: arturokohatsu@gmail.com

S. TINDEL
INSTITUT ÉLIE CARTAN LORRAINE
UNIVERSITÉ DE LORRAINE
B.P. 239
54506 VANDŒUVRE-LÈS-NANCY
FRANCE
E-MAIL: samy.tindel@univ-lorraine.fr