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NONOPTIMALITY OF CONSTANT RADII IN HIGH DIMENSIONAL CONTINUUM PERCOLATION

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Consider a Boolean model Σ in $\mathbb{R}^d$. The centers are given by a homogeneous Poisson point process with intensity λ and the radii of distinct balls are i.i.d. with common distribution ν. The critical covered volume is the proportion of space covered by Σ when the intensity λ is critical for percolation. Previous numerical simulations and heuristic arguments suggest that the critical covered volume may be minimal when ν is a Dirac measure. In this paper, we prove that it is not the case in sufficiently high dimension.

1. Introduction and statement of the main results. The Boolean model is a popular model for continuum percolation. It can be described in the following way. Let ν be a finite measure on $(0,+\infty)$, with positive mass. Let $d \geq 2$ be an integer, $\lambda > 0$ be a real number and ξ be a Poisson point process on $\mathbb{R}^d \times (0,+\infty)$ whose intensity measure is the Lebesgue measure on $\mathbb{R}^d$ times $\lambda\nu$. The Boolean model $\Sigma(\lambda\nu)$ in $\mathbb{R}^d$ driven by $\lambda\nu$ is the following random subset of $\mathbb{R}^d$:

$$\Sigma(\lambda\nu) = \bigcup_{(c,r)\in\xi} B(c,r),$$

where $B(c,r)$ is the open Euclidean ball centered at $c \in \mathbb{R}^d$ and with radius $r \in (0,+\infty)$. Note that the collection of centers of the balls of the Boolean model is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\lambda\nu((0,+\infty))$, and that the radii of the distinct balls are i.i.d. with law $\nu(\cdot)/\nu((0,+\infty))$, and independent of the point process of the centers. In our study, we focus on the Boolean model with deterministic radii (when ν is a Dirac mass δ_ρ, with $\rho > 0$) and on the Boolean model with two distinct radii (when ν is a weighted sum of two Dirac masses).

We say that $\Sigma(\lambda\nu)$ percolates if the probability that there is an unbounded connected component of $\Sigma(\lambda\nu)$ that contains the origin is positive. This is equivalent to the almost-sure existence of an unbounded connected component of $\Sigma(\lambda\nu)$. We refer to the book by Meester and Roy [11] for background on continuum percolation. The critical intensity is defined by

$$\lambda_d^c(\nu) = \inf\{\lambda > 0 : \Sigma(\lambda\nu) \text{ percolates}\}.$$

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One easily checks that $\lambda_d^c(\nu)$ is finite, and in [6] it is proven that $\lambda_d^c(\nu)$ is positive if and only if

$$\int r^d \nu(dr) < +\infty. \tag{1}$$

We assume that this assumption is fulfilled.

By ergodicity, the Boolean model $\Sigma(\lambda\nu)$ has a deterministic natural density. This is also the probability that a given point belongs to the Boolean model and it is given by

$$P\big(0 \in \Sigma(\lambda\nu)\big) = 1 - \exp\biggl(-\lambda \int v_d r^d \nu(dr)\biggr),$$

where v_d denotes the volume of the unit ball in $\mathbb{R}^d$. The critical covered volume $c_d^c(\nu)$ is the density of the Boolean model when the intensity is critical:

$$c_d^c(\nu) = 1 - \exp\biggl(-\lambda_d^c(\nu) \int v_d r^d \nu(dr)\biggr).$$

Unlike the critical intensity λ_d^c, the critical covered volume c_d^c is invariant under scaling. For all $a > 0$, let $H^a(\nu)$ be the image of ν under the map defined by $x \mapsto ax$. We have the following scaling property:

$$c_d^c\big(H^a \nu\big) = c_d^c(\nu). \tag{2}$$

Indeed, a critical Boolean model remains critical when rescaling and the density is invariant by rescaling.[1] More formally, this invariance of the critical covered volume c_d^c under rescaling is a consequence of Proposition 2.11 in [11]. Note, for example, that for any $\rho > 0$,

$$c_d^c(\delta_1) = c_d^c(\delta_\rho) \qquad \text{while } \lambda_d^c(\delta_1) = \rho^d \lambda_d^c(\delta_\rho).$$

One also easily checks the following invariance property: for all $a > 0$, $c_d^c(a\nu) = c_d^c(\nu)$.

Practically, we study the critical covered volume through the normalized critical intensity:

$$\widetilde{\lambda}_d^c(\nu) = \lambda_d^c(\nu) \int v_d (2r)^d \nu(dr).$$

We then have $c_d^c(\nu) = 1 - \exp(-\frac{\widetilde{\lambda}_d^c(\nu)}{2^d})$. The factor 2^d may seem arbitrary here; its interest will appear in the statement of the next theorems. Note also that the normalized critical intensity $\widetilde{\lambda}_d^c$ is also invariant under rescaling.

[1] By rescaling, we mean multiplying all coordinates and radii by the same scalar.

Normalized critical intensity as a function of ν. It has been conjectured by Kertész and Vicsek [9] that the normalized critical intensity should be independent of ν, as soon as the support of ν is bounded. Dhar and Phani [4] gave a heuristic argument suggesting that the conjecture were false. A rigorous proof was then given by Meester, Roy and Sarkar in [12]. More precisely, they gave examples of measures ν with two atoms such that

$$\widetilde{\lambda}_d^c(\nu) > \widetilde{\lambda}_d^c(\delta_1). \tag{3}$$

As a consequence of Theorem 1.1 in the paper by Menshikov, Popov and Vachkovskaia [13], we even get that $\widetilde{\lambda}_d^c(\nu)$ can be arbitrarily large.[2] On the contrary, Theorem 2.1 in [6] gives the existence of a positive constant C_d, that depends only on the dimension d, such that, for all ν satisfying (1):

$$\widetilde{\lambda}_d^c(\nu) \geq C_d.$$

To sum up, $\widetilde{\lambda}_d^c(\cdot)$ is not bounded from above but is bounded from below by a positive constant. In other words, the critical covered volume $c_d^c(\cdot) \in (0,1)$ can be arbitrarily close to 1 but is bounded from below by a positive constant. It is thus natural to seek optimal measures, that is, the ones which minimize the normalized critical intensity, or equivalently, the critical covered volume.

In the physical literature, it is strongly believed that, at least when $d = 2$ and $d = 3$, the critical covered volume is minimum in the case of a deterministic radius, that is, when the distribution of radii is a Dirac measure. This conjecture is supported by numerical evidence (to the best of our knowledge, the most accurate estimations are given in a paper by Quintanilla and Ziff [15] when $d = 2$ and in a paper by Consiglio, Baker, Paul and Stanley [2] when $d = 3$). On Figure 1, we plot the critical covered volume in dimension 2 as a function of α and for different values of ρ when $\nu = (1-\alpha)\delta_1 + \alpha\rho^{-2}\delta_\rho$. The data for finite values of ρ come from numerical estimations in [15], while the data for the limit of ρ going to infinity come from the study of the multiscale Boolean model in [7]. See Section 1.4 in [7] for further references. The conjecture is also supported by some heuristic arguments in any dimension (see, e.g., Dhar [3], and Balram and Dhar [1]). In [12], it is noted that the rigorous proof of (3) suggests that the deterministic case might be optimal for any $d \geq 2$.

In this paper, we show on the contrary that for all d large enough the critical covered volume is not minimized by the case of deterministic radii.

Normalized critical intensity in high dimension: The case of a deterministic radius. Assume here that the measure ν is a Dirac mass at 1, that is, that the radii of the balls are all equal to 1. Penrose proved the following result in [14]:

[2] Actually the result of [13] is a much stronger statement than the consequence we use here.

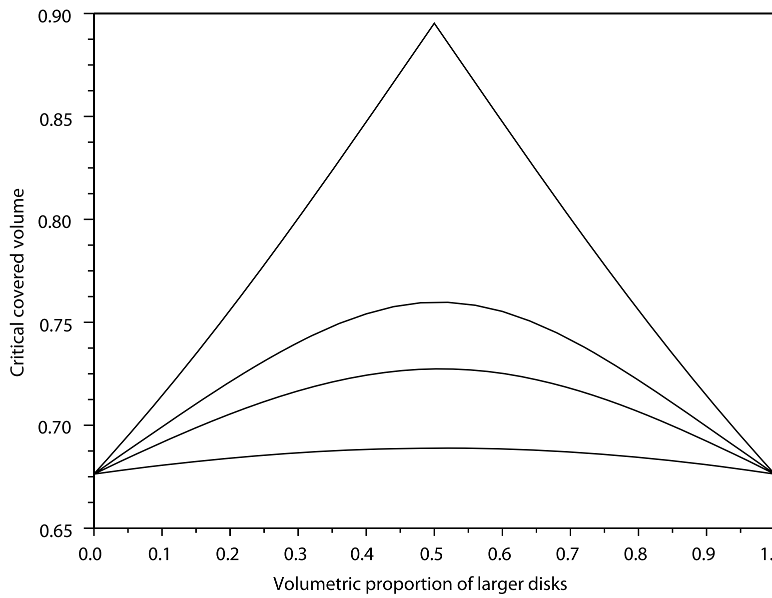

FIG. 1. *Critical covered volume as a function of* α *for different values of* ρ. *From bottom to top*: $\rho = 2$, $\rho = 5$, $\rho = 10$ *and the limit as* $\rho \to \infty$.

THEOREM 1.1 (Penrose).

$$\lim_{d\to\infty} \widetilde{\lambda}_d^c(\delta_1) = 1.$$

With the scale invariance of $\widetilde{\lambda}_d^c$, this limit can readily be generalized to any constant radius: for any $\rho > 0$,

$$\lim_{d\to\infty} \widetilde{\lambda}_d^c(\delta_\rho) = \lim_{d\to\infty} \widetilde{\lambda}_d^c(\delta_1) = 1.$$

Theorem 1.1 is the continuum analogue of a result of Kesten [10] for Bernoulli bond percolation on the nearest-neighbor integer lattice $\mathbb{Z}^d$, which says that the critical percolation parameter is asymptotically equivalent to $1/(2d)$.

Let us say a word about the ideas of the proof of Theorem 1.1.

The inequality $\widetilde{\lambda}_d^c(\delta_1) > 1$ holds for any $d \geq 2$. The proof is simple, and here is the idea. We consider the following natural genealogy. The deterministic ball $B(0, 1)$ is said to be the ball of generation 0. The random balls of $\Sigma(\lambda\delta_1)$ that touch $B(0, 1)$ are then the balls of generation 1. The random balls that touch one ball of generation 1 without being one of them are then the balls of generation 2 and so on. Let us denote by N_d the number of all balls that are descendants of $B(0, 1)$. There is no percolation if and only if N_d is almost surely finite.

Now denote by m the Poisson distribution with mean $\lambda v_d 2^d$: this is the law of the number of balls of $\Sigma(\lambda\delta_1)$ that touch a given ball of radius 1. Therefore, if there were no interference between children of different balls, N_d would be equal to Z, the total population in a Galton–Watson process with offspring distribution m. Because of the interferences due to the fact that the Boolean model lives in $\mathbb{R}^d$, this is not true: in fact, N_d is only stochastically dominated by Z. Therefore, if $\lambda v_d 2^d \leq 1$, then Z is finite almost surely, so N_d is finite almost surely and therefore there is no percolation. This implies

$$\widetilde{\lambda}_d^c(\delta_1) = v_d 2^d \lambda_d^c(\delta_1) > 1.$$

The difficult part of Theorem 1.1 is to prove that if d is large, then the interferences are small, so N_d is close to Z and, therefore, there is percolation for large d as soon as $v_d 2^d \lambda$ is a constant strictly larger than one.

To sum up, at first order, the asymptotic behavior of the critical intensity of the Boolean model with constant radius is given by the threshold of the associated Galton–Watson process, as in the case of Bernoulli percolation on $\mathbb{Z}^d$: roughly speaking, as the dimension increases, the geometrical constraints of the finite dimension space decrease and at the limit, we recover the nongeometrical case of the corresponding Galton–Watson process.

Normalized critical intensity in high dimension: *The case of radii taking two values*. Let $1 < \rho < 2$. Set

$$\mu = \delta_1 + \delta_\rho.$$

If $d \geq 1$ is an integer, we define the normalized measure μ_d on $(0, +\infty)$ by setting

$$\mu_d = \delta_1 + \frac{1}{\rho^d}\delta_\rho. \tag{4}$$

We will study the behavior of $\widetilde{\lambda}_d^c(\mu_d)$ as d tends to infinity. Let us motivate the definition of μ_d with the following two related properties:

(1) Consider the Boolean model $\Sigma(\lambda\mu_d)$ on $\mathbb{R}^d$ driven by $\lambda\mu_d$ where $\lambda > 0$. The number of balls of $\Sigma(\lambda\mu_d)$ with radius 1 that contains a given point is a Poisson random variable with intensity λv_d. The number of balls of $\Sigma(\lambda\mu_d)$ with radius ρ that contains a given point is also a Poisson random variable with intensity λv_d. Loosely speaking, this means that contrary to what happens in the Boolean model driven by $\lambda\mu$, the relative importance of the two types of radii does not depend on the dimension d in the Boolean model driven by $\lambda\mu_d$.

(2) A closely related property is the following one. Consider two independent Boolean model Σ and Σ', both driven by $\lambda\delta_1$. Then $\Sigma \cup \rho\Sigma'$ is a Boolean model driven by $\lambda\mu_d$.

THEOREM 1.2. *Let $1 < \rho < 2$. Set as before $\mu_d = \delta_1 + \rho^{-d}\delta_\rho$. Then*

$$\lim_{d\to+\infty} \frac{1}{d} \ln\bigl(\widetilde{\lambda}_d^c(\mu_d)\bigr) = \ln\bigl(\kappa_\rho^c\bigr) \qquad \textit{where } \kappa_\rho^c = \frac{2\sqrt{\rho}}{1+\rho}.$$

Note that as $1 < \rho < 2$, $\kappa_\rho^c < 1$. The following result is then an immediate consequence of Theorem 1.1 and Theorem 1.2.

COROLLARY 1.3. *If the dimension d is large enough, then there exists a probability measure ν on $(0, +\infty)$ such that*

$$c_d^c(\nu) < c_d^c(\delta_1).$$

In other words, the conjecture is false in high dimensions.

We end this section with some remarks:

- One can easily extend Theorem 1.2 as follows. Let $\alpha, \beta, a, b > 0$. Set $\rho = b/a$ and assume $1 < \rho < 2$. Then
$$\lim_{d\to+\infty} \frac{1}{d} \ln\big(\widetilde{\lambda}_d^c\big(\alpha a^{-d}\delta_a + \beta b^{-d}\delta_b\big)\big) = \ln\big(\kappa_\rho^c\big) < 0.$$
- As we will see in the proof, the critical threshold κ_ρ^c is given by the critical parameter of an associated two-types Galton–Watson process when $1 < \rho < 2$; we prove in a companion paper [8] that this is not the case for $\rho > 2$.
- If one does not normalize the distribution one has[3] $\widetilde{\lambda}_d^c(\alpha\delta_a + \beta\delta_b) \to 1$, and thus $\widetilde{\lambda}_d^c(\alpha\delta_a + \beta\delta_b) \sim \widetilde{\lambda}_d^c(\delta_1)$. This behavior is due to the fact that, without normalization, the influence of the small balls vanishes in high dimension.

2. Proofs.

2.1. *Notation*. Fix $1 < \rho < 2$ and $\kappa > 0$. Once the dimension $d \geq 1$ is given, we consider two independent stationary Poisson point processes on $\mathbb{R}^d$: χ_1 and χ_ρ, with respective intensities

$$\lambda_1 = \frac{\kappa^d}{v_d 2^d} \quad \text{and} \quad \lambda_\rho = \frac{\kappa^d}{v_d 2^d \rho^d}.$$

To χ_1 and χ_ρ, we, respectively, associate the two Boolean models

$$\Sigma_1 = \bigcup_{x\in\chi_1} B(x,1) \quad \text{and} \quad \Sigma_\rho = \bigcup_{x\in\chi_\rho} B(x,\rho).$$

We focus on the percolation properties of the following two-type Boolean model:

$$\Sigma = \Sigma_1 \cup \Sigma_\rho.$$

This Boolean model is driven by the measure

$$\lambda_1\delta_1 + \lambda_\rho\delta_\rho = \frac{\kappa^d}{v_d 2^d}\mu_d,$$

[3]The upper bound can be proven using $\lambda_d^c(\alpha\delta_a + \beta\delta_b) \le \lambda_d^c(\beta\delta_b)$. The lower bound can be proven using the easy part of the comparison with a two-type Galton–Watson process.

where μ_d is defined as before by (4). Remember that

$$\kappa_\rho^c = \frac{2\sqrt{\rho}}{1+\rho} < 1.$$

2.2. *Subcritical phase.* The aim of this subsection is to prove the following result.

PROPOSITION 2.1. *If $\kappa < \kappa_\rho^c$, then, as soon as the dimension d is large enough, percolation does not occur in the two-type Boolean model Σ.*

PROOF. The proof is very similar to the easy part of the proof of Theorem 1.1. The only difference is that we consider a two-types Galton–Watson process instead of a one-type Galton–Watson process. Therefore, we only sketch the proof and refer to [14] for a more detailed proof.

The idea is to consider the following natural genealogy. The deterministic ball $B(0,\rho)$ is said to be the ball of generation 0. The random balls of Σ that touch $B(0,\rho)$ are then the balls of generation 1. They can be of two different types: either of radius 1 or of radius ρ. The random balls that touch one ball of generation 1 without being one of them are then the balls of generation 2 and so on.

This genealogical process is stochastically dominated by a two-types Galton–Watson process. Basically, the Galton–Watson process is obtained by neglecting the geometrical constraints due to the fact that the Boolean model lives in $\mathbb{R}^d$. It is defined as follows. Start with one individual of type ρ. The offspring distribution of type 1 of an individual of type ρ is defined to be the distribution of the number of balls of Σ_1 that intersect a given deterministic ball of radius ρ. Therefore, it is a Poisson random variable with mean $\lambda_1 v_d(1+\rho)^d$. The other offspring distributions are defined similarly. The matrix of means of offspring distributions is thus given by

$$M_d = \begin{pmatrix} \lambda_1 v_d(1+1)^d & \lambda_\rho v_d(1+\rho)^d \\ \lambda_1 v_d(1+\rho)^d & \lambda_\rho v_d(\rho+\rho)^d \end{pmatrix} = \kappa^d \begin{pmatrix} 1 & \left(\dfrac{1+\rho}{2\rho}\right)^d \\ \left(\dfrac{1+\rho}{2}\right)^d & 1 \end{pmatrix}.$$

Let r_d denote the largest eigenvalue of M_d. The extinction probability of the two-types Galton–Watson process is 1 if and only if $r_d \le 1$. We have

$$r_d \sim \left(\frac{\kappa(1+\rho)}{2\sqrt{\rho}}\right)^d.$$

As $\kappa < \kappa_\rho^c$, we get that the Galton–Watson process is subcritical for large enough d. Therefore, for large enough d, the total progeny of the Galton–Watson process is almost surely finite. Thus, almost surely, there is no infinite cluster of the Boolean model Σ that touches $B(0,\rho)$. As a consequence, almost surely, there is no infinite cluster in the Boolean model Σ. $\square$

2.3. *Supercritical phase.*

2.3.1. *Result.* For every $n \geq 0$, we set $R_n = \rho$ if n is even and $R_n = 1$ otherwise. We say that alternating percolation occurs if there exists an infinite sequence of distinct points $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^d$ such that, for every $n \geq 0$:

- $x_n \in \chi_{R_n}$.
- $B(x_n, R_n) \cap B(x_{n+1}, R_{n+1}) \neq \varnothing$.

In other words, alternating percolation occurs if there exists an infinite path along which balls of radius 1 alternate with balls of radius ρ. The aim of this subsection is to prove the following proposition.

PROPOSITION 2.2. *Assume $\kappa > \kappa_\rho^c$. If the dimension d is large enough, then alternating percolation occurs in Σ with probability one.*

By a straightforward coupling argument, one sees that it is sufficient to prove the proposition under the following assumptions on κ:

$$\kappa > \kappa_\rho^c \quad \text{and} \quad \kappa < \frac{2\sqrt{2}}{1+\rho} \quad \text{and} \quad \kappa < 1.$$

We make this assumption in the remaining of this subsection.

We will prove that alternating percolation occurs in the two-type Boolean model in the supercritical case by embedding in the Boolean model a supercritical 2-dimensional oriented percolation process.

We thus specify the two first coordinates, and introduce the following notation. When $d \geq 3$, for any $x \in \mathbb{R}^d$, we write

$$x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{d-2}.$$

We write $B'(c, r)$ for the open Euclidean ball of $\mathbb{R}^2$ with center $c \in \mathbb{R}^2$ and radius $r > 0$. In the same way, we denote by $B''(c, r)$ the open Euclidean ball of $\mathbb{R}^{d-2}$ with center $c \in \mathbb{R}^{d-2}$ and radius $r > 0$.

2.3.2. *One step in the 2-dimensional oriented percolation model.* The point here is to define the event that will govern the opening of the edges in the 2-dimensional oriented percolation process: it is naturally linked to the existence of a finite path composed of a ball of radius 1 and a ball of radius ρ.

We define, for a given dimension $d \geq 3$, the two following subsets of $\mathbb{R}^d$:

$$W = d^{-1/2}\big((-1, 1) \times (-1, 0) \times \mathbb{R}^{d-2}\big),$$
$$W^+ = d^{-1/2}\big((0, 1) \times (0, 1) \times \mathbb{R}^{d-2}\big).$$

For $x_0 \in W$, we set

$$\mathcal{G}^+(x_0) = \left\{ \begin{array}{c} \text{There exist } x_1 \in \chi_1 \cap W^+ \text{ and } x_2 \in \chi_\rho \cap W^+ \\ \text{such that } B(x_0, \rho) \cap B(x_1, 1) \neq \varnothing \text{ and } B(x_1, 1) \cap B(x_2, \rho) \neq \varnothing \end{array} \right\}. \tag{5}$$

Our goal here is to prove that the probability of occurrence of this event is asymptotically large.

PROPOSITION 2.3. *Assume that $\kappa \in (\kappa_\rho^c, 1)$. Choose $p \in (0, 1)$. If the dimension d is large enough, then for every $x_0 \in W$,*

$$P(\mathcal{G}^+(x_0)) \geq p.$$

Note already that by translation invariance, $P(\mathcal{G}^+(x_0))$ does not depend on x_0'', so we can assume without loss of generality that $x_0'' = 0$. In Table 1, we introduce some subsets.

Finally, we set

$$\forall i \in \{0, 1, 2\} \qquad C_i = D_i' \times C_i''.$$

Note that for d large enough, $C_1'' \cap C_2'' = \varnothing$, and thus $C_1 \cap C_2 = \varnothing$. The next straightforward lemma controls the asymptotics in the dimension d of the volume of these sets. The proof is left to the reader.

LEMMA 2.4. *For $i \in \{1, 2\}$:*

$$\lim_{d \to +\infty} \frac{1}{d} \ln \frac{|C_i''|}{v_{d-2}} = \lim_{d \to +\infty} \frac{1}{d} \ln \frac{|C_i|}{v_d} = \ln(\sqrt{i}(1+\rho)).$$

We will seek the couple (x_1, x_2) involved in the event $\mathcal{G}^+(x_0)$ in $C_1 \times C_2$. But we also have to ensure that $B(x_0, \rho) \cap B(x_1, 1) \neq \varnothing$ and $B(x_1, 1) \cap B(x_2, \rho) \neq \varnothing$. We set, for $y \in C_1$,

$$D_2''(y'') = \left\{ z'' \in C_2'' : \langle z'', y'' \rangle \geq \|y''\| \cdot \|z''\| \frac{\sqrt{2}}{2} \right\} \subset \mathbb{R}^{d-2} \quad \text{and}$$
$$D_2(y) = D_2' \times D_2''(y'') \subset C_2.$$

The set $D_2''(y'')$ is the intersection of the annulus C_2'' and of a cone with axis y''.

TABLE 1
Definition of some subsets

Subsets of $\mathbb{R}^2$	**Subsets of $\mathbb{R}^{d-2}$**
$D_0' = d^{-1/2}(-1, 1) \times (-1, 0)$	$C_0'' = \{0\}$
$D_1' = d^{-1/2}(0, 1) \times (0, 1)$	$C_1'' = B''(0, (1+\rho) - \frac{6}{d}) \setminus B''(0, (1+\rho) - \frac{7}{d})$
$D_2' = d^{-1/2}(0, 1) \times (0, 1)$	$C_2'' = B''(0, \sqrt{2}(1+\rho) - \frac{6}{d}) \setminus B''(0, \sqrt{2}(1+\rho) - \frac{7}{d})$

LEMMA 2.5. 1. *If the dimension d is large enough*,

$$\forall y \in C_0 \qquad C_1 \subset B(y, 1+\rho), \tag{6}$$

$$\forall y \in C_1 \qquad D_2(y) \subset B(y, 1+\rho) \cap C_2. \tag{7}$$

2. *Let $x_0 \in C_0$, and take d large enough to have* (6) *and* (7). *If there exist $X_1 \in \chi_1 \cap C_1$ and $X_2 \in \chi_\rho \cap D_2(X_1)$, then the event $\mathcal{G}_+(x_0)$ occurs.*

PROOF. 1. Let $y \in C_0$ and $z \in C_1$. For d large enough,

$$\|z-y\|^2 = \|z'-y'\|^2 + \|z''-y''\|^2 \le \frac{8}{d} + \big((1+\rho) - 6d^{-1}\big)^2 < (1+\rho)^2.$$

Let now $y \in C_1$ and $z \in D_2(y)$. Then, as soon as d is large enough,

$$\begin{aligned}
\|z-y\|^2 &= \|z'-y'\|^2 + \|z''-y''\|^2 \\
&\le \frac{2}{d} + \|y''\|^2 + \|z''\|^2 - 2\langle y'', z''\rangle \\
&\le \frac{2}{d} + \left(1+\rho-\frac{6}{d}\right)^2 + \left(\sqrt{2}(1+\rho) - \frac{6}{d}\right)^2 \\
&\quad - 2\left(1+\rho-\frac{7}{d}\right)\left(\sqrt{2}(1+\rho)-\frac{7}{d}\right)\frac{\sqrt{2}}{2} \\
&\le (1+\rho)^2 + \frac{1}{d}\big(2 - (12-7\sqrt{2})(1+\rho)(1+\sqrt{2})\big) + O\big(d^{-2}\big) \\
&< (1+\rho)^2.
\end{aligned}$$

2. Let $x_0 \in D_0$. Assume there exist $X_1 \in \chi_1 \cap C_1$ and $X_2 \in \chi_\rho \cap D_2(X_1)$. Then:

- (6) ensures that $\|X_1 - x_0\| < 1+\rho$, and thus $B(x_0, \rho) \cap B(X_1, 1) \neq \varnothing$;
- (7) ensures that $\|X_2 - X_1\| < 1+\rho$, and thus $B(X_1, 1) \cap B(X_2, \rho) \neq \varnothing$.

Thus, $\mathcal{G}_+(x_0)$ occurs. $\square$

The volume $|D_2''(y'')|$ does not depend on $y \in C_1$, and is denoted by $|D_2''|$. We now give asymptotic estimates for $|D_2''|$.

LEMMA 2.6.

$$\lim_{d\to+\infty} \frac{1}{d} \ln \frac{|D_2''|}{v_{d-2}} = \lim_{d\to+\infty} \frac{1}{d} \ln \frac{|D_2|}{v_d} = \ln(1+\rho).$$

PROOF. We have, by homogeneity and isotropy,

$$\big|D_2''\big| = \big(\big(\sqrt{2}(1+\rho) - 6d^{-1}\big)^{d-2} - \big(\sqrt{2}(1+\rho) - 7d^{-1}\big)^{d-2}\big)|S|, \tag{8}$$

where $S = \{x = (x_1, \dots, x_{d-2}) \in B''(0,1) : x_1 \geq \|x\| \frac{\sqrt{2}}{2}\}$. But S is included in the cylinder

$$\left\{(x_i)_{1 \leq i \leq d-2} \in \mathbb{R}^{d-2} : x_1 \in [0,1], \|(x_2, \dots, x_{d-2})\| \leq \frac{\sqrt{2}}{2}\right\}$$

and S contains the cone

$$\left\{(x_i)_{1 \leq i \leq d-2} \in \mathbb{R}^{d-2} : x_1 \in \left[0, \frac{\sqrt{2}}{2}\right], \|(x_2, \dots, x_{d-2})\| \leq x_1\right\}.$$

Therefore,

$$v_{d-3}\left(\frac{\sqrt{2}}{2}\right)^{d-2}(d-2)^{-1} \leq |S| \leq v_{d-3}\left(\frac{\sqrt{2}}{2}\right)^{d-3}. \tag{9}$$

From (8) and (9), we get

$$\lim_{d \to +\infty} \frac{1}{d} \ln\left(\frac{|D_2''|}{v_{d-2}}\right) = \ln(1+\rho).$$

The lemma follows. Note that a direct calculus with spherical coordinates can also give the announced estimates. □

PROOF OF PROPOSITION 2.3. Choose $p < 1$ and $x_0 \in W$ such that $x_0'' = 0$.

We start with a single individual, encoded by its position $\zeta_0 = \{x_0\} \subset C_0$, and we set

$$\zeta_1 = \chi_1 \cap C_1 \quad \text{and} \quad \zeta_2 = \chi_\rho \cap \bigcup_{y \in \zeta_1} D_2(y) \subset C_2.$$

By Lemma 2.5, for d large enough, if $\zeta_2 \neq \varnothing$ then the event $\mathcal{G}^+(x_0)$ occurs. To bound from below, the probability that $\zeta_2 \neq \varnothing$, we build a simpler random set ξ, stochastically dominated by ζ_2.

We set $\alpha_1 = \lambda_1 |C_1|$ and $\alpha_2 = \lambda_\rho |D_2|$: thus, α_i is the mean number of children of a point in ζ_{i-1}.

Consider a random vector $X = (X_1, X_2)$ of points in $\mathbb{R}^d$ defined as follows: X_1 is taken uniformly in C_1, then X_2 is taken uniformly in $D_2(X_1)$. We think of X as a potential single branch of progeny of x_0. Let then $(X^j)_{j \geq 1}$ be independent copies of X. Let now N be an independent Poisson random variable with parameter α_1: this random variable gives the number $|\zeta_1|$ of children of x_0. We will use the N first X^j, one for each child of x_0.

We now take into account the fact that some individuals may have no children. We shall deal with geometric dependencies later. Let $Y = (Y^j)_{j \geq 1}$ be an independent family of independent random variables, such that Y^j follows the Bernoulli law with parameter $1 - \exp(-\alpha_2)$, which is the probability that a Poisson random variable with parameter α_2 is different from 0. We set $J_1 = \{1, \dots, N\}$ and

$$J_2 = \left\{1 \leq j \leq N : Y^j = 1\right\}.$$

Thus, the random set J_2 gives the superscripts of the individuals, among the N individuals of the first generation, that have at least one child in a process with no dependencies due to geometry.

To take into account the geometrical constraints between individuals, we set, for every $j \geq 1$,

$$Z^j = 1 \qquad \text{if } X_2^j \notin \bigcup_{j' \in J_1 \setminus \{j\}} D_2(X_1^{j'}) \quad \text{and} \quad Z^j = 0 \qquad \text{otherwise,}$$

$$\xi = \{X_2^j : j \in J_2 \text{ and } Z^j = 1\}.$$

We thus reject an individual X_2^j as soon as $Z^j = 0$. Recall that, when building generation 2 from generation 1, we explore the Poisson point processes in the area $\bigcup_{j \in J_1} D_2(X_1^j) \subset C_2$. Remember that by construction, C_1 and C_2 are disjoint. Therefore, one can check that the set ξ is stochastically dominated by ζ_2.[4] Thus, to prove Proposition 2.3, we now need to bound from below the probability that ξ is not empty.

Let T be the smallest integer j such that $Y^j = 1$: in other words, T is the smallest superscript of a branch that lives till generation 2. To ensure that $\xi \neq \varnothing$, it is sufficient that $T \leq N$ and that $Z^T = 1$. So,

$$1 - P(\mathcal{G}^+(x_0)) \leq P(\xi = \varnothing) \leq P(\# J_2 = 0) + P(\{T \leq N\} \cap \{Z^T = 0\}).$$

By construction,

$$\begin{aligned} P(T \leq N \text{ and } Z^T = 0) &= P(T \leq N, \exists j \in J_1 \setminus \{T\} \text{ such that } X_2^T \in D_2(X_1^j)) \\ &\leq \sum_{j \geq 1} P(T \leq N \text{ and } j \in J_1 \setminus \{T\} \text{ and } X_2^T \in D_2(X_1^j)) \\ &= \sum_{j \geq 1} E(1_{T \leq N} 1_{j \in J_1 \setminus \{T\}} P(X_2^T \in D_2(X_1^j) | Y, N)) \end{aligned}$$

[4]Note that the random set $\{X_1^1, \ldots, X_1^N\}$ has the same distribution as ζ_1. In order to build a random set with the same distribution as ζ_2, we could proceed as follows. Let $(N^j)_{j \geq 1}$ be independent random variable distributed according to the Poisson distribution with mean α_2. Throw N^1 random points uniformly in $D_2(X_1^1)$. Then throw N^2 random points uniformly in $D_2(X_1^2)$ and remove the points that fell in $D_2(X_1^1)$. Then throw N^2 random points uniformly in $D_2(X_1^3)$ and remove the points that fell in $D_2(X_1^1)$ or in $D_2(X_1^2)$, and so on. The random set of all the points thrown and not removed has the same distribution as ζ_2.

In the proof of Proposition 2.3, we reject more points than in this classical construction, thus only obtaining a stochastic domination:

First, we replace N^j by $\min(1, N^j)$ to keep at most one point X_2^j for each j (this is the role of Y^j).

Second, we reject this point X_2^j as soon as it falls into any of the $D_2(X_1^{j'})$ for $j' \neq j$ instead of only forbidding the $D_2(X_1^{j'})$ for $j' < j$ (this is the role of Z^j).

$$= \sum_{j\geq 1} E(1_{T\leq N} 1_{j\in J_1\setminus\{T\}}) P(X_2^1 \in D_2(X_1^2))$$
$$\leq E(\#J_1) P(X_2^1 \in D_2(X_1^2)) = E(N) P(X_2^1 \in D_2(X_1^2)).$$

Besides, as $(X_2^1)''$ is uniformly distributed on C_2'' and is independent of $(X_1^2)''$,

$$P(X_2^1 \in D_2(X_1^2)) = P((X_2^1)'' \in D_2''((X_1^2)'')) = \frac{|D_2''|}{|C_2''|}.$$

This leads to

$$1 - P(\mathcal{G}^+(x_0)) \leq P(\#J_2 = 0) + E(N)\frac{|D_2''|}{|C_2''|}. \tag{10}$$

N follows a Poisson law with parameter $\alpha_1 = \lambda_1 |C_1|$ with $\lambda_1 = \frac{\kappa^d}{v_d 2^d}$. Thus,

$$E(N) = \frac{\kappa^d}{2^d}\frac{|C_1|}{v_d}.$$

Lemmas 2.4 and 2.6 ensure that

$$\lim_{d\to+\infty}\frac{1}{d}\ln\left(\frac{|C_1|}{v_d}\right) = \ln(1+\rho) \quad \text{and} \quad \lim_{d\to+\infty}\frac{1}{d}\ln\left(\frac{|D_2''|}{|C_2''|}\right) = \ln\left(\frac{1}{\sqrt{2}}\right).$$

Thus, we have

$$\lim_{d\to+\infty}\frac{1}{d}\ln\left(E(N)\frac{|D_2''|}{|C_2''|}\right) \leq \ln\left(\frac{(1+\rho)\kappa}{2\sqrt{2}}\right) < 0 \qquad \text{since } \kappa < \frac{2\sqrt{2}}{1+\rho},$$

therefore,

$$\lim_{d\to+\infty} E(N)\frac{|D_2''|}{|C_2''|} = 0. \tag{11}$$

The cardinality of J_2 follows a Poisson law with parameter:

$$\eta = \alpha_1(1 - \exp(-\alpha_2)).$$

Remember that $\alpha_1 = \lambda_1|C_1|$, $\alpha_2 = \lambda_\rho|D_2|$, $\lambda_1 = \frac{\kappa^d}{v_d 2^d}$ and $\lambda_\rho = \frac{\kappa^d}{v_d 2^d \rho^d}$. By Lemma 2.6, we have the following limits:

$$\lim_{d\to+\infty}\frac{1}{d}\ln\alpha_1 = \ln\frac{\kappa(1+\rho)}{2} > 0,$$
$$\lim_{d\to+\infty}\frac{1}{d}\ln\alpha_2 = \ln\left(\kappa\frac{1+\rho}{2\rho}\right) < 0.$$

The first inequality is a consequence of $\kappa > \kappa_\rho^c$. The second inequality is a consequence of $\kappa < 1$. Consequently, we first see that

$$\lim_{d\to+\infty}\frac{1}{d}\ln(\eta) = \lim_{d\to+\infty}\frac{1}{d}\ln(\alpha_1\alpha_2) = \ln\left(\kappa^2\frac{(1+\rho)^2}{4\rho}\right) > 0,$$

therefore,

$$\lim_{d\to+\infty} P(\#J_2 = 0) = 0. \tag{12}$$

The inequality is a consequence of $\kappa > \kappa_\rho^c$.

To end the proof, we put estimates (12) and (11) in (10). □

2.3.3. *Several steps in the* 2*-dimensional oriented percolation model.* We prove here Proposition 2.2 by building the supercritical 2-dimensional oriented percolation process embedded in the two-type Boolean model.

PROOF OF PROPOSITION 2.2. We first define an oriented graph in the following manner: the set of sites is

$$S = \{(a,n) \in \mathbb{Z} \times \mathbb{N} : |a| \le n, a+n \text{ is even}\};$$

from any point $(a,n) \in S$, we put an oriented edge to $(a+1, n+1)$, and an oriented edge to $(a-1, n+1)$. We denote by $\vec{p}_c \in (0,1)$ the critical parameter for Bernoulli percolation on this oriented graph; see Durrett [5] for results on oriented percolation in dimension 2.

For any $(a,n) \in S$, we define the following subsets of $\mathbb{R}^d$:

$$\begin{aligned} W_{a,n} &= d^{-1/2}\big((a-1,a+1) \times (n-1,n) \times \mathbb{R}^{d-2}\big),\\ W_{a,n}^- &= d^{-1/2}\big((a-1,a) \times (n,n+1) \times \mathbb{R}^{d-2}\big),\\ W_{a,n}^+ &= d^{-1/2}\big((a,a+1) \times (n,n+1) \times \mathbb{R}^{d-2}\big). \end{aligned}$$

Note that the $(W_{a,n})_{(a,n)\in S}$ are disjoint and that $W_{a,n}^+ \cup W_{a+2,n}^- \subset W_{a+1,n+1}$.

We now fix $\kappa \in (\kappa_\rho^c, 1)$, and for $x_0 \in W_{a,n}$, we introduce the events:

$$\mathcal{G}_{a,n}^+(x_0) = \left\{ \begin{array}{c} \text{There exist } x_1 \in \chi_1 \cap W_{a,n}^+ \text{ and } x_2 \in \chi_\rho \cap W_{a,n}^+ \\ \text{such that } B(x_0,\rho) \cap B(x_1,1) \neq \varnothing \text{ and } B(x_1,1) \cap B(x_2,\rho) \neq \varnothing \end{array} \right\},$$

$$\mathcal{G}_{a,n}^-(x_0) = \left\{ \begin{array}{c} \text{There exist } x_1 \in \chi_1 \cap W_{a,n}^- \text{ and } x_2 \in \chi_\rho \cap W_{a,n}^- \\ \text{such that } B(x_0,\rho) \cap B(x_1,1) \neq \varnothing \text{ and } B(x_1,1) \cap B(x_2,\rho) \neq \varnothing \end{array} \right\}.$$

Note that $\mathcal{G}_{0,0}^+(x_0)$ is exactly the event $\mathcal{G}^+(x_0)$ introduced in (5), and that the other events are obtained from this one by symmetry and/or translation.

Next, we choose $p \in (\vec{p}_c, 1)$. With Proposition 2.3, and by translation and symmetry invariance, we know that for every large enough dimension d, for every $(a,n) \in S$, for every $x \in W_{a,n}$:

$$P\big(\mathcal{G}_{a,n}^\pm(x)\big) \ge p. \tag{13}$$

We fix then a dimension d large enough to satisfy (13). We can now construct the random states, open or closed, of the edges of our oriented graph. The aim is to build inductively some appropriate paths of balls from a ball centered at a point $x(0,0) \in W_{0,0}$ to balls centered at points $x(a,n) \in W_{a,n}$. In case of failure for a given (a,n), we find it convenient to set $x(a,n) = \infty$, where ∞ denotes a virtual site. In the end, useful paths will only use finite $x(a,n)$.

Definition of the site on level 0. Almost surely, $\chi_\rho \cap W_{0,0} \neq \varnothing$. We take then some $x(0,0) \in \chi_\rho \cap W_{0,0}$.

Definition of the edges between levels n and $n+1$. Fix $n \geq 0$ and assume we have built a site $x(a,n) \in W_{a,n} \cup \{\infty\}$ for every a such that $(a,n) \in S$. Consider $(a,n) \in S$:

- If $x(a,n) = \infty$: we decide that each of the two edges starting from (a,n) is open with probability p and closed with probability $1-p$, independently of everything else; we set $z^-(a,n) = z^+(a,n) = \infty$.
- Otherwise, $x(a,n) \in W_{a,n}$ and:
  - Edge to the left-hand side:
    * if the event $\mathcal{G}^-_{a,n}(x(a,n))$ occurs, we take for $z^-(a,n)$ some point $x_2 \in W^-_{a,n} \subset W_{a-1,n+1}$ given by the occurrence of the event, and we open the edge from (a,n) to $(a-1,n+1)$;
    * otherwise, we set $z^-(a,n) = \infty$ and we close the edge from (a,n) to $(a-1,n+1)$.
  - Edge to the right-hand side:
    * if the event $\mathcal{G}^+_{a,n}(x(a,n))$ occurs, we take for $z^+(a,n)$ some point $x_2 \in W^+_{a,n} \subset W_{a+1,n+1}$ given by the occurrence of the event, and we open the edge from (a,n) to $(a+1,n+1)$;
    * otherwise, we set $z^+(a,n) = \infty$ and we close the edge from (a,n) to $(a+1,n+1)$.

For (a,n) outside S, we set $z^\pm(a,n) = \infty$.

Definition of the sites at level $n+1$. Fix $n \geq 0$ and assume we determined the state of every edge between levels n and $n+1$. Consider $(a,n+1) \in S$:

- If $z^+(a-1,n) \neq \infty$: set $x(a,n+1) = z^+(a-1,n) \in W_{a,n+1}$.
- Otherwise:
  - if $z^-(a+1,n) \neq \infty$: set $x(a,n+1) = z^-(a+1,n) \in W_{a,n+1}$,
  - otherwise: set $x(a,n+1) = \infty$.

Assume that there exists an open path of length n starting from the origin in this oriented percolation. We can check that the leftmost open path of length n starting from the origin gives a path in the two-type Boolean model along which balls with radius 1 alternate with balls with radius ρ. Thus, percolation in this oriented percolation model implies alternating percolation in the two-type Boolean model. Let us check that percolation occurs indeed with positive probability.

For every n, denote by $\mathcal{F}_n$ the σ-field generated by the restrictions of the Poisson point processes χ_1 and χ_ρ to the set

$$d^{-1/2}\big(\mathbb{R} \times (-\infty, n) \times \mathbb{R}^{d-2}\big).$$

By definition of the events $\mathcal{G}$—remember that the $(W_{a,n})_{(a,n)\in S}$ are disjoint—and by (13), the states of the different edges between levels n and $n+1$ are independent conditionally to $\mathcal{F}_n$. Moreover, conditionally to $\mathcal{F}_n$, each edge between levels n and $n+1$ has a probability at least p to be open. Therefore, the oriented percolation model we built stochastically dominates Bernoulli oriented percolation with parameter p. As $p > \vec{p}_c$, with positive probability, there exists an infinite open path in the oriented percolation model we built; this completes the proof of Proposition 2.2. □

2.4. *Proof of Theorem* 1.2. If $\kappa < \kappa_\rho^c$ then, by Proposition 2.1, there is no percolation for d large enough. Therefore, for any such κ and for any large enough d we have

$$\lambda_d^c(\mu_d) \geq \frac{\kappa^d}{v_d 2^d} \quad \text{and then} \quad \widetilde{\lambda}_d^c(\mu_d) = \lambda_d^c(\mu_d) v_d 2^d \int r^d \mu_d(dr) \geq 2\kappa^d.$$

Letting d go to $+\infty$ and then κ go to κ_ρ^c, we then obtain

$$\liminf_{d\to+\infty} \frac{1}{d} \ln\big(\lambda_d^c(\mu_d)\big) \geq \ln\big(\kappa_\rho^c\big). \tag{14}$$

Choose now κ such that $\kappa_\rho^c < \kappa$. By Proposition 2.2, there is percolation for d large enough in Σ. Therefore, for any $\kappa > \kappa_\rho^c$ and for any large enough d we have, as before

$$\lambda_d^c(\mu_d) \leq \frac{\kappa^d}{v_d 2^d} \quad \text{and then} \quad \widetilde{\lambda}_d^c(\mu_d) \leq 2\kappa^d.$$

Letting d go to $+\infty$ and then κ go to κ_ρ^c, we then obtain

$$\limsup_{d\to+\infty} \frac{1}{d} \ln\big(\lambda_d^c(\mu_d)\big) \leq \ln\big(\kappa_\rho^c\big). \tag{15}$$

Bringing (14) and (15) together, we complete the proof of Theorem 1.2.

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