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SUPERMARTINGALES AS RADON–NIKODYM DENSITIES AND RELATED MEASURE EXTENSIONS

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Certain countably and finitely additive measures can be associated to a given nonnegative supermartingale. Under weak assumptions on the underlying probability space, existence and (non)uniqueness results for such measures are proven.

1. Introduction. It is a simple but very useful observation that a probability measure Q which is not absolutely continuous with respect to some reference measure P has a nonnegative P-supermartingale as its "Radon–Nikodym derivative." For instance, such supermartingales appear naturally in the generalization of Girsanov's theorem to measures without absolute continuity relation as in Yoeurp (1985), or when working with killed diffusions.

Conversely, given a nonnegative supermartingale, under suitable assumptions on the probability space, it is possible to reconstruct a measure associated to it, the so-called Föllmer measure. The behavior of the Föllmer measure characterizes the most important properties of the supermartingale; see Föllmer (1972, 1973); see also Ruf (2013a, 2013b), Cui (2013) and Larsson and Ruf (2014) for applications in the detection of strict local martingales. Further applications include, among others, potential theory [Airault and Föllmer (1974), Föllmer (1972)], simple proofs of the main semimartingale decomposition theorems [Föllmer (1973)], filtration enlargements [Kardaras (2012), Yoeurp (1985)], filtration shrinkage [Föllmer and Protter (2011), Larsson (2014)] and a simple approach to the study of conditioned measures [Perkowski and Ruf (2012)].

Measures associated to nonnegative supermartingales have also appeared naturally in the duality approach to stochastic control. The dual formulation has been developed for several important applications, such as utility maximization [see, among many others, Föllmer and Gundel (2006), Karatzas et al. (1991), Kramkov

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and Schachermayer (1999)] or super-replication of contingent claims [see, e.g., Jacka (1992), Ruf (2011)]. In many situations, the dual variables represent nonnegative supermartingales. As observed by Karatzas et al. (1991) [see also Kramkov and Schachermayer (1999)], only in very special situations do those supermartingales turn out to be martingales, in which case computations of the dual problem can be simplified via standard changes of measure.

In order to tackle the general case, more general changes of measure have been suggested. There are mainly two approaches, one relying on powerful arguments in functional analysis, and the other one relying on deep probabilistic insights. The functional analytic arguments identify supermartingales with the elements of the dual space of bounded measurable functions, the space of finitely additive measures; see, for example, Cvitanić, Schachermayer and Wang (2001) or Karatzas and Žitković (2003), but also Larsen and Žitković (2013). The probabilistic approach, on the contrary, relies on certain canonical assumptions on the underlying probability space but allows the identification of supermartingales with countably additive probability measures; see, for example, Föllmer and Gundel (2006).

Recently, there has been an increased interest in economically meaningful asset price models in which local martingales and supermartingales and their interpretation as putative changes of measure appear naturally. For example, there are models that allow for certain arbitrage opportunities but have associated to them a class of dual supermartingales [Karatzas and Kardaras (2007), Platen (2006), Platen and Heath (2006), Ruf and Rungaldier (2014)]. The dual supermartingales then correspond either to weakly equivalent finitely additive local martingale measures [Kardaras (2010)] or to dominating local martingale measures, which turn out to be the appropriate pricing operators in this context [Fernholz and Karatzas (2010), Imkeller and Perkowski (2013), Ruf (2013c)]. Furthermore, underlying asset prices have been modeled as strict local martingales under a pricing measure and the corresponding associated measures have been constructed in order to model certain phenomena, such as bubbles [Kardaras, Kreher and Nikeghbali (2015), Pal and Protter (2010)] or explosive exchange rates [Carr, Fisher and Ruf (2014, 2013)], and in order to compute actual quantities in such models. Short-selling constraints lead directly to models in which asset prices follow supermartingale dynamics [Pulido (2014)] and changes of numéraires in such models correspond to supermartingales as Radon–Nikodym derivatives.

It is thus of great interest to construct the measure associated to a given supermartingale Z. There are several different constructions that all require different assumptions, and some of which only work on an extended probability space:

- For a general filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with a nonnegative supermartingale Z, it is possible to construct a finitely additive measure on $(\Omega \times [0,\infty], \mathscr{A})$, where $\mathscr{A} \subset \mathscr{P}$ is a suitable algebra, and where $\mathscr{P}$ denotes the predictable sigma algebra; see Metivier and Pellaumail (1975). Without further assumptions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$, this measure can be extended to a countably additive measure on $(\Omega \times [0, \infty], \mathscr{P})$ if and only if Z is of class (D), in which case one obtains the measure of Doléans (1968); see also Meyer (1972).

- Under certain topological assumptions on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, it is possible to construct the Föllmer measure on the enlarged space $(\Omega \times [0, \infty], \mathscr{P})$. We refer to Föllmer (1972), Meyer (1972) and Stricker (1975) for three different constructions. Under appropriate conditions, it is also possible to construct the Föllmer measure on $(\Omega, \mathcal{F}_{\zeta-})$, where ζ is a certain stopping time, and not on an enlarged space [Azéma and Jeulin (1976), Delbaen and Schachermayer (1995), Meyer (1972), Moy (1953)].
- Taking a different approach, if Z is the pointwise limit of a family of uniformly integrable martingales, then there exists a finitely additive measure associated to it [Cvitanić, Schachermayer and Wang (2001), Karatzas and Žitković (2003)]. However, so far it seems to be not very well understood under which conditions the supermartingale Z is such a pointwise limit of martingales.

In this work, we contrast the last two approaches of associating countably and finitely additive measures to supermartingales. In the countably additive case, we prove the existence of a probability measure such that a given supermartingale Z can be interpreted as Radon–Nikodym density of this measure. In particular, we show that this probability measure can already be constructed on the canonical space itself and not only on the product space, even if the supermartingale Z is not a local martingale. Moreover, we provide precise necessary and sufficient conditions for the uniqueness of such a probability measure associated to the supermartingale Z.

In the finitely additive case, we show the existence of a finitely additive measure associated to the supermartingale Z, as long as the underlying filtered probability space is sufficiently rich, that is, as long as it supports a Brownian motion. Furthermore, we show that such a finitely additive measure is never unique. The argument for the existence of such finitely additive measures also yields the existence of uniformly integrable martingales that Fatou converge to a given supermartingale.

Structure of the paper. The paper is structured as follows: we conclude the Introduction by a short overview of the notation used in the following. Section 2 introduces the notion of the Föllmer measure associated to a supermartingale. Section 3 contains the main results concerning existence and (non)uniqueness of Föllmer measures. Sections 4 and 5 then consist of the proofs of those results and of some pedagogical examples.

Appendix A reviews modifications of processes if the filtration is not augmented by null sets. Appendix B recalls results concerning the multiplicative decomposition of a supermartingale, which will be used in the proofs of Sections 4 and 5. Appendix C provides a collection of definitions and results concerning relevant measure-theoretic spaces. Appendix D discusses results concerning the extension of measures and Appendix E lists important properties of the canonical path space.

Appendix F shows how the approximation techniques used to prove the existence of the Föllmer finitely additive measure can be modified to show that any supermartingale can be approximated, in the sense of Fatou, by uniformly integrable martingales, provided the underlying probability space supports a Brownian motion. Appendix G extends the discussion of nonuniqueness of the Föllmer finitely additive measure and illustrates that the uniqueness claim of the Carathéodory extension theorem does not hold among the class of finitely additive measures. Finally, Appendix H provides an alternative and insightful proof of one lemma concerning the nonuniqueness of the Föllmer finitely additive measure.

Basic notation. We shall use the convention that $\inf \varnothing = \infty$ and $\infty \times \mathbf{1}_A(\omega) = 0$ for all $\omega \notin A$, where A denotes some event. Expectations under a probability measure R are denoted by $\mathbb{E}_R[\cdot]$. Equalities between processes are to be understood up to indistinguishability, and statements such as "G is a càdlàg process" or "G is nonnegative" mean that these properties hold almost surely—unless explicitly stated otherwise. We shall assume that all considered semimartingales are (almost surely) càdlàg. If $G = (G_t)_{t\geq 0}$ is a làg process, then we denote by $G_- = (G_{t-})_{t\geq 0}$ its left limit process, that is, $G_{0-} = G_0$ and $G_{t-} = \limsup_{s\uparrow t} G_s \mathbf{1}_{\limsup_{s\uparrow t} G_s<\infty} + 0 \times \mathbf{1}_{\limsup_{s\uparrow t} G_s=\infty}$ for all $t > 0$, and by $\Delta G = G - G_-$ the jump process of the process G. Similarly, if G is a làd process, we denote by $G_+ = (G_{t+})_{t\geq 0}$ its right limit process.

Throughout this paper, we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where Ω denotes a nonempty set, $\mathcal{F}$ a sigma algebra on Ω, $(\mathcal{F}_t)_{t\geq 0}$ a right-continuous filtration with $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \geq 0$, and P a probability measure. We set $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t \subset \mathcal{F}$. Moreover, we are given a nonnegative, right-continuous P-supermartingale $Z = (Z_t)_{t\geq 0}$ with $E_P[Z_0] = 1$. We stress the fact that $(\mathcal{F}_t)_{t\geq 0}$ will not always be complete with respect to P; see Appendix A for a discussion of this critical point. We shall always silently assume that the notions of martingales, supermartingales, etc., hold with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

2. Föllmer measures associated to a supermartingale. We like to think of the P-supermartingale Z as the "Radon–Nikodym density" of a probability measure Q that is not necessarily absolutely continuous with respect to P; also P is not necessarily absolutely continuous with respect to Q. Our aim is to recover the measure Q.

On a general probability space, for example, one with a completed filtration, such a probability measure Q does not always exist on $(\Omega, \mathcal{F})$—except if Z is a uniformly integrable martingale. However, as we will show, if $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ is the space of (possibly explosive) right-continuous paths with left limits along with the canonical filtration, such a probability measure always exists. Moreover, without such a canonical assumption, but under the assumption that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ supports some Brownian motion, we shall see that it is still possible to associate a finitely additive measure Q to the P-supermartingale Z.

In the following, we make precise the meaning of "measures associated to supermartingales."

2.1. *Föllmer countably additive measures.* Here, we explain in which way a supermartingale can be interpreted as the Radon–Nikodym derivative of a countably additive probability measure. We begin with the following definition.

DEFINITION 2.1. If Q and τ are a probability measure and a stopping time on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$, then (Q, τ) is called a *Föllmer pair* for Z if

$$\begin{aligned} & P[\tau=\infty]=1 \quad \text{and} \\ (1) \qquad & Q\big[A \cap \{\rho<\tau\}\big] = \mathbb{E}_P[Z_\rho \mathbf{1}_A] \\ & \qquad \text{for all } A \in \mathcal{F}_\rho \text{ and finite stopping times } \rho. \end{aligned}$$

In that case, we also call Q a *Föllmer* (*countably additive*) *measure* for (Z, τ), or, slightly abusing notation, a *Föllmer* (*countably additive*) *measure* for Z.

Note that every Föllmer measure Q is defined on $(\Omega, \mathcal{F})$, despite the fact that (1) only involves the restriction of Q to $\mathcal{F}_\infty \subset \mathcal{F}$. If (Q, τ) is a Föllmer pair for the P-supermartingale Z then the pair (Z, τ) is called the *Kunita–Yoeurp decomposition* of Q with respect to P. In that case, the two measures $Q|_{\mathcal{F}_t}[\cdot \cap \{t \geq \tau\}]$ and $P|_{\mathcal{F}_t}[\cdot]$ are singular for each $t \geq 0$ as the second one has full mass on the event $\{\tau = \infty\}$ while the first one assigns measure zero to it. Hence, the Kunita–Yoeurp decomposition can be interpreted as a progressive Lebesgue decomposition on filtered probability spaces. The decomposition was introduced by Kunita (1976) for Markov processes. The general formulation is due to Yoeurp (1985), who used it to prove a generalized Girsanov theorem for probability measures that are not necessarily absolutely continuous with respect to each other.

Given two probability measures P and Q, where Q has the Kunita–Yoeurp decomposition (Z, τ) with respect to P, the stopping time τ is uniquely determined up to a Q-null set, and the P-supermartingale Z is uniquely determined up to a P-evanescent set; see Proposition 2 in Yoeurp (1985). As Theorem 3.1 below yields, it might well be possible, however, to associate two different Föllmer countably additive measures to a given P-supermartingale Z.

DEFINITION 2.2. We say that the Föllmer pair for Z is *unique* if given two probability measures Q and $\widetilde{Q}$ and two stopping times τ and $\widetilde{\tau}$ such that (Q, τ) and $(\widetilde{Q}, \widetilde{\tau})$ both satisfy (1), we have $Q = \widetilde{Q}$ (and $Q[\tau = \widetilde{\tau}] = 1$).

If τ is a stopping time, then we say that the Föllmer (countably additive) measure for (Z, τ) is *unique* if, given two probability measures Q and $\widetilde{Q}$ such that (Q, τ) and $(\widetilde{Q}, \tau)$ both satisfy (1), we have $Q = \widetilde{Q}$.

In order to verify whether a given probability measure is a Föllmer countably additive measure for the P-supermartingale Z, it suffices to verify (1) for deterministic times.

PROPOSITION 2.3. *If Q and τ are a probability measure and a stopping time on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ such that $P[\tau = \infty] = 1$ and $Q[A \cap \{t < \tau\}] = \mathbb{E}_P[Z_t \mathbf{1}_A]$ for all $t \geq 0$ and all $A \in \mathcal{F}_t$, then (Q, τ) is a* Föllmer pair *for Z. Moreover,*

$$\mathbb{E}_Q[G\mathbf{1}_{\{\rho<\tau\}}] = \mathbb{E}_P[Z_\rho G\mathbf{1}_{\{\rho<\infty\}\cap\{Z_\rho>0\}}] \tag{2}$$

holds for all $[0, \infty]$-valued, $\mathcal{F}_\rho$-measurable random variables G and for all stopping times ρ; in particular, $Q[\rho < \tau] = \mathbb{E}_P[Z_\rho]$ for all finite stopping times ρ.

PROOF. Using the linearity of the expectation operator and the monotone convergence theorem, it is sufficient to show (2) for a fixed stopping time ρ, with $G = \mathbf{1}_A$ for an arbitrary $A \in \mathcal{F}_\rho$. If ρ takes only countably many values, then this identity follows directly. For the general case, consider the nonincreasing sequence $(\rho_n)_{n\in\mathbb{N}}$ of stopping times where $\rho_n = \inf\{k2^{-n} : k2^{-n} \geq \rho, k \in \mathbb{N}\}$. Then (2) holds with G replaced by $\mathbf{1}_A$ and with ρ replaced by ρ_n for each $n \in \mathbb{N}$. Finally, by taking limits on both sides and using the fact that the nonnegative, discrete-time backward P-supermartingale $(Z_{\rho_n})_{n\in\mathbb{N}}$ is uniformly integrable, we may conclude. □

COROLLARY 2.4. *If (Q, τ) is a Föllmer pair for Z then the following two statements hold*:

- $P|_{\mathcal{F}_\infty} \ll Q|_{\mathcal{F}_\infty}$ *if and only if* $P[\lim_{t\uparrow\infty} Z_t > 0] = 1$;
- $Q|_{\mathcal{F}_\infty} \ll P|_{\mathcal{F}_\infty}$ *if and only if* $\mathbb{E}_P[\lim_{t\uparrow\infty} Z_t] = 1$.

PROOF. Let $A = \{\lim_{t\uparrow\infty} Z_t = 0\} \cap \{\tau = \infty\}$ and note that $Q[A] = 0$. This and the fact that $P[\tau = \infty] = 1$ yield the first "only if" implication. For the reverse direction, let $B \in \bigcup_{t\geq 0} F_t$ and observe that

$$Q[B] \geq \lim_{t\uparrow\infty} Q\big[B \cap \{\tau > t\}\big] = \lim_{t\uparrow\infty} \mathbb{E}_P[1_B Z_t] \geq \mathbb{E}_P\Big[1_B \lim_{t\uparrow\infty} Z_t\Big].$$

By the monotone class theorem, this extends to all $B \in \mathcal{F}_\infty$. Since

$$P\Big[\lim_{t\uparrow\infty} Z_t > 0\Big] = 1$$

by assumption, we deduce that $P|_{\mathcal{F}_\infty} \ll Q|_{\mathcal{F}_\infty}$. The second equivalence follows from Proposition III.3.5 Jacod and Shiryaev (2003). □

The following observation describes the dynamics of the P-supermartingale Z under the Föllmer measure.

PROPOSITION 2.5. *If (Q, τ) is a Föllmer pair for Z then the process $Y = (Y_t)_{t \geq 0}$, given by $Y_t = 1/Z_t \mathbf{1}_{\{t<\tau\}}$, is a (nonnegative) Q-supermartingale. Moreover, the following two statements hold*:

- *Y is a Q-local martingale if and only if Z does not jump to zero under P*;
- *Y is a Q-martingale if and only if $P[Z_t > 0] = 1$ for all $t \geq 0$.*

PROOF. The statement follows from (2) and a version of Bayes' rule; for details, see the proof of Theorem 2.1 in Carr, Fisher and Ruf (2014). □

2.2. *Föllmer finitely additive measures.* Recall that $\mathrm{ba}(\Omega, \mathcal{F})$ is the space of bounded, finitely additive set functions on $\mathcal{F}$ that take their values in $\mathbb{R}$. An element $Q \in \mathrm{ba}(\Omega, \mathcal{F})$ is called a *finitely additive probability measure* if it is nonnegative and satisfies $Q[\Omega] = 1$. In that case, Q can be uniquely decomposed into a regular part $Q^r \geq 0$ and a singular part $Q^s \geq 0$; see Theorem III.7.8 in Dunford and Schwartz (1958). Here, Q^r is a sigma-additive measure on $(\Omega, \mathcal{F})$, and Q^s is purely finitely additive, that is, any sigma-additive measure μ on $\mathcal{F}$ which satisfies $0 \leq \mu \leq Q^s$ is constantly 0.

If Q and R are two finitely additive measures, then Q is said to be *weakly absolutely continuous* with respect to R if for all $A \in \mathcal{F}$ we have that $R[A] = 0$ implies $Q[A] = 0$; see Remark 6.1.2 in Bhaskara Rao and Bhaskara Rao (1983). We shall write $\mathrm{ba}(\Omega, \mathcal{F}, P)$ for the space of all finitely additive measures on $\mathcal{F}$ that are weakly absolutely continuous with respect to P; we write $\mathrm{ba}_1(\Omega, \mathcal{F}, P)$ for all nonnegative elements of $\mathrm{ba}(\Omega, \mathcal{F}, P)$ that have total mass one.

DEFINITION 2.6. A weakly absolutely continuous, finitely additive probability measure $Q \in \mathrm{ba}_1(\Omega, \mathcal{F}, P)$, such that

$$(Q|_{\mathcal{F}_\rho})^r[A] = \mathbb{E}_P[Z_\rho \mathbf{1}_A] \qquad \text{for all } A \in \mathcal{F}_\rho \text{ and finite stopping times } \rho, \tag{3}$$

is called *Föllmer finitely additive measure* for Z.

Recall that the dual space $L^\infty(\Omega, \mathcal{F}, P)^*$ of $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ can be identified with the elements of $\mathrm{ba}(\Omega, \mathcal{F}, P)$; see Theorem IV.8.16 in Dunford and Schwartz (1958). This is the reason why finitely additive probability measures naturally appear in the dual approach to stochastic optimization problems. For further details, see also Cvitanić, Schachermayer and Wang (2001) and Karatzas and Žitković (2003).

In Example 5.11 below, we construct a P-supermartingale Z and two finitely additive probability measures $Q_1, Q_2 \in \mathrm{ba}_1(\Omega, \mathcal{F}, P)$ that satisfy $(Q_1|_{\mathcal{F}_t})^r = (Q_2|_{\mathcal{F}_t})^r$ for all $t \geq 0$. Moreover, Q_1 satisfies (3), and there exists a finite stopping time ρ such that

$$(Q_1|_{\mathcal{F}_\rho})^r = 0 \neq P|_{\mathcal{F}_\rho} = (Q_2|_{\mathcal{F}_\rho})^r.$$

Therefore, there is no result corresponding to Proposition 2.3 in the finitely additive case. To wit, if a finitely additive measure Q satisfies (3) for deterministic times, then this does not automatically imply that Q satisfies (3).

2.3. *Comparison of Föllmer countably and finitely additive measures.* We have introduced two different notions of Föllmer measures, and it is natural to ask how these two concepts are related. As it turns out, in most situations they are mutually exclusive, despite their apparent similarity.

PROPOSITION 2.7. *If Z is a uniformly integrable P-martingale and if $\mathcal{F} = \mathcal{F}_\infty$, then each Föllmer countably additive measure for Z is a Föllmer finitely additive measure for Z. If Z is not a uniformly integrable P-martingale, then the sets of Föllmer countably additive measures for Z and of Föllmer finitely additive measures for Z are disjoint.*

PROOF. The statement follows from the second equivalence in Corollary 2.4. □

We shall see in Theorem 3.7 below and also in Appendix G that, in the case of a uniformly integrable martingale Z, the class of Föllmer finitely additive measures is strictly larger than the class of Föllmer countably additive measures, as long as the probability space is sufficiently rich. However, in general, the existence of a Föllmer countably additive measure does not imply the existence of a Föllmer finitely additive measure, nor does the opposite implication hold.

EXAMPLE 2.8. Assume that Z is a P-local martingale which is not a uniformly integrable P-martingale and assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is augmented by all P-null sets in $\mathcal{F}$. In Example 5.3 below, we show that there exists a Föllmer finitely additive measure for Z. However, since $\mathbb{E}_P[Z_0] = 1$ holds, any Föllmer countably additive measure Q for Z is absolutely continuous with respect to P on the sigma algebra $\mathcal{F}_0$. Since $\mathcal{F}_0$ contains all P-null sets, Q is absolutely continuous with respect to P, which is only possible if Z is a uniformly integrable P-martingale. Thus, there exists no Föllmer countably additive measure for the P-local martingale Z.

This example illustrates why we shall assume an incomplete filtration when constructing Föllmer countably additive measures below. If the P-supermartingale Z is a martingale, we could also assume a filtration that is enlarged in a progressive manner; see Bichteler (2002) or Najnudel and Nikeghbali (2011) for details. If Z is a local martingale, then it is still possible, by using a localization sequence, to perform such a progressive enlargement; see Kreher and Nikeghbali (2013); however, in that case the filtration depends on the local martingale Z itself. If Z is only a P-supermartingale and not a local martingale, then finding a progressive completion that would allow us to construct Föllmer countably additive measures for Z seems impossible. We continue this discussion on the issue of completing the filtration in Remark 3.2 below.

EXAMPLE 2.9. Assume that $\Omega = \{0, 1\}$, $\mathcal{F} = \sigma(\{0\}) = \mathcal{F}_t$ for all $t \geq 1$, $\mathcal{F}_t = \{\emptyset, \Omega\}$ for all $t \in [0, 1)$ and $P[\{0\}] = 1$. Moreover, assume that Z satisfies $Z_t = \mathbf{1}_{t<1}$ for all $t \geq 0$. Consider the probability measure Q that satisfies $Q[\{1\}] = 1$ and the stopping time $\tau = 1 + \infty \mathbf{1}_{\{0\}}$, which satisfies $P[\tau = \infty] = 1 = Q[\tau = 1]$. Then (Q, τ) is a Föllmer pair for the P-supermartingale Z. However, there is no Föllmer finitely additive probability measure for the P-supermartingale Z since P is the only finitely additive probability measure on $(\Omega, \mathcal{F})$ which is weakly absolutely continuous with respect to P.

3. Existence and (non)uniqueness. We next collect the main results of this paper concerning existence and (non)uniqueness of Föllmer countably and finitely additive measures associated to the nonnegative P-supermartingale Z.

3.1. *The countably additive case.* In the countably additive case, we shall rely on a specific choice of a canonical probability space. This motivates us to formulate the following assumption (we recall the definition of several measure-theoretic notions such as "state space" in Appendix C).

ASSUMPTION $(\mathcal{P})$. Let E be a state space, and let $\Delta \notin E$ be a cemetery state. For all $\omega \in (E \cup \{\Delta\})^{[0,\infty)}$ define

$$\zeta(\omega) = \inf\{t \geq 0 : \omega(t) = \Delta\}.$$

Let $\Omega \subset (E \cup \{\Delta\})^{[0,\infty)}$ be the space of paths $\omega : [0, \infty) \to E \cup \{\Delta\}$, for which ω is càdlàg on $[0, \zeta(\omega))$, and for which $\omega(t) = \Delta$ for all $t \geq \zeta(\omega)$. For all $t \geq 0$ define $X_t(\omega) = \omega(t)$ and the sigma algebras $\mathcal{F}_t^0 = \sigma(X_s : s \in [0, t])$ and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^0$. Moreover, set $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t^0 = \bigvee_{t\geq 0} \mathcal{F}_t = \mathcal{F}_\infty$.

Thus, under Assumption $(\mathcal{P})$, the states of the world $\omega \in \Omega$ are paths taking values in a state space up to a certain time when they get absorbed in a cemetery state Δ. Before the time of absorption, those paths are assumed to be càdlàg. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is the right-continuous modification of the canonical filtration $(\mathcal{F}_t^0)_{t\geq 0}$.

If ρ is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time, then the sigma algebra $\mathcal{F}_{\rho-}$ is defined as

$$\mathcal{F}_{\rho-} = \mathcal{F}_0^0 \vee \sigma\big(A \cap \{\rho > t\} : A \in \mathcal{F}_t, t \geq 0\big). \tag{4}$$

For later use, note that ρ is $\mathcal{F}_{\rho-}$-measurable. This definition is slightly different from the usual one, where $\mathcal{F}_0^0$ would be replaced by $\mathcal{F}_0$ in (4). The definition in (4), taken from Föllmer (1972), has the advantage that $\mathcal{F}_{\rho-}$ is countably generated, as Lemma E.1 will show, which collects several properties of the probability space in Assumption $(\mathcal{P})$.

We define the nondecreasing sequence $(\widehat{\tau}_n^Z)_{n\in\mathbb{N}}$ of stopping times and the stopping time $\widehat{\tau}^Z$ by

$$\widehat{\tau}_n^Z = \inf\{t \geq 0 : Z_t \geq n\} \wedge n; \qquad \widehat{\tau}^Z = \lim_{n\uparrow\infty} \widehat{\tau}_n^Z. \tag{5}$$

Before we get to the (somewhat subtle) precise formulation of our results, let us describe them informally. We show that, under Assumption ($\mathcal{P}$), it is possible to construct a Föllmer countably additive measure for the P-supermartingale Z on the space $(\Omega, \mathcal{F})$, as long as $\mathbb{E}_P[Z_\zeta \mathbf{1}_{\{\zeta<\infty\}}] = 0$ or Z is a local martingale. In particular, this is the case if P satisfies $P[\zeta < \infty] = 0$. Essentially, the Föllmer countably additive measure of Z is unique if Z is a martingale (not necessarily uniformly integrable), or if Z is a local martingale which explodes at time ζ and not before.

If the P-supermartingale Z has a nontrivial part of finite variation, then we have to artificially make Q lose mass to obtain a Föllmer countably additive measure for Z. Since we have a degree of freedom here in choosing where to send the mass of Q, it is not surprising that in this case we never have uniqueness—except possibly if the state space E is countable. Of course, if we fix the stopping time τ in the Föllmer pair for the P-supermartingale Z, and hereby implicitly specify where we send the mass of Q, then it is also possible to have uniqueness if Z is a P-supermartingale. In particular, once the stopping time τ is fixed, Q is always uniquely determined on $\mathcal{F}_{\tau-}$ and we only have to study under which conditions there exists a unique extension to $\mathcal{F}$.

THEOREM 3.1. *Under Assumption* ($\mathcal{P}$), *suppose that one* (*or both*) *of the following conditions hold*:

- *the P-supermartingale Z is a P-local martingale*;
- *the probability measure P satisfies $\mathbb{E}_P[Z_\zeta \mathbf{1}_{\{\zeta<\infty\}}] = 0$.*

Then there exists a Föllmer pair (Q, τ) for Z. If the P-supermartingale Z is a P-local martingale, then we can take $\widehat{\tau}^Z$, defined in (5), *as the stopping time*; *to wit*, *in that case there exists a Föllmer countably additive measure $\widehat{Q}^Z$ for Z such that $(\widehat{Q}^Z, \widehat{\tau}^Z)$ is a Föllmer pair. Moreover, the following statements always hold*:

(I) *The following conditions are equivalent*:

(a) *the set $\{\tau < \zeta\}$ is $Q|_{\mathcal{F}_{\tau-}}$-negligible*;
(b) *there is a unique Föllmer countably additive measure for (Z, τ).*

(II) *If $\widetilde{\tau}$ is a stopping time such that the pair $(Q, \widetilde{\tau})$ also satisfies* (1), *then $Q[\tau = \widetilde{\tau}] = 1$.*

(III) *The following statement in* (c) *always implies the one in* (d). *The reverse implication holds provided that the state space E is uncountable.*

(c) *The P-supermartingale Z is a P-local martingale and the set $\{\widehat{\tau}^Z < \zeta\}$ is $\widehat{Q}^Z|_{\mathcal{F}_{\widehat{\tau}^Z-}}$-negligible*;
(d) *there is a unique Föllmer pair for Z.*

REMARK 3.2. Azéma and Jeulin (1976) also show the existence of a Föllmer countably additive measure Q for (Z, ζ) on $(\Omega, \mathcal{F})$. Their construction is quite different from the one presented below and does not address the question of uniqueness: after fixing the stopping time ζ, there exists at most one probability measure Q for which (Q, ζ) is a Föllmer pair; see point (I) in the previous theorem.

Azéma and Jeulin (1976) construct the countably additive Föllmer measure directly on the universal completion $(\Omega, \mathcal{F}^u)$. Indeed, if we construct Q according to Theorem 3.1 and then augment $\mathcal{F}_t$ with the intersection of P- and Q-nullsets for all $t \geq 0$ to obtain a filtration $(\mathcal{F}_t^{P+Q})_{t\geq 0}$ and also a sigma algebra $\mathcal{F}^{P+Q}$, the two probability measures Q and P can be uniquely extended to $(\Omega, \mathcal{F}^{P+Q})$ by Lemma E.1 and Theorem D.4 in the Appendix D. Thus, in particular, Theorem 3.1 also yields the existence of a Föllmer countably additive measure on the universally augmented space $(\Omega, \mathcal{F}^u)$. Note, however, that the universally completed filtration $(\mathcal{F}_t^u)_{t\geq 0}$ still misses some of the nice properties of complete filtrations: for example, it is not clear that supermartingales have identically càdlàg modifications under $(\mathcal{F}_t^u)_{t\geq 0}$.

The proof of the uniqueness statement in Theorem 3.1 relies on the following observation.

LEMMA 3.3. *Assume that Z is a nonnegative P-local martingale. Then the Föllmer pair $(\widehat{Q}^Z, \widehat{\tau}^Z)$ from Theorem 3.1 is minimal in the following sense. If (Q, τ) is another Föllmer pair then $Q|_{\mathcal{F}_{(\widehat{\tau}^Z \vee \tau)-}}$ is uniquely determined by $Q|_{\mathcal{F}_{\tau-}}$, and $Q[\widehat{\tau}^Z = \tau] = 1$. In particular, we have $Q|_{\mathcal{F}_{\widehat{\tau}^Z-}} = \widehat{Q}^Z|_{\mathcal{F}_{\widehat{\tau}^Z-}}$.*

Section 4 contains the proofs of Theorem 3.1 and Lemma 3.3.

REMARK 3.4. The pair $(\widehat{Q}^Z, \widehat{\tau}^Z)$ of Lemma 3.3 is minimal in the sense of Lemma 3.3, but usually not unique. For example, consider the canonical probability space of Assumption $(\mathcal{P})$ with $E = [0, \infty)$ equipped with two measures Q_1 and Q_2 where Q_1 makes the canonical process a Brownian motion stopped when hitting zero and Q_2 makes the canonical process a Brownian motion killed when hitting zero; that is, if ρ_1 denotes the first hitting time of zero by the canonical process, then $Q_1[\rho_1 < \infty] = 1 = Q_2[\zeta < \infty]$ and $Q_1[\zeta < \infty] = 0 = Q_2[\rho_1 < \infty]$.

Now, if the canonical process is a three-dimensional Bessel process under the probability measure P and if Z denotes its reciprocal, then it is easily verified that both (Q_1, ρ_1) and (Q_2, ζ) satisfy (1). However, those two pairs clearly do not agree. The minimal pair $(\widehat{Q}^Z, \widehat{\tau}^Z)$ of Lemma 3.3, where $\widehat{Q}^Z$ is a-priori only defined on $\mathcal{F}_{\widehat{\tau}^Z-}$, can be extended to $\mathcal{F}$ either by Q_1 or Q_2 (or other measures).

The following proposition provides an important sufficient criterion for the uniqueness of the Föllmer countably additive measure in Theorem 3.1.

PROPOSITION 3.5. *In the setup of Theorem 3.1, if the nonnegative P-supermartingale Z is a P-martingale, then $\widehat{Q}^Z[\widehat{\tau}^Z = \infty] = 1$; in particular, then the set $\{\widehat{\tau}^Z < \zeta\}$ is $\widehat{Q}^Z|_{\mathcal{F}_{\widehat{\tau}^Z -}}$-negligible and there is a unique Föllmer pair for Z.*

PROOF. If Z is a P-martingale, then

$$\begin{aligned}\widehat{Q}^Z[\widehat{\tau}^Z < \infty] &= \lim_{n\uparrow\infty} \widehat{Q}^Z[\widehat{\tau}^Z \le n] = \lim_{n\uparrow\infty}\big(1 - \widehat{Q}^Z[\widehat{\tau}^Z > n]\big) \\ &= \lim_{n\uparrow\infty}\big(1 - \mathbb{E}_P[Z_n]\big) = 0,\end{aligned}$$

which completes the proof. □

The next result contains a discussion of the missing implication from (d) to (c) in Theorem 3.1 if the state space E is countable.

PROPOSITION 3.6. *Under Assumption ($\mathcal{P}$), suppose that the state space E is countable. Then we can distinguish the following cases:*

(A) *If E has exactly one element and if $P[\zeta < \infty] = 0$ then there is a unique Föllmer pair for each nonnegative P-supermartingale Z. However, if $P[\zeta < \infty] > 0$ then the Föllmer pair is not necessarily unique.*

(B) *If E has more than one element, then:*

(i) *there exists a probability measure P on the sigma algebra $\mathcal{F}$, with $P[\zeta < \infty] = 0$, such that for each P-supermartingale Z that is not a P-local martingale there are at least two different Föllmer pairs for Z;*

(ii) *there exists a probability measure P on the sigma algebra $\mathcal{F}$, with $P[\zeta < \infty] = 0$, and a P-supermartingale Z that is not a P-local martingale such that there is a unique Föllmer pair for Z.*

The proof of Proposition 3.6 can be found in Section 4. The proposition implies, in particular, that the implication from (d) to (c) in Theorem 3.1 requires E to be uncountable.

3.2. *The finitely additive case.* In the finitely additive case, we assume that the underlying probability space is sufficiently large to support a Brownian motion. If that assumption holds then it is possible to associate a Föllmer finitely additive measure to any nonnegative P-supermartingale.

ASSUMPTION ($\mathcal{B}$). The filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ supports a Brownian motion $W = (W_t)_{t\ge 0}$.

An assumption that the underlying probability space is sufficiently large, such as Assumption ($\mathcal{B}$), is clearly necessary. For example, if the sigma algebra $\mathcal{F}$ is of finite cardinality, then any finitely additive probability measure is automatically countably additive. So if P charges every nonempty element of $\mathcal{F}$, then one cannot have a Föllmer finitely additive measure for a P-supermartingale Z that is not a P-martingale.

THEOREM 3.7. *Under Assumption* ($\mathcal{B}$), *there exists a Föllmer finitely additive measure for the* P*-supermartingale* Z. *The Föllmer finitely additive measure is never unique.*

Section 5 contains the proof of Theorem 3.7. Note that a similar proof also yields that, under Assumption ($\mathcal{B}$), the P-supermartingale Z can be approximated, in the sense of Fatou convergence, by a sequence of uniformly integrable nonnegative martingales; see Theorem F.2 in the Appendix F.

Observe that the stopping times in Definition 2.6 were assumed to be finite. We might also consider the extended P-supermartingale $\overline{Z} = (\overline{Z}_t)_{t\in\infty}$ with $\overline{Z}_t = Z_t$ for all $t \geq 0$ and with an $\mathcal{F}_\infty$-measurable $\overline{Z}_\infty \in [0, \lim_{t\uparrow\infty} Z_t]$; note that the limit exists by the supermartingale convergence theorem. This observation then motivates the following definition.

DEFINITION 3.8. A weakly absolutely continuous, finitely additive probability measure $Q \in \mathrm{ba}_1(\Omega, \mathcal{F}, P)$, such that

$$(Q|_{\mathcal{F}_\rho})^r[A] = \mathbb{E}_P[\overline{Z}_\rho \mathbf{1}_A] \qquad \text{for all } A \in \mathcal{F}_\rho \tag{6}$$
$$\text{and (possibly infinitely-valued) stopping times } \rho,$$

is called *extended Föllmer finitely additive measure* for $\overline{Z}$.

We obtain a similar statement as in Theorem 3.7; again, the proof of the following theorem is provided in Section 5.

THEOREM 3.9. *Under Assumption* ($\mathcal{B}$), *there exists an extended Föllmer finitely additive measure for the extended* P*-supermartingale* $\overline{Z}$. *The extended Föllmer finitely additive measure is not unique if* $\mathbb{E}_P[\overline{Z}_\infty] < 1$. *The extended Föllmer finitely additive measure is unique if* $\mathbb{E}_P[\overline{Z}_\infty] = 1$ *and* $\mathcal{F} = \mathcal{F}_\infty$.

Note that an extended Föllmer finitely additive measure for the extended P-supermartingale $\overline{Z}$ is automatically a Föllmer finitely additive measure for the P-supermartingale Z. As a corollary of the existence result in Theorem 3.9, we make the following observation.

COROLLARY 3.10. *Under Assumption* ($\mathcal{B}$), *there exists a Föllmer purely finitely additive measure* Q *for the* P*-supermartingale* Z; *to wit, there exists* $Q \in \mathrm{ba}_1(\Omega, \mathcal{F}, P)$, *such that* (3) *holds and such that* $Q^r = 0$.

PROOF. Define the extended P-supermartingale $\overline{Z}$ as above, now with $\overline{Z}_\infty = 0$. The existence result in Theorem 3.9 then yields an extended Föllmer finitely additive measure Q for the extended P-supermartingale $\overline{Z}$. The statement now follows from the simple observation that $\mathrm{d}Q^r/\mathrm{d}P \le \mathrm{d}(Q|_{\mathcal{F}_\infty})^r/\mathrm{d}P = 0$ from (6) with $\rho = \infty$. $\square$

Note that Corollary 3.10 includes the case that Z is a uniformly integrable P-local martingale; for example, consider $Z_t = 1$ for all $t \ge 0$. As a consequence, Corollary 3.10 illustrates that a sequence of probability measures, given on the sigma algebras $\mathcal{F}_t$ for all $t \ge 0$, cannot uniquely be extended to the sigma algebra $\mathcal{F}_\infty$ within the class of finitely additive probability measures; however, uniqueness holds within the class of countably additive probability measures due to a pi-lambda argument. We elaborate further on this point by discussing the case of the Lebesgue measure on [0, 1] in Appendix G.

4. Proofs: Föllmer countably additive measure on the path space. This section contains the proofs of the existence and uniqueness results for the countably additive case in Section 3. Before providing the proofs, we discuss some motivating examples to outline the construction of the Föllmer countably additive measure. Then we first give the proof of existence, and afterward the proof of the assertions concerning uniqueness.

4.1. *Motivating examples.* We start by discussing two illustrative examples.

EXAMPLE 4.1. Let Q be a probability measure on the sigma algebra $\mathcal{F}$ and let $Y = (Y_t)_{t\ge 0}$ be a uniformly integrable nonnegative Q-martingale that starts in 1 and jumps to 0 with positive probability; that is, assume that $Q[\tau < \infty, Y_{\tau-} \neq 0] > 0$, where $\tau = \inf\{t \ge 0 : Y_t = 0\}$. Next, define the probability measure P by $P(\mathrm{d}\omega) = Y_\infty(\omega)Q(\mathrm{d}\omega)$. Then the process $Z = (Z_t)_{t\ge 0}$ with $Z_t = \mathbf{1}_{\{t<\tau\}}/Y_t$ is a strictly positive P-supermartingale, but it is not a P-local martingale: fix $s < t$ and $A \in \mathcal{F}_s$. Then the inequalities

$$\mathbb{E}_P[\mathbf{1}_A Z_t] = \mathbb{E}_Q\left[\mathbf{1}_A \frac{1}{Y_t}\mathbf{1}_{\{t<\tau\}} Y_t\right] \le \mathbb{E}_Q\left[\mathbf{1}_A \frac{1}{Y_s}\mathbf{1}_{\{s<\tau\}} Y_s\right] = \mathbb{E}_P[\mathbf{1}_A Z_s]$$

show that Z is a P-supermartingale. Now let $(\tau_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of stopping times such that $1 = \mathbb{E}_P[Z_{\tau_n}] = Q[\tau_n < \tau]$ holds for each $n \in \mathbb{N}$. Let us show that $P[\lim_{n\uparrow\infty} \tau_n < \infty] > 0$, which then implies that the P-supermartingale Z is not a P-local martingale. Toward this end, let $C > 0$ be such that $Q[\tau \le C, Y_{\tau-} \neq 0] > 0$. Observe that

$$\begin{aligned} P\Big[\lim_{n\uparrow\infty} \tau_n \le C\Big] &= \lim_{n\uparrow\infty} P[\tau_n \le C] = \lim_{n\uparrow\infty} \mathbb{E}_Q[Y_{\tau_n}\mathbf{1}_{\{\tau_n \le C\}}] \\ &\ge \mathbb{E}_Q\Big[\Big(\inf_{t<\tau} Y_t\Big)\mathbf{1}_{\{\tau\le C, Y_{\tau-}\neq 0\}}\Big] > 0, \end{aligned}$$

where we used that $(\inf_{t<\tau} Y_t) > 0$ on the event $\{Y_{\tau-} \neq 0\}$, which holds because Y is a nonnegative Q-supermartingale.

The P-supermartingale Z fails to be a P-local martingale exactly because the Q-martingale Y jumps to zero with positive probability under the probability measure Q. If the Q-martingale Y did not jump to zero, it would be possible to stop Y upon crossing the level $1/n$ for each $n \in \mathbb{N}$, and this approach would provide us with a localizing sequence of stopping times for the process Z under the probability measure P. Note that, despite Z not being a local martingale, it is of course possible to construct its Föllmer countably additive measure on the original space $(\Omega, \mathcal{F})$: the pair (Q, τ) satisfies the conditions in (1).

EXAMPLE 4.2. Let $Z = (Z_t)_{t\geq 0}$ be the P-supermartingale defined by $Z_t = e^{-t}$ for all $t \geq 0$. We want to interpret Z as $1/Y$, where Y is a martingale under the Föllmer countably additive measure Q, exactly as in Example 4.1. Since Z is not a local martingale, Y must jump to zero with positive probability under Q. Furthermore, Q must be equivalent to P before Y hits zero. This indicates that $Y_t = e^t \mathbf{1}_{\{t<\tau\}}$ under Q, where τ is the stopping time when Y hits zero.

Note that Y is a martingale exactly if τ is standard exponentially distributed and P needs to satisfy $P[\tau = \infty] = 1$. In general, it is not possible to find such a stopping time τ on $(\Omega, \mathcal{F})$, think, for example, of the space $\Omega = \{0\}$ consisting only of one singleton. However, let $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\geq 0})$ denote an extended filtered space with $\overline{\Omega} = \Omega \times [0, \infty]$, $\overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{B}([0, \infty])$, and $\overline{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{B}([0, t])$, where $\mathcal{B}$ denotes the Borel sigma algebra, and let $\overline{\tau}(\overline{\omega})$ denote the second component of $\overline{\omega}$ for all $\overline{\omega} \in \overline{\Omega}$. Then we can define the probability measures $\overline{P} = P \otimes \delta_\infty$ and $\overline{Q} = P \otimes \mu$ on this extended space, where δ_∞ is the Dirac measure in infinity and μ is a standard exponential distribution. It is not hard to check that the pair $(\overline{Q}, \overline{\tau})$ satisfies the conditions in (1) with P being replaced by $\overline{P}$.

Now the crucial point is that even though a general $(\Omega, \mathcal{F})$ might not be large enough to support an exponential time τ, the path space of Assumption $(\mathcal{P})$ is always large enough to support τ—as long as we allow for explosions to a cemetery state in finite time. In general we will not need an exponential time, but a stopping time τ with distribution $Q[\tau > t] = \mathbb{E}_P[Z_t]$. However, this can be easily reduced to the exponential case (or to the case of a uniform variable on $[0, 1]$) by a time change.

The insights gained from these guiding examples allow us to construct a Föllmer countably additive measure on the path space $(\Omega, \mathcal{F})$ itself, rather than on the extended probability space $(\Omega \times (0, \infty], \mathscr{P})$ used in Föllmer (1972), where $\mathscr{P}$ denotes the predictable sigma algebra. The crucial observation is that the Föllmer countably additive measure of a local martingale can be constructed on $(\Omega, \mathcal{F})$ without enlarging the space, and that a supermartingale fails to be a local martingale if and only if under its Föllmer countably additive measure, its inverse jumps to zero with positive probability. Thus, if $(\Omega, \mathcal{F})$ is large enough to allow for such a jump to zero, and if we are able to describe what happens under the Föllmer countably additive measure Q once $1/Z$ jumps to zero, then we should be able to construct the probability measure Q on the sigma algebra $\mathcal{F}$.

In order to construct such a jump to zero, we proceed in a similar manner as in the classical construction of killed diffusions, as presented, for example, in Chapter 5 of Itô and McKean (1965): we first introduce an independent random variable that triggers exactly when the supermartingale loses mass, and we later forget about this independent random variable when we project the constructed solution down to the path space.

4.2. *Föllmer countably additive measure*: *Proof of existence*. In this subsection, we provide the proof of the existence statement in Theorem 3.1.

Let $Z = MD$ be the multiplicative decomposition given in Proposition B.1. Define the stopping times $(\widehat{\tau}_n^M)_{n\in\mathbb{N}}$ and $\widehat{\tau}^M$ exactly as in (5), with Z replaced by M, and note that the stopped process $M^{\widehat{\tau}_n^M}$ is a uniformly integrable martingale for each $n \in \mathbb{N}$. In particular, we can define a sequence of measures $(Q^{(n)})_{n\in\mathbb{N}}$ by setting $Q^{(n)}(\mathrm{d}\omega) = M_{\widehat{\tau}_n^M(\omega)}(\omega)P(\mathrm{d}\omega)$. It is straightforward to check that $(Q^{(n)})_{n\in\mathbb{N}}$ is consistent on the filtration $(\mathcal{F}_{\widehat{\tau}_n^M})_{n\in\mathbb{N}}$, that is, that $Q^{(n)}(A) = Q^{(m)}(A)$ for all $A \in \mathcal{F}_{\widehat{\tau}_m}$ and $m, n \in \mathbb{N}$ with $m \le n$. Since the set inclusion $\mathcal{F}_{\widehat{\tau}_n^M-} \subset \mathcal{F}_{\widehat{\tau}_n^M}$ holds, the measures $(Q^{(n)})_{n\in\mathbb{N}}$ are also consistently defined on $(\mathcal{F}_{\widehat{\tau}_n^M-})_{n\in\mathbb{N}}$.

According to Lemma E.1, the filtration $(\mathcal{F}_{\widehat{\tau}_n^M-})_{n\in\mathbb{N}}$ is a standard system, a condition that allows to apply Parthasarathy's extension theorem, provided in Theorem D.3, which then yields the existence of a unique probability measure Q^M on $\bigvee_{n\ge 0}\mathcal{F}_{\widehat{\tau}_n^M-} = \mathcal{F}_{\widehat{\tau}^M-}$, such that $Q^M|_{\mathcal{F}_{\widehat{\tau}_n^M-}} = Q^{(n)}|_{\mathcal{F}_{\widehat{\tau}_n^M-}}$ for all $n \in \mathbb{N}$. Note that $P[\widehat{\tau}^M = \infty] = 1$ and that

$$\begin{aligned} Q^M[A \cap \{t < \widehat{\tau}^M\}] &= \lim_{n\uparrow\infty} Q^M[A \cap \{t < \widehat{\tau}_n^M\}] = \lim_{n\uparrow\infty} Q^{(n)}[A \cap \{t < \widehat{\tau}_n^M\}] \\ &= \lim_{n\uparrow\infty} \mathbb{E}_P[M_{\widehat{\tau}_n^M}\mathbf{1}_{A\cap\{t<\widehat{\tau}_n^M\}}] = \lim_{n\uparrow\infty} \mathbb{E}_P[M_t\mathbf{1}_{A\cap\{t<\widehat{\tau}_n^M\}}] \\ &= \mathbb{E}_P[M_t\mathbf{1}_A] \end{aligned}$$

for all $t \ge 0$ and $A \in \mathcal{F}_t$. Proposition 2.3 now yields that (1) holds with Q, τ, and Z replaced by Q^M, $\widehat{\tau}^M$ and M, respectively. In particular, if Z is a local martingale, that is, if $Z \equiv M$, we are done, as we may take any extension $\widehat{Q}^M$ of Q^M to $\mathcal{F}$ by Theorem E.2. Note that, in this case, we have $\widehat{\tau}^M = \widehat{\tau}^Z$, as defined in (5).

For the general case, we will now apply the ideas developed in Section 4.1 to construct a Föllmer countably additive measure for the P-supermartingale Z. Toward this end, we define the auxiliary space $\overline{\Omega} = \Omega \times [0, 1]$ and equip it with the sigma algebra $\overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{B}([0, 1])$, where $\mathcal{B}$ denotes the Borel sets. Let $\overline{Q} = \widehat{Q}^M \otimes \mu$ denote the product measure of $\widehat{Q}^M$ and μ, where μ is the uniform distribution on $[0, 1]$. We will define a measurable map $\theta : \overline{\Omega} \to \Omega$ and an $(\mathcal{F}_t)_{t\ge 0}$-stopping time τ, such that $Q = \overline{Q} \circ \theta^{-1}$ and τ satisfy (1).

Before we continue, let us select a good version of the process D: we may always suppose that D is right-continuous and nonincreasing for *all* $\omega \in \Omega$, see Lemma A.3. Since D starts at 1 and is nonnegative, $1 - D$ is the (random) distribution function of a measure on $[0, \infty)$ that has mass less or equal to 1. The "quantile function" $\mathcal{Q} : \Omega \times [0, 1] \to [0, \infty)$ of $1 - D$ is defined as

$$\mathcal{Q}_z(\omega) = \inf\{s \geq 0 : 1 - D_s(\omega) \geq z\} = \inf\{s \geq 0 : D_s(\omega) \leq 1 - z\}$$

for all $z \in [0, 1]$ and $\omega \in \Omega$. Note that

$$\begin{aligned}\{(\omega, z) : \mathcal{Q}_z(\omega) > t\} &= \{(\omega, z) : D_t(\omega) > 1 - z\} \\ &= \bigcup_{q \in \mathbb{Q} \cap [0,1]} \{\omega : D_t(\omega) > 1 - q\} \times [q, 1] \in \overline{\mathcal{F}}.\end{aligned} \tag{7}$$

Next, consider the map $\theta : \overline{\Omega} \to \Omega$ with

$$\theta(\omega, z)(t) = \begin{cases} \omega(t), & t < \mathcal{Q}_z(\omega), \\ \Delta, & t \geq \mathcal{Q}_z(\omega). \end{cases} \tag{8}$$

To see that the map θ is measurable, it suffices to note that

$$\begin{aligned}\{(\omega, z) : \theta(\omega, z)(t) \in B\} &= \{(\omega, z) : \omega(t) \in B, \mathcal{Q}_z(\omega) > t\}; \\ \{(\omega, z) : \theta(\omega, z)(t) \in B \cup \{\Delta\}\} &= \{(\omega, z) : \omega(t) \in B \cup \{\Delta\}, \mathcal{Q}_z(\omega) > t\} \\ &\quad \cup \{(\omega, z) : \mathcal{Q}_z(\omega) \leq t\}\end{aligned}$$

hold for each Borel subset $B \subset E$, so that (7) implies $\{(\omega, z) : \theta(\omega, z)(t) \in B\} \in \overline{\mathcal{F}}$.

Next, define the stopping time $\tau = \widehat{\tau}^M \wedge \zeta$ and the probability measure $Q = \overline{Q} \circ \theta^{-1}$. Note that

$$\begin{aligned}Q[A \cap \{\tau > t\}] &= \overline{Q}[\{(\omega, z) : \omega \in A, \widehat{\tau}^M(\omega) > t, \zeta(\omega) > t, \mathcal{Q}_z(\omega) > t\}] \\ &= \int_\Omega \left(\mathbf{1}_{A \cap \{\widehat{\tau}^M > t\} \cap \{\zeta > t\}}(\omega) \int_{[0,1]} \mathbf{1}_{(1 - D_t(\omega), 1]}(z) \mu(\mathrm{d}z)\right) \widehat{Q}^M(\mathrm{d}\omega) \\ &= \mathbb{E}_{\widehat{Q}^M}[D_t \mathbf{1}_{A \cap \{\widehat{\tau}^M > t\} \cap \{\zeta > t\}}] = \mathbb{E}_P[D_t M_t \mathbf{1}_{A \cap \{\zeta > t\}}] \\ &= \mathbb{E}_P[Z_t \mathbf{1}_A]\end{aligned}$$

for all $t \geq 0$, using (2) in the second to last step and

$$0 \leq \mathbb{E}_P[Z_t \mathbf{1}_{A \cap \{\zeta \leq t\}}] \leq \mathbb{E}_P[Z_\zeta \mathbf{1}_{\{\zeta \leq t\}}] \leq \mathbb{E}_P[Z_\zeta \mathbf{1}_{\{\zeta < \infty\}}] = 0$$

in the last step. Another application of Proposition 2.3 then completes the proof of the existence statement in Theorem 3.1.

4.3. *Föllmer countably additive measure*: *Proof of* (*non*)*uniqueness*. Here, we provide the proofs of Lemma 3.3, of the uniqueness statements of Theorem 3.1 and of Proposition 3.6.

PROOF OF LEMMA 3.3. Let (Q, τ) also satisfy (1). By Theorem D.4 in conjunction with Lemma E.1 there exists an extension $\overline{Q}$ of Q from $\mathcal{F}_{\tau-}$ to $\mathcal{F}_{\overline{\tau}-}$, where $\overline{\tau} = \widehat{\tau}^Z \vee \tau$. Note that

$$\overline{Q}[\widehat{\tau}^Z < \tau] = Q[\widehat{\tau}^Z < \tau] = \mathbb{E}_P[Z_{\widehat{\tau}^Z} \mathbf{1}_{\{\widehat{\tau}^Z < \infty\}}] = 0$$

by Proposition 2.3 and that $\overline{Q}[\widehat{\tau}_n^Z < \tau] = Q[\widehat{\tau}_n^Z < \tau] = \mathbb{E}_P[Z_{\widehat{\tau}_n^Z}] = 1$. These computations imply that $\overline{Q}[\widehat{\tau}^Z = \tau] = 1$ since $\widehat{\tau}_n^Z(\omega) \uparrow \widehat{\tau}^Z(\omega)$ for all $\omega \in \Omega$. Therefore, $(\overline{Q}, \overline{\tau})$ also satisfies (1). This again yields the uniqueness of the extension. □

Now, let us prove the uniqueness statement in Theorem 3.1.

Concerning the equivalence in (I), note that Q is uniquely determined on $\mathcal{F}_{\tau-}$ by (1). The statement then follows from the uniqueness result of Theorem E.2. The statement in (II) is proven in Proposition 2 in Yoeurp (1985).

We next show that the statement in (c) implies the one in (d). Thus, assume that (c) holds and let (Q, τ) also satisfy (1). Then Lemma 3.3 implies that also $(Q, \widehat{\tau}^Z)$ satisfies (1). This yields that $\widehat{Q}^Z$ and Q agree on $\mathcal{F}_{\widehat{\tau}^Z-}$ and we may apply the implication from (a) to (b).

For the reverse implication from (d) to (c), we assume that the state space E is uncountable. Thanks to the implication from (b) to (a), we only need to show that if Z is not a P-local martingale, then there are two different Föllmer pairs for Z. Toward this end, consider the family of stopping times $(\rho_x)_{x \in E}$, defined by

$$\rho_x = \inf\{t \geq 0 : \text{there exists } \varepsilon > 0 \text{ s.t. } \omega|_{[t,t+\varepsilon)} \equiv x\}; \tag{9}$$

here, the right-continuity of the filtration $(\mathcal{F}_t)_{t \geq 0}$ is used to guarantee that "peeking into the future is allowed," and thus each ρ_x is indeed a stopping time. Since the state space E is uncountable, by Lemma E.3, there must exist some $x^* \in E$ for which $P[\rho_{x^*} < \infty] = 0$. We define $\widetilde{\theta}$ as in (8) but with Δ replaced by x^* and construct, exactly as in the construction of the existence proof in Section 4.2, a Föllmer pair (Q^*, τ^*) with stopping time $\tau^* = \widehat{\tau}^M \wedge \rho_{x^*}$, where $\widehat{\tau}^M$ is as in Section 4.2. If (Q, τ) is the Föllmer pair constructed in Section 4.2, then we have $Q[\rho_{x*} < \tau_M] = 0$ but $Q^*[\rho_{x*} < \tau_M] > 0$ and, therefore, $Q^* \neq Q$. This completes the proof of Theorem 3.1.

REMARK 4.3. The proof that (d) implies (c) in Theorem 3.1 leads to the following observation: if the state space E is uncountable and if the P-local martingale M in the multiplicative decomposition of Z is a true P-martingale then a Föllmer countably additive measure Q can be defined so that $Q[\zeta = \infty] = 1$. In particular, in such a case, the state space of Assumption $(\mathcal{P})$ would not need to be enlarged with the cemetery state Δ.

PROOF OF PROPOSITION 3.6. To show (A), assume first that $P[\zeta < \infty] = 0$ and let the two pairs (Q, τ) and $(\widetilde{Q}, \widetilde{\tau})$ both satisfy (1). We obtain $Q[\tau \leq \zeta] = 1$ from (2) with $G \equiv 1$ and $\rho = \zeta$. Assume now that $\{\zeta \leq t\} \subsetneq \{\tau \wedge \zeta \leq t\}$ for some $t > 0$. Since $\{\zeta > t\}$ is an atom in $\mathcal{F}_t$ we then have $\{\tau \wedge \zeta \leq t\} = \Omega$, which contradicts $P[\tau \wedge \zeta < \infty] = 0$. Thus, we have $Q[\tau = \zeta] = 1$ and, similarly, $\widetilde{Q}[\widetilde{\tau} = \zeta] = 1$, which, in particular, implies that $Q|_{\mathcal{F}_{\zeta-}} = \widetilde{Q}|_{\mathcal{F}_{\zeta-}}$ and an application of Lemma E.1 concludes.

Next, consider a probability measure P such that $P[\zeta = 1] = 1$ and the P-supermartingale Z given by $Z_t = \mathbf{1}_{1>t}$ for all $t \geq 0$. As candidate for a stopping time, consider $\tau = 1 + \infty \mathbf{1}_{\{\zeta \leq 1\}}$. It is clear that $P[\tau = \infty] = 1$. Consider next two measures Q_1 and Q_2 such that $Q_1[\zeta = 2] = 1$ and $Q_2[\zeta = 3] = 1$. Then (Q_1, τ) and (Q_2, τ) are two different Föllmer pairs for the P-supermartingale Z.

For the claim of existence in (i), just fix $x^* \in E$ and consider a probability measure P under which $P[\rho_{x^*} = \infty] = 1$ holds, where ρ_{x^*} is defined as in (9) and then proceed as in the proof of the implication from (d) to (c) in Theorem 3.1.

For the claim in (ii), assume that $E = \{0, \dots, n\}$ or $E = \mathbb{N}_0$. Let ρ denote the infimum of the jump times of the canonical process ω to another state in the state space E and let P denote a probability measure on the sigma algebra $\mathcal{F}$ so that $P[\omega(0) = 0] = 1$, and such that at time 1 (but not before), the coordinate process jumps to any other state in $E \setminus \{0\}$ with strictly positive probability or stays in 0 with strictly positive probability. Then we have, in particular, $P[\rho \geq 1] = 1$ and $P[\rho > 1] > 0$ as well as $P[\rho = 1] > 0$. We now consider the P-supermartingale Z, given by $Z_t = \mathbf{1}_{1>t}$ for all $t \geq 0$.

Assume that the pair (Q, τ) is a Föllmer pair. We want to show that $Q[\tau = \zeta] = 1$ holds, which yields the uniqueness of the Föllmer pair, as in the proof of (A). Toward this end, note that $Q[\tau = 1] = 1 = Q[\tau \leq \zeta]$, again, as in the proof of (A), and that

$$\Omega = \{\rho \leq 1\} \cup \{\zeta \leq 1\} \cup \{\zeta \wedge \rho > 1\}.$$

Thus, if we can show that $Q[\zeta \wedge \rho > 1] = 0$ and $Q[\rho_i = 1] = 0$ for all $i \in E \setminus \{0\}$, where ρ_i denotes the first hitting time of level i by the canonical process, then we have $Q[\zeta \leq 1] = Q[\zeta \geq 1] = 1$ and, therefore, $Q[\zeta = 1 = \tau] = 1$.

Assume first that $Q[\{\tau = 1\} \cap \{\zeta \wedge \rho > 1\}] = Q[\zeta \wedge \rho > 1] > 0$. Then, by Problem 1.2.2 in Karatzas and Shreve (1991), we have $\tau(\omega) = 1$ for all $\omega \in \Omega$ with $\omega(0) = 0$ and $\zeta(\omega) \wedge \rho(\omega) > 1$; however, this would lead to the contradiction

$$0 < P[\zeta \wedge \rho > 1] \leq P[\tau = 1] \leq P[\tau < \infty] = 0.$$

Next, fix $i \in E \setminus \{0\}$ and assume that $Q[\rho_i = \tau] = Q[\rho_i = 1] > 0$. Observe that

$$\begin{aligned} &\{(\omega, t) \in \Omega \times [0, \infty) : \omega(0) = 0, \tau(\omega) = \rho(\omega) = \rho_i(\omega) = t\} \\ &= \{(\omega, t) \in \Omega \times B : \omega(0) = 0, \rho(\omega) = \rho_i(\omega) = t\} \end{aligned}$$

for some Borel set $B \subset [0, \infty)$; see again Problem 1.2.2 in Karatzas and Shreve (1991). By assumption, we have that $1 \in B$. This then yields that

$$0 < P[\rho_i = 1] \leq P[\rho_i \in B] = P[\tau = \rho_i \in B] \leq P[\tau \in B] \leq P[\tau < \infty] = 0.$$

This contradiction completes the proof. □

5. Proofs: Föllmer finitely additive measure.

5.1. *Föllmer finitely additive measure*: *Proof of existence*. We now prove the existence statement of Theorem 3.7 with the help of several lemmas. The idea of the proof is a combination of approximating the P-supermartingale Z, approximating those approximations again, and using a compactness argument. We split the results up in three subsections. First, we recall some fundamental observations that will be the key component of the proof, then we collect several useful approximations, and finally, we will put everything together in the proof of existence of a Föllmer finitely additive measure for the P-supermartingale Z.

5.1.1. *Fundamental observations*.

PROPOSITION 5.1 [Cvitanić, Schachermayer and Wang (2001), Proposition A.1]. *Consider a sequence $(Q^{(n)})_{n\in\mathbb{N}}$ of finitely additive probability measures in $\mathrm{ba}_1(\Omega, \mathcal{F}, P)$. Assume that $\mathrm{d}(Q^{(n)})^r/\mathrm{d}P$ converges almost surely to a nonnegative random variable G. Then any cluster point Q of $(Q^{(n)})_{n\in\mathbb{N}}$ in $L^\infty(\Omega, \mathcal{F}, P)^*$ satisfies $Q^r(\mathrm{d}\omega) = G(\omega)P(\mathrm{d}\omega)$.*

This powerful result will enable us to approximate the P-supermartingale Z, step by step, with processes for which it is relatively simple to construct the corresponding finitely additive probability measure. Toward this end, we shall rely on the following consequence of the Banach–Alaoglu theorem:

COROLLARY 5.2. *Let $(Q^{(n)})_{n\in\mathbb{N}}$ be a sequence of finitely additive probability measures in $\mathrm{ba}_1(\Omega, \mathcal{F}, P)$. Assume that the Radon–Nikodym derivatives $\mathrm{d}(Q^{(n)}|_{\mathcal{F}_\rho})^r/\mathrm{d}P|_{\mathcal{F}_\rho}$ converge to Z_ρ as n tends to infinity almost surely, for each finite stopping time ρ (resp., to $\overline{Z}_\rho$ for each stopping time). Then there exists a Föllmer finitely additive measure for the P-supermartingale Z (resp., an extended Föllmer finitely additive measure for the extended P-supermartingale $\overline{Z}$).*

PROOF. First, note that we may identify $\mathrm{ba}_1(\Omega, \mathcal{F}, P)$ with a subset of the unit ball of $L^\infty(\Omega, \mathcal{F}, P)^*$. The Banach–Alaoglu theorem then implies that $(Q^{(n)})_{n\in\mathbb{N}}$ has a cluster point Q. Next, observe that $Q|_{\mathcal{F}_\rho}$ is also a cluster point of the sequence $(Q^{(n)}|_{\mathcal{F}_\rho})_{n\in\mathbb{N}}$. We conclude the argument with an application of Proposition 5.1. □

To illustrate the approach we shall follow, and for later use, we now discuss the case that Z is a P-local martingale:

EXAMPLE 5.3. Assume that Z is a P-local martingale. Then there exists a Föllmer finitely additive measure for Z. To see this, let $(\rho_n)_{n\in\mathbb{N}}$ denote a sequence of localizing stopping times for Z. Then, for each $n \in \mathbb{N}$, the uniformly integrable P-martingale Z^{ρ_n} defines a probability measure $Q^{(n)}$ that is absolutely continuous with respect to P. Since $Q^{(n)}$ has no singular part, for each finite stopping time ρ the Radon–Nikodym derivative $\mathrm{d}(Q^{(n)}|_{\mathcal{F}_\rho})^r/\mathrm{d}P|_{\mathcal{F}_\rho}$ is given by $Z_\rho^{\rho_n}$ and, therefore, converges almost surely to Z_ρ as n tends to ∞ (indeed the null set outside of which convergence takes place does not depend on ρ). Corollary 5.2 now implies the existence of a Föllmer finitely additive measure for Z.

5.1.2. *Approximations*. Recall that the Doob–Meyer decomposition of the P-supermartingale Z is given by $Z = M + D$, where M is a P-local martingale and D is a predictable nonincreasing process with $D_0 = 0$. This decomposition is unique up to indistinguishability. Example 5.3 indicates that the local martingale component of Z can be handled easily. Thus, in the following, we shall focus mostly on approximating the nonincreasing process D. Toward this end, we introduce the notion of a simple process, which, in particular, has càdlàg paths.

DEFINITION 5.4. A process $G = (G_t)_{t\geq 0}$ is called *simple process* if there exists a strictly increasing sequence of stopping times $(\rho_n)_{n\in\mathbb{N}_0}$ and a sequence of random variables $(H_n)_{n\in\mathbb{N}}$ such that $\rho_0(\omega) = 0$ and $\lim_{n\uparrow\infty}\rho_n(\omega) = \infty$ for all $\omega \in \Omega$, H_n is $\mathcal{F}_{\rho_{n-1}}$-measurable for all $n \in \mathbb{N}$, H_0 is $\mathcal{F}_0$-measurable, and

$$G_t(\omega) = H_0(\omega)\mathbf{1}_{t=0} + \sum_{n=1}^{\infty} H_n(\omega)\mathbf{1}_{(\rho_{n-1}(\omega),\rho_n(\omega)]}(t)$$

holds for all $\omega \in \Omega$ and $t \geq 0$.

LEMMA 5.5. *Let $G = (G_t)_{t\geq 0}$ be a nonincreasing adapted process with càdlàg paths. Then there exists a sequence of nonincreasing simple processes $(G^{(k)})_{k\in\mathbb{N}}$ with $G^{(k)} = (G_t^{(k)})_{t\geq 0}$ such that almost surely $G_0^{(k)} = G_0$, $\lim_{k\uparrow\infty} G_t^{(k)} = G_{t-}$, and $G_t^{(k)} \geq G_t$ for all $n \in \mathbb{N}$ and $t \geq 0$.*

PROOF. It suffices to set

$$G_t^{(k)}(\omega) = G_0(\omega)\mathbf{1}_{t=0} + \sum_{n=0}^{k2^k-1} G_{n2^{-k}}(\omega)\mathbf{1}_{(n2^{-k},(n+1)2^{-k}]}(t) + G_k(\omega)\mathbf{1}_{(k,\infty)}(t)$$

for all $\omega \in \Omega$ and $t \geq 0$. □

The crucial observation now is that every nonincreasing simple process is the limit of a sequence of local martingales, at least as long as the filtered probability space is rich enough to support a Brownian motion. Before we discuss the general result, the next example illustrates that such an approximation is possible.

EXAMPLE 5.6. Under Assumption ($\mathcal{B}$), let Z be a deterministic process with $Z_t = 1$ for all $t \in [0, 1)$ and $Z_t = a \in [0, 1]$ for all $t \in [1, \infty)$. Then there exists a Föllmer finitely additive measure for Z. To see this, define the continuous P-local martingales $(\mathcal{E}^{(m)})_{m\in\mathbb{N}}$ with $\mathcal{E}^{(m)} = (\mathcal{E}_t^{(m)})_{t\geq 0}$ and $(N^{(m)})_{m\in\mathbb{N}}$ with

$N^{(m)} = (N_t^{(m)})_{t\geq 0}$ by

$$\mathcal{E}_t^{(m)} = \begin{cases} 1, & t \in [0, 1-2^{-m}), \\ \exp\left(\int_{1-2^{-m}}^t \frac{1}{\sqrt{1-s}}\,\mathrm{d}W_s - \frac{1}{2}\int_{1-2^{-m}}^t \frac{1}{1-s}\,\mathrm{d}s\right), & t \in [1-2^{-m}, 1), \\ 0, & t \in [1, \infty) \end{cases}$$

and

$$N_t^{(m)} = 1 + \int_0^t (1-a)\frac{\mathcal{E}_s^{(m)}}{\sqrt{1-s}}\mathbf{1}_{[1-2^{-m},1)}(s)\,\mathrm{d}W_s.$$

Then $N^{(m)} = a + (1-a)\mathcal{E}^{(m)}$ for each $n \in \mathbb{N}$; in particular $N_t^{(m)} = 1$ for all $t \in [0, 1-2^{-m}]$, and $N_t^{(m)} = a$ for all $t \in [1, \infty)$.

Therefore, if ρ is a finite stopping time, then $N_\rho^{(m)}$ converges almost surely to Z_ρ, and the null set outside of which the convergence holds does not depend on ρ. By Corollary 5.2 in conjunction with Example 5.3, there exists a Föllmer finitely additive measure for Z.

LEMMA 5.7. *Under Assumption ($\mathcal{B}$), let $G = (G_t)_{t\geq 0}$ be a nonincreasing simple process. Then there exists a sequence of continuous local martingales $(N^{(m)})_{m\in\mathbb{N}}$ with $N^{(m)} = (N_t^{(m)})_{t\geq 0}$ such that almost surely $\lim_{m\uparrow\infty} N_\rho^{(m)} = G_\rho$, $N_0^{(m)} = G_0$ and $N_\rho^{(m)} \geq G_\rho$ for all finite stopping times ρ and $m \in \mathbb{N}$.*

PROOF. The sequence of local martingales $(N^{(m)})_{m\in\mathbb{N}}$ can be constructed in the same manner as described in Example 5.6. Toward this end, define again certain stochastic exponentials as follows. Let the sequence of stopping times $(\rho_n)_{n\in\mathbb{N}}$ denote the (well-ordered) jump times of G, set $\rho_0 = 0$, and define the continuous P-local martingales $(\mathcal{E}^{(m,n)})_{m,n\in\mathbb{N}_0}$ with $\mathcal{E}^{(m,n)} = (\mathcal{E}_t^{(m,n)})_{t\geq 0}$ by $\mathcal{E}_t^{(m,n)} = 1$ for all $t \in [0, \rho_n)$,

$$\mathcal{E}_t^{(m,n)} = \exp\left(\int_{\rho_n}^t \frac{1}{\sqrt{\rho_n + 2^{-m} - s}}\,\mathrm{d}W_s - \frac{1}{2}\int_{\rho_n}^t \frac{1}{\rho_n + 2^{-m} - s}\,\mathrm{d}s\right)$$

for all $t \in [\rho_n, \rho_n + 2^{-m})$, and $\mathcal{E}_t^{(m,n)} = 0$ for all $t \geq \rho_n + 2^{-m}$.

Next, for each $m \in \mathbb{N}_0$, define the "suicide strategy" $H^{(m)} = (H_t^{(m)})_{t\geq 0}$ by

$$H_t^{(m)} = \sum_{n=0}^{\infty}(G_{\rho_n} - G_{\rho_n+})\frac{\mathcal{E}_t^{(m,n)}}{\sqrt{\rho_n + 2^{-m} - t}}\mathbf{1}_{[\rho_n, \rho_n+2^{-m})}(t)$$

and construct the local martingale $N^{(m)} = (N_t^{(m)})_{t\geq 0}$ by

$$N_t^{(m)} = G_0 + \int_0^t H_s^{(m)}\,\mathrm{d}W_s.$$

Almost surely, for each $m, n \in \mathbb{N}_0$, on the event $\{\rho_{n+1} - \rho_n > 2^{-m}\}$ we have $N_t^{(m)} = G_t$ for all $t \in [\rho_n + 2^{-m}, \rho_{n+1}]$. This implies that almost surely $\lim_{m\uparrow\infty} N_\rho^{(m)} = G_\rho$ holds for all finite stopping times ρ. $\square$

So far, we have approximated the process D_-, where D is the nondecreasing process in the Doob–Meyer decomposition of the P-supermartingale Z, by simple processes, and we approximated those simple processes by local martingales. What remains to be shown is how to pass from the left-continuous process D_- to the right-continuous process D. In the following lemma, we will provide the key component for this step, using the fact that the process D is predictable. The proof of the next lemma is tedious but the underlying idea for it is very simple.

To illustrate that simple idea, let $i \in \mathbb{N}$ be a positive constant and let σ_1 denote the first time that D jumps down by more than $1/i$. This jump time is predictable, thus, in particular, there exists an announcing sequence $(\sigma_1^{(j)})_{j\in\mathbb{N}}$ for σ_1. With the help of these stopping times, we define the P-supermartingale $Z^{(j)}$ as

$$Z_t^{(j)} = \begin{cases} \mathbb{E}[Z_{\sigma_1}|\mathcal{F}_t], & t \in [\sigma_1^{(j)}, \sigma_1]; \\ Z_t, & \text{otherwise}, \end{cases}$$

for each $j \in \mathbb{N}$. Then the expectation of $Z^{(j)}$ is constant on $[\sigma_1^{(j)}, \sigma_1]$, and, in particular, the P-supermartingale $Z^{(j)}$ can be decomposed in a local martingale and a nonincreasing process that stays constant on an interval before the stopping time σ_1. One can now show, and that is what the proof of Lemma 5.8 will do, that this local martingale plus the left-continuous version of the nonincreasing process converges to the P-supermartingale Z at time σ_1 and to $M + D_-$ at all other times, as j tends to infinity.

LEMMA 5.8. *Let Z have Doob–Meyer decomposition $Z = M + D$ and fix $i \in \mathbb{N}$. Then there exists a sequence of P-local martingales $(M^{(i,j)})_{j\in\mathbb{N}}$ with $M^{(i,j)} = (M_t^{(i,j)})_{t\geq 0}$ and $M_0^{(i,j)} = 1$ and a sequence of càdlàg, adapted, nonincreasing processes $(D^{(i,j)})_{j\in\mathbb{N}}$ with $D^{(i,j)} = (D_t^{(i,j)})_{t\geq 0}$ and $D_0^{(i,j)} = 0$ such that the P-supermartingales $M^{(i,j)} + D^{(i,j)}$ are nonnegative and such that almost surely*

$$\lim_{j\uparrow\infty} (M_\rho^{(i,j)} + D_{\rho-}^{(i,j)}) = M_\rho + (\mathbf{1}_{\{\Delta D_\rho \leq -1/i\}} D_\rho + \mathbf{1}_{\{\Delta D_\rho > -1/i\}} D_{\rho-})$$

for each finite stopping time ρ.

PROOF. We shall work on the completion of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, so that we can assume D to be predictable and càdlàg for *all* $\omega \in \Omega$. Once we constructed $M^{(i,j)}$ and $D^{(i,j)}$ on this completion, we may switch to indistinguishable versions that are adapted to $(\mathcal{F}_t)_{t\geq 0}$; see Lemma A.3 in the Appendix A. Define $\sigma_0 = 0$ and the sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ iteratively by

$$\sigma_n = \inf\left\{t > \sigma_{n-1} : \Delta D_t \leq -\frac{1}{i}\right\}.$$

Since the process D is predictable, the jump time σ_n is a predictable time for each $n \in \mathbb{N}$; thus, there exists an announcing sequence $(\sigma_n^{(j)})_{j\in\mathbb{N}}$, such that $\lim_{j\uparrow\infty} \sigma_n^{(j)}(\omega) = \sigma_n(\omega)$ and $\sigma_n^{(j)}(\omega) \leq \sigma_n^{(j+1)}(\omega) < \sigma_n(\omega)$ for all $j \in \mathbb{N}$ and $\omega \in \Omega$; see 1.2.16 in Jacod and Shiryaev (2003).

Next, since $\sigma_{n-1} < \sigma_n$ holds on the event $\{\sigma_{n-1} < \infty\}$ for all $n \in \mathbb{N}$, we may assume, without loss of generality, that $\sigma_n^{(j)} \geq \sigma_{n-1}$ holds with strict inequality on the event $\{\sigma_{n-1} < \infty\}$ for all $n, j \in \mathbb{N}$; if not, we just replace $\sigma_n^{(j)}$ by

$$\inf\{t > \sigma_{n-1} \vee \sigma_n^{(j-1)} : t = \sigma_n^{(k)} \text{ for some } k \in \mathbb{N}\}.$$

Thus, we have $\sigma_n^{(j)} \in (\sigma_{n-1}, \sigma_n)$ on the event $\{\sigma_{n-1} < \infty\}$ for all $n, j \in \mathbb{N}$.

For all $j \in \mathbb{N}$, define now the processes $M^{(i,j)}$ and $D^{(i,j)}$ by

$$M_t^{(i,j)} = M_t + \sum_{n=1}^{\infty} \left(\mathbf{1}_{\{\sigma_n^{(j)} \leq t\}} \left(\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_t] - \mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_{\sigma_n^{(j)}}] - M_{\sigma_n \wedge t} + M_{\sigma_n^{(j)}}\right)\right);$$

$$D_t^{(i,j)} = D_t + \sum_{n=1}^{\infty} \left(\mathbf{1}_{\{\sigma_n^{(j)} \leq t\}} \left(\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_{\sigma_n^{(j)}}] - M_{\sigma_n^{(j)}} - D_{\sigma_n \wedge t}\right)\right)$$

for all $t \geq 0$, where we always take the same version of the conditional expectations for the two processes and a càdlàg modification of the processes $(\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_t])_{t\geq 0}$ for all $n \in \mathbb{N}$. Clearly, the processes $M^{(i,j)}$ and $D^{(i,j)}$ are càdlàg, satisfy $M_0^{(i,j)} = 1$ and $D_0^{(i,j)} = 0$, and are P-local martingales, or nonincreasing, respectively, for all $j \in \mathbb{N}$. We compute

$$M_t^{(i,j)} + D_t^{(i,j)} = Z_t + \sum_{n=1}^{\infty} \left(\mathbf{1}_{\{\sigma_n^{(j)} \leq t\}} \left(\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_t] - Z_{\sigma_n \wedge t}\right)\right)$$

for all $t \geq 0$, which, in particular, yields that $M^{(i,j)} + D^{(i,j)}$ is nonnegative for each $j \in \mathbb{N}$.

Moreover, fix a finite stopping time ρ and note that

$$\begin{aligned}
&M_\rho^{(i,j)} + D_{\rho-}^{(i,j)} \\
&\quad = M_\rho + D_{\rho-} \\
&\qquad + \sum_{n=1}^{\infty} \left(\mathbf{1}_{\{\sigma_n^{(j)} \leq \rho\}} \left(\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_\rho] - M_{\sigma_n \wedge \rho} - D_{\rho-} \mathbf{1}_{\{\rho \leq \sigma_n\}} - D_{\sigma_n} \mathbf{1}_{\{\rho > \sigma_n\}}\right)\right) \\
&\qquad - \sum_{n=1}^{\infty} \left(\mathbf{1}_{\{\sigma_n^{(j)} = \rho\}} \left(\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_{\sigma_n^{(j)}}] - M_{\sigma_n^j} - D_{\rho-}\right)\right)
\end{aligned}$$

for each $j \in \mathbb{N}$. For each $\omega \in \Omega$, there exists maximally finitely many $j \in \mathbb{N}$ such that the identity $\sigma_n^{(j)}(\omega) = \rho(\omega)$ holds for some $n \in \mathbb{N}$. Thus, a-fortiori, the last sum converges to zero as j tends to infinity. For studying the first sum, fix $n \in \mathbb{N}$. Then we want to show that

$$\lim_{j \uparrow \infty} \bigl(\mathbf{1}_{\{\sigma_n^{(j)} \le \rho\}} \bigl(\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_\rho] - Z_{\sigma_n \wedge \rho} + \Delta D_\rho \mathbf{1}_{\{\rho \le \sigma_n\}}\bigr)\bigr) = \mathbf{1}_{\{\rho = \sigma_n\}} \Delta D_\rho \tag{10}$$

almost surely, where the null set on which the equality does not hold, can be chosen independently of the stopping time ρ. This then proves the statement. Path-by-path, (10) holds on the event $\{\rho < \sigma_n\}$; thus, we only need to argue that $\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_\rho] = Z_{\sigma_n}$ holds on the event $\{\rho \ge \sigma_n\}$ almost surely (independently of the choice of ρ). To see this, note that almost surely $\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_q] \mathbf{1}_{\sigma_n \le q} = Z_{\sigma_n} \mathbf{1}_{\sigma_n \le q}$ holds for all $q \in \mathbb{Q} \cap [0, \infty)$ and recall that we chose a right-continuous modification of $(\mathbb{E}_P[Z_{\sigma_n} | \mathcal{F}_t])_{t \ge 0}$. □

The next result summarizes the approximation results that we obtained so far in this subsection.

PROPOSITION 5.9. *Under Assumption ($\mathcal{B}$), there exists a family $(L^{(i,j,k,m,n)})_{i,j,k,m,n \in \mathbb{N}}$ of uniformly integrable nonnegative P-martingales with $L^{(i,j,k,m,n)} = (L_t^{(i,j,k,m,n)})_{t \ge 0}$ and $\mathbb{E}_P[L_0^{(i,j,k,m,n)}] = 1$ for all $i, j, k, m, n \in \mathbb{N}$, such that almost surely*

$$\lim_{i \uparrow \infty} \lim_{j \uparrow \infty} \lim_{k \uparrow \infty} \lim_{m \uparrow \infty} \lim_{n \uparrow \infty} L_\rho^{(i,j,k,m,n)} = Z_\rho \tag{11}$$

for all finite stopping times ρ.

PROOF. The statement follows by first applying Lemma 5.8, then approximating the corresponding processes $D^{(i,j)}$ via Lemmas 5.5 and 5.7 for each $i, j \in \mathbb{N}$, and finally localizing the approximating local martingales. □

We emphasize that there exists one P-null set outside of which (11) holds for all stopping times ρ. Theorem F.2 in the Appendix F provides a similar statement as Proposition 5.9, but with one limit (instead of five limits) only, thus yielding the existence of a sequence of uniformly integrable nonnegative martingales Fatou converging to the P-supermartingale Z.

Now observe that if there exists a deterministic time $T > 0$ such that $Z_{T+t} = Z_T$ for all $t \ge 0$, and if we set $Z_\infty = \lim_{t \uparrow \infty} Z_t = Z_T$, then by construction of the martingales $L^{(i,j,k,m,n)}$, the convergence in (11) extends to general stopping times, not necessarily finite. This allows us to approximate extended P-supermartingales by an additional limit procedure.

COROLLARY 5.10. *Under Assumption* ($\mathcal{B}$), *let* $(\overline{Z}_t)_{t\in[0,\infty]}$ *be an extended nonnegative* P*-supermartingale with* $\mathbb{E}_P[\overline{Z}_0] = 1$. *There exists a family* $(\overline{L}^{(h,i,j,k,m,n)})_{h,i,j,k,m,n\in\mathbb{N}}$ *of uniformly integrable nonnegative* P*-martingales with* $\overline{L}^{(h,i,j,k,m,n)} = (\overline{L}_t^{(h,i,j,k,m,n)})_{t\in[0,\infty]}$ *and* $\mathbb{E}_P[\overline{L}_0^{(h,i,j,k,m,n)}] = 1$ *for all* $h, i, j, k, m, n \in \mathbb{N}$, *such that almost surely*

$$\lim_{h\uparrow\infty}\lim_{i\uparrow\infty}\lim_{j\uparrow\infty}\lim_{k\uparrow\infty}\lim_{m\uparrow\infty}\lim_{n\uparrow\infty}\overline{L}_\rho^{(h,i,j,k,m,n)} = \overline{Z}_\rho \tag{12}$$

for all stopping times ρ.

PROOF OF COROLLARY 5.10. For fixed $h \in \mathbb{N}$, consider the family $(\widetilde{L}^{(h,i,j,k,m,n)})_{i,j,k,m,n\in\mathbb{N}}$ of uniformly integrable nonnegative P-martingales with $\mathbb{E}[\widetilde{L}_0^{(h,i,j,k,m,n)}] = 1$, which is given by Proposition 5.9 and which approximates the P-supermartingale

$$\widetilde{Z}_t^{(h)} = \mathbf{1}_{t<h}\frac{Z_t - \mathbb{E}_P[\overline{Z}_\infty|\mathcal{F}_t]}{\mathbb{E}_P[Z_0 - \overline{Z}_\infty]}$$

for all $t \geq 0$, where we set $0/0 = 1$. As remarked above, the convergence in (11) extends to general stopping times if we set $\widetilde{Z}_\infty^{(h)} = \widetilde{Z}_h^{(h)} = 0$. Since $\mathbb{E}_P[\overline{Z}_\infty|\mathcal{F}_\infty] = \overline{Z}_\infty$, it now suffices to set

$$\overline{L}_t^{(h,i,j,k,m,n)} = \mathbb{E}_P[\overline{Z}_\infty|\mathcal{F}_t] + \mathbb{E}_P[Z_0 - \overline{Z}_\infty]\widetilde{L}_t^{(h,i,j,k,m,n)}$$

for all $t \in [0,\infty]$ and all $h, i, j, k, m, n \in \mathbb{N}$. □

5.1.3. *Proofs of Theorems* 3.7 *and* 3.9, *existence*. With the help of the auxiliary results of the last two subsections, the construction of Föllmer finitely additive measures is now simple. We start by using the Radon–Nikodym derivatives $(L^{(i,j,k,m,n)})_{i,j,k,m,n\in\mathbb{N}}$ from Proposition 5.9 to construct a family of probability measures. Applying Corollary 5.2 then five times yields the existence of a Föllmer finitely additive measure for the P-supermartingale Z.

In the same manner, Corollary 5.10 implies the existence of an extended Föllmer finitely additive measure for the extended P-supermartingale $\overline{Z}$.

5.2. *Föllmer finitely additive measure*: *Proof of nonuniqueness*. Before we get to the question of uniqueness, let us first illustrate that a Föllmer finitely additive measure needs to satisfy (3) for all stopping times, it does not suffice to verify (3) only for deterministic times.

EXAMPLE 5.11. Let $(\Omega, \mathcal{F}, P)$ be a probability space that supports a Brownian motion $W = (W_t)_{t\geq 0}$ and an independent random variable ρ with uniform distribution on $[1, 2]$. Define a filtration $(\mathcal{F}_t)_{t\geq 0}$ by $\mathcal{F}_t = \bigcap_{s>t}(\sigma(W_r : r \leq s) \vee \sigma(\rho))$

for all $t \geq 0$. Since ρ and W are independent, W is a Brownian motion in the filtration $(\mathcal{F}_t)_{t\geq 0}$. Moreover, ρ is $\mathcal{F}_0$-measurable and therefore a stopping time. Define now the P-supermartingale $Z = (Z_t)_{t\geq 0}$ by $Z_t = \mathbf{1}_{[0,\rho)}(t)$ for all $t \geq 0$.

In the same way as in Example 5.6, we can now construct two sequences $(M^{(m)})_{m\in\mathbb{N}}$ and $(N^{(m)})_{m\in\mathbb{N}}$ of continuous, nonnegative local martingales with $M^{(m)} = (M_t^{(m)})_{t\geq 0}$ and $N^{(m)} = (N_t^{(m)})_{t\geq 0}$ for all $m \in \mathbb{N}$. Toward this end, note that $\rho - 1/m$ is a stopping time for each $m \in \mathbb{N}$. Then, for each $m \in \mathbb{N}$, let $M^{(m)}$ be a local martingale that is constant 1 until time $\rho - 1/m$, fluctuates in the interval $[\rho - 1/m, \rho]$, and is constant 0 after time ρ, and let $N^{(m)}$ be a local martingale that is constant 1 until time ρ, fluctuates in the interval $[\rho, \rho + 1/m]$, and is constant 0 after time $\rho + 1/m$.

Since ρ is absolutely continuous, we have almost surely $\lim_{m\uparrow\infty} M_t^{(m)} = Z_t = \lim_{m\uparrow\infty} N_t^{(m)}$ for all $t \geq 0$. So if Q_1 is a cluster point of the Föllmer finitely additive measures for $(M^{(m)})_{m\in\mathbb{N}}$, and if Q_2 is a cluster point for the Föllmer finitely additive measures for $(N^{(m)})_{m\in\mathbb{N}}$, then we have

$$(Q_1|_{\mathcal{F}_t})^r[A] = \mathbb{E}_P[\mathbf{1}_A Z_t] = (Q_2|_{\mathcal{F}_t})^r[A]$$

for all $A \in \mathcal{F}_t$ and $t \geq 0$. However, we have $\lim_{m\uparrow\infty} M_\rho^{(m)} = 0 \neq 1 = \lim_{m\uparrow\infty} N_\rho^{(m)}$ and, therefore, $(Q_1|_{\mathcal{F}_\rho})^r \neq (Q_2|_{\mathcal{F}_\rho})^r$.

In order to prove the (non)uniqueness results of Theorems 3.7 and 3.9, we first prove some important special cases in auxiliary lemmas. We shall use the convention $Z_\infty = \lim_{t\uparrow\infty} Z_t$, but warn the reader that it is possible that $\overline{Z}_\infty \neq Z_\infty$; however, we always have $\overline{Z}_\infty \in [0, Z_\infty]$.

LEMMA 5.12. *Under Assumption ($\mathcal{B}$), suppose that $P[Z_\infty > 0] > 0$. Then there exist two Föllmer finitely additive measures Q_1, Q_2 for the P-supermartingale Z.*

PROOF. Observe that under the assumption the extension $\overline{Z}$ of the P-supermartingale Z is not unique; for example, we may set $\overline{Z}_\infty = 0$ or $\overline{Z}_\infty = Z_\infty$. However, for each extension there exists an extended Föllmer finitely additive measure, which is also a Föllmer finitely additive measure for the P-supermartingale Z. Since the measures corresponding to different extensions of Z do not agree, the statement is proven. □

The proof of Lemma 5.12 is short but not very insightful. We thus provide an alternative, more "constructive" proof in Appendix H to illustrate where the lack of uniqueness comes into play.

LEMMA 5.13. *Under Assumption ($\mathcal{B}$), suppose that there exists $c \in [0, 1)$ for which $P[\rho < \infty] = 1$, where $\rho = \inf\{t \geq 0 : Z_t \leq c\}$. Then there exist two extended Föllmer finitely additive measures Q_1, Q_2 for the extended P-supermartingale $\overline{Z}$.*

PROOF. Recall the family $(\overline{L}^{(h,i,j,k,m,n)})_{h,i,j,k,m,n\in\mathbb{N}}$ of uniformly integrable nonnegative martingales from Corollary 5.10. For sake of notation, fix $h,i,j,k,m,n \in \mathbb{N}$ and set $\widehat{L} = \overline{L}^{(h,i,j,k,m,n)}$. Define the stopping time $\widehat{\sigma}$ by ρ on the event $\{\widehat{L}_\rho > (1+c)/2\}$ and ∞ on its complement. Note that the convergence in (11) implies that, for P-almost all $\omega \in \Omega$, there exists $h^*(\omega)$, such that for all $h \geq h^*(\omega)$ there exists $i^*(\omega, h)$, such that for all $i \geq i^*(\omega, h)$ there exists $j^*(\omega, h, i), \dots,$ such that $\widehat{\sigma}(\omega) = \infty$ as long as $h \geq h^*(\omega), i \geq i^*(\omega, h), \dots, n \geq n^*(\omega, h, i, j, k, m)$.

Define now the events $(B_l)_{l\in\mathbb{N}}$ by

$$B_l = \{W_{\rho+1} \in (l, l+1)\}$$

and note that $\mathbb{E}_P[\mathbf{1}_{B_l}|\mathcal{F}_\rho] > 0$ almost surely using the fact that $\rho < \infty$. For each $l \in \mathbb{N}$, consider the right-continuous, uniformly integrable P-martingale $\widehat{L}^{(l)}$ with $\widehat{L}^{(l)} = (\widehat{L}^{(l)})_{t\geq 0}$, defined by

$$\widehat{L}^{(l)}_\infty = \widehat{L}_{\widehat{\sigma}}\left(\mathbf{1}_{\{\widehat{\sigma}=\infty\}} + \mathbf{1}_{\{\widehat{\sigma}=\rho\}} \frac{\mathbf{1}_{B_l}}{\mathbb{E}_P[\mathbf{1}_{B_l}|\mathcal{F}_\rho]}\right).$$

Note that, for each fixed $l \in \mathbb{N}$, (12) holds with $\widehat{L}$ replaced by $\widehat{L}^{(l)}$, for each $h, i, j, k, m, n \in \mathbb{N}$. Thus, for each $l \in \mathbb{N}$, we obtain an extended Föllmer finitely additive measure $Q^{(l)}$ for the extended P-supermartingale $\overline{Z}$; see the proof of the existence statement of Theorem 3.9.

Now, for each $l \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{E}_P[\widehat{L}^{(l)}_\infty \mathbf{1}_{B_l}] &\geq \mathbb{E}_P\left[\widehat{L}_{\widehat{\sigma}} \mathbf{1}_{\{\widehat{\sigma}=\rho\}} \frac{\mathbf{1}_{B_l}}{\mathbb{E}_P[\mathbf{1}_{B_l}|\mathcal{F}_\rho]}\right] = \mathbb{E}_P[\widehat{L}_{\widehat{\sigma}} \mathbf{1}_{\{\widehat{\sigma}=\rho\}}] \\ &= 1 - \mathbb{E}_P[\widehat{L}_\rho \mathbf{1}_{\{\widehat{\sigma}>\rho\}}] \geq 1 - \frac{1+c}{2} = \frac{1-c}{2} > 0;\end{aligned}$$

thus, we also have $Q^{(l)}[B_l] \geq (1-c)/2$. Since the events $(B_l)_{l\in\mathbb{N}}$ are disjoint, there must exist more than one extended Föllmer finitely additive measure for the extended P-supermartingale $\overline{Z}$. $\square$

The previous two lemmas now yield the proof of the nonuniqueness assertion of Theorem 3.7.

First, note that the P-supermartingale Z always satisfies one (or both) of the following two conditions:

(A) $P[Z_\infty > 0] > 0$;
(B) $P[\rho < \infty] = 1$, where $\rho = \inf\{t \geq 0 : Z_t \leq 1/2\}$.

That is, either the P-supermartingale Z has positive probability to have a positive limit or it crosses $1/2$ almost surely in finite time, or both. Then Lemmas 5.12 and 5.13 yield the nonuniqueness, in both of those cases.

The corresponding statement of Theorem 3.9 needs one more lemma.

LEMMA 5.14. *Under Assumption* ($\mathcal{B}$), *there exist two extended Föllmer finitely additive measures* Q_1, Q_2 *for the extended* P*-supermartingale* $\overline{Z} = (\mathbf{1}_{[0,\infty)(t)})_{t\in[0,\infty]}$ *such that* $Q_1 \neq Q_2$.

PROOF. Consider the P-local martingale $G = (G_t)_{t\geq 0}$, defined by $G_t = \int_0^t \exp(-s)\,\mathrm{d}W_s$, and note that if we set $G_\infty = \lim_{t\uparrow\infty} G_t$, then for every $t \geq 0$ the random variable $G_\infty - G_t$ is normally distributed with nontrivial variance, and independent of $\mathcal{F}_t$. In particular, the two disjoint events $A^{(+)} = \{G_\infty > 0\}$ and $A^{(-)} = \{G_\infty < 0\}$ satisfy $\mathbb{E}_P[A^{(\dagger)}|\mathcal{F}_n] > 0$ almost surely for $\dagger \in \{-,+\}$ and for all $n \in \mathbb{N}$.

We now construct, "by hand," two sequences $(L^{(+,n)})_{n\in\mathbb{N}}$, $(L^{(-,n)})_{n\in\mathbb{N}}$ of nonnegative uniformly integrable martingales with $L^{(\dagger,n)} = (L_t^{(\dagger,n)})_{t\geq 0}$ such that $\lim_{n\uparrow\infty} L_\rho^{(\dagger,n)} = \mathbf{1}_{\{\rho<\infty\}}$ and that $\mathbb{E}_P[L_\infty^{(\dagger,n)}\mathbf{1}_{A^{(\dagger)}}] = 1$ for $\dagger \in \{-,+\}$ and for all $n \in \mathbb{N}$. This then shows the statement, by the same arguments in the proof of the existence statement of Theorem 3.7.

To construct such sequences of nonnegative uniformly integrable martingales, fix $\dagger \in \{-,+\}$ and let $\mathcal{E}^{(n)} = (\mathcal{E}_t^{(n)})_{t\geq 0}$ be a continuous nonnegative local martingale that stays constant at one up to time $n-1$ and is zero almost surely at time n for each $n \in \mathbb{N}$; for instance, such a local martingale can easily be obtained by modifying the process given in Example 5.6. Now, let ρ_n denote the first hitting time of 2^n by $\mathcal{E}^{(n)}$ for each n. The Borel–Cantelli lemma yields that $\sum_{n=1}^\infty \mathbf{1}_{\{\rho_n<\infty\}} < \infty$ almost surely. Now, for each $n \in \mathbb{N}$ define the random variable

$$L_\infty^{(\dagger,n)} = \mathcal{E}_{\rho_n\wedge n}^{(n)} \frac{\mathbf{1}_{A^{(\dagger)}}}{\mathbb{E}_P[A^{(\dagger)}|\mathcal{F}_n]},$$

note that $\mathbb{E}_P[L_\infty^{(\dagger,n)}] = 1$, and define the uniformly integrable martingale $L^{(\dagger,n)}$ as the right-continuous modification of the process $(\mathbb{E}_P[L_\infty^{(\dagger,n)}|\mathcal{F}_t])_{t\geq 0}$. It is simple to see that both sequences $(L^{(+,n)})_{n\in\mathbb{N}}$, $(L^{(-,n)})_{n\in\mathbb{N}}$ of nonnegative uniformly integrable martingales, constructed in this way, satisfy the claimed conditions, which completes the proof. $\square$

We are now ready to prove the uniqueness claims of Theorem 3.9.

It is clear that the extended Föllmer finitely additive measure is unique whenever the P-supermartingale Z is a uniformly integrable martingale, $\mathbb{E}_P[\overline{Z}_\infty] = 1$, and $\mathcal{F} = \mathcal{F}_\infty$. Thus, let us now assume that $\mathbb{E}_P[\overline{Z}_\infty] < 1$.

To make headway, write $\overline{Z}$ as the sum of a uniformly integrable P-martingale $N = (N_t)_{t\in[0,\infty]}$ and an extended P-supermartingale $G = (G_t)_{t\in[0,\infty]}$; here N is just the right-continuous modification of the process $(\mathbb{E}_P[\overline{Z}_\infty|\mathcal{F}_t])_{t\geq 0}$. Next, note that there exist two extended Föllmer finitely additive measures Q^N and Q^G corresponding to the two P-supermartingales $N/\mathbb{E}_P[N_0]$ (if $\mathbb{E}_P[N_0] > 0$, otherwise just use the null measure) and $G/\mathbb{E}_P[G_0]$. An extended Föllmer finitely additive measure Q for the extended P-supermartingale $\overline{Z}$ can then be constructed by setting $Q = \mathbb{E}_P[N_0]Q^N + \mathbb{E}_P[G_0]Q^G$. Thus, to show nonuniqueness of the extended Föllmer finitely additive measure Q for $\overline{Z}$, it is sufficient to show nonuniqueness of the extended Föllmer finitely additive measure Q^G for the extended P-supermartingale $G = (G_t)_{t\in[0,\infty]}$. For sake of notation, we thus assume from now on that $\overline{Z} \equiv G$; that is, that $\overline{Z}_\infty = 0$.

We now first consider the case that Z is a P-uniformly integrable martingale. Then there exists a (countably additive) probability measure P', defined by $P'(\mathrm{d}\omega) = Z_\infty(\omega)P(\mathrm{d}\omega)$. For the extended P'-supermartingale $(1_{[0,\infty)}(t))_{t\in[0,\infty]}$ there exist two different extended Föllmer finitely additive measures Q_1 and Q_2 according to Lemma 5.14. However, note that Q_1 and Q_2 are also extended Föllmer finitely additive measures for the extended P-supermartingale $\overline{Z}$.

We next consider the case that Z is not a P-uniformly integrable martingale. In particular, the extended P-supermartingale $\overline{Z}$ can be written as a sum of a uniformly integrable martingale and a nonzero potential. With the same arguments as above, in order to show nonuniqueness, we may assume, without loss of generality, that $\overline{Z}$ is a potential; that is, in particular, that $Z_\infty = 0 = \overline{Z}_\infty$. However, then Lemma 5.13 yields the nonuniqueness of the extended Föllmer finitely additive measure and the proof is complete.

APPENDIX A: INCOMPLETE FILTRATIONS

To dispel possible concerns about the fact that we are working with incomplete filtrations, here we collect some observations which allow us to transfer results from complete filtrations to our setting. Note that there are at least two important monographs which avoid the use of complete filtrations as far as possible, Jacod (1979) and Jacod and Shiryaev (2003).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. Write $\mathcal{F}^P$ for the P-completion of $\mathcal{F}$, and $\mathcal{N}^P$ for the P-null sets of $\mathcal{F}^P$. Then the filtration $(\mathcal{F}_t^P)_{t\geq 0} = (\mathcal{F}_t \vee \mathcal{N}^P)_{t\geq 0}$ satisfies the usual conditions. For every random variable X on $(\Omega, \mathcal{F}^P)$, there exists a random variable Y on $(\Omega, \mathcal{F})$ with $P[X = Y] = 1$.

The first result relates stopping times under $(\mathcal{F}_t)_{t\geq 0}$ and under $(\mathcal{F}_t^P)_{t\geq 0}$.

LEMMA A.1 [Jacod and Shiryaev (2003), Lemma I.1.19]. *Any stopping time on $(\Omega, \mathcal{F}^P, (\mathcal{F}_t^P)_{t\geq 0})$ is almost surely equal to a stopping time on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$.*

Comparable results hold on the level of processes.

LEMMA A.2. *Any predictable (resp., optional) process on the completion $(\Omega, \mathcal{F}^P, (\mathcal{F}_t^P)_{t\geq 0})$ is indistinguishable from a predictable (resp., optional) process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$.*

PROOF. The predictable case is Lemma I.2.17 of Jacod and Shiryaev (2003). The optional case is shown in the same way. □

LEMMA A.3. *Let $G = (G_t)_{t\geq 0}$ be an $(\mathcal{F}_t^P)_{t\geq 0}$-adapted process that it almost surely càdlàg. Then G is indistinguishable from an $(\mathcal{F}_t)_{t\geq 0}$-adapted process $\widetilde{G} = (\widetilde{G}_t)_{t\geq 0}$ which is right-continuous for* all *$\omega \in \Omega$ and which possesses left limits everywhere except at a stopping time τ with $P[\tau = \infty] = 1$. If G is almost surely nondecreasing and bounded from above, then $\widetilde{G}$ can be chosen nondecreasing and bounded from above for all $\omega \in \Omega$.*

PROOF. For each $q \in \mathbb{Q} \cap [0, \infty)$, consider an $\mathcal{F}_q$-measurable random variable $\overline{G}_q$ with $P[\overline{G}_q = G_q] = 1$. Then there exists a nondecreasing sequence $(\mathcal{N}_t)_{t\geq 0}$ of null sets with $\mathcal{N}_t \in \mathcal{F}_t$ such that the process $(\overline{G}_q)_{q\in\mathbb{Q}\cap[0,t]}$ has left and right limits for all $\omega \in \Omega \setminus \mathcal{N}_t$; see, for example, page 59 in Ethier and Kurtz (1986) for details. We then define the stopping time $\tau(\omega) = \inf\{t \geq 0 : \omega \in \mathcal{N}_t\}$ and take $\widetilde{G}$ as the right limit process of $(\overline{G}_q^\tau)_{q\in\mathbb{Q}\cap[0,\infty)}$. If G is almost surely nondecreasing, define

$$\mathcal{M}_t = \{\omega \in \Omega : \exists 0 \leq q < q' \leq t \in \mathbb{Q} \text{ such that } \overline{G}_q(\omega) > \overline{G}_{q'}(\omega)\}$$

for all $t \geq 0$ and note that $\mathcal{M}_t \in \mathcal{F}_t$ is a P-null set. We now define the stopping time τ as before, but now with $\mathcal{N}_t \cup \mathcal{M}_t$ replacing $\mathcal{N}_t$, for all $t \geq 0$, and set $\widetilde{G}_t = (G_t \wedge K)\mathbf{1}_{\{\tau>t\}} + K\mathbf{1}_{\{\tau\leq t\}}$ for all $t \geq 0$, where K is an almost sure upper bound of the process G. □

APPENDIX B: MULTIPLICATIVE DECOMPOSITION OF A SUPERMARTINGALE

In this Appendix, we discuss the multiplicative decomposition of a nonnegative supermartingale. For the nonnegative P-supermartingale Z, we shall write ${}^pZ$ to denote its predictable projection, which is the unique predictable process that is characterized by the identity ${}^pZ_\rho = \mathbb{E}_P[{}^pZ_\rho|\mathcal{F}_{\rho-}]$ on the event $\{\rho < \infty\}$ for all predictable stopping times ρ; see also Theorem 1.2.28 in Jacod and Shiryaev (2003).

Let ρ_0 denote the first hitting time of zero by the P-supermartingale Z. According to Théorème 1 in Jacod (1978) there exists a nondecreasing sequence of finite stopping times $(\rho_n)_{n\in\mathbb{N}}$ such that $(Z_{\rho_n-}) \wedge ({}^pZ_{\rho_n}) \geq 1/n$ and $\lim_{n\uparrow\infty} \rho_n = \rho_0$. Define the event $B = \bigcup_{n\in\mathbb{N}}\{\rho_n = \rho_0\}$ and denote its complement by B^c. This allows us to write $\rho_0 = \rho_0^P \wedge \rho_0^S$ for the two stopping times $\rho_0^P = \rho_0\mathbf{1}_{B^c} + \infty\mathbf{1}_{B^c}$ and $\rho_0^S = \rho_0\mathbf{1}_B + \infty\mathbf{1}_B$ It is clear that the stopping time ρ_0^P is predictable, announced by the sequence $(\rho_n + n\mathbf{1}_{\{\rho_n=\rho_0\}})_{n\in\mathbb{N}}$.

To obtain some intuition, note that ρ_0^P is finite if and only if either one of two events occurs: either Z hits zero continuously, that is, for each $n \in \mathbb{N}$ it crosses the level $1/n$ strictly before it hits zero, or Z jumps to zero, but with an announced jump. On the other side, ρ_0^S is the time when Z jumps to zero "by surprise;" by which we mean that one has not almost sure knowledge about that jump just before it occurs.

We are now ready to state an existence and uniqueness result for a multiplicative decomposition of the P-supermartingale Z.

PROPOSITION B.1 [Yoeurp (1976), Théorème 3.9]. *There exist a nonnegative local martingale $M = (M_t)_{t\geq 0}$ and a nonnegative, nonincreasing and predictable process $D = (D_t)_{t\geq 0}$ with $D_0 = 1$ and càdlàg paths such that*:

- $Z \equiv MD$;
- $M \equiv M^{\rho_0}$ *and* $D \equiv D^{\rho_0}$;
- $D_{\rho_0^P} = 0$ *on the event* $\{\rho_0^P < \infty\}$;
- M *is continuous at time* ρ_0^P *on the event* $\{\rho_0^P < \infty\}$.

These properties determine the processes M and D uniquely up to indistinguishability.

In the setting of the last proposition, D^{ρ_0} is a predictable process despite the fact that ρ_0 is in general not a predictable time; see Proposition I.2.4 in Jacod and Shiryaev (2003). Thus, assuming that $D \equiv D^{\rho_0}$ does not lead to problems.

PROOF OF PROPOSITION B.1. The assertion follows from Théorème 3.9 in Yoeurp (1976) and Corollaire 8 in Jacod (1978), which yield the existence of two nonnegative processes N and H with càdlàg paths, such that N is a local martingale on the stochastic interval $[0, \rho_0^P)$ and H is a nonincreasing and predictable process such that $Z \equiv NH$ holds. Note that we may replace the process H by $D = (H_t \mathbf{1}_{\{\rho_0^P > t\}})_{t\geq 0}$, which is also a predictable process; see, for example, Theorems 2.15.a and 2.28.c in Jacod and Shiryaev (2003). It is easy to check that we still have the decomposition $Z = ND$.

Next, an application of Proposition A.4 in Carr, Fisher and Ruf (2014) allows us to extend N to a local martingale M on the whole positive half line such that $M \equiv M^{\rho_0^P}$ and M is continuous at time ρ_0^P. We still have the decomposition $Z = MD$ since $Z_{\rho_0^P} = 0 = D_{\rho_o^P}$ on the event $\rho_0^P < \infty$. Moreover, note that we may assume, without loss of generality, that $M \equiv M^{\rho_0}$ and $D \equiv D^{\rho_0}$.

To see the asserted uniqueness, consider two processes M' and D' as in the theorem. Then, Corollaire 8 in Jacod (1978) yields that $M' \equiv M$ on $[0, \rho_0^P)$. By the required continuity of M' at time ρ_0^P we obtain that $M' \equiv M$. Corollaire 8 in Jacod (1978) also yields that $D' \equiv D$ on $[0, \rho_0)$ and that $D'_{\rho_0^S} = D_{\rho_0^S}$ on the event $\{\rho_0^S < \infty\}$. Thus, $D' \equiv D$ follows from the assumption that $D_{\rho_0^P} = 0$ and that $D' \equiv D'^{\rho_0}$. $\square$

We remark that Yoeurp (1976) does not mention a condition that corresponds to $D \equiv D^{\rho_0}$ in the formulation of Théorème 3.9. However, simple counterexamples illustrate that such a condition is needed to obtain uniqueness of the processes M and D. Both Yoeurp (1976) and Jacod (1978) assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. However, using the observations made in Appendix A, we can easily dispense with that assumption.

For another multiplicative decomposition of a given nonnegative supermartingale, see also Theorem 4.2 and Remark 4.5 in Penner and Reveillac (2015). In that decomposition, however, the nonincreasing process D is not necessarily predictable.

APPENDIX C: CERTAIN SPACES IN MEASURE THEORY

In this Appendix, we recall some measure-theoretic concepts.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces. A bijection $f: X \to Y$ is called *isomorphism between* $(X, \mathcal{X})$ *and* $(Y, \mathcal{Y})$ if both f and f^{-1} are measurable. The spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are called *isomorphic* if there exists an isomorphism between them. A bijection $\varphi: \mathcal{X} \to \mathcal{Y}$ is called *σ-isomorphism* if it preserves countable set operations, that is, if $\varphi(\bigcup_{n\in\mathbb{N}} A_n) = \bigcup_{n\in\mathbb{N}} \varphi(A_n)$ and $\varphi(\bigcap_{n\in\mathbb{N}} A_n) = \bigcap_{n\in\mathbb{N}} \varphi(A_n)$ for each sequence $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathcal{X}$ for all $n \in \mathbb{N}$. The sigma algebras $\mathcal{X}$ and $\mathcal{Y}$ are called *σ-isomorphic* if there exists a σ-isomorphism between them. Note that if the function $f: X \to Y$ is an isomorphism between the measurable spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ then the mapping $\mathcal{X} \to \mathcal{Y}$ with $A \mapsto \{f(x): x \in A\}$ is a σ-isomorphism between $\mathcal{X}$ and $\mathcal{Y}$.

A measurable space $(X, \mathcal{X})$ is called *countably generated* if there exists a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$, such that $\mathcal{X} = \sigma(A_n : n \in \mathbb{N})$. If X is a separable metrizable space, then its Borel sigma algebra $\mathcal{B}(X)$ is countably generated: if $B_r(x)$ denotes the open ball around $x \in X$ with radius $r \geq 0$ with respect to a metric that generates the topology, and if $(x_n)_{n\in\mathbb{N}}$ is a countable dense subset, then $\{B_q(x_n): n \in \mathbb{N}, q \in \mathbb{Q}, q \geq 0\}$ is a countable base for the topology, and, in particular, generates $\mathcal{B}(X)$.

DEFINITION C.1 [Parthasarathy (1967), Definition V.2.2]. A measurable space $(X, \mathcal{X})$ is called *standard Borel space* if there exists a Polish space Y such that $\mathcal{X}$ is σ-isomorphic to $\mathcal{B}(Y)$, where $\mathcal{B}(Y)$ denotes the Borel sigma algebra of Y.

LEMMA C.2. *Any standard Borel space is countably generated.*

PROOF. Let $(X, \mathcal{X})$ denote a standard Borel space and $(Y, \mathcal{B}(Y))$ the corresponding Polish space of Definition C.1. As we remarked above, $\mathcal{B}(Y)$ is countably generated. Let $(A_n)_{n\in\mathbb{N}}$ generate $\mathcal{B}(Y)$, let $\varphi: \mathcal{X} \to \mathcal{B}(Y)$ denote a σ-isomorphism, and note that

$$\mathcal{X} = \sigma\big(\{\varphi^{-1}(A_n) : n \in \mathbb{N}\}\big)$$

holds, which proves the statement. □

DEFINITION C.3. A *Lusin space* is a topological space E for which there exists a Polish space Y and a continuous bijection $f: Y \to E$. A *state space* is a metrizable Lusin space.

It can easily be seen that E is a Lusin space if and only if there exists a finer topology on E under which E becomes Polish. Each Polish space is a Lusin space and a state space. An example of a state space that is not Polish is the set $\mathbb{Q} \subset \mathbb{R}$ of rational numbers, equipped with the Euclidean metric. For the corresponding Polish space we may choose $Y = \mathbb{Q}$, equipped with the discrete topology.

We also need the notion of a standard system, introduced by Föllmer. Recall that if $\mathcal{X}$ is a sigma algebra, then a set $A \in \mathcal{X}$ is called *atom* in $\mathcal{X}$ if $B \in \mathcal{X}$ and $B \subset A$ implies $B = \varnothing$ or $B = A$.

DEFINITION C.4 [Föllmer (1972), Appendix]. Let X be a set and let $\mathcal{T}$ be a partially ordered nonempty index set. Assume that $(\mathcal{X}_t)_{t \in \mathcal{T}}$ is an nondecreasing sequence of sigma algebras on X. The "filtration" $(\mathcal{X}_t)_{t \in \mathcal{T}}$ is called *standard system* if it satisfies the following conditions:

(i) The space $(X, \mathcal{X}_t)$ is a standard Borel space for each $t \in \mathcal{T}$.

(ii) If $(t_n)_{n \in \mathbb{N}}$ is a nondecreasing sequence in $\mathcal{T}$, and if $(A_n)_{n \in \mathbb{N}}$ is a nonincreasing sequence of atoms with $A_n \in \mathcal{X}_{t_n}$ for each $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$.

The following criterion is useful for verifying whether a given sub-sigma algebra of a standard Borel space corresponds to a standard Borel space.

LEMMA C.5 [Parthasarathy (1967), Theorem V.2.4]. *Let $(X, \mathcal{X})$ be a standard Borel space, and let $\mathcal{W} \subset \mathcal{X}$ be countably generated. Then $(X, \mathcal{W})$ is a standard Borel space.*

Lemma C.5 yields, in particular, that each state space E is a standard Borel space: first note that E is separable, because if Y is a Polish space, $(y_n)_{n \in \mathbb{N}}$ is dense in Y, and $f: Y \to E$ is a continuous bijection, then $(f(y_n))_{n \in \mathbb{N}}$ is dense in E. Therefore, the Borel sigma algebra $\mathcal{B}(E)$ of E is countably generated. If now $\widetilde{\mathcal{B}}(E)$ is the Borel sigma algebra corresponding to a finer topology on E under which E is Polish, then $\mathcal{B}(E) \subset \widetilde{\mathcal{B}}(E)$, and therefore E is a standard Borel space according to Lemma C.5.

APPENDIX D: EXTENSION OF MEASURES

In this Appendix, we recall the extension theorems needed to construct Föllmer countably additive measures on the path space, and to prove their (non)uniqueness. We start with the simplest extension, when we just want to add one set to the sigma algebra.

DEFINITION D.1. Let $(X, \mathcal{X}, \mu)$ be a probability space and let $A \subset X$. The inner and outer content $\mu_*[A]$ and $\mu^*[A]$ of the set A are defined by

$$\begin{aligned}
\mu_*[A] &= \sup\{\mu[B] : B \in \mathcal{X}, B \subset A\} = \max\{\mu[B] : B \in \mathcal{X}, B \subset A\};\\
\mu^*[A] &= \inf\{\mu[B] : B \in \mathcal{X}, B \supset A\} = \min\{\mu[B] : B \in \mathcal{X}, B \supset A\},
\end{aligned} \tag{13}$$

respectively.

In (13), for example, the minimum is attained since the intersection of the events $(B_n)_{n\in\mathbb{N}}$ satisfying $B_n \supset A$ and $\mu^*[B_n] \le \mu^*[A] + 1/n$ is again in the sigma algebra $\mathcal{X}$.

LEMMA D.2 [Bierlein (1962), Satz 1A]. *Let $(X, \mathcal{X}, \mu)$ be a probability space and let $A \subset X$. There exists an extension ν of μ to $\mathcal{X} \vee \sigma\{A\}$ such that $\nu[A] = \mu^*[A]$.*

PROOF. Observe that there exists an event $\widehat{A} \in \mathcal{X}$ so that $\widehat{A} \supset A$ and $\widehat{A}^c \subset A^c$ with $\mu^*[A] = \mu[\widehat{A}]$ and $\mu_*[A^c] = \mu[\widehat{A}^c]$, where we denote complements by the superscript ^{c}. Moreover, we have

$$\mathcal{X} \vee \sigma\{A\} = \{(A \cap B_1) \cup (A^c \cap B_2) : B_1, B_2 \in \mathcal{F}\}.$$

Now, for any set $(A \cap B_1) \cup (A^c \cap B_2) \in \mathcal{X} \vee \sigma\{A\}$ define

$$\nu[(A \cap B_1) \cup (A^c \cap B_2)] = \mu[\widehat{A} \cap B_1] + \mu[\widehat{A}^c \cap B_2].$$

It is easy to see that ν is indeed a probability measure that extends μ and satisfies $\nu(A) = \mu^*[A]$. □

Next, we state Parthasarathy's extension theorem.

THEOREM D.3 [Parthasarathy (1967), Theorem V.4.1]. *Let X be a set equipped with a standard system $(\mathcal{X}_n)_{n\in\mathbb{N}}$. Let $(\mu_n)_{n\in\mathbb{N}}$ be a consistent family of probability measures on $(\mathcal{X}_n)_{n\in\mathbb{N}}$, that is, $\mu_{n+1}|_{\mathcal{X}_n} = \mu_n$ for all $n \in \mathbb{N}$. Then there exists a unique probability measure on $\bigvee_{n\in\mathbb{N}} \mathcal{X}_n$, such that $\mu|_{\mathcal{X}_n} = \mu_n$ for all $n \in \mathbb{N}$.*

THEOREM D.4. *Let $(X, \mathcal{X})$ be a standard Borel space, and let $\mathcal{W} \subset \mathcal{X}$ be a countably generated sigma algebra. Let μ be a probability measure on $\mathcal{W}$. Then there exists a probability measure ν on $\mathcal{X}$, such that $\nu|_{\mathcal{W}} = \mu$. The extension ν is unique if and only if the sigma algebra $\mathcal{X}$ is contained in the completion, with respect to the probability measure μ, of the sigma algebra $\mathcal{W}$.*

PROOF. Let $(Y, \mathcal{B}(Y))$ be a Polish space along with its Borel sigma algebra, and let $\varphi: \mathcal{X} \to \mathcal{B}(Y)$ be a σ-isomorphism between $\mathcal{X}$ and $\mathcal{B}(Y)$. Define $\mathcal{G} = \{\varphi(A) : A \in \mathcal{W}\} \subset \mathcal{B}(Y)$. It is not hard to check that $\mathcal{G}$ is a sigma algebra, that $\mathcal{G}$ is countably generated, and that $\mathcal{W} = \varphi^{-1}(\mathcal{G})$. Define the measure $m = \mu \circ \varphi^{-1}$ on $\mathcal{G}$. If we can extend m to a measure n on $\mathcal{B}(Y)$, then the proof is complete, because then we can set $\nu = n \circ \varphi$.

The existence of an extension of m from $\mathcal{G}$ to $\mathcal{B}(Y)$ is shown, for example, in Theorem 5 in Lubin (1974). The result in Lubin (1974) is formulated for Blackwell spaces rather than Polish spaces. However, each Polish space is a Blackwell space, as defined in Lubin (1974).

The uniqueness result follows from Theorem 2 in Ascherl and Lehn (1977). We only need to show that if $A \in \mathcal{X}$, and if $\widetilde{\mu}$ is an extension of μ to $\sigma(\mathcal{W} \cup \{A\})$, then there exists an extension $\widetilde{\nu}$ of μ on $\mathcal{X}$, such that $\widetilde{\nu}|_{\sigma(\mathcal{W}\cup\{A\})} = \widetilde{\mu}$. However, the existence of such an extension is an immediate consequence of the existence result since $\sigma(\mathcal{W} \cup \{A\})$ is again countably generated. □

APPENDIX E: PROPERTIES OF THE CANONICAL PATH SPACE

We now collect some properties of the path space $(\Omega, \mathcal{F})$ of Assumption $(\mathcal{P})$.

LEMMA E.1 [Föllmer (1972), Appendix]. *Under Assumption $(\mathcal{P})$, we have the following statements*:

1. *The probability space $(\Omega, \mathcal{F})$ is standard Borel.*
2. *Let ρ denote a stopping time. Then the sigma algebra $\mathcal{F}_{\rho-}$ is countably generated.* [*Recall the definition of $\mathcal{F}_{\rho-}$ in* (4).]
3. *Let $(\rho_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of $(\mathcal{F}_t)_{t\geq 0}$-stopping times. Then the filtration $(\mathcal{F}_{\rho_n-})_{n\in\mathbb{N}}$ is a standard system.*
4. *The set identity $\mathcal{F}_{\zeta-} = \mathcal{F}$ holds.*

PROOF. Meyer has shown [see page 100 in Dellacherie (1969)] that there exists a Polish space Y, with Borel sigma algebra $\mathcal{B}(Y)$, such that $(\Omega, \mathcal{F})$ is isomorphic to $(Y, \mathcal{B}(Y))$. This implies the first part of the statement.

For the second statement, note that the set equality

$$\{\rho > t\} = \bigcup_{q\in\mathbb{Q}: q>t} \{\rho > q\}$$

holds and $\mathcal{F}_t^0$ is countably generated for each $t \geq 0$; see also the observations after Lemma C.5. Thus, if $(A_m^{(t)})_{m\in\mathbb{N}}$ is a countable generating system for $\mathcal{F}_t^0$ for each $t \geq 0$, then

$$\mathcal{F}_{\rho-} = \sigma\big(A_m^{(0)}, A_m^{(q)} \cap \{\rho > q\} : q \in \mathbb{Q} \cap [0, \infty), m \in \mathbb{N}\big).$$

For the third statement, property (i) in the definition of a standard system follows directly from Lemma C.5. Property (ii) is easy to check.

For the fourth statement, it suffices to show that for each Borel set $B \subset E \cup \{\Delta\}$ and each $t \geq 0$ we have $\{\omega : \omega(t) \in B\} \in \mathcal{F}_{\zeta-}$. Toward this end, note that

$$\{\omega(t) \in B\} = (\{\omega(t) \in B\} \cap \{\zeta \leq t\}) \cup (\{\omega(t) \in B\} \cap \{\zeta > t\}).$$

The first event on the right-hand side equals $\{\zeta \leq t\}$ if B contains Δ, and it is the empty set otherwise. Therefore, that event is in $\mathcal{F}_{\zeta-}$. The second event is in $\mathcal{F}_{\zeta-}$ by definition. □

THEOREM E.2. *Under Assumption ($\mathcal{P}$), let ρ be a stopping time and μ a probability measure on $(\Omega, \mathcal{F}_{\rho-})$. Then the measure μ can be extended to a probability measure ν on $(\Omega, \mathcal{F})$ such that $\nu|_{\mathcal{F}_{\rho-}} = \mu$. Moreover, that extension ν is unique if and only if the set $\{\rho < \zeta\}$ is μ-negligible.*

PROOF. Since $(\Omega, \mathcal{F})$ is a standard Borel space and $\mathcal{F}_{\rho-}$ is countably generated by Lemma E.1, Theorem D.4 implies the existence of an extension ν of μ from $\mathcal{F}_{\rho-}$ to $\mathcal{F}$. Moreover, the extension ν is unique if and only if $\mathcal{F}_{\zeta-} = \mathcal{F} \subset \mathcal{F}^{\mu}_{\rho-}$, where the first equality follows from Lemma E.1 and where the completion of a sigma algebra $\mathcal{X}$ with respect to the probability measure μ is denoted by $\mathcal{X}^{\mu}$.

Assume now that the set $\{\rho < \zeta\}$ is μ-negligible. To prove that $\mathcal{F}_{\zeta-} \subset \mathcal{F}^{\mu}_{\rho-}$, it suffices to show that $A \cap \{\zeta > t\} \in \mathcal{F}^{\mu}_{\rho-}$ for all $A \in \mathcal{F}_t$ and $t \geq 0$. Toward this end, note that

$$A \cap \{\zeta > t\} = (A \cap \{\zeta > t\} \cap \{\rho > t\}) \cup (A \cap \{\zeta > t\} \cap \{\rho \leq t\})$$

for all $A \in \mathcal{F}_t$ and $t \geq 0$. The first event on the right-hand side is in $\mathcal{F}_{\rho-}$ since $\{\zeta > t\} \in \mathcal{F}_t$ holds for all $t \geq 0$. The second event on the right-hand side is contained in the μ-negligible set $\{\rho < \zeta\}$ and, therefore, it is an element of $\mathcal{F}^{\mu}_{\rho-}$.

For the reverse direction, assume that the set $\{\rho < \zeta\}$ is not μ-negligible, which implies that its outer content is strictly positive, that is, $\mu^*[\rho < \zeta] > 0$. By Lemma D.2, there exists an extension $\widehat{\nu}$ from $\mathcal{F}_{\rho-}$ to $\mathcal{F}_{\rho-} \vee \sigma(\{\rho < \zeta\})$, such that $\widehat{\nu}[\rho < \zeta] > 0$. Since $\mathcal{F}_{\rho-} \vee \sigma(\{\rho < \zeta\})$ is countably generated, Theorem D.4 yields an extension ν of $\widehat{\nu}$ from $\mathcal{F}_{\rho-} \vee \sigma(\{\rho < \zeta\})$ to $\mathcal{F}$.

Next, fix a sufficiently large $n \in \mathbb{N}$ so that the stopping time $\widetilde{\rho} = \rho + 2/n$ satisfies $\nu[\widetilde{\rho} < \zeta] > 0$ and define the measurable function $\phi : \Omega \to \Omega$ by $\phi(\omega)(t) = \omega(t)$ for all $t \in [0, \widetilde{\rho})$ and $\phi(\omega)(t) = \Delta$ for all $t \in [\widetilde{\rho}, \infty)$. This mapping introduces a new probability measure $\widetilde{\nu} = \nu \circ \phi^{-1}$, such that $\widetilde{\nu}[\widetilde{\rho} < \zeta] = 0$.

Observe furthermore that $\rho + 1/n$ is an $(\mathcal{F}^0_t)_{t \geq 0}$-stopping time and that $\mathcal{F}_{\rho-} \subset \mathcal{F}^0_{\rho+1/n} = \sigma\{\omega(t \wedge (\rho + 1/n)) : t \geq 0\}$, where the last equality can be shown, for example, as in Lemma 1.3.3 in Stroock and Varadhan (2006). We conclude that $\widehat{\nu}|_{\mathcal{F}_{\rho-}} = \nu|_{\mathcal{F}_{\rho-}}$ but $\widetilde{\nu}[\widetilde{\rho} < \zeta] = 0 < \nu[\widetilde{\rho} < \zeta]$, and thus the extension is not unique. □

LEMMA E.3. *Under Assumption* ($\mathcal{P}$), *let* μ *be a probability measure on* $(\Omega, \mathcal{F})$. *Then the set*

$$A = \{x \in E : \mu[\{\omega : \exists t \geq 0, \varepsilon > 0, \text{ s.t. } \omega(s) = x \text{ for all } s \in [t, t+\varepsilon)\}] > 0\} \tag{14}$$

is at most countable.

PROOF. The proof is an adaption of the arguments in Lemma 3.7.7 of Ethier and Kurtz (1986). Let $T, \varepsilon, \delta > 0$. We claim that the set

$$A_{T,\varepsilon,\delta} = \{x \in E : \mu[\{\omega : \exists t \in [0, T] \text{ s.t. } \omega(s) = x \text{ for all } s \in [t, t+\varepsilon)\}] > \delta\}$$

is finite. If the claim holds, then the set inclusion $A \subset \bigcup_{n\in\mathbb{N}} A_{n,1/n,1/n}$ yields the statement.

To prove this claim, assume that there exists an infinite sequence $(x_n)_{n\in\mathbb{N}}$ in $A_{T,\varepsilon,\delta}$, where $x_n \neq x_m$ whenever $n \neq m$ for all $n, m \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\mu\Bigg[\bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} \{\omega : \exists t \in [0, T] \text{ s.t. } \omega(s) = x_n \text{ for all } s \in [t, t+\varepsilon)\}\Bigg] \\
&\qquad = \lim_{k\uparrow\infty} \mu\Bigg[\bigcup_{n\geq k} \{\omega : \exists t \in [0, T] \text{ s.t. } \omega(s) = x_n \text{ for all } s \in [t, t+\varepsilon)\}\Bigg] > \delta.
\end{aligned}$$

However, for each $\omega \in \Omega$ there are at most $\lfloor (T+\varepsilon)/\varepsilon \rfloor$ values of $x \in E$ for which there exists $t \in [0, T]$ such that $\omega(s) = x$ for all $s \in [t, t+\varepsilon)$, a contradiction. Here, $\lfloor \cdot \rfloor$ denotes the Gauss bracket. □

APPENDIX F: SUPERMARTINGALES AS FATOU LIMITS OF MARTINGALES

Similar techniques as developed in Section 5.1.2 allow us to show that each nonnegative P-supermartingale is the Fatou limit of a sequence of uniformly integrable martingales, provided that the probability space supports a Brownian motion. Toward this end, recall the definition of Fatou convergence.

DEFINITION F.1. A sequence of processes $(G^{(n)})_{n\in\mathbb{N}}$ with $G^{(n)} = (G_t^{(n)})_{t\geq 0}$ *Fatou converges* to a process $G = (G_t)_{t\geq 0}$ if there exists a dense subset $\mathcal{T}$ of $[0, \infty)$, such that

$$\liminf_{s\downarrow t, s\in\mathcal{T}} \Big(\liminf_{n\uparrow\infty} G_s^{(n)}\Big) = \limsup_{s\downarrow t, s\in\mathcal{T}} \Big(\limsup_{n\uparrow\infty} G_s^{(n)}\Big) = G_t$$

holds for all $t \geq 0$ almost surely.

THEOREM F.2. *Under Assumption* ($\mathcal{B}$), *let the* P*-supermartingale* Z *have Doob–Meyer decomposition* $Z = M + D$. *Then there exists a sequence of uniformly integrable nonnegative martingales* $(Z^{(m)})_{m\in\mathbb{N}}$ *with* $Z^{(m)} = (Z_t^{(m)})_{t\geq 0}$ *such that there exists a dense subset* $\mathcal{T}$ *of* $[0,\infty)$, *whose complement is a Lebesgue null set, such that* $\lim_{m\uparrow\infty} Z_t^{(m)} = M_t + D_{t-}$ *for all* $t \in \mathcal{T}$ *almost surely. In particular,* $(Z^{(m)})_{m\in\mathbb{N}}$ *Fatou converges to the* P*-supermartingale* Z.

PROOF. Let us start by approximating the left-continuous process D_- by a sequence $(N^{(m)})_{m\in\mathbb{N}}$ of local martingales with $N^{(m)} = (N_t^{(m)})_{t\geq 0}$, similarly to Lemmas 5.5 and 5.7. Toward this end, fix $m \in \mathbb{N}$, set $N_0^{(m)} = D_0$, and keep $N^{(m)}$ constant up to time 2^{-m}. Initiate then a "mass loss phase" such that $N^{(m)}$ fluctuates on $(2^{-m}, 2^{-m} + 2^{-3m})$, until it reaches $D_{2^{-m}}$ at time $2^{-m} + 2^{-3m}$. Now, $N^{(m)}$ stays constant on $[2^{-m} + 2^{-3m}, 2 \times 2^{-m}]$, until at time 2×2^{-m} we initiate the next mass loss phase, so that $N^{(m)}$ fluctuates again on an interval of length 2^{-3m}, until it reaches $D_{2\times 2^{-m}}$. For $k = 3, \ldots, m2^m$ we continue by having mass loss phases on the interval $(k2^{-m}, k2^{-m} + 2^{-3m})$, at the end of which we reach $D_{m2^{-m}}$. From time $m + 2^{-3m}$ on, the process $N^{(m)}$ stays constant.

Next, set

$$\mathcal{S} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \bigcup_{k=0}^{m2^m} (k2^{-m}, k2^{-m} + 2^{-3m}) \tag{15}$$

and note that $\mathcal{T} = [0,\infty) \setminus \mathcal{S}$ is a dense subset of $[0,\infty)$ since the set $\mathcal{S}$ has Lebesgue measure zero. The set $\mathcal{S}$ can be interpreted as the set of all points that lie in infinitely many "mass loss intervals." Now, fix $t \in \mathcal{T}$ and note that $(N_t^{(m)})_{m\in\mathbb{N}}$ is eventually a nonincreasing sequence with $\lim_{m\uparrow\infty} N_t^{(m)} = D_{t-}$. Moreover, observe that $N^{(m)}$ almost surely attains a finite maximal absolute value, and therefore there exists a constant $K_m > 0$ such that for $\rho_m = \inf\{t \geq 0 : |N_t^{(m)}| \geq K_m\}$ we have $P[\rho_m < \infty] < 2^{-m}$ for each $m \in \mathbb{N}$.

Next, define the uniformly integrable nonnegative martingales

$$Z^{(m)} = M^{\tau_m^M \wedge \rho_m} + (N^{(m)})^{\tau_m^M \wedge \rho_m},$$

for all $m \in \mathbb{N}$, where $(\tau_m^M)_{m\in\mathbb{N}}$ is a localizing sequence for the P-local martingale M. An application of the Borel–Cantelli lemma then yields that $\lim_{m\uparrow\infty} Z_t^{(m)} = M_t + D_{t-}$ for all $t \in \mathcal{T}$ almost surely. The Fatou convergence follows directly from the right-continuity of the P-supermartingale Z. □

REMARK F.3. In the proof of Theorem F.2, we used the fact that $\mathcal{S}$, the set of all points that lie in infinitely many "mass loss intervals" given in (15), has Lebesgue measure zero. One might suspect that the set $\mathcal{S}$ is countable or even empty. However, this is not true. There exists a suitable strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ with $m_n \in \mathbb{N}$ for each $n \in \mathbb{N}$ such that the map $\varphi: \{0,1\}^{\mathbb{N}} \to [0,2)$,

$(a_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} a_n 2^{-m_n}$ satisfies $\varphi((a_n)_{n\in\mathbb{N}}) \in \mathcal{S}$ for each sequence $(a_n)_{n\in\mathbb{N}}$ that satisfies $\sum_{n=1}^{\infty} a_n = \infty$. As a consequence, $\mathcal{S}$ is a Cantor-like set, in the sense that it is an uncountable Lebesgue-null set.

APPENDIX G: FINITELY ADDITIVE MEASURES ON THE DYADIC FILTRATION

Here, we show that on the unit interval equipped with the dyadic filtration, no finitely additive probability measure is uniquely determined by its values on the dyadic algebra generating the Borel σ-algebra.

Let $\Omega = (0, 1]$ be equipped with the Borel sigma field $\mathcal{F} = \mathcal{B}(\Omega)$, let

$$\mathcal{F}_n = \sigma\big(\big(k2^{-n}, (k+1)2^{-n}\big] : 0 \le k \le 2^n - 1\big)$$

for all $n \in \mathbb{N}$, and let P be a finitely additive probability measure on $(\Omega, \mathcal{F})$. We will construct a finitely additive measure $\widetilde{P} \neq P$ such that $\widetilde{P}$ agrees with P on the algebra $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$.

Let $A = \{x_m : m \in \mathbb{N}\}$ denote a countable dense subset of $(0, 1]$, such that $P[A] < 1$. Such a set has to exist since there are disjoint countable dense subsets of $(0, 1]$, for example, $\mathbb{Q} \cap (0, 1]$ and $(\pi + \mathbb{Q}) \cap (0, 1]$. Next, for each $k, n \in \mathbb{N}$ such that $k \le 2^n - 1$ we define $y_k^{(n)} = x_{m(n,k)}$, where $m(n, k)$ is the smallest integer m with $x_m \in A \cap (k2^{-n}, (k+1)2^{-n}]$, and define the set function $\widetilde{P}^{(n)} : \mathcal{F} \to [0, 1]$ by

$$\widetilde{P}^{(n)}[B] = \sum_{k=0}^{2^n-1} P\big[\big(k2^{-n}, (k+1)2^{-n}\big]\big] \mathbf{1}_B\big(y_k^{(n)}\big)$$

for all $B \in \mathcal{F}$. By construction, we have $\widetilde{P}^{(n)}|_{\mathcal{F}_n} = P|_{\mathcal{F}_n}$. Note that $\widetilde{P}^{(n)}$ is a countably additive probability measure and, therefore,

$$\widetilde{Q} = \sum_{n=0}^{\infty} 2^{-n-1} \widetilde{P}^{(n)}$$

is a countably additive probability measure such that $\widetilde{P}^{(n)}$ is absolutely continuous with respect to $\widetilde{Q}$ for each $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, the set function $\widetilde{P}^{(n)}$ can be identified with an element of the unit ball of $(L^\infty(\widetilde{Q}))^*$ and, therefore, the Banach–Alaoglu theorem implies the existence of a subsequence $(\widetilde{P}^{(n_k)})_{k\in\mathbb{N}}$ that converges in $(L^\infty(\widetilde{Q}))^*$ to a finitely additive probability measure $\widetilde{P}$. By construction, $\widetilde{P}|_{\mathcal{F}_n} = P|_{\mathcal{F}_n}$ for all $n \in \mathbb{N}$, and $\widetilde{P}[A] = 1$, whereas $P[A] < 1$.

For instance, we could take P as the Lebesgue measure. Carathéodory's extension theorem then implies the uniqueness of the extension of P from $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ to $\mathcal{F}$. However, as we just illustrated, this extension is only unique among the sigma additive measures, not among the larger class of finitely additive measures.

APPENDIX H: ALTERNATIVE PROOF OF LEMMA 5.12

We here provide a proof of Lemma 5.12 that is more "constructive" than the one in Section 5.2. It relies on the following lemma.

LEMMA H.1. *Let $Q = Q^r + Q^s$ be a finitely additive probability measure, where Q^r is the regular part and Q^s is the singular part. If $Q^r \neq 0$ is absolutely continuous with respect to P, then there exists $\varepsilon > 0$, such that $Q[A] < 1$ for each event $A \in \mathcal{F}$ with $P[A] < \varepsilon$.*

PROOF. Assume that there exists no such $\varepsilon > 0$ as in the statement. Then, for each $n \in \mathbb{N}$, there exists an event $A_n \in \mathcal{F}$ such that $P[A_n] \leq 1/n$ but $Q[A_n] = 1$. The dominated convergence theorem then implies that

$$1 = \lim_{n\uparrow\infty} Q[A_n] = \lim_{n\uparrow\infty}\left(\mathbb{E}_P\left[\frac{\mathrm{d}Q^r}{\mathrm{d}P}\mathbf{1}_{A_n}\right] + Q^s[A_n]\right) = \lim_{n\uparrow\infty} Q^s[A_n],$$

so that $Q^s[\Omega] = 1 = Q[\Omega]$, a contradiction to $Q^r \neq 0$. □

PROOF OF LEMMA 5.12. Define the nonnegative martingale $Z^{(1)} = (Z_t^{(1)})_{t\geq 0}$ as the right-continuous modification of the uniformly integrable P-martingale $(\mathbb{E}_P[Z_\infty|\mathcal{F}_t])_{t\geq 0}$ and the P-potential $Z^{(2)} = (Z_t^{(2)})_{t\geq 0}$ by $Z^{(2)} = Z - Z^{(1)}$. There exist a Föllmer finitely additive measure $Q^{(2)}$ for the P-potential $Z^{(2)}/\mathbb{E}_P[Z_0^{(2)}]$ (assuming that $\mathbb{E}_P[Z_0^{(2)}] > 0$, otherwise set the measure to zero) and a Föllmer countably additive measure $Q^{(1)}$ for the uniformly integrable P-martingale $Z^{(1)}/\mathbb{E}_P[Z_0^{(1)}]$, yielding a Föllmer finitely additive measure $Q = \mathbb{E}_P[Z_0^{(1)}]Q^{(1)} + \mathbb{E}_P[Z_0^{(2)}]Q^{(2)}$ for the P-supermartingale Z. Note that $Q^r > 0$ since $Q^r(\mathrm{d}\omega) = Z_\infty(\omega)P(\mathrm{d}\omega) \neq 0$.

Next, for all $n \in \mathbb{N}$ and $\varepsilon > 0$, choose $k_{n,\varepsilon} \in \mathbb{N}$, such that $P[A_{n,\varepsilon}] \leq \varepsilon 2^{-n}$, where

$$A_{n,\varepsilon} = \{|W_t - W_n| < 1 \text{ for all } t \in [n, k_{n,\varepsilon}]\}.$$

Set $A_\varepsilon = \bigcup_{n\in\mathbb{N}} A_{n,\varepsilon}$ and note that $P[A_\varepsilon] \leq \varepsilon$. By Lemma H.1 we have $Q[A_\varepsilon] < 1$ for some $\varepsilon > 0$. Fix such an ε and recall Proposition 5.9. We now replace the nonnegative P-martingale $L^{(i,j,k,m,n)}$ in (11) by the process $\widehat{L}^{(i,j,k,m,n)} = (\widehat{L}_t^{(i,j,k,m,n)})_{t\geq 0}$, given by

$$\widehat{L}_t^{(i,j,k,m,n)} = L_{t\wedge n}^{(i,j,k,m,n)}\mathbb{E}_P\left[\frac{\mathbf{1}_{A_{n,\varepsilon}}}{\mathbb{E}_P[\mathbf{1}_{A_{n,\varepsilon}}|\mathcal{F}_n]}\middle|\mathcal{F}_t\right]$$

for all $t \geq 0$, for each $i, j, k, m, n \in \mathbb{N}$. Note that (11) holds after this replacement and the same argument as in the existence proof of Theorem 3.7 yields a Föllmer finitely additive measure $\widehat{Q}$, but now we have $\widehat{Q}[A_\varepsilon] = 1$, which completes the proof. □

This proof illustrates that it is always possible to "destroy" some possibly remaining regular part of a Föllmer finitely additive measure at infinity.

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