The Annals of Probability
2015, Vol. 43, No. 5, 2511–2544
DOI: 10.1214/14-AOP942
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EXACT ROSENTHAL-TYPE BOUNDS

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It is shown that, for any given $p \geq 5$, $A > 0$ and $B > 0$, the exact upper bound on $\mathsf{E}|\sum X_i|^p$ over all independent zero-mean random variables (r.v.'s) $X_1, \dots, X_n$ such that $\sum \mathsf{E}X_i^2 = B$ and $\sum \mathsf{E}|X_i|^p = A$ equals $c^p \mathsf{E}|\Pi_\lambda - \lambda|^p$, where $(\lambda, c) \in (0, \infty)^2$ is the unique solution to the system of equations $c^p\lambda = A$ and $c^2\lambda = B$, and Π_λ is a Poisson r.v. with mean λ. In fact, a more general result is obtained, as well as other related ones. As a tool used in the proof, a calculus of variations of moments of infinitely divisible distributions with respect to variations of the Lévy characteristics is developed.

1. Introduction, summary, and discussion. Let $\mathscr{X}$ denote the class of all finite sequences $\mathbf{X} = (X_1, \dots, X_n)$ of independent zero-mean random variables (r.v.'s). For any $\mathbf{X} = (X_1, \dots, X_n) \in \mathscr{X}$, let

$$S_{\mathbf{X}} := X_1 + \cdots + X_n. \tag{1}$$

Take any real number

$$p > 2 \tag{2}$$

and any positive real numbers A and B. Consider

$$\mathscr{X}_{p;A,B} := \left\{\mathbf{X} = (X_1, \dots, X_n) \in \mathscr{X} : \sum_1^n \mathsf{E}X_i^2 = B, \sum_1^n \mathsf{E}|X_i|^p = A\right\}, \tag{3}$$

$$\mathscr{X}_{p;\leq A,\leq B} := \left\{\mathbf{X} = (X_1, \dots, X_n) \in \mathscr{X} : \sum_1^n \mathsf{E}X_i^2 \leq B, \sum_1^n \mathsf{E}|X_i|^p \leq A\right\}, \tag{4}$$

$$\mathscr{E}_{p;A,B} := \sup\bigl\{\mathsf{E}|S_{\mathbf{X}}|^p : \mathbf{X} \in \mathscr{X}_{p;A,B}\bigr\}, \tag{5}$$

$$\mathscr{E}_{p;\leq A,\leq B} := \sup\bigl\{\mathsf{E}|S_{\mathbf{X}}|^p : \mathbf{X} \in \mathscr{X}_{p;\leq A,\leq B}\bigr\}. \tag{6}$$

Rosenthal's upper bound (Theorem 3 in [30]) can be presented by the inequality

$$\mathscr{E}_{p;A,B} \leq C_p \max\bigl(A, B^{p/2}\bigr), \tag{7}$$

Received June 2013; revised May 2014.
MSC2010 subject classifications. Primary 60E15; secondary 60E07.
Key words and phrases. Rosenthal inequality, bounds on moments, sums of independent random variables, probability inequalities, calculus of variations, infinitely divisible distributions, Lévy characteristics.

with $C_p := (p/2)^{p/2} 2^{p+p^2/4}$. In particular, this implies that $\mathscr{E}_{p;A,B} < \infty$. For some of the subsequent developments, see, for example, Sections 4 and 5 in [11], [8, 12] and references therein.

PROPOSITION 1.1. *One has* $\varnothing \neq \mathscr{X}_{p;A,B} \subseteq \mathscr{X}_{p;\le A,\le B}$. *Moreover, one has the homogeneity property* $\mathscr{E}_{p;\kappa^p A,\kappa^2 B} = \kappa^p \mathscr{E}_{p;A,B}$ *for all real* $\kappa > 0$. *Furthermore,* $\mathscr{E}_{p;A,B}$ *is nondecreasing in* A *and in* B *and hence*

$$\mathscr{E}_{p;\le A,\le B} = \mathscr{E}_{p;A,B}. \tag{8}$$

All the necessary proofs are deferred to Sections 2 and 3. In particular, Proposition 1.1 will be proved in Section 3.

Using Proposition 1.1, one can easily see (cf. [8]) that the problem of finding a good expression of $\mathscr{E}_{p;A,B}$ is equivalent to that of finding, for an arbitrary balancing parameter $\gamma \in (0,\infty)$, a good expression of the best constant $C_{p;\gamma}$ in the Rosenthal-type inequality

$$\mathscr{E}_{p;A,B} \le C_{p;\gamma} \max(\gamma A, B^{p/2}); \tag{9}$$

cf. (7). Indeed, one has:

PROPOSITION 1.2. $C_{p;\gamma} = \mathscr{E}_{p;1/\gamma,1}$ *and* $\mathscr{E}_{p;A,B} = B^{p/2} C_{p;B^{p/2}/A}$.

The idea of balancing the contributions of the terms A and $B^{p/2}$ in the Rosenthal-type bounds, depending on the relative sizes of these terms, goes back at least to the Corollary in [26]; see also Sections 4 and 5, Remark 6.8, and Theorem 8.3 in [11], Proposition 9.2 in [19], Corollaries 3.1, 3.2 in [23], and Corollaries 2, 3, 4 in [20].

For any real $\lambda > 0$, let Π_λ denote a r.v. with the Poisson distribution with mean λ, and then introduce the corresponding centered r.v.

$$\tilde{\Pi}_\lambda := \Pi_\lambda - \lambda.$$

Using Theorem 4 by Utev [32], Bestsennaya and Utev [1] showed that

$$\mathscr{E}_{p;A,B} = c^p \mathsf{E}|\tilde{\Pi}_\lambda|^p \qquad \text{if } p = 4, 6, \dots, \tag{10}$$

where

$$\lambda := \lambda_p(A,B) := \left(\frac{B^{p/2}}{A}\right)^{2/(p-2)} \quad \text{and} \quad c := c_p(A,B) := \left(\frac{A}{B}\right)^{1/(p-2)}, \tag{11}$$

so that the pair $(\lambda, c) \in (0,\infty)^2$ is the unique solution to the system of equations

$$c^2\lambda = B \quad \text{and} \quad c^p\lambda = A.$$

Obviously, if p is an even natural number, then the absolute pth moment $\mathsf{E}|X|^p$ of a r.v. X is the same as its pth moment $\mathsf{E}X^p$. This fact allows the proof in [1] to

be based on the well-known representation of moments in terms of cumulants and the log-convexity of $\int_{\mathbb{R}} |x|^r G(\mathrm{d}x)$ in $r > 0$, for any nonnegative measure G.

Under the additional restriction that the X_i's be symmetric(ally distributed), exact Rosenthal-type bounds were obtained in [4, 7, 8, 32]. In particular, it was shown by Utev [32] that

$$\sup_{\substack{\mathbf{X}\in\mathscr{X}_{p;\le A,\le B},\\ \mathbf{X}\text{ is symmetric}}} \mathsf{E}|S_{\mathbf{X}}|^p = \sup_{\substack{\mathbf{X}\in\mathscr{X}_{p;A,B},\\ \mathbf{X}\text{ is symmetric}}} \mathsf{E}|S_{\mathbf{X}}|^p = c^p \mathsf{E}\big|\Pi_{\lambda/2} - \Pi^{\diamond}_{\lambda/2}\big|^p \tag{12}$$

if $p > 4$, where λ and c are as in (11), and $\Pi^{\diamond}_{\lambda/2}$ is an independent copy of $\Pi_{\lambda/2}$.

Take any

$$q \in (2, p] \tag{13}$$

and then take any r.v. X such that

$$\mathsf{E}|X|^q < \infty. \tag{14}$$

Consider

$$\begin{aligned} \mathscr{X}_{p;X;A,B} &:= \{\mathbf{X} \in \mathscr{X}_{p;A,B} : \mathbf{X} \text{ is independent of } X\}, \\ \mathscr{X}_{p;X;\le A,\le B} &:= \{\mathbf{X} \in \mathscr{X}_{p;\le A,\le B} : \mathbf{X} \text{ is independent of } X\}. \end{aligned} \tag{15}$$

The main result of the present paper is:

THEOREM 1.3. *Suppose that $p \ge q \ge 5$ and $\mathsf{E}X = 0$. Then*

$$\begin{aligned} \sup_{\mathbf{X}\in\mathscr{X}_{p;X;\le A,\le B}} \mathsf{E}|X + S_{\mathbf{X}}|^q &= \sup_{\mathbf{X}\in\mathscr{X}_{p;X;A,B}} \mathsf{E}|X + S_{\mathbf{X}}|^q \\ &= \max\big(\mathsf{E}|X + c\tilde{\Pi}_\lambda|^q, \mathsf{E}|X - c\tilde{\Pi}_\lambda|^q\big), \end{aligned} \tag{16}$$

where λ and c are as in (11), *and the r.v. $\tilde{\Pi}_\lambda$ is independent of X.*

In the special case when $X = 0$ and $q = p$, Theorem 1.3 yields

$$\mathscr{E}_{p;A,B} = \mathscr{E}_{p;\le A,\le B} = c^p \mathsf{E}|\tilde{\Pi}_\lambda|^p \qquad \text{if } p \ge 5;$$

cf. (10).

Allowing q in Theorem 1.3 to differ from p not only provides a more general result, but also helps with the proof. Indeed, Theorem 1.3 will be first proved in the case when $p > q > 5$ [see (114)], and then the proof will be completed by limit transitions in q and in p.

REMARK 1.4. It is of substantial interest to obtain exact Rosenthal-type inequalities for moment functions more general than the function $|\cdot|^p$ used in Theorem 1.3; cf., for example, [3, 4]. In fact, one can indeed easily extend the result of Theorem 1.3 to the class of all moment functions of the form

$$x \longmapsto \int_{[5,p]\times[0,\infty)} (a+x)_+^r \nu_1(\mathrm{d}r \times \mathrm{d}a) + \int_{[5,p]\times[0,\infty)} (a-x)_+^r \nu_2(\mathrm{d}r \times \mathrm{d}a), \tag{17}$$

where ν_1 and ν_2 are any nonnegative Borel measures on the set $[5, p] \times [0, \infty)$ such that the resulting moment function is real-valued; of course, the moment function $x \mapsto |x|^p (= x_+^p + (-x)_+^p)$ is just one member of this class; as usual, we let $x_+ := 0 \vee x$ and $x_+^r := (x_+)^r$ for all real x and all real $r > 0$. To see why this extension of Theorem 1.3 is valid, one needs to look at the place in the proof of the theorem that imposes the narrowest restriction on the moment function—which is the condition that the difference $h''(u\alpha s) - u^{p-4}h''(\alpha s)$, considered in (72), be strictly positive for all u, α, and s in $(0, 1)$. The class of functions given by (17) may be compared with classes of moment functions considered, for example, in [10, 13, 15].

In what follows, to avoid repetitiveness, it is assumed that the different instances of

$$\boxed{\text{all r.v.'s entering the same expression are independent.}}$$

Thus, conditions such as that of the independence of the r.v.'s $\tilde{\Pi}_\lambda$ and X in Theorem 1.3 may not be explicitly stated in the sequel.

Theorem 1.3 is complemented by:

THEOREM 1.5. *Suppose that* $p \in (2, 3]$ *and* $\mathsf{E}|X|^p < \infty$ (*the condition* $\mathsf{E}X = 0$ *is not needed here*). *Then*

$$\sup_{\mathbf{X}\in\mathscr{X}_{p;X;\le A,\le B}} \mathsf{E}|X + S_{\mathbf{X}}|^p = \sup_{\mathbf{X}\in\mathscr{X}_{p;X;A,B}} \mathsf{E}|X + S_{\mathbf{X}}|^p = A + \mathsf{E}\big|X + B^{1/2}Z\big|^p. \tag{18}$$

Here and in what follows, $Z \sim N(0, 1)$, *unless specified otherwise*.

Theorem 1.5 is based on a result by Tyurin [31]. In the case $p = 3$, important for applications to Berry–Esseen bounds, a certain refinement of (18) was obtained in Corollary 2 from [22], based on the main result in the paper [16], a shorter version of which appeared in [24].

One has the following interpretation of the last expression in (18), in terms of centered Poisson r.v.'s $\tilde{\Pi}_{\lambda_1}$ and $\tilde{\Pi}_{\lambda_2}$ (such that the r.v.'s X, $\tilde{\Pi}_{\lambda_1}$, and $\tilde{\Pi}_{\lambda_2}$ are independent).

PROPOSITION 1.6. *Suppose that* $p \in (2,3]$ *and* $\mathsf{E}|X|^p < \infty$. *Then*

$$\begin{aligned} &A + \mathsf{E}\big|X + B^{1/2}Z\big|^p \\ &\quad = \lim\big(\mathsf{E}|X + c_1\tilde{\Pi}_{\lambda_1} + c_2\tilde{\Pi}_{\lambda_2}|^p : \\ &\qquad\qquad (c_1, c_2, \lambda_1, \lambda_2) \in Q_{p;A,B}, c_1 \to 0, |c_2| \to \infty\big), \end{aligned} \tag{19}$$

where

$$\begin{aligned} Q_{p;A,B} := \big\{&(c_1, c_2, \lambda_1, \lambda_2) \in \mathbb{R}^2 \times (0,\infty)^2 : \\ &c_1^2\lambda_1 + c_2^2\lambda_2 = B, |c_1|^p\lambda_1 + |c_2|^p\lambda_2 = A\big\}. \end{aligned} \tag{20}$$

Proposition 1.6 will be useful in the proof of Theorem 1.5.

Now one can present a unified form of the exact upper bounds in (16) and (18):

COROLLARY 1.7. *Suppose that* $p \in (2,3] \cup [5,\infty)$ *and* $\mathsf{E}|X|^p < \infty$. *For* $p \in [5,\infty)$, *also suppose that* $\mathsf{E}X = 0$. *Then*

$$\begin{aligned} &\sup_{\mathbf{X} \in \mathscr{X}_{p;X;\le A,\le B}} \mathsf{E}|X + S_{\mathbf{X}}|^p \\ &\quad = \sup_{\mathbf{X} \in \mathscr{X}_{p;X;A,B}} \mathsf{E}|X + S_{\mathbf{X}}|^p \\ &\quad = \sup\big\{\mathsf{E}|X + c_1\tilde{\Pi}_{\lambda_1} + c_2\tilde{\Pi}_{\lambda_2}|^p : (c_1, c_2, \lambda_1, \lambda_2) \in Q_{p;A,B}\big\}. \end{aligned} \tag{21}$$

By Theorem 1.5 and Proposition 1.6, for $p \in (2,3]$ the last supremum in (21) is "attained in the limit" as $c_1 \to 0$ and $|c_2| \to \infty$, whereas, by Theorem 1.3, for $p \ge 5$ the same supremum is (actually) attained at $(c_1, c_2, \lambda_1, \lambda_2) = (c, 0, \lambda, 0)$ or at $(c_1, c_2, \lambda_1, \lambda_2) = (-c, 0, \lambda, 0)$, where λ and c are as in (11).

The cases $p \in (3,4)$ and $p \in (4,5)$ remain open. Certain considerations suggest that Theorem 1.5 should hold for $p \in (3,4)$ as well, whereas Theorem 1.3 should hold for $p \in (4,5)$—at least when $q = p$. For $q = p = 4$, it is easy to see that the "answers" in (16) and (18) coincide with each other:

$$\mathsf{E}|X + c\tilde{\Pi}_\lambda|^4 = \mathsf{E}|X - c\tilde{\Pi}_\lambda|^4 = A + \mathsf{E}\big|X + B^{1/2}Z\big|^4.$$

This situation may be compared with the one concerning the exact Khinchin-type upper bound. There the summands are weighted independent Rademacher r.v.'s $X_1 = a_1\varepsilon_1, \dots, X_n = a_n\varepsilon_n$, where $\mathsf{P}(\varepsilon_i = \pm 1) = 1/2$, and the weights $a_1, \dots, a_n$ are real numbers subject to the restriction $\sum_1^n a_i^2 = 1$. Since these summands have each a simplest symmetric distribution, and there is only one restriction here on the sum of the moments, $\sum_1^n \mathsf{E}X_i^2 = \sum_1^n a_i^2$, it appears that the problem of the exact Khinchin-type upper bound is significantly simpler than its Rosenthal-type counterpart. Indeed, in 1960 Whittle [33] gave a very simple proof of the exact Khinchin-type upper bound, $\mathsf{E}|Z|^p$, for the case $p \ge 3$. The proof in [33] was

based on the fact that, again for $p \geq 3$, the second derivative of $|x|^p$ in x is convex in $x \in \mathbb{R}$. It was claimed in [33] that the result holds for all real $p \geq 2$, but that was not supported by the proof. Actually, the problem of the exact Khinchin-type upper bound in the case $p \in (2, 3)$ turned to be very difficult and was solved only in 1981 by Haagerup [6]. Haagerup's proof was somewhat simplified in [9]; see also [25]. One may speculate that the case $p \geq 5$ in the Rosenthal-type context is parallel to the case $p \geq 3$ in the Khinchin-type one, whereas the Rosenthal-type case of a small noninteger $p \in (3, 4) \cup (4, 5)$ is parallel to the Khinchin-type case of $p \in (2, 3)$. If so, the remaining Rosenthal-type case of $p \in (3, 4) \cup (4, 5)$ may be exceedingly difficult, on comparing the treatment of the Rosenthal-type case of $p \geq 5$ in the present paper with that of the Khinchin-type case of $p \geq 3$ in [33]. One may also note here that the condition $p \geq 5$ will be used twice, and in rather different ways, in the proof of Theorem 1.3, namely in the proofs of Propositions 2.9 and 2.11.

For the symmetric case, one has:

THEOREM 1.8. *Suppose that $p \geq q \geq 5$ and $\mathsf{E}X = 0$. Then*

$$\begin{aligned} &\sup_{\substack{\mathbf{X} \in \mathscr{X}_{p;X;\leq A,\leq B},\\ \mathbf{X} \text{ is symmetric}}} \mathsf{E}|X + S_{\mathbf{X}}|^q \\ &\quad = \sup_{\substack{\mathbf{X} \in \mathscr{X}_{p;X;A,B},\\ \mathbf{X} \text{ is symmetric}}} \mathsf{E}|X + S_{\mathbf{X}}|^q = \mathsf{E}\big|X + c\Pi_{\lambda/2} - c\Pi^{\diamond}_{\lambda/2}\big|^q, \end{aligned} \tag{22}$$

where λ and c are as in (11) *and, as in* (12), $\Pi^{\diamond}_{\lambda/2}$ *is an independent copy of* $\Pi_{\lambda/2}$.

Theorem 1.8 generalizes (12), but only for $p \geq 5$. The generalization has two aspects: (i) letting q differ from p and (ii) introducing the extra summand X. Note that X is not required to be symmetric in Theorem 1.8.

An advantage of having the extra summand X is illustrated by the following straightforward combination of Theorems 1.3 and 1.8.

COROLLARY 1.9. *Suppose that $p \geq q \geq 5$ and $\mathsf{E}X = 0$. Take any positive real numbers A_0, B_0, A_1, B_1. For each $j \in \{0, 1\}$, let $\lambda_j := \lambda_p(A_j, B_j)$ and $c_j := c_p(A_j, B_j)$, in accordance with* (11). *Then*

$$\begin{aligned} \sup_{\substack{\mathbf{X} \in \mathscr{X}_{p;\leq A_0,\leq B_0},\\ \mathbf{Y} \in \mathscr{X}_{p;\leq A_1,\leq B_1},\\ \mathbf{X} \text{ is symmetric},\\ X,\mathbf{X},\mathbf{Y} \text{ are independent}}} \mathsf{E}|X + S_{\mathbf{X}} + S_{\mathbf{Y}}|^q &= \sup_{\substack{\mathbf{X} \in \mathscr{X}_{p;A_0,B_0},\\ \mathbf{Y} \in \mathscr{X}_{p;A_1,B_1},\\ \mathbf{X} \text{ is symmetric},\\ X,\mathbf{X},\mathbf{Y} \text{ are independent}}} \mathsf{E}|X + S_{\mathbf{X}} + S_{\mathbf{Y}}|^q \\ &= \mathsf{E}\big|X + c_0\Pi_{\lambda_0/2} - c_0\Pi^{\diamond}_{\lambda_0/2} + c_1\tilde{\Pi}_{\lambda_1}\big|^q \\ &\quad \vee \mathsf{E}\big|X + c_0\Pi_{\lambda_0/2} - c_0\Pi^{\diamond}_{\lambda_0/2} - c_1\tilde{\Pi}_{\lambda_1}\big|^q. \end{aligned} \tag{23}$$

This follows immediately from Theorems 1.8 and 1.3, by taking first the supremum in $\mathbf{X}$ (say) and then in $\mathbf{Y}$.

Note that, in the case when X is symmetric (or, in particular, zero), the maximum in (23) simplifies to $\mathsf{E}|X + c_0\Pi_{\lambda_0/2} - c_0\Pi^{\diamond}_{\lambda_0/2} + c_1\tilde{\Pi}_{\lambda_1}|^q$.

Corollary 1.9 may be useful when some, but not all, of the independent summands are known to be symmetric.

For the calculation of absolute moments, especially such more complicated ones as in the maximum expression in (23), Fourier- or Fourier–Laplace-type identities such as those given in [18] can be effective; one of such identities will be reproduced in the present paper as (40).

2. Proof of Theorem 1.3.

2.1. *Domination by the accompanying compound Poisson distribution.*

THEOREM A. *Let $f:\mathbb{R}\to\mathbb{R}$ be any twice continuously differentiable function such that f and f'' are convex. Let G be any finite nonnegative Borel measure on $\mathbb{R}$ such that $G(\{0\}) = 0$ and $\int_{\mathbb{R}} xG(\mathrm{d}x) = 0$, and then let X_G be any r.v. with the characteristic function $t \mapsto \exp\int_{\mathbb{R}}(e^{itx} - 1)G(\mathrm{d}x)$. Then*

$$\sup\bigl\{\mathsf{E}f(S_{\mathbf{X}}) : \mathbf{X} \in \mathscr{X}, G_{\mathbf{X}} = G\bigr\} = \mathsf{E}f(X_G),$$

where $S_{\mathbf{X}}$ is as in (1) and $G_{\mathbf{X}}$ is the "sum of the tails" measure defined by

$$G_{\mathbf{X}}(E) := \sum \mathsf{P}\bigl(X_i \in E \setminus \{0\}\bigr)$$

for all Borel subsets E of $\mathbb{R}$. In particular, for all $x \in \mathbb{R}$ and all real $p \geq 3$,

$$\sup\bigl\{\mathsf{E}|S_{\mathbf{X}} - x|^p : \mathbf{X} \in \mathscr{X}, G_{\mathbf{X}} = G\bigr\} = \mathsf{E}|X_G - x|^p,$$
$$\sup\bigl\{\mathsf{E}(S_{\mathbf{X}} - x)_+^p : \mathbf{X} \in \mathscr{X}, G_{\mathbf{X}} = G\bigr\} = \mathsf{E}(X_G - x)_+^p.$$

Theorem A is essentially the same as the mentioned Theorem 4 by Utev [32]; cf. [14, 27, 29]. The assumptions on f in Theorem 4 from [32], were slightly different; namely, it was assumed there that f'' is convex whereas f is nonnegative and satisfies a certain limited growth condition, which latter may be dropped, by Proposition 1 and Lemma 4 in [15], provided that f and f'' are convex, as in Theorem A.

REMARK. If a r.v. X has a finite expectation and a function $f:\mathbb{R}\to\mathbb{R}$ is convex, then, by Jensen's inequality, $\mathsf{E}f(X)$ always exists in $(-\infty, \infty]$.

Let us complement Theorem A by the following standard lemma; cf., for example, [27, 28] or the paragraphs containing formulas (6.1) and (6.2) in [11].

LEMMA 2.1. *Let G be any finite nonnegative Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x|^p G(\mathrm{d}x) = A$ and $\int_{\mathbb{R}} x^2 G(\mathrm{d}x) = B$; such a measure G exists. Let then X_G be any r.v. with the characteristic function $t \mapsto \exp \int_{\mathbb{R}} (e^{itx} - 1 - itx) G(\mathrm{d}x)$. Then there exists a sequence $(\mathbf{Z}_n)$ in $\mathscr{X}_{p;A,B}$ such that $S_{\mathbf{Z}_n} \xrightarrow{\mathrm{D}} X_G$, where $\xrightarrow{\mathrm{D}}$ denotes the convergence in distribution. In particular, it follows that $\mathscr{X}_{p;A,B} \neq \varnothing$.*

The conditions on G in Lemma 2.1 are different from those in Theorem A. In particular, the conditions $G(\{0\}) = 0$ and $\int_{\mathbb{R}} x G(\mathrm{d}x) = 0$ are not required in Lemma 2.1. However, when the condition $\int_{\mathbb{R}} x G(\mathrm{d}x) = 0$ does hold, the definition of the r.v. X_G in Lemma 2.1 is consistent with that in Theorem A. Also, the condition $p \geq 5$ imposed in Theorem 1.3 is not needed in Lemma 2.1; rather, it is enough to assume there that the general condition (2) holds.

PROOF OF LEMMA 2.1. First, concerning the existence of G, note that all the conditions on G imposed in Lemma 2.1 are satisfied by the measure $\lambda \delta_c$, where λ and c are as in (11) and δ_u denotes the Dirac probability measure at u.

Next, for each natural n and all $j \in \{1, \dots, n\}$, let

$$Z_{j,n} := W_{j,n} - \mathsf{E} W_{j,n}, \tag{24}$$

where the $W_{j,n}$'s are independent identically distributed r.v.'s with the distribution determined by the condition that

$$\mathsf{E} f(W_{j,n}) = f(0) + \frac{\kappa_n}{n} \int_{\mathbb{R}} [f(\gamma_n x) - f(0)] G(\mathrm{d}x) \tag{25}$$

for all (say) bounded or nonnegative Borel functions $f : \mathbb{R} \to \mathbb{R}$, and where in turn κ_n and γ_n are positive real numbers such that $\frac{\kappa_n}{n} \int_{\mathbb{R}} G(\mathrm{d}x) \leq 1$; the latter condition is precisely what is needed for formula (25) to define a probability distribution. It follows that for $r \in \{2, p\}$,

$$\sum_1^n \mathsf{E}|Z_{j,n}|^r = n \mathsf{E}|Z_{1,n}|^r = F_r\left(\frac{1}{n}, \kappa_n, \gamma_n\right),$$

where

$$F_r(\alpha, \kappa, \gamma) := |\kappa \gamma m_G|^r |\alpha|^{r-1} \operatorname{sign} \alpha + \kappa \gamma^r \int_{\mathbb{R}} (|x - \kappa \alpha m_G|^r - |\kappa \alpha m_G|^r) G(\mathrm{d}x)$$

and $m_G := \int_{\mathbb{R}} x G(\mathrm{d}x)$. Introducing now the vector function $\mathbf{F} := (F_2, F_p)$, we see that it is continuously differentiable on $\mathbb{R} \times (0, \infty)^2$, and the Jacobian matrix $\begin{pmatrix} \frac{\partial F_2}{\partial \kappa} & \frac{\partial F_2}{\partial \gamma} \\ \frac{\partial F_p}{\partial \kappa} & \frac{\partial F_p}{\partial \gamma} \end{pmatrix}$ at the point $(\alpha, \kappa, \gamma) = (0, 1, 1)$ is $\begin{pmatrix} B & 2B \\ A & pA \end{pmatrix}$, which is nonsingular. Moreover, $\mathbf{F}(0, 1, 1) = (B, A)$. So, by the implicit function theorem, there exist a positive real number α_0 and continuously differentiable functions $\tilde{\kappa} : (-\alpha_0, \alpha_0) \to \mathbb{R}$ and $\tilde{\gamma} : (-\alpha_0, \alpha_0) \to \mathbb{R}$ such that $\tilde{\kappa}(0) = \tilde{\gamma}(0) = 1$ and $\mathbf{F}(\alpha, \tilde{\kappa}(\alpha), \tilde{\gamma}(\alpha)) = (B, A)$

for all $\alpha \in (-\alpha_0, \alpha_0)$. For all natural $n > 1/\alpha_0$, letting now $\kappa_n := \tilde{\kappa}(\frac{1}{n})$ and $\gamma_n := \tilde{\gamma}(\frac{1}{n})$, one sees that $\sum_1^n \mathsf{E}|Z_{j,n}|^2 = B$ and $\sum_1^n \mathsf{E}|Z_{j,n}|^p = A$, so that $\mathbf{Z}_n := (Z_{1,n}, \dots, Z_{n,n}) \in \mathscr{X}_{p;A,B}$. Thus, indeed $\mathscr{X}_{p;A,B} \neq \varnothing$.

Moreover, $\kappa_n \to \tilde{\kappa}(0) = 1$ and $\gamma_n \to \tilde{\gamma}(0) = 1$ (the convergence in this context is of course as $n \to \infty$). So, by (24) and (25),

$$\mathsf{E}\exp(itS_{\mathbf{Z}_n}) = \left[1 + \frac{\kappa_n}{n}\int_{\mathbb{R}} \big(e^{it\gamma_n x} - 1\big)G(\mathrm{d}x)\right]^n e^{-it\kappa_n\gamma_n m_G}$$
$$\longrightarrow \quad \exp\int_{\mathbb{R}} \big(e^{itx} - 1 - itx\big)G(\mathrm{d}x) = \mathsf{E}\exp(itX_G)$$

for all real t, so that indeed $S_{\mathbf{Z}_n} \xrightarrow{\mathrm{D}} X_G$. $\square$

2.2. *Zero-mean truncation of zero-mean r.v.'s.*

PROPOSITION 2.2. *Let Y be any zero-mean r.v. Then for any real $M > 0$ there is an r.v. Y_M with the following properties*:

(i) $\mathsf{E}Y_M = 0$;
(ii) $|Y_M| \le M \wedge |Y|$;
(iii) $\mathsf{E}f(Y_M) \le \mathsf{E}f(Y)$ *for all convex functions* $f : \mathbb{R} \to \mathbb{R}$;
(iv) $Y_M \to Y$ *almost surely* (*a.s.*) *as* $M \to \infty$.

This follows immediately from Proposition 3.15 in [17], and Jensen's inequality on letting

$$Y_M := Y\,\mathrm{I}(E_M) = \mathsf{E}(Y|\mathscr{F}_M),$$

where $E_M := \{|Y| \le M, |\mathsf{r}(Y, U)| \le M\}$, U is any r.v. which is independent of Y and uniformly distributed on the unit interval $[0, 1]$, r stands for the reciprocating function of (the distribution of) the r.v. Y in accordance with the definition (formula (2.6) in [17]), and $\mathscr{F}_M$ is the σ-algebra generated by all events of the form $E_M \cap \{Y \le y, U \le u\}$ with any real y and u. Note that, by Proposition 3.6 in [17], $|\mathsf{r}(Y, U)| < \infty$ a.s. The r.v. U, which may be referred to as a randomizing r.v., is used to split atoms of the distribution of Y, as such splitting may be needed to satisfy the condition $\mathsf{E}Y_M = 0$.

2.3. *Differentiation under the integral sign.* Take any measurable space $(\Omega, \mathscr{F})$ with a measure $\mu : \mathscr{F} \to \mathbb{C}$. Take also any $t_* \in (0, \infty)$. Let $f : \Omega \times [0, t_*) \to \mathbb{R}$. Suppose that for each $t \in [0, t_*)$ the function $\Omega \ni \omega \mapsto f(\omega, t)$ is μ-integrable, and let

$$F(t) := \int_\Omega \mu(\mathrm{d}\omega) f(\omega, t).$$

Suppose also that, for each $\omega \in \Omega$, the function $[0, t_*) \ni t \mapsto f(\omega, t)$ is continuous and has a right-continuous right-hand side derivative $[0, t_*) \ni t \mapsto (\partial_2 f)(\omega, t) \in [-\infty, \infty]$ such that the function $\Omega \ni \omega \mapsto (\partial_2 f)(\omega, t)$ is $\mathscr{F}$-measurable, for each $t \in [0, t_*)$.

LEMMA 2.3. *Suppose that for each pair $(t, \varepsilon) \in [0, t_*) \times (0, \infty)$ there exist a set $\Omega_{t,\varepsilon} \in \mathscr{F}$, a measurable function $g_{t,\varepsilon} : \Omega_{t,\varepsilon} \to [0, \infty]$ and a real number $h_{t,\varepsilon} \in (0, t_* - t)$ such that $\int_{\Omega_{t,\varepsilon}} |\mathrm{d}\mu| g_{t,\varepsilon} < \infty$,*

$$\big|(\partial_2 f)(\omega, v)\big| \le g_{t,\varepsilon}(\omega) \qquad \textit{for all } (\omega, v) \in \Omega_{t,\varepsilon} \times [t, t + h_{t,\varepsilon}) \tag{26}$$

and

$$\sup_{v \in [t, t+h_{t,\varepsilon})} \int_{\Omega \setminus \Omega_{t,\varepsilon}} \big|\mu(\mathrm{d}\omega)(\partial_2 f)(\omega, v)\big| \underset{\varepsilon \downarrow 0}{\longrightarrow} 0. \tag{27}$$

Then

$$F'(t+) := \lim_{h \downarrow 0} \frac{F(t+h) - F(t)}{h} = I(t) := \int_\Omega \mu(\mathrm{d}\omega)(\partial_2 f)(\omega, t) \in \mathbb{C}. \tag{28}$$

The following lemma is a special case of Lemma 2.3.

LEMMA 2.4. *Suppose that there exists a μ-integrable function $g : \Omega \to [0, \infty]$ such that*

$$\big|(\partial_2 f)(\omega, t)\big| \le g(\omega) \qquad \textit{for all } (\omega, t) \in \Omega \times [0, t_*). \tag{29}$$

Then (28) *holds.*

Lemma 2.4 is apparently rather common; cf., for example, Theorem (2.27)(b) in [5]. Lemma 2.3 will be used in the proof of Proposition 2.11. More generally, this lemma should be useful in certain situations when the condition (29) of the boundedness of $(\partial_2 f)(\omega, t)$ in t for each ω is violated. More specifically, in such situations (i) $(\partial_2 f)(\omega, t)$ could have blow-up singularities and hence be unbounded in t for each ω in a somewhat "small" exceptional set $\Omega \setminus \Omega_\varepsilon$, and yet (ii) the integration of $|(\partial_2 f)(\omega, t)|$ with respect to $|\mu(\mathrm{d}\omega)|$ would smooth out the singularities, resulting in a small value of the integral over the "small" set $\Omega \setminus \Omega_\varepsilon$—as is assumed in (27). Even though such situations seem rather natural and their treatment is rather straightforward, I have been unable to find in the literature a statement similar enough to Lemma 2.3. So, for the readers' convenience, a proof of Lemma 2.3 is provided below.

PROOF OF LEMMA 2.3. Take any $t \in [0, t_*)$, $\varepsilon \in (0, \infty)$, and $h \in (0, h_{t,\varepsilon})$. Then

$$\begin{aligned} \frac{F(t+h) - F(t)}{h} &= \int_\Omega \mu(\mathrm{d}\omega) \frac{f(\omega, t+h) - f(\omega, t)}{h} \\ &= \int_\Omega \mu(\mathrm{d}\omega) \int_0^1 \mathrm{d}s (\partial_2 f)(\omega, t + sh) = I_{1,\varepsilon,h}(t) + I_{2,\varepsilon,h}(t), \end{aligned} \tag{30}$$

where

$$I_{1,\varepsilon,h}(t) := \int_{\Omega_{t,\varepsilon}} \mu(\mathrm{d}\omega) \int_0^1 \mathrm{d}s (\partial_2 f)(\omega, t + sh)$$

and

$$I_{2,\varepsilon,h}(t) := \int_{\Omega\setminus\Omega_{t,\varepsilon}} \mu(\mathrm{d}\omega) \int_0^1 \mathrm{d}s (\partial_2 f)(\omega, t + sh).$$

In view of the right continuity of $(\partial_2 f)(\omega, t)$ in t and the condition (26), by the dominated convergence theorem,

$$I_{1,\varepsilon,h}(t) \underset{h\downarrow 0}{\longrightarrow} I_{1,\varepsilon,0}(t) = \int_{\Omega_{t,\varepsilon}} \mu(\mathrm{d}\omega)(\partial_2 f)(\omega, t). \tag{31}$$

It also follows that the integral $I_{1,\varepsilon,0}(t)$ exists in the Lebesgue sense (and is finite). Next,

$$\sup_{h\in[0,h_{t,\varepsilon})} \left|I_{2,\varepsilon,h}(t)\right| \le \sup_{\substack{h\in[0,h_{t,\varepsilon})\\ s\in(0,1)}} \int_{\Omega\setminus\Omega_{t,\varepsilon}} \left|\mu(\mathrm{d}\omega)(\partial_2 f)(\omega, t + sh)\right| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0 \tag{32}$$

by (27). It also follows from (27) that the integral

$$I_{2,\varepsilon,0}(t) = \int_{\Omega\setminus\Omega_{t,\varepsilon}} \mu(\mathrm{d}\omega)(\partial_2 f)(\omega, t)$$

exists in the Lebesgue sense (and is finite) provided that ε is small enough. So, the integral $I(t)$, defined in (28), exists in the Lebesgue sense (and is finite), since $I(t) = I_{1,\varepsilon,0}(t) + I_{2,\varepsilon,0}(t)$. Moreover, (32) implies $I_{2,\varepsilon,0}(t) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0$. Hence, $I_{1,\varepsilon,0}(t) = I(t) - I_{2,\varepsilon,0}(t) \underset{\varepsilon\downarrow 0}{\longrightarrow} I(t)$. Combining now (30), (31) and (32), one completes the proof of the lemma. □

2.4. *A calculus of variations of moments of infinitely divisible distributions with respect to variations of the Lévy characteristics.* For any finite nonnegative Borel measure H on $\mathbb{R}$, let Y_H denote any r.v. such that

$$\mathsf{E}e^{itY_H} = \exp\left\{-t^2 \int_{\mathbb{R}} H(\mathrm{d}u)(R_1 \exp)(0; itu)\right\} \tag{33}$$

for all $t \in \mathbb{R}$, where, for any $m \in \{0, 1, \ldots\}$, any $(m+1)$-times continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$, and any real x and u,

$$(R_m g)(x; u) := \begin{cases} \dfrac{1}{u^{m+1}}\left(g(x+u) - \displaystyle\sum_{j=0}^{m} \frac{u^j}{j!} g^{(j)}(x)\right), & \text{if } u \neq 0, \\ \dfrac{1}{(m+1)!} g^{(m+1)}(x), & \text{if } u = 0 \end{cases} \tag{34}$$

$$= \frac{1}{m!} \int_0^1 \mathrm{d}s (1-s)^m g^{(m+1)}(x + su). \tag{35}$$

This definition of Y_H is valid, as the right-hand side expression in (33) does define a characteristic function (c.f.) of (an infinitely divisible) probability distribution, which is the weak limit of a sequence of centered compound Poisson distributions. Let

$$\binom{z}{j} := \frac{z(z-1)\cdots(z-j+1)}{j!} \qquad \text{for all } z \in \mathbb{C} \text{ and all } j \in \{0, 1, \dots\}.$$

LEMMA 2.5. *Take any $q \in (2, \infty)$, $t_0 \in (0, \infty)$ and $\sigma \in (0, \infty)$. Let H be a nonnegative Borel measure on $\mathbb{R}$, and let Δ be a real-valued Borel measure on $\mathbb{R}$ such that the measure*

$$H_t := H + t\Delta \tag{36}$$

is nonnegative for all $t \in [0, t_0]$. Let Y be a r.v. independent of Y_{H_t}, where Y_{H_t} is defined according to (33). Suppose also that

$$\int_{\mathbb{R}} \big(\mathsf{P}(Y \in \mathrm{d}u) + H(\mathrm{d}u) + |\Delta(\mathrm{d}u)|\big) e^{\sigma|u|} < \infty. \tag{37}$$

Then for all $t \in [0, t_0)$

$$\begin{aligned} &\left(\frac{\partial}{\partial t}\right)^+ \mathsf{E}|Y + Y_{H_t}|^q \\ &\qquad = 2!\binom{q}{2} \int_{\mathbb{R}} \Delta(\mathrm{d}u) \int_0^1 \mathrm{d}s(1-s)\mathsf{E}|su + Y + Y_{H_t}|^{q-2}, \end{aligned} \tag{38}$$

where $(\frac{\partial}{\partial t})^+$ denotes the right-hand side partial derivative in t.

Moreover, if $q > 4$, then for all $t \in [0, t_0)$

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2}\right)^+ \mathsf{E}|Y + Y_{H_t}|^q \\ &\qquad = 4!\binom{q}{4} \int_{\mathbb{R}^2} \Delta(\mathrm{d}u_1)\Delta(\mathrm{d}u_2) \\ &\qquad\quad \times \int_{(0,1)^2} \mathrm{d}s_1\,\mathrm{d}s_2(1-s_1)(1-s_2)\mathsf{E}|s_1u_1 + s_2u_2 + Y + Y_{H_t}|^{q-4}, \end{aligned} \tag{39}$$

where $(\frac{\partial^2}{\partial t^2})^+ := (\frac{\partial}{\partial t})^+(\frac{\partial}{\partial t})^+$ denotes the second right-hand side partial derivative in t.

Identities (38) and (39) hold if the four instances therein of the absolute-value function $x \mapsto |x|$ are replaced by the four instances of the positive-part function $x \mapsto x_+ := 0 \vee x$ or the four instances of the negative-part function $x \mapsto x_- := (-x)_+$.

PROOF. By Theorem 1 in [18],

$$\mathsf{E}(Y + Y_{H_t})_+^q = \kappa_q \int_{\mathfrak{Re}\, z = \sigma} \frac{\mathrm{d}z}{z^{q+1}} \mathsf{E}e^{zY}\, \mathsf{E}\exp\{zY_{H_t}\}, \tag{40}$$

where $x_+^q := (x_+)^q$ for all $x \in \mathbb{R}$ and

$$\kappa_q := \frac{\Gamma(q+1)}{2\pi i}; \tag{41}$$

here and below in this proof, by default, $t \in [0, t_0)$. By (33) and analytic continuation, for all $z \in \mathbb{C}$ with $\mathfrak{Re}\, z = \sigma$

$$\mathsf{E}\exp\{zY_{H_t}\} = \exp\left\{z^2 \int_{\mathbb{R}} H_t(\mathrm{d}u)(R_1 \exp)(0; zu)\right\}, \tag{42}$$

whence, by (36),

$$\left(\frac{\partial}{\partial t}\right)^+ \mathsf{E}\exp\{zY_{H_t}\} = \mathsf{E}\exp\{zY_{H_t}\} z^2 \int_{\mathbb{R}} \Delta(\mathrm{d}u)(R_1 \exp)(0; zu). \tag{43}$$

In view of (34), $\mathfrak{Re}[z^2(R_1 \exp)(0; zu)] \le \sigma^2 (R_1 \exp)(0; \sigma u) \le \sigma^2 e^{\sigma|u|}/2$ and $|(R_1 \exp)(0; zu)| \le e^{\sigma|u|}/2$ for all $u \in \mathbb{R}$ and all $z \in \mathbb{C}$ with $\mathfrak{Re}\, z = \sigma > 0$. It follows by (42) and (37) that, again for all $z \in \mathbb{C}$ with $\mathfrak{Re}\, z = \sigma > 0$,

$$\sup_{t \in [0,t_0]} \left|\mathsf{E}\exp\{zY_{H_t}\}\right| \le \exp\left\{\frac{\sigma^2}{2} \int_{\mathbb{R}} \left(H(\mathrm{d}u) + t_0 |\Delta(\mathrm{d}u)|\right) e^{\sigma|u|}\right\} < \infty$$

and

$$\left|\int_{\mathbb{R}} \Delta(\mathrm{d}u)(R_1 \exp)(0; zu)\right| \le \int_{\mathbb{R}} |\Delta(\mathrm{d}u)| e^{\sigma|u|}/2 < \infty.$$

Also, $\int_{\mathfrak{Re}\, z=\sigma} |z^2 \frac{\mathrm{d}z}{z^{q+1}}| < \infty$, since $q > 2$ and $\sigma \in (0, \infty)$. So, by Lemma 2.4,

$$\left(\frac{\partial}{\partial t}\right)^+ \mathsf{E}(Y + Y_{H_t})_+^q = \kappa_q \int_{\mathfrak{Re}\, z=\sigma} \frac{\mathrm{d}z}{z^{q+1}} \mathsf{E}e^{zY} \left(\frac{\partial}{\partial t}\right)^+ \mathsf{E}\exp\{zY_{H_t}\}. \tag{44}$$

Further, by (35), $(R_1 \exp)(0; zu) = \int_0^1 \mathrm{d}s (1-s) e^{szu}$. Hence, by (44), (43), the Fubini theorem, (41), and (40),

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}\right)^+ \mathsf{E}(Y + Y_{H_t})_+^q \\
&\quad = \kappa_q \int_{\mathfrak{Re}\, z=\sigma} \frac{\mathrm{d}z}{z^{q+1}} \mathsf{E}e^{zY} \mathsf{E}e^{zY_{H_t}} z^2 \int_{\mathbb{R}} \Delta(\mathrm{d}u) \int_0^1 \mathrm{d}s (1-s) e^{szu} \\
&\quad = \frac{\kappa_q}{\kappa_{q-2}} \int_{\mathbb{R}} \Delta(\mathrm{d}u) \int_0^1 \mathrm{d}s (1-s) \kappa_{q-2} \int_{\mathfrak{Re}\, z=\sigma} \frac{\mathrm{d}z}{z^{q-1}} \mathsf{E}e^{z(su+Y)} \mathsf{E}e^{zY_{H_t}} \\
&\quad = 2! \binom{q}{2} \int_{\mathbb{R}} \Delta(\mathrm{d}u) \int_0^1 \mathrm{d}s (1-s) \mathsf{E}(su + Y + Y_{H_t})_+^{q-2}.
\end{aligned}$$

This proves (38) for the function $x \mapsto x_+$ in place of the function $x \mapsto |x|$. Now (39), again for the function $x \mapsto x_+$, follows by Lemma 2.4.

The case of the function $x \mapsto x_-$ can be considered quite similarly. Alternatively, this case can be simply reduced to the case of the function $x \mapsto x_+$ by observing that, with $H_t^-(\mathrm{d}u) := H_t(-\mathrm{d}u)$, one has $-Y_{H_t} \overset{\mathrm{D}}{=} Y_{H_t^-}$, where $\overset{\mathrm{D}}{=}$ denotes the equality in distribution.

Finally, the case of the function $x \mapsto |x|$ follows immediately from the considered two cases by the obvious identity $|x|^r = x_+^r + x_-^r$ for all $r \in (0, \infty)$ and $x \in \mathbb{R}$. $\square$

Results similar to Lemma 2.5, but for general moment functions f in place of the power-like moment functions $|\cdot|^q$, $\cdot_+^q$ and $\cdot_-^q$ in Lemma 2.5, were obtained by lengthier direct probabilistic arguments in earlier versions of this paper [21]. It is possible to obtain such more general results by the Fourier–Laplace method as well, by decomposing f into harmonics, the way this was done in [18] for the function $\cdot_+^p$. However, this possibility will not be pursued here.

2.5. *Main propositions in the proof of Theorem* 1.3. Let $\mathscr{H}$ denote the set of all nonnegative Borel measures on $\mathbb{R}$. Take any real numbers $p > 3$, $A > 0$, $B > 0$, and $M > 0$, and introduce the following subsets of the set $\mathscr{H}$:

$$\mathscr{H}_{p;A,B} := \left\{ H \in \mathscr{H} : \int H(\mathrm{d}x) = B, \int |x|^{p-2} H(\mathrm{d}x) = A \right\}, \tag{45}$$

$$\mathscr{H}_{p;\le A,\le B} := \left\{ H \in \mathscr{H} : \int H(\mathrm{d}x) \le B, \int |x|^{p-2} H(\mathrm{d}x) \le A \right\}, \tag{46}$$

$$\mathscr{H}_{p;A,B;M} := \left\{ H \in \mathscr{H}_{p;A,B} : \operatorname{supp} H \subseteq [-M, M] \right\}, \tag{47}$$

$$\mathscr{H}_{p;\le A,\le B;M} := \left\{ H \in \mathscr{H}_{p;\le A,\le B} : \operatorname{supp} H \subseteq [-M, M] \right\}, \tag{48}$$

where $\operatorname{supp} H$ stands for the support set of the measure H; we also write $\int$ for $\int_{\mathbb{R}}$. Note that the set $\mathscr{H}_{p;\le A,\le B}$ obviously contains the other three of the above four sets.

REMARK 2.6. Given any positive real A, B, and M, for the condition $\mathscr{H}_{p;A,B;M} \ne \varnothing$ to hold it is clearly necessary that

$$A \le B M^{p-2} \tag{49}$$

or, equivalently, $B \ge A/M^{p-2}$ or, equivalently,

$$M \ge c, \tag{50}$$

where $c = c_p(A, B)$ as in (11).

Therefore, in the statements concerning $\mathscr{H}_{p;A,B;M}$, let us assume by default that this restriction on A, B, and M holds.

In Propositions 2.7–2.11 below, let X be any bounded zero-mean r.v.

Let then

$$\mathscr{S}_{p,q;A,B;X;M} := \sup\{\mathsf{E}|X+Y_H|^q : H \in \mathscr{H}_{p;A,B;M}\}, \tag{51}$$

$$\mathscr{S}_{p,q;\le A,\le B,X;M} := \sup\{\mathsf{E}|X+Y_H|^q : H \in \mathscr{H}_{p;\le A,\le B;M}\}, \tag{52}$$

where Y_H and q are as in (33) and (13), respectively.

PROPOSITION 2.7. *The supremum $\mathscr{S}_{p,q;\le A,\le B,X;M}$ is finite and attained. If $A \le BM^{p-2}$ (recall Remark 2.6), then the supremum $\mathscr{S}_{p,q;A,B;X;M}$ is finite and attained as well.*

PROPOSITION 2.8. *Suppose that $p \ge q > 4$. Then the supremum $\mathscr{S}_{p,q;A,B;X;M}$ is (strictly) increasing in $B \in [A/M^{p-2}, \infty)$ for each $A \in (0,\infty)$ and in $A \in (0, BM^{p-2}]$ for each $B \in (0,\infty)$; in particular, it follows that*

$$\mathscr{S}_{p,q;A,B;X;M} = \mathscr{S}_{p,q;\le A,\le B,X;M} \tag{53}$$

for any positive real A, B, and M such that $A \le BM^{p-2}$.

Introduce the set

$$\begin{aligned}&\mathscr{H}_{*,p,q;A,B;X;M}\\ &\quad := \{H \in \mathscr{H}_{p;A,B;M} : \mathsf{E}|X+Y_H|^q = \mathscr{S}_{p,q;A,B;X;M} = \mathscr{S}_{p,q;\le A,\le B,X;M}\}\end{aligned} \tag{54}$$

of the maximizers of $\mathsf{E}|X+Y_H|^q$ over all $H \in \mathscr{H}_{p;A,B;M}$ or, equivalently, over all $H \in \mathscr{H}_{p;\le A,\le B;M}$. According to Propositions 2.7 and 2.8,

$$\mathscr{H}_{*,p,q;A,B;X;M} \ne \varnothing.$$

PROPOSITION 2.9. *Suppose that $p \ge q > 5$. Take any $H \in \mathscr{H}_{*,p,q;A,B;X;M}$. Then*

$$\operatorname{card}\big((0,\infty) \cap \operatorname{supp} H\big) \le 1 \quad \textit{and} \quad \operatorname{card}\big((-\infty,0) \cap \operatorname{supp} H\big) \le 1, \tag{55}$$

where card *denotes the cardinality of the set.*

PROPOSITION 2.10. *Suppose that $p \ge q > 4$. Take any $H \in \mathscr{H}_{*,p,q;A,B;X;M}$. Then $H(\{0\}) = 0$.*

PROPOSITION 2.11. *Suppose that $p \ge q > 5$. Take any $H \in \mathscr{H}_{*,p,q;A,B;X;M}$. Suppose also that the set* supp H *is contained in the* open *interval $(-M, M)$. Then* $\operatorname{card}\operatorname{supp} H = 1$.

PROPOSITION 2.12. *Suppose that $p \ge q > 2$. Let the quadruple $(c_1, c_2, w_1, w_2) \in \mathbb{R}^2 \times [0,\infty)^2$ vary so that $w_1 + w_2 = B$, $c_1 \to b$, $|c_2| \to \infty$, and $|c_2|^{q-2} w_2 \to a$, for some $a \in [0, A]$ and $b \in [-c, c]$, where c is as in (11). Then, for $H := H_{c_1,c_2,w_1,w_2} := w_1\delta_{c_1} + w_2\delta_{c_2}$,*

$$\mathsf{E}|X+Y_H|^q \longrightarrow a + \mathsf{E}|X+Y_{B\delta_b}|^q.$$

PROOF OF PROPOSITION 2.7. Let us only show that the supremum $\mathscr{S}_{p,q;A,B;X;M}$ is finite and attained; that $\mathscr{S}_{p,q;\le A,\le B,X;M}$ is so is shown similarly and even a bit more easily. Let (H_m) be a sequence in $\mathscr{H}_{p;A,B;M}$ such that $\mathsf{E}|X+Y_{H_m}|^q \to \mathscr{S}_{p,q;A,B;X;M}$. Because the interval $[-M,M]$ is compact and the functions 1 and $|\cdot|^{p-2}$ are continuous and bounded on $[-M,M]$, without loss of generality (w.l.o.g.) the sequence (H_m) converges weakly to some $H \in \mathscr{H}_{p;A,B;M}$. So, by (33) and (35), $Y_{H_m} \xrightarrow{\mathrm{D}} Y_H$, since $(R_1\exp)(0;itu)$ is continuous and bounded in $u \in [-M,M]$. Moreover, by the analytic extension of (33), for any $\tilde{H} \in \mathscr{H}_{p;A,B;M}$

$$\begin{aligned}\mathsf{E}\cosh(kY_{\tilde{H}}) &= \frac{1}{2}\exp\biggl\{k^2\int \tilde{H}(\mathrm{d}u)(R_1\exp)(0;ku)\biggr\}\\ &\quad+\frac{1}{2}\exp\biggl\{k^2\int \tilde{H}(\mathrm{d}u)(R_1\exp)(0;-ku)\biggr\}\\ &\le \exp\bigl\{k^2B(R_1\exp)\bigl(0;|k|M\bigr)\bigr\}<\infty\end{aligned} \tag{56}$$

for all real k—because, by (35), $(R_1\exp)(0;u)$ is increasing in $u \in \mathbb{R}$. Also, $|X+Y_{H_m}|^q \le 2^{q-1}(|X|^q+|Y_{H_m}|^q)$. So, by [2], Theorem 5.4,

$$\mathscr{S}_{p,q;A,B;X;M} = \lim_m \mathsf{E}|X+Y_{H_m}|^q = \mathsf{E}|X+Y_H|^q < \infty. \tag{57}$$

□

PROOF OF PROPOSITION 2.8. Let us show that $\mathscr{S}_{p,q;A,B;X;M}$ is increasing in A and in B; then (53) follows immediately.

In accordance with Proposition 2.7, take any $H \in \mathscr{H}_{p;A,B;M}$ such that $\mathsf{E}|X+Y_H|^q = \mathscr{S}_{p,q;A,B;X;M}$. Then $H_t := H + t\delta_0 \in \mathscr{H}_{p;A,B+t;M}$ for all real $t \ge 0$, where, as before, δ_u denotes the Dirac probability measure at u. So, by Lemma 2.5, the right derivative of $\mathsf{E}|X+Y_{H_t}|^q$ in t at $t=0$ is $\binom{q}{2}\mathsf{E}|X+Y_H|^{q-2} > 0$; the last inequality is strict because the measure H is in $\mathscr{H}_{p;A,B;M}$ and hence nonzero, which in turn implies that the r.v. Y_H is nondegenerate. Therefore, for the lower right derivative of $\mathscr{S}_{p,q;A,B;X;M}$ in B one has

$$\begin{aligned}\liminf_{t\downarrow 0}\frac{\mathscr{S}_{p,q;A,B+t,X;M}-\mathscr{S}_{p,q;A,B;X;M}}{t} &\ge \liminf_{t\downarrow 0}\frac{\mathsf{E}|X+Y_{H_t}|^q-\mathsf{E}|X+Y_H|^q}{t}\\ &= \lim_{t\downarrow 0}\frac{\mathsf{E}|X+Y_{H_t}|^q-\mathsf{E}|X+Y_H|^q}{t}\\ &= \binom{q}{2}\mathsf{E}|X+Y_H|^{q-2}>0.\end{aligned}$$

Next, note that $\mathscr{S}_{p,q;A,B;X;M}$ is left-upper semi-continuous in $B \in (A/M^{p-2},\infty)$; that is,

$$\limsup_{\tilde{B}\uparrow B}\mathscr{S}_{p,q;A,\tilde{B},X;M} \le \mathscr{S}_{p,q;A,B;X;M}.$$

Indeed, take any sequence (B_m) such that $B_m \uparrow B$ and

$$\lim_{m\to\infty} \mathscr{S}_{p,q;A,B_m,X;M} > \mathscr{S}_{p,q;A,B;X;M}.$$

By Proposition 2.7, for each large enough m there is some measure $H_m \in \mathscr{H}(p, A, B_m; M)$ such that $\mathsf{E}|X + Y_{H_m}|^q = \mathscr{S}_{p,q;A,B_m,X;M}$. Passing to a subsequence of the sequence (B_m), w.l.o.g. one may assume that H_m converges weakly on the compact set $[-M, M]$ to some measure H_*. Since the functions 1, $|\cdot|^{p-2}$, and $|\cdot|^q$ are continuous, it follows that $H_* \in \mathscr{H}(p, A, B; M)$ and $\mathscr{S}_{p,q;A,B_m,X;M} = \mathsf{E}|X + Y_{H_m}|^q \longrightarrow \mathsf{E}|X + Y_{H_*}|^q \le \mathscr{S}_{p,q;A,B;X;M}$ as $m \to \infty$, which contradicts the assumption on the sequence (B_m). This completes the proof that $\mathscr{S}_{p,q;A,B;X;M}$ is increasing in B.

To show that $\mathscr{S}_{p,q;A,B;X;M}$ is increasing in A, take any $A \in (0, BM^{p-2})$; cf. (49). Then

$$H\big((-M, M)\big) > 0, \tag{58}$$

because otherwise $\operatorname{supp} H \subseteq \{-M, M\}$ and hence $A = BM^{p-2}$. So, there exists some $b \in (-M, M) \cap \operatorname{supp} H$. For $\delta \in (0, \infty)$ and $t \in [0, \infty)$, let now

$$H_t := H_{\delta,t} := H + t\Delta, \tag{59}$$

where $\Delta = \Delta_\delta$ is the real-valued Borel measure on $\mathbb{R}$ defined by the condition that

$$\begin{aligned}\int_{\mathbb{R}} f(u)\Delta(\mathrm{d}u) &= \frac{M+b}{2M} f(M) + \frac{M-b}{2M} f(-M) \\ &\quad - \frac{1}{H([b-\delta, b+\delta])} \int_{[b-\delta,b+\delta]} f(u) H(\mathrm{d}u)\end{aligned} \tag{60}$$

for all locally bounded (say) Borel functions $f : \mathbb{R} \to \mathbb{R}$; note that $H([b-\delta, b+\delta]) > 0$, by the condition $b \in \operatorname{supp} H$. Also, then the measure H_t is nonnegative for all $t \in [0, t_0]$, where $t_0 := H([b-\delta, b+\delta]) > 0$. So, letting

$$h(x) := 2!\binom{q}{2} \mathsf{E}|x + X + Y_H|^{q-2} \tag{61}$$

for all $x \in \mathbb{R}$, by Lemma 2.5 one has

$$\begin{aligned}&\lim_{\delta\downarrow 0}\lim_{t\downarrow 0} \frac{\mathsf{E}|X + Y_{H_t}|^q - \mathsf{E}|X + Y_H|^q}{t} \\ &\quad = \int_0^1 \mathrm{d}s(1-s)\Big(\frac{M+b}{2M} h(Ms) + \frac{M-b}{2M} h(-Ms) - h(bs)\Big) > 0,\end{aligned} \tag{62}$$

because $q > 4$, and the r.v. Y_H is nondegenerate, whence the function h is strictly convex. Thus, eventually

$$\mathscr{S}_{p,q;A,B;X;M} = \mathsf{E}|X + Y_H|^q < \mathsf{E}|X + Y_{H_t}|^q. \tag{63}$$

In this context, we say that an assertion $\mathscr{A} = \mathscr{A}_{\delta,t}$ holds "eventually" if $\exists \delta_* \in (0,\infty)\ \forall \delta \in (0,\delta_*)\ \exists t_\delta \in (0,t_0)\ \forall t \in (0,t_\delta)\ \mathscr{A}_{\delta,t}$ holds; recall here that, in view of (59) and (60), H_t depends not only on t but also on δ.

On the other hand, for all $t \in (0,t_0)$ one has $\int_{\mathbb{R}} H_t(\mathrm{d}x) = \int_{\mathbb{R}} H(\mathrm{d}x) + t\int_{\mathbb{R}} \Delta(\mathrm{d}x) = B + t\int_{\mathbb{R}} \Delta(\mathrm{d}x) = B$ and $\int_{\mathbb{R}} |x|^{p-2} H_t(\mathrm{d}x) = A + ta$ and hence $H_t \in \mathscr{H}(p, A+ta, B; M)$, where $a := \int_{\mathbb{R}} |x|^{p-2}\Delta(\mathrm{d}x) \ge (M^{p-2} - (|b|+\delta)^{p-2}) > 0$ for all small enough $\delta > 0$. So, by (51), eventually $\mathscr{S}_{p,q;A+ta,B,X;M} \ge \mathsf{E}|X + Y_{H_t}|^q$, whence, by (63), $\mathscr{S}_{p,q;\cdot,B,X;M}$ is increasing in a right neighborhood of the previously chosen value of $A \in (0, BM^{p-2})$.

Since A was chosen arbitrarily in the interval $(0, BM^{p-2})$, to complete the proof of Proposition 2.8, it remains to note that $\mathscr{S}_{p,q;A,B;X;M}$ is left-upper semi-continuous in $A \in (0, BM^{p-2}]$; this semi-continuity property is established quite similarly to the left-upper semi-continuity in B, proved earlier. $\square$

PROOF OF PROPOSITION 2.9. To obtain a contradiction, suppose that there exist b and b_1 such that $0 < b < b_1 < \infty$ and $\{b, b_1\} \subseteq \operatorname{supp} H$. In view of possible rescaling [i.e., replacing X, A, B, M and $H(\mathrm{d}x)$ by X/b_1, A/b_1^p, B/b_1^2, M/b_1, and $H(b_1\,\mathrm{d}y)/b_1^2$, resp.], w.l.o.g. assume that $b_1 = 1$, so that

$$0 < b < 1.$$

By (54),

$$\mathscr{S}_{p,q;\le A,\le B,X;M} = \mathsf{E}|X + Y_H|^q. \tag{64}$$

Introduce now

$$k := \frac{b^2(1-b^{p-3})}{p-3}, \tag{65}$$

take any

$$a \in (0, 1/k) \tag{66}$$

and then also introduce

$$\begin{aligned} \varepsilon &:= a\big(b - b^{p-1} - (p-2)k\big), \\ \tilde{a} &:= 1 + a\big(b^{p-1} + (p-1)k\big), \quad \text{and} \\ \tilde{b} &:= 1 - ka. \end{aligned} \tag{67}$$

Note that the conditions (67) and (66) imply $\tilde{b} \in (0,1)$. Observe also that $\varepsilon = abr(b)/(p-3)$, where $r(b) := p-3-(p-2)b+b^{p-2}$, and $r(1) = 0$ and $r'(b) = -(p-2)(1-b^{p-3}) < 0$ for $b \in (0,1)$, so that $\varepsilon > 0$.

Define the real-valued measure $\Delta = \Delta_{a,\delta}$ by the condition

$$\begin{aligned}\int_{\mathbb{R}} f(u)\Delta(\mathrm{d}u) &= \varepsilon f(0) + \tilde{a}\tilde{b} f(\tilde{b}) - \frac{ab}{H([b-\delta, b+\delta])}\int_{[b-\delta,b+\delta]} f(u)H(\mathrm{d}u) \\ &\quad - \frac{1}{H([1-\delta, 1+\delta])}\int_{[1-\delta,1+\delta]} f(u)H(\mathrm{d}u)\end{aligned} \tag{68}$$

for all locally bounded (say) Borel functions $f:\mathbb{R}\to\mathbb{R}$, where δ is any real number in the interval $(0, \frac{1-b}{2})$, so that the denominators $H([b-\delta, b+\delta])$ and $H([1-\delta, 1+\delta])$ are strictly positive, and the intervals $[b-\delta, b+\delta]$ and $[1-\delta, 1+\delta]$ are disjoint, in view of the assumptions $\{b, b_1\} \subseteq \operatorname{supp} H$ and $b_1 = 1$. For $t \in [0, \infty)$, let now

$$H_t := H_{a,\delta,t} := H + t\Delta. \tag{69}$$

This measure is nonnegative for all $t \in [0, t_0]$, where

$$t_0 := \min\biggl(\frac{1}{ab}H\bigl([b-\delta, b+\delta]\bigr), H\bigl([1-\delta, 1+\delta]\bigr)\biggr) > 0.$$

By Lemma 2.5,

$$\begin{aligned}&\lim_{\delta\downarrow 0}\lim_{t\downarrow 0}\frac{\mathsf{E}|X + Y_{H_{a,\delta,t}}|^q - \mathsf{E}|X + Y_H|^q}{t} \\ &\qquad = \int_0^1 \mathrm{d}s(1-s)\bigl[\varepsilon h(0) + \tilde{a}\tilde{b}h(s\tilde{b}) - abh(sb) - h(s)\bigr],\end{aligned}$$

where the function h is still defined by (61).

Letting further $a \downarrow 0$ and using Lemma 2.4, one obtains

$$\begin{aligned}\mathscr{L} &:= \frac{1}{b^2}\lim_{a\downarrow 0}\frac{1}{a}\lim_{\delta\downarrow 0}\lim_{t\downarrow 0}\frac{\mathsf{E}|X + Y_{H_{a,\delta,t}}|^q - \mathsf{E}|X + Y_H|^q}{t} \\ &= \int_0^1 \mathrm{d}s(1-s)F(b,s),\end{aligned} \tag{70}$$

where, in view of (67), (65), (34) and (35),

$$\begin{aligned}F(b,s) &:= \frac{1}{b^2}\frac{\mathrm{d}}{\mathrm{d}a}\bigl[\varepsilon h(0) + \tilde{a}\tilde{b}h(s\tilde{b}) - abh(sb) - h(s)\bigr]\Big|_{a=0} \\ &= \frac{1}{b^2}\bigl\{\bigl(b^{p-1} + (p-2)k\bigr)\bigl(h(s) - h(0)\bigr) - b\bigl(h(sb) - h(0)\bigr) - ksh'(s)\bigr\} \\ &= h(s) - h(0) - \frac{h(sb) - h(0)}{b} + \frac{1 - b^{p-3}}{p-3}\bigl[h(s) - h(0) - sh'(s)\bigr]\end{aligned} \tag{71}$$

$$= \int_0^1 \mathrm{d}\alpha\, s\big[h'(\alpha s) - h'(\alpha s b)\big] - \int_b^1 \mathrm{d}u\, u^{p-4} s^2 (R_1 h)(s; -s)$$
$$= \int_0^1 \mathrm{d}\alpha\, s\alpha s \int_b^1 \mathrm{d}u\, h''(u\alpha s) - \int_b^1 \mathrm{d}u\, u^{p-4} s^2 \int_0^1 \mathrm{d}\theta (1-\theta) h''(s - \theta s)$$
$$= s^2 \int_b^1 \mathrm{d}u \int_0^1 \mathrm{d}\alpha\, \alpha\big[h''(u\alpha s) - u^{p-4} h''(\alpha s)\big].$$

By (61) and Lemma 2.4, for $x \in \mathbb{R}$ and $u \in (0, \infty)$

$$\frac{u^{q-4} h''(x)}{q(q-1)(q-2)(q-3)} = \psi_{ux}(u),$$

where $\psi_v(u) := \mathsf{E}|v + uW|^{q-4}$ for all $v \in \mathbb{R}$ and $W := X + Y_H$. Note that, in view of the condition $\mathsf{E}X = 0$ and the definition (33), $\mathsf{E}W = 0$. Also, $\mathsf{E}W^2 > 0$, because $\operatorname{card}\operatorname{supp} H \geq 2 > 0$ and hence $H \neq 0$ and thus the r.v. Y_H is nondegenerate. Also, clearly $h'' \geq 0$. Therefore and because $p \geq q > 5$, for all u, α and s in $(0, 1)$

$$\begin{aligned} \frac{h''(u\alpha s) - u^{p-4} h''(\alpha s)}{q(q-1)(q-2)(q-3)} &\geq \frac{h''(u\alpha s) - u^{q-4} h''(\alpha s)}{q(q-1)(q-2)(q-3)} \\ &= \psi_{u\alpha s}(1) - \psi_{u\alpha s}(u). \end{aligned} \tag{72}$$

Recalling that $q > 5$, one sees that for each $v \in (0, \infty)$ the function ψ_v is convex, with $\psi_v'(0) = 0$ and $\psi_v''(0) = (q-4)(q-5)\mathsf{E}W^2 v^{q-6} > 0$. This implies that $\psi_v(u)$ is strictly increasing in $u \geq 0$, which shows that the expression $\psi_{u\alpha s}(1) - \psi_{u\alpha s}(u)$ in (72) is strictly positive. Thus, by (70), (71) and (72), $\mathscr{L} > 0$. Now (64) implies that eventually

$$\mathscr{S}_{p,q;\leq A,\leq B,X;M} = \mathsf{E}|X + Y_H|^q < \mathsf{E}|X + Y_{H_{a,\delta,t}}|^q.$$

In this context, we say that an assertion $\mathscr{A} = \mathscr{A}_{a,\delta,t}$ holds "eventually" if $\exists a_0 \in (0, \infty)$ $\forall a \in (0, a_0)$ $\exists \delta_a^* \in (0, \infty)$ $\forall \delta \in (0, \delta_a^*)$ $\exists t_{a,\delta} \in (0, t_0)$ $\forall t \in (0, t_{a,\delta})$ $\mathscr{A}_{a,\delta,t}$ holds.

Thus, we obtain a contradiction with the definition of $\mathscr{S}_{p,q;\leq A,\leq B,X;M}$ in (52), because, as we shall check in moment, $H_{a,\delta,t} \in \mathscr{H}_{p;\leq A,\leq B;M}$ eventually. Indeed, by (68), (67), and (65),

$$\begin{aligned} \int_{\mathbb{R}} \Delta(\mathrm{d}x) &= \varepsilon + \tilde{a}\tilde{b} - ab - 1 \\ &= a\big(b - b^{p-1} - (p-2)k\big) \\ &\quad + (1 - ka)\big(1 + a\big[b^{p-1} + (p-1)k\big]\big) - ab - 1 \\ &< ab - a\big(b^{p-1} + (p-2)k\big) + 1 + \big[b^{p-1} + (p-2)k\big]a - ab - 1 \\ &= 0, \end{aligned}$$

so that

$$\int_{\mathbb{R}} H_{a,\delta,t}(\mathrm{d}x) = \int_{\mathbb{R}} H(\mathrm{d}x) + t\int_{\mathbb{R}} \Delta(\mathrm{d}x) < \int_{\mathbb{R}} H(\mathrm{d}x) = B. \tag{73}$$

Similarly,

$$\lim_{\delta\downarrow 0}\int_{\mathbb{R}} |x|^{p-2}\Delta(\mathrm{d}x) = \tilde{a}\tilde{b}^{p-1} - ab^{p-1} - 1 < 0,$$

where the inequality holds eventually, for all small enough $a > 0$. Indeed, in view of (67), this inequality can be rewritten as

$$f_\gamma(u) := \big[1 + (\gamma + r)u\big](1-u)^r - (1+\gamma u) < 0, \tag{74}$$

with $r := p - 1 > 0$, $u := ka$, and $\gamma := b^r/k \geq 0$. Note that eventually $u \in (0,1)$. To verify inequality (74) for such u, note that $f_\gamma(u)$ decreases in γ, so that w.l.o.g. $\gamma = 0$. The inequality $f_0(u) < 0$ is equivalent to $\ln(1+ru) + r\ln(1-u) < 0$, which is easy to check for $u \in (0,1)$ by differentiation. It follows that [cf. (73)] $\int_{\mathbb{R}} |x|^{p-2} H_{a,\delta,t}(\mathrm{d}x) < \int_{\mathbb{R}} |x|^{p-2} H(\mathrm{d}x) = A$ eventually.

Also, the conditions $H \in \mathscr{H}_{p;\leq A,\leq B;M}$, $\{b, b_1\} \subseteq \operatorname{supp} H$ and $b_1 = 1$ imply $\operatorname{supp} H \subseteq [-M, M]$ and hence $M \geq 1$. So, $\operatorname{supp} H_{a,\delta,t} \subseteq \operatorname{supp} H \cup \{0, \tilde{b}\} \subseteq [-M, M]$ eventually, in view of (67).

By (48), we conclude that indeed $H_{a,\delta,t} \in \mathscr{H}_{p;\leq A,\leq B;M}$ eventually. Thus, indeed the assumption that there exist b and b_1 such that $0 < b < b_1 < \infty$ and $\{b, b_1\} \subseteq \operatorname{supp} H$ leads to a contradiction, which proves the first inequality in (55). The second inequality there can be proved quite similarly or, alternatively, quickly obtained from the first one by a reflection. $\square$

PROOF OF PROPOSITION 2.10. The proof is somewhat similar to that Proposition 2.9. Suppose that, to the contrary,

$$\sigma := \sqrt{H(\{0\})} > 0. \tag{75}$$

On the other hand, recalling definition (47) of $\mathscr{H}_{p;A,B;M}$ and the conditions $H \in \mathscr{H}_{p;A,B;M}$ and $A > 0$, one sees that necessarily $\operatorname{supp} H \setminus \{0\} \neq \varnothing$. So, in view of possible rescaling and reflection, w.l.o.g.

$$1 \in \operatorname{supp} H.$$

Take now any $\beta \in (0, (\frac{p-2}{p-1})^{1/(p-2)})$, so that

$$\varepsilon := \frac{\beta^{p-2}}{p-2} \in \left(0, \frac{1}{p-1}\right) \subset (0,1). \tag{76}$$

Introduce then

$$\tilde{a} := \frac{1-(p-1)\varepsilon}{(1-\varepsilon)^p} \quad \text{and} \quad \tilde{b} := 1 - \varepsilon. \tag{77}$$

Define the real-valued measure $\Delta := \Delta_{\beta,\delta}$ by the condition

$$\int_{\mathbb{R}} f(u)\Delta(\mathrm{d}u) = \frac{1}{2}f(\beta) + \frac{1}{2}f(-\beta) - f(0) + \tilde{a}\tilde{b}f(\tilde{b}) - \frac{1}{H([1-\delta, 1+\delta])}\int_{[1-\delta,1+\delta]} f(u)H(\mathrm{d}u) \tag{78}$$

for all locally bounded (say) Borel functions $f:\mathbb{R}\to\mathbb{R}$, where δ is any positive real number, so that $H([1-\delta, 1+\delta]) > 0$. For σ as in (75), let

$$t_0 := \sigma^2 \wedge H\big([1-\delta, 1+\delta]\big).$$

Then $t_0 > 0$ and for all $t \in [0, t_0]$ the measure

$$H_t := H_{\beta,\delta,t} := H + t\Delta \tag{79}$$

is nonnegative. By Lemma 2.5,

$$\begin{aligned}\mathscr{L}(\beta) &:= \lim_{\delta\downarrow 0}\lim_{t\downarrow 0}\frac{\mathsf{E}|X + Y_{H_{\beta,\delta,t}}|^q - \mathsf{E}|X + Y_H|^q}{t} \\ &= \int_0^1 \mathrm{d}s(1-s)\Big[\frac{1}{2}h(s\beta) + \frac{1}{2}h(-s\beta) - h(0) + \tilde{a}\tilde{b}h(s\tilde{b}) - h(s)\Big],\end{aligned} \tag{80}$$

where h is still defined by (61). Let now $\beta \downarrow 0$. Then, in view of (77) and (76), the expression $\tilde{a}\tilde{b}h(s\tilde{b}) - h(s)$ in (80) is $O(\varepsilon) = O(\beta^{p-2}) = o(\beta^2)$ uniformly over $s \in [0, 1]$. Concerning the other part of the expression in the brackets in (80), by (34) and (35),

$$\begin{aligned}\frac{1}{2}h(u) + \frac{1}{2}h(-u) - h(0) &= \frac{u^2}{2}\big[(R_1h)(0;u) + (R_1h)(0;-u)\big] \\ &= \frac{u^2}{2}\int_{-1}^1 \mathrm{d}v\big(1-|v|\big)h''(vu)\end{aligned}$$

for all $u \in \mathbb{R}$. So,

$$\mathscr{L}(\beta) = \frac{\beta^2}{2}\int_0^1 \mathrm{d}s(1-s)s^2\int_{-1}^1 \mathrm{d}v\big(1-|v|\big)h''(vs\beta) + o\big(\beta^2\big),$$

whence

$$\lim_{\beta\downarrow 0}\lim_{\delta\downarrow 0}\lim_{t\downarrow 0}\frac{\mathsf{E}|X + Y_{H_{\beta,\delta,t}}|^q - \mathsf{E}|X + Y_H|^q}{\beta^2 t/24} = h''(0) = 4!\binom{q}{4}\mathsf{E}|W|^{q-4},$$

where $W = X + Y_H$, which is a nondegenerate r.v., so that $\mathsf{E}|W|^{q-4} > 0$.

Now (64) implies that eventually

$$\mathscr{S}_{p,q;\le A,\le B,X;M} = \mathsf{E}|X + Y_H|^q < \mathsf{E}|X + Y_{H_{\beta,\delta,t}}|^q.$$

In this context, we say that an assertion $\mathscr{A} = \mathscr{A}_{\beta,\delta,t}$ holds "eventually" if $\exists \beta_0 \in (0, (\frac{p-2}{p-1})^{1/(p-2)})$ $\forall \beta \in (0, \beta_0)$ $\exists \delta^*_\beta \in (0, \infty)$ $\forall \delta \in (0, \delta^*_\beta)$ $\exists t_{\beta,\delta} \in (0, t_0)$ $\forall t \in (0, t_{\beta,\delta})$ $\mathscr{A}_{\beta,\delta,t}$ holds.

Thus, we obtain a contradiction with the definition of $\mathscr{S}_{p,q;\le A,\le B,X;M}$ in (52), because, as we shall check in moment, $H_{\beta,\delta,t} \in \mathscr{H}_{p;\le A,\le B;M}$ eventually. Indeed, by (78) and (77),

$$\int_{\mathbb{R}} \Delta(\mathrm{d}x) = \tilde{a}\tilde{b} - 1 = \frac{1-(p-1)\varepsilon}{(1-\varepsilon)^{p-1}} - 1 < 0,$$

so that

$$\int_{\mathbb{R}} H_{\beta,\delta,t}(\mathrm{d}x) = \int_{\mathbb{R}} H(\mathrm{d}x) + t\int_{\mathbb{R}} \Delta(\mathrm{d}x) < \int_{\mathbb{R}} H(\mathrm{d}x) \le B,$$

by (79) and (48). Next, by (77),

$$\begin{aligned}
&\lim_{\delta\downarrow 0} \int_{\mathbb{R}} |x|^{p-2}\Delta(\mathrm{d}x) \\
&\qquad = \beta^{p-2} + \tilde{a}\tilde{b}^{p-1} - 1 = (p-2)\varepsilon + \frac{1-(p-1)\varepsilon}{1-\varepsilon} - 1 \\
&\qquad = -\frac{(p-2)\varepsilon^2}{1-\varepsilon} < 0.
\end{aligned}$$

It follows that eventually

$$\begin{aligned}
&\int_{\mathbb{R}} |x|^{p-2} H_{\beta,\delta,t}(\mathrm{d}x) \\
&\qquad = \int_{\mathbb{R}} |x|^{p-2} H(\mathrm{d}x) + t\int_{\mathbb{R}} |x|^{p-2}\Delta(\mathrm{d}x) < \int_{\mathbb{R}} |x|^{p-2} H(\mathrm{d}x) \le A,
\end{aligned}$$

again by (79) and (48).

Also, the conditions $H \in \mathscr{H}_{p;\le A,\le B;M}$ and $1 \in \operatorname{supp} H$ imply $M \ge 1$. So, eventually $\operatorname{supp} H_{\beta,\delta,t} \subseteq \operatorname{supp} H \cup \{\beta, -\beta, \tilde{b}\} \subseteq [-M, M]$ in view of (77).

By (48), we conclude that indeed $H_{\beta,\delta,t} \in \mathscr{H}_{p;\le A,\le B;M}$ eventually. Thus, assumption (75) leads to a contradiction. $\square$

PROOF OF PROPOSITION 2.11. By Propositions 2.9 and 2.10 and the condition $\operatorname{supp} H \subseteq (-M, M)$,

$$H = w_1\delta_{c_1} + w_2\delta_{-c_2} \tag{81}$$

for some c_1 and c_2 in the interval $(0, M)$ and some nonnegative real w_1 and w_2 such that

$$w_1 + w_2 = B \quad \text{and} \quad c_1^{p-2}w_1 + c_2^{p-2}w_2 = A. \tag{82}$$

It is enough to show that $w_1 \wedge w_2 = 0$. To obtain a contradiction, suppose the contrary,

$$w := w_1 \wedge w_2 > 0. \tag{83}$$

Then, by the implicit function theorem, there exist a real number $\tau_* > 0$ and an infinitely differentiable mapping $(-\tau_*, \tau_*) \ni \tau \mapsto (\tilde{c}_1(\tau), \tilde{c}_2(\tau))$ such that

$$\tilde{c}_1(0) = c_1, \qquad \tilde{c}_2(0) = c_2, \tag{84}$$

and for each $\tau \in (-\tau_*, \tau_*)$ one has $\tilde{c}_1'(\tau)\tilde{c}_2'(\tau) \neq 0$,

$$0 < \tilde{c}_1(\tau), \tilde{c}_2(\tau) < M \quad \text{and} \quad \tilde{c}_1(\tau)^{p-2} + \tilde{c}_2(\tau)^{p-2} = c_1^{p-2} + c_2^{p-2}. \tag{85}$$

(In this case, this mapping could also be defined explicitly, e.g., by the formulas $\tilde{c}_1(\tau) = (c_1^{p-2} + \tau)^{1/(p-2)}$ and $\tilde{c}_2(\tau) = (c_2^{p-2} - \tau)^{1/(p-2)}$, with $\tau_* = \frac{1}{2}\min[M^{p-2} - c_1^{p-2}, M^{p-2} - c_2^{p-2}, c_1^{p-2}, c_2^{p-2}]$.) Note that the condition $\tilde{c}_1'(\tau)\tilde{c}_2'(\tau) \neq 0$, taken together with (85), implies

$$\tilde{c}_1'(\tau)\tilde{c}_2'(\tau) < 0. \tag{86}$$

By choosing a possibly smaller real $\tau_* > 0$, let us assume w.l.o.g. that, on the interval $(-\tau_*, \tau_*)$, the derivatives of any order of the functions $\tilde{c}_1$ and $\tilde{c}_2$ are each uniformly continuous and hence bounded, and also that the functions $\tilde{c}_1$ and $\tilde{c}_2$ are each positive and bounded away from 0.

For each $\tau \in (-\tau_*, \tau_*)$, introduce the real-valued measure

$$\Delta_\tau := \delta_{\tilde{c}_1(\tau)} + \delta_{-\tilde{c}_2(\tau)} - \delta_{c_1} - \delta_{-c_2} \tag{87}$$

and then the measures

$$H_{t,\tau} := H + t\Delta_\tau \qquad \text{for } t \in (-w, w), \tag{88}$$

where w is as in (83). By (47), (45), (81), (82) and (85), these measures are all in $\mathscr{H}_{p;A,B;M}$.

In the rest of this proof, it is assumed that $\tau \in (-\tau_*, \tau_*)$, $t \in (-w, w)$, $\{j, k, \ell\} \subset \{1, 2, 3, 4\}$, and $x \in \mathbb{R}$—unless otherwise indicated.

Letting now

$$g_{t,\tau}(x) := \mathsf{E}|x + X + Y_{H_{t,\tau}}|^q, \tag{89}$$

then using Lemma 2.5 and recalling (87), one has

$$\begin{aligned} \mathscr{D}(\tau) &:= \frac{\partial^2 g_{t,\tau}(0)}{\partial t^2}\bigg|_{t=0} \\ &= \int_\Omega \mu(\mathrm{d}\omega) \int_{\mathbb{R}^2} \Delta_\tau(\mathrm{d}u_1)\Delta_\tau(\mathrm{d}u_2) h(x + su_1 + tu_2) \\ &= \int_\Omega \mu(\mathrm{d}\omega) F_\omega(\tau), \end{aligned} \tag{90}$$

where

$$\Omega := (0,1)^2 \times \mathbb{R}, \qquad \omega := (s,t,x) \in \Omega,$$

$$\mu(\mathrm{d}\omega) := \mathrm{d}t\,\mathrm{d}s(1-t)(1-s)\mathsf{P}(X + Y_H \in \mathrm{d}x), \tag{91}$$

$$F_\omega(\tau) := \sum_{j,k=1}^{4} v_j v_k h\big(x + sb_j(\tau) + tb_k(\tau)\big), \tag{92}$$

$$h(x) := 4!\binom{q}{4}|x|^{q-4}, \tag{93}$$

$$(v_1, v_2, v_3, v_4) := (1, 1, -1, -1), \tag{94}$$

$$\big(b_1(\tau), b_2(\tau), b_3(\tau), b_4(\tau)\big) := \big(\tilde{c}_1(\tau), -\tilde{c}_2(\tau), c_1, -c_2\big). \tag{95}$$

Next,

$$\mathscr{D}(\tau) = \sum_{j,k=1}^{4} v_j v_k \mathscr{D}_{j,k}(\tau), \tag{96}$$

where

$$\mathscr{D}_{j,k}(\tau) := \int_\Omega \mu(\mathrm{d}\omega) h\big(x + sb_j(\tau) + tb_k(\tau)\big). \tag{97}$$

By Lemma 2.4,

$$\mathscr{D}'_{j,k}(\tau) = \int_\Omega \mu(\mathrm{d}\omega) f(\omega,\tau), \tag{98}$$

where

$$f(\omega,\tau) := f_{j,k}(\omega,\tau) := h'\big(x + sb_j(\tau) + tb_k(\tau)\big)\big[sb'_j(\tau) + tb'_k(\tau)\big], \tag{99}$$

which is clearly bounded in $(\omega,\tau) \in \Omega \times (-\tau_*, \tau_*)$. For each $\varepsilon \in [0,\infty)$, introduce the set

$$\Omega_\varepsilon := \Omega_{j,k;\varepsilon} := \big\{\omega = (s,t,x) \in \Omega : \big|x + sb_j(0) + tb_k(0)\big| > \varepsilon\big\}. \tag{100}$$

Since $b_j(\tau)$ is uniformly continuous in $\tau \in (-\tau_*, \tau_*)$ for each j, one sees that $|x + sb_j(\tau) + tb_k(\tau)|$ is continuous in τ uniformly over all $(\omega,\tau,j,k) \in \Omega \times (-\tau_*,\tau_*) \times \{1,2,3,4\} \times \{1,2,3,4\}$. So, by further decreasing (if necessary) the value of $\tau_* > 0$, let us assume, again w.l.o.g., that

$$\big|x + sb_j(\tau) + tb_k(\tau)\big| > \varepsilon/2 \qquad \text{for all } (\omega,\tau) \in \Omega_\varepsilon \times (-\tau_*,\tau_*), \tag{101}$$

$$\big|x + sb_j(\tau) + tb_k(\tau)\big| \le 2\varepsilon \qquad \text{for all } (\omega,\tau) \in (\Omega \setminus \Omega_\varepsilon) \times (-\tau_*,\tau_*). \tag{102}$$

By (99), for $(\omega,\tau) \in \Omega_0 \times (-\tau_*,\tau_*)$, the partial derivative of $f(\omega,\tau)$ in τ is

$$(\partial_2 f)(\omega,\tau) = D_1(\omega,\tau) + D_2(\omega,\tau), \tag{103}$$

where

$$D_1(\omega,\tau) := h'\big(x + sb_j(\tau) + tb_k(\tau)\big)\big[sb_j''(\tau) + tb_k''(\tau)\big],$$

$$D_2(\omega,\tau) := h''\big(x + sb_j(\tau) + tb_k(\tau)\big)\big[sb_j'(\tau) + tb_k'(\tau)\big]^2.$$

In view of the condition $q > 5$, definition (93), inequality (101), and the boundedness of all the derivatives of the functions b_j on the interval $(-\tau_*, \tau_*)$,

$$\left.\begin{aligned} \big|D_1(\omega,\tau)\big| &\le K\big(1 + |x|^{q-5}\big) \quad \text{and} \\ \big|D_2(\omega,\tau)\big| &\le K\big(1 + |x|^{(q-6)_+} + \varepsilon^{-(6-q)_+}\big) \end{aligned}\right\} \tag{104}$$
$$\text{for all } (\omega,\tau) \in \Omega_\varepsilon \times (-\tau_*, \tau_*);$$

here and in the rest of this proof, K denotes various positive real constants which do not depend on ω, τ, or ε. So, by (103),

$$\big|(\partial_2 f)(\omega,\tau)\big| \le g_\varepsilon(\omega) := K\big(1 + |x|^{q-5} + \varepsilon^{-(6-q)_+}\big) \tag{105}$$
$$\text{for all } (\omega,\tau) \in \Omega_\varepsilon \times (-\tau_*, \tau_*).$$

By (88), (81), (87), (56), and (14), $\int_{\Omega_\varepsilon} |\mathrm{d}\mu| g_\varepsilon < \infty$, where μ is still as in (91).

Next, by (104) and dominated convergence,

$$\sup_{\tau\in(-\tau_*,\tau_*)} \int_{\Omega\setminus\Omega_\varepsilon} \big|\mu(\mathrm{d}\omega) D_1(\omega,\tau)\big| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0. \tag{106}$$

Further, $|D_2(\omega,\tau)| \le K|x + sb_j(\tau) + tb_k(\tau)|^{q-6}$ for all $(\omega,\tau) \in \Omega_0 \times (-\tau_*, \tau_*)$, whence, by (102), with $\nu(\mathrm{d}x) := \mathsf{P}(X + Y_H \in \mathrm{d}x)$,

$$\begin{aligned} &\int_{\Omega\setminus\Omega_\varepsilon} \big|\mu(\mathrm{d}\omega) D_2(\omega,\tau)\big| \\ &\quad \le K \int_0^1 \mathrm{d}t \int_{\mathbb{R}} \nu(\mathrm{d}x) \int_0^1 \mathrm{d}s \big|x + sb_j(\tau) + tb_k(\tau)\big|^{q-6} \\ &\qquad\qquad \times \mathrm{I}\big\{\big|x + sb_j(\tau) + tb_k(\tau)\big| \le 2\varepsilon\big\} \\ &\quad \le K \int_0^1 \mathrm{d}t \int_{\mathbb{R}} \nu(\mathrm{d}x) \frac{1}{b_j(\tau)} \int_{\mathbb{R}} \mathrm{d}v |v|^{q-6} \mathrm{I}\big\{|v| \le 2\varepsilon\big\} = \frac{2K(2\varepsilon)^{q-5}}{b_j(\tau)(q-5)} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0 \end{aligned}$$

uniformly in $\tau \in (-\tau_*, \tau_*)$, since the functions b_j are bounded away from 0 on $(-\tau_*, \tau_*)$. Combining this with (103) and (106), one has

$$\sup_{\tau\in(-\tau_*,\tau_*)} \int_{\Omega\setminus\Omega_\varepsilon} \big|\mu(\mathrm{d}\omega)(\partial_2 f)(\omega,\tau)\big| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0.$$

Therefore and by (105), one may use Lemma 2.3 together with (98) and (99) to conclude that $\mathscr{D}_{j,k}''(0) = \int_\Omega \mu(\mathrm{d}\omega) \frac{\partial^2}{\partial\tau^2} h(x + sb_j(\tau) + tb_k(\tau))|_{\tau=0}$ and hence, by (96), (97) and (92),

$$\mathscr{D}''(0) = \int_\Omega \mu(\mathrm{d}\omega) F_\omega''(0); \tag{107}$$

that is, we have shown that the second integral expression of $\mathscr{D}(\tau)$ in (90) can be twice differentiated (at least at $\tau = 0$) under the integral sign to obtain the corresponding integral expression of $\mathscr{D}''(0)$. Note here that $F''_\omega(0)$ is defined only for $\omega \in \bigcap_{j,k=1}^4 \Omega_{j,k;0}$, where $\Omega_{j,k;0}$ is understood according to (100). However, this causes no problem, since $\mu(\Omega \setminus \bigcap_{j,k=1}^4 \Omega_{j,k;0}) = 0$.

In view of (92), (93), (94), (95), and (84), it is straightforward but tedious to check that

$$F_\omega(0) = 0, \tag{108}$$

$$F'_\omega(0) = 0, \tag{109}$$

$$\begin{aligned} F''_\omega(0) = 2st\big\{&h''(x-(s+t)c_2)\tilde{c}'_2(0)^2 + h''(x+(s+t)c_1)\tilde{c}'_1(0)^2 \\ &- \big[h''(x+sc_1-tc_2) + h''(x-sc_2+tc_1)\big]\tilde{c}'_1(0)\tilde{c}'_2(0)\big\} \end{aligned} \tag{110}$$

for all $\omega \in \bigcap_{j,k=1}^4 \Omega_{j,k;0}$. The equality in (109) in fact holds for all $\omega \in \Omega$ and any continuously differentiable function h, not necessarily the one defined by (93), whereas the equality in (108) holds for any function $h : \mathbb{R} \to \mathbb{R}$ whatsoever.

By (93), $h''(z) > 0$ for all real $z \neq 0$. So, by (110) and (86), $F''_\omega(0) > 0$ for all $\omega \in \bigcap_{j,k=1}^4 \Omega_{j,k;0}$. It follows by (90), (108), (109), and Lemma 2.4 that $\mathscr{D}(0) = \mathscr{D}'(0) = 0$, whereas, by (107), $\mathscr{D}''(0) > 0$ and hence $\mathscr{D}(\tau) > 0$ for some $\tau \in (-\tau_*, \tau_*)$ (in fact for all nonzero τ close enough to 0). Take any such τ. Then, by (90),

$$\left.\frac{\partial^2 g_{t,\tau}(0)}{\partial t^2}\right|_{t=0} = \mathscr{D}(\tau) > 0,$$

which implies that $g_{0,\tau}(0) < g_{-t,\tau}(0) \vee g_{t,\tau}(0)$ if $|t|$ is small enough. In view of (88) and (89), this means that for all $t \in (-w, w)$ with small enough $|t|$,

$$\mathsf{E}|X + Y_H|^p = \mathsf{E}|X + Y_{H_{0,\tau}}|^p < \mathsf{E}|X + Y_{H_{-t,\tau}}|^p \vee \mathsf{E}|X + Y_{H_{t,\tau}}|^p,$$

which is a contradiction, in view of the conditions $H \in \mathscr{H}_{*,p,q;A,B;X;M}$ and $H_{t,\tau} \in \mathscr{H}_{p;A,B;M}$ for all $(\tau, t) \in (-\tau_*, \tau_*) \times (-w, w)$, and the definition (54) of $\mathscr{H}_{*,p,q;A,B;X;M}$. $\square$

PROOF OF PROPOSITION 2.12. Since $|c_2| \to \infty$, w.l.o.g. $c_2 \neq 0$. So, the definition $\lambda_2 := w_2/c_2^2$ makes sense, and $\lambda_2 \in [0, \infty)$. If $\lambda_2 = 0$, let $\tilde{\Pi}_{\lambda_2} := 0$. So, $Y_H \overset{\mathrm{D}}{=} Y_{w_1\delta_{c_1}} + c_2\tilde{\Pi}_{\lambda_2}$ and hence

$$e^{\lambda_2}\mathsf{E}|X + Y_H|^q = \sum_{j=0}^\infty T_j, \tag{111}$$

where

$$T_j := \frac{\lambda_2^j}{j!}\mathsf{E}\big|X + Y_{w_1\delta_{c_1}} + c_2(j - \lambda_2)\big|^q,$$

letting $\lambda_2^0 := 1$ even if $\lambda_2 = 0$. So, in view of the conditions $|c_2| \to \infty$, $|c_2|^{q-2} w_2 \to a$, $w_1 + w_2 = B$, and $c_1 \to b$, one has $w_2 \to 0$, $w_1 \to B$, $\lambda_2 \to 0$, $Y_{w_1 \delta_{c_1}} \xrightarrow{\mathrm{D}} Y_{B\delta_b}$, $c_2 \lambda_2 \to 0$, $|c_2|^q \lambda_2 \to a$, $|c_2|^q \lambda_2^2 \to 0$, whence, by dominated convergence,

$$\begin{aligned} &T_0 \to \mathsf{E}|X + Y_{B\delta_b}|^q, \\ &T_1 = \mathsf{E}|\lambda_2^{1/q}(X + Y_{w_1\delta_{c_1}}) + \big(|c_2|^q \lambda_2\big)^{1/q} \operatorname{sign} c_2 \,(1 - \lambda_2)|^q \to a. \end{aligned} \tag{112}$$

Also, eventually $\lambda_2 \in [0, 1]$ and hence

$$2^{1-q} \sum_{j=2}^{\infty} |T_j| \le \lambda_2^2 \sum_{j=2}^{\infty} \frac{1}{j!} \mathsf{E}|X + Y_{w_1\delta_{c_1}}|^q + |c_2|^q \lambda_2^2 \sum_{j=2}^{\infty} \frac{j^q}{j!} \longrightarrow 0. \tag{113}$$

Combining (111), (112), (113) and recalling that $\lambda_2 \to 0$, one completes the proof. □

2.6. *Conclusion of the proof of Theorem* 1.3. Consider first the case when X is bounded and

$$5 < q < p. \tag{114}$$

Recall definition (54) of $\mathscr{H}_{*,p,q;A,B;X;M}$. By Propositions 2.11 and 2.10, for each real M as in (50), either $\mathscr{H}_{*,p,q;A,B;X;M} \subseteq \{B\delta_c, B\delta_{-c}\}$ or there is some $H_{*,M} \in \mathscr{H}_{*,p,q;A,B;X;M}$ such that $\operatorname{supp} H_{*,M} = \{-c_{2,M}, c_{1,M}\}$ for some real $c_{1,M}$ and $c_{2,M}$ such that $0 < c_{1,M} \wedge c_{2,M} \le c_{1,M} \vee c_{2,M} = M$. So, w.l.o.g. one of the following two cases holds:

Case 1. $\mathscr{H}_{*,p,q;A,B;X;M} \subseteq \{B\delta_c, B\delta_{-c}\}$ for all real $M \ge c$.

Case 2. There exist sequences (M_k) in $[c, \infty)$, (b_k) in $[0, c]$, $(w_{1,k})$ in $[0, B]$, and $(w_{2,k})$ in $[0, B]$ such that $M_k \uparrow \infty$, and for all k one has $H_k := w_{1,k}\delta_{M_k} + w_{2,k}\delta_{-b_k} \in \mathscr{H}_{*,p,q;A,B;X;M_k}$, $w_{1,k} + w_{2,k} = B$, and $b_k^{p-2} w_{1,k} + M_k^{p-2} w_{2,k} = A$.

In case 1, by (54),

$$\mathscr{S}_{p,q;A,B;X;M} = \max\big(\mathsf{E}|X + Y_{B\delta_c}|^q, \mathsf{E}|X + Y_{B\delta_{-c}}|^q\big) \quad \text{for all real } M \ge c. \tag{115}$$

Let us show that (115) holds in case 2 as well. W.l.o.g., $b_k \to b$ for some $b \in [0, c]$. Also, $0 \le M_k^{q-2} w_{2,k} \le M_k^{q-2} \frac{A}{M_k^{p-2}} \to 0$, by the condition $q < p$ in (114). So, by Proposition 2.12,

$$\mathsf{E}|X + Y_{H_k}|^q \to \mathsf{E}|X + Y_{B\delta_{-b}}|^q. \tag{116}$$

Since $H_k \in \mathscr{H}_{*,p,q;A,B;X;M_k}$ and $\mathscr{S}_{p,q;A,B;X;M}$ is obviously nondecreasing in $M > 0$, it now follows that

$$\begin{aligned}\sup_{M>0} \mathscr{S}_{p,q;A,B;X;M} &= \mathsf{E}|X + Y_{B\delta_{-b}}|^q \\ &\le \mathscr{S}_{p,q;Bb^{p-2},B;X;c} \le \mathscr{S}_{p,q;A,B;X;c}.\end{aligned} \tag{117}$$

The last inequality follows by Proposition 2.8, because $b \in [0, c]$ and hence $Bb^{p-2} \le Bc^{p-2} = A$. Moreover, if $b \in [0, c)$ then, again by Proposition 2.8, the last inequality in (117) is strict, which is a contradiction. Thus, necessarily $b = c$, and so, by the equality in (117), (115) holds in case 2 as well, because obviously $\mathscr{S}_{p,q;A,B;X;M} \ge \mathsf{E}|X + Y_{B\delta_c}|^q$ for all real $M \ge c$.

Take now any real $M \ge c$ and any $H \in \mathscr{H}_{p;\le A,\le B;M}$. Then, by (115), (53), and (52),

$$\mathsf{E}|X + Y_H|^q \le \max\bigl(\mathsf{E}|X + Y_{B\delta_c}|^q, \mathsf{E}|X + Y_{B\delta_{-c}}|^q\bigr) \tag{118}$$

—provided that $q \in (5, p)$. Since $\mathsf{E}|X + Y_H|^q$ is continuous in $q \in (0, \infty)$ [cf. the second equality in (57)], inequality (118) holds for all $q \in [5, p]$—provided that $p > 5$.

Let us show that (118) holds when $p = 5$ (and then $q = 5$ as well). Take any $H \in \mathscr{H}_{5,\le A,\le B;M}$ and any sequence (p_n) in $(5, \infty)$ such that $p_n \downarrow 5$ as $n \to \infty$. Then $|x|^{p_n-2} \to |x|^{5-2}$ uniformly in $x \in [-M, M]$ and hence

$$A_n := A \vee \int_{\mathbb{R}} |x|^{p_n-2} H(\mathrm{d}x) \quad \longrightarrow \quad A \vee \int_{\mathbb{R}} |x|^{5-2} H(\mathrm{d}x) = A.$$

So, recalling (11) and letting $b_n := c_{p_n}(A_n, B)$, one has $b_n \to c$. Also, clearly $H \in \mathscr{H}_{p_n;\le A_n,\le B;M}$ for all n. Therefore, by (118) with $q = 5$,

$$\begin{aligned}\mathsf{E}|X + Y_H|^5 &\le \max\bigl(\mathsf{E}|X + Y_{B\delta_{b_n}}|^5, \mathsf{E}|X + Y_{B\delta_{-b_n}}|^5\bigr) \\ &\to \max\bigl(\mathsf{E}|X + Y_{B\delta_c}|^5, \mathsf{E}|X + Y_{B\delta_{-c}}|^5\bigr).\end{aligned} \tag{119}$$

Thus, indeed (118) holds when $p = q = 5$.

Take now any $\mathbf{X} = (X_1, \dots, X_n) \in \mathscr{X}_{p;X;\le A,\le B}$ and abandon the assumption that the r.v. X is bounded. Let $X_0 := X$. By Proposition 2.2, for each $i \in \{0, \dots, n\}$ and each real $M > 0$ there is a truncated version $X_{i,M}$ of X_i such that:

(i) $\mathsf{E}X_{i,M} = 0$;
(ii) $|X_{i,M}| \le M \wedge |X_i|$;
(iii) $\mathsf{E}f(X_{i,M}) \le \mathsf{E}f(X_i)$ for all convex functions $f : \mathbb{R} \to \mathbb{R}$;
(iv) $X_{i,M} \to X_i$ a.s. as $M \to \infty$;
(v) $X_{0,M}, \dots, X_{n,M}$ are independent.

Then obviously

$$(X_{1,M}, \dots, X_{n,M}) \in \mathscr{X}_{p;X;\le A,\le B}. \tag{120}$$

Letting now $S_M := X_{1,M} + \cdots + X_{n,M}$, one also has $|X_{0,M} + S_M|^q \le (n+1)^{q-1}(|X_{0,M}|^q + \sum_1^n |X_{i,M}|^q) \le (n+1)^{q-1}(|X|^q + \sum_1^n |X_i|^q)$. So, by dominated convergence,

$$\mathsf{E}|X_{0,M} + S_M|^q \underset{M\to\infty}{\longrightarrow} \mathsf{E}|X + S_{\mathbf{X}}|^q. \tag{121}$$

On the other hand, by Theorem A (with $\mathsf{E}|X_{0,M} + \cdot|^p$ and $X_{i,M}$ in place of f and X_i) and (33),

$$\mathsf{E}|X_{0,M} + S_M|^q \le \mathsf{E}|X_{0,M} + Y_{H_{*,M}}|^q, \tag{122}$$

where

$$H_{*,M}(E) := \int_E x^2 \sum_1^n \mathsf{P}(X_{i,M} \in \mathrm{d}x)$$

for all Borel sets $E \subseteq \mathbb{R}$. It follows from (120) that the measure $H_{*,M}$ is in $\mathscr{H}_{p;\le A,\le B;M}$. By (122), (118) (proved for bounded X and $H \in \mathscr{H}_{p;\le A,\le B;M}$) and item (iii) on page 2539,

$$\begin{aligned}
\mathsf{E}|X_{0,M} + S_M|^q &\le \mathsf{E}|X_{0,M} + Y_{H_{*,M}}|^q \\
&\le \max\bigl(\mathsf{E}|X_{0,M} + Y_{B\delta_c}|^q, \mathsf{E}|X_{0,M} + Y_{B\delta_{-c}}|^q\bigr) \\
&\le \max\bigl(\mathsf{E}|X + Y_{B\delta_c}|^q, \mathsf{E}|X + Y_{B\delta_{-c}}|^q\bigr) \\
&= \max\bigl(\mathsf{E}|X + c\tilde{\Pi}_\lambda|^q, \mathsf{E}|X - c\tilde{\Pi}_\lambda|^q\bigr),
\end{aligned}$$

where again λ and c are as in (11).

Now (121) yields

$$\mathsf{E}|X + S_{\mathbf{X}}|^q \le \max\bigl(\mathsf{E}|X + c\tilde{\Pi}_\lambda|^q, \mathsf{E}|X - c\tilde{\Pi}_\lambda|^q\bigr). \tag{123}$$

Thus, the first supremum in (16) is no greater than the right-hand side of (123).

To complete the proof of Theorem 1.3, it remains to note that the second supremum in (16) is no less than the right-hand side of (123). Indeed, by Lemma 2.1 with $G = \lambda\delta_c$, one has a sequence $(\mathbf{Z}_n)$ in $\mathscr{X}_{p;A,B}$ such that $S_{\mathbf{Z}_n} \xrightarrow{\mathrm{D}} c\tilde{\Pi}_\lambda$. Now, by the Fatou lemma for the convergence in distribution (Theorem 5.3 in [2]), $\liminf_n \mathsf{E}|X + S_{\mathbf{Z}_n}|^q \ge \mathsf{E}|X + c\tilde{\Pi}_\lambda|^q$, so that the second supremum in (16) is no less than $\mathsf{E}|X + c\tilde{\Pi}_\lambda|^q$. Quite similarly, that supremum is no less than $\mathsf{E}|X - c\tilde{\Pi}_\lambda|^q$, and thus it is indeed no less than the right-hand side of (123).

3. Other proofs.

PROOF OF PROPOSITION 1.1. That $\varnothing \ne \mathscr{X}_{p;A,B}$ is part of Lemma 2.1, and the inclusion $\mathscr{X}_{p;A,B} \subseteq \mathscr{X}_{p;\le A,\le B}$ is trivial. The homogeneity property holds because for any $\mathbf{X} \in \mathscr{X}_{p;A,B}$ and any real $\kappa > 0$, one has $\kappa\mathbf{X} \in \mathscr{X}_{p;\kappa^p A,\kappa^2 B}$.

Now it follows easily by Jensen's inequality that $\mathscr{E}_{p;A,B}$ is nondecreasing in A and in B. Indeed, let us first take any $\tilde{A} \in (0, A)$ and $\tilde{B} \in (0, B)$. Take then any independent finite sequences $\mathbf{X} = (X_1, \dots, X_n) \in \mathscr{X}_{p;\tilde{A},\tilde{B}}$ and $\mathbf{Y} = (Y_1, \dots, Y_m) \in \mathscr{X}_{p;A-\tilde{A},B-\tilde{B}}$; by the already verified first sentence of Proposition 1.1, such $\mathbf{X}$ and $\mathbf{Y}$ exist. Then $\mathbf{Z} := (X_1, \dots, X_n, Y_1, \dots, Y_m) \in \mathscr{X}_{p;A,B}$. Moreover, by Jensen's inequality, $\mathsf{E}|S_{\mathbf{X}}|^p \le \mathsf{E}|S_{\mathbf{X}} + S_{\mathbf{Y}}|^p = \mathsf{E}|S_{\mathbf{Z}}|^p$. Thus, $\mathscr{E}_{p;\tilde{A},\tilde{B}} \le \mathscr{E}_{p;A,B}$, for any $\tilde{A} \in (0, A)$ and $\tilde{B} \in (0, B)$.

This and the homogeneity property in turn imply that $\mathscr{E}_{p;A,\tilde{B}} \le \mathscr{E}_{p;\kappa^p A,\kappa^2 B} = \kappa^p \mathscr{E}_{p;A,B}$ for any $\tilde{B} \in (0, B]$ and any real $\kappa > 1$. Letting now $\kappa \downarrow 1$ and recalling that, by (7), $\mathscr{E}_{p;A,B} < \infty$, one concludes that $\mathscr{E}_{p;A,\tilde{B}} \le \mathscr{E}_{p;A,B}$ for any $\tilde{B} \in (0, B]$. Similarly, $\mathscr{E}_{p;\tilde{A},B} \le \mathscr{E}_{p;A,B}$ for any $\tilde{A} \in (0, A]$. Thus, indeed $\mathscr{E}_{p;A,B}$ is nondecreasing in A and in B. Now (8) immediately follows. $\square$

PROOF OF PROPOSITION 1.2. For brevity, let $K_{A,B} := \max(\gamma A, B^{p/2})^{1/p}$. Then $A/K_{A,B}^p \le 1/\gamma$, $B/K_{A,B}^2 \le 1$, and, in view of (9) and the homogeneity and monotonicity properties of $\mathscr{E}_{p;A,B}$ presented in Proposition 1.1,

$$C_{p;\gamma} = \sup_{A,B>0} K_{A,B}^{-p} \mathscr{E}_{p;A,B} = \sup_{A,B>0} \mathscr{E}_{p;A/K_{A,B}^p, B/K_{A,B}^2} \le \mathscr{E}_{p;1/\gamma,1}.$$

On the other hand, by (9), $\mathscr{E}_{p;1/\gamma,1} \le C_{p;\gamma}$. Thus, the first equality in Proposition 1.2 is verified.

The second equality there easily follows from (and in fact is equivalent to) the first one. Indeed, choosing $\gamma = B^{p/2}/A$ and using again the homogeneity property, one has $\mathscr{E}_{p;A,B} = B^{p/2}\mathscr{E}_{p;1/\gamma,1} = B^{p/2}C_{p;\gamma} = B^{p/2}C_{p;B^{p/2}/A}$. $\square$

PROOF OF THEOREM 1.5. Take any $\mathbf{X} \in \mathscr{X}_{p;X;\le A,\le B}$. Let $\sigma := \sqrt{\mathsf{Var} S_{\mathbf{X}}}$, so that $\sigma \in [0, \sqrt{B}]$. If $\sigma = 0$ then, by Jensen's inequality, $\mathsf{E}|X + S_{\mathbf{X}}|^p = \mathsf{E}|X|^p \le \mathsf{E}|X + B^{1/2}Z|^p \le A + \mathsf{E}|X + B^{1/2}Z|^p$, whence

$$\mathsf{E}|X + S_{\mathbf{X}}|^p \le A + \mathsf{E}\big|X + B^{1/2}Z\big|^p. \tag{124}$$

Suppose now that $\sigma \ne 0$. Define the function f by the formula $f(x) := \frac{\mathsf{E}|X/\sigma + x|^p}{p(p-1)}$ for all $x \in \mathbb{R}$. Using Lemma 2.4, it is easy to see that $f''(x) = \mathsf{E}|X/\sigma + x|^{p-2}$ for all $x \in \mathbb{R}$, and hence the function f is in the class $\mathscr{F}_p$ defined on page 515 from [31]. It follows by Theorem 2 in [31], and Jensen's inequality that $\mathsf{E}|X + S_{\mathbf{X}}|^p \le \mathsf{E}|X + \sigma Z|^p + A \le \mathsf{E}|X + \sigma Z + \sqrt{B - \sigma^2} Z_1|^p + A = \mathsf{E}|X + B^{1/2}Z|^p + A$, where $Z_1 \sim N(0, 1)$. So, inequality (124) holds as well in the case $\sigma \ne 0$. Thus, the first supremum in (18) is no greater than $A + \mathsf{E}|X + B^{1/2}Z|^p$.

It remains to show that the second supremum in (18) is no less than $A + \mathsf{E}|X + B^{1/2}Z|^p$. Recall (20) and take any quadruple $(c_1, c_2, \lambda_1, \lambda_2) \in Q_{p;A,B}$. By Lemma 2.1 with $G = \lambda_1 \delta_{c_1} + \lambda_2 \delta_{c_2}$, one has a sequence $(\mathbf{Z}_n)$ in $\mathscr{X}_{p;A,B}$

such that $S_{\mathbf{Z}_n} \xrightarrow{\mathrm{D}} c_1\tilde{\Pi}_{\lambda_1} + c_2\tilde{\Pi}_{\lambda_2}$. By the Fatou lemma (Theorem 5.3 in [2]), $\liminf_n \mathsf{E}|X + S_{\mathbf{Z}_n}|^p \geq \mathsf{E}|X + c_1\tilde{\Pi}_{\lambda_1} + c_2\tilde{\Pi}_{\lambda_2}|^p$, so that the second supremum in (18) is no less than $\mathsf{E}|X + c_1\tilde{\Pi}_{\lambda_1} + c_2\tilde{\Pi}_{\lambda_2}|^p$, for any $(c_1, c_2, \lambda_1, \lambda_2) \in Q_{p;A,B}$. So, by Proposition 1.6 (whose proof does not rely on Theorem 1.5), this supremum is indeed no less than $A + \mathsf{E}|X + B^{1/2}Z|^p$. $\square$

PROOF OF PROPOSITION 1.6. Let the quadruple $(c_1, c_2, \lambda_1, \lambda_2) \in Q_{p;A,B}$ vary as in (19), so that $c_1 \to 0$ and $|c_2| \to \infty$. For $j \in \{1, 2\}$, let $w_j := c_j^2\lambda_j$, so that $w_1 + w_2 = B$, $|c_1|^{p-2}w_1 + |c_2|^{p-2}w_2 = A$ and $c_1\tilde{\Pi}_{\lambda_1} + c_2\tilde{\Pi}_{\lambda_2} \overset{\mathrm{D}}{=} Y_H$ with $H := w_1\delta_{c_1} + w_2\delta_{c_2}$. It follows that $|c_1|^{p-2}w_1 \leq |c_1|^{p-2}B \to 0$ and hence $|c_2|^{p-2}w_2 \to A$. It remains to refer to Proposition 2.12 (with $q = p$), since $Y_{B\delta_0} \overset{\mathrm{D}}{=} B^{1/2}Z$. $\square$

PROOF OF COROLLARY 1.7. The first equality in (21) follows immediately by Theorems 1.3 and 1.5. Also, by Lemma 2.1 with $G = \lambda_1\delta_{c_1} + \lambda_2\delta_{c_2}$ and the Fatou lemma (Theorem 5.3 in [2]), $\mathsf{E}|X + c_1\tilde{\Pi}_{\lambda_1} + c_2\tilde{\Pi}_{\lambda_2}|^p$ is no greater than the second supremum in (21), for each $(c_1, c_2, \lambda_1, \lambda_2) \in Q_{p;A,B}$. So, the last supremum in (21) is no greater than the first two ones there.

On the other hand, the last supremum in (21) is obviously no less than the maximum in (16), and, by Proposition 1.6, this supremum is no less than $A + \mathsf{E}|X + B^{1/2}Z|^p$. So, by Theorems 1.3 and 1.5, the last supremum in (21) is no less than the first two ones there. $\square$

PROOF OF THEOREM 1.8. This proof is analogous to that of Theorem 1.3 and even significantly simpler overall, since analogues of Propositions 2.11 and 2.12 are not needed here. In the proofs of the analogues of Propositions 2.8, 2.9, and 2.10, one should use the symmetrized real-valued measure $\Delta(\mathrm{d}u) + \Delta(-\mathrm{d}u)$ in place of $\Delta(\mathrm{d}u)$. $\square$

Acknowledgement. I am pleased to thank the referees for useful and stimulating comments.

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