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2D COULOMB GASES AND THE RENORMALIZED ENERGY

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We study the statistical mechanics of classical two-dimensional "Coulomb gases" with general potential and arbitrary β, the inverse of the temperature. Such ensembles also correspond to random matrix models in some particular cases. The formal limit case $\beta = \infty$ corresponds to "weighted Fekete sets" and also falls within our analysis.

It is known that in such a system points should be asymptotically distributed according to a macroscopic "equilibrium measure," and that a large deviations principle holds for this, as proven by Petz and Hiai [In *Advances in Differential Equations and Mathematical Physics (Atlanta, GA, 1997)* (1998) Amer. Math. Soc.] and Ben Arous and Zeitouni [*ESAIM Probab. Statist.* **2** (1998) 123–134].

By a suitable splitting of the Hamiltonian, we connect the problem to the "renormalized energy" W, a Coulombian interaction for points in the plane introduced in [*Comm. Math. Phys.* **313** (2012) 635–743], which is expected to be a good way of measuring the disorder of an infinite configuration of points in the plane. By so doing, we are able to examine the situation at the microscopic scale, and obtain several new results: a next order asymptotic expansion of the partition function, estimates on the probability of fluctuation from the equilibrium measure at microscale, and a large deviations type result, which states that configurations above a certain threshhold of W have exponentially small probability. When $\beta \to \infty$, the estimate becomes sharp, showing that the system has to "crystallize" to a minimizer of W. In the case of weighted Fekete sets, this corresponds to saying that these sets should microscopically look almost everywhere like minimizers of W, which are conjectured to be "Abrikosov" triangular lattices.

1. Introduction.

Our goal in this paper is to improve the understanding of the $n \to +\infty$ asymptotics of the probability law

$$d\mathbb{P}_n^\beta(x_1, \ldots, x_n) = \frac{1}{Z_n^\beta} e^{-(\beta/2) w_n(x_1,\ldots,x_n)}\, dx_1 \cdots dx_n, \tag{1.1}$$

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where Z_n^β is the associated partition function and

$$w_n(x_1, \dots, x_n) = -\sum_{i \neq j} \log|x_i - x_j| + n \sum_{i=1}^n V(x_i), \tag{1.2}$$

is the Hamiltonian. Here, the x_i's belong to $\mathbb{R}^2$ (identified with the complex plane $\mathbb{C}$), $\beta > 0$ is a parameter corresponding to the inverse of the temperature and V is a potential satisfying some growth and regularity assumptions, which we will detail below.

When $V(x) = |x|^2$ and $\beta = 2$, this probability law happens to be the one obeyed by the eigenvalues of random matrices with independent complex Gaussian entries—the so-called Ginibre ensemble—as shown in [27] (see also [32], Chapter 15). In statistical mechanics, for any value of β and possibly different potentials, this is the law for the two-dimensional Coulomb gas with confining potential V, also known as the two-dimensional one-component plasma, the Gaussian β-ensemble or the Dyson gas.

The fact that Coulomb gases are naturally related to random matrices was first pointed out by Wigner [50] and later exploited by Dyson [20]. For the general background and references to the literature, we refer to the book by Forrester [22]. Particularly relevant references in the physics literature are [2, 29, 43]. Note that one aspect of the current research on the random matrix aspect in the complex case is to study the more general case of random matrices with entries that are not necessarily Gaussian and to show that the average behavior is the same as for the Ginibre ensemble, see [8, 48, 49]. Our results only apply to Gaussian ensembles, but allow a large class of potentials, and any value of β.

The problem of minimizing w_n, which we call the $\beta = +\infty$ case, also falls within the scope of our results. In this case, the minimizers of w_n are known as weighted Fekete sets, which are of interest for interpolation and have been studied for about a century (cf. [38, 39]).

Known results. The first category of results regarding the limit of $\mathbb{P}_n^\beta$ (or the limits of minimizers of w_n in the case $\beta = +\infty$) takes as the converging object the empirical measure

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i},$$

and in this sense can be dubbed macroscopic, mean-field, or density results. In the case of finite β, the measure μ_n is random and its law is $\mathbb{P}_n^\beta$, modulo a change of variables. In the case $\beta = +\infty$, the measure μ_n is almost surely of the form $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$, where $(x_i)_{1 \le i \le n}$ minimizes w_n.

In the $\beta = +\infty$ case, it was shown by Fekete [21] and Szegö [47] at the beginning of the twentieth century that, in modern language, the functions w_n/n^2

Gamma converge as $n \to +\infty$ to a functional defined on probability measures over $\mathbb{R}^2$, that one may call the *mean-field limit* in the language of statistical mechanics:

$$(1.3)\qquad I(\mu) = \int_{\mathbb{R}^2\times\mathbb{R}^2} -\log|x-y|\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} V(x)\,d\mu(x).$$

The minimization of I was first studied by Gauss, who noted that the unique minimizing measure μ_0, is characterized by the fact that the potential it generates achieves its minimum on the support of μ_0. This characterization was made rigourous, and generalized to other interaction potentials by Frostman [25].

The main consequence of the Gamma-convergence (for a definition we refer to [15, 18]) statement is that, as $n \to +\infty$, the minimum of w_n/n^2 converges to $I(\mu_0) = \min I$ and that μ_n converges to μ_0. We refer to the book [39] for more on this subject. The measure μ_0 is commonly referred to as the *equilibrium measure*, and its support as the *droplet*.

In the case of finite β, the corresponding result is that the limit of μ_n, which can be seen as a point process with law $\mathbb{P}_n^\beta$, is in fact deterministic and equal to μ_0. In the case $V(x) = |x|^2$ and $\beta = 2$, then $\mu_0 = \frac{1}{\pi}\mathbf{1}_B$, where B is the unit disc in $\mathbb{R}^2$ and $\mathbf{1}$ denotes a characteristic function. This is the celebrated *circle law* or circular law, attributed to Ginibre, Mehta, an unpublished paper of Silverstein in 1984, and then Girko [28]. Moreover, a corresponding large deviation statement was established by Petz and Hiai [33] and Ben Arous and Zeitouni (in a slightly different setting) [10]; see Theorem 5.1 below, in particular, the following holds, as $n \to +\infty$:

$$\mathbb{P}_n^\beta\left(\frac{w_n(x_1,\ldots,x_n)}{n^2} > I(\mu_0) + \delta\right) \le \exp\bigl(-\delta\beta n^2 + o(n^2)\bigr).$$

For the case of a general V and a general β, a similar result holds, adapting the proof of [10]. Such results are proven in possibly higher complex dimensions in [11].

The second category of results concerns the microscopic or local behavior. In this case, one is interested in studying the behavior of the point process, or the minimizers of w_n, at a smaller scale—ideally the scale $1/\sqrt{n}$ which is the scale at which points are at finite distance from their neighbors. In the $\beta = +\infty$ case, precise bounds on the discrepancy between the measure μ_n minimizing w_n an μ_0 can be found in [6] the paper [36] improves them using techniques introduced in [42] and in the present paper. In the so-called determinantal case $\beta = 2$, it is proved in [4, 5] that the law of the linear statistics of the fluctuations is a Gaussian with specific variance and mean; see also [34] for related results. Also, a local version of the circle law is proved in [14] for matrices with i.i.d. entries which are not necessarily Gaussian.

We finally note that much more precise results are known in the much better studied—and generally considered easier—case where the x_i's belong to $\mathbb{R}$ instead of $\mathbb{R}^2$, or where one considers cases in which μ_0 is a measure with a one-dimensional support. The application of our techniques to that case is the object of [40] which also contains references to the vast literature on this topic. Let us also mention that the analogue of the present study is undertaken for higher dimensions ($x_i \in \mathbb{R}^d$ for any $d \geq 2$, and the corresponding Coulomb interaction kernel) in [37], using a partly different approach.

Sketch of our results. Our point of view is to start from an object whose limit will be more discriminating then that of the measures $\{\mu_n\}_n$, and to obtain results valid for arbitrary β (the nondeterminantal case) and quite general V's. To simplify the exposition, let us consider first the case $V(x) = |x|^2$ and $\beta = +\infty$, that is, the minimization of w_n. In this case, we have seen that $\mu_n \to \frac{1}{\pi}\mathbf{1}_B$. To study μ_n at the scale $n^{-1/2}$, we introduce a measure on the blow-ups of $(x_i)_{1\leq i\leq n}$ at this scale. This scheme is similar to ideas developed in [3] for abstract two-scale Gamma-convergence, and was in fact suggested to us in a conversation with Varadhan; we first used it in [42]. The idea is to associate to any point x in the droplet B the set of points blown up with origin at x:

$$\Lambda_{n,x} = \{\sqrt{n}(x_i - x) \mid 1 \leq i \leq n\}.$$

We then define P_n to be the probability measure which makes the $\Lambda_{n,x}$'s equiprobable with respect to $x \in B$. We write this as

$$P_n = \fint_B \delta_{\Lambda_{n,x}}\, dx.$$

In the limit $n \to +\infty$, the measures P_n converge to a probability measure P on the set of discrete subsets of $\mathbb{R}^2$, or a point process. In the case of finite β, the limiting measure P is itself random, and obeys a certain law that we wish to describe, as much as possible.

The description of P (in the case $\beta = +\infty$) or its law (in the case $\beta < +\infty$) is done in terms of the renormalized Coulombian interaction energy W introduced in [42], which corresponds to total Coulomb interaction of the system known in the physics literature as a jellium. It is defined below precisely, but for now suffice it to say that it is defined for any discrete subset Λ of $\mathbb{R}^2$ which has bounded density at infinity, and corresponds to the average interaction energy on large balls of a set of unit singular charges placed at the points of Λ with a uniform negative background. It is proved in [42] that W is bounded below and that its infimum is achieved. It is also proved there that among Bravais lattices, the unique minimizer of W is the triangular lattice, consisting of the vertices of identical equilateral triangles tiling the plane.

Our results are as follows: In the case $\beta = +\infty$, we prove that P is supported on the set of minimizers of W. To be more precise, we compute a second term in the expansion by Gamma-convergence of w_n, which implies the following statement:

$$\frac{1}{n}\left(w_n - n^2 I(\mu_0) + \frac{n}{2}\log n\right) \xrightarrow{\Gamma} \widetilde{W} \qquad \text{where } \widetilde{W}(P) = \frac{|\Sigma|}{\pi}\int W(\Lambda)\, dP(\Lambda).$$

In the case $\beta < +\infty$, where P is random, we show that there exists $C_\beta > 0$ such that, almost surely, $\widetilde{W}(P) \le \alpha + C_\beta$, where α is the minimum of $\widetilde{W}$, equal to $\frac{|\Sigma|}{\pi}$ times the minimum of W. Moreover, we prove that $C_\beta \to 0$ as $\beta \to +\infty$, so that we recover the $\beta = +\infty$ statement. In fact, this will follow (after some work) from a large deviations type result, in particular using the previous Gamma-convergence result. As further consequences, we will obtain the following asymptotics:

$$\log \mathbb{P}_n^\beta\left(\frac{1}{n}\left(w_n - n^2 I(\mu_0) + \frac{n}{2}\log n\right) > \alpha + \eta\right) \le -\frac{n\beta}{2}(\eta - C_\beta) + o(n),$$
$$\left|\log Z_n^\beta + \frac{n\beta}{2}\left(nI(\mu_0) - \frac{1}{2}\log n + \alpha\right)\right| \le C_\beta n\beta, \tag{1.4}$$

where C_β tends to 0 as $\beta \to +\infty$.

Inequality (1.4) improves on known results which only give the expansion $\log Z_n^\beta \sim \frac{\beta}{2}n^2 I(\mu_0)$. It can also be compared to a formula derived by nonrigorous arguments in [51]: our results contradicts their formula, however, it seems that the contradiction can be resolved by seeing their formula as an expansion for Z_n^β relative to the expansion for a reference potential. Let us recall that an exact value for Z_n^β is only known for the Ginibre ensemble case $\beta = 2$ and $V(x) = |x|^2$: it is $Z_n^2 = n^{-(1/2)n(n+1)}\pi^n \prod_{k=1}^n k!$ (see [32], Chapter 15). For comparison, known asymptotics allow us to deduce (cf. [22], equation (4.184))

$$\log Z_n^2 = -\frac{3n^2}{4} + \frac{n}{2}\log n + n\left(-1 + \frac{1}{2}\log 2 + \frac{3}{2}\log \pi\right) + O(\log n) \quad \text{as } n \to \infty, \tag{1.5}$$

where we note that the value $\frac{3}{4}$ indeed coincides with $I(\mu_0)$ for this potential. On the other hand, no exact formula exists for general potentials,[3] nor for quadratic potentials if $\beta \ne 2$. This is in contrast with the one-dimensional situation for which, at least in the case of quadratic V, Z_n^β has an explicit expression for every β, given by the famous Selberg integral formulas (see, e.g., [7]). In statistical mechanics language, the existence of an exact asymptotic expansion up to order n for $\log Z_n^\beta$ is essentially the existence of a thermodynamic limit. This is established in a nonrigorous way in [43] in two dimensions. The existence of the thermodynamic limit here remains to be completed by getting upper and lower bounds which match up to $o(n)$ in (2.9).

The above results will be stated as theorems in the next section, for a class of potentials V which includes smooth strictly convex potentials which grow sufficiently fast at infinity. Some differences from the above statements will arise from the fact that, under our assumptions, the equilibrium measure μ_0 is in general not

[3] An exception is the result of [19] for a quadrupole potential.

uniform on the droplet. Also, the probability measures P_n and P will in fact be defined on the space of electric fields generated by the charges at Λ rather than on the sets Λ themselves.

Connection to the crystallization problem. The definition of W in [42] arose in the study of the Ginzburg–Landau model of superconductivity (for general reference on the topic, cf. [41]) where superconducting vortices in certain regimes can be proven to be accurately approximated by points. These points, when densely packed, form perfect triangular *Abrikosov lattices*,[4] named after the physicist who predicted them [1]. These lattices are indeed observed in experiments on superconductors. In [42], we partly explained their appearance starting from the Ginzburg–Landau model, by showing that minimizers of the Ginzburg–Landau energy have vortices that minimize the renormalized energy W after blow-up by a factor $\sqrt{n}$. The conjecture made in [42], also supported by some mathematical evidence (see Section 3.2), is that the minimal value of W is achieved by the triangular lattice—proving this conjecture is a crystallization question, and there are very few instances in which such questions are solved. If proven true, this would indeed explain why vortices form these patterns.

The results of this paper say that if indeed W was minimized by the triangular lattice, then the same conclusion on the local behavior of weighted Fekete sets would hold, that is, their blow-ups at the scale $1/\sqrt{n}$ would look like the triangular lattice. The picture for finite β would be that, as $\beta \to +\infty$, the blow-ups tend to minimize W, hence in some sense tend to crystallize onto a triangular lattice. To our knowledge, this is the first time Coulomb gases and Fekete sets are rigorously connected to triangular lattices—although in an averaged way—in agreement with predictions in the physics literature (see [2] and references therein).

2. Statement of results.

We now state our results precisely.

Assumptions on V. Our assumptions on V are made to ensure that the droplet is a compact set with smooth boundary and that the equilibrium measure has a strictly positive smooth density with respect to the restriction of the Lebesgue measure to this set. They are mostly technical—we use them to simplify our constructions—and could certainly be somewhat relaxed.

We begin by defining the equilibrium measure μ_0. Assuming that $\lim_{|x|\to+\infty} \frac{V(x)}{2} - \log|x| = +\infty$, that V is lower semicontinuous and bounded below and that V is finite on a set of positive capacity, there exists a unique minimizer (see [25] or [39], Chapter 1) of (1.3) among probability measures on $\mathbb{R}^2$, denoted μ_0 (the equilibrium measure), and its support, the droplet, will be denoted by Σ.

[4] For photos one can see http://www.fys.uio.no/super/vortex/.

The following characterization of μ_0 is in essence due to Gauss, see [25] or [39] for a modern treatment. Defining the electrostatic potential generated by a measure μ to be

$$U^{\mu}(x) = -\int_{\mathbb{R}^2} \log|x-y|\,d\mu(y), \tag{2.1}$$

μ_0 is characterized by the fact that there exists $c \in \mathbb{R}$ such that

$$\begin{aligned} U^{\mu_0} + \frac{V}{2} &\geq c \qquad \text{q.e. in } \mathbb{R}^2 \quad \text{and} \\ U^{\mu_0} + \frac{V}{2} &= c \qquad \text{q.e. in } \operatorname{Supp}(\mu_0) = \Sigma, \end{aligned} \tag{2.2}$$

where q.e. means "quasi-everywhere" or outside of a set of zero capacity. Below we will sometimes denote by Δ^{-1} the operator of convolution by $\frac{1}{2\pi}\log|\cdot|$, so that $\Delta \circ \Delta^{-1} = \mathrm{Id}$, where Δ is the usual Laplacian, and $U^{\mu} = -2\pi\Delta^{-1}\mu$.

Note that another way to characterize U^{μ_0} is as the solution of the following *obstacle problem*:[5] It is the minimal superharmonic function bounded below by the function $c - V/2$ and harmonic outside the set

$$\omega := \{U^{\mu_0} = c - V/2\} \tag{2.3}$$

called the *coincidence set* and containing the droplet Σ. This implies in particular that U^{μ_0} is $C^{1,1}_{\mathrm{loc}}$ if V is (see [23]). For more details and references on this correspondence, see [44], Chapter 2.

We may now state our assumptions on V:

$$\lim_{|x|\to+\infty} \frac{V(x)}{2} - \log|x| = +\infty, \tag{2.4}$$

$$\begin{aligned} &V \text{ is } C^3 \text{ and s.t. } \partial\Sigma \text{ is } C^1 \text{ and } \mu_0 = m_0 \mathbf{1}_{\Sigma}\,dx \\ &\qquad\qquad \text{where } m_0 \text{ is } C^1 \text{ and strictly positive in } \Sigma, \end{aligned} \tag{2.5}$$

$$\text{there exists } \beta_1 > 0 \text{ such that} \qquad \int_{\mathbb{R}^2} e^{-\beta_1(V/2(x) - \log|x|)}\,dx < +\infty. \tag{2.6}$$

Assumption (2.4) on the growth of V is what is needed to apply the results from [39] and to guarantee that (1.3) has a minimizer with compact support. Assumption (2.6) is a supplementary assumption on the growth of V at infinity, needed for the case with temperature, to show that the partition function is well defined. It is only slightly more restrictive than (2.4): It is satisfied, for example, if $\lim_{|x|\to\infty} \frac{V}{2}(x) - (1+\varepsilon)\log|x| = +\infty$.

[5]The obstacle problem is a free-boundary problem and a much studied classical problem in the calculus of variations; for general reference, see [24, 30].

Assumption (2.5) is technical and certainly not optimal, it is needed essentially in the construction of recovery sequences (in the language of Gamma-convergence). A class of potentials satisfying (2.5) are the smooth strictly convex (in the sense that $D^2 V > 0$ everywhere) potentials satisfying the growth assumption of (2.4). Indeed in this case, as discussed in [42], Section 7.1, the coincidence set has a smooth boundary. Moreover, as soon as V is $C^{1,1}_{\mathrm{loc}}$, it follows from $U^{\mu_0} \in C^{1,1}_{\mathrm{loc}} = W^{2,\infty}_{\mathrm{loc}}$, that we may write

$$d\mu_0 = m_0(x)\, dx \qquad \text{where } m_0(x) = \frac{\Delta V(x)}{4\pi} \mathbf{1}_\omega(x), \tag{2.7}$$

so that m_0 is indeed strictly positive and C^1 if V is strictly convex and C^3, and ω and Σ then coincide.

Finally, we denote by $\underline{m}$ (resp., $\overline{m}$) the minimum (resp., the maximum) of m_0 on Σ so that, on Σ, we have

$$0 < \underline{m} \le m_0 \le \overline{m}. \tag{2.8}$$

Asymptotics for $\mathbb{P}_n^\beta$, Z_n^β. Our main results are stated in terms of mathematical objects like the renormalized energy, which will take some space to define, but we can already state some consequences in terms of a constant α related to the minimum of the renormalized energy and defined below.

THEOREM 2.1. *Let V satisfy* (2.4)–(2.6). *For any $\beta > 0$, there exists $C_\beta > 0$ such that* $\lim_{\beta\to+\infty} C_\beta = 0$ *and*

$$\limsup_{n\to+\infty} \frac{1}{n\beta} \left| \log Z_n^\beta + \frac{n\beta}{2}\left(nI(\mu_0) - \frac{1}{2}\log n + \alpha \right) \right| \le C_\beta, \tag{2.9}$$

where α is defined in (2.25). *More generally, if*

$$A_n = \left\{ (x_1,\dots,x_n) \in \mathbb{C}^n \mid w_n(x_1,\dots,x_n) - \left(n^2 I(\mu_0) - \frac{n}{2}\log n \right) \ge n(\alpha+\eta) \right\},$$

then we have

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}_n^\beta(A_n) \le -\frac{\beta}{2}(\eta - C_\beta). \tag{2.10}$$

As noted above, estimate (2.9) improves on known expansions for $\log Z_n^\beta$ (except for some special cases) in the following way: in addition to the leading term $n^2 I(\mu_0)$ we have a term $\frac{\beta}{4} n \log n$ which comes from a scaling argument, and then the term $\frac{\beta}{2}\alpha n$ with the constant α precisely characterized below. The error term in the expansion is small compared to $n\beta$ in the limit of large n and β. Thus, (2.9) may be seen as a low-temperature asymptotic expansion of the partition function, even though it also provides information at finite temperature, namely that fixing $\beta > 0$, as $n \to +\infty$ we have

$$\log Z_n^\beta = -\frac{\beta}{2}\left(n^2 I(\mu_0) - \frac{1}{2} n \log n\right) + O(n).$$

When $\beta = +\infty$ the result should be read as the following next order asymptotic expansion of the minimum of the Coulomb gas energy:

$$\min w_n = n^2 I(\mu_0) - \tfrac{1}{2} n \log n + \alpha n + o(n).$$

We now define the mathematical objects which will allow us to define α and state the results from which Theorem 2.1 will follow.

Definition of P_n and existence of its limit. We choose some $p \in [1, 2)$ once and for all, and construct, from a given n-tuple $(x_1, \dots, x_n) \in \mathbb{C}^n$, a probability measure P_n on $X := \Sigma \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, as follows.

First, we let

$$\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}, \qquad H_n = U^{\mu_n - \mu_0}, \tag{2.11}$$

where U^μ is defined in (2.1). Note that, by elliptic regularity, H_n belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ since $p \in [1, 2)$.

As already mentioned, a crucial point in our work is that we use blown-up coordinates at the scale $1/\sqrt{n}$. We denote these by a prime, so that $x' = \sqrt{n} x$. We then let $m_0'(x') = m_0(x)$ where $m_0(x)$ is as in (2.5), and define rescaled versions of the measures μ_n and μ_0, which are no longer probability measures, but rather, positive measures of mass n, by letting

$$\mu_n' = \sum_{i=1}^n \delta_{x_i'}, \qquad \mu_0' = m_0'(x')\, dx'. \tag{2.12}$$

Finally, we define the (electric) field in blown-up coordinates associated to $(x_1, \dots, x_n)$ to be

$$E_n(x') = -\sqrt{n} \nabla H_n\left(\frac{x'}{\sqrt{n}}\right), \tag{2.13}$$

so that $E_n \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for $p \in [1, 2)$ and E_n is the only solution tending to 0 at infinity of

$$\operatorname{div} E_n = 2\pi(\mu_n' - \mu_0'), \qquad \operatorname{curl} E_n = 0. \tag{2.14}$$

The probability measure P_n, which plays the main role in our results is the push-forward of the normalized Lebesgue measure on Σ by the map $x \mapsto (x, E_n(x' + \cdot))$. Alternatively, we write it as

$$P_n = \fint_\Sigma \delta_{(x, E_n(\sqrt{n}x + \cdot))}\, dx, \tag{2.15}$$

where $\fint_A$ will denote the integral average over A. P_n is a probability measure on $X := \Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ [couples of (blow-up centers, blown-up electric field around this center)]. We emphasize that P_n should not be confused with $\mathbb{P}_n^\beta$. Each realization or configuration $(x_1, \dots, x_n)$ gives rise in a deterministic fashion to its own P_n, which encodes all the blown-up profiles of associated electric fields. We denote by i_n this encoding mapping (or embedding):

$$\begin{aligned} i_n : \mathbb{C}^n &\to \mathcal{P}(X) \\ (x_1, \dots, x_n) &\mapsto P_n, \end{aligned} \tag{2.16}$$

where $\mathcal{P}(X)$ denotes the space of probability measures on $X = \Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, endowed with the topology of weak convergence.

Our first result states that if the law of $(x_1, \dots, x_n)$ is $\mathbb{P}_n^\beta$ (including the case $\beta = +\infty$) then, as $n \to +\infty$, the law of P_n converges to a probability law on a space which we define now.

DEFINITION 2.1. Let m be a nonnegative number and E be a vector field in $\mathbb{R}^2$. We say E belongs to the admissible class $\mathcal{A}_m$ if

$$\operatorname{div} E = 2\pi(\nu - m), \qquad \operatorname{curl} E = 0, \tag{2.17}$$

where ν has the form

$$\nu = \sum_{p \in \Lambda} \delta_p \qquad \text{for some discrete set } \Lambda \subset \mathbb{R}^2, \tag{2.18}$$

and if

$$\frac{\nu(B_R)}{|B_R|} \qquad \text{is bounded by a constant independent of } R > 1. \tag{2.19}$$

DEFINITION 2.2. We say $P \in \mathcal{P}(X)$ is $T_{\lambda(x)}$-invariant if, for any C^1 function $\lambda : \Sigma \to \mathbb{R}^2$, P is invariant under $(x, E) \mapsto (x, E(\lambda(x) + \cdot))$.

We have:

THEOREM 2.2 (First properties of the limiting object). *Let V satisfy (2.4)–(2.6). Assume that for each n, the n-tuple $(x_1, \dots, x_n)$ minimizes w_n or, more generally, that for some $C > 0$ independent of n*

$$w_n(x_1, \dots, x_n) \le n^2 I(\mu_0) - \frac{n}{2} \log n + Cn.$$

Then, modulo a subsequence, the associated probability measures $P_n \in \mathcal{P}(X)$ converge to $P \in \mathcal{P}(X)$ such that:

- *The first marginal of P is the normalized Lebesgue measure on Σ.*

- *It holds for P-a.e. (x, E) that $E \in \mathcal{A}_{m_0(x)}$.*
- *P is $T_{\lambda(x)}$-invariant.*

On the other hand, if $\beta > 0$ is finite, then the law of P_n—that is, the push-forward of $\mathbb{P}_n^\beta$ by i_n—converges weakly to a probability measure $\widetilde{\mathbb{P}}^\beta$ on $\mathcal{P}(\mathcal{P}(X))$, and $\widetilde{\mathbb{P}}^\beta$-almost every P satisfies the above properties.

The above result motivates the following definition.

DEFINITION 2.3. We say $P \in \mathcal{P}(X)$ is *admissible* if the first marginal of P is the normalized Lebesgue measure on Σ, if it holds for P-a.e. (x, E) that $E \in \mathcal{A}_{m_0(x)}$ and if P is $T_{\lambda(x)}$-invariant.

REMARK 2.4. Let

$$\mu'_{n,x} = \sum_{i=1}^n \delta_{\sqrt{n}(x_i - x)}$$

and define

$$Q_n = \fint_\Sigma \delta_{\mu'_{n,x}}\, dx.$$

This is equivalent to letting Q_n be the push-forward of P_n by the map $(x, E) \mapsto \frac{1}{2\pi} \operatorname{div} E + m'_0(\sqrt{n}x + \cdot)\, dx'$, which is continuous for a suitable topology on the target space.

Then Theorem 2.2 implies for any $\beta > 0$ the existence of a limiting point process $\widetilde{\mathbb{Q}}^\beta$, that is, a probability on the limiting Q's, which themselves encode all the (x, ν)'s.

Renormalized energy. We now wish to describe further the law $\widetilde{\mathbb{P}}^\beta$ of P. This is done in terms of the *renormalized energy* W introduced in [42], which is a way of computing the Coulomb interaction between an infinite number of point charges in the plane with a uniform neutralizing background of density m, in other words a jellium of density m. We point out that, to our knowledge, each of the analogous Coulomb systems studied in the physics literature (e.g., [2, 43]) comprise a finite number of point charges, and hence implicitly extend only to a bounded domain on which there is charge neutrality. Here, we do not assume any local charge neutrality.

The point of the definition of W below—the main properties will be described in Section 3.2—is that we would like to define $W(\nabla H)$ for H solving $-\Delta H = 2\pi(\sum_p \delta_p - m)$ as $\limsup_{R\to\infty} \frac{1}{2} \fint_{B_R} |\nabla H|^2$, however, these integrals diverge because of the logarithmic divergence of H near each point. Instead, we compute $\int |\nabla H|^2$ in a "renormalized" way or in "finite parts," by cutting out holes around each p and subtracting off the corresponding divergence, in the manner of [12], from which the name "renormalized energy" is borrowed.

We denote by $B(x,R)$ or $B_R(x)$ the ball centered at x with radius R and let $B_R = B(0,R)$. In all the paper, when U is a measurable set, $|U|$ will denote its Lebesgue measure, and when U is a finite set, $\#U$ will denote its cardinal.

DEFINITION 2.5. Let m be a nonnegative number. For any continuous function χ and any vector-field E in $\mathbb{R}^2$ satisfying (2.17) where ν has the form (2.18), we let

$$W(E,\chi) = \lim_{\eta\to 0}\left(\frac{1}{2}\int_{\mathbb{R}^2\setminus\bigcup_{p\in\Lambda}B(p,\eta)} \chi|E|^2 + \pi\log\eta\sum_{p\in\Lambda}\chi(p)\right). \tag{2.20}$$

To see that the limit $\eta\to 0$ exists, it suffices to observe that in view of (2.17)–(2.18), E is a gradient and near each $p\in\Lambda$ we may write $E = \nabla\log|\cdot - p| + \nabla f(\cdot)$ where f is C^1 by elliptic regularity. The existence of the limit follows easily. It also follows that E belongs to L^q_{loc} for any $q<2$.

In the sequel, K_R will denote the two-dimensional squares $[-R,R]^2$. We also use the notation χ_{K_R} for positive cutoff functions satisfying, for some constant C independent of R,

$$\begin{aligned} &|\nabla\chi_{K_R}|\le C, \qquad \operatorname{Supp}(\chi_{K_R})\subset K_R,\\ &\chi_{K_R}(x)=1 \qquad \text{if } d(x,K_R^c)\ge 1. \end{aligned} \tag{2.21}$$

DEFINITION 2.6. The renormalized energy W is defined, for $E\in\mathcal{A}_m$, by

$$W(E) = \limsup_{R\to\infty}\frac{W(E,\chi_{K_R})}{|K_R|}, \tag{2.22}$$

with $\{\chi_{K_R}\}_R$ satisfying (2.21). If $E\notin\mathcal{A}_m$ for any $m\ge 0$ then we let $W(E)=+\infty$.

Finally, for any probability measure $P\in\mathcal{P}(X)$ we let $\widetilde{W}(P)=+\infty$ if P is not admissible (see Definition 2.3) and, if P is admissible,

$$\widetilde{W}(P) = \frac{|\Sigma|}{\pi}\int W(E)\,dP(x,E). \tag{2.23}$$

We note that we have taken a slightly different definition from [42]: first, the vector-fields in (2.17) have been rotated by $\pi/2$, second $\mathcal{A}_m$ here corresponds to $\mathcal{A}_{2\pi m}$ in [42] and finally in [42] we presented the definition with averages over general sets; here, we have chosen for simplicity to introduce it only with square averages.

We may guess the minimal value of $\widetilde{W}(P)$ for an admissible P: It is easy to check that if E belongs to $\mathcal{A}_m$, $m>0$, then $E' = \frac{1}{\sqrt{m}}E(\cdot/\sqrt{m})$ belongs to $\mathcal{A}_1$ and that

$$W(E) = m\left(W(E') - \frac{\pi}{2}\log m\right). \tag{2.24}$$

In particular, we have

$$\alpha_m := \min_{\mathcal{A}_m} W = m\left(\min_{\mathcal{A}_1} W - \frac{\pi}{2}\log m\right).$$

Therefore, if $E \in \mathcal{A}_{m_0(x)}$ then $W(E) \geq \alpha_m$, and it follows that if $E \in \mathcal{A}_{m_0(x)}$ for P-a.e. (x, E) and if the first marginal of P is the normalized Lebesgue measure on Σ then, using the fact that, from (2.7), $\int_\Sigma m_0 = 1$, we find

$$\widetilde{W}(P) \geq \fint_\Sigma \alpha_{m_0(x)}\,dx = \frac{\alpha_1}{\pi} - \frac{1}{2}\int_\Sigma m_0(x)\log m_0(x)\,dx.$$

We may now define the constant α which appeared in our results above:

$$\alpha := \frac{|\Sigma|}{\pi}\fint_\Sigma \alpha_{m_0(x)}\,dx = \frac{\alpha_1}{\pi} - \frac{1}{2}\int_\Sigma m_0(x)\log m_0(x)\,dx, \tag{2.25}$$

and α is in fact really the minimum of $\widetilde{W}$. Note that α only depends on V, via the integral term involving the density of the equilibrium measure, and on the (so far) unknown constant α_1, conjectured to be the value of W at the triangular lattice of volume 1.

We may now give more information about the law $\widetilde{\mathbb{P}}^\beta$ of P. We begin with the case $\beta = +\infty$.

THEOREM 2.3 (Microscopic behavior of weighted Fekete sets). *Let V satisfies (2.4)–(2.6). Assume that for each n, the n-tuple $(x_1, \ldots, x_n)$ minimizes w_n. Then*

$$\lim_{n\to\infty} \frac{1}{n}\left(w_n(x_1, \ldots, x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right) = \alpha.$$

Moreover, the limit P of $P_n = i_n(x_1, \ldots, x_n)$ is such that for P-almost every (x, E), E minimizes W over $\mathcal{A}_{m_0(x)}$.

This is an averaged statement about the microscopic behavior of minimizers. In informal terms, it says that when blowing up around a point chosen uniformly at random in Σ, the configurations converge to minimizers of W almost surely. This result is improved in [36], where it is shown that this in fact holds in some suitable sense when blowing up around *any* point in Σ, and, as in [6], that the number of points in any ball which is large at the microscopic scale coincides with the corresponding mass of the equilibrium measure.

The two theorems above are of course closely linked. They both follow from the stronger statement (Theorem 4.1 below) that

$$\frac{1}{n}\left(w_n(x_1, \ldots, x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right)$$

Gamma-converges to $\widetilde{W}$, whose minimum over admissible P's is α. This theorem is the analogue of the main result of [42] but for w_n rather than the Ginzburg–Landau energy. It is technically simpler to prove, except for the possibility of a nonconstant weight $m_0(x)$ which was absent from [42].

We now turn to the case of finite β. Assuming the law of $(x_1, \dots, x_n)$ is $\mathbb{P}_n^\beta$, then by Theorem 2.2 the law of $i_n(x_1, \dots, x_n)$ converges to $\widetilde{\mathbb{P}}^\beta$ and we have:

THEOREM 2.4. *For any $\beta > 0$, there exists $C_\beta > 0$ such that* $\lim_{\beta\to+\infty} C_\beta = 0$ *and, for $\widetilde{\mathbb{P}}^\beta$-almost every P, it holds that*

$$\widetilde{W}(P) \le \alpha + C_\beta.$$

This statement says that P, which is a random object if β is finite, might not be a minimizer of $\widetilde{W}$ with probability one, but the value of $\widetilde{W}(P)$ is bounded by a value which depends on β and tends to the minimal value α as $\beta \to +\infty$, that is, when the temperature tends to 0, which agrees with the $\beta = +\infty$ statement of Theorem 4.1. This will follow from a more precise large-deviation type result (Theorem 5.2 below).

REMARK 2.7. One could wonder whether the error C_β does indeed exist or if a more clever proof would show that $C_\beta = 0$, which would in some sense show that P is after all a deterministic object. In fact we do not expect that, at nonzero temperature, P is concentrated with probability one on minimizers of W: Indeed numerical simulations of the Ginibre ensemble[6] corresponding to $\beta = 2$ show patterns of points with a certain microscopic disorder, which are certainly not crystalline. This is probably explained by the fact that at finite temperature, and at this scale, an entropy term should come into the computation of the law of P and compete with the minimization of W.

The rest of the paper is organized as follows: Section 3 contains the proof of the splitting formula which relates w_n and W, and properties of W. Section 4 contains the proof of the main Gamma-convergence statement (Theorem 4.1), less the recovery sequence construction which is the object of Section 6. In Section 5, we prove the large deviations statement Theorem 5.2 as well as Theorem 2.1, and in Section 5.2 we deduce from Theorems 4.1 and 5.2 the theorems of this section.

[6]Cf., for example, Benedek Valko's webpage http://www.math.wisc.edu/~valko/courses/833/833.html.

3. Preliminary results.

3.1. *The splitting formula.* Set $\zeta = U^{\mu_0} + \frac{V}{2} - c$ where c is the constant in (2.2) and (2.3). Thanks to our assumption (2.5), we have seen that the droplet Σ and the coincidence set ω coincide (up to sets of measure zero). It follows that ζ satisfies

$$\begin{cases} \Delta\zeta = \frac{1}{2}\Delta V \mathbf{1}_{\mathbb{R}^2\setminus\Sigma}, & \\ \zeta = 0, & \text{quasi-everywhere in } \Sigma, \\ \zeta > 0, & \text{quasi-everywhere in } \mathbb{R}^2 \setminus \Sigma. \end{cases} \tag{3.1}$$

It also follows from [16], Lemma 5, that there exists a constant $\kappa > 0$ such that for every $x \in \mathbb{R}^2$,

$$\zeta(x) \geq \kappa \operatorname{dist}(x, \Sigma)^2, \tag{3.2}$$

and such a rate is in fact optimal [16], Lemma 2. The quantities introduced so far, μ_0, Σ, ζ, only depend on V.

The connection between w_n and W originates in the following (simple but crucial) identity.

LEMMA 3.1. *For any n and any $x_1, \dots, x_n \in \mathbb{R}^2$, we have*

$$\begin{aligned} &\frac{1}{n}\left(w_n(x_1, \dots, x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right) \\ &\qquad = \frac{1}{n\pi} W(E_n, \mathbf{1}_{\mathbb{R}^2}) + 2\sum_{i=1}^n \zeta(x_i), \end{aligned} \tag{3.3}$$

where W is defined as in (2.20), and E_n is defined in (2.13).

PROOF. Let $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$, and let, as in (2.11),

$$H_n = U^{\mu_n - \mu_0} = -2\pi\Delta^{-1}(\mu_n - \mu_0). \tag{3.4}$$

We note that since μ_n and μ_0 have same mass and compact support for any n we have $H_n(x) = O(1/|x|)$ and $\nabla H_n(x) = O(1/|x|^2)$ as $|x| \to +\infty$.

We prove that, denoting by D the diagonal in $\mathbb{R}^2 \times \mathbb{R}^2$, we have

$$\begin{aligned} &\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus D} -\log|x-y| \, d(\mu_n - \mu_0)(x) \, d(\mu_n - \mu_0)(y) \\ &\qquad = \frac{1}{n^2\pi} W(n\nabla H_n, \mathbf{1}_{\mathbb{R}^2}). \end{aligned} \tag{3.5}$$

First, using Green's formula, we have

$$\begin{aligned} \int_{B_R\setminus\bigcup_{i=1}^n B(x_i,\eta)} |\nabla H_n|^2 &= \int_{\partial B_R} H_n \nabla H_n \cdot \vec{\nu} + \sum_{i=1}^n \int_{\partial B(x_i,\eta)} H_n \nabla H_n \cdot \vec{\nu} \\ &\quad + 2\pi \int_{B_R\setminus\bigcup_{i=1}^n B(x_i,\eta)} H_n d(\mu_n - \mu_0). \end{aligned} \tag{3.6}$$

Here, and throughout the paper, $\vec{\nu}$ denotes the outer unit normal vector.

Let $H^i(x) := H_n(x) + \frac{1}{n}\log|x - x_i|$. We have $H^i = -\log *(\mu^i - \mu_0)$, with $\mu^i = \mu_n - \frac{1}{n}\delta_{x_i}$, and near x_i, H^i is C^1. Therefore, using (3.4) and the boundedness of m_0 in L^∞, we have that, as $\eta \to 0$

$$\int_{\partial B(x_i,\eta)} H_n \nabla H_n \cdot \vec{\nu} = -\frac{2\pi}{n^2}\log\eta + \frac{2\pi}{n} H^i(x_i) + o(1),$$

while the integral on ∂B_R tends to 0 as $R \to +\infty$ from the decay properties of H_n. We thus obtain, as $\eta \to 0$ and $R \to +\infty$,

$$\int_{B_R\setminus\bigcup_i B(x_i,\eta)} |\nabla H_n|^2 = -\frac{2\pi}{n}\log\eta + \frac{2\pi}{n}\sum_{i=1}^n H^i(x_i) - 2\pi\int_{\mathbb{R}^2} H_n\, d\mu_0 + o(1),$$

and, therefore, by the definition of W in (2.20),

$$W(n\nabla H_n, \mathbf{1}_{\mathbb{R}^2}) = \pi n \sum_{i=1}^n H^i(x_i) - \pi n^2 \int H_n\, d\mu_0. \tag{3.7}$$

Second, we note that

$$\int_{\mathbb{R}^2\setminus\{x_i\}} -\log|x_i - y|\, d(\mu_n - \mu_0)(y) = H^i(x_i),$$

and if $x \notin \{x_i\}$ then

$$\int_{\mathbb{R}^2\setminus\{x\}} -\log|x - y|\, d(\mu_n - \mu_0)(y) = H_n(x).$$

It follows that

$$\begin{aligned}\int_{D^c} &-\log|x - y|\, d(\mu_n - \mu_0)(x)\, d(\mu_n - \mu_0)(y)\\ &= \frac{1}{n}\sum_{i=1}^n H^i(x_i) - \int_{\mathbb{R}^2} H_n(x)\, d\mu_0(x),\end{aligned}$$

which together with (3.7) proves (3.5).

On the other hand, we may rewrite w_n as

$$w_n(x_1,\dots,x_n) = n^2\left(\int_{D^c} -\log|x - y|\, d\mu_n(x)\, d\mu_n(y) + \int V(x)\, d\mu_n(x)\right)$$

and, splitting μ_n as $\mu_0 + (\mu_n - \mu_0)$ and using the fact that $(\mu_0 \otimes \mu_0)(D) = 0$, we obtain

$$\begin{aligned}w(x_1,\dots,x_n) = n^2 I(\mu_0) &+ 2n^2 \int U^{\mu_0}(x)\, d(\mu_n - \mu_0)(x)\\ &+ n^2 \int V(x)\, d(\mu_n - \mu_0)(x)\\ &+ n^2 \int_{D^c} -\log|x - y|\, d(\mu_n - \mu_0)(x)\, d(\mu_n - \mu_0)(y).\end{aligned}$$

Since $U^{\mu_0}+\frac{V}{2}=c+\zeta$ and since μ_n and μ_0 have same mass 1, we have

$$2n^2\int U^{\mu_0}(x)\,d(\mu_n-\mu_0)(x)+n^2\int V(x)\,d(\mu_n-\mu_0)(x)$$
$$=2n^2\int \zeta\,d(\mu_n-\mu_0)=2n^2\int \zeta\,d\mu_n,$$

where we used the fact that $\zeta=0$ on the support of μ_0. Therefore, in view of (3.5) we have found

$$w(x_1,\dots,x_n)=n^2 I(\mu_0)+2n^2\int \zeta\,d\mu_n+\frac{1}{\pi}W(n\nabla H_n,\mathbf{1}_{\mathbb{R}^2}). \tag{3.8}$$

But, changing variables, we find in view of (2.13) that

$$\frac{1}{2}\int_{\mathbb{R}^2\setminus\bigcup_{i=1}^n B(x_i,\eta)}|n\nabla H_n|^2=\frac{1}{2}\int_{\mathbb{R}^2\setminus\bigcup_{i=1}^n B(x_i',\sqrt{n}\eta)}|E_n|^2,$$

and by adding $\pi n\log\eta$ on both sides and letting $\eta\to 0$ we deduce that $W(n\nabla H_n,\mathbf{1}_{\mathbb{R}^2})=W(E_n,\mathbf{1}_{\mathbb{R}^2})-\frac{\pi}{2}n\log n$. Together with (3.8) this proves (3.3). □

DEFINITION 3.2. Given a measure μ of the form $\frac{1}{n}\sum_i\delta_{x_i}$, where $(x_1,\dots,x_n)\in\mathbb{C}^n$, we define

$$F_n(\mu):=\frac{1}{n\pi}W(E_n,\mathbf{1}_{\mathbb{R}^2})+2\sum_{i=1}^n\zeta(x_i). \tag{3.9}$$

If μ is not a measure of this type, we let $F_n(\mu)=+\infty$.

Now the relation (3.3) can be rewritten, using the notation (2.11),

$$w_n(x_1,\dots,x_n)=n^2 I(\mu_0)-\frac{n}{2}\log n+nF_n(\mu_n). \tag{3.10}$$

This allows us to separate orders as announced since we will see that $F_n(\mu_n)$ is typically of order 1. We may next cancel out leading order terms and rewrite the probability law (1.1) as

$$d\mathbb{P}_n^\beta(x_1,\dots,x_n)=\frac{1}{K_n^\beta}e^{-n(\beta/2)F_n(\mu_n)}\,dx_1\cdots dx_n, \tag{3.11}$$

where $\mu_n=\frac{1}{n}\sum_{i=1}^n\delta_{x_i}$ and

$$K_n^\beta:=Z_n^\beta e^{(\beta/2)(n^2 I(\mu_0)-(n/2)\log n)}. \tag{3.12}$$

As we will see below $\log K_n^\beta$ is of order $n\beta$, which leads to Theorem 2.1.

We will also denote

$$\widehat{F}_n(\mu)=F_n(\mu)-2n\int\zeta\,d\mu=\frac{1}{n\pi}W(E_n,\mathbf{1}_{\mathbb{R}^2}). \tag{3.13}$$

Since $\zeta \geq 0$, the term in ζ will not play any major role, other than act as an effective potential which confines the points to Σ. In view of (3.3), the main task in our proof is to pass to the limit $n \to \infty$ in $\frac{1}{\pi} W(E_n, \mathbf{1}_{\mathbb{R}^2})$ and obtain $\widetilde{W}$ as a limiting energy. Passing to the limit in (2.14) will lead to a curl-free E such that $\operatorname{div} E = 2\pi(\sum_p \delta_p - cste)$ where the sum is now infinite. The passage to the limit $\widetilde{W}$ is not obvious for several reasons. The first is the lack of local charge neutrality, and the fact that the energy density associated to $W(E_n, \mathbf{1}_{\mathbb{R}^2})$ is not pointwise bounded below. The second is the need of the "averaged formulation" alluded to above; this will be provided by an abstract method relying on the ergodic theorem, and inspired by Varadhan.

3.2. *Further properties of the renormalized energy.* We list here some properties of the energy W defined in Section 2. These are proven in [42].

First, we note that W was defined as a function of the electric field E satisfying (2.17) rather than of the measure $\nu = \sum_{p\in\Lambda} \delta_p$. The fields satisfying (2.17) for a given ν differ by the gradient of a harmonic function, but as it turns out, they only differ by a constant if we consider only those fields for which W is finite.

LEMMA 3.3. *Let $m \geq 0$ and $\nu = \sum_{p\in\Lambda} \delta_p$, where $\Lambda \subset \mathbb{R}^2$ is discrete, and assume there exists E such that*

$$\operatorname{div} E = 2\pi(\nu - m), \qquad \operatorname{curl} E = 0 \text{ and } W(E) < +\infty. \tag{3.14}$$

Then any other E' satisfying (3.14) is such that $E - E'$ is constant.

If there exists E such that (3.14) holds and such that

$$\lim_{R\to\infty} \fint_{K_R} E = 0, \tag{3.15}$$

then any other E' satisfying (3.14) is such that $W(E') > W(E)$.

PROOF. Let E, E' be as above. We may view them as complex functions of a complex variable. From (3.14), we have $\operatorname{div}(E - E') = \operatorname{curl}(E - E') = 0$, and thus $E - E'$ is holomorphic. We can write it as a power series $\sum_{n=0}^{\infty} a_n z^n$ with infinite radius of convergence. On the other hand, from the finiteness of $W(E)$ and $W(E')$ we deduce from [45], Corollary 1.2 (see Lemma 3.9 below) that there exists $C > 0$ such that

$$\forall R > 1 \qquad \int_{B_{R+1}\setminus B_R} |E - E'| \leq C R^{3/2} \log^{1/2} R. \tag{3.16}$$

But by Cauchy's formula, we have, for any $R > 0$ and $t \in [R, R+1]$

$$a_n = \frac{1}{2i\pi} \int_{\partial B(0,t)} \frac{(E - E')(z)}{z^{n+1}}\, dz = \frac{1}{2i\pi} \int_R^{R+1} \int_{\partial B(0,t)} \frac{(E - E')(z)}{z^{n+1}}\, dz\, dt.$$

It follows with (3.16) that

$$|a_n| \le \frac{1}{2\pi R^{n+1}} \int_{B(0,R+1)\setminus B(0,R)} |E - E'| \le \frac{C}{R^{n+1}} R^{3/2} \log^{1/2} R.$$

Letting $R \to \infty$, we find that $a_n = 0$ for any $n \ge 1$, and thus $E - E'$ is constant. For the second statement, we deduce from the first statement that $E' = E + \vec{C}$ for some constant vector $\vec{C} \neq 0$, and then

$$W(E', \chi_{K_R}) = W(E, \chi_{K_R}) + \vec{C} \cdot \int E\chi_{K_R} + \frac{|c|^2}{2} \int \chi_{K_R},$$

so that dividing by $|K_R|$, passing to the limit as $R \to +\infty$ and in view of (3.15), we find $W(E') = W(E) + \frac{1}{2}|\vec{C}|^2$. $\square$

Note that given ν, the above lemma shows that either for all E's satisfying (3.14) the limit $\lim_{R\to\infty} \fint_{K_R} E$ exists, or it exists for none of them. Both cases may occur.

The following additional facts and remarks about W are mostly from [42]:

- In [42], we introduced W as being computed with averages over general shapes (say balls, squares, etc.). We showed that the minimum of W over $\mathcal{A}_m$ does not depend on the shape used. Since squares are the most useful ones, we restricted to them here for the sake of simplicity.
- It was shown in [42], Theorem 1, that the value of W does not depend on the choice of $\{\chi_{K_R}\}_R$ as long as it satisfies (2.21).
- W is bounded below and admits a minimizer over $\mathcal{A}_1$; cf. [42], Theorem 1.
- Because the number of points is in general infinite, the interaction over large balls needs to be normalized by the volume, as in a thermodynamic limit. Thus, W does not feel compact perturbations of the configuration of points. Even though the interactions are long-range, this is not difficult to justify rigorously.
- In [26], some necessary and some sufficient conditions on the configuration of points for which $W(E) < \infty$ are given.
- We may define W as a function of the point measure ν only, by setting for every ν satisfying (2.18),

$$\mathbb{W}(\nu) = \inf_{E \text{ such that (2.17) holds}} W(E), \tag{3.17}$$

  and $\mathbb{W}(\nu) = +\infty$ if ν is not of the form $\sum_{p\in\Lambda} \delta_p$. This definition is somehow "relaxed" since $\mathbb{W}(\nu) \le W(E)$ for any E satisfying (2.17). The main point to check is the measurability of $\mathbb{W}$, which we will discuss below in Section 5.3.
- In the case $m = 1$ and when the set of points Λ is periodic with respect to some lattice $\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$, then it can be viewed as a set of n points $a_1, \dots, a_n$ over the torus $\mathbb{T}_{(\vec{u},\vec{v})} := \mathbb{R}^2/(\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v})$ with $|\mathbb{T}_{(\vec{u},\vec{v})}| = n$. In this case, the infimum of $W(E)$ among E's which satisfy (3.14) is achieved by $E_{\{a_i\}} = -\nabla h$, where h is the periodic solution to $-\Delta h = 2\pi(\sum_i \delta_{a_i} - 1)$, and

$$W(E_{\{a_i\}}) = \frac{\pi}{|\mathbb{T}_{(\vec{u},\vec{v})}|} \sum_{i\neq j} G(a_i - a_j) + \pi \lim_{x\to 0}\bigl(G(x) + \log|x|\bigr), \tag{3.18}$$

where G is the Green function of the torus with respect to its volume form, that is, the solution to

$$-\Delta G(x) = 2\pi\left(\delta_0 - \frac{1}{|\mathbb{T}_{(\vec{u},\vec{v})}|}\right) \qquad \text{in } \mathbb{T}_{(\vec{u},\vec{v})}.$$

An explicit expression for G can be found via Fourier series and this leads to an explicit expression for W of the form $\sum_{i\neq j} \mathrm{E}(a_i - a_j)$ where E is an Eisenstein series (for more details, see [42], Lemma 1.3 and also [13]). In this periodic setting, the expression of W is thus much simpler than (2.22) and reduces to the computation of a sum of explicit pairwise interaction.

– When the set of points Λ is itself exactly a lattice $\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$ then W can be expressed explicitly through the Epstein Zeta function of the lattice. Moreover, using results from number theory, it is proved in [42], Theorem 2, that the unique minimizer of W over lattice configurations of fixed volume is the triangular lattice. This supports the conjecture that the Abrikosov triangular lattice is a global minimizer of W, with a slight abuse of language since W is here not a function of the points, but of their associated "electric fields" $E_{\{a_i\}}$.

This last fact allows us to think of W as a way of measuring the disorder and lack of homogeneity of a configuration of points in the plane (this point of view is pursued in [13] with explicit computations for random point processes). Another way to see it is to view W as measuring the distance between $\sum_{p\in\Lambda} \delta_p$ and the constant m in H^{-1}, the dual space to the Sobolev space H_0^1 (with $\|f\|_{H_0^1} = \|\nabla f\|_{L^2}$) which only makes sense modulo the "renormalization" as $\eta \to 0$ and modulo normalizing by the volume.

3.3. *Mass spreading and upper bound on Z_n^β.* The main technical problem in dealing with the limit of the functional F_n is that it involves the renormalized energy W which is the finite part of a divergent integral and thus corresponds to an energy density which is not bounded from below. The following result—proved in [42], Proposition 4.9 and Remark 4.10 (with slightly different notation)—allows us to replace this finite part with the integral of a density bounded below by a universal constant, which also implies a lower bound for F_n.

For any set Ω, $\widehat{\Omega}$ denotes its 1-tubular neighborhood, that is, $\{x \in \mathbb{R}^2, \operatorname{dist}(x,\Omega) < 1\}$.

PROPOSITION 3.4. *Assume $\Omega \subset \mathbb{R}^2$ is open and (ν, E) are such that $\nu = 2\pi \sum_{p\in\Lambda} \delta_p$ for some finite subset Λ of $\widehat{\Omega}$ and* $\operatorname{div} E = 2\pi(\nu - a(x)\,dx)$, $\operatorname{curl} E = 0$ *in $\widehat{\Omega}$, where $a \in L^\infty(\widehat{\Omega})$. Then, given any $\rho > 0$, there exists a measure g supported on $\widehat{\Omega}$ and such that*:

– *there exists a family $\mathcal{B}_\rho$ of disjoint closed balls covering* $\operatorname{Supp}(\nu)$, *with the sum of the radii of the balls in $\mathcal{B}_\rho$ intersecting with any ball of radius* 1 *bounded* *by ρ, and such that*

$$g \geq -C\big(\|a\|_{L^\infty}+1\big)+\tfrac{1}{4}|E|^2\mathbf{1}_{\Omega\setminus\mathcal{B}_\rho} \qquad \text{in } \widehat{\Omega}, \tag{3.19}$$

where C depends only on ρ.

–

$$g=\frac{1}{2}|E|^2 \qquad \text{outside } \bigcup_{p\in\Lambda} B(p,\lambda) \tag{3.20}$$

where λ depends only on ρ.

– *For any function χ compactly supported in Ω, we have*

$$\left|W(E,\chi)-\int \chi\, dg\right| \leq CN\big(\log N+\|a\|_{L^\infty}\big)\|\nabla\chi\|_\infty, \tag{3.21}$$

where $N=\#\{p\in\Lambda : B(p,\lambda)\cap \operatorname{Supp}(\nabla\chi)\neq\varnothing\}$ for some λ and C depending only on ρ.

– *For any $U\subset\Omega$,*

$$\#(\Lambda\cap U)\leq C\big(1+\|a\|^2_{L^\infty}|\widehat{U}|+g(\widehat{U})\big). \tag{3.22}$$

Note that the result in [42] is not stated for any ρ but a careful inspection of the proof there allows us to show that it can be readapted to make ρ arbitrarily small. From now on, we fix some $\rho<1/8$.

DEFINITION 3.5. Assume $(x_1,\dots,x_n)\in\mathbb{C}^n$. Letting $\mu'_n=\sum_{i=1}^n\delta_{x'_i}$ be the measure in blown-up coordinates (2.12), and E_n, be defined by (2.13), we denote by g_n the result of applying the previous proposition to (μ'_n,E_n) in $\mathbb{R}^2$.

Even though we will not use the following result in the sequel, we state it to show how we can quickly derive a first upper bound on Z_n^β from what precedes.

PROPOSITION 3.6. *We have*

$$\log K_n^\beta \leq Cn\beta+n\big(\log|\Sigma|+o(1)\big), \tag{3.23}$$

where we recall $\Sigma=\operatorname{Supp}(\mu_0)$, and

$$\log Z_n^\beta \leq -\frac{\beta}{2}n^2 I(\mu_0)+\frac{\beta n}{4}\log n+Cn\beta+n\big(\log|\Sigma|+o(1)\big), \tag{3.24}$$

where $o(1)\to 0$ as $n\to\infty$ uniformly with respect to $\beta>\beta_0$, for any $\beta_0>0$, and C depends only on V.

The proof uses two lemmas.

LEMMA 3.7. *For any $(x_1,\dots,x_n)\in\mathbb{C}^n$, and letting $\mu_n=\frac{1}{n}\sum_i\delta_{x_i}$ we have*

$$F_n(\mu_n)=\frac{1}{n\pi}\int_{\mathbb{R}^2} dg_n+2\sum_{i=1}^n\zeta(x_i), \tag{3.25}$$

where F_n is as in (3.9) and g_n is defined above.

PROOF. This follows from (3.21) applied to χ_{K_R}, where χ_{K_R} is as in (2.21). If R is large enough then, λ being the constant of Proposition 3.4 and letting $\mu_n' = \sum_i \delta_{x_i'}$, we have $\#\{p \in \mathrm{Supp}(\mu_n') : B(p,\lambda) \cap \mathrm{Supp}(\nabla \chi_{K_R}) \neq \varnothing\} = 0$ and, therefore, (3.21) reads

$$W(E_n, \chi_{K_R}) = \int \chi_{K_R}\, dg_n.$$

Letting $R \to +\infty$ yields $W(E_n, \mathbf{1}_{\mathbb{R}^2}) = \int dg_n$ and the result, in view of (3.9). $\square$

LEMMA 3.8. *For any constant $\gamma > 0$ and uniformly w.r.t. β greater than any arbitrary positive constant β_0, we have*

$$\lim_{n\to\infty} \left(\int_{\mathbb{C}^n} e^{-\gamma\beta n \sum_{i=1}^n \zeta(x_i)}\, dx_1 \cdots dx_n \right)^{1/n} = |\Sigma|. \tag{3.26}$$

PROOF. This is where we use assumption (2.6). We recall that $\zeta = U^{\mu_0} + \frac{V}{2} - c$, and note that since μ_0 is a compactly supported probability measure $U^{\mu_0}(x) = -\int \log|x-y|\, d\mu_0(y)$ behaves asymptotically like $-\log|x|$ when $|x| \to \infty$, more precisely one can easily show that there exists C such that $|U^{\mu_0}(x) + \log|x|| \le C$ for $|x|$ large enough. It thus follows that $\zeta(x) \ge -\log|x| + \frac{V}{2}(x) - C$ for $|x|$ large enough, and in view of (2.6), this implies that for some $\beta_2 > 0$, $\int_{\mathbb{C}} e^{-\beta_2 \zeta(x)}\, dx$ converges.

Next, by separation of variables, we have

$$\int_{\mathbb{C}^n} e^{-\gamma\beta n \sum_{i=1}^n \zeta(x_i)}\, dx_1 \cdots dx_n = \left(\int_{\mathbb{C}} e^{-\gamma\beta n \zeta(x)}\, dx \right)^n.$$

On the other hand, we have $\zeta \ge 0$ and $\{\zeta = 0\} = \Sigma$ by (3.1), hence we have $e^{-\gamma\beta n\zeta(x)} \to \mathbf{1}_\Sigma$ pointwise, as $\beta n \to \infty$. In addition, if $\beta \ge \beta_0 > 0$, for n large enough depending on β_0, $e^{-\gamma\beta n\zeta(x)}$ is dominated by $e^{-\beta_2\zeta(x)}$ which is integrable. Therefore, by dominated convergence theorem, it follows that (3.26) holds uniformly w.r.t. $\beta \ge \beta_0$, for any $\beta_0 > 0$. $\square$

PROOF OF PROPOSITION 3.6. From (3.20), we have $g_n \ge 0$ outside $\bigcup_i B(x_i, \lambda)$ and from (3.19) we have $g_n \ge -C$ (depending only on $\|m_0\|_{L^\infty}$ hence on V) in $\bigcup_i B(x_i, \lambda)$. Inserting into (3.25), we deduce that

$$F_n(\mu_n) \ge -C + 2\sum_{i=1}^n \zeta(x_i),$$

where C depends only on V. Inserting into (3.11) and integrating over $\mathbb{C}^n$, we find

$$1 \le \frac{1}{K_n^\beta} e^{Cn\beta} \int_{\mathbb{C}^n} e^{-n\beta \sum_{i=1}^n \zeta(x_i)}\, dx_1 \cdots dx_n.$$

Inserting (3.26) and taking logarithms, it follows that

$$\log K_n^\beta \le Cn\beta + n\big(\log|\Sigma| + o(1)\big).$$

The relation (3.24) follows using (3.12). $\square$

3.4. *Control of E by $W(E)$.* In this section, we show, via tools from [42, 45], how $\widehat{F}_n$ or W control the discrepancy between μ_n and μ_0.

LEMMA 3.9. *Let $\mu_n = \sum_{i=1}^n \delta_{x_i}$ and E_n be associated through* (2.11), (2.13). *Let B_R be any ball of radius R (not necessarily centered at 0). Assume χ is a smooth nonnegative function compactly supported in U. Then for any $1 < q < 2$, we have*

$$\begin{aligned}
&\|\sqrt{\chi} E_n\|_{L^q(U)} \\
&\quad \le C_q |U|^{1/q-1/2} \qquad (3.27)\\
&\qquad \times \big(W(E_n,\chi) + \mu_n'(\widehat{U})\big(\|\chi\|_{L^\infty} + \|\nabla\chi\|_{L^\infty}\big) + N\log N\big)^{1/2},
\end{aligned}$$

where $N = \#\{i \in [1,n] \mid 0 < \chi(x_i) \le \frac{1}{2}\|\chi\|_{L^\infty}\}$ and $\mu_n' = \sum_{i=1}^n \delta_{x_i'}$. Thus

$$\int_{B_R} |E_n|^q \le C_q \big(n + R^2\big)^{1-q/2} n^{q/2} \big(\widehat{F}_n(\mu_n) + 1\big)^{q/2} \tag{3.28}$$

and

$$\|\mu_n - \mu_0\|_{W^{-1,q}(B_R)} \le \frac{C_q}{\sqrt{n}} \big(1 + R^2\big)^{1/q-1/2} \big(\widehat{F}_n(\mu_n) + 1\big)^{1/2}. \tag{3.29}$$

PROOF. The first item is a rewriting of [45], Corollary 1.2. We then choose χ such that $\chi = 1$ on $U := \Sigma' \cup (\bigcup_{i=1}^n B(x_i', \frac{1}{2})) \cup B_R$ and $\|\chi\|_\infty, \|\nabla\chi\|_\infty \le 1$, compactly supported on $\widehat{U} = \{x : d(x,U) \le 1\}$. Using the fact that $|\widehat{U}| \le C(n + R^2)$ where C only depends on Σ, from (3.27) we find

$$\|\sqrt{\chi} E_n\|_{L^q(U)} \le C_q \big(n + R^2\big)^{1/q-1/2} \big(W(E_n,\chi) + n\big)^{1/2}. \tag{3.30}$$

Since $\mu_n' = 0$ in the support of $1 - \chi$, we have

$$W(E_n, 1-\chi) = \frac{1}{2} \int (1-\chi)|E_n|^2 \ge 0.$$

In particular, $W(E_n,\chi) \le W(E_n,\chi) + W(E_n,1-\chi) = W(E_n,\mathbf{1}_{\mathbb{R}^2})$. It then follows from (3.30) and the fact that $\widehat{F}_n(\mu_n) = \frac{1}{\pi n} W(E_n,\mathbf{1}_{\mathbb{R}^2})$ [cf. (3.13)] that (3.28) holds.

By scaling, given a domain Ω and denoting $\Omega' = \sqrt{n}\Omega$ the rescaled domain, we have in view of (2.13) that

$$\int_{\Omega'} |E_n|^q = n^{q/2+1} \int_\Omega |\nabla H_n|^q,$$

where $H_n = -2\pi\Delta^{-1}(\mu_n - \mu_0)$, while $\|\mu_n - \mu_0\|_{W^{-1,q}(\Omega)} \le C\|\nabla H_n\|_{L^q(\Omega)}$. Thus, (3.29) follows from (3.28). $\square$

4. Gamma-convergence. We start by stating, and proving, the Gamma-convergence statement mentioned in the Introduction and which is an important ingredient in the proof of the other results.

THEOREM 4.1 (Gamma convergence of F_n to $\widetilde{W}$). *As above, let the potential V satisfy assumptions* (2.4)–(2.6) *and m_0 be the density of the equilibrium measure μ_0. Fix from now on $1 < p < 2$ and let $X = \Sigma \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$.*

A. Lower bound. *Let $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ be a sequence such that $\widehat{F}_n(\mu_n) \le C$. Then P_n defined by* (2.15) *is a probability measure on X and*:

1. *Any subsequence of $\{P_n\}_n$ has a convergent subsequence converging to some $P \in \mathcal{P}(X)$ as $n \to \infty$.*
2. *P is admissible in the sense of Definition* 2.3.
3. *Defining α as in* (2.25), *it holds that*

$$\liminf_{n\to\infty} F_n(\mu_n) \ge \liminf_{n\to\infty} \widehat{F}_n(\mu_n) \ge \widetilde{W}(P) \ge \alpha. \tag{4.1}$$

B. Recovery sequence. *Conversely, assume P is admissible. Then there exists a sequence $\{\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\}_n$ with $x_i \in \Sigma$ and a sequence $\{\overline{E}_n\}_n$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that*

$$\operatorname{div} \overline{E}_n = 2\pi\left(\sum_i \delta_{x'_i} - m'_0\right)$$

and such that, defining $\overline{P}_n$ as in (2.15) *with $\overline{E}_n$ replacing E_n, we have $\overline{P}_n \to P$ as $n \to \infty$ and*

$$\limsup_{n\to\infty} F_n(\mu_n) \le \widetilde{W}(P). \tag{4.2}$$

C. Consequences for minimizers. *If $(x_1, \dots, x_n)$ minimizes w_n for every n and $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$, then the limit P of P_n as defined in* (2.15) *satisfies the following.*

1. *For P-almost every (x, E), E minimizes W over $\mathcal{A}_{m_0(x)}$.*
2. *We have*

$$\lim_{n\to\infty} F_n(\mu_n) = \lim_{n\to\infty} \widehat{F}_n(\mu_n) = \widetilde{W}(P) = \alpha, \qquad \lim_{n\to\infty} \sum_{i=1}^n \operatorname{dist}^2(x_i, \Sigma) = 0.$$

Note that part B of the theorem is only a partial converse to part A because the constructed $\overline{E}_n$ need not be curl free.

REMARK 4.1. Defining Q_n as in Remark 2.4, we can also express this limiting result in terms of the limit Q to Q_n, which is the push-forward of P by $(x, E) \mapsto \frac{1}{2\pi} \operatorname{div} E + m_0(x)$. The limiting energy for both the upper bound and the lower bound is then

$$\frac{|\Sigma|}{\pi} \int \mathbb{W}(\nu)\, dQ(x, \nu).$$

Of course such a statement is a bit weaker than Theorem 4.1 since some information is lost: namely we do not keep the information of which E corresponded to ν.

4.1. *Abstract result via the ergodic theorem.* In this section, we present the ergodic framework introduced in [42] for obtaining "lower bounds for 2-scale energies" and inspired by Varadhan. We cannot directly use the result there because it is written for a uniform "macroscopic environment," which would correspond to the case where $m_0(x)$ is constant on its support (as in the circle law). To account for the possibility of varying environment or weight at the macroscopic, we can however adapt Theorem 3 of [42] and easily prove the following variant.

Let X denote a Polish metric space, when we speak of measurable functions on X we will always mean Borel-measurable. We assume there is a d-parameter group of transformations θ_λ acting continuously on X. More precisely, we require that:

- For all $u \in X$ and $\lambda, \mu \in \mathbb{R}^d$, $\theta_\lambda(\theta_\mu u) = \theta_{\lambda+\mu} u$, $\theta_0 u = u$.
- The map $(\lambda, u) \mapsto \theta_\lambda u$ is continuous with respect to each variable (hence measurable with respect to both).

Typically we think of X as a space of functions defined on $\mathbb{R}^d$ and θ as the action of translations, that is, $\theta_\lambda u(x) = u(x + \lambda)$. Then we consider the following d-parameter group of transformations T_λ^ε acting continuously on $\mathbb{R}^d \times X$ by $T_\lambda^\varepsilon(x, u) = (x + \varepsilon\lambda, \theta_\lambda u)$. We also define $T_\lambda(x, u) = (x, \theta_\lambda u)$.

For a probability measure P on $\mathbb{R}^d \times X$, we say that P is translation-invariant if it is invariant under the action T, and we say it is $T_{\lambda(x)}$-invariant if for every function $\lambda(x)$ of class C^1, it is invariant under the mapping $(x, u) \mapsto (x, \theta_{\lambda(x)} u)$. Note that $T_{\lambda(x)}$-invariant implies translation-invariant.

Let G denote a compact set in $\mathbb{R}^d$ such that

$$|G| > 0, \qquad \lim_{\varepsilon \to 0} \frac{|(G + \varepsilon x) \triangle G|}{|G|} = 0, \tag{4.3}$$

for every $x \in \mathbb{R}^2$, where $\triangle$ denotes the symmetric difference of sets. We let $\{f_\varepsilon\}_\varepsilon$ and f be measurable nonnegative functions on $G \times X$, and assume that for any family $\{(x_\varepsilon, u_\varepsilon)\}_\varepsilon$ such that

$$\forall R > 0 \qquad \limsup_{\varepsilon \to 0} \int_{B_R} f_\varepsilon\big(T_\lambda^\varepsilon(x_\varepsilon, u_\varepsilon)\big)\, d\lambda < +\infty$$

the following hold.

1. (Coercivity). $\{(x_\varepsilon, u_\varepsilon)\}_\varepsilon$ admits a convergent subsequence (note that $\{x_\varepsilon\}_\varepsilon$ subsequentially converges since G is compact).
2. (Γ-liminf). If $\{(x_\varepsilon, u_\varepsilon)\}_\varepsilon$ converges to (x, u) then

$$\liminf_{\varepsilon \to 0} f_\varepsilon(x_\varepsilon, u_\varepsilon) \ge f(x, u).$$

Then for the sake of generality, we consider an increasing family of bounded open sets $\{\mathbf{U}_R\}_{R>0}$ such that

$$(4.4)\quad \text{(i)}\quad \{\mathbf{U}_R\}_{R>0} \text{ is a Vitali family,} \qquad \text{(ii)}\quad \lim_{R\to+\infty} \frac{|(\lambda+\mathbf{U}_R)\bigtriangleup \mathbf{U}_R|}{|\mathbf{U}_R|}=0$$

for any $\lambda\in\mathbb{R}^d$, where Vitali means (see [35]) that the intersection of the closures is $\{0\}$, that $R\mapsto|\mathbf{U}_R|$ is left continuous, and that $|\mathbf{U}_R-\mathbf{U}_R|\le C|\mathbf{U}_R|$.

We have the following theorem.

THEOREM 4.2. *Let G, X, $\{\theta_\lambda\}_\lambda$, and $\{f_\varepsilon\}_\varepsilon$, f be as above. For any $u\in X$, let*

$$F_\varepsilon(u)=\fint_G f_\varepsilon(x,\theta_{x/\varepsilon}u)\,dx$$

and let $\phi_\varepsilon(u)$ be the probability on $G\times X$ which is the image of the normalized Lebesgue measure on G under the map $x\mapsto(x,\theta_{x/\varepsilon}u)$.

A. *Assume that $\{u_\varepsilon\}_\varepsilon$, a family of elements of X, is such that $\{F_\varepsilon(u_\varepsilon)\}_\varepsilon$ is bounded, and let $P_\varepsilon=\phi_\varepsilon(u_\varepsilon)$. Then P_ε converges to a Borel probability measure P on $G\times X$ whose first marginal is the normalized Lebesgue measure on G, which is $T_{\lambda(x)}$-invariant, such that P-a.e. (x,u) is of the form $\lim_{\varepsilon\to0}(x_\varepsilon,\theta_{x_\varepsilon/\varepsilon}u_\varepsilon)$ and such that*

$$(4.5)\qquad \liminf_{\varepsilon\to0}F_\varepsilon(u_\varepsilon)\ge\int f(x,u)\,dP(x,u).$$

Moreover,

$$(4.6)\qquad \int f(x,u)\,dP(x,u)=\mathbf{E}^P\Big(\lim_{R\to+\infty}\fint_{\mathbf{U}_R} f(x,\theta_\lambda u)\,d\lambda\Big),$$

where $\mathbf{E}^P$ denotes the expectation under the probability P.

B. *Let $\mathbb{P}_\varepsilon$ be a probability on X such that $\lim_{M\to+\infty}\lim_{\varepsilon\to0}\mathbb{P}_\varepsilon(\{F_\varepsilon(u)\ge M\})=0$, then $\{\phi_\varepsilon\#\mathbb{P}_\varepsilon\}_\varepsilon$ is tight, that is, converges up to a subsequence to a probability measure on $\mathcal{P}(G\times X)$.*

The proof uses the following simple lemma, whose statement and proof can be found in [42], Lemma 2.1.

LEMMA 4.2. *Assume $\{P_n\}_n$ are Borel probability measures on a Polish metric space X and that for any $\delta>0$ there exists $\{K_n\}_n$ such that $P_n(K_n)\ge1-\delta$ for every n and such that if $\{x_n\}_n$ satisfies for every n that $x_n\in K_n$, then any subsequence of $\{x_n\}_n$ admits a convergent subsequence (note that we do not assume K_n to be compact). Then P_n admits a subsequence which converges tightly, that is, converges weakly to a probability measure P.*

PROOF OF THEOREM 4.2. It follows the steps of [42], Section 2:

1. P_ε is tight hence has a limit P. This follows from the coercivity property of f_ε as in [42], Section 2, step 1 and uses Lemma 4.2.

2. P is $T_{\lambda(x)}$-invariant. Let Φ be bounded and continuous, and let P_λ be the push-forward of P by $(x,u)\mapsto(x,\theta_{\lambda(x)}u)$. Then from the definition of P_λ, P, P_ε, we have

$$\begin{aligned}\int \Phi(x,u)\,dP_\lambda(x,u) &= \int \Phi(x,\theta_{\lambda(x)}u)\,dP(x,u)\\ &= \lim_{\varepsilon\to 0}\int \Phi(x,\theta_{\lambda(x)}u)\,dP_\varepsilon(x,u)\\ &= \lim_{\varepsilon\to 0}\fint_G \Phi(x,\theta_{x/\varepsilon+\lambda(x)}u_\varepsilon)\,dx\\ &= \lim_{\varepsilon\to 0}\fint_{(I+\varepsilon\lambda)(G)} \frac{\Phi((I+\varepsilon\lambda)^{-1}(y),\theta_{y/\varepsilon}u_\varepsilon)}{|\det(I+\varepsilon D\lambda((I+\varepsilon\lambda)^{-1}(y))|}\,dy,\end{aligned}$$

where the last equality follows by the change of variables $y=(I+\varepsilon\lambda)(x)$. Using the boundedness of Φ, the C^1 character of λ, the compactness of G and (4.3), we may replace $(I+\varepsilon\lambda)(G)$ by G and the denominator by 1 in the last integral and we find, using the definition of P_ε

$$\int \Phi(x,u)\,dP_\lambda(x,u) = \lim_{\varepsilon\to 0}\int \Phi\big((I+\varepsilon\lambda)^{-1}(x),u\big)\,dP_\varepsilon(x,u). \tag{4.7}$$

Since $\{P_\varepsilon\}_\varepsilon$ is tight, for any $\delta>0$ there exists K_δ such that $P_\varepsilon(K_\delta^c)<\delta$ for every ε. Then by uniform continuity of Φ on K_δ the map $(x,u)\mapsto\Phi((I+\varepsilon\lambda)^{-1}(x),u)$ converges uniformly on K_δ to $(x,u)\mapsto\Phi(x,u)$, and thus

$$\lim_{\varepsilon\to 0}\int_{K_\delta}\Phi\big((I+\varepsilon\lambda)^{-1}(x),u\big)\,dP_\varepsilon(x,u) = \lim_{\varepsilon\to 0}\int_{K_\delta}\Phi(x,u)\,dP_\varepsilon(x,u).$$

Since this is true for any $\delta>0$, and using the boundedness of Φ, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int \Phi\big((I+\varepsilon\lambda)^{-1}(x),u\big)\,dP_\varepsilon(x,u) &= \lim_{\varepsilon\to 0}\int\Phi(x,u)\,dP_\varepsilon(x,u)\\ &= \int\Phi(x,u)\,dP(x,u),\end{aligned}$$

by definition of P. Thus, in view of (4.7), we have $P_\lambda=P$ and P is thus $T_{\lambda(x)}$-invariant.

3. $\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)\ge\int f\,dP$. This follows from [42], Lemma 2.2, since $F_\varepsilon(u_\varepsilon)=\int f_\varepsilon\,dP_\varepsilon$.

To conclude, as in [42], Section 2, the fact that P is $T_{\lambda(x)}$-invariant (which implies T_λ-invariant) and Wiener's multiparametric ergodic theorem (see, e.g., [9]) implies that

$$\begin{aligned}\int f(x,u)\,dP(x,u) &= \mathbf{E}^P\Big(\lim_{R\to+\infty}\fint_{\mathbf{U}_R} f\big(T_\lambda(x,u)\big)\,d\lambda\Big)\\ &= \mathbf{E}^P\Big(\lim_{R\to+\infty}\fint_{\mathbf{U}_R} f(x,\theta_\lambda u)\,d\lambda\Big).\end{aligned}$$

We now turn to the proof of B. Let $A_{M,\varepsilon} = \{u \in X, F_\varepsilon(u) \le M\}$. Then we have

$$\phi_\varepsilon \# \mathbb{P}_\varepsilon(\phi_\varepsilon(A^c_{M,\varepsilon})) = \mathbb{P}_\varepsilon(A^c_{M,\varepsilon}) \to 0$$

as $\varepsilon \to 0$ and $M \to \infty$. In view of Lemma 4.2 applied with $K_n = \phi_\varepsilon(A_{M,\varepsilon})$, in order to prove the tightness of $\phi_\varepsilon \# \mathbb{P}_\varepsilon$ it suffices to take M large enough and check that if $P_\varepsilon \in \phi_\varepsilon(A_{M,\varepsilon})$ then P_ε has a convergent subsequence. But this is a direct application of what we have established in part A, since such a P_ε is the image by ϕ_ε of a family u_ε for which $F_\varepsilon(u_\varepsilon) \le M$. Therefore, P_ε is tight and $\phi_\varepsilon \# \mathbb{P}_\varepsilon$ as well by the lemma. □

We now apply this abstract framework to our specific situation to obtain the lower bound on $\widehat{F}_n$.

4.2. *Proof of Theorem* 4.1, *part* A. The proof follows essentially [42], Proposition 4.1 and below. Let $\{\mu_n\}_n$ and P_n be as in the statement of Theorem 4.1. We need to prove that any subsequence of $\{P_n\}_n$ has a convergent subsequence and that the limit P is a $T_{\lambda(x)}$-invariant probability measure such that P-almost every (x, E) is such that $E \in \mathcal{A}_{m_0(x)}$ and (4.1) holds. Note that the fact that the first marginal of P is $dx_{|\Sigma}/|\Sigma|$ follows from the fact that, by definition, this is true of P_n.

We thus take a subsequence of $\{\mu_n\}$ (which we do not relabel). We may assume that it has a subsequence, denoted $\bar{\mu}_n$, such that $\widehat{F}_n(\bar{\mu}_n) \le C$, otherwise there is nothing to prove. This implies in particular that $\bar{\mu}_n$ is of the form $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$. We let $\bar{E}_n$ denote the current and $\bar{g}_n$ the measures associated to $\bar{\mu}_n$ as in Definition 3.5 and note that $\int d\bar{g}_n = W(\bar{E}_n, \mathbf{1}_{\mathbb{R}^2})$. As usual, $\bar{\mu}'_n = \sum_{i=1}^n \delta_{\sqrt{n}x_i}$.

A first consequence of $\widehat{F}_n(\bar{\mu}_n) \le C$ is that, in view of (3.29), we have

$$\bar{\mu}_n \to \mu_0, \tag{4.8}$$

in the weak sense of measures.

Step 1: *We set up the framework of Section* 4.1. We will use integers n instead of ε to label sequences, and the correspondence will be $\varepsilon = 1/\sqrt{n}$. We let $G = \Sigma$ and $X = \mathcal{M}_+ \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2) \times \mathcal{M}$, where $p \in (1, 2)$, where $\mathcal{M}_+$ denotes the set of positive Radon measures on $\mathbb{R}^2$ and $\mathcal{M}$ the set of those which are bounded below by the constant $-C(\|m_0\|_\infty + 1)$ of Proposition 3.4, both equipped with the topology of weak convergence.

For $\lambda \in \mathbb{R}^2$ and abusing notation we let θ_λ denote both the translation $x \to x + \lambda$ and the action

$$\theta_\lambda(\nu, E, g) = (\theta_\lambda \# \nu, E \circ \theta_\lambda, \theta_\lambda \# g).$$

Accordingly, the action T^n is defined for $\lambda \in \mathbb{R}^2$ by

$$T^n_\lambda(x, \nu, E, g) = \left(x + \frac{\lambda}{\sqrt{n}}, \theta_\lambda \# \nu, E \circ \theta_\lambda, \theta_\lambda \# g\right).$$

Then we let χ be a smooth cut-off function with integral 1 and support in $B(0,1)$ and define

$$\mathbf{f}_n(x,\nu,E,g) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}^2}\chi(y)\,dg(y), & \text{if } (\nu,E,g)=\theta_{\sqrt{n}x}(\bar{\mu}'_n,\bar{E}_n,\bar{g}_n),\\ +\infty, & \text{otherwise.}\end{cases} \tag{4.9}$$

Finally, we let in agreement with Section 4.1

$$\mathbf{F}_n(\nu,E,g) = \fint_{\Sigma}\mathbf{f}_n\bigl(x,\theta_{x\sqrt{n}}(\nu,E,g)\bigr)\,dx. \tag{4.10}$$

We have the following relation between $\mathbf{F}_n$ and $\widehat{F}_n$, as $n\to+\infty$:

$$\mathbf{F}_n(\nu,E,g) \text{ is } \begin{cases} \le \dfrac{1}{|\Sigma|}\widehat{F}_n(\bar{\mu}_n)+o(1), & \text{if } (\nu,E,g)=(\bar{\mu}'_n,\bar{E}_n,\bar{g}_n)\\ =+\infty, & \text{otherwise.}\end{cases} \tag{4.11}$$

Indeed it is obvious from (4.9) that if $(\nu,E,g)\neq(\bar{\mu}'_n,\bar{E}_n,\bar{g}_n)$ then $\mathbf{F}_n(\nu,E,g)=+\infty$. On the other hand, if $(\nu,E,g)=(\bar{\mu}'_n,\bar{E}_n,\bar{g}_n)$, then from the definition of the image measure $\theta_\lambda\#\bar{g}_n$,

$$\mathbf{F}_n(\nu,E,g)=\frac{1}{\pi}\fint_{\Sigma}\int\chi(y-x\sqrt{n})\,d\bar{g}_n(y)\,dx=\frac{1}{\pi|\Sigma'|}\int\chi*\mathbf{1}_{\Sigma'}\,d\bar{g}_n.$$

Since $\chi*\mathbf{1}_{\Sigma'}$ is bounded above by 1 and is equal to 1 on $U:=\{x':\operatorname{dist}(x',\mathbb{R}^2\setminus\Sigma')\ge 1\}$, we deduce that

$$\begin{aligned}\pi\mathbf{F}_n(\nu,E,g) &\le \frac{\bar{g}_n{}^+(\mathbb{R}^2)-\bar{g}_n^-(U)}{|\Sigma'|}=\frac{\bar{g}_n(\mathbb{R}^2)+\bar{g}_n^-(U^c)}{n|\Sigma|}\\ &=\frac{\pi\widehat{F}_n(\bar{\mu}_n)}{|\Sigma|}+\frac{\bar{g}_n^-(U^c)}{n|\Sigma|}.\end{aligned} \tag{4.12}$$

Then we note that from (3.19)–(3.20) in Proposition 3.4 the measure $\bar{g}_n^-$ is supported in the union of balls $B(x',C)$ for $x'\in\operatorname{Supp}(\bar{\mu}'_n)$, and bounded above by a constant. Thus, $\bar{g}_n^-(U^c)$ is bounded by a constant times the number of balls intersecting U^c, hence by $C\bar{\mu}'_n\{x':\operatorname{dist}(x',U^c)\le C\}$. From (4.8), this is equal to

$$Cn\mu_0\{x:\operatorname{dist}(x,\partial\Sigma)\le C/\sqrt{n}\}+o(n)\le Cn\bigl|\{x:\operatorname{dist}(x,\partial\Sigma)\le C/\sqrt{n}\}\bigr|+o(n)$$

since m_0 is bounded. Using standard estimates on the volumes of tubular neighborhoods, since $\partial\Sigma$ is C^1 by assumption (2.5), we conclude that this is $o(n)$. Plugging this into (4.12) proves (4.11).

Step 2: *We check the hypotheses in Section* 4.1. We must now check the Γ-liminf and coercivity properties of $\{\mathbf{f}_n\}_n$. The main point is again that $\widehat{F}_n$ controls $\mu_n - \mu_0$ by Lemma 3.9.

LEMMA 4.3. *Assume that* $\{(x_n, \nu_n, E_n, g_n)\}_n$ *converges to* (x, ν, E, g). *Then*

$$\liminf_n \mathbf{f}_n(x_n, \nu_n, E_n, g_n) \geq \mathbf{f}(x, \nu, E, g) := \frac{1}{\pi}\int \chi\, dg.$$

PROOF. We may assume that the left-hand side is finite, in which case $\mathbf{f}_n(x_n, \nu_n, E_n, g_n) = \frac{1}{\pi}\int \chi\, dg_n$ for every large enough n, from which the result follows by passing to the limit. $\square$

LEMMA 4.4. *Assume that for any* $R > 0$ *we have*

$$\limsup_{n\to+\infty} \int_{B_R} \mathbf{f}_n\left(x_n + \frac{\lambda}{\sqrt{n}}, \theta_\lambda(\nu_n, E_n, g_n)\right) d\lambda < +\infty. \tag{4.13}$$

Then a subsequence of $\{(x_n, \nu_n, E_n, g_n)\}_n$ *converges to some* $(x, \nu, E, g) \in \Sigma \times X$.

PROOF. Assume (4.13). Then the integrand there is bounded for a.e. λ and from the definition (4.9) we deduce that

$$\theta_\lambda(\nu_n, E_n, g_n) = \theta_{\sqrt{n}x_n+\lambda}(\bar{\mu}_n', \bar{E}_n, \bar{g}_n)$$

and then that $(\nu_n, E_n, g_n) = \theta_{\sqrt{n}x_n}(\bar{\mu}_n', \bar{E}_n, \bar{g}_n)$. Thus, (4.13) gives, in view of (4.9), that for every $R > 0$ there exists $C_R > 0$ such that for any n

$$\int_{B_R}\int \chi(y - \sqrt{n}x_n - \lambda)\, d\bar{g}_n(y)\, d\lambda = \int \chi * \mathbf{1}_{B_R(\sqrt{n}x_n)}\, d\bar{g}_n < C_R.$$

This and the fact that $\bar{g}_n$ is bounded below implies that $\bar{g}_n(B_R(\sqrt{n}x_n))$ is bounded independently of n and then, using (3.22), that the same is true of $\bar{\mu}_n'(B_R(\sqrt{n}x_n))$. In other words, $\{\nu_n = \theta_{\sqrt{n}x_n}\bar{\mu}_n'\}_n$ is a locally bounded sequence of (positive) measures hence converges weakly after taking a subsequence, and the same is true of $\{g_n = \theta_{\sqrt{n}x_n}\bar{g}_n\}_n$. On the other hand $\{x_n\}_n$ is a sequence in the compact set Σ hence converges modulo a subsequence.

It remains to study the convergence of $\{E_n = \bar{E}_n \circ \theta_{\sqrt{n}x_n+\lambda}\}_n$. From (3.21) in Proposition 3.4 and the local boundedness of $\{\nu_n\}_n$, we get that $W(\bar{E}_n, \chi * \mathbf{1}_{B_R(\sqrt{n}x_n)}) = W(E_n, \chi * \mathbf{1}_{B_R})$ is bounded independently of n for any $R > 0$ and then, using (3.27), that $\{E_n\}_n$ is locally bounded in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, for any $1 \leq p < 2$ hence a subsequence locally weakly converges in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. Moreover, $\operatorname{curl} E_n = 0$ and by the above $\operatorname{div} E_n$ is locally bounded in the sense of measures, hence weakly compact in $W^{-1,p}_{\mathrm{loc}}$ for $p < 2$. By elliptic regularity, it follows that the convergence of E_n is strong in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. This concludes the proof of coercivity. $\square$

Step 3: *Conclusion*. From the previous steps, we may apply Theorem 4.2 in this setting (choosing $\mathbf{U}_R = K_R$) and we deduce in view of (4.11) that, temporarily denoting Q_n the push-forward of the normalized Lebesgue measure on Σ by the map $x \mapsto (x, \theta_{\sqrt{n}x}(\bar{\mu}'_n, \bar{E}_n, \bar{g}_n))$, and $Q = \lim_n Q_n$,

$$\begin{aligned}\liminf_n \frac{1}{|\Sigma|}\widehat{F}_n(\bar{\mu}_n) &\geq \liminf_n \mathbf{F}_n(\bar{\mu}'_n, \bar{E}_n, \bar{g}_n)\\ &\geq \int \left(\frac{1}{\pi}\int \chi\, dg\right) dQ(x, \nu, E, g)\\ &= \int \lim_{R\to+\infty} \fint_{K_R} \int \frac{1}{\pi}\chi(y-\lambda)\, dg(y)\, d\lambda\, dQ(x, \nu, E, g)\\ &= \int \lim_{R\to+\infty}\left(\frac{1}{\pi|K_R|}\int \chi * \mathbf{1}_{K_R}\, dg\right) dQ(x, \nu, E, g).\end{aligned} \tag{4.14}$$

Now we use the fact that for Q-almost every (x, ν, E, g):

(i) There exists a sequence $\{x_n\}_n$ in Σ such that

$$(x, \nu, E, g) = \lim_n \bigl(x_n, \theta_{\sqrt{n}x_n}(\bar{\mu}'_n, \bar{E}_n, \bar{g}_n)\bigr).$$

(ii) As a consequence of the above $\frac{1}{\pi|K_R|}\int \chi * \mathbf{1}_{K_R}\, dg$ converges to a finite limit as $R \to +\infty$.

The first point implies, since $\operatorname{div} \bar{E}_n = \bar{\mu}'_n - m'_0$ and $\operatorname{curl} \bar{E}_n = 0$, that by passing to the limit $n \to \infty$ we have $\operatorname{div} E = \nu - m_0(x)$ and $\operatorname{curl} E = 0$. The second point implies in particular using (3.22) that $\nu(B_R) \leq CR^2$, proving that $(E, \nu) \in \mathcal{A}_{m_0(x)}$.

Moreover, the second point implies that for any $C > 0$ we have $g(K_{R+C} \setminus K_{R-C}) = o(R^2)$ as $R \to +\infty$, and thus from point (i) above

$$\lim_{R\to+\infty}\lim_{n\to+\infty}\frac{1}{R^2}\bar{g}_n\bigl(K_{R+C}(\sqrt{n}x_n)\bigr) \setminus \bigl(K_{r-C}(\sqrt{n}x_n)\bigr) = 0.$$

Using (3.22), we deduce that

$$\lim_{R\to+\infty}\lim_{n\to+\infty}\frac{1}{R^2}\bar{\mu}'_n\bigl(K_{R+C}(\sqrt{n}x_n)\bigr) \setminus \bigl(K_{r-C}(\sqrt{n}x_n)\bigr) = 0$$

and then from (3.21),

$$\lim_{R\to+\infty}\lim_{n\to+\infty}\frac{1}{R^2}\left|W(\bar{E}_n, \chi * \mathbf{1}_{K_R(\sqrt{n}x_n)}) - \int \chi * \mathbf{1}_{K_R(\sqrt{n}x_n)}\, d\bar{g}_n\right| = 0.$$

Thus, using [42], Lemma 4.8, we may take the limit $n \to \infty$ and deduce

$$\lim_{R\to+\infty}\frac{1}{R^2}\left|W(E, \chi * \mathbf{1}_{K_R}) - \int \chi * \mathbf{1}_{K_R}\, dg\right| = 0.$$

Together with (4.14) this yields, by definition of W,

$$\liminf_n \frac{1}{|\Sigma|}\widehat{F}_n(\bar{\mu}_n) \geq \frac{1}{\pi}\int W(E)\,dQ(x,\nu,E,g) \tag{4.15}$$

and, we recall, Q-a.e. $(E,\nu)\in\mathcal{A}_{m_0(x)}$.

Now we let P_n (resp., P) be the marginal of Q_n (resp., Q) with respect to the variables (x,E). Then the first marginal of P is the normalized Lebesgue measure on Σ and P-a.e. we have $E\in\mathcal{A}_{m_0(x)}$, in particular

$$W(E)\geq \min_{\mathcal{A}_{m_0(x)}} W = m_0(x)\left(\min_{\mathcal{A}_1} W - \frac{\pi}{2}\log m_0(x)\right).$$

Integrating with respect to P and noting that since only x appears on the right-hand side we may replace P by its first marginal there we find, in view of (2.25) that the lower bound (4.1) holds.

4.3. *Proof of Theorem* 4.1, *part B*. In this section, we construct a set of explicit configurations whose W is not too large, and show that their probability is not too small, which will lead to a lower bound on Z_n^β. This is the longest part of our proof. The method is borrowed from [42] but requires various adjustments that we shall detail in Section 6. We will need (2.5) in order to simplify the construction and estimates near the boundary.

The following proves Theorem 4.1, part B and contains a bit more information useful for proving Theorem 5.2 below.

PROPOSITION 4.5. *Let $P\in\mathcal{P}(X)$ be admissible. Then, for any $\eta>0$, there exists $\delta>0$ and, for any n, a subset $A_n\subset\mathbb{C}^n$ such that $|A_n|\geq n!(\pi\delta^2/n)^n$ and for every sequence $\{\mu_n=\frac{1}{n}\sum_{i=1}^n\delta_{y_i}\}_n$ with $(y_1,\dots,y_n)\in A_n$ the following hold:*

(i) *We have the upper bound*

$$\limsup_{n\to\infty}\frac{1}{n}\left(w_n(y_1,\dots,y_n)-n^2I(\mu_0)+\frac{n}{2}\log n\right)\leq \widetilde{W}(P)+\eta. \tag{4.16}$$

(ii) *There exists $\{\overline{E}_n\}_n$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ such that $\operatorname{div}\overline{E}_n=2\pi(\mu_n'-m_0')$, where $\mu_n'=\sum_i\delta_{y_i'}$ and such that the image $\overline{P}_n$ of $dx_{|\Sigma}/|\Sigma|$ by the map $x\mapsto(x,\overline{E}_n(\sqrt{n}x+\cdot))$ is such that*

$$\limsup_{n\to\infty}\operatorname{dist}(\overline{P}_n,P)\leq\eta, \tag{4.17}$$

where dist *is a distance which metrizes the topology of weak convergence on $\mathcal{P}(X)$.*

Respectively, the image $\overline{Q}_n$ of $dx_{|\Sigma}/|\Sigma|$ by the map $x\mapsto(x,\mu_n'(\sqrt{n}x+\cdot))$ is such that

$$\limsup_{n\to\infty}\operatorname{dist}(\overline{Q}_n,Q)\leq\eta, \tag{4.18}$$

where dist *is a distance which metrizes the topology of weak convergence, and Q is the push-forward of P by $(x,E)\mapsto\frac{1}{2\pi}\operatorname{div}E+m_0'(x)$.*

Applying the above proposition with $\eta = 1/k$, we get a subset $A_{n,k}$ in which we choose any n-tuple $(y_{i,k})_{1\le i\le n}$. This yields in turn a family $\{P_{n,k}\}$ of probability measures on X. A standard diagonal extraction argument then yields the following corollary.

COROLLARY 4.6 (Theorem 4.1, part B). *Let $P \in \mathcal{P}(X)$ be admissible. Then there exists a sequence $\{\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\}_n$ and a sequence $\{\overline{E}_n\}_n$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that* $\operatorname{div} \overline{E}_n = 2\pi(\mu_n' - m_0'(x')\,dx')$ *and*

$$\limsup_{n\to\infty} \frac{1}{n}\left(w_n(x_1,\dots,x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right) \le \widetilde{W}(P). \tag{4.19}$$

Moreover, denoting $\overline{P}_n$ the image of $dx_{|\Sigma}/|\Sigma|$ by the map $x \mapsto (x, \overline{E}_n(\sqrt{n}x + \cdot))$, we have $\overline{P}_n \to P$ as $n \to +\infty$.

Another consequence of Proposition 4.5 is recalling (2.25) and (3.12).

COROLLARY 4.7. *For any $\beta > 0$, there exists $C_\beta > 0$ such that $\lim_{\beta\to+\infty} C_\beta = 0$ and*

$$\liminf_{n\to+\infty} \frac{\log K_n^\beta}{n} \ge -\frac{\beta}{2}(\alpha + C_\beta). \tag{4.20}$$

PROOF. Choose $E_0 \in \mathcal{A}_1$ such that $W(E_0) = \alpha_1 = \min_{\mathcal{A}_1} W$. Such an E_0 exists by [42], Theorem 1. Let P be the image of the normalized Lebesgue measure on Σ by the map $x \mapsto (x, \sigma_{m_0(x)} E_0)$, where

$$\sigma_m E(y) := \sqrt{m}E(\sqrt{m}y). \tag{4.21}$$

Then by construction P-almost every (x, E) satisfies $E \in \mathcal{A}_{m_0(x)}$ and the first marginal of P is $dx_{|\Sigma}/|\Sigma|$.

Given $\eta > 0$, applying Proposition 4.5 and using the notation there, we have $|A_n| \ge n!(\delta^2/n)^n$, and from (3.11) we have

$$1 \ge \int_{A_n} \frac{1}{K_n^\beta} e^{-n(\beta/2)F_n(\mu_n)}\,dy_1\cdots dy_n, \tag{4.22}$$

where $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{y_i}$. From (3.10) and (4.16), when $(y_1,\dots,y_n) \in A_n$ we have

$$F_n(\mu_n) \le \eta + \widetilde{W}(P) = \eta + \frac{1}{\pi}\int_\Sigma W(\sigma_{m_0(x)}E_0)\,dx.$$

From (2.24), and since $\int_\Sigma m_0 = 1$, we obtain

$$\frac{1}{\pi}\int_\Sigma W(\sigma_{m_0(x)}E_0)\,dx = \frac{\alpha_1}{\pi} - \frac{1}{2}\int_\Sigma m_0(x)\log m_0(x)\,dx = \alpha,$$

by definition (2.25). We deduce

$$F_n(\mu_n) \le \alpha + \eta.$$

Together with (4.22), we find $1 \ge \frac{|A_n|}{K_n^\beta} e^{-n(\beta/2)(\eta+\alpha)}$. Taking logarithms, we are led to

$$\log K_n^\beta \ge \log n! + n \log \delta^2 - n \log n - \tfrac{1}{2} n\beta(\eta + \alpha).$$

From Stirling's formula, $\log n! \ge n \log n - Cn$ and we deduce that

$$\liminf_{n\to\infty} \frac{\log K_n^\beta}{n} \ge -\frac{\beta}{2}(\alpha + \eta) - C_\eta,$$

with $C_\eta = -\log \delta^2 + C$. But, for any $\eta > 0$, if β is large enough the right-hand side is greater than $-\frac{\beta}{2}(\alpha + 2\eta)$. This proves the corollary. $\square$

4.4. *Proof of Theorem* 4.1, *completed*. As mentioned above, part B of the theorem is a direct consequence of Proposition 4.5, see Corollary 4.6.

Part C follows from the comparison of parts A and B: for minimizers, the chains of inequalities (4.1) and (4.2) are in fact equalities and $\widetilde{W}$ must be minimized hence equal to α. Also we must have $\lim_{n\to\infty}(F_n - \widehat{F}_n)(\mu_n) = \lim_{n\to\infty} \sum_{i=1}^n \zeta(x_i) = 0$, which in view of (3.2), implies that $\lim \sum_i \operatorname{dist}(x_i, \Sigma)^2 = 0$.

This completes the proof of Theorem 4.1.

5. Large deviations and proofs of the remaining results. We now turn to Coulomb gases, that is, to the case with temperature. Our next result mostly expresses Theorem 4.1 in a "moderate" deviations language. Before stating it, let us recall for comparison the result of [10, 33].

THEOREM 5.1. *Let $\beta = 2$ and $V(x) = |x|^2$. Denote by $\widetilde{\mathbb{P}}_n^\beta$ the image of the law* (1.1) *by the map $(x_1, \dots, x_n) \mapsto \mu_n$, where $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$. Then for any subset A of the set of probability measures on $\mathbb{R}^2$* (*endowed with the topology of weak convergence*), *we have*

$$-\inf_{\mu \in \mathring{A}} \widetilde{I}(\mu) \le \liminf_{n\to\infty} \frac{1}{n^2} \log \widetilde{\mathbb{P}}_n^\beta(A) \le \limsup_{n\to\infty} \frac{1}{n^2} \log \widetilde{\mathbb{P}}_n^\beta(A) \le -\inf_{\mu \in \bar{A}} \widetilde{I}(\mu),$$

where $\widetilde{I} = I - \min I$.

Our result is the following theorem.

THEOREM 5.2. *Let V satisfy assumptions* (2.4)–(2.6). *For any $\beta > 0$, there exists $C_\beta > 0$ such that $\lim_{\beta\to+\infty} C_\beta = 0$ and the following holds.*

For any $n > 0$ let $A_n \subset \mathbb{C}^n$. Denote

$$A_\infty = \bigcap_{m>0} \overline{\bigcup_{n>m} i_n(A_n)}, \tag{5.1}$$

where i_n is as in (2.16), and the topology is the weak convergence on $\mathcal{P}(X)$ [in other words, A_∞ is the set of weak limits, up to subsequences, of $P_n \in i_n(A_n)$]. Then, α being as in (2.25),

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}_n^\beta(A_n)}{n} \le -\frac{\beta}{2}\Big(\inf_{P\in A_\infty} \widetilde{W}(P) - \alpha - C_\beta\Big). \tag{5.2}$$

Conversely, let $A \subset \mathcal{P}(X)$ be a set of $T_{\lambda(x)}$-invariant probability measures on X and let $\mathring{A}$ be the interior of A. Then there exists a sequence of subsets $A_n \subset \Sigma^n$ such that

$$-\frac{\beta}{2}\Big(\inf_{P\in \mathring{A}} \widetilde{W}(P) - \alpha + C_\beta\Big) \le \liminf_{n\to\infty} \frac{\log \mathbb{P}_n^\beta(A_n)}{n}, \tag{5.3}$$

and such that for any sequence $\{\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\}_n$ such that $(x_1, \dots, x_n) \in A_n$ for every n there exists a sequence of fields $\overline{E}_n \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\operatorname{div} \overline{E}_n = 2\pi(\mu_n' - \mu_0')$ and such that—defining $\overline{P}_n$ as in (2.15) with $\overline{E}_n$ replacing E_n—we have

$$\lim_n \overline{P}_n \in \mathring{A}. \tag{5.4}$$

Note that if $\overline{P}_n$ was P_n, then (5.4) would be equivalent to saying that $\bigcap_m \overline{\bigcup_{n>m} i_n(A_n)} \subset \mathring{A}$. The difference between $\overline{P}_n$ and P_n is that the latter is generated by a field $\overline{E}_n$ which is not necessarily a gradient.

Compared to Theorem 5.1, this result can be seen as a next order (speed n instead of n^2) deviations result, where $\widetilde{W}$ plays the role of a rate function, with a margin which becomes small as $\beta \to \infty$. While Theorem 5.1 said that empirical measures at macroscopic scale converge to μ_0, except for a set of exponentially decaying probability, Theorem 5.2 says that within the empirical measures which do converge to μ_0, the ones with large $\widetilde{W}$ (computed after blow-up) also have exponentially decaying probability, but at the slower rate e^{-n} instead of e^{-n^2}. More precisely, there is a threshhold C_β such that configurations satisfying

$$\widetilde{W}(P) \ge \alpha + C_\beta$$

have exponentially small probability, where we recall α is also the minimal value of $\widetilde{W}(P)$. Since we believe that W measures the disorder of a (limit) configuration of (blown up) points in the plane, this means that most configurations have a certain order. The threshhold, or gap, C_β tends to 0 as β tends to ∞, hence in this limit, configurations have to be closer and closer to the minimum of the average of

W, or have more and more order. Modulo the conjecture that the minimum of W is achieved by the perfect “Abrikosov” triangular lattice, this constitutes a crystallization result. Note that to solve this conjecture, it would suffice to evaluate α, which in view of Theorem 2.1 is equivalent to being able to compute the asymptotics of Z_n^β as $\beta \to \infty$.

5.1. *Proofs of Theorems 5.2 and 2.1.* We start with the upper bound on $\log \mathbb{P}_n^\beta$. Let A_n be a subset of $\mathbb{C}^n$. We denote as usual $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$. From (3.11), we have

$$\mathbb{P}_n^\beta(A_n) = \frac{1}{K_n^\beta}\int_{A_n} e^{-(1/2)\beta n F_n(\mu_n)}\, dx_1 \cdots dx_n$$

hence

$$\frac{\log \mathbb{P}_n^\beta(A_n)}{n} = -\frac{\log K_n^\beta}{n} + \frac{1}{n}\log \int_{A_n} e^{-(1/2)\beta n F_n(\mu_n)}\, dx_1 \cdots dx_n. \tag{5.5}$$

We deduce, since $\widehat{F}_n(\mu_n) = F_n(\mu_n) - 2\sum_i \zeta(x_i)$, that

$$\begin{aligned}\frac{\log \mathbb{P}_n^\beta(A_n)}{n} &\le -\frac{\log K_n^\beta}{n} \\ &\quad + \frac{1}{n}\log\biggl(e^{-(1/2)\beta n \inf_{A_n}\widehat{F}_n}\int_{A_n} e^{-\beta n\sum_i \zeta(x_i)}\, dx_1\cdots dx_n\biggr).\end{aligned} \tag{5.6}$$

Let μ_n such that $\widehat{F}_n(\mu_n) \le \inf_{A_n}\widehat{F}_n + 1/n$. Then from (4.1) in Theorem 4.1 we have, using the notation there, $\liminf_{n\to\infty}\widehat{F}_n(\mu_n) \ge \widetilde{W}(P)$ where $P = \lim_n P_n$ and $P_n = i_n(x_1,\ldots,x_n)$. Since $P_n \in i_n(A_n)$, by definition we have $P \in A_\infty$, since by definition $A_\infty = \bigcap_{m>0}\overline{\bigcup_{n>m} i_n(A_n)}$. We may thus write

$$\liminf_{n\to+\infty}\widehat{F}_n(\mu_n) \ge \inf_{P\in A_\infty}\widetilde{W}(P). \tag{5.7}$$

Inserting into (5.6) we are led to

$$\begin{aligned}\frac{\log \mathbb{P}_n^\beta(A_n)}{n} &\le -\frac{\beta}{2}\inf_{P\in A_\infty}\widetilde{W}(P) - \frac{\log K_n^\beta}{n} \\ &\quad + \frac{1}{n}\log\biggl(\int_{\mathbb{C}^n} e^{-\beta n\sum_i \zeta(x_i)}\, dx_1\cdots dx_n\biggr) + o(1)\end{aligned} \tag{5.8}$$

thus in view of Lemma 3.8 and (4.20), we have established (5.2). An immediate corollary of (5.8), choosing A_n to be the full space and using $\min \widetilde{W} = \alpha$ and Lemma 3.8, is that

$$\limsup_{n\to\infty}\frac{\log K_n^\beta}{n} \le -\frac{\beta\alpha}{2} + \log|\Sigma|. \tag{5.9}$$

We next turn to the lower bound. Fix $\eta > 0$. Given A, let $P \in \mathring{A}$ be such that

$$\widetilde{W}(P) \leq \inf_{P\in\mathring{A}} \widetilde{W}(P) + \frac{\eta}{2}. \tag{5.10}$$

Since $P \in \mathring{A}$, if η is chosen small enough (which we assume) then $B(P, 2\eta) \subset A$, where the ball is for a distance metrizing weak convergence as in Proposition 4.5.

We then apply Proposition 4.5 to P and η. We find $\delta > 0$ and for any n large enough a set A_n such that $|A_n| \geq n!(\pi\delta^2/n)^n$ and, rewriting (4.16) with (3.3),

$$\limsup_{n\to\infty} \sup_{A_n} F_n \leq \widetilde{W}(P) + \eta. \tag{5.11}$$

Moreover, for every $(y_1, \dots, y_n) \in A_n$ and letting $\{\mu_n = \frac{1}{n}(\delta_{y_1} + \cdots + \delta_{y_n})\}_n$, there exists $\{\overline{E}_n\}_n$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\operatorname{div} \overline{E}_n = 2\pi(\mu_n' - \mu_0')$ and such that the image $\overline{P}_n$ of $dx_{|\Sigma}/|\Sigma|$ by the map $x \mapsto (x, E_n(\sqrt{n}x + \cdot))$ is such that

$$\limsup_{n\to\infty} \operatorname{dist}(\overline{P}_n, P) \leq \eta. \tag{5.12}$$

In particular, (5.4) holds. Moreover, inserting (5.11) and (5.10) into (3.11), we find that

$$\frac{\log \mathbb{P}_n^\beta(A_n)}{n} \geq -\frac{\log K_n^\beta}{n} - \frac{\beta}{2} \inf_{P\in\mathring{A}} \widetilde{W}(P) - \frac{1}{2}\beta\eta + \frac{1}{n}\log\left|\frac{A_n}{\sqrt{n}}\right| + o(1).$$

On the other hand, using $|A_n| \geq n!(\pi\delta^2/n)^n$ and Stirling's formula, we have $\log|A_n| \geq 2n\log\delta - Cn$. Combining with (5.9), we find

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}_n^\beta(A_n)}{n} \geq -\frac{\beta}{2}\left(\inf_{P\in\mathring{A}} \widetilde{W}(P) + \eta + \frac{C_\eta}{\beta}\right),$$

with $C_\eta = -2\log\delta + C + \log|\Sigma|$. The right-hand side is greater than $-\frac{\beta}{2}(\inf_{\mathring{A}} \widetilde{W} + 2\eta)$ if β is large enough. Since this is true for any $\eta > 0$, this proves (5.3).

Statement (2.9) of Theorem 2.1 immediately follows by combining (5.9), Corollary 4.7 and (3.12). Statement (2.10) in Theorem 2.1 is a consequence of (2.9): Let A_n be the set of n-tuples $(x_1, \dots, x_n)$ such that

$$w_n(x_1, \dots, x_n) \geq n^2 I(\mu_0) - \frac{n}{2}\log n + n(\alpha + \eta).$$

From (3.11), we have

$$\mathbb{P}_n^\beta(A_n) = \frac{1}{K_n^\beta}\int_{A_n} \exp\left(-n\frac{\beta}{2}F_n(\mu_n)\right) dx_1 \cdots dx_n. \tag{5.13}$$

Now we use Lemma 3.7 and the fact that $g_n \geq -C$, where C is a universal constant: we split A_n as $A_n' \cup A_n''$, where A_n' denotes the n-tuples in A_n for which $\sum_i \zeta(x_i) > C(|\Sigma| + 1)/\pi$ and A_n'' those for which $\sum_i \zeta(x_i) \leq C(|\Sigma| + 1)/\pi$.

Using (3.25) and the fact that $g_n = 0$ outside $\Sigma' \bigcup_i B(x_i', 1)$, so that $\frac{1}{n\pi}\int dg_n \geq -C(|\Sigma|+1)/\pi$, we have if $(x_1, \dots, x_n) \in A_n'$ that $F_n(\mu_n) \geq \sum_i \zeta(x_i)$. Then, using Lemma 3.8, we deduce

$$\limsup_n \frac{1}{n} \log \mathbb{P}_n^\beta(A_n') \leq |\Sigma| - \limsup_n \left(\frac{1}{n} \log K_n^\beta\right).$$

Now if $(x_1, \dots, x_n) \in A_n''$ and writing $C' = C(|\Sigma|+1)/\pi$ we have

$$F_n(\mu_n) \geq F_n(\mu_n) + \sum_i \zeta(x_i) - C' \geq \alpha + \eta + \sum_i \zeta(x_i) - C'.$$

Inserting into (5.13) and in view of Lemma 3.8, we find

$$\limsup_n \frac{1}{n} \log \mathbb{P}_n^\beta(A_n'') \leq -\frac{\beta}{2}(\alpha + \eta) + C - \limsup_n \left(\frac{1}{n} \log K_n^\beta\right),$$

where C is independent of $\beta > 1$. Combining the asymptotics for $\mathbb{P}_n^\beta(A_n')$ and $\mathbb{P}_n^\beta(A_n'')$ with (2.9), we deduce

$$\limsup_n \frac{1}{n} \log \mathbb{P}_n^\beta(A_n) \leq -\frac{\beta}{2}(\eta + C_\beta),$$

where $\lim_{\beta\to+\infty} C_\beta = 0$. This proves (2.10).

5.2. *Proofs of Theorems* 2.2, 2.3, 2.4. The $\beta = +\infty$ part of Theorem 2.2 is clearly contained in Theorem 4.1. So is Theorem 2.3.

For the finite β case of Theorem 2.2, we apply the method of Theorem 4.2 part B: Let $A_{n,M} = \{(x_1, \dots, x_n) \mid \widehat{F}_n(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}) \leq M\}$. In view of (5.6), Corollary 4.7 and Lemma 3.8, if M is chosen large enough we have $\mathbb{P}_n^\beta(A_{n,M}^c) \to 0$ as $n \to \infty$. In view of Lemma 4.2, to prove the tightness of $i_n \# \mathbb{P}_n^\beta$ it thus suffices to check that if $P_n \in i_n(A_{n,M})$ then P_n has a convergent subsequence. But this is the case from Theorem 4.1, part A. Therefore, $i_n \# \mathbb{P}_n^\beta$ does indeed converge, modulo a subsequence, to a certain law $\widetilde{\mathbb{P}}^\beta$.

It remains to prove Theorem 2.4, and the fact that $\widetilde{\mathbb{P}}^\beta$-almost every P is admissible. For this, we use Theorem 2.1 and let, for a given $\beta > 0$, A_n be the set of n-tuples $(x_1, \dots, x_n)$ such that

$$w_n(x_1, \dots, x_n) > n^2 I(\mu_0) - \frac{n}{2}\log n + n(\alpha + 2C_\beta),$$

and $\tilde{A}_n = i_n(A_n)$. Then from (2.10) we have $i_n \# \mathbb{P}_n^\beta(\tilde{A}_n) = \mathbb{P}_n^\beta(A_n) \to 0$ as $n \to +\infty$. Therefore, letting $\tilde{A} = \bigcap_m \bigcup_{n>m} \tilde{A}_n^c$, we have that $\widetilde{\mathbb{P}}^\beta(\tilde{A}) = 1$. On the other hand, if $P \in \tilde{A}$ then there exists a sequence $P_n \in \tilde{A}_n$ such that $P_n \to P$, and $P_n = i_n(x_1, \dots, x_n)$ where $w_n(x_1, \dots, x_n) \leq n^2 I(\mu_0) - \frac{n}{2}\log n + n(\alpha + 2C_\beta)$ by definition of A_n. Using Theorem 4.1, part A, we deduce that P is admissible, and that $\widetilde{W}(P) \leq \alpha + 2C_\beta$. This proves Theorem 2.4 and the fact that $\widetilde{\mathbb{P}}^\beta$-almost every P is admissible.

5.3. *Definition of* $\mathbb{W}$. In this subsection, we briefly examine how to define the renormalized energy as a function of the points only, via (3.17). We prove the following.

LEMMA 5.1. *The function* $\mathbb{W}$ *be defined by* (3.17) *is Borel-measurable on the set of locally finite measures.*

PROOF. First we show that there exists a measurable map $\nu \mapsto E_\nu$ where E_ν satisfies (2.17). The set

$$A = \{E \in \mathcal{A}_m, W(E) < \infty\}$$

is Borel measurable, since W is (as proven in [42], Theorem 1). We may partition A into equivalence classes for the relation $E \sim E'$ if $\operatorname{div} E = \operatorname{div} E'$. In view of Lemma 3.3, denoting by E^* the equivalence class of $E \in A$, we have $E^* = \{E + \vec{C}, \vec{C} \in \mathbb{R}^2\}$. In particular, this implies that if U is an open set in A, then $U^* = \bigcup_{E \in U} E^*$ is open too in $A/\sim$. By Effros's theorem (cf., e.g., [46], Theorem 5.4.3) there thus exists a Borel section B of A which contains exactly one element of each equivalence class. The map $E^* \mapsto \frac{1}{2\pi} \operatorname{div} E + m$ is then a Borel measurable and injective map from B to $\{\nu \in \mathcal{M}_+ : \mathbb{W}(\nu) < \infty\}$ where $\mathcal{M}_+$ is the set of positive Radon measures on $\mathbb{R}^2$. By [17], Proposition 8.3.5, its inverse is also Borel measurable and injective. This provides a measurable selection $\psi : \nu \mapsto E$ satisfying (2.17) on $\{\nu \in \mathcal{M}_+ : \mathbb{W}(\nu) < \infty\}$. Since $E^* = \{E + \vec{C}, \vec{C} \in \mathbb{R}^2\}$ we may write

$$\mathbb{W}(\nu) = \inf_{\vec{C} \in \mathbb{R}^2} W(\psi(\nu) + \vec{C}).$$

Using again the fact that W is Borel measurable and $\nu \mapsto \psi(\nu) + \vec{C}$, too, it follows that $\mathbb{W}$ is measurable as claimed. $\square$

We may then without too much difficulty translate the results of Theorems 4.1, 5.2 with $\int \mathbb{W}(\nu)\, dQ(x, \nu)$ instead of $\int W(E)\, dP(x, E)$.

6. Proof of Proposition 4.5. The construction consists of the following. We are given $\varepsilon > 0$, which is the error we can afford. First, we select a finite set of vector fields $J_1, \dots, J_N$ (N will depend on ε) which will represent the probability $P(x, E)$ with respect to its E dependence, and whose renormalized energies are well controlled. Since P is $T_{\lambda(x)}$-invariant, we need it to be well approximated by measures supported on the orbits of the J_i's under translations. Second, we work in blown-up coordinates and split the region Σ' (whose diameter is order $\sqrt{n}$) into many rectangles K with centers x_K and side-lengths of order $\overline{R}$ large enough. Even though we choose $\overline{R}$ to be large, it will still be very small compared to the size of Σ', as $n \to \infty$, so that the Diracs at $x_K/\sqrt{n}$ approximate $P(x, E)$ with respect to its x dependence. On each rectangle K, the weight m_0' is temporarily replaced by its average m_K. Then we split each rectangle K into q^2 identical rectangles, with side-lengths of order $2R = \overline{R}/q$, where both R and q will be sufficiently large. We then select the proportion of the rectangles that corresponds to the weight that the orbit of each J_i carries in the approximation of P. In these rectangles, we paste a (translated) copy of J_i at the scale m_K and suitably modified near the boundary according to a construction of [42] (Proposition 6.4 below) so that its tangential component on the boundary is 0 (this can be done while inducing only an error ε on W). In the few rectangles that may remain unfilled, we paste a copy of an arbitrary J_0 whose renormalized energy is finite. We perform the construction above provided we are far enough from $\partial\Sigma'$. The layer near the boundary must be treated separately, and there again an arbitrary (translated and rescaled) current can be pasted. Finally, we add a vector field to correct the discrepancy between m_K and m'_0 in each of the rectangles.

To complete the proof of Proposition 4.5, we collect all of the estimates on the constructed vector field to show that its energy w_n is bounded above in terms of $\widetilde{W}$ and that the probability measures associated to the construction have remained close to P.

6.1. *Estimates on distances between probabilities.* First, we choose distances which metrize the topologies of $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $\mathcal{B}(X)$, the set of finite Borel measures on $X = \Sigma \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. For $E_1, E_2 \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ we let

$$d_p(E_1, E_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|E_1 - E_2\|_{L^p(B(0,k))}}{1 + \|E_1 - E_2\|_{L^p(B(0,k))}},$$

and on X we use the sum of the Euclidean distance on Σ and d_p, which we denote d_X. It is a distance on X. On $\mathcal{B}(X)$ we define a distance by choosing a sequence of bounded continuous functions $\{\varphi_k\}_k$ which is dense in $C_b(X)$ and we let, for any $\mu_1, \mu_2 \in \mathcal{B}(X)$,

$$d_{\mathcal{B}}(\mu_1, \mu_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{|\langle \varphi_k, \mu_1 - \mu_2 \rangle|}{1 + |\langle \varphi_k, \mu_1 - \mu_2 \rangle|},$$

where we have used the notation $\langle \varphi, \mu \rangle = \int \varphi \, d\mu$.

We have the following general facts.

LEMMA 6.1. *For any $\varepsilon > 0$, there exists $\eta_0 > 0$ such that if $P, Q \in \mathcal{B}(X)$ and $\|P - Q\| < \eta_0$, then $d_{\mathcal{B}}(P, Q) < \varepsilon$. Here $\|P - Q\|$ denotes the total variation of the signed measure $P - Q$, that is, the supremum of $\langle \varphi, P - Q \rangle$ over measurable functions φ such that $|\varphi| \leq 1$.*

In particular, if $P = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ and $Q = \sum_{i=1}^{\infty} \beta_i \delta_{x_i}$ with $\sum_i |\alpha_i - \beta_i| < \eta_0$, then $d_{\mathcal{B}}(P, Q) < \varepsilon$.

LEMMA 6.2. *Let $K \subset X$ be compact. For any $\varepsilon > 0$, there exists $\eta_1 > 0$ such that if $x \in K$, $y \in X$ and $d_X(x, y) < \eta_1$ then $d_{\mathcal{B}}(\delta_x, \delta_y) < \varepsilon$.*

LEMMA 6.3. *Let $0 < \varepsilon < 1$. If μ is a probability measure on a set A and $f, g : A \to X$ are measurable and such that $d_{\mathcal{B}}(\delta_{f(x)}, \delta_{g(x)}) < \varepsilon$ for every $x \in A$, then*

$$d_{\mathcal{B}}\big(f^{\#}\mu, g^{\#}\mu\big) < C\varepsilon\big(|\log\varepsilon| + 1\big).$$

PROOF. Take any bounded continuous function φ_k defining the distance on $\mathcal{B}(X)$. Then if $d_{\mathcal{B}}(\delta_{f(x)}, \delta_{g(x)}) < \varepsilon$ for any $x \in X$ we have in particular

$$\frac{|\varphi_k(f(x)) - \varphi_k(g(x))|}{1 + |\varphi_k(f(x)) - \varphi_k(g(x))|} \le 2^k\varepsilon.$$

It follows that

$$\begin{aligned} d_{\mathcal{B}}\big(f^{\#}\mu, g^{\#}\mu\big) &\le \sum_k 2^{-k}\min\big(\varepsilon 2^k, 1\big) \\ &\le \varepsilon\big([\log_2\varepsilon] + 1\big) + \sum_{k=[\log_2\varepsilon]+1}^{\infty} 2^{-k} \le C\varepsilon\big(|\log\varepsilon| + 1\big). \end{aligned}$$

□

6.2. *Preliminary results*. In what follows $\Sigma' = \sqrt{n}\Sigma$, $m_0'(x) = m_0(x/\sqrt{n})$: we work in blown-up coordinates. We consider a probability measure P on $\Sigma \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ which is as in the proposition. We let $\widetilde{P}$ be the probability measure on $\Sigma \times \mathcal{A}_1$ which is the image of P under $(x, E) \mapsto (x, \sigma_{1/m_0(x)}E)$, so that

$$\widetilde{P} = \int \delta_x \otimes \delta_{\sigma_{1/m_0(x)}E}\, dP(x, E), \qquad P = \int \delta_x \otimes \delta_{\sigma_{m_0(x)}E}\, d\widetilde{P}(x, E). \tag{6.1}$$

It is easy to check that since P is $T_{\lambda(x)}$-invariant, $\widetilde{P}$ is as well, and in particular it is translation-invariant.

The construction is based on the following statement which is a rewriting of Proposition 4.2 in [42] and the remark following it.

PROPOSITION 6.4 (Screening of an arbitrary vector field). *Let $K_R = [-R, R]^2$, let $\{\chi_R\}_R$ satisfy (2.21).*

Let $G \subset \mathcal{A}_1$ be such that there exists $C > 0$ such that for any $E \in G$ we have

$$\forall R > 1 \qquad \frac{\nu(K_R)}{|K_R|} < C, \tag{6.2}$$

for the associated ν's, and

$$\lim_{R\to+\infty} \frac{W(E, \chi_R)}{|K_R|} = W(E), \tag{6.3}$$

where the convergence is uniform w.r.t. $E \in G$.

Then for any $\varepsilon > 0$ there exists $R_0 > 0$, $\eta_2 > 0$ such that if $R > R_0$ and L is a rectangle centered at 0 whose side-lengths belong to $[2R, 2R(1+\eta_2)]$ and such that $|L| \in \mathbb{N}$, then for every $E \in G$ there exists a $E_L \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that the following hold:

(i) $E_L = 0$ *in* L^c,

(ii) *There is a discrete subset $\Lambda \subset L$ such that*

$$\operatorname{div} E_L = 2\pi\left(\sum_{p\in\Lambda} \delta_p - \mathbf{1}_L\right).$$

In particular, $E \cdot \vec{\nu} = 0$ on ∂L, there is no singular part of the divergence on ∂L, and thus $\#\Lambda = |L|$.

(iii) *If $d(x, L^c) > R^{3/4}$ then $E_L(x) = E(x)$.*

(iv)

$$\frac{W(E_L, \mathbf{1}_L)}{|L|} \le W(E) + \varepsilon. \tag{6.4}$$

We note that if E is such that $\operatorname{div} E = 2\pi(\sum_p \delta_p - 1)$ and we have $\operatorname{curl} E = 0$ in a neighborhood of each $p \in \Lambda$, then the definition (2.20) still makes sense, in particular the limit exists. This is what is meant by W in (6.4), as well as in the rest of the section.

The next lemma explains how to partition Σ into rectangles. The main point is to cut Σ' into stripes and then each stripe into rectangles in such a way that $\int m_0'$ over each rectangle is a large integer.

LEMMA 6.5. *There exists a constant $C_0 > 0$ such that, given any $R > 1$ and $q \in \mathbb{N}^*$, there exists for any $n \in \mathbb{N}^*$ a collection $\mathcal{K}_n$ of closed rectangles in Σ' with disjoint interiors, whose side-lengths are between $\overline{R} = 2qR$ and $\overline{R} + C_0\overline{R}/R^2$, and which are such that*

$$\{x \in \Sigma' : d(x, \partial\Sigma') \le \overline{R}\} \subset \Sigma' \setminus \bigcup_{K\in\mathcal{K}_n} K \subset \{x \in \Sigma' : d(x, \partial\Sigma') \le C_0\overline{R}\}$$

and, for all $K \in \mathcal{K}_n$,

$$\int_K m_0' \in q^2\mathbb{N}. \tag{6.5}$$

PROOF. For each $j \in \mathbb{Z}$, we let

$$m_j(t) = \int_{x=-\infty}^{t} \int_{y=j\overline{R}}^{(j+1)\overline{R}} m_0'(x, y)\, dy\, dx.$$

Then each strip $\{j\overline{R} \le y < (j+1)\overline{R}\}$ is cut into rectangles $[t_{ij}, t_{(i+1),j}] \times [j\overline{R}, (j+1)\overline{R}]$ where $t_{0j} = -\infty$ and

$$t_{i+1,j} = \min\{t \ge t_{ij} + \overline{R} : m_j(t_{ij}) \in q^2\mathbb{N}\}.$$

Since by assumption (2.5), we have $m_0'(x) \in [\underline{m}, \overline{m}]$ for any $x \in \Sigma'$, it is not difficult to check that if such a rectangle is included in Σ' then

$$t_{ij} + \overline{R} \le t_{i+1,j} \le t_{ij} + \overline{R} + \frac{q^2}{\underline{m}\overline{R}},$$

and thus its side-lengths are between $\overline{R}$ and $\overline{R} + C\overline{R}/R^2$ since $\overline{R}/R^2 = 4q^2/\overline{R}$. We let $\mathcal{K}_n$ be the set of rectangles of the form $[t_{ij}, t_{(i+1),j}] \times [j\overline{R}, (j+1)\overline{R}]$ which are included in $\{x : d(x, \partial\Sigma') > \overline{R}\}$. From the above, it follows that these rectangles in fact cover the set $\{x : d(x, \partial\Sigma') > C\overline{R}\}$ for some $C > 0$ independent of $R > 1$, q. By construction, each $K \in \mathcal{K}_n$ is such that

$$\int_K m_0' = m_j(t_{(i+1),j}) - m_j(t_{ij}) \in q^2\mathbb{N}. \qquad \square$$

The next lemma explains how to select a good subset of $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$.

LEMMA 6.6. *Let $\widetilde{P}$ be a translation invariant measure on X such that $\widetilde{P}$-a.e. $E \in \mathcal{A}_1$ and $W(E) < \infty$. Then for any $\varepsilon > 0$, for any $R_\varepsilon > 0$, there exist subsets $H_\varepsilon \subset G_\varepsilon$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ which are compact and such that*:

(i) *η_0 being given by Lemma 6.1 we have*

$$\widetilde{P}(\Sigma \times G_\varepsilon^c) < \min(\eta_0^2, \eta_0\varepsilon), \qquad \widetilde{P}(\Sigma \times H_\varepsilon^c) < \min(\eta_0, \varepsilon). \tag{6.6}$$

(ii) *For every $E \in H_\varepsilon$, there exists $\Gamma(E) \subset K_{\overline{m}R_\varepsilon}$ such that*

$$|\Gamma(E)| < CR_\varepsilon^2\eta_0 \quad \textit{and} \quad \lambda \notin \Gamma(E) \Longrightarrow \theta_\lambda E \in G_\varepsilon. \tag{6.7}$$

(iii) *The convergence in the definition of $W(E)$ is uniform w.r.t. $(x, E) \in G_\varepsilon$ and, writing* $\operatorname{div} E = 2\pi(\nu - 1)$,

$$W(E) \textit{ and } \frac{\nu(K_R)}{R^2} \textit{ are bounded uniformly w.r.t. } (x, E) \in G_\varepsilon \textit{ and } R > 1. \tag{6.8}$$

(iv) *We have*

$$\begin{gathered} d_{\mathcal{B}}(P, P'') < C\varepsilon(|\log\varepsilon| + 1) \\ \textit{where } P'' = \int_{\Sigma \times H_\varepsilon} \frac{1}{m_0(x)|K_{R_\varepsilon}|} \int_{\sqrt{m_0(x)}K_{R_\varepsilon} \setminus \Gamma(E)} \delta_x \otimes \delta_{\sigma_{m_0(x)}\theta_\mu E} \, d\mu \, d\widetilde{P}(x, E). \end{gathered} \tag{6.9}$$

Moreover, there exists a partition of H_ε *into* $\bigcup_{i=1}^{N_\varepsilon} H_\varepsilon^i$ *satisfying* $\operatorname{diam}(H_\varepsilon^i) < \eta_3$, *where* η_3 *is such that*

$$E \in H_\varepsilon, d_p(E, E') < \eta_3, m \in [\underline{m}, \overline{m}], \mu \in \sqrt{\overline{m}} K_{R_\varepsilon} \setminus \Gamma(E) \\ \implies \quad d_{\mathcal{B}}(\delta_{\sigma_m \theta_\mu E}, \delta_{\sigma_m \theta_\mu E'}) < \varepsilon; \tag{6.10}$$

and there exists for all i, $J_i \in H_\varepsilon^i$ *such that*

$$W(J_i) < \inf_{H_\varepsilon^i} W + \varepsilon. \tag{6.11}$$

At this point, denoting $\tilde{Q}$ the projection of $\tilde{P}$ under $E \mapsto \frac{1}{2\pi} \operatorname{div} E + 1$, we may always choose J_i such that $W(J_i) < \inf_{H_\varepsilon^i} \mathbb{W} + \varepsilon$.

PROOF OF LEMMA 6.6. *Step* 1: *Choice of* G_ε. Since $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ is Polish we can always find a compact set G_ε satisfying (6.6) and $P(G_\varepsilon^c) < \eta_0$. Then from Lemma 6.1, $P \llcorner G_\varepsilon$ (the restriction of P to G_ε) satisfies $d_{\mathcal{B}}(P, P \llcorner G_\varepsilon) < \varepsilon$.

From the translation invariance of $\widetilde{P}$ and for any λ, we have $\widetilde{P}(\Sigma \times \theta_\lambda G_\varepsilon) > 1 - \eta_0$ and therefore $d_{\mathcal{B}}(\widetilde{P}, \widetilde{P} \llcorner \theta_\lambda G_\varepsilon) < \varepsilon$. In view of (6.1), it follows that for any $\lambda \in \mathbb{R}^2$ we have $\|P - P_\lambda\| < \eta_0$ and then $d_{\mathcal{B}}(P, P_\lambda) < \varepsilon$, where

$$P_\lambda = \int_{\Sigma \times \theta_\lambda G_\varepsilon} \delta_x \otimes \delta_{\sigma_{m_0(x)} E} \, d\widetilde{P}(x, j) = \int_{\Sigma \times G_\varepsilon} \delta_x \otimes \delta_{\theta_\lambda \sigma_{m_0(x)} E} \, d\widetilde{P}(x, E).$$

Then using Lemma 6.3 we deduce that if $A \subset \mathbb{R}^2$ is any measurable set of positive measure, then

$$d_{\mathcal{B}}(P, P') < C\varepsilon\bigl(|\log \varepsilon| + 1\bigr) \\ \text{where } P' = \int_{\Sigma \times G_\varepsilon} \fint_A \delta_x \otimes \delta_{\theta_\lambda \sigma_{m_0(x)} E} \, d\lambda \, d\widetilde{P}(x, E). \tag{6.12}$$

Moreover, since P is $T_{\lambda(x)}$-invariant, choosing χ to be a smooth positive function with integral 1 supported in $B(0, 1)$, the ergodic theorem (as in Section 4.1 or see again [9]) ensures that for P-almost every (x, E) the limit

$$\lim_{R \to +\infty} \frac{1}{|K_R|} \int_{K_R} W\bigl(E(\lambda + \cdot), \chi(\lambda + \cdot)\bigr) \, d\lambda$$

exists. Then $\mathbf{1}_{K_R} * \chi$ is a family of functions which satisfies (2.21) with respect to the family of squares $\{K_R\}_R$, and from the definition of the renormalized energy relative to $\{K_R\}_R$ we may rewrite the limit above as

$$W(E) = \lim_{R \to +\infty} \frac{1}{|K_R|} W(E, \mathbf{1}_{K_R} * \chi). \tag{6.13}$$

By Egoroff's theorem, we may choose the compact set G_ε above to be such that, in addition to (6.12), the convergence in (6.13) is uniform on G_ε. In fact, since

$W(E) < +\infty$ and $\limsup_R \nu(K_R)/R^2 < +\infty$ for P-a.e. (x, E), where $\operatorname{div} E = 2\pi(\nu - 1)$, we may choose G_ε such that (6.8) holds.

The arguments above show that the properties (6.12), (6.8) can be satisfied for a compact set G_ε of measure arbitrarily close to 1. We choose G_ε such that (6.6) holds.

The next difficulty we have to face is that $\theta_\lambda E$ need not belong to G_ε if E does.

Step 2: *Choice of* H_ε. For $E \in G_\varepsilon$, let $\Gamma(E)$ be the set of λ's in $\sqrt{\overline{m}} K_{R_\varepsilon}$ such that $\theta_\lambda E \notin G_\varepsilon$. Since, from (6.6) and the translation-invariance of $\widetilde{P}$, for any $\lambda \in \mathbb{R}^2$ we have $\widetilde{P}(\Sigma \times \theta_\lambda(G_\varepsilon)^c) < \eta_0^2$, it follows from Fubini's theorem that

$$\int_{G_\varepsilon} |\Gamma(E)|\, d\widetilde{P}(x, E) = \int_{\sqrt{\overline{m}} K_{R_\varepsilon}} \widetilde{P}(\Sigma \times (\theta_\lambda G_\varepsilon)^c)\, d\lambda < 4\overline{m} R_\varepsilon^2 \min(\eta_0^2, \eta_0 \varepsilon).$$

Therefore, letting

$$H_\varepsilon = \{E \in G_\varepsilon : |\Gamma(E)| < 4\overline{m} R_\varepsilon^2 \eta_0\}, \tag{6.14}$$

we have that (6.6) holds.

Combining (6.6) and (6.14) with Lemma 6.1, we deduce from (6.12) that (6.9) holds, where we have used the fact that $\theta_\lambda \sigma_m E = \sigma_m \theta_{\sqrt{m}\lambda} E$ to change the integration variable to $\mu = \sqrt{m_0(x)}\lambda$ in (6.12).

Step 3: *Choice of* $J_1, \dots, J_{N_\varepsilon}$. We use the fact that G_ε is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, Lemma 6.2, and the fact that $(m, E) \mapsto \sigma_m E$ is continuous to find that there exists $\eta_4 > 0$ such that for any $m \in [\underline{m}, \overline{m}]$ and any $E \in G_\varepsilon$ it holds that

$$d_p(E, E') < \eta_4 \quad \Longrightarrow \quad d_{\mathcal{B}}(\delta_{\sigma_m E}, \delta_{\sigma_m E'}) < \varepsilon. \tag{6.15}$$

Moreover, from the continuity of $(\mu, E) \mapsto \theta_\mu E$, there exists $\eta_3 > 0$ such that

$$\begin{aligned} &E \in G_\varepsilon, d_p(E, E') < \eta_3, \\ &\mu \in \sqrt{\overline{m}} K_{R_\varepsilon} \quad \Longrightarrow \quad d_p(\theta_\mu E, \theta_\mu E') < \eta_4. \end{aligned} \tag{6.16}$$

If $E \in H_\varepsilon$ and $\mu \in K \setminus \Gamma(E)$, then $\theta_\mu E \in G_\varepsilon$ hence applying (6.15), we get (6.10).

Now we cover the relatively compact set H_ε by a finite number of balls $B_1, \dots, B_{N_\varepsilon}$ of radius $\eta_3/2$, derive from it a partition of H_ε by sets with diameter less than η_3, by letting $H_\varepsilon^1 = B_1 \cap H_\varepsilon$ and

$$H_\varepsilon^{i+1} = B_{i+1} \cap H_\varepsilon \setminus (B_1 \cup \cdots \cup B_i)$$

we then have

$$H_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} H_\varepsilon^i, \qquad \operatorname{diam}(H_\varepsilon^i) < \eta_3, \tag{6.17}$$

where the union is disjoint. Then we may choose $J_i \in H_\varepsilon^i$ such that (6.11) holds. $\square$

6.3. *Completing the construction.* *Step* 1: *Choice of* R_ε. We apply Proposition 6.4 with $G = G_\varepsilon$ and $\sqrt{m}R$, where $m \in [\underline{m}, \overline{m}]$. The proposition yields $\eta_2 > 0$, $R_\varepsilon > 1$ such that for any $E \in G_\varepsilon$ and any $m \in [\underline{m}, \overline{m}]$ and any rectangle L centered at 0 with side-lengths in $[2\sqrt{m}R_\varepsilon, 2\sqrt{m}R_\varepsilon(1+\eta_2)]$, (6.4) is satisfied for some E_L, with R replaced by $\sqrt{m}R_\varepsilon$. The reason for including $\sqrt{m}$ is that we will need to scale the construction to account for the varying weight $m_0(x)$.

Since our rectangles will be obtained from Lemma 6.5 and we wish to use the approximation by E_L in them, we choose R_ε large enough so that if $m \in [\underline{m}, \overline{m}]$ and L is a rectangle centered at zero with side-lengths in $[2\sqrt{m}R_\varepsilon, 2\sqrt{m}R_\varepsilon(1 + C_0/R_\varepsilon^2)]$ then

$$\frac{C_0}{R_\varepsilon^2} < \eta_2, \qquad \frac{C_1}{R_\varepsilon^2} < \eta_0, \tag{6.18}$$
$$K_{\sqrt{m}R_\varepsilon(1-\eta_0)} \subset \{x : d(x, L^c) > \sqrt{m}R_\varepsilon^{3/4}\} \subset K_{\sqrt{m}R_\varepsilon(1+\eta_0)},$$

where C_0 is the constant in Lemma 6.5, $C_1 \geq 1$ is to be determined later, and η_0 is the constant in Lemma 6.1.

If $\lambda \in K_{\sqrt{m}R_\varepsilon(1-\eta_0)}$ and since $E = E_L$ if $d(x, L^c) > \sqrt{m}R_\varepsilon^{3/4}$, we deduce from (6.18) that $\theta_\lambda E_L = \theta_\lambda E$ in $B(0, \sqrt{m}R_\varepsilon^{3/4})$, so that from the definition of d_p, taking R_ε larger if necessary,

$$\forall E \in G_\varepsilon, m \in [\underline{m}, \overline{m}], \lambda \in K_{\sqrt{m}R_\varepsilon(1-\eta_0)}, \tag{6.19}$$
$$d_p(\theta_\lambda \sigma_m E, \theta_\lambda \sigma_m E_L) < \frac{\eta_1}{2},$$

where η_1 comes from Lemma 6.2 applied on $\{\sigma_m E : m \in [\underline{m}, \overline{m}], E \in G_\varepsilon\}$, that is, such that

$$m \in [\underline{m}, \overline{m}], E \in G_\varepsilon, E' \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2) \quad \text{and} \tag{6.20}$$
$$d_p(E, E') < \eta_1 \quad \Longrightarrow \quad d_{\mathcal{B}}(\delta_{\sigma_m E}, \delta_{\sigma_m E'}) < \varepsilon.$$

Step 2: *Choice of* q_ε *and the rectangles.* We choose an integer q_ε large enough so that

$$\frac{N_\varepsilon}{C_1 q_\varepsilon^2} < \eta_0, \qquad \frac{N_\varepsilon}{q_\varepsilon^2} \times \max_{\substack{0 \leq i \leq N_\varepsilon \\ \underline{m} \leq m \leq \overline{m}}} W_K(\sigma_m J_i) < \varepsilon, \tag{6.21}$$

where $C_1 > 1$ is to be determined later. We apply Lemma 6.5 with R_ε, q_ε and N_ε to obtain for any n a collection $\mathcal{K}_n$ of rectangles (we omit to mention the ε dependence) which cover most of Σ', and we also apply Lemma 6.6. We rewrite P'' given by (6.9) as

$$P'' = \sum_{K \in \mathcal{K}_n} \int_{K/\sqrt{n} \times H_\varepsilon} \frac{1}{m_0(x)|K_{R_\varepsilon}|} \tag{6.22}$$
$$\times \int_{\sqrt{m_0(x)}K_{R_\varepsilon} \setminus \Gamma(E)} \delta_x \otimes \delta_{\sigma_{m_0(x)}\theta_\mu E}\, d\mu\, d\widetilde{P}(x, E).$$

Now we claim that if n is large enough and $x \in K/\sqrt{n}$, $E \in H_\varepsilon^i$, $\mu \in \sqrt{m_0(x)}K_{R_\varepsilon} \setminus \Gamma(E)$, then

$$d_{\mathcal{B}}(\delta_x \otimes \delta_{\sigma_{m_0(x)}\theta_\mu E}, \delta_{x_K} \otimes \delta_{\sigma_{m_K}\theta_\mu J_i}) < 2\varepsilon, \tag{6.23}$$

where x_K is the center of $K/\sqrt{n}$ and m_K is the average of m_0 over $K/\sqrt{m}$. Indeed, since m_0 is C^1 we have $|x - x_K| < C/\sqrt{n}$, $|m_0(x) - m_K| < C/\sqrt{n}$ thus if n is large enough, since $\theta_\mu E \in G_\varepsilon$ we find

$$d_{\mathcal{B}}(\delta_x \otimes \delta_{\sigma_{m_0(x)}\theta_\mu E}, \delta_{x_K} \otimes \delta_{\sigma_{m_K}\theta_\mu E}) < \varepsilon.$$

Moreover, since $d_p(E, J_i) < \eta_3$, we deduce from (6.10) that

$$d_{\mathcal{B}}(\delta_{x_K} \otimes \delta_{\sigma_{m_K}\theta_\mu E}, \delta_{x_K} \otimes \delta_{\sigma_{m_K}\theta_\mu J_i}) < \varepsilon,$$

which together with the previous estimate proves (6.23).

Using (6.23) together with Lemmas 6.2, 6.1 and (6.7), we deduce from (6.9) and (6.22) that $d_{\mathcal{B}}(P, P''') < C\varepsilon(|\log\varepsilon| + 1)$, where

$$\begin{aligned} P''' &= \sum_{\substack{K\in\mathcal{K}_n \\ 1\le i\le N_\varepsilon}} \int_{K/\sqrt{n}\times H_\varepsilon^i} \fint_{\sqrt{m_K}K_{R_\varepsilon}} \delta_{x_K} \otimes \delta_{\sigma_{m_K}\theta_\mu J_i}\, d\mu\, d\widetilde{P}(x, E) \\ &= \sum_{\substack{K\in\mathcal{K}_n \\ 1\le i\le N_\varepsilon}} p_{i,K} \fint_{\sqrt{m_K}K_{R_\varepsilon}} \delta_{x_K} \otimes \delta_{\sigma_{m_K}\theta_\mu J_i}\, d\mu, \end{aligned} \tag{6.24}$$

where

$$p_{i,K} = \widetilde{P}\left(\frac{K}{\sqrt{n}} \times H_\varepsilon^i\right). \tag{6.25}$$

Step 3: *Choice of subrectangles and vector field E_n*. We now replace $p_{i,K}$ in the definition (6.24) by

$$\frac{|K|}{q_\varepsilon^2|\Sigma'|} n_{i,K} \qquad \text{where } n_{i,K} = \left[\frac{q_\varepsilon^2|\Sigma'|}{|K|} p_{i,K}\right]. \tag{6.26}$$

We have, since $\widetilde{P}(\frac{K}{\sqrt{n}} \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)) = |K|/|\Sigma'|$,

$$\sum_{k=1}^{N_\varepsilon} n_{i,K} \le \frac{q_\varepsilon^2|\Sigma'|}{|K|} \widetilde{P}\left(\frac{K}{\sqrt{n}} \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)\right) = q_\varepsilon^2 \tag{6.27}$$

and

$$\left|\frac{|K_{R_\varepsilon}|}{|\Sigma'|} n_{i,K} - p_{i,K}\right| < C\left(\frac{|K|}{q_\varepsilon^2|E'|} + \frac{n_{i,K}}{R_\varepsilon^2|\Sigma'|}\right).$$

Summing with respect to i and K, using the facts that $\sum_{K\in\mathcal{K}_n}|K|<|\Sigma'|$, (6.27), and the fact that the cardinal of $\mathcal{K}_n$ is $\frac{|\Sigma'|}{4q_\varepsilon^2 R_\varepsilon^2}$, we find

$$\sum_{1\le i\le N_\varepsilon, K\in\mathcal{K}_n}\left|\frac{|K_{R_\varepsilon}|}{|\Sigma'|}n_{i,K}-p_{i,K}\right|<C\left(\frac{N_\varepsilon}{q_\varepsilon^2}+\frac{1}{R_\varepsilon^4}\right).$$

We may always choose C_1 large enough in (6.18) and (6.21) so that the right-hand side is $<\eta_0$. Then Lemma 6.1 implies that $d_{\mathcal{B}}(P,P^{(4)})<C\varepsilon(|\log\varepsilon|+1)$ is still true after replacing $p_{i,K}$ by $\frac{|K_{R_\varepsilon}|}{|\Sigma'|}n_{i,K}$ in (6.24), that is, where

$$P^{(4)}=\frac{1}{|\Sigma'|}\sum_{\substack{K\in\mathcal{K}_n\\1\le i\le N_\varepsilon}}\frac{n_{i,K}}{m_K}\int_{\sqrt{m_K}K_{R_\varepsilon}}\delta_{x_K}\otimes\delta_{\sigma_{m_K}\theta_\mu J_i}\,d\mu. \tag{6.28}$$

Next, we divide each $K\in\mathcal{K}_n$ into a collection $\mathcal{L}_K$ of q_ε^2 identical subrectangles in the obvious way and we partition $\mathcal{L}_K$ into collections $\mathcal{L}_{K,i}$, $0\le i\le N_\varepsilon$ such that if $k\ge 1$ then $\mathcal{L}_{K,i}$ contains $n_{i,K}$ subrectangles. This is clearly possible from (6.27). If the inequality is strict we put the extra subrectangles in $\mathcal{L}_{K,0}$, there will be $n_{0,K}$ of them and then

$$\sum_{k=0}^{N_\varepsilon}n_{k,K}=q_\varepsilon^2. \tag{6.29}$$

We rewrite (6.28) as

$$P^{(4)}=\frac{1}{|\Sigma'|}\sum_{\substack{K\in\mathcal{K}_n\\1\le i\le N_\varepsilon\\ \tilde{L}\in\mathcal{L}_{K,i}}}\frac{1}{m_K}\int_{\sqrt{m_K}K_{R_\varepsilon}}\delta_{x_K}\otimes\delta_{\sigma_{m_K}\theta_\mu J_i}\,d\mu. \tag{6.30}$$

Now, for $\tilde{L}\in\mathcal{L}_{K,i}$, let $L=\sqrt{m_K}(\tilde{L}-x_{\tilde{L}})$, where $x_{\tilde{L}}$ denotes the center of $\tilde{L}$. From Lemma 6.5, a rectangle $K\in\mathcal{K}_n$ has side-lengths between $2q_\varepsilon R_\varepsilon$ and $2q_\varepsilon R_\varepsilon(1+C_0/R_\varepsilon^2)$. Therefore, L is a rectangle centered at zero with side-lengths between $2\sqrt{m_K}R_\varepsilon$ and $2\sqrt{m_K}R_\varepsilon(1+C_0/R_\varepsilon^2)$, and (6.19) holds.

This, and the results of Lemma 6.6, allow us to apply Proposition 6.4 on L to any J_i, $1\le i\le N_\varepsilon$. Note that $|L|\in\mathbb{N}$ follows from the fact that

$$|L|=m_K|\tilde{L}|=\fint_K m_0'\frac{|K|}{q_\varepsilon^2}=\frac{1}{q_\varepsilon^2}\int_K m_0'$$

and (6.5). In this way, we define currents $J_{i,L}$ which satisfy (6.4) and (6.19). We claim that, as a consequence of the latter, we have

$$E'=J_{i,L}\text{ on }L$$

$$\Longrightarrow\quad d_{\mathcal{B}}\left(\fint_{\sqrt{m_K}K_{R_\varepsilon}}\delta_{x_K}\otimes\delta_{\sigma_{m_K}\theta_\mu J_i}\,d\mu,\right. \tag{6.31}$$

$$\frac{1}{m_K|K_{R_\varepsilon}|}\int_L \delta_{x_K}\otimes\delta_{\sigma_{m_K}\theta_\mu E'}\,d\mu\Big)$$
$$< C\varepsilon.$$

This goes as follows: (i) using Lemma 6.1 and (6.7), (6.18), we find that integrating on $\sqrt{m_K}K_{(1-\eta_0)R_\varepsilon}\setminus\Gamma(J_i)$ instead of $\sqrt{m_K}K_{R_\varepsilon}$ and L induces an error of $C\varepsilon$. (ii) From (6.19), and (6.20) applied to $\theta_\mu J_i$ and $\theta_\mu E'$ we have $d_{\mathcal{B}}(\delta_{\theta_\mu J_i},\delta_{\theta_\mu E'})<\varepsilon$, and thus in view of Lemma 6.3 we may replace $\theta_\mu J_i$ by $\theta_\mu E'$ in the integral with an error of $C\varepsilon|\log\varepsilon|$ at most. (iii) Using (6.18), (6.7) and Lemma 6.1 again, we may integrate back on $\sqrt{m_K}K_{R_\varepsilon}$ and L rather than on $K_{(1-\eta_0)R_\varepsilon}\setminus\Gamma(J_i)$, with an additional error of $C\varepsilon$. This proves (6.31).

Combining (6.31) with (6.30) and $d_{\mathcal{B}}(P,P^{(4)})<C\varepsilon(|\log\varepsilon|+1)$, using Lemma 6.3 we find $d_{\mathcal{B}}(P,P^{(5)})<C\varepsilon(|\log\varepsilon|+1)$, where

$$\begin{aligned}P^{(5)}&=\frac{1}{|\Sigma'|}\sum_{\substack{K\in\mathcal{K}_n\\1\le i\le N_\varepsilon\\ \tilde{L}\in\mathcal{L}_{K,i}}}\frac{1}{m_K}\int_L\delta_{x_K}\otimes\delta_{\sigma_{m_K}\theta_\mu\tilde{J}_{i,L}}\,d\mu\\&=\frac{1}{|\Sigma'|}\sum_{\substack{K\in\mathcal{K}_n\\1\le i\le N_\varepsilon\\ \tilde{L}\in\mathcal{L}_{K,i}}}\int_{L/\sqrt{m_K}}\delta_{x_K}\otimes\delta_{\theta_\lambda\sigma_{m_K}\tilde{J}_{i,L}}\,d\lambda,\end{aligned}\tag{6.32}$$

where the last equality follows by changing variables to $\lambda=\mu/\sqrt{m_K}$, and where $\tilde{J}_{i,L}$ denotes an arbitrarily chosen element of $L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ such that $\tilde{J}_{i,L}=J_{i,L}$ on L, the constant C being independent of this choice.

If we choose an arbitrary J_0 in $\mathcal{A}_1$ and let the sum in (6.32) range over $0\le i\le N_\varepsilon$ instead of $1\le i\le N_\varepsilon$ we obtain a measure $P^{(6)}$ such that, by (6.21),

$$\|P^{(5)}-P^{(6)}\|\le\frac{1}{|\Sigma'|}\sum_{K\in\mathcal{K}_n}\frac{N_\varepsilon|K|}{q_\varepsilon^2}\le\eta_0,$$

hence using Lemma 6.1 we have $d_{\mathcal{B}}(P^{(5)},P^{(6)})<\varepsilon$ and then $d_{\mathcal{B}}(P,P^{(6)})<C\varepsilon(|\log\varepsilon|+1)$.

We now define the vector field $E_n^{\mathrm{int}}:\mathbb{R}^2\to\mathbb{R}^2$ by letting $E_n^{\mathrm{int}}(x)=\sigma_{m_K}J_{i,L}(x-x_{\tilde{L}})$ on $\tilde{L}=x_{\tilde{L}}+L/\sqrt{m_K}$, for every $K\in\mathcal{K}_n$, $0\le i\le N_\varepsilon$ and $\tilde{L}\in\mathcal{L}_{K,i}$. Then, for every $L\in\mathcal{L}_{K,i}$ we have $E_n^{\mathrm{int}}(x_{\tilde{L}}+\cdot)=\sigma_{m_K}J_{i,L}$ on $\tilde{L}$, therefore, we may choose $\tilde{J}_{i,L}=\sigma_{1/m_K}E_n^{\mathrm{int}}(x_{\tilde{L}}+\cdot)$ in (6.32), and then we may summarize the above by writing

$$\begin{gathered}d_{\mathcal{B}}\big(P,P^{(6)}\big)<C\varepsilon\big(|\log\varepsilon|+1\big),\\P^{(6)}=\frac{1}{|\Sigma'|}\sum_{K\in\mathcal{K}_n}\int_K\delta_{x_K}\otimes\delta_{\theta_\lambda E_n^{\mathrm{int}}}\,d\lambda.\end{gathered}\tag{6.33}$$

Note that since $J_{i,L} = 0$ outside L, we also have

$$E_n^{\rm int} = \sum_{\substack{K\in\mathcal{K}_n\\ 1\le i\le N_\varepsilon\\ \tilde L\in\mathcal{L}_{K,i}}} \sigma_{m_K} J_{i,L}(\cdot - x_{\tilde L}),$$

$$\operatorname{div} E_n^{\rm int} = 2\pi \sum_{\substack{K\in\mathcal{K}_n\\ p\in\Lambda_K}} (\delta_p - m_K), \tag{6.34}$$

where Λ_K is a finite subset of the interior of K. The second equation is satisfied in the sense of distributions on $\mathbb{R}^2$.

Step 4: *Treating the boundary.* Let $\hat\Sigma' := \Sigma' \setminus \bigcup_{K\in\mathcal{K}_n} K$. We let $t\in[0,\ell\sqrt{n}]$ denote arc length on $\partial\Sigma'$—where ℓ is the length of $\partial\Sigma$—and s denote the distance to $\partial\Sigma'$, so that (t,s) is a C^1 coordinate system on $\{x\in\Sigma' : d(x,(\Sigma')^c) < c\sqrt{n}\}$, if $c>0$ is small enough, since the boundary of Σ is C^1 by (2.5). We let C_t denote the curvilinear rectangle of points with coordinates in $[0,t]\times[0,C\overline{R}_\varepsilon]$, where $\overline{R}_\varepsilon = q_\varepsilon R_\varepsilon$ and C is large enough so that $\hat\Sigma' \subset \{x\in\Sigma' : d(x,\partial\Sigma') < C\overline{R}_\varepsilon\}$, and define $m(t) = \int_{C_t\cap\hat\Sigma'} m_0'$. Since the distance of $\bigcup_{K\in\mathcal{K}_n} K$ to a given $x\in\partial\Sigma'$ is between $\overline{R}_\varepsilon$ and $C_0\overline{R}_\varepsilon$ from Lemma 6.5 and since m_0' is bounded above and below by (2.8), the derivative of $t\mapsto m(t)$ is between $\overline{R}_\varepsilon/C$ and $C\overline{R}_\varepsilon$ for some $C>0$ large enough.

We let

$$k_\varepsilon = \left[\frac{\ell\sqrt{n}}{\overline{R}_\varepsilon}\right] \tag{6.35}$$

and choose $0 = t_0,\dots,t_{k_\varepsilon} = \ell\sqrt{n}$ to be such that

$$m(t_l) = \left[\frac{l}{k_\varepsilon} m(\ell\sqrt{n})\right].$$

We note that indeed $t_{k_\varepsilon} = \ell\sqrt{n}$: Since the integral of m_0' on each square $K\in\mathcal{K}_n$ is an integer as well as the integral on Σ', we have $\int_{\hat\Sigma'} m_0' \in \mathbb{N}$ and, therefore, $m(\ell\sqrt{n})\in\mathbb{N}$.

From the above remark about the derivative of $t\to m(t)$, we deduce that $\frac{m(\ell\sqrt{n})}{\ell\sqrt{n}}$ belongs to the interval $[\overline{R}_\varepsilon/C, C\overline{R}_\varepsilon]$ for some $C>0$ and then it is easy to deduce that if $\sqrt{n}$ is large enough compared to $\overline{R}_\varepsilon$ then

$$n_l := m(t_{l+1}) - m(t_l) \in [\overline{R}_\varepsilon^2/C, C\overline{R}_\varepsilon^2], \qquad t_{l+1} - t_l \in [\overline{R}_\varepsilon/C, C\overline{R}_\varepsilon].$$

This means that the side-lengths of the curvilinear rectangle $C_{t_{l+1}}\setminus C_{t_l}$ are comparable to $\overline{R}_\varepsilon$, and that the number of points n_l to put there in is of order $\overline{R}_\varepsilon^2$.

We may then include each of the sets $K_l := \hat\Sigma' \cap (C_{t_{l+1}}\setminus C_{t_l})$ in a ball B_l with radius in $[\overline{R}_\varepsilon/C, C\overline{R}_\varepsilon]$ and we may also choose a set of n_l points Λ_l which are at distance at least $1/C$ from each other and the complement of K_l. Let $E_l = -\nabla H$, where H solves $-\Delta H = 2\pi(\sum_{p\in\Lambda_l}\delta_p - m_l)$ in B_l and $\nabla H\cdot\vec{\nu} = 0$ on ∂B_l, where

$$m_l = \frac{n_l}{|K_l|}\mathbf{1}_{K_l}.$$

Then we have $\operatorname{div} E_l = 2\pi(\sum_{p\in\Lambda_l}\delta_p - m_l)$ in B_l and $E_l\cdot\vec{\nu} = 0$ on ∂B_l and we claim that for any $q \geq 1$,

$$W(E_l, \mathbf{1}_{B_l}) \leq C_\varepsilon, \qquad \|E_l\|_{L^q(B_l\setminus K_l)} \leq C_{\varepsilon,q}, \tag{6.36}$$

where the constants do not depend on n, but do depend on ε through $\overline{R}_\varepsilon$. This is proved by noting that these quantities are finite, and that a compactness argument shows that the bound is uniform for any choice of points which are at distance at least $1/C$ from each other and the complement of some $K_l \subset B_l$, using, for instance, the explicit formulas for W in [31]. Note that because the sets $\{K_l\}$ and the rectangles $\{K\}$ are disjoint, have measure between $\overline{R}_\varepsilon^2/C$ and $C\overline{R}_\varepsilon^2$ and diameter between $\overline{R}_\varepsilon/C$ and $C\overline{R}_\varepsilon$, we know that their overlap is bounded by a constant C independent of ε, n.

Step 5: *Rectification of the weight.* We rectify the weights m_K, m_l: For $K \in \mathcal{K}_n$ we let H_K solve $-\Delta H_K = 2\pi(m_0' - m_K)$ on K and $\nabla H_K\cdot\vec{\nu} = 0$ on ∂K. Similarly we let H_l solve $-\Delta H_l = 2\pi(m_0'\mathbf{1}_{K_l} - m_l)$, $\nabla H_l\cdot\vec{\nu} = 0$. By elliptic regularity, we deduce for any $q > 1$ that $\|\nabla H_K\|_{L^q(K)}$ (resp., $\|\nabla H_l\|_{L^q(B_l)}$) is bounded by $C_{q,\varepsilon}\|m_0' - m_K\|_{L^\infty(K)}$ (resp., $C_{q,\varepsilon}\|m_0' - m_l\|_{L^\infty(B_l)}$). Since m_0 is C^1, we have $|\nabla m_0'| \leq C/\sqrt{n}$, therefore, $\|m_0' - m_K\|_{L^\infty(K)} \leq C\overline{R}_\varepsilon/\sqrt{n}$, while $\|m_0' - m_l\|_{L^\infty(B_l)} \leq C$. We deduce that

$$\|\nabla H_K\|_{L^q} \leq \frac{C_{q,\varepsilon}}{\sqrt{n}}, \qquad \|\nabla H_l\|_{L^q} \leq C_{q,\varepsilon}. \tag{6.37}$$

We let

$$E_K = E_n^{\mathrm{int}}{}_{|K} \tag{6.38}$$

and

$$\begin{aligned}\overline{E}_n &= E_n^{\mathrm{int}} + \sum_{K\in\mathcal{K}_n} -\nabla H_K + \sum_{i=1}^{k_\varepsilon} -\nabla H_l\\ &= \sum_{K\in\mathcal{K}_n} E_K - \nabla H_K + \sum_{l=1}^{k_\varepsilon} E_l - \nabla H_l,\end{aligned} \tag{6.39}$$

$$\Lambda_n = \bigcup_{K\in\mathcal{K}} \Lambda_K \bigcup_{l=1}^{k_\varepsilon} \Lambda_l,$$

where E_K and ∇H_K are set to 0 outside K and similarly for E_l, ∇H_l outside B_l. Then $\operatorname{div}\overline{E}_n = 2\pi(\sum_{p\in\Lambda_n}\delta_p - m_0')$ in $\mathbb{R}^2$. This completes the construction of $\overline{E}_n$.

6.4. *Estimating the energy.* *Step* 1: *Energy estimate.* We have

$$W(E_K, \mathbf{1}_K) = \sum_{\substack{0 \le i \le N_\varepsilon \\ \tilde{L} \in \mathcal{L}_{K,i}}} W\big(\sigma_{m_K} J_{i,L}(\cdot - x_{\tilde{L}}), \tilde{L}\big).$$

From (6.4) we find, letting $L = \sqrt{m_K}(\tilde{L} - x_{\tilde{L}})$, using (6.29) and $|L| = |K|/q_\varepsilon^2$ that

$$\begin{aligned} W(E_K, \mathbf{1}_K) &= \sum_{\substack{0 \le i \le N_\varepsilon \\ \tilde{L} \in \mathcal{L}_{K,i}}} W(\sigma_{m_K} J_{i,L}, \mathbf{1}_{\tilde{L} - x_{\tilde{L}}}) \\ &\le |K| \left(\sum_{i=0}^{N_\varepsilon} \frac{n_{i,K}}{q_\varepsilon^2} W(\sigma_{m_K} J_i) + C\varepsilon \right). \end{aligned} \tag{6.40}$$

We estimate the integral of $|\overline{E}_n|^2$ on $\mathbb{R}^2 \setminus \bigcup_{p \in \Lambda_n} B(p, \eta)$. From (6.39), this integral involves on the one hand the square terms

$$\sum_{l=1}^{k_\varepsilon} \int_{(B_l)_\eta} |E_l - \nabla H_l|^2 + \sum_{K \in \mathcal{K}_n} \int_{K_\eta} |E_K - \nabla H_K|^2, \tag{6.41}$$

where $K_\eta = K \setminus \bigcup_{p \in \Lambda_n} B(p, \eta)$ and similarly for $(B_l)_\eta$, and on the other hand the rectangle terms

$$\sum_{\substack{K, K' \in \mathcal{K}_n \\ K \ne K'}} \int_{K_\eta \cap K'_\eta} (E_K - \nabla H_K) \cdot (E_{K'} - \nabla H_{K'}) + \sum_{1 \le l \ne i \le k_\varepsilon} + \cdots + \sum_{\substack{K \in \mathcal{K}_n \\ 1 \le l \le k_\varepsilon}} + \cdots.$$

We estimate the latter as follows: Since the rectangles in $\mathcal{K}_n$ do not overlap, the first sum is equal to zero. A nonzero rectangle term must involve some B_l, and moreover a given B_l can only be present in a number of terms bounded independently of n, ε because the overlap of the balls B_l and the rectangles K is bounded. Thus, from (6.35) we have at most $C\sqrt{n}/\overline{R}_\varepsilon$ nonzero rectangle terms. Moreover, since the K_l's are disjoint, and disjoint from the K's, in a rectangle term involving $B_l \cap K$ the integral can be taken over $B_l \setminus K_l$, and in a term involving $B_l \cap B_i$ it can be taken over $(B_l \cap B_i \setminus K_i) \cup (B_i \cap B_l \setminus K_l)$.

In any case, we use Hölder's inequality and the bound $\|E_l - \nabla H_l\|_{L^q(B_l \setminus K_l)} \le C_{\varepsilon,q}$ for some $q > 2$, which follows from (6.36), (6.37), together with the bound

$$\|E_l - \nabla H_l\|_{L^{q'}(B_l)}, \qquad \|E_K - \nabla H_K\|_{L^{q'}(K)} \le C_{\varepsilon,q},$$

which follows from (6.40), (6.36) using Lemma 3.9, to conclude that each rectangle term is bounded by C_ε and then that their sum is $O(\sqrt{n})$, meaning a quantity bounded by a constant depending on ε times $\sqrt{n}$.

The limit as $\eta \to 0$ of the terms in (6.41) is estimated as above by expanding the squares and using Hölder's inequality with (6.37), (6.36), (6.40), together with the bound (6.35) to show that

$$\lim_{\eta\to 0}\frac{1}{2}\left(\sum_{l=1}^{k_\varepsilon}\int_{(B_l)_\eta}|E_l-\nabla H_l|^2+\sum_{K\in\mathcal{K}_n}\int_{K_\eta}|E_K-\nabla H_K|^2+\pi\#\Lambda_n\log\eta\right)$$
$$\le\sum_{K\in\mathcal{K}_n}W(E_K,\mathbf{1}_K)+O(\sqrt{n}).$$

In view of the bound $O(\sqrt{n})$ for the rectangle terms and (6.40) we find using (6.29) that

$$W(\overline{E}_n,\mathbf{1}_{\mathbb{R}^2})\le\sum_{\substack{K\in\mathcal{K}_n\\0\le i\le N_\varepsilon}}|K|\frac{n_{i,K}}{q_\varepsilon^2}W(\sigma_{m_K}J_i)+Cn\varepsilon+O(\sqrt{n}).\tag{6.42}$$

Step 2: *We proceed to estimating* $W(\overline{E}_n,\mathbf{1}_{\mathbb{R}^2})$. We have, using (6.26), (6.25), (6.21), then the fact that $m_0'-m_K\le C\overline{R}_\varepsilon/\sqrt{n}$ on K, then (6.11) with (2.24), then (6.11) and finally (6.1), that

$$\begin{aligned}
&\sum_{i=1}^{N_\varepsilon}\frac{|K|n_{i,K}}{q_\varepsilon^2}W(\sigma_{m_K}J_i)\\
&\quad\le|\Sigma'|\sum_{i=1}^{N_\varepsilon}\widetilde{P}\left(\frac{K}{\sqrt{n}}\times H_\varepsilon^i\right)W(\sigma_{m_K}J_i)+|K|\varepsilon\\
&\quad\le|\Sigma'|\sum_{i=1}^{N_\varepsilon}\int_{(K/\sqrt{n})\times H_\varepsilon^i}W(\sigma_{m_0'(x)}J_i)\,d\widetilde{P}(x,E)+|K|\left(\frac{C}{\sqrt{n}}+\varepsilon\right) \qquad (6.43)\\
&\quad\le|\Sigma'|\sum_{i=1}^{N_\varepsilon}\int_{(K/\sqrt{n})\times H_\varepsilon^i}W(\sigma_{m_0'(x)}E)\,d\widetilde{P}(x,E)+|K|\left(\frac{C}{\sqrt{n}}+C\varepsilon\right)\\
&\quad=|\Sigma'|\int_{(K/\sqrt{n})\times H_\varepsilon}W(E)\,dP(x,E)+|K|\left(\frac{C}{\sqrt{n}}+C\varepsilon\right).
\end{aligned}$$

Here, we have used the fact that W is bounded below by some (negative) constant, a fact proved in [42] that we use below several times.

We proceed by estimating $n_{0,K}$. From (6.26), we deduce that

$$\sum_{i=1}^{N_\varepsilon}(n_{i,K}+1)\ge\frac{q_\varepsilon^2|\Sigma'|}{|K|}\widetilde{P}\left(\frac{K}{\sqrt{n}}\times H_\varepsilon\right)\ge\frac{q_\varepsilon^2|\Sigma'|}{|K|}\left(\frac{|K|}{|\Sigma'|}-\widetilde{P}\left(\frac{K}{\sqrt{n}}\times H_\varepsilon^c\right)\right),$$

and then it follows from (6.29) that

$$n_{0,K}=q_\varepsilon^2-\sum_{i=1}^{N_\varepsilon}n_{i,K}\le N_\varepsilon+\frac{q_\varepsilon^2|\Sigma'|}{|K|}\widetilde{P}\left(\frac{K}{\sqrt{n}}\times H_\varepsilon^c\right).$$

Summing over $K \in \mathcal{K}_n$, using the fact that

$$\Big|\Sigma' \setminus \bigcup_{\mathcal{K}_n} K\Big| < C_\varepsilon \sqrt{n} \tag{6.44}$$

and then (6.21), (6.6), we find that

$$\sum_{K \in \mathcal{K}} \frac{|K|}{q_\varepsilon^2} n_{0,K} W(\sigma_{m_K} J_0) \le C|\Sigma'|\Big(\tilde{P}(\Sigma \times H_\varepsilon^c) + \frac{1}{\sqrt{n}} + \varepsilon\Big) \le Cn\Big(\frac{C_\varepsilon}{\sqrt{n}} + \varepsilon\Big).$$

Summing (6.43) with respect to $K \in \mathcal{K}_n$ and adding the above estimate we find, in view of (6.44), (6.42) and (6.6) that

$$W(\overline{E}_n, \mathbf{1}_{\mathbb{R}^2}) \le n|\Sigma| \int_{\Sigma \times L^p_{\mathrm{loc}}} W(E)\, dP(x, E) + Cn\Big(\varepsilon + \frac{C_\varepsilon}{\sqrt{n}}\Big). \tag{6.45}$$

Note that at this point if we had chosen J_i such that $W(J_i) < \inf_{H_\varepsilon^i} \mathbb{W} + \varepsilon$, we obtain

$$W(\overline{E}_n, \mathbf{1}_{\mathbb{R}^2}) \le n|\Sigma| \int_{\Sigma \times \mathcal{M}_+} \mathbb{W}(\nu)\, dQ(x, \nu) + Cn\Big(\frac{C_\varepsilon}{\sqrt{n}} + \varepsilon\Big).$$

Step 3: *Energy bound for* $(x_1, \dots, x_n)$. From (6.45), the constructed fields $\{\overline{E}_n\}$ and points $\{\Lambda_n\}_n$ satisfy $\operatorname{div} \overline{E}_n = 2\pi(\sum_{p \in \Lambda_n} \delta_p - m_0')$ in $\mathbb{R}^2$ with $\#\Lambda_n = n$ [cf. item (iii) in Proposition 6.4] and

$$\limsup_n \frac{W(\overline{E}_n, \mathbf{1}_{\mathbb{R}^2})}{n} \le |\Sigma| \int W(E)\, dP(x, E) + C\varepsilon. \tag{6.46}$$

Now let $\{x_i\}_i = \{p/\sqrt{n}\}_{p \in \Lambda_n}$ be the points in Λ_n in the initial scale. The next step is to show that we can project $\overline{E}_n$ onto curl-free vectors and decrease its energy. This is well known is we replace the renormalized energy with the square of the L^2 norm, there is only a minor difference due to the fact that we take the finite part. To see this, let H_n' be the unique solution of $-\Delta H_n' = \operatorname{div} \overline{E}_n$ which decays at infinity. Then $\overline{E}_n = -\nabla H_n' + \nabla^\perp f_n$ for some function f_n. But $\overline{E}_n = 0$ outside of Σ, by construction, while H_n' decays fast at infinity since $\operatorname{div} E_n$ has integral 0. Letting $U_\eta = \bigcup_{p \in \Lambda_n} B(p, \eta)$, we first have

$$\begin{aligned}
&\int_{B_R \setminus U_\eta} |\nabla^\perp f_n - \nabla H_n'|^2 - \int_{B_R \setminus U_\eta} |\nabla H_n'|^2 \\
&\quad = -2 \int_{B_R \setminus U_\eta} \nabla^\perp f_n \cdot \nabla H_n' + \int_{B_R \setminus U_\eta} |\nabla f_n|^2 \\
&\quad \ge -2 \int_{B_R \setminus U_\eta} \nabla^\perp f_n \cdot \nabla H_n'.
\end{aligned}$$

Since $\overline{E}_n \in L^q_{\mathrm{loc}}$ for any $q < 2$ and since $f_n \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2)$ for all q, the last term on the right-hand side converges as $\eta \to 0$ to the integral over B_R. Also integrating by parts, using the Jacobian structure and the decay of f_n and H_n', we have $\int_{B_R} \nabla^\perp f_n \cdot \nabla H_n' \to 0$ as $R \to +\infty$. Therefore, letting $\eta \to 0$ then $R \to +\infty$ in the above yields

$$W(\overline{E}_n, \mathbf{1}_{\mathbb{R}^2}) - W(\nabla H_n', \mathbf{1}_{\mathbb{R}^2}) \geq 0.$$

Since $\Lambda_n \subset \Sigma'$ by construction, we have $\sum_i \zeta(x_i) = 0$. Together with (6.46), we deduce in view of (3.3) that

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\left(w_n(x_1,\ldots,x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right)\\ \leq \frac{|\Sigma|}{\pi}\int W(E)\,dP(x,E) + C\varepsilon.\end{aligned} \tag{6.47}$$

[Resp., $\leq \frac{|\Sigma|}{\pi}\int \mathbb{W}(\nu)\,dQ(x,\nu) + C\varepsilon$.]

Step 4: *Existence of* A_n. We claim that if n is large enough and if $E \in L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ is such that

$$d_p\big(E(\sqrt{n}x + \cdot), E_n^{\mathrm{int}}(\sqrt{n}x + \cdot)\big) < \eta_1/2 \tag{6.48}$$

for any $x \in \Sigma \setminus \Xi$ for some set Ξ satisfying $|\Xi| < \eta_0|\Sigma|$, then

$$d_{\mathcal{B}}\left(\fint_\Sigma \delta_x \otimes \delta_{\theta_{\sqrt{n}x}E}\,dx, \fint_\Sigma \delta_x \otimes \delta_{\theta_{\sqrt{n}x}E_n^{\mathrm{int}}}\,dx\right) < C\varepsilon\big(|\log\varepsilon| + 1\big). \tag{6.49}$$

This would follow immediately from Lemmas 6.1, 6.2 and 6.3 if $\theta_{\sqrt{n}x}E_n^{\mathrm{int}}$ belonged to some compact set independent of $x \notin \Xi$ and n. In our case, we note that if x belongs to some $\tilde{L} \in \mathcal{L}_{K,i}$, where $K \in \mathcal{K}_n$ and $0 \leq i \leq N_\varepsilon$, then

$$E_n^{\mathrm{int}}(x + \cdot) = \sigma_{m_K} J_{i,L}(\cdot + x - x_{\tilde{L}}).$$

Moreover, since $J_i \in H_\varepsilon$, from (6.14) it follows that if $x - x_{\tilde{L}} \notin \Gamma(J_i)/\sqrt{m_K}$ then $E' := \sigma_{m_K} J_i(\cdot + x - x_{\tilde{L}}) \in G_\varepsilon$. If in addition, $\mathrm{dist}(x, \partial\tilde{L}) > \eta_0 R_\varepsilon$, then we deduce from (6.19) that $d_p(E_n^{\mathrm{int}}(x+\cdot), E') < \eta_1/2$ and $d_p(E(x+\cdot), E') < \eta_1$. Then, from Lemma 6.2, we deduce that $d_{\mathcal{B}}(\delta_{E_n^{\mathrm{int}}(x+\cdot)}, \delta_{E'}) < \varepsilon$ and $d_{\mathcal{B}}(\delta_{E(x+\cdot)}, \delta_{E'}) < \varepsilon$ thus

$$d_{\mathcal{B}}(\delta_{E_n^{\mathrm{int}}(x+\cdot)}, \delta_{E(x+\cdot)}) < 2\varepsilon.$$

In view of Lemma 6.3, we find

$$d_{\mathcal{B}}\left(\frac{1}{|\Sigma|}\int_{\Sigma\setminus\tilde{\Xi}} \delta_x \otimes \delta_{\theta_{\sqrt{n}x}E}\,dx, \frac{1}{|\Sigma|}\int_{\Sigma\setminus\tilde{\Xi}} \delta_x \otimes \delta_{\theta_{\sqrt{n}x}E_n^{\mathrm{int}}}\,dx\right) < C\varepsilon\big(|\log\varepsilon| + 1\big),$$

where $\tilde{\Xi}$ is the union of Ξ and of the union with respect to $0 \leq i \leq N_\varepsilon$, $K \in \mathcal{K}_n$ and $\tilde{L} \in \mathcal{L}_{K,i}$ of $\frac{1}{\sqrt{n}}(x_{\tilde{L}} + \Gamma(J_i)/\sqrt{m_K})$, of $\frac{1}{\sqrt{n}}\{x \in \tilde{L} : \mathrm{dist}(x, \partial\tilde{L}) \leq \eta_0 R_\varepsilon\}$, and of $\Sigma \setminus \bigcup_{\mathcal{K}_n} \frac{K}{\sqrt{n}}$. It turns out that $|\tilde{\Xi}| < C\eta_0$ if n is large enough, C being of course independent of ε, and thus using Lemma 6.1 we deduce (6.49). The claim is proved.

To prove the existence of the set A_n, we note that the vector fields J_i used in constructing E_n^{int} depend on ε but are independent on n. Then they are truncated to obtain $J_{i,K}$ where the side-lengths of L are in $[R_\varepsilon/C, CR_\varepsilon]$, that is, in an interval independent of n. It follows at once that there exists $\delta > 0$ such that the points in L may be perturbed by an amount δ so that for every i, K and $\tilde{L} \in \mathcal{L}_{K,i}$ the perturbed $J_{i,L}^{\mathrm{pert}}$ is at a distance at most $\eta_1/4$ of $J_{i,K}$, for every n. Then in view of (6.44) and (6.37) it follows that for n large enough the resulting E_n^{pert} will satisfy (6.48) for x far enough from $\partial\Sigma'$, that is, outside a set of proportion relative to $|\Sigma'|$ tending to 0 as $n \to \infty$. We deduce that E_n^{pert} satisfies (6.49), hence if n is large enough

$$d_{\mathcal{B}}(P_{E_n^{\mathrm{pert}}}, P) < C\varepsilon(|\log\varepsilon| + 1).$$

The same reasoning implies that if we let $\{x_1, \dots, x_n\}$ be the points in Λ_n in original coordinates, then perturbing the points in Λ_n by an amount $\delta > 0$ small enough, that is, perturbing the x_i's by an amount $\delta/\sqrt{n}$ at most we obtain points y_i such that $w_n(y_i) \le w_n(x_i) + n\varepsilon$. Since the ordering of the points is irrelevant, we let S_n denote the set of permutations of $1, \dots, n$ and define

$$A_n = \{(y_1, \dots, y_n) : \exists\sigma \in S_n, |x_i - y_{\sigma(i)}| < \delta\}.$$

Then, given $\eta > 0$, from the previous discussion and choosing $\varepsilon > 0$ small enough we have for any n and any $(y_1, \dots, y_n) \in A_n$ that (4.16) is satisfied and the existence of $\overline{E}_n$ such that $\operatorname{div} \overline{E}_n = 2\pi(\sum_i \delta_{y_i'} - m_0')$ and such that the associated $\{\overline{P}_n\}_n$ satisfies (4.17).

This completes the proof of Proposition 4.5.

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