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BSDES WITH WEAK TERMINAL CONDITION

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We introduce a new class of backward stochastic differential equations in which the T-terminal value Y_T of the solution (Y, Z) is not fixed as a random variable, but only satisfies a weak constraint of the form $E[\Psi(Y_T)] \geq m$, for some (possibly random) nondecreasing map Ψ and some threshold m. We name them *BSDEs with weak terminal condition* and obtain a representation of the minimal time t-values Y_t such that (Y, Z) is a supersolution of the BSDE with weak terminal condition. It provides a non-Markovian BSDE formulation of the PDE characterization obtained for Markovian stochastic target problems under controlled loss in Bouchard, Elie and Touzi [*SIAM J. Control Optim.* **48** (2009/10) 3123–3150]. We then study the main properties of this minimal value. In particular, we analyze its continuity and convexity with respect to the m-parameter appearing in the weak terminal condition, and show how it can be related to a dual optimal control problem in Meyer form. These last properties generalize to a non-Markovian framework previous results on quantile hedging and hedging under loss constraints obtained in Föllmer and Leukert [*Finance Stoch.* **3** (1999) 251–273; *Finance Stoch.* **4** (2000) 117–146], and in Bouchard, Elie and Touzi (2009/10).

1. Introduction. Solving a backward stochastic differential equation (hereafter BSDE), with terminal data $\xi \in \mathbf{L}_2(\mathcal{F}_T)$ and driver g, consists in finding a pair of predictable processes (Y, Z), with certain integrability properties, such that the dynamics of Y satisfies $dY_t = -g(t, Y_t, Z_t)\,dt + Z_t\,dW_t$ and $Y_T = \xi$ (where W denotes a standard Brownian motion). It can be rephrased in: find an initial data Y_0 and a control process Z such that the solution Y^Z of the controlled stochastic differential equation

$$Y_t^Z = Y_0 - \int_0^t g\big(s, Y_s^Z, Z_s\big)\,ds + \int_0^t Z_s\,dW_s, \qquad 0 \leq t \leq T, \tag{1.1}$$

satisfies $Y_T^Z = \xi$. In cases where the previous problem does not admit a solution, a weaker formulation is to find an initial data Y_0 and a control Z such that

$$Y_T^Z \geq \xi, \qquad \mathbb{P}\text{-a.s.} \tag{1.2}$$

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In most applications, one is interested in the minimal initial condition Y_0 and in the associated control Z. This is for instance the case in the financial literature in which Y_0 represents the cost of the cheapest super-replication strategy for the contingent claim ξ, and Z provides the associated hedging strategy; see, for example, El Karoui, Peng and Quenez (1997).

Motivated by situations where this minimal value Y_0 is too large for practical applications, it was suggested to relax the strong constraint (1.2) into a weaker one of the form

$$E\big[\ell\big(Y_T^Z - \xi\big)\big] \geq m, \tag{1.3}$$

where m is a given threshold and ℓ is a nondecreasing map. For $\ell(x) = \mathbf{1}_{\{x\geq 0\}}$, this corresponds to matching the criteria $Y_T^Z \geq \xi$ at least with probability m. In financial terms, this is the so-called quantile hedging problem; see Föllmer and Leukert (1999).[3] More generally, ℓ is viewed as a loss function, one typical example being $\ell(x) := -(x^-)^q$ with $q \geq 1$; see Föllmer and Leukert (2000) for general non-Markovian but linear dynamics. Such problems were coined "stochastic target problems with controlled loss" by Bouchard, Elie and Touzi (2009/10) who consider a nonlinear Markovian formulation in a Brownian diffusion setting; see also Moreau (2011) for the jump diffusions setting.

The aim of this paper is to study the nonlinear non-Markovian setting in which the terminal constraint is of the form

$$E\big[\Psi\big(Y_T^Z\big)\big] \geq m. \tag{1.4}$$

In the above, $m \in \mathbb{R}$ and Ψ is a (possibly random) nondecreasing real-valued map. Our problem can then be written as the following:

(1.5) Find the minimal Y_0 such that (1.1) and (1.4) hold for some Z.

This leads to the Introduction of a new class of BSDEs which we call BSDEs with weak terminal condition. More precisely, we refer to this problem by saying that we want to solve the BSDE with driver g and weak terminal condition (Ψ, m) to insist on the fact that the terminal condition Y_T^Z is not fixed as a random variable, but only has to satisfy the weak constraint (1.4).

The first step in our analysis lies in a reformulation based on the martingale representation theorem, as suggested in Bouchard, Elie and Touzi (2009/10). More precisely, if Y_0 and Z are such that (1.4) holds, then the martingale representation theorem implies that we can find an element α in the set $\mathbf{A}_0$, of predictable square integrable processes, such that

$$\Psi\big(Y_T^Z\big) \geq M_T^\alpha := m + \int_0^T \alpha_s \, dW_s.$$

[3] In fact, their original formulation also imposes a budget constraint $Y_T^Z \geq 0$, $\mathbb{P}$-a.s., which can be taken into account by imposing a criteria of the form (1.4) with $\Psi(Y_T^Z) := \mathbf{1}_{\{Y_T^Z - \xi \geq 0\}} - \infty \mathbf{1}_{\{Y_T^Z < 0\}}$.

On the other hand, since Ψ is nondecreasing, one can introduce its left-continuous inverse Φ and note that the solution (Y^α, Z^α) of the BSDE

$$Y_t^\alpha = \Phi(M_T^\alpha) + \int_t^T g(s, Y_s^\alpha, Z_s^\alpha)\,ds - \int_t^T Z_s^\alpha\,dW_s, \qquad 0 \le t \le T, \tag{1.6}$$

actually solves (1.1) and (1.4). We indeed show that the solution of (1.5) is given by

$$\inf\{Y_0^\alpha, \alpha \in \mathbf{A}_0\}. \tag{1.7}$$

This leads to study its dynamical counterpart

$$\mathcal{Y}_\tau^\alpha := \operatorname{ess\,inf}\{Y_\tau^{\alpha'}, \alpha' \in \mathbf{A}_0 \text{ s.t. } \alpha' = \alpha \text{ on } [\![0, \tau]\!]\}, \qquad 0 \le \tau \le T. \tag{1.8}$$

We verify that the family $\{\mathcal{Y}^\alpha, \alpha \in \mathbf{A}_0\}$ satisfies a dynamic programming principle which can be seen as a counterpart of the geometric dynamic programming principle of Soner and Touzi (2002) used in Bouchard, Elie and Touzi (2009/10). In particular, this implies that $\{\mathcal{Y}^\alpha, \alpha \in \mathbf{A}_0\}$ is a g-submartingale family to which we can apply the nonlinear Doob–Meyer decomposition of Peng (1999). This provides a representation of the family $\{\mathcal{Y}^\alpha, \alpha \in \mathbf{A}_0\}$ in terms of minimal supersolutions to a family of BSDEs with driver g and (strong) terminal conditions $\{\Phi(M_T^\alpha), \alpha \in \mathbf{A}_0\}$. This representation allows in particular to characterize the family $\{\mathcal{Y}^\alpha, \alpha \in \mathbf{A}_0\}$ uniquely. Under additional convexity assumptions on the coefficients g and Φ, we observe that the essential infimum in (1.8) is attained. Hence, there exists an optimal $\hat{\alpha} \in \mathbf{A}_0$ such that solving the BSDE with weak terminal condition (Ψ, m) boils down to solving the BSDE with dynamics (1.6) and strong terminal condition $\Phi(M_T^{\hat{\alpha}})$. In a Markovian framework, our approach provides in particular a BSDE formulation for the PDEs derived in Bouchard, Elie and Touzi (2009/10).

We then study in details important properties of this family and focus in particular on the regularity of $\mathcal{Y}^\alpha$ with respect to the threshold parameter m. We exhibit, under weak conditions, a stability property of the solution with respect to the variations of the parameter m. We also observe that $\mathcal{Y}^\alpha$ is convex with respect to the threshold parameter. This observation allows us in particular to conclude that Φ (whenever it is deterministic) can be replaced by its more regular convex envelope in order to compute $\mathcal{Y}^\alpha$ on $[0, T)$. This was already observed in the restrictive Markovian setting of Bouchard, Elie and Touzi (2009/10), in which it is proved by using PDE technics. We provide here a pure probabilistic argument. Similarly, it was also observed in Föllmer and Leukert (1999, 2000) and Bouchard, Elie and Touzi (2009/10) that (1.5) admits a dual linear problem when g is linear. We extend this result via probabilistic arguments to the semilinear setting, for which the dual formulation takes the form of a stochastic control problem in Meyer form.

The rest of the paper is organized as follows. In Section 2, we provide a precise formulation for (1.5) and relate this problem to a g-submartingale family satisfying

a dynamic programming principle. Attainability of the optimal control $\hat{\alpha} \in \mathbf{A}_0$ is also discussed. Section 3 collects the continuity and convexity properties as well as the dual formulation of the problem. Finally, Section 4 contains the proof of the BSDE representation for $\{\mathcal{Y}^\alpha, \alpha \in \mathbf{A}_0\}$.

We close this Introduction with a series of notation that will be used all over this paper. Let $d \geq 1$ and $T > 0$ be fixed. We denote by $W := (W_t)_{t\in[0,T]}$ a d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$-augmented natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. The components of W will be denoted by $W = (W^1, \ldots, W^d)$ and E will stand for the expectation with respect to $\mathbb{P}$. For simplicity, we assume that $\mathcal{F} = \mathcal{F}_T$. Throughout the paper, we will make use of the following spaces.

- $\mathbf{L}_p(U, \mathcal{G})$ denotes the set of p-integrable $\mathcal{G}$-measurable random variables with values in U, $p \geq 0$, U a Borel set of $\mathbb{R}^n$ for some $n \geq 1$ and $\mathcal{G} \subset \mathcal{F}$. When U and $\mathcal{G}$ can be clearly identified by the context, we omit them. This will be in particular the case when $\mathcal{G} = \mathcal{F}$.
- $\mathcal{T}$ denotes the set of $\mathbb{F}$-stopping times in $[0, T]$. For $\tau_1 \in \mathcal{T}$, $\mathcal{T}_{\tau_1}$ is the set of stopping times τ_2 in $\mathcal{T}$ such that $\tau_2 \geq \tau_1, \mathbb{P}$-a.s. The notation $E_\tau[\cdot]$ stands for the conditional expectation given $\mathcal{F}_\tau$, $\tau \in \mathcal{T}$.
- $\mathbf{S}_2$ denotes the set of $\mathbb{R}$-valued, *càdlàg*[4] and $\mathbb{F}$-adapted stochastic processes $X = (X_t)_{t\in[0,T]}$ such that $\|X\|_{\mathbf{S}_2} := E[\sup_{t\in[0,T]} |X_t|^2]^{1/2} < \infty$.
- $\mathbf{H}_2$ denotes the set of $\mathbb{R}^n$-valued, $\mathbb{F}$-predictable stochastic processes $X = (X_t)_{t\in[0,T]}$ such that $\|X\|_{\mathbf{H}_2} := E[\int_0^T |X_t|^2\, dt]^{1/2} < \infty$. In the following, the dimension n will be given by the context.
- $\mathbf{K}_2$ denotes the set of nondecreasing $\mathbb{R}$-valued and $\mathbb{F}$-adapted stochastic processes $X = (X_t)_{t\in[0,T]}$ such that $\|X\|_{\mathbf{S}_2} < \infty$.

Inequalities between random variables are understood in the $\mathbb{P}$-a.s.-sense.

2. BSDE with weak terminal condition.

2.1. *Definitions and problem reformulation.* We first define the main object of this paper.

DEFINITION 2.1 (Solution to a BSDE with weak terminal condition). Given a measurable map $\Psi : \mathbb{R} \times \Omega \mapsto U$, with $U \subset \mathbb{R} \cup \{-\infty\}$, $\tau \in \mathcal{T}$ and $\mu \in \mathbf{L}_0(U, \mathcal{F}_\tau)$, we say that $(Y, Z) \in \mathbf{S}_2 \times \mathbf{H}_2$ is a supersolution of the BSDE with generator $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and weak terminal condition (Ψ, μ, τ), in short BSDE(g, Ψ, μ, τ), if for any $0 \leq s \leq t \leq T$,

$$Y_s \geq Y_t + \int_s^t g(r, Y_r, Z_r)\, dr - \int_s^t Z_r\, dW_r \quad \text{and} \tag{2.1}$$

$$E_\tau\big[\Psi(Y_T)\big] \geq \mu. \tag{2.2}$$

[4]Right-continuous with left limits.

Before discussing the well-posedness of equations (2.1)–(2.2), let us emphasize that the difference with classical BSDEs lies in the fact that we do not prescribe a terminal condition to Y in the classical $\mathbb{P}$-a.s.-sense but only impose a weak condition in expectation form (which justifies the terminology of *BSDE with weak terminal condition*). Even if we were asking for equalities in (2.1)–(2.2), this would obviously be too weak to expect uniqueness, as any random variable ξ satisfying $E_\tau[\Psi(\xi)] = \mu$ could serve as a terminal condition.

However, when Ψ is nondecreasing, the set

$$\begin{aligned}&\Gamma(\tau,\mu)\\ &:= \big\{Y_\tau : (Y,Z) \in \mathbf{S}_2 \times \mathbf{H}_2 \text{ is a supersolution of BSDE}(g,\Psi,\mu,\tau)\big\},\end{aligned} \tag{2.3}$$

defined for any $\tau \in \mathcal{T}$ and $\mu \in \mathbf{L}_0(U, \mathcal{F}_\tau)$, can be characterized by its lower-bound, whenever it is achieved.

Throughout the paper, we shall restrict to the case where g is Lipschitz continuous with linear growth, Ψ^+ is bounded, and the domain of Ψ is bounded from below, in order to avoid unnecessary technicalities.

Standing assumption ($\mathbf{H}_\Psi$): For $\mathbb{P}$-a.e. $\omega \in \Omega$, the map $y \in \mathbb{R} \mapsto \Psi(\omega, y)$ is nondecreasing, right-continuous, valued in $[0,1] \cup \{-\infty\}$ and its left-continuous inverse $\Phi(\omega,\cdot)$ satisfies $\Phi : \Omega \times [0,1] \mapsto [0,1]$ is measurable.

By left-continuous inverse, we mean the left-continuous map defined for ω fixed by

$$\Phi(\omega, x) := \inf\big\{y \in \mathbb{R}, \Psi(\omega, y) \geq x\big\},$$

which satisfies

$$\Phi \circ \Psi \leq \mathrm{Id} \leq \Psi \circ \Phi. \tag{2.4}$$

The left-hand side follows from the definition of Φ, the right-hand side holds by right-continuity of Ψ. Note that the above assumption implies $\Psi(\omega,\cdot) = -\infty$ on $(-\infty, 0)$ and $\Psi(\omega,\cdot) = 1$ on $[1,\infty)$. In particular, the constraint in expectation (2.2) implies $Y_T \geq 0, \mathbb{P}$-a.s. Obviously, the set $[0,1]$ is chosen for ease of notation and can be replaced by any closed interval.

Standing assumption ($\mathbf{H}_g$): g is a measurable map from $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R}$ and $g(\cdot, y, z)$ is $\mathbb{F}$-predictable, for each $(y,z) \in \mathbb{R} \times \mathbb{R}^d$. There exists a constant $K_g > 0$ and a random variable $\chi_g \in \mathbf{L}_2(\mathbb{R}_+)$, such that

$$\begin{aligned}\big|g(t,0,0)\big| &\leq \chi_g, \qquad \mathbb{P}\text{-a.s.},\\ \big|g(t,y_1,z_1) - g(t,y_2,z_2)\big| &\leq K_g\big(|y_1 - y_2| + |z_1 - z_2|\big),\\ &\qquad \mathbb{P}\text{-a.s. } \forall (t,y_i,z_i) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d, i = 1,2.\end{aligned}$$

Let $\mathbf{A}_{\tau,\mu}$ denote the set elements $\alpha \in \mathbf{H}_2$ such that

$$M^{(\tau,\mu),\alpha} := \mu + \int_\tau^{\tau\vee\cdot} \alpha_s \, dW_s \qquad \text{takes values in } [0,1]. \tag{2.5}$$

Then (2.2) is equivalent to $\Psi(Y_T) \geq M_T^{(\tau,\mu),\alpha}$ for some $\alpha \in \mathbf{A}_{\tau,\mu}$. In view of (2.4), this is equivalent to $Y_T \geq \Phi(M_T^{(\tau,\mu),\alpha})$ for some $\alpha \in \mathbf{A}_{\tau,\mu}$. This implies that supersolutions of BSDE(g, Ψ, μ, τ) can be characterized in terms of g-expectations whose definition is recalled below.

DEFINITION 2.2 (g-expectation). Given $\tau_2 \in \mathcal{T}$ and $\xi \in \mathbf{L}_2(\mathbb{R}, \mathcal{F}_{\tau_2})$, let $(Y, Z) \in \mathbf{S}_2 \times \mathbf{H}_2$ denote the solution of

$$Y = \xi + \int_{\cdot\wedge\tau_2}^{\tau_2} g(s, Y_s, Z_s)\, ds - \int_{\cdot\wedge\tau_2}^{\tau_2} Z_s\, dW_s.$$

Then we define the (conditional) g-expectation of ξ at the stopping time $\tau_1 \leq \tau_2$ as $\mathcal{E}^g_{\tau_1,\tau_2}[\xi] := Y_{\tau_1}$. When $\tau_2 \equiv T$, we only write $\mathcal{E}^g_{\tau_1}[\xi]$, and say that (Y, Z) solves BSDE(g, ξ).

Note that existence and uniqueness hold under assumption $(\mathbf{H}_g)$. In the following, we shall adopt the terminology of Peng (2004) and call g-martingale (resp., g-submartingale) a process Y such that $\mathcal{E}^g_{t,s}[Y_s] = Y_t$ (resp., $\mathcal{E}^g_{t,s}[Y_s] \geq Y_t$), for all $t \leq s \leq T$.

PROPOSITION 2.1. *Fix $\tau \in \mathcal{T}$, $\mu \in \mathbf{L}_0([0,1], \mathcal{F}_\tau)$. Then, $(Y, Z) \in \mathbf{S}_2 \times \mathbf{H}_2$ is a supersolution of* BSDE(g, Ψ, μ, τ) *if and only if (Y, Z) satisfies (2.1) and there exists $\alpha \in \mathbf{A}_{\tau,\mu}$ such that $Y_t \geq \mathcal{E}^g_t[\Phi(M_T^{(\tau,\mu),\alpha})]$ for $t \in [0, T]$, $\mathbb{P}$-a.s.*

PROOF. Let (Y, Z) be a supersolution of BSDE(g, Ψ, μ, τ). Then there exists some element ρ in $\mathbf{L}_0([0,1], \mathcal{F}_\tau)$ with $\rho \geq \mu$, $\mathbb{P}$-a.s. and $\tilde{\alpha}$ in $\mathbf{A}_{\tau,\rho}$ such that $\Psi(Y_T) = M_T^{(\tau,\rho),\tilde{\alpha}}$. Set $\theta^{\tilde{\alpha}} := \inf\{s \geq \tau,\ M_s^{(\tau,\mu),\tilde{\alpha}} = 0\}$. It is clear that $\theta^{\tilde{\alpha}}$ belongs to $\mathcal{T}$ and that $\alpha := \tilde{\alpha}\mathbf{1}_{[0,\theta^{\tilde{\alpha}})}$ belongs to $\mathbf{A}_{\tau,\mu}$ and satisfies $M_T^{(\tau,\rho),\tilde{\alpha}} \geq M_T^{(\tau,\mu),\alpha}$, $\mathbb{P}$-a.s., since $M_T^{(\tau,\rho),\tilde{\alpha}} \geq 0$ by definition of $\mathbf{A}_{\tau,\rho}$. The monotonicity of Φ and (2.4) imply that

$$Y_T \geq (\Phi \circ \Psi)(Y_T) \geq \Phi\big(M_T^{(\tau,\mu),\alpha}\big).$$

By comparison for Lipschitz BSDEs, we obtain $Y_t \geq \mathcal{E}^g_t[\Phi(M_T^{(\tau,\mu),\alpha})]$ for $t \in [0, T]$. Conversely, let $\alpha \in \mathbf{A}_{\tau,\mu}$ be such that $Y_t \geq \mathcal{E}^g_t[\Phi(M_T^{(\tau,\mu),\alpha})]$ for $t \in [0, T]$ and assume that (Y, Z) satisfies (2.1). Then (2.4) implies

$$\Psi(Y_T) \geq (\Psi \circ \Phi)\big(M_T^{(\tau,\mu),\alpha}\big) \geq M_T^{(\tau,\mu),\alpha}.$$

Taking the conditional expectation on both sides leads to (2.2). □

In view of Proposition 2.1, the lower bound of $\Gamma(\tau, \mu)$ [which we recall, has been defined in (2.3)] can be expressed in terms of

$$\mathcal{Y}_\tau(\mu) := \operatorname*{ess\,inf}_{\alpha \in \mathbf{A}_{\tau,\mu}} \mathcal{E}^g_\tau\big[\Phi\big(M_T^{(\tau,\mu),\alpha}\big)\big], \qquad \tau \in \mathcal{T}, \mu \in \mathbf{L}_0\big([0,1], \mathcal{F}_\tau\big). \tag{2.6}$$

This is the statement of the next corollary.

COROLLARY 2.1. $\operatorname{ess\,inf}\Gamma(\tau,\mu)=\mathcal{Y}_\tau(\mu)$, $\forall\tau\in\mathcal{T}$, $\mu\in\mathbf{L}_0([0,1],\mathcal{F}_\tau)$.

PROOF. The fact that $Y_\tau\in\Gamma(\tau,\mu)$ implies $Y_\tau\geq\mathcal{Y}_\tau(\mu)$ follows from Proposition 2.1. On the other hand, the same proposition implies that each $\mathcal{E}^g_\tau[\Phi(M_T^{(\tau,\mu),\alpha})]$ with $\alpha\in\mathbf{A}_{\tau,\mu}$ belongs to $\Gamma(\tau,\mu)$. □

REMARK 2.1. For later use, note that the assumptions $(\mathbf{H}_g)$ and $(\mathbf{H}_\Psi)$ ensure that we can find $\eta\in\mathbf{S}_2$ such that $|\mathcal{E}^g_t[\Phi(M)]|\vee|\mathcal{Y}_t(\mu)|\leq\eta_t$, for all $t\leq T$ and $\mu\in\mathbf{L}_0([0,1],\mathcal{F}_t)$, $M\in\mathbf{L}_0([0,1])$. See (i) of Proposition A.2 in the Appendix.

REMARK 2.2. Note that $\mathcal{Y}_\tau(\mu)=\mathcal{Y}_\tau(\mu_1)\mathbf{1}_A+\mathcal{Y}_\tau(\mu_2)\mathbf{1}_{A^c}$ whenever $\mu:=\mu_1\mathbf{1}_A+\mu_2\mathbf{1}_{A^c}$ for $A\in\mathcal{F}_\tau$, $\mu_1,\mu_2\in\mathbf{L}_0([0,1],\mathcal{F}_\tau)$, and $\tau\in\mathcal{T}$. Indeed, $\alpha:=\mathbf{1}_{[\tau,T]}(\alpha_1\mathbf{1}_A+\alpha_2\mathbf{1}_{A^c})\in\mathbf{A}_{\tau,\mu}$ for all $\alpha_i\in\mathbf{A}_{\tau,\mu_i}$ with $i=1,2$. Since $\mathcal{E}^g_\tau[\Phi(M_T^{(\tau,\mu),\alpha})]=\mathcal{E}^g_\tau[\Phi(M_T^{(\tau,\mu_1),\alpha_1})]\mathbf{1}_A+\mathcal{E}^g_\tau[\Phi(M_T^{(\tau,\mu_2),\alpha_2})]\mathbf{1}_{A^c}$, this implies $\mathcal{Y}_\tau(\mu)\leq\mathcal{Y}_\tau(\mu_1)\mathbf{1}_A+\mathcal{Y}_\tau(\mu_2)\mathbf{1}_{A^c}$. The converse inequality follows from the previous identity applied with $\alpha_1:=\alpha\mathbf{1}_A$ and $\alpha_2:=\alpha\mathbf{1}_{A^c}$ for any $\alpha\in\mathbf{A}_{\tau,\mu}$ so that $\alpha_i\in\mathbf{A}_{\tau,\mu_i}$ for $i=1,2$.

REMARK 2.3. Before going on with the study of the set Γ, let us notice that a similar analysis can be carried out for weak constraints of the form $\mathcal{E}^h_\tau[\Psi(Y_T)]\geq\mu$ in place of $E_\tau[\Psi(Y_T)]\geq\mu$ in (2.2), with $\mathcal{E}^h$ defined as the h-expectation associated to some random map h satisfying similar conditions as g. In finance, the latter condition interprets as a risk-measure constraints [see, e.g., Peng (2004)], while our condition is more related to expected loss constraints, see Föllmer and Leukert (2000). Again, we try to avoid un-necessary additional technicalities and stick to the case $h\equiv 0$.

2.2. *BSDE characterization of the minimal initial condition.* The main result of this section is a BSDE characterization for the lower bound of the set $\Gamma(\tau,\mu)$ of time-τ initial conditions of supersolutions of BSDE(g,Ψ,μ,τ). In particular, this extends to a non-Markovian framework the PDE characterization of Bouchard, Elie and Touzi (2009/10).

For ease of notation, we now fix $m_o\in[0,1]$ and set

$$\begin{cases} M^\alpha_t:=M^{(0,m_o),\alpha}_t, & \mathbf{A}^\alpha_\tau:=\{\alpha'\in\mathbf{A}_{\tau,M^\alpha_\tau}:\alpha'=\alpha\,dt\times d\mathbb{P}\text{ on }[\![0,\tau]\!]\},\\ \mathbf{A}_0:=\mathbf{A}_{0,m_o} \quad\text{and}\quad \mathcal{Y}^\alpha_t:=\mathcal{Y}_t(M^\alpha_t) & \text{for }\alpha\in\mathbf{A}_0,\ t\in[0,T],\end{cases}$$

where we recall that $M^{(0,m_o),\alpha}$ and $\mathbf{A}_{0,m_o}$ are given in (2.5).

THEOREM 2.1. *For any $\alpha\in\mathbf{A}_0$, $\mathcal{Y}^\alpha$ is a g-submartingale, it is làdlàg* [5] *on countable sets, and the following dynamic programming principle holds*:

[5] Left- and right-limited according to the french celebrated acronym.

(i) $\mathcal{Y}^{\alpha}_{\tau_1} = \operatorname{ess\,inf}_{\bar{\alpha}\in\mathbf{A}^{\alpha}_{\tau_1}} \mathcal{E}^{g}_{\tau_1,\tau_2}[\mathcal{Y}^{\bar{\alpha}}_{\tau_2}]$, *for each* $\tau_1 \in \mathcal{T}$, $\tau_2 \in \mathcal{T}_{\tau_1}$. *Under the additional assumption that*

$$m \in [0,1] \mapsto \Phi(\omega, m) \qquad \text{is continuous for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{2.7}$$

the following holds:

(ii) $\mathcal{Y}^{\alpha}$ *is indistinguishable from a càdlàg* g*-submartingale, for each* $\alpha \in \mathbf{A}_0$.

(iii) *There exists a family* $(\mathcal{Z}^{\alpha}, \mathcal{K}^{\alpha})_{\alpha\in\mathbf{A}_0} \subset \mathbf{H}_2 \times \mathbf{K}_2$ *satisfying*

$$\sup_{\alpha\in\mathbf{A}_0} \big\|(\mathcal{Y}^{\alpha}, \mathcal{Z}^{\alpha}, \mathcal{K}^{\alpha})\big\|_{\mathbf{S}_2\times\mathbf{H}_2\times\mathbf{K}_2} < \infty, \tag{2.8}$$

and such that, for all $\alpha \in \mathbf{A}_0$, *we have*

$$\mathcal{Y}^{\alpha} = \Phi(M^{\alpha}_T) + \int_{\cdot}^{T} g(s, \mathcal{Y}^{\alpha}_s, \mathcal{Z}^{\alpha}_s)\, ds - \int_{\cdot}^{T} \mathcal{Z}^{\alpha}_s\, dW_s + \mathcal{K}^{\alpha} - \mathcal{K}^{\alpha}_T, \tag{2.9}$$

$$\mathcal{K}^{\alpha}_{\tau_1} = \operatorname*{ess\,inf}_{\bar{\alpha}\in\mathbf{A}^{\alpha}_{\tau_1}} E\big[\mathcal{K}^{\bar{\alpha}}_{\tau_2} | \mathcal{F}_{\tau_1}\big] \qquad \forall \tau_1 \in \mathcal{T},\ \tau_2 \in \mathcal{T}_{\tau_1} \tag{2.10}$$

and

$$(\mathcal{Y}^{\alpha}, \mathcal{Z}^{\alpha}, \mathcal{K}^{\alpha})\mathbf{1}_{[\![0,\tau]\!]} = (\mathcal{Y}^{\bar{\alpha}}, \mathcal{Z}^{\bar{\alpha}}, \mathcal{K}^{\bar{\alpha}})\mathbf{1}_{[\![0,\tau]\!]} \qquad \forall \tau \in \mathcal{T}, \bar{\alpha} \in \mathbf{A}^{\alpha}_{\tau}. \tag{2.11}$$

(iv) $(\mathcal{Y}^{\alpha}, \mathcal{Z}^{\alpha}, \mathcal{K}^{\alpha})_{\alpha\in\mathbf{A}_0}$ *is the unique family of* $\mathbf{S}_2 \times \mathbf{H}_2 \times \mathbf{K}_2$ *satisfying* (2.8)–(2.11) *for all* $\alpha \in \mathbf{A}_0$.

The proof of this theorem is reported in Section 4.

REMARK 2.4. (i) The precise continuity assumption needed in the proof is: $\Phi(M^{\alpha_n}_T)$ converges in $\mathbf{L}_2$ to $\Phi(M^{\alpha}_T)$ whenever $\|M^{\alpha_n}_T - M^{\alpha}_T\|_{\mathbf{L}_2}$ tends to 0, for any sequence $(\alpha_n)_n \subset \mathbf{A}_0$. However, this condition implies that Φ is continuous, as soon as random variables with nonabsolute continuous law with respect to the Lebesgue measure might be considered (which is the case here).

(ii) We shall see in Proposition 3.3 below that Φ can be replaced by its m-convex envelope, under mild assumptions. In this case, the continuity assumption of the second part of Theorem 2.1 is not required anymore because the convex envelope of Φ is continuous; see Remark 3.1 below.

2.3. *Representation as a BSDE with strong terminal condition.* The previous section raises in particular one natural question: Does there exist an admissible control $\hat{\alpha}$ on the whole time interval $[0, T]$ allowing to match all time t-values of the minimal solution of a BSDE with weak terminal condition? Rephrasing, we wonder about the existence of a control $\hat{\alpha}$ in $\mathbf{A}_0$ such that

$$\mathcal{Y}^{\hat{\alpha}}_t = \mathcal{E}^{g}_t[\Phi(M^{\hat{\alpha}}_T)], \qquad 0 \le t \le T.$$

Hereby, solving the BSDE with weak terminal condition $(\Psi, m_o, 0)$ boils down to solving the classical BSDE with the optimal strong terminal one $\Phi(M_T^{\hat{\alpha}})$: along the optimal path $\hat{\alpha}$, the compensator $\mathcal{K}^{\hat{\alpha}}$ of the BSDE (2.9) must degenerate to 0.

Not surprisingly, the existence of an optimal control requires the addition of convexity assumptions on the coefficients of the BSDE. We shall therefore assume that:

($\mathbf{H}_{\mathrm{conv}}$). For all $(\lambda, m_1, m_2, t, y_1, y_2, z_1, z_2) \in [0,1] \times [0,1]^2 \times [0,T] \times \mathbb{R}^2 \times [\mathbb{R}^d]^2$, the following holds $\mathbb{P}$-a.s.:

$$\Phi\big(\lambda m_1 + (1-\lambda)m_2\big) \le \lambda\Phi(m_1) + (1-\lambda)\Phi(m_2),$$

$$g\big(t, \lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2\big) \le \lambda g(t, y_1, z_1) + (1-\lambda)g(t, y_2, z_2).$$

REMARK 2.5. We recall the following result which is based on standard comparison arguments; see, for example, Rosazza Gianin (2006), Proposition 7. For any $\tau \in \mathcal{T}$, the map $\mathcal{E}_\tau^g[\Phi(\cdot)] : \mathbf{L}_0([0,1]) \to \mathbf{L}_0$ is convex under assumption ($\mathbf{H}_{\mathrm{conv}}$).

PROPOSITION 2.2. *Assume that assumptions* ($\mathbf{H}_{\mathrm{conv}}$) *and* (2.7) *hold. Then, for any* $(\tau, \alpha) \in \mathcal{T} \times \mathbf{H}_2$, *there exists* $\hat{\alpha}^{\tau,\alpha} \in \mathbf{A}_\tau^\alpha$ *such that*

$$\mathcal{Y}_\tau^\alpha = \mathcal{E}_\tau^g\big[\Phi\big(M_T^{\hat{\alpha}^{\tau,\alpha}}\big)\big] = \mathcal{E}_{\tau,\tau'}^g\big[\mathcal{Y}_{\tau'}^{\hat{\alpha}^{\tau,\alpha}}\big] \qquad \forall \tau' \in \mathcal{T}_\tau.$$

REMARK 2.6. As detailed in Remark 3.2 below, the convexity assumption on the terminal map Φ can be avoided in some cases. In particular, if Φ is deterministic, then it can be replaced by its convex envelope. Then, only the convexity assumption on g, has to hold.

PROOF OF PROPOSITION 2.2. Lemma 4.1 below provides a sequence $(\alpha^n)_n$ valued in $\mathbf{A}_\tau^\alpha$ such that

$$\mathcal{Y}_\tau^\alpha = \lim_{n\to\infty} \downarrow \mathcal{E}_\tau^g\big[\Phi\big(M_T^{\alpha^n}\big)\big], \qquad \mathbb{P}\text{-a.s.} \tag{2.12}$$

Since the sequence $(M_T^{\alpha^n})_n$ is bounded in $[0,1]$, we can find sequences of non-negative real numbers $(\lambda_i^n)_{i\ge n}$ with $\sum_{i\ge n} \lambda_i^n = 1$, such that only a finite number of λ_i^n do not vanish, for each n, and such that the sequence of convex combinations $(\tilde{M}_T^n)_n$ given by

$$\tilde{M}_T^n := \sum_{i\ge n} \lambda_i^n M_T^{\alpha^i} \tag{2.13}$$

converges $\mathbb{P}$-a.s. to some $\hat{M}_T \in \mathbf{L}_0([0,1])$. By dominated convergence, the convergence holds in $\mathbf{L}_2$, in particular $E_\tau[\hat{M}_T] = M_\tau^\alpha$, and the martingale representation

theorem implies that we can find $\hat{\alpha} \in \mathbf{A}_\tau^\alpha$ such that $\hat{M}_T = M_T^{\hat{\alpha}}$. Using the convexity of Φ and g (see Remark 2.5), we deduce that

$$\tilde{Y}_\tau^n := \sum_{i \geq n} \lambda_i^n \mathcal{E}_\tau^g\big[\Phi\big(M_T^{\alpha^i}\big)\big] \geq \mathcal{E}_\tau^g\big[\Phi\big(\tilde{M}_T^n\big)\big].$$

By (2.12), $\tilde{Y}_\tau^n \to \mathcal{Y}_\tau^\alpha$, $\mathbb{P}$-a.s. On the other hand, the convergence $\tilde{M}_T^n \to M_T^{\hat{\alpha}}$ in $\mathbf{L}_2$ combined with the boundedness and a.s. continuity of Φ implies that $\Phi(\tilde{M}_T^n) \to \Phi(M_T^{\hat{\alpha}})$ in $\mathbf{L}_2$, after possibly passing to a subsequence. Therefore, the convergence $\mathcal{E}_\tau^g[\Phi(\tilde{M}_T^n)] \to \mathcal{E}_\tau^g[\Phi(M_T^{\hat{\alpha}})]$, $\mathbb{P}$-a.s. follows by Proposition A.1 below. This gives $\mathcal{Y}_\tau^\alpha \geq \mathcal{E}_\tau^g[\Phi(M_T^{\hat{\alpha}})]$, while the converse holds by definition of $\mathcal{Y}_\tau^\alpha$.

It remains to show that $\mathcal{Y}_\tau^\alpha = \mathcal{E}_{\tau,\tau'}^g[\mathcal{Y}_{\tau'}^{\hat{\alpha}}]$, for $\tau' \in \mathcal{T}_\tau$. To see this, first note that the above implies that $\mathcal{Y}_\tau^\alpha = \mathcal{E}_{\tau,\tau'}^g[\mathcal{E}_{\tau'}^g[\Phi(M_T^{\hat{\alpha}})]] \geq \mathcal{E}_{\tau,\tau'}^g[\mathcal{Y}_{\tau'}^{\hat{\alpha}}]$ by standard comparison arguments and the fact that $\mathcal{E}_{\tau'}^g[\Phi(M_T^{\hat{\alpha}})] \geq \mathcal{Y}_{\tau'}^{\hat{\alpha}}$ by definition. As above, we can find a sequence $(\hat{\alpha}^n) \in \mathbf{A}_{\tau'}^{\hat{\alpha}}$ such that $\mathcal{E}_{\tau'}^g[\Phi(M_T^{\hat{\alpha}^n})] \to \mathcal{Y}_{\tau'}^{\hat{\alpha}}$, $\mathbb{P}$-a.s. In view of Remark 2.1, the convergence holds in $\mathbf{L}_2$ and Proposition A.1 below implies

$$\mathcal{Y}_\tau^\alpha \leq \mathcal{E}_{\tau,\tau'}^g\big[\mathcal{E}_{\tau'}^g\big[\Phi\big(M_T^{\hat{\alpha}^n}\big)\big]\big] \to \mathcal{E}_{\tau,\tau'}^g\big[\mathcal{Y}_{\tau'}^{\hat{\alpha}}\big],$$

where we used the fact that $\hat{\alpha}^n \in \mathbf{A}_{\tau'}^{\hat{\alpha}} \subset \mathbf{A}_\tau^\alpha$ to obtain the left-hand side. □

3. Main properties of the minimal initial condition process. In this section, we emphasize remarkable properties of the map $\mathcal{Y}_t : \mu \in \mathbf{L}_0([0,1], \mathcal{F}_t) \mapsto \mathcal{Y}_t(\mu)$, for $t \in [0, T)$. We first derive the continuity of this map under a weak continuity assumption on $\mathcal{E}^g[\Phi(\cdot)]$. Then we verify that this map (or more precisely its l.s.c. envelope) is convex, and discuss the propagation of the convexity property to the time boundary $T-$. Finally, we retrieve, in this non-Markovian setting, a dual representation of the map $\mathcal{Y}_0$, using solely probabilistic arguments.

3.1. *Continuity.* Our continuity result is stated in terms of the quantities

$$\mathrm{Err}_t(\eta) := \operatorname{ess\,sup}\big\{\mathcal{R}_t\big(M, M'\big) : M, M' \in \mathbf{L}_0\big([0,1]\big),\, E_t\big[\big|M - M'\big|^2\big] \leq \eta\big\},$$

defined for $\eta \in \mathbf{L}_0([0,1])$, in which

$$\mathcal{R}_t\big(M, M'\big) := \big|\mathcal{E}_t^g\big[\Phi(M)\big] - \mathcal{E}_t^g\big[\Phi\big(M'\big)\big]\big|.$$

Observe that classical a priori estimates on BSDEs ensure that $\mathrm{Err}_t(\eta_n) \to 0$ as $\eta_n \to 0$, $\mathbb{P}$-a.s. with $(\eta_n)_n \subset \mathbf{L}_0([0,1])$, whenever Φ is a deterministic Lipschitz map; see, for example, Proposition A.1 below. This observation remains valid when Φ is simply continuous, via a classical convolution density argument for Lipschitz maps on bounded domains. The next result indicates that this property ensures the regularity of the map: $\mu \mapsto \mathcal{Y}_t(\mu)$.

PROPOSITION 3.1. *Let $t < T$, $\mu_1, \mu_2 \in \mathbf{L}_0([0,1], \mathcal{F}_t)$. Then*

$$|\mathcal{Y}_t(\mu_1) - \mathcal{Y}_t(\mu_2)| \le \mathrm{Err}_t\big(\Delta(\mu_1, \mu_2)\big) + \mathrm{Err}_t\big(\Delta(\mu_2, \mu_1)\big),$$

where

$$\Delta(\mu_i, \mu_j) := \left(1 - \frac{\mu_i}{\mu_j}\right)\mathbf{1}_{\{\mu_i < \mu_j\}} + \frac{\mu_i - \mu_j}{1 - \mu_j}\mathbf{1}_{\{\mu_i > \mu_j\}}, \qquad i, j = 1, 2.$$

Moreover,

$$|\mathcal{Y}_t(\mu_1) - \mathcal{Y}_t(\mu_2)|\mathbf{1}_{\{\mu_1 = 0\}} \le \mathcal{R}_t(\mu_2, 0)$$

and

$$\begin{aligned}&|\mathcal{Y}_t(\mu_1) - \mathcal{Y}_t(\mu_2)|\mathbf{1}_{\{\mu_1 = 1\}}\\ &\quad\le \operatorname{ess\,sup}\big\{\mathcal{R}_t(1, M) : M \in \mathbf{L}_0\big([0,1]\big),\, E_t\big[|1 - M|^2\big] \le 1 - \mu_2\big\}.\end{aligned}$$

In particular, if $\mathrm{Err}_t(\eta_n) \to 0$, $\mathbb{P}$-a.s. as $\eta_n \to 0$, $\mathbb{P}$-a.s., for all $(\eta_n)_n \subset \mathbf{L}_0([0,1])$, then $\mu \in \mathbf{L}_0((0,1), \mathcal{F}_t) \mapsto \mathcal{Y}_t(\mu)$ is continuous for the sequential $\mathbb{P}$-a.s. convergence and the strong $\mathbf{L}_2$ convergence.

PROOF. *Step* 1. Fix $\mu_1, \mu_2 \in \mathbf{L}_0([0,1], \mathcal{F}_t)$. Given $\alpha_2 \in \mathbf{A}_{t,\mu_2}$, we define

$$\lambda := \frac{1 - \mu_1}{1 - \mu_2}\mathbf{1}_{\{\mu_2 < \mu_1\}} + \frac{\mu_1}{\mu_2}\mathbf{1}_{\{\mu_1 < \mu_2\}} + \mathbf{1}_{\{\mu_1 = \mu_2\}},$$

which is by construction valued in $[0,1]$. Since $M^{(t,\mu_2),\alpha_2}$ takes values in $[0,1]$,

$$M^{(t,\mu_1),\lambda\alpha_2} = \mu_1 - \lambda\mu_2 + \lambda M^{(t,\mu_2),\alpha_2} \in \big[\mu_1 - \lambda\mu_2, \mu_1 + \lambda(1 - \mu_2)\big] \subset [0,1].$$

In particular, $\lambda\alpha_2 \in \mathbf{A}_{t,\mu_1}$. Thus, (2.6) leads to

$$\begin{aligned}\mathcal{Y}_t(\mu_1) &\le \mathcal{E}_t^g\big[\Phi\big(M_T^{(t,\mu_2),\alpha_2}\big)\big]\\ &\quad + \big(\mathcal{E}_t^g\big[\Phi\big(M_T^{(t,\mu_1),\lambda\alpha_2}\big)\big] - \mathcal{E}_t^g\big[\Phi\big(M_T^{(t,\mu_2),\alpha_2}\big)\big]\big).\end{aligned} \tag{3.1}$$

Besides,

$$M_T^{(t,\mu_1),\lambda\alpha_2} - M_T^{(t,\mu_2),\alpha_2} = \mu_1 - \lambda\mu_2 + (\lambda - 1)M_T^{(t,\mu_2),\alpha_2}$$

so that, since $M_T^{(t,\mu_2),\alpha_2}$ belongs to $[0,1]$, we have

$$\mu_1 - 1 + \lambda(1 - \mu_2) \le M_T^{(t,\mu_1),\lambda\alpha_2} - M_T^{(t,\mu_2),\alpha_2} \le \mu_1 - \lambda\mu_2.$$

In addition,

$$\begin{aligned}\mu_1 - \lambda\mu_2 &= 0 \qquad \text{if } \mu_1 < \mu_2 \quad \text{and}\\ \mu_1 - 1 + \lambda(1 - \mu_2) &= 0 \qquad \text{if } \mu_1 \ge \mu_2.\end{aligned}$$

This directly leads to

$$E_t\big[\big|M_T^{(t,\mu_1),\lambda\alpha_2} - M_T^{(t,\mu_2),\alpha_2}\big|\big] \le \Delta(\mu_1,\mu_2).$$

Since these two processes belong to $[0,1]$, we get

$$E_t\big[\big|M_T^{(t,\mu_1),\lambda\alpha_2} - M_T^{(t,\mu_2),\alpha_2}\big|^2\big] \le \Delta(\mu_1,\mu_2).$$

Hence, the arbitrariness of $\alpha_2 \in \mathbf{A}_{t,\mu_2}$ together with (2.6) and (3.1) provides

$$\mathcal{Y}_t(\mu_1) \le \mathcal{Y}_t(\mu_2) + \mathrm{Err}_t\big(\Delta(\mu_1,\mu_2)\big).$$

Interchanging the roles of μ_1 and μ_2 leads to

$$\mathcal{Y}_t(\mu_2) \le \mathcal{Y}_t(\mu_1) + \mathrm{Err}_t\big(\Delta(\mu_2,\mu_1)\big).$$

Step 2. We next consider the case where $\mathbb{P}[\mu_1 = 0] > 0$. Without loss of generality, we can assume that $\mu_1 \equiv 0$. Fix $\alpha \in \mathbf{A}_{t,\mu_2}$. Since $\mathbf{A}_{t,\mu_1} = \{0\}$, $M_T^{(t,\mu_2),\alpha} \ge 0$ and Φ is nondecreasing, comparison implies that

$$\mathcal{Y}_t(0) = \mathcal{E}_t^g\big[\Phi(0)\big] \le \mathcal{E}_t^g\big[\Phi\big(M_T^{(t,\mu_2),\alpha}\big)\big].$$

In particular, $\mathcal{Y}_t(0) = \mathcal{E}_t^g[\Phi(0)] \le \mathcal{Y}_t(\mu_2) \le \mathcal{E}_t^g[\Phi(M_T^{(t,\mu_2),0})] = \mathcal{E}_t^g(\Phi(\mu_2))$.

Step 3. We now consider the case where $\mathbb{P}[\mu_1 = 1] > 0$. Again, we can assume that $\mu_1 \equiv 1$ so that $\mathbf{A}_{t,\mu_1} = \{0\}$. By comparison as above, one has

$$\mathcal{Y}_t(1) = \mathcal{E}_t^g\big[\Phi(1)\big] \ge \mathcal{Y}_t(\mu_2).$$

On the other hand, since $M^{(t,\mu_2),\alpha}$ is a martingale taking values in $[0,1]$, we have

$$E_t\big[\big|1 - M_T^{(t,\mu_2),\alpha}\big|^2\big] \le E_t\big[1 - M_T^{(t,\mu_2),\alpha}\big] = 1 - \mu_2, \qquad \alpha \in \mathbf{A}_{t,\mu_2},$$

from which the result follows. □

3.2. *Convexity.* In Bouchard, Elie and Touzi (2009/10) and Moreau (2011), it is shown that the map $m \in [0,1] \mapsto \mathcal{Y}_0(m)$ is convex. This is done in a Markovian framework using PDE arguments. In this section, we provide a probabilistic proof of this result which hereby extends to our setting. The result is stated for the lower-semicontinuous envelope $\mathcal{Y}_{t*}$ of $\mathcal{Y}_t$ defined as

$$\mathcal{Y}_{t*}(\mu) := \lim_{\varepsilon\to 0} \operatorname{ess\,inf}\big\{\mathcal{Y}_t(\mu') : |\mu' - \mu| \le \varepsilon,\ \mu' \in \mathbf{L}_0\big([0,1],\mathcal{F}_t\big)\big\}, \tag{3.2}$$

for any $t \in [0,T]$. We refer to Proposition 3.1, the discussion before it and to (ii) of Remark 2.4 for conditions ensuring that $\mathcal{Y}_* = \mathcal{Y}$.

We first make precise the notion of convexity adapted to our non-Markovian setting. Fix a time $t \in [0,T]$.

DEFINITION 3.1 ($\mathcal{F}_t$-convexity).

(i) In the following, we say that a subset $D \subset \mathbf{L}_\infty(\mathbb{R},\mathcal{F}_t)$ is $\mathcal{F}_t$-convex if $\lambda\mu_1 + (1-\lambda)\mu_2 \in D$, for all $\mu_1, \mu_2 \in D$ and $\lambda \in \mathbf{L}_0([0,1],\mathcal{F}_t)$.

(ii) Let D be an $\mathcal{F}_t$-convex subset of $\mathbf{L}_\infty(\mathbb{R}, \mathcal{F}_t)$. A map $\mathcal{J} : D \mapsto \mathbf{L}_2(\mathbb{R}, \mathcal{F}_t)$ is said to be $\mathcal{F}_t$-convex if

$$\mathrm{Epi}(\mathcal{J}) := \big\{(\mu, Y) \in D \times \mathbf{L}_2(\mathbb{R}, \mathcal{F}_t) : Y \geq \mathcal{J}(\mu)\big\}$$

is $\mathcal{F}_t$-convex.

(iii) Let $\mathrm{Epi}^c(\mathcal{Y}_t)$ be the set of elements of the form $\sum_{n\leq N} \lambda_n(\mu_n, Y_n)$ with $(\mu_n, Y_n, \lambda_n)_{n\leq N} \subset \mathrm{Epi}(\mathcal{Y}_t) \times \mathbf{L}_0([0,1], \mathcal{F}_t)$ such that $\sum_{n\leq N} \lambda_n = 1$, for some $N \geq 1$. We then denote by $\overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$ its closure in $\mathbf{L}_2$. Finally, the $\mathcal{F}_t$-convex envelope of $\mathcal{Y}_t$ is defined as

$$\mathcal{Y}_t^c(\mu) := \operatorname{ess\,inf}\big\{Y \in \mathbf{L}_2(\mathbb{R}, \mathcal{F}_t) : (\mu, Y) \in \overline{\mathrm{Epi}}^c(\mathcal{Y}_t)\big\}. \tag{3.3}$$

We can now state the convexity property. It requires a right-continuity property in time, which holds under the conditions of Theorem 2.1(ii); also recall (ii) of Remark 2.4.

PROPOSITION 3.2. *Assume that $\mathcal{Y}_t(\mu) = \mathcal{Y}_{t+}(\mu)$ for any $\mu \in \mathbf{L}_0([0,1], \mathcal{F}_t)$ and $t < T$. Then the map $\mu \in \mathbf{L}_0([0,1], \mathcal{F}_t) \mapsto \mathcal{Y}_{t*}(\mu)$ is $\mathcal{F}_t$-convex, for all $t < T$.*

PROOF. Fix $t \in [0, T)$ and set $D := \mathbf{L}_0([0,1], \mathcal{F}_t)$ for ease of notation. The proof is divided in several steps.

Step 1. $(\mu, \mathcal{Y}_t^c(\mu)) \in \overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$, *for all* $\mu \in D$.

Indeed, the family $F := \{Y \in \mathbf{L}_2(\mathbb{R}, \mathcal{F}_t) : (\mu, Y) \in \overline{\mathrm{Epi}}^c(\mathcal{Y}_t)\}$ is directed downward (for every fixed element μ in D) since $Y^1\mathbf{1}_{\{Y^1\leq Y^2\}} + Y^2\mathbf{1}_{\{Y^1>Y^2\}} \in F$, by $\mathcal{F}_t$-convexity of $\overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$, for all $Y^1, Y^2 \in F$. It then follows from [Neveu (1975), Proposition VI.1.1] that there exists a sequence $(Y^n)_{n\geq 1} \subset F$ such that $Y^n \downarrow \mathcal{Y}_t^c(\mu)$, $\mathbb{P}$-a.s. Since Y^1 and $\mathcal{Y}_t^c(\mu) \in \mathbf{L}_2$, the monotone convergence theorem implies that $Y^n \to \mathcal{Y}_t^c(\mu)$ in $\mathbf{L}_2$, as n goes to infinity. The set $\overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$ being closed in $\mathbf{L}_2$, this proves our claim.

Step 2. *Let $\eta \in \mathbf{S}_2$ be as in Remark 2.1. Then $|\mathcal{Y}_t^c(\mu)| \leq \eta_t$, for all $t \leq T$ and $\mu \in D$.*

We first observe that $\mathcal{Y} \geq \mathcal{Y}^c$ by construction. Remark 2.1 thus implies that $\mathcal{Y}_t^c(\mu) \leq \eta_t$. On the other hand, let $(Y^n)_{n\geq 1}$ be as in the step above. We claim that it satisfies $Y^n \geq -\eta_t$, for each $n \geq 1$. Then the lower bound $\mathcal{Y}_t^c(\mu) \geq -\eta_t$ is obtained by passing to the limit. To see this, it suffices to prove this property for any $Y \in \mathbf{L}_2(\mathbb{R}, \mathcal{F}_t)$ such that $(\mu, Y) \in \overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$. But, such an element (μ, Y) is obtained by taking the $\mathbf{L}_2$ limit of elements of the form $\sum_{n\leq N} \lambda_n(\mu_n, Y_n)$ with $(\mu_n, Y_n, \lambda_n)_{n\leq N} \subset \mathrm{Epi}(\mathcal{Y}_t) \times \mathbf{L}_0([0,1], \mathcal{F}_t)$, such that $\sum_{n\leq N} \lambda_n = 1$. Each Y_n of the latter family is bounded from below by $-\eta_t$ by Remark 2.1, and hence so is Y.

Step 3. *The map $\mu \in D \mapsto \mathcal{Y}_t^c(\mu)$ is $\mathcal{F}_t$-convex.*

Fix $\mu^1, \mu^2 \in D$ and $\lambda \in \mathbf{L}_0([0,1], \mathcal{F}_t)$. Step 1 implies that $(\mu^i, \mathcal{Y}_t^c(\mu^i)) \in \overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$ for $i = 1, 2$. Clearly, $\overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$ is $\mathcal{F}_t$-convex. It follows that $(\lambda\mu^1 + (1 -$

$\lambda)\mu^2, \lambda\mathcal{Y}_t^c(\mu^1)+(1-\lambda)\mathcal{Y}_t^c(\mu^2)) \in \overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$, so that $\lambda\mathcal{Y}_t^c(\mu^1)+(1-\lambda)\mathcal{Y}_t^c(\mu^2) \geq \mathcal{Y}_t^c(\lambda\mu^1+(1-\lambda)\mu^2)$. Now, for any Y^i such that $(\mu^i, Y^i) \in \mathrm{Epi}(\mathcal{Y}_t^c)$, one has $Y^i \geq \mathcal{Y}_t^c(\mu^i)$, $i=1,2$. This fact combined with the previous inequality thus implies $\lambda Y^1+(1-\lambda)Y^2 \geq \mathcal{Y}_t^c(\lambda\mu^1+(1-\lambda)\mu^2)$. This means that $\mathrm{Epi}(\mathcal{Y}_t^c)$ is $\mathcal{F}_t$-convex.

Step 4. $\mathcal{Y}_{t*}(\mu) \geq \mathcal{Y}_t^c(\mu)$, *for all* $\mu \in D$.

Fix $\varepsilon > 0$ and set $D_\mu^\varepsilon := \{\mu' \in \mathbf{L}_0([0,1], \mathcal{F}_t), |\mu'-\mu| \leq \varepsilon\}$. It follows from Remark 2.2 that the family $\{\mathcal{Y}_t(\mu') : \mu' \in D_\mu^\varepsilon\}$ is directed downward. Then we can find a sequence $(\mu_n^\varepsilon)_{n\geq 1} \subset D_\mu^\varepsilon$ such that

$$\mathcal{Y}_t(\mu_n^\varepsilon) \to Z_\varepsilon(\mu) := \operatorname{ess\,inf}\{\mathcal{Y}_t(\mu') : \mu' \in D_\mu^\varepsilon\}, \qquad \mathbb{P}\text{-a.s.}$$

Since $(Z_\varepsilon(\mu))_{\varepsilon>0}$ is nondecreasing, $\lim_{N\to\infty} Z_{1/N}(\mu) = \mathcal{Y}_{t*}(\mu)$, recall (3.2). Note that Remark 2.1 implies that $(\mathcal{Y}_t(\mu_n^{1/N}))_{n\geq 1} \to_n Z_{1/N}(\mu)$ in $\mathbf{L}^2$ and define

$$k_N := \min\{n \geq 1 : \|\mathcal{Y}_t(\mu_n^{1/N}) - Z_{1/N}(\mu)\|_{\mathbf{L}^2} \leq 1/N\}.$$

Then $(\mu_{k_N}^{1/N}, \mathcal{Y}_t(\mu_{k_N}^{1/N})) \to (\mu, \mathcal{Y}_{t*}(\mu))$ in $\mathbf{L}^2$ as $N\to\infty$. Since $\mathrm{Epi}(\mathcal{Y}_t) \subset \overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$ and the latter is closed under $\mathbf{L}^2$-convergence, this implies that $(\mu, \mathcal{Y}_{t*}(\mu)) \in \overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$. We conclude by appealing to the definition of $\mathcal{Y}_t^c$ in (3.3).

Step 5. $\mathcal{Y}_t^c(\mu) \geq \mathcal{Y}_{t*}(\mu)$, *for all* $\mu \in D$.

In view of steps 3 and 4, the result of step 5 actually proves that $\mathcal{Y}_{t*} = \mathcal{Y}_t^c$ is $\mathcal{F}_t$-convex.

We now proceed to the proof of step 5 which is itself divided in two parts.

Step 5(a). It follows from step 1 that there exists a sequence

$$(\mu_n, Y_n, \lambda_n^N)_{n\geq 1, N\geq 1} \subset \mathrm{Epi}(\mathcal{Y}_t) \times \mathbf{L}_0([0,1], \mathcal{F}_t) \tag{3.4}$$

such that $\sum_{n\leq N} \lambda_n^N = 1$, for all N, and

$$(\hat{\mu}_N, \hat{Y}_N) := \sum_{n\leq N} \lambda_n^N(\mu_n, Y_n) \to (\mu, \mathcal{Y}_t^c(\mu)) \qquad \text{in } \mathbf{L}_2. \tag{3.5}$$

Fix $N \geq 1$ and $\varepsilon > 0$. Let $\hat{\alpha}^N \in \mathbf{H}_2$ be such that $\hat{\mu}_N = m_o + \int_0^t \hat{\alpha}_s^N\, dW_s$. Since the family $(\lambda_n^N)_{n\leq N}$ is composed of $\mathcal{F}_t$-measurable random variables summing to 1, one can find $\alpha^N \in \mathbf{H}_2$ and a random variable $\xi_N^\varepsilon \in \mathbf{L}_2(\mathcal{F}_{t+\varepsilon})$ such that

$$\hat{\mu}_N + \int_t^{t+\varepsilon} \alpha_s^N\, dW_s = \xi_N^\varepsilon \quad \text{and} \quad \mathbb{P}[\xi_N^\varepsilon = \mu_n | \mathcal{F}_t] = \lambda_n^N \qquad \text{for } n \leq N. \tag{3.6}$$

Without loss of generality, we can assume that $\alpha^N = \hat{\alpha}^N\, dt \times d\mathbb{P}$ on $[0,t]$. Then (i) of Theorem 2.1 and Remark 2.2 yield

$$\begin{aligned} \mathcal{Y}_t(\hat{\mu}_N) = \mathcal{Y}_t^{\hat{\alpha}^N} &\leq \mathcal{E}_{t,t+\varepsilon}^g(\mathcal{Y}_{t+\varepsilon}^{\alpha^N}) = \mathcal{E}_{t,t+\varepsilon}^g(\mathcal{Y}_{t+\varepsilon}(\xi_N^\varepsilon)) \\ &= \mathcal{E}_{t,t+\varepsilon}^g\Big(\sum_{n\leq N} 1_{\xi_N^\varepsilon=\mu_n} \mathcal{Y}_{t+\varepsilon}(\mu_n)\Big). \end{aligned} \tag{3.7}$$

We claim that

$$\liminf_{\varepsilon\to 0} \mathcal{E}^g_{t,t+\varepsilon}\Big(\sum_{n\le N} 1_{\xi^\varepsilon_N=\mu_n}\mathcal{Y}_{t+\varepsilon}(\mu_n)\Big) \le \sum_{n\le N}\lambda^N_n \mathcal{Y}_t(\mu_n). \tag{3.8}$$

Then (3.7), (3.8), (3.4) and (3.5) lead to

$$\mathcal{Y}_t(\hat\mu_N) \le \sum_{n\le N}\lambda^N_n \mathcal{Y}_t(\mu_n) \le \sum_{n\le N}\lambda^N_n Y_n = \hat Y_N.$$

Appealing to (3.5), we deduce that

$$\liminf_{N\to\infty} \mathcal{Y}_t(\hat\mu_N) \le \mathcal{Y}^c_t(\mu).$$

Since $\hat\mu_N \to \mu$, $\mathbb{P}$-a.s., this together with Remark 2.2 implies that

$$\begin{aligned} Z_\varepsilon(\mu) &\le \liminf_{N\to\infty} \mathcal{Y}_t(\bar\mu_N) = \liminf_{N\to\infty}\big(\mathcal{Y}_t(\hat\mu_N)\mathbf{1}_{\{|\hat\mu_N-\mu|\le\varepsilon\}} + \mathcal{Y}_t(\mu)\mathbf{1}_{\{|\hat\mu_N-\mu|>\varepsilon\}}\big)\\ &\le \mathcal{Y}^c_t(\mu), \end{aligned}$$

for all $\varepsilon > 0$, where

$$\bar\mu_N := \hat\mu_N \mathbf{1}_{\{|\hat\mu_N-\mu|\le\varepsilon\}} + \mu\mathbf{1}_{\{|\hat\mu_N-\mu|>\varepsilon\}} \in D^\varepsilon_\mu,$$

see step 4 for the definitions of $Z_\varepsilon(\mu)$ and D^ε_μ. Since $Z_\varepsilon(\mu) \uparrow \mathcal{Y}_{t*}(\mu)$ as ε goes to 0 by (3.2), this shows the required result.

Step 5(b). It finally remains to prove the claim (3.8).

Remark 2.1 and (ii) of Proposition A.2 in the Appendix imply that

$$\begin{aligned} \mathcal{E}^g_{t,t+\varepsilon}\Big(\sum_{n\le N} 1_{\xi^\varepsilon_N=\mu_n}\mathcal{Y}_{t+\varepsilon}(\mu_n)\Big) &\le E_t\Big[\sum_{n\le N} 1_{\xi^\varepsilon_N=\mu_n}\mathcal{Y}_{t+\varepsilon}(\mu_n)\Big] + \eta_\varepsilon\\ &\le E_t\Big[\sum_{n\le N} 1_{\xi^\varepsilon_N=\mu_n}\mathcal{Y}_t(\mu_n)\Big] + \eta_\varepsilon\\ &\quad + \sum_{n\le N} E_t\big[|\mathcal{Y}_{t+\varepsilon}(\mu_n) - \mathcal{Y}_t(\mu_n)|\big], \end{aligned}$$

where $\eta_\varepsilon \to 0$, $\mathbb{P}$-a.s. as $\varepsilon \to 0$. The right-hand side of (3.6) then leads to

$$\begin{aligned} \mathcal{E}^g_{t,t+\varepsilon}\Big(\sum_{n\le N} 1_{\xi^\varepsilon_N=\mu_n}\mathcal{Y}_{t+\varepsilon}(\mu_n)\Big) &\le \sum_{n\le N}\lambda^N_n\mathcal{Y}_t(\mu_n) + \eta_\varepsilon\\ &\quad + \sum_{n\le N} E_t\big[|\mathcal{Y}_{t+\varepsilon}(\mu_n) - \mathcal{Y}_t(\mu_n)|\big]. \end{aligned}$$

Recall that $\mathcal{Y}_{t+}(\mu_n) = \mathcal{Y}_t(\mu_n)$ by assumption, and that $(\mathcal{Y}(\mu_n))_n$ is bounded by some $\eta \in \mathbf{S}_2$; see Remark 2.1. Sending $\varepsilon \to 0$ in the above inequality and appealing to the Lebesgue dominated convergence theorem proves (3.8). □

In the context of PDEs, convexity in the domain propagates up to the boundary, which leads to a boundary layer phenomenon. In Bouchard, Elie and Touzi (2009/10) and Moreau (2011), this translates in the fact that the natural T-time boundary condition should be stated in terms of the m-convex envelope of Φ. We observe hereafter that this property extends to our non-Markovian setting, whenever Φ is deterministic.

We recall from Theorem 2.1(i) that $\mathcal{Y}$ is làdlàg on countable sets. Under the following condition, it will actually be càdlàg up to undistinguishability. As opposed to Proposition 3.2, we shall not need to impose any right-continuity for the following.

PROPOSITION 3.3. *Assume that Φ is deterministic and let $\hat{\Phi}$ denote its convex envelope. Then*

$$\lim_{t\uparrow T} \mathcal{Y}_t^\alpha = \hat{\Phi}\big(M_T^\alpha\big) \quad \text{and} \quad \mathcal{Y}_\tau^\alpha = \operatorname*{ess\,inf}_{\alpha' \in \mathbf{A}_\tau^\alpha} \mathcal{E}_\tau^g\big[\hat{\Phi}\big(M_T^{\alpha'}\big)\big],$$

for all $\alpha \in \mathbf{A}_0$ and $\tau \in \mathcal{T}$ such that $\tau < T$.

Before proving this result, let us make some observations.

REMARK 3.1. Since Φ is nondecreasing, its convex envelope is continuous on $[0, 1)$. Moreover, Φ is left-continuous, so that $\hat{\Phi}$ has to be continuous at 1 as well.

REMARK 3.2. In Section 2.3, we observed that the essential infimum in the dynamic programming principle is attained whenever Φ and g are convex. Hence, the previous proposition allows straightforwardly to avoid the convexity requirement on Φ, whenever it is deterministic.

REMARK 3.3. The proof below can easily be adapted to the case where $\Phi(\omega, m) = \phi(m)\xi(\omega)$ for some nonnegative random variable ξ and a deterministic map ϕ. This is due to the fact that the m-convex envelope of Φ is fully characterized by the convex envelope $\hat{\phi}$ of ϕ : $\hat{\Phi}(\omega, m) = \hat{\phi}(m)\xi(\omega)$. This allows one to follow the construction used in our proof. In particular, in the quantile hedging problem of Föllmer and Leukert (1999), one has $\Phi(\omega, m) = \mathbf{1}_{\{m>0\}}\xi(\omega)$ $(m \in [0, 1])$, with ξ taking nonnegative values, so that $\hat{\Phi}(\omega, m) = m\xi(\omega)$; see also Bouchard, Elie and Touzi (2009/10).

PROOF OF PROPOSITION 3.3. We prove each assertion separately.

Step 1. By definition of the convex envelope, we can find a measurable map $m \in [0, 1] \mapsto (\underline{\wp}(m), \overline{\wp}(m), \varepsilon(m)) \in [0, 1]^3$ such that $\underline{\wp}(m) \le m \le \overline{\wp}(m)$, $\varepsilon(m)\underline{\wp}(m) + (1 - \varepsilon(m))\overline{\wp}(m) = m$ and

$$\hat{\Phi}(m) = \varepsilon(m)\Phi\big(\underline{\wp}(m)\big) + \big(1 - \varepsilon(m)\big)\Phi\big(\overline{\wp}(m)\big),$$

for any $m \in [0,1]$. Let $t_n \uparrow T$. Then one can find $\alpha^n \in \mathbf{A}_{t_n}^{\alpha}$ and $\xi^n \in \mathbf{L}_0([0,1])$ such that $M_T^{\alpha^n} = M_{t_n}^{\alpha} + \int_{t_n}^T \alpha_s^n \, dW_s = \xi^n$, where $\mathbb{P}[\xi^n = \underline{\wp}(M_{t_n}^{\alpha})|\mathcal{F}_{t_n}] = \varepsilon(M_{t_n}^{\alpha})$ and $\mathbb{P}[\xi^n = \overline{\wp}(M_{t_n}^{\alpha})|\mathcal{F}_{t_n}] = 1 - \varepsilon(M_{t_n}^{\alpha})$. It follows from the above and (iii) of Proposition A.2 in the Appendix that

$$\mathcal{Y}_{t_n}^{\alpha} \le E_{t_n}\big[\Phi(\xi^n)\big] + \eta_n = \hat{\Phi}\big(M_{t_n}^{\alpha}\big) + \eta_n,$$

where $\eta_n \to 0$ as $n \to \infty$. Since $\mathcal{Y}$ is làdlàg on countable sets (by Proposition 4.2), passing to the limit implies that

$$\lim_{n\to\infty} \mathcal{Y}_{t_n}^{\alpha} \le \hat{\Phi}\big(M_T^{\alpha}\big). \tag{3.9}$$

We now prove the converse inequality. We use (iii) in Proposition A.2 in the Appendix and Jensen's inequality to deduce that

$$Y_{t_n}^{\alpha'} := \mathcal{E}_{t_n,T}^{g}\big[\Phi\big(M_T^{\alpha'}\big)\big] \ge E_{t_n}\big[\hat{\Phi}\big(M_T^{\alpha'}\big)\big] - \bar{\eta}_n \ge \hat{\Phi}\big(M_{t_n}^{\alpha}\big) - \bar{\eta}_n, \qquad \alpha' \in \mathbf{A}_{t_n}^{\alpha},$$

where $\bar{\eta}_n \to 0$ as $n \to \infty$. Combining the arbitrariness of $\alpha' \in \mathbf{A}_{t_n}^{\alpha}$ with the làdlàg property of $\mathcal{Y}$ on countable sets, we get that

$$\lim_{n\to\infty} \mathcal{Y}_{t_n}^{\alpha} \ge \liminf_{n\to\infty} \operatorname*{ess\,inf}_{\alpha' \in \mathbf{A}_{t_n}^{\alpha}} Y_{t_n}^{\alpha'} \ge \hat{\Phi}\big(M_T^{\alpha}\big).$$

Step 2. It follows from Theorem 2.1(i) that

$$\mathcal{Y}_{\tau}^{\alpha} = \operatorname*{ess\,inf}_{\alpha' \in \mathbf{A}_{\tau}^{\alpha}} \mathcal{E}_{\tau,t_n\vee\tau}^{g}\big[\mathcal{Y}_{t_n\vee\tau}^{\alpha'}\big], \qquad n \in \mathbb{N}.$$

The process $\mathcal{Y}_{\cdot\vee\tau}^{\alpha'}$ being làdlàg on the set $\{t_n, n \ge 1\}$, $\lim_{n\to\infty} \mathcal{Y}_{t_n\vee\tau}^{\alpha'}$ is well defined and coincides with $\lim_{n\to\infty} \mathcal{Y}_{t_n}^{\alpha'}$. Moreover, it follows from the bound in Remark 2.1 that the convergence holds in $\mathbf{L}_2$. In view of the stability result of Proposition A.1 and step 1. above, passing to the limit as $n \to \infty$ leads to

$$\mathcal{Y}_{\tau}^{\alpha} \le \operatorname*{ess\,inf}_{\alpha' \in \mathbf{A}_{\tau}^{\alpha}} \mathcal{E}_{\tau}^{g}\Big[\lim_{n\to\infty} \mathcal{Y}_{t_n}^{\alpha'}\Big] = \operatorname*{ess\,inf}_{\alpha' \in \mathbf{A}_{\tau}^{\alpha}} \mathcal{E}_{\tau}^{g}\big[\hat{\Phi}\big(M_T^{\alpha'}\big)\big].$$

Since $\Phi \ge \hat{\Phi}$, the reverse inequality holds by definition of $\mathcal{Y}_{\tau}^{\alpha}$ in (2.6). Since $\hat{\Phi}$ is continuous by Remark 3.1, we can now appeal to the second statement of Proposition 4.2 to assert that, up to indistinguishability, $\mathcal{Y}^{\alpha}$ is càdlàg, so that $\lim_{t\uparrow T} \mathcal{Y}_t^{\alpha} = \lim_{n\to\infty} \mathcal{Y}_{t_n}^{\alpha}$. □

3.3. *Dual representation*. In this section, we provide a dual formulation for the minimal initial condition at time 0, $m \mapsto \mathcal{Y}_0(m)$. It requires the introduction of the Fenchel transforms of g and Φ.

We therefore define

$$\tilde{\Phi} : (\omega, l) \in \Omega \times \mathbb{R} \mapsto \sup_{m\in[0,1]} \big(ml - \Phi(\omega, m)\big)$$

and

$$\tilde{g}:(\omega,t,u,v)\in\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\mapsto \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d}\bigl(yu+z^\top v-g(\omega,t,y,z)\bigr).$$

REMARK 3.4. It follows from the assumption $(\mathbf{H}_g)$ that the domain of $\tilde{g}(\omega,t,\cdot)$, $\mathrm{dom}(\tilde{g}(\omega,t,\cdot))$, is contained in $[-K_g,K_g]^{d+1}$ for $\mathbb{P}$-a.e. $\omega\in\Omega$ and all $t\le T$. The assumption $(\mathbf{H}_\Psi)$ ensures that the domain of $\tilde{\Phi}(\omega,\cdot)$ is the all real line, $\mathbb{P}$-a.s.

In the following, we denote by $\mathbf{\Lambda}$ the set of predictable processes λ with values in $\mathbb{R}\times\mathbb{R}^d$ such that $\lambda_t(\omega)\in\mathrm{dom}(\tilde{g}(\omega,t,\cdot))$ for Leb $\times\,\mathbb{P}$-a.e. $(\omega,t)\in\Omega\times[0,T]$.

To $\lambda=(\nu,\vartheta)\in\mathbf{\Lambda}$, we associate the process L^λ defined by

$$L_t^\lambda=1+\int_0^t L_s^\lambda\nu_s\,ds+\int_0^t L_s^\lambda\vartheta_s\,dW_s,\qquad t\in[0,T].$$

Our dual formulation for $\mathcal{Y}_0$ is stated in terms of

$$\mathcal{X}_0(l):=\inf_{\lambda\in\mathbf{\Lambda}}\mathrm{X}_0^{l,\lambda},\qquad l>0,$$

where

$$\mathrm{X}_0^{l,\lambda}:=E\biggl[\int_0^T L_s^\lambda\tilde{g}(s,\lambda_s)\,ds+L_T^\lambda\tilde{\Phi}\bigl(l/L_T^\lambda\bigr)\biggr],\qquad \lambda\in\mathbf{\Lambda},\,l>0.$$

The fact that the Fenchel transform of $\mathcal{X}_0$ provides a lower bound for $\mathcal{Y}_0$ is straightforward, and detailed in Proposition 3.4 below for the convenience of the reader. For ease of notation, we now write $\mathbf{A}_m$ for $\mathbf{A}_{0,m}$, $M^{m,\alpha}$ for $M^{(0,m),\alpha}$, and denote by $(Y^{m,\alpha},Z^{m,\alpha})$ the solution of the BSDE$(g,\Phi(M_T^{m,\alpha}))$, $\alpha\in\mathbf{A}_m$.

PROPOSITION 3.4. *$\mathcal{Y}_0(m)\ge\sup_{l>0}(lm-\mathcal{X}_0(l))$, for all $m\in[0,1]$.*

PROOF. Fix $\alpha\in\mathbf{A}_m$ and $\lambda=(\nu,\vartheta)\in\mathbf{\Lambda}$. Then it follows from the definition of $\tilde{\Phi}$ and $\tilde{g}$ that

$$\begin{aligned}E\bigl[Y_T^{m,\alpha}L_T^\lambda\bigr]&=Y_0^{m,\alpha}+E\biggl[\int_0^T L_s^\lambda\bigl(\nu_sY_s^{m,\alpha}+\vartheta_s^\top Z_s^{m,\alpha}-g\bigl(s,Y_s^{m,\alpha},Z_s^{m,\alpha}\bigr)\bigr)\,ds\biggr]\\&\le Y_0^{m,\alpha}+E\biggl[\int_0^T L_s^\lambda\tilde{g}(s,\lambda_s)\,ds\biggr]\end{aligned}$$

and

$$Y_T^{m,\alpha}L_T^\lambda=\Phi\bigl(M_T^{m,\alpha}\bigr)L_T^\lambda\ge lM_T^{m,\alpha}-L_T^\lambda\tilde{\Phi}\bigl(l/L_T^\lambda\bigr),$$

for $l>0$. Note that, in the above, we have cancelled the expectation of the local martingale part $\int_0^T(L_s^\lambda Z_s^{m,\alpha}+Y_s^{m,\alpha}L_s^\lambda\vartheta_s)\,dW_s$ although $L^\lambda Z^{m,\alpha}$ might not belong to $\mathbf{H}_2$. If not, one may use a localization argument since all other terms belongs to

L^1 uniformly in time. Combining the above and using the martingale property of $M^{m,\alpha}$ yields

$$Y_0^{m,\alpha} \geq lm - E\left[\int_0^T L_s^\lambda \tilde{g}(s,\lambda_s)\,ds + L_T^\lambda \tilde{\Phi}\big(l/L_T^\lambda\big)\right] = lm - X_0^{l,\lambda}.$$

The result follows from the arbitrariness of $l > 0$, $\lambda \in \mathbf{\Lambda}$ and $\alpha \in \mathbf{A}_m$. $\square$

We now show that equality is satisfied in Proposition 3.4 whenever existence holds in the dual problem. This is proved under the following assumptions. Let C_b^1 be the set of continuously differentiable maps with bounded first derivatives.

Assumption ($\mathbf{H}_d^1$). The following holds for Leb $\times\, \mathbb{P}$-a.e. $(t,\omega) \in [0,T] \times \Omega$:

(a) the maps $\tilde{\Phi}(\omega,\cdot)$ and $\tilde{g}(\omega,\cdot)$ are C_b^1 on their domain, and $\mathrm{dom}(\tilde{g}(\omega,t,\cdot))$ is closed;

(b) $|\nabla\tilde{\Phi}(\omega,\cdot)| + |\nabla\tilde{g}(\omega,t,\cdot)| \leq \chi_{\tilde{\Phi},\tilde{g}}(\omega)$, for some $\chi_{\tilde{\Phi},\tilde{g}} \in \mathbf{L}_2(\mathbb{R})$;

(c) $\Phi(\omega,m) = \sup_{l>0}(lm - \tilde{\Phi}(\omega,l))$, for all $m \in [0,1]$;

(d) $g(\omega,t,y,z) = \max_{(u,v)\in\mathrm{dom}(\tilde{g}(\omega,t,\cdot))}(yu + z^\top v - \tilde{g}(\omega,t,u,v))$, for all $(y,z) \in \mathbb{R}\times\mathbb{R}^d$.

In the above, $\nabla\tilde{\Phi}$ and $\nabla\tilde{g}$ stands for the gradient with respect to l and (u,v), respectively.

Note that (a) and (b) are of technical nature, while (c) and (d) mean that Φ and g are convex, that is, coincide with their bidual. The latter is a minimal requirement if one wants the duality to hold.

PROPOSITION 3.5. *Let assumption* ($\mathbf{H}_d^1$) *hold. Assume further that there exists* $\hat{l} > 0$ *and* $\hat{\lambda} \in \mathbf{\Lambda}$ *such that*

$$\sup_{l>0}\big(lm - \mathcal{X}_0(l)\big) = \hat{l}m - \mathcal{X}_0(\hat{l}) = \hat{l}m - \mathrm{X}_0^{\hat{l},\hat{\lambda}}. \tag{3.10}$$

Then there exists $\hat{\alpha} \in \mathbf{A}_m$ *such that*

$$\mathcal{Y}_0(m) = Y_0^{m,\hat{\alpha}} = \hat{l}m - \mathcal{X}_0(\hat{l}).$$

It satisfies

$$\begin{aligned} g\big(\cdot, Y^{m,\hat{\alpha}}, Z^{m,\hat{\alpha}}\big) &= \hat{\lambda}^\top\big(Y^{m,\hat{\alpha}}, Z^{m,\hat{\alpha}}\big) - \tilde{g}(\cdot,\hat{\lambda}), \\ \Phi\big(M_T^{m,\hat{\alpha}}\big) &= \frac{M_T^{m,\hat{\alpha}}\hat{l}^{\hat{\lambda}}}{L}_T - \tilde{\Phi}\big(\hat{l}/L_T^{\hat{\lambda}}\big). \end{aligned} \tag{3.11}$$

Before to provide the proof, let us make the following observation which pertains for the case of a linear driver g.

REMARK 3.5. Assume that g is linear, that is, there exist bounded predictable processes A^Y and A^Z such that $g:(\omega,t,y,z)\mapsto g(\omega,t,0,0)+A_t^Y(\omega)y+A_t^Z(\omega)z$. In this case, $\mathbf{\Lambda}=\{(A^Y,A^Z)\}$ and, therefore,

$$\mathcal{X}_0(l)=E\bigg[\int_0^T L_s\tilde{g}\big(s,A_s^Y,A_s^Z\big)\,ds+L_T\tilde{\Phi}(l/L_T)\bigg],$$

with L given by

$$L_t=1+\int_0^t L_sA_s^Y\,ds+\int_0^t L_sA_s^Z\,dW_s,\qquad t\in[0,T].$$

Then the dual formulation of Proposition 3.5 above drops down to finding $\hat{l}$ which maximizes $lm-\mathcal{X}_0(l)$. This generalizes the result of Föllmer and Leukert (1999) and Bouchard, Elie and Touzi (2009/10) obtained for quantile hedging problems in linear models of financial markets.

PROOF OF PROPOSITION 3.5. We split the proof in two steps.

Step 1. For ease of notation, we set $\hat{L}:=L^{\hat{\lambda}}$. By optimality of $\hat{l}$, one has

$$\hat{l}m-E\big[\hat{L}_T\tilde{\Phi}(\hat{l}/\hat{L}_T)\big]\geq m(\hat{l}+\iota)-E\big[\hat{L}_T\tilde{\Phi}\big((\hat{l}+\iota)/\hat{L}_T\big)\big],$$

for all $\iota>-\hat{l}$. Since $\tilde{\Phi}$ is by construction $\mathbb{P}$-a.s. convex, this implies that $\zeta_\iota:=\nabla\tilde{\Phi}((\hat{l}+\iota)/\hat{L}_T)$ satisfies $m\iota\leq E[\zeta_\iota]\iota$, for all $\iota>-\hat{l}$, recall $(\mathbf{H}_d^1)$(a) and (b). Taking ι of the form $-1/n$ and then $1/n$, for $n\to\infty$, and using $(\mathbf{H}_d^1)$(a) and (b) then leads to

$$m=E[\zeta]\qquad\text{where }\zeta:=\nabla\tilde{\Phi}(\hat{l}/\hat{L}_T). \tag{3.12}$$

We now appeal to $(\mathbf{H}_d^1)$(c) to deduce that

$$\Phi(\zeta)=\zeta(\hat{l}/\hat{L}_T)-\tilde{\Phi}(\hat{l}/\hat{L}_T). \tag{3.13}$$

By construction, $\tilde{\Phi}$ is $\mathbb{P}$-a.s. 1-Lipschitz and nondecreasing, that is, $\zeta\in\mathbf{L}_0([0,1])$. In view of (3.12), the martingale representation theorem then implies that we can find $\hat{\alpha}\in\mathbf{A}_m$ such that $\hat{M}_T:=M_T^{m,\hat{\alpha}}=\zeta$.

Step 2. We now write $(\hat{\nu},\hat{\vartheta}):=\hat{\lambda}$ and fix $\lambda=(\nu,\theta)\in\mathbf{\Lambda}$ to be chosen later on. Clearly, $\mathbf{\Lambda}$ is convex. Hence, $\lambda^\varepsilon:=(1-\varepsilon)(\hat{\nu},\hat{\vartheta})+\varepsilon(\nu,\vartheta)\in\mathbf{\Lambda}$, $\varepsilon\in[0,1]$. Moreover, direct computations show that

$$\frac{\partial}{\partial\varepsilon}L^{\lambda^\varepsilon}\bigg|_{\varepsilon=0}=\hat{L}\hat{R}\qquad\text{where }\hat{R}:=\int_0^\cdot(\delta\nu_s-\delta\vartheta_s\hat{\vartheta}_s)\,ds+\int_0^\cdot\delta\vartheta_s\,dW_s,$$

in which we use the notation $\delta\lambda:=(\delta\nu,\delta\vartheta):=(\nu-\hat{\nu},\vartheta-\hat{\vartheta})$.

Recalling that elements of $\mathbf{\Lambda}$ take bounded values (see Remark 3.4, and arguing as in step 1), one easily checks that the optimality condition $\mathrm{X}_0^{\hat{l},\lambda^\varepsilon}\geq\mathrm{X}_0^{\hat{l},\hat{\lambda}}$, for all

$\varepsilon \in [0,1]$, implies that $\hat{\eta} := \nabla \tilde{g}(\cdot, \hat{\lambda})$ satisfies

$$\begin{aligned} 0 &\le E\bigg[\int_0^T \hat{L}_s\big(\hat{R}_s \tilde{g}(s,\hat{\lambda}_s) + \hat{\eta}_s^\top \delta\lambda_s\big)\,ds + \hat{R}_T \hat{L}_T \tilde{\Phi}(\hat{I}/\hat{L}_T)\bigg] \\ &\quad + E\big[-\hat{R}_T \hat{L}_T (\hat{I}/\hat{L}_T) \nabla \tilde{\Phi}(\hat{I}/\hat{L}_T)\big] \\ &= E\bigg[\int_0^T \hat{L}_s\big(\hat{R}_s \tilde{g}(s,\hat{\lambda}_s) + \hat{\eta}_s^\top \delta\lambda_s\big)\,ds - \hat{R}_T \hat{L}_T \Phi(\hat{M}_T)\bigg], \end{aligned} \tag{3.14}$$

in which we used (3.12), (3.13) and the relation $\zeta = \hat{M}_T$ to deduce the second equality. Let $(\hat{Y}, \hat{Z})$ be defined by

$$\hat{Y} := \hat{L}^{-1} E_\cdot\bigg[\hat{L}_T \Phi(\hat{M}_T) - \int_\cdot^T \hat{L}_s \tilde{g}(s,\hat{\lambda}_s)\,ds\bigg] \quad \text{and} \quad \hat{Z} := \bar{Z} - \hat{Y}\hat{\vartheta}, \tag{3.15}$$

where $\bar{Z} \in \mathbf{H}_2$ is implicitly given by

$$\hat{L}_t \hat{Y}_t = \hat{L}_T \Phi(\hat{M}_T) - \int_t^T \hat{L}_s \tilde{g}(s,\hat{\lambda}_s)\,ds - \int_t^T \hat{L}_s \bar{Z}_s\,dW_s, \qquad 0 \le t \le T. \tag{3.16}$$

The above combined with (3.14) implies

$$0 \le E\bigg[\int_0^T \hat{L}_s\big(\hat{R}_s \tilde{g}(s,\hat{\lambda}_s) + \hat{\eta}_s^\top \delta\lambda_s\big)\,ds - \hat{R}_T \hat{L}_T \hat{Y}_T\bigg].$$

Recalling the definition of $\hat{R}$ and $\hat{\eta}$ and applying Itô's lemma, this leads to

$$\begin{aligned} 0 &\le E\bigg[\int_0^T \hat{L}_s\big(\hat{\eta}_s - (\hat{Y}_s, \hat{Z}_s)\big)^\top \delta\lambda_s\,ds\bigg] \\ &= E\bigg[\int_0^T \hat{L}_s\big(\nabla \tilde{g}(s,\hat{\lambda}_s) - (\hat{Y}_s, \hat{Z}_s)\big)^\top \delta\lambda_s\,ds\bigg]. \end{aligned} \tag{3.17}$$

By assumption $(\mathbf{H}_d^1)$(a), Remark 3.4 and [Aliprantis and Border (2006), Theorem 18.19, page 605], one can choose $\bar{\lambda} \in \mathbf{\Lambda}$ such that

$$\bar{\lambda} = \operatorname{argmin}\big\{f(\cdot,u,v),\,(u,v) \in \operatorname{dom}\big(\tilde{g}(\cdot)\big)\big\}, \qquad \text{Leb} \times \mathbb{P}\text{-a.e.},$$

where

$$f : (\omega, s, u, v) \mapsto \big(\nabla \tilde{g}\big(\omega, s, \hat{\lambda}_s(\omega)\big) - \big(\hat{Y}_s(\omega), \hat{Z}_s(\omega)\big)\big)^\top \big(u - \hat{v}_s(\omega), v - \hat{\vartheta}_s(\omega)\big).$$

Considering now relation (3.17) with λ chosen to be equal to $\bar{\lambda}\mathbf{1}_{\{f(\cdot,\bar{\lambda})<0\}}$, we see that, for Leb $\times\, \mathbb{P}$-a.e. $(\omega, t) \in \Omega \times [0,T]$, the gradient $\Delta_t(\omega)$ at $\hat{\lambda}_t(\omega)$ of the convex map

$$(u,v) \in \operatorname{dom}\big(\tilde{g}(\omega,t,\cdot)\big) \mapsto F(\omega,t,u,v) := \tilde{g}(\omega,t,u,v) - u\hat{Y}_t(\omega) - v^\top \hat{Z}_t(\omega)$$

satisfies

$$\Delta_t(\omega)^\top\big(b-\hat\lambda_t(\omega)\big)\ge 0 \qquad \text{for all } b\in \operatorname{dom}\big(\tilde g(\omega,t,\cdot)\big).$$

This implies that $\hat\lambda_t(\omega)$ minimizes $F(\omega,t,\cdot)$ for Leb $\times\,\mathbb{P}$-a.e. $(\omega,t)\in\Omega\times[0,T]$ and, therefore, we compute

$$\tilde g(\cdot,\hat\lambda)=\hat\lambda^\top(\hat Y,\hat Z)-g(\cdot,\hat Y,\hat Z), \qquad \text{Leb}\times\mathbb{P}\text{-a.e.}$$

by $(\mathbf{H}_d^1)$(d). Combining the above identity with (3.16) leads to $(\hat Y,\hat Z)=(Y^{m,\hat\alpha},Z^{m,\hat\alpha})$. Then, by using (3.12), (3.13) and (3.15), in which $\hat L_0=1$, we obtain

$$\begin{aligned}
Y_0^{m,\hat\alpha}&=E\bigg[\hat L_T\Phi(\hat M_T)-\int_0^T\hat L_s\tilde g(s,\hat\lambda_s)\,ds\bigg]\\
&=E\bigg[\hat L_T\big(\zeta\hat l/\hat L_T-\tilde\Phi(\hat l/\hat L_T)\big)-\int_0^T\hat L_s\tilde g(s,\hat\lambda_s)\,ds\bigg]\\
&=\hat l m-E\bigg[\hat L_T\tilde\Phi(\hat l/\hat L_T)+\int_0^T\hat L_s\tilde g(s,\hat\lambda_s)\,ds\bigg].
\end{aligned}$$

In view of Proposition 3.4, this completes the proof. □

We now state the reciprocal statement: existence in the primal problem provides existence in the dual one. Here again, we need to impose some additional technical conditions.

Assumption $(\mathbf{H}_d^2)$. The following holds for Leb $\times\,\mathbb{P}$-a.e. $(t,\omega)\in[0,T]\times\Omega$:

(a) the maps $\Phi(\omega,\cdot)$ and $g(\omega,t,\cdot)$ are C_b^1 on $[0,1]$ and $\mathbb{R}\times\mathbb{R}^d$, respectively;
(b) $|\nabla\Phi(\omega,\cdot)|\le\chi_\Phi(\omega)$, for some $\chi_\Phi\in\mathbf{L}_2(\mathbb{R})$.

PROPOSITION 3.6. *Let assumption $(\mathbf{H}_d^2)$ hold. Let $l>0$ be fixed and assume that there exists $\hat m\in[0,1]$ and $\hat\alpha\in\mathbf{A}_{\hat m}$ such that*

$$\sup_{m\in[0,1]}\sup_{\alpha\in\mathbf{A}_m}\big(ml-\mathcal{Y}_0(m)\big)=\hat m l-Y_0^{\hat m,\hat\alpha}. \tag{3.18}$$

Then there exists $\hat\lambda\in\mathbf{\Lambda}$ such that

$$\mathcal{Y}_0(\hat m)=\hat m l-\mathcal{X}_0(l)=\hat m l-\mathrm{X}_0^{l,\hat\lambda},$$

and $\hat\lambda$ satisfies (3.11) with $m=\hat m$ and $\hat l=l$.

PROOF. Given $\varepsilon\in[0,1]$, a martingale M with values in $[0,1]$, $m:=M_0$, we set $m_\varepsilon:=\hat m+\varepsilon(m-\hat m)$, $M^\varepsilon:=\hat M+\varepsilon(M-\hat M)$, where $\hat M:=M^{\hat m,\hat\alpha}$. For ease of notation, we set $(\hat Y,\hat Z):=(Y^{\hat m,\hat\alpha},Z^{\hat m,\hat\alpha})$ and denote by $(Y^\varepsilon,Z^\varepsilon)$ the solution of BSDE$(g,\Phi(M_T^\varepsilon))$, $\delta m:=m-\hat m$, $(\delta M,\delta Y^\varepsilon,\delta Z^\varepsilon):=(M-\hat M,Y^\varepsilon-\hat Y,Z^\varepsilon-\hat Z)$.

Step 1. *We first show that* $\varepsilon^{-1}(\delta Y_s^\varepsilon, \delta Z_s^\varepsilon)$ *converges in* $\mathbf{S}_2 \times \mathbf{H}_2$ *as* $\varepsilon \to 0$ *to the solution* $(\nabla Y, \nabla Z)$ *of*

$$\begin{aligned}\nabla Y_t &= \nabla\Phi(\hat{M}_T)\delta M_T + \int_t^T \nabla g(s, \hat{Y}_s, \hat{Z}_s)^\top (\nabla Y_s, \nabla Z_s)\, ds \\ &\quad - \int_t^T \nabla Z_s\, dW_s.\end{aligned} \tag{3.19}$$

First note that existence and uniqueness of the solution to the above BSDE in guaranteed by assumption $(\mathbf{H}_d^2)$.

Letting $\xi^\varepsilon := \varepsilon^{-1}(\Phi(M_T^\varepsilon) - \Phi(\hat{M}_T))$, one easily checks that $\varepsilon^{-1}(\delta Y_s^\varepsilon, \delta Z_s^\varepsilon)$ solves

$$\frac{\delta Y_s^\varepsilon}{\varepsilon} = \xi^\varepsilon - \int_s^T \frac{\delta Z_r^\varepsilon}{\varepsilon}\, dW_r + \int_s^T \left(A_r^{Y,\varepsilon} \frac{\delta Y_r^\varepsilon}{\varepsilon} + A_r^{Z,\varepsilon} \frac{\delta Z_r^\varepsilon}{\varepsilon} \right) dr,$$

where

$$A_r^{Y,\varepsilon} := \int_0^1 \partial_y g\big(r, \hat{Y}_r + \theta \delta Y_r^\varepsilon, \hat{Z}_r\big)\, d\theta \quad \text{and}$$

$$A_r^{Z,\varepsilon} := \int_0^1 \partial_z g\big(r, Y_r^\varepsilon, \hat{Z}_r + \theta \delta Z_r^\varepsilon\big)\, d\theta.$$

In the above, $\partial_y g$ and $\partial_z g$ denotes respectively the partial gradients of g with respect to y and z, recall $(\mathbf{H}_d^2)$. Assumption $(\mathbf{H}_g)$ implies $|A^{Y,\varepsilon}| + |A^{Z,\varepsilon}| \le K_g$.

We now set $U^\varepsilon := \varepsilon^{-1}\delta Y_s^\varepsilon - \nabla Y$, $V^\varepsilon := \varepsilon^{-1}\delta Z_s^\varepsilon - \nabla Z$ and $\zeta^\varepsilon := \xi^\varepsilon - \nabla\Phi(\hat{M}_T)\delta M$. The pair $(U^\varepsilon, V^\varepsilon)$ is an element of $\mathbf{S}_2 \times \mathbf{H}_2$ and solves

$$U_s^\varepsilon = \zeta^\varepsilon - \int_s^T V_r^\varepsilon\, dW_r + \int_s^T \big(A_r^{Y,\varepsilon} U_r^\varepsilon + A_r^{Z,\varepsilon} V_r^\varepsilon + R_r^\varepsilon\big)\, dr, \qquad 0 \le s \le T,$$

with

$$R_r^\varepsilon := \nabla Z_r \big(A_r^{Z,\varepsilon} - \partial_z g(r, \hat{Y}_r, \hat{Z}_r)\big) + \nabla Y_r \big(A_r^{Y,\varepsilon} - \partial_y g(r, \hat{Y}_r, \hat{Z}_r)\big), \qquad 0 \le r \le T.$$

Hence, by stability for Lipschitz BSDEs (see Proposition A.1 in the Appendix), there exists a constant $C > 0$ (which does not depend on ε) such that

$$\big\|U^\varepsilon\big\|_{\mathbf{S}_2}^2 + \big\|V^\varepsilon\big\|_{\mathbf{H}_2}^2 \le C\big(\big\|\zeta^\varepsilon\big\|_{\mathbf{L}_2}^2 + \big\|R_r^\varepsilon\big\|_{\mathbf{H}_2}^2\big). \tag{3.20}$$

The result of step 1 will follow if we prove that the right-hand side of the inequality (3.20) vanishes as ε tends to zero. The convergence of $\|R_r^\varepsilon\|_{\mathbf{H}_2}^2$ to 0 follows from assumption $(\mathbf{H}_d^2)$ and the convergence of M_T^ε to M_T. As for the second term, it suffices to prove that $(Y^\varepsilon, Z^\varepsilon)_\varepsilon$ converges in $\mathbf{S}_2 \times \mathbf{H}_2$ to $(\hat{Y}, \hat{Z})$, and to appeal to $(\mathbf{H}_g)$ and $(\mathbf{H}_d^2)$. The latter is obtained by standard stability results (see Proposition A.1 below), which imply the existence of a constant $C > 0$ (which does not depend on ε) such that

$$\big\|Y^\varepsilon - \hat{Y}\big\|_{\mathbf{S}_2}^2 + \big\|Z^\varepsilon - \hat{Z}\big\|_{\mathbf{H}_2}^2 \le C\big\|\Phi(M_T^\varepsilon) - \Phi(\hat{M}_T)\big\|_{\mathbf{L}_2}^2 \longrightarrow_{\varepsilon\to 0} 0.$$

In the latter, the convergence follows from Lebesgue's dominated convergence theorem and assumption $(\mathbf{H}_d^2)$.

Step 2. By optimality of $(\hat{m}, \hat{\alpha})$, $Y_0^\varepsilon - m_\varepsilon l - \hat{Y}_0 + \hat{m} l \geq 0$, for any $\varepsilon > 0$. In view of step 1, dividing by $\varepsilon > 0$ and sending $\varepsilon \to 0$ leads to

$$0 \leq \nabla\Phi(\hat{M}_T)\delta M_T - l\delta m + \int_0^T \nabla g(s, \hat{Y}_s, \hat{Z}_s)^\top (\nabla Y_s, \nabla Z_s)\, ds - \int_0^T \nabla Z_s\, dW_s$$
$$= \nabla Y_0 - l\delta m,$$

after possibly passing to a subsequence.

Set $\hat{L} := L^{\hat{\lambda}}$ where $\hat{\lambda} := \nabla g(\cdot, \hat{Y}, \hat{Z})$. Observe that the latter belongs to $\mathbf{\Lambda}$. For later use, also notice that

$$g(\cdot, \hat{Y}, \hat{Z}) = (\hat{\nu}, \hat{\vartheta})^\top (\hat{Y}, \hat{Z}) - \tilde{g}(\cdot, \hat{\nu}, \hat{\vartheta}), \tag{3.21}$$

see, for example, Rockafellar (1997). Then it follows from (3.19) that $\hat{L}\nabla Y$ is a martingale. The previous inequality thus implies that

$$0 \leq \hat{L}_0 \nabla Y_0 - l\delta m = E[\hat{L}_T \nabla Y_T] - l\delta m = E\big[\hat{L}_T \delta M_T \big(\nabla\Phi(\hat{M}_T) - l/\hat{L}_T\big)\big],$$

in which we used the fact that $\hat{L}_0 = 1$ and $E[\delta M_T] = \delta m$. Since M_T can be any arbitrary random variable with values in $[0, 1]$, this shows that, $\mathbb{P}$-a.s., $\hat{M}_T(\omega)$ minimizes $m \in [0, 1] \mapsto \Phi(\omega, m) - ml/\hat{L}_T(\omega)$. Hence,

$$\hat{M}_T l - \hat{L}_T \Phi(\hat{M}_T) = \hat{L}_T \tilde{\Phi}(l/\hat{L}_T),$$

see, for example, Rockafellar (1997). Combining the above identity together with (3.21) and using Itô's lemma leads to $l\hat{m} - \hat{Y}_0 = \mathrm{X}_0^{\hat{l},\hat{\lambda}}$. One concludes by appealing to Proposition 3.4. □

4. Proof of Theorem 2.1. In all of this section, we use the notation introduced at the beginning of Section 2.2. The first main result provides a dynamic programming principle for the family $\{\mathcal{Y}_\tau^\alpha, \tau \in \mathcal{T}, \alpha \in \mathbf{A}_0\}$.

PROPOSITION 4.1. *For all* $(\tau_1, \tau_2, \alpha) \in \mathcal{T} \times \mathcal{T} \times \mathbf{A}_0$ *such that* $\tau_1 \leq \tau_2$, *we have*

$$\mathcal{Y}_{\tau_1}^\alpha = \operatorname*{ess\,inf}_{\alpha' \in \mathbf{A}_{\tau_1}^\alpha} \mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{Y}_{\tau_2}^{\alpha'}].$$

PROOF. We prove the two corresponding inequalities separately.

Step 1. $\mathcal{Y}_{\tau_1}^\alpha \geq \operatorname{ess\,inf}_{\alpha' \in \mathbf{A}_{\tau_1}^\alpha} \mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{Y}_{\tau_2}^{\alpha'}]$.

It follows from Lemma 4.1 below that there exists $(\alpha^n)_n$ in $\mathbf{A}_{\tau_1}^\alpha$ such that the sequence $(\mathcal{E}_{\tau_1,T}^g[\Phi(M_T^{\alpha^n})])_n$ is nonincreasing and

$$\lim_{n\to\infty} \mathcal{E}_{\tau_1,T}^g\big[\Phi\big(M_T^{\alpha^n}\big)\big] = \mathcal{Y}_{\tau_1}^\alpha, \qquad \mathbb{P}\text{-a.s.} \tag{4.1}$$

Since $\alpha^n \in \mathbf{A}_{\tau_2}^{\alpha^n}$ for every $n \geq 1$, we deduce that

$$\mathcal{Y}_{\tau_2}^{\alpha^n} \leq \mathcal{E}_{\tau_2,T}^g[\Phi(M_T^{\alpha^n})].$$

By comparison for BSDEs with Lipschitz continuous drivers on the time interval $[\tau_1, \tau_2]$, this implies

$$\mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{Y}_{\tau_2}^{\alpha^n}] \leq \mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{E}_{\tau_2,T}^g[\Phi(M_T^{\alpha^n})]] = \mathcal{E}_{\tau_1,T}^g[\Phi(M_T^{\alpha^n})],$$

leading to

$$\operatorname*{ess\,inf}_{\alpha' \in \mathbf{A}_{\tau_1}^{\alpha}} \mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{Y}_{\tau_2}^{\alpha'}] \leq \mathcal{E}_{\tau_1,T}^g[\Phi(M_T^{\alpha^n})].$$

Letting n go to infinity in the above inequality, (4.1) provides directly

$$\operatorname*{ess\,inf}_{\alpha' \in \mathbf{A}_{\tau_1}^{\alpha}} \mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{Y}_{\tau_2}^{\alpha'}] \leq \mathcal{Y}_{\tau_1}^{\alpha}.$$

Step 2. $\mathcal{Y}_{\tau_1}^{\alpha} \leq \operatorname{ess\,inf}_{\alpha' \in \mathbf{A}_{\tau_1}^{\alpha}} \mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{Y}_{\tau_2}^{\alpha'}]$.

Fix α' in $\mathbf{A}_{\tau_1}^{\alpha}$. Lemma 4.1 below ensures the existence of a sequence $(\alpha'_n)_n$ in $\mathbf{A}_{\tau_2}^{\alpha'}$ such that $(\mathcal{E}_{\tau_2,T}^g[\Phi(M_T^{\alpha'^n})])_n$ is nonincreasing and

$$\lim_{n\to\infty} \mathcal{E}_{\tau_2,T}^g[\Phi(M_T^{\alpha'_n})] = \mathcal{Y}_{\tau_2}^{\alpha'}, \qquad \mathbb{P}\text{-a.s.}$$

In view of Remark 2.1, the convergence holds in $\mathbf{L}_2$ as well. Thus, the stability result of Proposition A.1 below indicates that $\mathcal{E}_{\tau_1,T}^g[\Phi(M_T^{\alpha'_n})]$ converges to $\mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{Y}_{\tau_2}^{\alpha'}]$ in $\mathbf{L}_2$. In addition, $\alpha'_n \in \mathbf{A}_{\tau_2}^{\alpha'} \subset \mathbf{A}_{\tau_1}^{\alpha}$ by construction. Combining the above leads to

$$\mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{Y}_{\tau_2}^{\alpha'}] = \lim_{n\to\infty} \mathcal{E}_{\tau_1,T}^g[\Phi(M_T^{\alpha'_n})] \geq \mathcal{Y}_{\tau_1}^{\alpha}.$$

The arbitrariness of $\alpha' \in \mathbf{A}_{\tau_1}^{\alpha}$ allows one to conclude

$$\operatorname*{ess\,inf}_{\alpha' \in \mathbf{A}_{\tau_1}^{\alpha}} \mathcal{E}_{\tau_1,\tau_2}^g[\mathcal{Y}_{\tau_2}^{\alpha'}] \geq \mathcal{Y}_{\tau_1}^{\alpha}.$$

□

LEMMA 4.1. *Fix $\theta, \tau \in \mathcal{T}$, with $\theta \geq \tau$, $\mu \in \mathbf{L}_0([0,1], \mathcal{F}_\tau)$ and $\alpha \in \mathbf{A}_{\tau,\mu}$. Then, there exists a sequence $(\alpha'_n) \subset \mathbf{A}_{\tau,\mu}^{\theta,\alpha} := \{\alpha' \in \mathbf{A}_{\tau,\mu}, \alpha'\mathbf{1}_{[0,\theta)} = \alpha\mathbf{1}_{[0,\theta)}\}$ such that $\lim_n \downarrow \mathcal{E}_{\theta,T}^g[\Phi(M_T^{\tau,\mu,\alpha'_n})] = \mathcal{Y}_\theta^\alpha(M_\theta^{\tau,\mu,\alpha})$, $\mathbb{P}$-a.s.*

PROOF. It suffices to show that the family $\{J(\alpha') := \mathcal{E}_{\theta,T}^g[\Phi(M_T^{\tau,\mu,\alpha'})], \alpha' \in \mathbf{A}_{\tau,\mu}^{\theta,\alpha}\}$ is directed downward; see, for example, Neveu (1975). Fix α'_1, α'_2 in $\mathbf{A}_{\tau,\mu}^{\theta,\alpha}$ and set

$$\tilde{\alpha}' := \alpha\mathbf{1}_{[0,\theta)} + \mathbf{1}_{[\theta,T]}\big(\alpha'_1\mathbf{1}_A + \alpha'_2\mathbf{1}_{A^c}\big),$$

where $A := \{J(\alpha_1') \le J(\alpha_2')\} \in \mathcal{F}_\theta$, so that $\tilde{\alpha}' \in \mathbf{A}_{\tau,\mu}^{\theta,\alpha}$ and

$$J(\tilde{\alpha}') = \mathcal{E}_{\theta,T}^{g}\big[\Phi\big(M_T^{\tau,\mu,\alpha_1'}\big)\mathbf{1}_A + \Phi\big(M_T^{\tau,\mu,\alpha_2'}\big)\mathbf{1}_{A^c}\big] = \min\big\{J(\alpha_1'), J(\alpha_2')\big\}. \qquad \square$$

We now observe that the family $(\mathcal{Y}^\alpha)_{\alpha\in\mathbf{H}_2}$ is làdlàg on countable sets. If in addition Φ is assumed to be continuous, the process $(\mathcal{Y}^\alpha)_{\alpha\in\mathbf{H}_2}$ is even indistinguishable from a càdlàg process.

PROPOSITION 4.2. *Fix $\alpha \in \mathbf{A}_0$. Then $\mathcal{Y}^\alpha$ is làdlàg on countable sets. Besides, if $m \in [0,1] \mapsto \Phi(\omega, m)$ is continuous for $\mathbb{P}$-a.e. $\omega \in \Omega$, then $\mathcal{Y}^\alpha$ is indistinguishable from a càdlàg process.*

PROOF. Fix $\alpha \in \mathbf{A}_0$. Proposition 4.1 and Remark 2.1 imply that $-\mathcal{Y}^\alpha$ is a $-g(-\cdot)$-supermartingale in the sense of Chen and Peng (2000) [a g-submartingale in the sense of Peng (1999)]. It follows from the nonlinear up-crossing lemma [see Chen and Peng (2000), Theorem 6][6] that the following limits

$$\lim_{s\in \mathrm{D}\cap(t,T]\downarrow t} \mathcal{Y}_s^\alpha \quad \text{and} \quad \lim_{s\in \mathrm{D}\cap[0,t)\uparrow t} \mathcal{Y}_s^\alpha$$

are well defined for every t in $[0,T]$, $\mathbb{P}$-a.s., and for all countable set D. So is the process

$$\bar{\mathcal{Y}}_t^\alpha := \lim_{s\in\mathbb{Q}\cap(t,T]\downarrow t} \mathcal{Y}_s^\alpha, \qquad t \in [0,T].$$

Besides, $\bar{\mathcal{Y}}^\alpha$ is by definition càd. Assuming that Φ is continuous, we will prove that, for every stopping time τ, it holds that

$$\bar{\mathcal{Y}}_\tau^\alpha = \operatorname*{ess\,inf}_{\alpha'\in\mathbf{A}_\tau^\alpha} \mathcal{E}_{\tau,T}^{g}\big[\Phi\big(M_T^{\alpha'}\big)\big] (= \mathcal{Y}_\tau^\alpha), \qquad \mathbb{P}\text{-a.s.} \tag{4.2}$$

By Delacherie and Meyer (1978), Chapter IV. (86), page 220, the relation (4.2) entails that $\mathcal{Y}^\alpha$ and $\bar{\mathcal{Y}}^\alpha$ are undistinguishable showing that $\mathcal{Y}^\alpha$ is undistinguishable from a càdlàg process. The rest of the proof is devoted to prove (4.2).

For this purpose, let us introduce $(\tau_n)_n$, a decreasing sequence of stopping times with values in $[0,T]\cap\mathbb{Q}$ such that $\tau \le \tau_n \le \tau + n^{-1}$ and $\bar{\mathcal{Y}}_\tau^\alpha = \lim_{n\to\infty} \mathcal{Y}_{\tau_n}^\alpha$.

Step 1. $\bar{\mathcal{Y}}_\tau^\alpha \le \operatorname{ess\,inf}_{\alpha'\in\mathbf{A}_\tau^\alpha} \mathcal{E}_{\tau,T}^{g}[\Phi(M_T^{\alpha'})]$.

[6]Note that Chen and Peng (2000), Theorem 6, restricts to positive g-supermartingales. However, the proof can be reproduced without difficulty under the integrability condition of Remark 2.1. In addition, Chen and Peng (2000), Theorem 6, implies that $E^{\mathbb{Q}}[D_a^b(\mathcal{Y}^\alpha, n)] \le \mathcal{Y}_0^\alpha \wedge b \le b$, where $D_a^b(\mathcal{Y}^\alpha, n)$ denotes the number of down crossing of $\mathcal{Y}^\alpha$ from an interval $[a,b]$ on a discrete time-grid $0 = t_0 \le t_1 \le \cdots \le t_n = T$ and $\mathbb{Q}$ is a particular measure absolutely continuous with respect to $\mathbb{P}$. To conclude, it is enough to reproduce the proof of [Delacherie and Meyer (1978), Chapter VI Theorem (2) point 1].

(a) Fix $\alpha' \in \mathbf{A}_\tau^\alpha$ and set

$$\lambda_n := \Big(\frac{M_{\tau_n}^\alpha}{M_{\tau_n}^{\alpha'}} \wedge \frac{1 - M_{\tau_n}^\alpha}{1 - M_{\tau_n}^{\alpha'}} \Big) \mathbf{1}_{\{M_{\tau_n}^\alpha \notin \{0,1\}\}} \in [0,1],$$

with the convention $a/0 = \infty$ for $a > 0$. Using the fact that $M_{\tau_n}^{\alpha'} + \int_{\tau_n}^T \alpha'_s \, dW_s = M_T^{\alpha'} \in [0,1]$, direct computations lead to

$$0 \le M_{\tau_n}^\alpha - \lambda_n M_{\tau_n}^{\alpha'} \le M_{\tau_n}^\alpha + \lambda_n \int_{\tau_n}^T \alpha'_s \, dW_s \le M_{\tau_n}^\alpha + \lambda_n (1 - M_{\tau_n}^{\alpha'}) \le 1.$$

We set $\alpha'_n := \alpha \mathbf{1}_{[0,\tau_n)} + \lambda_n \alpha' \mathbf{1}_{[\tau_n, T]}$. The above implies that α'_n belongs to $\mathbf{A}_{\tau_n}^\alpha$.

(b) Now we prove that $M_T^{\alpha'_n}$ converges $M_T^{\alpha'}$ in $\mathbf{L}_2$ as n goes to infinity, possibly up to a subsequence. Since both have norms bounded by 1, it suffices to show the $\mathbb{P}$-a.s. convergence, possibly up to a subsequence. To see this, first note that

$$M_T^{\alpha'_n} - M_T^{\alpha'} = M_{\tau_n}^\alpha - M_{\tau_n}^{\alpha'} + \int_{\tau_n}^T (\lambda_n - 1) \alpha'_s \, dW_s,$$

from which we deduce that

$$M_T^{\alpha'_n} - M_T^{\alpha'} = M_{\tau_n}^\alpha - M_{\tau_n}^{\alpha'} + (\lambda_n - \mathbf{1}_{\{M_{\tau_n}^\alpha \notin \{0,1\}\}}) \int_{\tau_n}^T \alpha'_s \, dW_s$$
$$- \mathbf{1}_{\{M_{\tau_n}^\alpha \in \{0,1\}\}} \int_{\tau_n}^T \alpha'_s \, dW_s.$$

Since $\tau_n \to \tau$, $\mathbb{P}$-a.s. and $\alpha' = \alpha$ on $[\![0, \tau]\!]$, the above construction implies that $\lim_{n\to\infty} M_{\tau_n}^\alpha - M_{\tau_n}^{\alpha'} = 0$, $\mathbb{P}$-a.s. and $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \mathbf{1}_{\{M_{\tau_n}^\alpha \notin \{0,1\}\}}$, $\mathbb{P}$-a.s. It thus only remains to prove that $\mathbf{1}_{\{M_{\tau_n}^\alpha \in \{0,1\}\}} \int_{\tau_n}^T \alpha'_s \, dW_s \to 0$, $\mathbb{P}$-a.s. First note that $\alpha' \mathbf{1}_{[\tau_n, T]} = 0$ on $\{M_{\tau_n}^{\alpha'} \in \{0,1\}\}$. This follows from the martingale property of this process with values in $[0,1]$. Hence, it suffices to consider $\mathbf{1}_{\{M_{\tau_n}^{\alpha'} \neq M_{\tau_n}^\alpha \in \{0,1\}\}} \int_{\tau_n}^T \alpha'_s \, dW_s$. But, since $M_\tau^{\alpha'} = M_\tau^\alpha$,

$$\mathbb{P}\big[M_{\tau_n}^{\alpha'} \neq M_{\tau_n}^\alpha \in \{0,1\}\big] \le \mathbb{P}\big[M_{\tau_n}^{\alpha'} \neq M_{\tau_n}^\alpha\big] = \mathbb{P}\bigg[\bigg| \int_\tau^{\tau_n} (\alpha_s - \alpha'_s) \, dW_s \bigg| > 0\bigg]$$
$$\to_{n\to\infty} 0.$$

(c) Now, since Φ is continuous and $M_T^{\alpha'_n} \in \mathbf{L}_0([0,1])$, we get that $\Phi(M_T^{\alpha'_n}) \to \Phi(M_T^{\alpha'})$ in $\mathbf{L}_2$, after possibly passing to a subsequence. The stability property for Lipschitz BSDEs given in Proposition A.1 implies that

$$\big\| \mathcal{E}^g_{\tau^n, T}\big[\Phi(M_T^{\alpha'_n})\big] - \mathcal{E}^g_{\tau^n, T}\big[\Phi(M_T^{\alpha'})\big] \big\|_{\mathbf{L}_2} \to_{n\to\infty} 0. \tag{4.3}$$

On the other hand, the bound of Remark 2.1 implies that

$$\big\| \mathcal{E}^g_{\tau^n, T}\big[\Phi(M_T^{\alpha'})\big] - \mathcal{E}^g_{\tau, T}\big[\Phi(M_T^{\alpha'})\big] \big\|_{\mathbf{L}_2} \to_{n\to\infty} 0, \tag{4.4}$$

by Lebesgue's dominated convergence theorem and by continuity of the process $\mathcal{E}^g_{\cdot,T}[\Phi(M_T^{\alpha'})]$. Combining (4.3) and (4.4) leads to

$$\bar{\mathcal{Y}}^\alpha_\tau = \lim_{n\to\infty} \mathcal{Y}^\alpha_{\tau^n} \le \lim_{n\to\infty} \mathcal{E}^g_{\tau^n,T}\big[\Phi\big(M_T^{\alpha'_n}\big)\big] = \mathcal{E}^g_{\tau,T}\big[\Phi\big(M_T^{\alpha'}\big)\big].$$

We conclude by arbitrariness of $\alpha' \in \mathbf{A}^\alpha_\tau$.

Step 2. $\bar{\mathcal{Y}}^\alpha_\tau \ge \operatorname{ess\,inf}_{\alpha'\in\mathbf{A}^\alpha_\tau} \mathcal{E}^g_{\tau,T}[\Phi(M_T^{\alpha'})]$.

Applying on $[\tau,\tau^n]$ the stability result of Proposition A.1 for the BSDEs with parameters $(\bar{\mathcal{Y}}^\alpha_\tau, 0)$ and $(\mathcal{Y}^\alpha_{\tau^n}, g\mathbf{1}_{[0,\tau^n)})$, we get

$$\begin{aligned}\big\|\bar{\mathcal{Y}}^\alpha_\tau - \mathcal{E}^g_{\tau,\tau^n}[\mathcal{Y}^\alpha_{\tau^n}]\big\|_{\mathbf{L}_2} &\le C\left(\big\|\bar{\mathcal{Y}}^\alpha_\tau - \mathcal{Y}^\alpha_{\tau^n}\big\|_{\mathbf{L}_2} + E\left[\int_\tau^{\tau^n} \big|g\big(s,\bar{\mathcal{Y}}^\alpha_\tau,0\big)\big|^2\,ds\right]\right)\\ &\le C\big\|\bar{\mathcal{Y}}^\alpha_\tau - \mathcal{Y}^\alpha_{\tau^n}\big\|_{\mathbf{L}_2} + \frac{C}{n}, \qquad n\in\mathbb{N},\end{aligned}$$

for some $C>0$, since the bound of Remark 2.1 holds for $\bar{\mathcal{Y}}^\alpha_\tau$, recall that assumption $(\mathbf{H}_g)$ is in force. Therefore, $\mathcal{E}^g_{\tau,\tau^n}[\mathcal{Y}^\alpha_{\tau^n}]$ converges to $\bar{\mathcal{Y}}^\alpha_\tau$ as n goes to infinity. Proposition 4.1 implies $\mathcal{E}^g_{\tau,\tau^n}[\mathcal{Y}^\alpha_{\tau^n}] \ge \mathcal{Y}^\alpha_\tau$. Passing to the limit leads to the required inequality: $\bar{\mathcal{Y}}^\alpha_\tau \ge \mathcal{Y}^\alpha_\tau = \operatorname{ess\,inf}_{\alpha'\in\mathbf{A}^\alpha_\tau} \mathcal{E}^g_{\tau,T}[\Phi(M_T^{\alpha'})]$. □

In the rest of this section, we complete the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. Items (i) and (ii) are already proved in Propositions 4.1 and 4.2, it remains to prove (iii) and (iv). For $\alpha\in\mathbf{A}_0$, it follows from Propositions 4.1, 4.2 and standard comparison results for BSDEs that $\mathcal{Y}^\alpha$ is a càdlàg strong g-submartingale in the sense of Peng (1999). Hence, the existence of a process $(\mathcal{Z}^\alpha,\mathcal{K}^\alpha)\in\mathbf{H}_2\times\mathbf{K}_2$ such that (2.9) holds follows from Peng (1999), Theorem 3.3. We now verify successively that the family $(\mathcal{Y}^\alpha,\mathcal{Z}^\alpha,\mathcal{K}^\alpha)_{\alpha\in\mathbf{H}_2}$ satisfies (2.8), (2.10), (2.11) and the uniqueness of solution for (2.8)–(2.11).

The bound (2.8) follows directly from Remark 2.1 and the representation Theorem 3.3 in Peng (1999); note that the driver function g does not depend on $\alpha\in\mathbf{A}_0$.

Step 1. *The irrelevance of future property* (2.11). For $(\alpha,\tau)\in\mathbf{A}_0\times\mathcal{T}$, observe that $\mathbf{A}^{\alpha'}_\cdot = \mathbf{A}^\alpha_\cdot$ on $[0,\tau]$ when $\alpha'\in\mathbf{A}^\alpha_\tau$. The definition of $\mathcal{Y}$ thus implies that $\mathcal{Y}^\alpha\mathbf{1}_{[0,\tau]} = \mathcal{Y}^{\alpha'}\mathbf{1}_{[0,\tau]}$ for $\alpha'\in\mathbf{A}^\alpha_\tau$. Hence, (2.11) follows from the uniqueness of the representation provided in Peng (1999), Theorem 3.3.

Step 2. *The minimality property* (2.10). We follow the arguments in the proof [Soner, Touzi and Zhang (2012), Theorem 4.6]. We fix $(\alpha,\tau_1,\tau_2)\in\mathbf{H}_2\times\mathcal{T}\times\mathcal{T}$ such that $\tau_1\le\tau_2$. For any $\alpha'\in\mathbf{A}^\alpha_{\tau_1}$, we denote by $(Y^{\alpha'},Z^{\alpha'})$ the solution of the classical BSDE

$$Y^{\alpha'}_t = \Phi\big(M_T^{\alpha'}\big) + \int_t^T g\big(s,Y^{\alpha'}_s,Z^{\alpha'}_s\big)\,ds - \int_t^T Z^{\alpha'}_s\,dW_s, \qquad 0\le t\le T.$$

Let $L^{\alpha'}$ be the process whose dynamics is given by

$$L_t^{\alpha'} = \exp\Big(\int_{\tau_1}^t \Lambda_s^z \, dW_s + \int_{\tau_1}^t \Big(\Lambda_s^y - \frac{|\Lambda_s^z|^2}{2}\Big) ds\Big), \qquad \tau_1 \le t \le T,$$

where (Λ^y, Λ^z) is the linearization process given by

$$\Lambda^y := \frac{g(\mathcal{Y}_s^{\alpha'}, \mathcal{Z}_s^{\alpha'}) - g(Y_s^{\alpha'}, \mathcal{Z}_s^{\alpha'})}{\mathcal{Y}_s^{\alpha'} - Y_s^{\alpha'}} \mathbf{1}_{\{\mathcal{Y}^{\alpha'} \neq Y^{\alpha'}\}},$$

$$\Lambda^z := \frac{g(Y_s^{\alpha'}, \mathcal{Z}_s^{\alpha'}) - g(Y_s^{\alpha'}, Z_s^{\alpha'})}{|\mathcal{Z}_s^{\alpha'} - Z_s^{\alpha'}|^2} (\mathcal{Z}^{\alpha'} - Z^{\alpha'}) \mathbf{1}_{\{\mathcal{Z}^{\alpha'} \neq Z^{\alpha'}\}}.$$

This linearization procedure implies that $Y_{\tau_1}^{\alpha'} - \mathcal{Y}_{\tau_1}^{\alpha'}$ rewrites as

$$\begin{aligned} Y_{\tau_1}^{\alpha'} - \mathcal{Y}_{\tau_1}^{\alpha'} &= E_{\tau_1}\big[L_{\tau_2}^{\alpha'}(Y_{\tau_2}^{\alpha'} - \mathcal{Y}_{\tau_2}^{\alpha'})\big] + E_{\tau_1}\Big[\int_{\tau_1}^{\tau_2} L_s^{\alpha'} d\mathcal{K}_s^{\alpha'}\Big] \\ &\ge E_{\tau_1}\Big[(\mathcal{K}_{\tau_2}^{\alpha'} - \mathcal{K}_{\tau_1}^{\alpha'}) \inf_{[\tau_1,\tau_2]} L^{\alpha'}\Big], \end{aligned} \tag{4.5}$$

where we used the fact that $Y^\alpha - \mathcal{Y}^\alpha \ge 0$. Using Hölder inequality, this implies

$$\begin{aligned} &E_{\tau_1}\big[(\mathcal{K}_{\tau_2}^{\alpha'} - \mathcal{K}_{\tau_1}^{\alpha'})\big]^3 \\ &\quad \le E_{\tau_1}\Big[(\mathcal{K}_{\tau_2}^{\alpha'} - \mathcal{K}_{\tau_1}^{\alpha'}) \inf_{[\tau_1,\tau_2]} L^{\alpha'}\Big] E_{\tau_1}\Big[\sup_{[\tau_1,\tau_2]} (1/L^{\alpha'})\Big] E_{\tau_1}\big[(\mathcal{K}_{\tau_2}^{\alpha'} - \mathcal{K}_{\tau_1}^{\alpha'})^2\big] \\ &\quad \le C E_{\tau_1}\big[(\mathcal{K}_{\tau_2}^{\alpha'} - \mathcal{K}_{\tau_1}^{\alpha'})^2\big](Y_{\tau_1}^{\alpha'} - \mathcal{Y}_{\tau_1}^{\alpha'}), \end{aligned}$$

for some $C > 0$ that depends on the uniform bounds on (Λ^y, Λ^z), recall $(\mathbf{H}_g)$. Hence, the estimate (2.8) together with the monotonicity of $\mathcal{K}$ implies

$$0 \le E_{\tau_1}\big[(\mathcal{K}_{\tau_2}^{\alpha'} - \mathcal{K}_{\tau_1}^{\alpha'})\big] \le C \eta'_{\tau_1} (Y_{\tau_1}^{\alpha'} - \mathcal{Y}_{\tau_1}^{\alpha'})^{1/3}, \qquad \alpha' \in \mathbf{A}_{\tau_1}^{\alpha}, \tag{4.6}$$

where

$$\eta'_{\tau_1} := \operatorname*{ess\,sup}_{\bar\alpha \in \mathbf{A}_{\tau_1}^{\alpha}} E_{\tau_1}\big[(\mathcal{K}_{\tau_2}^{\bar\alpha} - \mathcal{K}_{\tau_1}^{\bar\alpha})^2\big]^{1/3}.$$

By the same arguments as in Lemma 4.1, we can find a sequence $(\alpha'_n)_n \subset \mathbf{A}_{\tau_1}^{\alpha}$ such that

$$\eta'_{\tau_1} = \lim_{n\to\infty} \uparrow E_{\tau_1}\big[(\mathcal{K}_{\tau_2}^{\alpha'_n} - \mathcal{K}_{\tau_1}^{\alpha'_n})^2\big]^{1/3}.$$

The monotone convergence theorem together with Jensen's inequality and relation (2.8) imply that

$$E\big[\eta'_{\tau_1}\big] = \lim_{n\to\infty} \uparrow E\big[(\mathcal{K}_{\tau_2}^{\alpha'_n} - \mathcal{K}_{\tau_1}^{\alpha'_n})^2\big]^{1/3} < \infty.$$

Since η'_{τ_1} is in addition nonnegative, it is a.s. bounded. Hence, combining (2.11) and (4.6), we obtain for $\alpha' \in \mathbf{A}^{\alpha}_{\tau_1}$

$$\begin{aligned} 0 &\le E_{\tau_1}\big[\mathcal{K}^{\alpha'}_{\tau_2}\big] - \mathcal{K}^{\alpha'}_{\tau_1} \\ &\le C\big(\mathcal{E}^{g}_{\tau_1,\tau_2}\big[Y^{\alpha'}_{\tau_2}\big] - \mathcal{Y}^{\alpha}_{\tau_1}\big)^{1/3} \\ &= C\big(\mathcal{E}^{g}_{\tau_1}\big[\Phi\big(M^{\alpha'}_T\big)\big] - \mathcal{Y}^{\alpha}_{\tau_1}\big)^{1/3}. \end{aligned}$$

Taking the essential infimum in the above inequality and appealing to (2.6) leads to (2.10).

Step 3. *The uniqueness property for* (2.8)–(2.11). Let us now consider a family $(\tilde{Y}^{\alpha}, \tilde{Z}^{\alpha}, \tilde{K}^{\alpha})_{\alpha\in\mathbf{A}_0}$ of $\mathbf{S}_2 \times \mathbf{H}_2 \times \mathbf{K}_2$ satisfying (2.8)–(2.11). Then (2.6) together with (2.9)–(2.11) applied to $(\tilde{Y}^{\alpha}, \tilde{Z}^{\alpha}, \tilde{K}^{\alpha})_{\alpha\in\mathbf{A}_0}$ imply via a direct comparison argument that

$$\mathcal{Y}^{\alpha}_t = \operatorname*{ess\,inf}_{\alpha'\in\mathbf{A}^{\alpha}_t} \mathcal{E}^{g}_t\big[\Phi\big(M^{\alpha'}_T\big)\big] \ge \tilde{Y}^{\alpha}_t, \qquad \alpha \in \mathbf{A}_0, 0 \le t \le T. \tag{4.7}$$

On the other hand, following the exact same line of arguments as the one developed in step 2 in order to derive (4.5), one easily shows that there exists a $\mathbf{S}_2$-uniformly bounded family of processes $(\tilde{L}^{\alpha})_{\alpha\in\mathbf{A}_0}$ such that

$$\mathcal{E}^{g}_t\big[\Phi\big(M^{\alpha}_T\big)\big] - \tilde{Y}^{\alpha}_t = E_t\bigg[\int_t^T \tilde{L}^{\alpha}_s \, d\tilde{K}^{\alpha}_s\bigg] \le C\,E_t\big[\big|\tilde{K}^{\alpha}_T - \tilde{K}^{\alpha}_t\big|^2\big]^{1/2}$$

for all $\alpha \in \mathbf{A}_0, 0 \le t \le T$, for some $C > 0$.

Now observe that (2.10), applied to $\tilde{K}^{\alpha}$, and the same arguments as in Lemma 4.1 provide the existence of $(\hat{\alpha}^n)_n \subset \mathbf{A}^{\alpha}_t$ such that $E_t[\tilde{K}^{\hat{\alpha}^n}_T - \tilde{K}^{\alpha}_t] \to 0$, $\mathbb{P}$-a.s. Hence, (2.8) ensures that $E_t[|\tilde{K}^{\hat{\alpha}^n}_T - \tilde{K}^{\alpha}_t|^2] \to 0$. Since (2.11) implies $(\tilde{Y}^{\hat{\alpha}^n}_t, \tilde{K}^{\hat{\alpha}^n}_t) = (\tilde{Y}^{\alpha}_t, \tilde{K}^{\alpha}_t)$ for $n \in \mathbb{N}$, we deduce

$$\mathcal{E}^{g}_t\big[\Phi\big(M^{\hat{\alpha}^n}_T\big)\big] - \tilde{Y}^{\alpha}_t \le C\,E_t\big[\big|\tilde{K}^{\hat{\alpha}^n}_T - \tilde{K}^{\alpha}_t\big|^2\big]^{1/2} \to_{n\to\infty} 0.$$

Combined with (4.7), this shows that

$$\tilde{Y}^{\alpha}_t = \operatorname*{ess\,inf}_{\alpha'\in\mathbf{A}^{\alpha}_t} \mathcal{E}^{g}_t\big[\Phi\big(M^{\alpha'}_T\big)\big] = \mathcal{Y}^{\alpha}_t, \qquad \alpha \in \mathbf{H}_2, 0 \le t \le T.$$

The fact that $(\tilde{Z}^{\alpha}, \tilde{K}^{\alpha})_{\alpha\in\mathbf{A}_0} = (\mathcal{Z}^{\alpha}, \mathcal{K}^{\alpha})_{\alpha\in\mathbf{A}_0}$ then follows from the uniqueness of the nonlinear Doob–Meyer decomposition of Peng (1999), Theorem 3.3. □

APPENDIX

We report here some standard results for Lipschitz BSDEs. The first one can be found in, for example, Theorem 1.5 in Pardoux (1998). The second one is proved for completeness, and by lack of a good reference.

PROPOSITION A.1 (Stability for Lipschitz BSDEs). *Let* (Y^1, Z^1) *and* (Y^2, Z^2) *in* $\mathbf{S}_2 \times \mathbf{H}_2$ *be solutions on* $[0, T]$ *of Lipschitz BSDEs associated to parameters* (ξ^1, g^1) *and* (ξ^2, g^2). *Then the following stability result holds*:

$$\begin{aligned}&\|Y^1 - Y^2\|_{\mathbf{S}_2}^2 + \|Z^1 - Z^2\|_{\mathbf{H}_2}^2 \\ &\quad \le C\left(\|\xi^1 - \xi^2\|_{\mathbf{L}_2}^2 + \int_0^T E|g^1 - g^2|^2(t, Y_t^1, Z_t^1)\,dt\right),\end{aligned}$$

for some constant $C > 0$ *depending only on* T *and on the Lipschitz constants of* g^1 *and* g^2.

PROPOSITION A.2. *Let the conditions* $(\mathbf{H}_g)$ *hold. Then*:

(i) *There exists* $C > 0$ *which only depends on* K_g *and* T *such that*

$$\operatorname*{ess\,sup}_{\xi \in \mathbf{L}_0([0,1])} |\mathcal{E}_t^g[\xi]| \le C\big(1 + E_t\big[|\chi_g|^2\big]^{1/2}\big), \qquad 0 \le t \le T.$$

(ii) *For some* $\xi \in \mathbf{L}_2$ *and* $t \in [0, T]$, *consider a family* $(\xi^\varepsilon)_{\varepsilon \ge 0} \subset \mathbf{L}_0(\mathbb{R}^d)$ *satisfying* $|\xi^\varepsilon| \le \xi$ *and* $\xi^\varepsilon \in L^0(\mathcal{F}_{(t+\varepsilon)\wedge T})$, *for any* $\varepsilon > 0$. *Then there exists a family* $(\eta_\varepsilon)_{\varepsilon > 0} \subset \mathbf{L}_0(\mathbb{R})$ *which converges to* 0, $\mathbb{P}$*-a.s. as* $\varepsilon \to 0$ *such that*

$$|\mathcal{E}_{t,t+\varepsilon}^g[\xi^\varepsilon] - E_t[\xi^\varepsilon]| \le \eta_\varepsilon \qquad \forall \varepsilon \in [0, T - t].$$

(iii) *Let* $(\xi^\varepsilon)_{\varepsilon > 0}$ *and* $t \in [0, T]$ *be as in* (ii). *Then there exists a family* $(\eta_\varepsilon)_{\varepsilon > 0} \subset \mathbf{L}_0(\mathbb{R})$ *which converges to* 0, $\mathbb{P}$*-a.s. as* $\varepsilon \to 0$ *such that*

$$|\mathcal{E}_{t-\varepsilon,t}^g[\xi^\varepsilon] - E_t[\xi^\varepsilon]| \le \eta_\varepsilon \qquad \forall \varepsilon \in [0, t].$$

PROOF. (a) We first prove (ii) [property (iii) being similar] using the standard linearization argument. Fix $t \in [0, T]$ and set $Y^\varepsilon := \mathcal{E}_{\cdot,t+\varepsilon}^g[\xi^\varepsilon]$. Assumption $(\mathbf{H}_g)$ implies that we can find a family of predictable processes $(\rho^\varepsilon, \gamma^\varepsilon)$ with values in $[-K_g, K_g]^{d+1}$ such that

$$L^\varepsilon Y^\varepsilon + \int_t^\cdot L_r^\varepsilon g(r, 0, 0)\,dr$$

is a martingale on $[t, t + \varepsilon]$, with

$$L_s^\varepsilon = 1 + \int_t^s \rho_r^\varepsilon L_r^\varepsilon\,dr + \int_t^s \gamma_r^\varepsilon L_r^\varepsilon\,dW_r, \qquad t \le s \le t + \varepsilon.$$

In particular,

$$\mathcal{E}_{t,t+\varepsilon}^g[\xi^\varepsilon] = L_t^\varepsilon Y_t^\varepsilon = E_t\left[L_{t+\varepsilon}^\varepsilon \xi^\varepsilon + \int_t^{t+\varepsilon} L_r^\varepsilon g(r, 0, 0)\,dr\right].$$

Condition $(\mathbf{H}_g)$ and the assumption on $(\xi^\varepsilon)_{\varepsilon > 0}$ thus leads to

$$|\mathcal{E}_{t,t+\varepsilon}^g[\xi^\varepsilon] - E_t[\xi^\varepsilon]| \le \eta_\varepsilon,$$

in which

$$\eta_\varepsilon := E_t\Big[\xi|L^\varepsilon_{t+\varepsilon} - L^\varepsilon_t| + \chi_g \int_t^{t+\varepsilon} L^\varepsilon_r\,dr\Big].$$

We have

$$\begin{aligned}|\eta_\varepsilon| &\le E_t\big[|\xi|^2\big]^{1/2} E_t\big[|L^\varepsilon_{t+\varepsilon} - L^\varepsilon_t|^2\big]^{1/2}\\ &\quad + E_t\big[|\chi_g|^2\big]^{1/2} E_t\Big[\Big|\int_t^{t+\varepsilon} L^\varepsilon_r\,dr\Big|^2\Big]^{1/2} \qquad \text{(A.1)}\\ &\le E_t\big[|\xi|^2\big]^{1/2} E_t\big[|L^\varepsilon_{t+\varepsilon} - L^\varepsilon_t|^2\big]^{1/2} + \varepsilon E_t\big[|\chi_g|^2\big]^{1/2} E_t\Big[\sup_{t\le s\le t+\varepsilon}|L^\varepsilon_s|^2\Big]^{1/2}.\end{aligned}$$

In addition,

$$\begin{aligned}E_t\big[|L^\varepsilon_{t+\varepsilon} - L^\varepsilon_t|^2\big] &\le C E_t\Big[\int_t^{t+\varepsilon} |L^\varepsilon_r|^2\,dr\Big]\\ &\le \varepsilon C E_t\Big[\sup_{t\le r\le t+\varepsilon}|L^\varepsilon_r|^2\Big].\end{aligned}$$

Hence,

$$E_t\big[|L^\varepsilon_{t+\varepsilon} - L^\varepsilon_t|^2\big] \le \varepsilon C\Big(1 + E_t\Big[\sup_{t\le r\le t+\varepsilon}|L^\varepsilon_r - L^\varepsilon_t|^2\Big]\Big).$$

Since γ^ε and ρ^ε are bounded, the quantity $\sup_{t\le\tau\le t+\varepsilon} E_\tau[|L^\varepsilon_{t+\varepsilon} - L^\varepsilon_\tau|^2]$ is uniformly bounded. Plugging back this estimate in (A.1) and recalling that $\sup_{t\in[0,T]} E_t[\xi^2]$ is finite $\mathbb{P}$-a.s. we get that $E_t[|\xi|^2]^{1/2} E_t[|L^\varepsilon_{t+\varepsilon} - L^\varepsilon_t|^2]^{1/2}$ tends to 0 uniformly in t, $\mathbb{P}$-a.s. as ε goes to 0. The second term of (A.1) can be estimated in the same way.

(b) We now prove (i). Pick any $t \in [0, T]$ and $\xi \in \mathbf{L}_0([0, 1])$. The same arguments as above yield

$$|\mathcal{E}^g_t[\xi]| \le \Big|E_t\Big[L^\xi_T\xi + \int_t^T L^\xi_r g(r, 0, 0)\,dr\Big]\Big| \le E_t\Big[|L^\xi_T| + T|\chi_g| \sup_{r\le T}|L^\xi_r|\,dr\Big],$$

where L^ξ solves

$$L^\xi_s = 1 + \int_t^s \rho^\xi_r L^\xi_r\,dr + \int_t^s \gamma^\xi_r L^\xi_r\,dW_r, \qquad t \le s \le T,$$

for some predictable processes (ρ^ξ, γ^ξ) with values in $[-K_g, K_g]^{d+1}$. Hence,

$$|\mathcal{E}^g_t[\xi]| \le E_t\Big[|L^\xi_T| + T|\chi_g| \sup_{t\le r\le T}|L^\xi_r|\,dr\Big].$$

Since (ρ^ξ, γ^ξ) are valued in $[-K_g, K_g]^{d+1}$, standard estimates imply that we can find $C > 0$, which only depends on K_g such that $E_t[\sup_{t\le r\le T}|L^\xi_r|^2] \le C^2$, $\mathbb{P}$-a.s. The above leads to

$$|\mathcal{E}^g_t[\xi]| \le \big(C + TC E_t\big[|\chi_g|^2\big]^{1/2}\big),$$

and the arbitrariness of $\xi \in \mathbf{L}_0([0, 1])$ concludes the proof. □

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