

COEXISTENCE OF GRASS, SAPLINGS AND TREES IN THE STAVER–LEVIN FOREST MODEL

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In this paper, we consider two attractive stochastic spatial models in which each site can be in state 0, 1 or 2: Krone’s model in which 0 = vacant, 1 = juvenile and 2 = a mature individual capable of giving birth, and the Staver–Levin forest model in which 0 = grass, 1 = sapling and 2 = tree. Our first result shows that if $(0, 0)$ is an unstable fixed point of the mean-field ODE for densities of 1’s and 2’s then when the range of interaction is large, there is positive probability of survival starting from a finite set and a stationary distribution in which all three types are present. The result we obtain in this way is asymptotically sharp for Krone’s model. However, in the Staver–Levin forest model, if $(0, 0)$ is attracting then there may also be another stable fixed point for the ODE, and in some of these cases there is a nontrivial stationary distribution.

1. Introduction. In a recent paper published in *Science* [17], Carla Staver, Sally Archibald and Simon Levin argued that tree cover does not increase continuously with rainfall but rather is constrained to low (<50%, “savanna”) or high (>75%, “forest”) levels. In follow-up work published in *Ecology* [16], the *American Naturalist* [18] and *Journal of Mathematical Biology* [15], they studied the following ODE for the evolution of the fraction of land covered by grass G , saplings S and trees T :

$$(1) \quad \begin{aligned} \frac{dG}{dt} &= \mu S + \nu T - \beta GT, \\ \frac{dS}{dt} &= \beta GT - \omega(G)S - \mu S, \\ \frac{dT}{dt} &= \omega(G)S - \nu T. \end{aligned}$$

Here, $\mu \geq \nu$ are the death rates for saplings and trees, and $\omega(G)$ is the rate at which saplings grow into trees. Fires decrease this rate of progression, and the incidence of fires is an increasing function of the fraction of grass, so $\omega(G)$ is decreasing. Studies suggest (see [18] for references) that regions with tree cover below about

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40% burn frequently but fire is rare above this threshold, so they used an ω that is close to a step function.

The ODE in (1) has very interesting behavior: it may have two stable fixed points, changing the values of parameters may lead to Hopf bifurcations, and if the system has an extra type of savanna trees, there can be periodic orbits. In this paper, we will begin the study of the corresponding spatial model. The state at time t is $\chi_t: \mathbb{Z}^d \rightarrow \{0, 1, 2\}$, where 0 = grass, 1 = sapling and 2 = tree. Given the application, it would be natural to restrict our attention to $d = 2$, but since the techniques we develop will be applicable to other systems we consider the general case.

In the forest model, it is natural to assume that dispersal of seeds is long range. To simplify our calculations, we will not use a continuous dispersal distribution for tree seeds, but instead let $f_i(x, L)$ denote the fraction of sites of type i in the box $x + [-L, L]^d$ and declare that site x changes:

- $0 \rightarrow 1$ at rate $\beta f_2(x, L)$,
- $1 \rightarrow 2$ at rate $\omega(f_0(x, \kappa L))$,
- $1 \rightarrow 0$ at rate μ ,
- $2 \rightarrow 0$ at rate ν .

The configuration with all sites 0 is an absorbing state. This naturally raises the question of finding conditions that guarantee that coexistence occurs, i.e., there is a stationary distribution in which all three types are present. Our model has three states but it is “attractive,” that is, if $\chi_0(x) \leq \chi'_0(x)$ for all x then we can construct the two processes on the same space so that this inequality holds for all time. From this, it follows from the usual argument that if we start from $\chi_0^2(x) \equiv 2$ then χ_t^2 converges to a limit χ_∞^2 that is a translation invariant stationary distribution, and there will be a nontrivial stationary distribution if and only if $P(\chi_\infty^2(0) = 0) < 1$. Since 2’s give birth to 1’s and 1’s grow into 2’s, if χ_∞^2 is nontrivial then both species will be present with positive density in χ_∞^2 .

If $\omega \equiv \gamma$ is constant, $\mu = 1 + \delta$, and $\nu = 1$, then our system reduces to one studied by Krone [12]. In his model, 1’s are juveniles who are not yet able to reproduce. Krone proved the existence of nontrivial stationary distributions in his model by using a simple comparison between the sites in state 2 and a discrete time finite-dependent oriented percolation. In the percolation process, we have an edge from $(x, n) \rightarrow (x + 1, n + 1)$ if a 2 at x at time $n\varepsilon$ will give birth to a 1 at $x + 1$, which then grows to a 2 before time $(n + 1)\varepsilon$, and there are no deaths at x or $x + 1$ in $[n\varepsilon, (n + 1)\varepsilon]$. As the reader can imagine, this argument produces a very crude result about the parameter values for which coexistence occurs.

A simple comparison shows that if we replace the decreasing function $\omega(G)$ in the Stayer–Levin model, χ_t , by the constant $\omega = \omega(1)$, to obtain a special case η_t of Krone’s model, then χ_t dominates η_t in the sense that given $\chi_0 \geq \eta_0$ the two processes can be coupled so that $\chi_t \geq \eta_t$ for all t . Because of this, we can prove

existence of nontrivial stationary distribution in the Staver–Levin model by studying Krone’s model. To do this under the assumption of long range interactions, we begin with the mean field ODE:

$$\begin{aligned}
 \frac{dG}{dt} &= \mu S + \nu T - \beta GT, \\
 \frac{dS}{dt} &= \beta GT - (\omega + \mu)S, \\
 \frac{dT}{dt} &= \omega S - \nu T.
 \end{aligned}
 \tag{2}$$

Here, ω is a constant. When it is a function, we will write $\omega(G)$.

Since $G + S + T = 1$, we can set $G = 1 - S - T$ and reduce the system to two equations for S and T . To guess a sufficient condition for coexistence in the long range limit we note that:

LEMMA 1.1. *For the mean-field ODE (2), $S = 0, T = 0$ is not an attracting fixed point if*

$$\mu\nu < \omega(\beta - \nu).
 \tag{3}$$

On the other hand, $S = 0, T = 0$ is attracting if

$$\mu\nu > \omega(\beta - \nu).
 \tag{4}$$

PROOF. When $(S, T) \approx (0, 0)$, and hence $G \approx 1$, the mean-field ODE is approximately

$$\begin{pmatrix} dS/dt \\ dT/dt \end{pmatrix} \approx A \begin{pmatrix} S \\ T \end{pmatrix} \quad \text{where } A = \begin{pmatrix} -(\omega + \mu) & \beta \\ \omega & -\nu \end{pmatrix}.$$

The trace of A , which is the sum of its eigenvalues is negative, so $(0, 0)$ is not attracting if the determinant of A , which is the product of the eigenvalues is negative. Since $(\omega + \mu)\nu - \beta\omega < 0$ if and only if $\mu\nu < (\beta - \nu)\omega$, we have proved the desired result. Similarly, $(0, 0)$ is attracting if the determinant of A is positive, which implies (4). \square

THEOREM 1. *Let η_t be Krone’s model with parameters that satisfy (3). Then when L is large enough, η_t survives with positive probability starting from a finite number of nonzero sites and η_t has a nontrivial stationary distribution.*

Foxall [9] has shown that for Krone’s model the existence of nontrivial stationary distribution is equivalent to survival for a finite set of nonzero sites, so we only have to prove one of these conclusions. However, our proof is via a block construction, so we get both conclusions at the same time.

Our next result is a converse, which does not require the assumption of long range.

THEOREM 2. *Suppose $\mu\nu \geq \omega(\beta - \nu)$. Then for any $L > 0$, Krone’s model η_t dies out, that is, for any initial configuration η_0 with finitely many nonzero sites:*

$$\lim_{t \rightarrow \infty} P(\eta_t(x) \equiv 0) = 1.$$

If $\mu\nu > \omega(\beta - \nu)$, then for any initial configuration η_0 and any $x \in \mathbb{Z}^d$, the probability

$$P(\exists t > 0, \text{ s.t. } \eta_s(x) = 0, \forall s \geq t) = 1.$$

The second conclusion implies that there is no nontrivial stationary distribution. Comparing with Krone’s model, we see that if $\mu\nu > \omega(0)(\beta - \nu)$ then the Staver–Levin model dies out.

1.1. *Survival when zero is stable.* When $\mu\nu > \omega(1)(\beta - \nu)$, the Staver–Levin ODE (1) may have another stable fixed point in the positive density region (and also an unstable fixed point in between), the Staver–Levin model, like the quadratic contact process studied by [6, 14] and [1] may have a nontrivial stationary distribution when $(0, 0)$ is attracting.

Based on the observation in [16] mentioned above, it is natural to assume that $\omega(\cdot)$ is a step function. In our proof, we let

$$(5) \quad \omega(G) = \begin{cases} \omega_0, & G \in [0, 1 - \delta_0), \\ \omega_1, & G \in [1 - \delta_0, 1], \end{cases}$$

where $\omega_0 > \omega_1$ and $\delta_0 \in (0, 1)$. However, according to the monotonicity of χ_t , our result about the existence of a nontrivial stationary distribution will also hold if one replaces the “=” in (5) by “ \geq ” since the new process dominates the old one.

To prove the existence of a nontrivial stationary distribution under the assumption of long range, a natural approach would be to show that when $L \rightarrow \infty$ and space is rescaled by diving by L , the Staver–Levin model converges weakly to the solution of following integro-differential equation:

$$(6) \quad \begin{aligned} \frac{dS(x, t)}{dt} &= \beta D_1^T(x, t)G - \mu S - \omega[D_\kappa^G(x, t)]S, \\ \frac{dT(x, t)}{dt} &= \omega[D_\kappa^G(x, t)]S - \nu T, \end{aligned}$$

where $G = 1 - S - T$ and

$$\begin{aligned} D_1^T(x, t) &= \frac{\int_{x+[-1,1]^d} T(y, t) dy_1 \cdots dy_d}{2^d}, \\ D_\kappa^G(x, t) &= \frac{\int_{x+[-\kappa, \kappa]^d} G(y, t) dy_1 \cdots dy_d}{(2\kappa)^d}, \end{aligned}$$

are the local densities of trees and grass on the rescaled lattice. The first problem with this approach is that since the density is computed by examining all sites in a

square, there is not a good dual process, which was the key to proofs in [6, 14, 19] and [1]. The second problem is that one does not know much about the limiting IDE. Results of Weinberger [20] show the existence of wave speeds and provide a convergence theorem in the case of a single equation, but we do not know of results for a pair of equations.

To avoid these difficulties, we will construct test functions S_{test} and T_{test} , so that under (6), the derivatives will always be positive for all x in $\{\overline{T_{\text{test}}} > 0\}$ and $\{\overline{S_{\text{test}}} > 0\}$, where \overline{A} stands for the closure of set A . The positive derivative implies that after a positive time the solution will dominate translates of the initial condition by positive and negative amounts. Monotonicity then implies that the solution will expand linearly and the result follows from a block construction. Details can be found in Section 8.

THEOREM 3. *Recall the definition of $\omega(G)$ in (5). Under condition*

$$(7) \quad \beta\omega_0 > 2^d v(\mu + \omega_0),$$

there is a constant $S_0(\beta, \omega_0, v, \mu)$ so that if

$$\delta_0 \in (0, 2^{-d} S_0)$$

the Staver–Levin forest model χ_t survives when L is large for any $\omega_1 \geq 0$ and $\kappa > 0$.

Combining Theorems 1–3, we have the following results for the Staver–Levin model:

1. When $\mu v < \omega(1)(\beta - v)$, χ_t survives from a finite set of nonzero sites when L is large.
2. When $\mu v \geq \omega(0)(\beta - v)$, χ_t dies out from a finite set of nonzero sites for all $L \geq 1$.
3. When $\mu v \geq \omega(1)(\beta - v)$, under the hypotheses of Theorem 3, χ_t can still survive from a finite set of nonzero sites when L is large, no matter how small is ω_1 .

1.2. *Sketch of the proof of Theorem 1.* Most of the remainder of the paper is devoted to the proof of Theorem 1. We will now describe the main ideas and then explain where the details can be found.

(i) *The key idea* is due to Grannan and Swindle [10]. They consider a model of a catalytic surface in which atoms of type $i = 1, 2$ land at vacant sites (0's) at rate p_i , while adjacent 1, 2 pairs turn into 0, 0 at rate ∞ . If after a landing event, several 1, 2 pairs are created, one is chosen at random to be removed. The first type of event is the absorption of an atom onto the surface of the catalyst, while the second is a chemical reaction, for example, carbon monoxide CO and oxygen O reacting to

produce CO₂. The last reaction occurs in the catalytic converted in your car, but the appropriate model for that system is more complicated. An oxygen molecule O₂ lands and dissociates to two O bound to the surface when a pair of adjacent sites is vacant. See Durrett and Swindle [7] for more details about the phase transition in the system.

Suppose without loss of generality that $p_1 + p_2 = 1$. In this case, Grannan and Swindle [10] showed that if $p_1 \neq p_2$ the only possible stationary distributions concentrate configurations that are $\equiv 1$ or $\equiv 2$. Mountford and Sudbury [13] later improved this result by showing that if $p_1 > 1/2$ and the initial configuration has infinitely many 1’s then the system converges to the all 1’s state.

The key to the Grannan–Swindle argument was to consider

$$Q(\eta_t) = \sum_x e^{-\lambda \|x\|} q[\eta_t(x)],$$

where $\|x\| = \sup_i |x_i|$ is the L^∞ norm, $q(0) = 0$, $q(1) = 1$, and $q(2) = -1$. If λ is small enough then $dEQ/dt \geq 0$ so Q is a bounded submartingale, and hence converges almost surely to a limit. Since an absorption or chemical reaction in $[-K, K]^d$ changes Q by an amount $\geq \delta_K$, it follows that such events eventually do not occur.

(ii) *Recovery from small density* is the next step. We will pick $\varepsilon_0 > 0$ small, let $\ell = [\varepsilon_0 L]$ be the integer part of $\varepsilon_0 L$ and divide space into *small boxes* $\hat{B}_x = 2\ell x + (-\ell, \ell]^d$. To make the number of 1’s and 2’s in the various small boxes sufficient to describe the state of the process, we declare two small boxes to be neighbors if all of their points are within an L^∞ distance L . For the “truncated process,” which is stochastically bounded by η_t , and in which births of trees can only occur between sites in neighboring small boxes, we will show that if $\kappa \in (d/2, d)$ and we start with a configuration that has L^κ nonzero sites in \hat{B}_0 and 0 elsewhere, then the system will recover and produce a small box \hat{B}_x at time τ in which the density of nonzero sites is $a_0 > 0$ and $P(\tau > t_0 \log L) < L^{d/2-\kappa}$. See Lemma 3.1. To prove this, we use an analogue of Grannan and Swindle’s Q . The fact that $(0, 0)$ is an unstable fixed point implies $dEQ/dt > 0$ as long as the density in all small boxes is $\leq a_0$.

(iii) *Bounding the location of the positive density box* is the next step. To do this, we use a comparison with branching random walk to show that the small box \hat{B}_x with density a_0 constructed in step (ii) is not too far from 0. Random walk estimates will later be used to control how far it will wander as we iterate the construction. For this step, it is important that the truncated process is invariant under reflection, so the mean displacement is 0. If we try to work directly with the original interacting particle system η_t then it is hard to show that the increments between box locations are independent and have mean 0. It is for this reason we introduced the truncated process.

(iv) *Moving particles*. The final ingredient in the block construction is to show that given a small block \hat{B}_x with positive density and any y with $\|y - x\|_1 \leq$

$[c \log L]$ then if c is small enough it is very likely that there will be $\geq L^k$ particles in \hat{B}_y at time $[c \log L]$. Choosing y appropriately and then using the recovery lemma, we can get lower bounds on the spread of the process.

(v) *Block construction.* Once we have completed steps (ii), (iii) and (iv), it is straightforward to show that our system dominates a one-dependent oriented percolation. This shows that the system survives from a finite set with positive probability and proves the existence of nontrivial stationary distribution.

The truncated process is defined in Section 2 and a graphical representation is used to couple it, Krone’s model and the Staver–Levin model. In Section 3, we use the Grannan–Swindle argument to do step (ii). The dying out result, Theorem 2, is proved in Section 4. In Sections 5, 6, and 7, we take care of steps (iii), (iv) and (v). In Section 8, we prove Theorem 3.

2. Box process and graphical representation. For some fixed $\varepsilon_0 > 0$ which will be specified in (11), let $l = [\varepsilon_0 L]$ and divide space \mathbb{Z}^d into small boxes:

$$\hat{B}_x = 2lx + (-l, l]^d, \quad x \in \mathbb{Z}^d.$$

For any $x \in \mathbb{Z}^d$, there is a unique x' such that $x \in \hat{B}_{x'}$. Define the new neighborhood of interaction as follows: for any $y \in \hat{B}_{y'}$, $y \in \mathcal{N}(x)$ if and only if

$$\sup_{z_1 \in \hat{B}_{x'}, z_2 \in \hat{B}_{y'}} \|z_1 - z_2\| \leq L.$$

It is easy to see that $\mathcal{N}(x) \subset B_x(L)$ where $B_x(L)$ is the L^∞ neighborhood centered at x with range L . To show that

$$(8) \quad \mathcal{N}(x) \supset B_x((1 - 4\varepsilon_0)L)$$

we note that if $\|x - y\| \leq (1 - 4\varepsilon_0)L$, $z_1 \in \hat{B}_{x'}$, and $z_2 \in \hat{B}_{y'}$, where $\hat{B}_{x'}$ and $\hat{B}_{y'}$ are the small boxes containing x and y :

$$\|z_1 - z_2\| \leq \|z_1 - x\| + \|x - y\| + \|y - z_2\| \leq 4([\varepsilon_0 L]) + (1 - 4\varepsilon_0)L \leq L.$$

Given the new neighborhood $\mathcal{N}(x)$, we define the truncated version of Krone’s model ξ_t by its transition rates:

| transition | at rate |
|-------------------|---|
| $1 \rightarrow 0$ | μ |
| $2 \rightarrow 0$ | ν |
| $1 \rightarrow 2$ | ω |
| $0 \rightarrow 1$ | $\beta N_2(\mathcal{N}(x)) / (2L + 1)^d,$ |

where $N_i(S)$ stands for the number of i ’s in the set S .

For any $x \in \mathbb{Z}^d$ and $\xi \in \{0, 1, 2\}^{\mathbb{Z}^d}$, define $n_i(x, \xi)$ to be the number of type i ’s in the small box \hat{B}_x in the configuration ξ . The box process is defined by

$$\zeta_t(x) = (n_1(x, \xi_t), n_2(x, \xi_t)) \quad \forall x \in \mathbb{Z}^d.$$

Then ζ_t is a Markov process on $\{(n_1, n_2) : n_1, n_2 \geq 0, n_1 + n_2 \leq |\hat{B}_0|\}^{\mathbb{Z}^d}$ in which

| transition | at rate |
|---|--|
| $\zeta_t(x) \rightarrow \zeta_t(x) - (1, 0)$ | $\mu\zeta_t^1(x)$ |
| $\zeta_t(x) \rightarrow \zeta_t(x) - (0, 1)$ | $\nu\zeta_t^2(x)$ |
| $\zeta_t(x) \rightarrow \zeta_t(x) + (-1, 1)$ | $\omega\zeta_t^1(x)$ |
| $\zeta_t(x) \rightarrow \zeta_t(x) + (1, 0)$ | $\zeta_t^0(x) \sum_{y: \hat{B}_y \subset \mathcal{N}(x)} \beta\zeta_t^2(y),$ |

where

$$\zeta_t^0(x) = |\hat{B}_0| - \zeta_t^1(x) - \zeta_t^2(x)$$

be the number of 0’s in that small box.

Because ζ_t only records the number of particles in any small box, and the neighborhood is defined so that all sites in the same small box have the same neighbors, the distribution of ζ_t is symmetric under reflection in any axis. The main use for this observation is that the displacement of the location of the positive density box produced by the recovery lemma in Section 3 has mean 0.

2.1. *Graphical representation.* We will use the graphical representation similar as in [12] to construct Krone’s model η_t and the truncated version ξ_t on the same probability space, so that:

(*) If $\eta_0 \geq \xi_0$, then we will have $\eta_t \geq \xi_t$ for all t .

Note that $\mu \geq \nu$. We use independent families of Poisson processes for each $x \in \mathbb{Z}^d$, as follows:

- $\{V_n^x : n \geq 1\}$ with rate ν . We put an \times at space–time point (x, V_n^x) and write a δ_{12} next to it to indicate a death will occur if x is occupied by a 1 or a 2.
- $\{U_n^x : n \geq 1\}$ with rate $\mu - \nu$. We put an \times at space–time point (x, U_n^x) and write a δ_1 next to it to indicate a death will occur if x is occupied by 1.
- $\{W_n^x : n \geq 1\}$ with rate ω . We put an \bullet at space–time point (x, W_n^x) which indicates that if x is in state 1, it will become a 2.
- $\{T_n^{x,y} : n \geq 1\}$ with rate $\beta/|B_0|$ for all $y \in \mathcal{N}(x)$. We draw a solid arrow from $(x, T_n^{x,y})$ to $(y, T_n^{x,y})$ to indicate that if x is occupied by a 2 and y is vacant, then a birth will occur at x in either process.
- $\{T_n^{x,y} : n \geq 1\}$ with rate $\beta/|B_0|$ for all $y \in B_x(L) - \mathcal{N}(x)$. We draw a dashed arrow from $(x, T_n^{x,y})$ to $(y, T_n^{x,y})$ to indicate that if x is occupied by a 2 and y is vacant then a birth will occur at x in the process ξ_t .

Standard arguments that go back to Harris [11] over forty years ago guarantee that we have constructed the desired processes. Since each flip preserves $\eta_s \geq \xi_s$, the stochastic order (*) is satisfied.

To finish the construction of the Staver–Levin model, χ_t , we add another family of Poisson process $\{\hat{W}_n^x : n \geq 1\}$ with rate $1 - \omega$, and independent random variables

$w_{x,n}$ uniform on $(0, 1)$. At any time \hat{W}_n^x if x is in state 1, it will increase to state 2 if

$$w_{n,x} > \frac{\omega(f_0(x, \kappa L)) - \omega}{1 - \omega}.$$

These events take care of the extra growth of 1's into 2's in χ_t . Again every flip preserves $\chi_s \geq \eta_s$ so we have:

(**) If $\chi_0 \geq \eta_0$ then we will have $\chi_t \geq \eta_t$ for all t .

3. Recovery lemma. Given (3), one can pick a θ , which must be >1 , such that

$$(9) \quad \frac{\mu + \omega}{\omega} < \theta < \frac{\beta}{\nu}$$

so we have

$$\theta\omega - (\omega + \mu) > 0, \quad \beta - \theta\nu > 0$$

and since the inequalities above are strict, we can pick some $a_0 > 0$ and $\rho \in (0, 1)$ such that

$$(10) \quad \theta\omega - (\omega + \mu) \geq \rho, \quad \beta(1 - 4a_0) - \theta\nu \geq \theta\rho.$$

Now we can let the undetermined ε_0 in the definition of ξ_t in Section 2 be a positive constant such that

$$(11) \quad (1 - 4\varepsilon_0)^d > 1 - 2a_0.$$

Fix some $\alpha \in (d/2, d)$. We start with an initial configuration in Ξ_0 , the ξ_0 that have $\xi_0(x) = 0$ for all $x \notin \hat{B}_0$ and the number of nonzero sites in \hat{B}_0 is at least L^α . We define a stopping time τ :

$$(12) \quad \tau = \inf\{t : \exists x \in \mathbb{Z}^d \text{ such that } n_1(x, \xi_t) + n_2(x, \xi_t) \geq a_0|\hat{B}_0|\}.$$

LEMMA 3.1 (Recovery lemma). *Suppose we start the truncated version of Krone's model from a $\xi_0 \in \Xi_0$. Let $t_0 = 2d/\rho$. When L is large,*

$$(13) \quad P(\tau > t_0 \log L) < L^{d/2-\alpha}.$$

PROOF. As mentioned in the [Introduction](#), we consider

$$Q(\xi_t) = \lambda^d \sum_{x \in \mathbb{Z}^d} e^{-\lambda\|x\|} w[\xi_t(x)],$$

where $\lambda = L^{-1}a_0/2$ and

$$w[\xi(x)] = \begin{cases} 0, & \text{if } \xi(x) = 0, \\ 1, & \text{if } \xi(x) = 1, \\ \theta, & \text{if } \xi(x) = 2. \end{cases}$$

If we imagine \mathbb{R}^d divided into cubes with centers at $\lambda\mathbb{Z}^d$ and think about sums approximating an integral, then we see that

$$(14) \quad \lambda^d \sum_{x \in \mathbb{Z}^d} e^{-\lambda\|x\|} \leq e^{\lambda/2} \int_{\mathbb{R}^d} e^{-\|z\|} dz \leq U_{(14)}$$

for all $\lambda \in (0, 1]$. From this, it follows that

$$(15) \quad Q(\xi_t) \leq \theta U_{(14)}.$$

REMARK 1. Here, and in what follows, we subscript important constants by the lemmas or formulas where they were first introduced, so it will be easier for the reader to find where they are defined. U 's are upper bounds that are independent of $\lambda \in (0, 1]$.

Our next step toward Lemma 3.1 is to study the infinitesimal mean

$$\mu(\xi) = \lim_{\delta t \downarrow 0} \frac{E[Q(\xi_{t+\delta t}) - Q(\xi_t) | \xi_t = \xi]}{\delta t}.$$

LEMMA 3.2. For all ξ such that $n_1(x, \xi) + n_2(x, \xi) \leq a_0|\hat{B}_0|$ and for all $x \in \mathbb{Z}^d$, $\mu(\xi) \geq \rho Q(\xi)$ where ρ is defined in (10).

PROOF. Straightforward calculation gives

$$(16) \quad \begin{aligned} \frac{\mu(\xi)}{\lambda^d} &= \sum_{\xi(x)=1} [(\theta - 1)\omega - \mu] e^{-\lambda\|x\|} \\ &+ \sum_{\xi(x)=0} \beta \frac{N_2[\mathcal{N}(x)]}{(2L + 1)^d} e^{-\lambda\|x\|} - \sum_{\xi(x)=2} \theta v e^{-\lambda\|x\|}. \end{aligned}$$

For the second term in the equation above, we interchange the roles of x and y then rearrange the sum:

$$\sum_{\xi(x)=0} \beta \frac{N_2[\mathcal{N}(x)]}{(2L + 1)^d} e^{-\lambda\|x\|} = \sum_{\xi(x)=2} (2L + 1)^{-d} \sum_{y \in \mathcal{N}(x), \xi(y)=0} \beta e^{-\lambda\|y\|}.$$

Noting that $\lambda = L^{-1}a_0/2$, and that for any x and $y \in \mathcal{N}(x) \subset B_x$, $-L \leq \|y\| - \|x\| \leq L$, we have

$$e^{-\lambda\|y\|} \geq e^{-a_0/2} e^{-\lambda\|x\|} \geq (1 - a_0) e^{-\lambda\|x\|}.$$

Using this with $n_1(x, \xi) + n_2(x, \xi) < a_0|\hat{B}_0|$, and $B_x[(1 - 4\varepsilon_0)L] \subset \mathcal{N}(x)$ from (8),

$$\begin{aligned} \sum_{\xi(x)=0} \beta \frac{N_2[\mathcal{N}(x)]}{(2L + 1)^d} e^{-\lambda\|x\|} &\geq (1 - a_0) \sum_{\xi(x)=2} \beta \frac{N_0(B_x[(1 - 4\varepsilon_0)L])}{(2L + 1)^d} e^{-\lambda\|x\|} \\ &\geq (1 - a_0)[(1 - 4\varepsilon_0)^d - a_0] \sum_{\xi(x)=2} \beta e^{-\lambda\|x\|}. \end{aligned}$$

Recall that by (11), ε_0 is small enough so that $(1 - 4\varepsilon_0)^d > 1 - 2a_0$. This choice implies

$$\begin{aligned} \sum_{\xi(x)=0} \beta \frac{N_2[\mathcal{N}(x)]}{(2L + 1)^d} e^{-\lambda\|x\|} &> (1 - a_0)(1 - 3a_0) \sum_{\xi(x)=2} \beta e^{-\lambda\|x\|} \\ &> (1 - 4a_0) \sum_{\xi(x)=2} \beta e^{-\lambda\|x\|}. \end{aligned}$$

Combining inequality above with (16) and (10) gives

$$\begin{aligned} \mu(\xi) &\geq \lambda^d \sum_{\xi(x)=1} [(\theta - 1)\omega - \mu] e^{-\lambda\|x\|} + \lambda^d \sum_{\xi(x)=2} [(1 - 4a_0)\beta - \theta\nu] e^{-\lambda\|x\|} \\ &\geq \rho Q(\xi), \end{aligned}$$

which proves the desired result. \square

Then for any initial configuration ξ_0 , define

$$(17) \quad M_t = Q(\xi_t) - Q(\xi_0) - \int_0^t \mu(\xi_s) ds.$$

According to Dynkin’s formula, M_t is a martingale with $EM_t = 0$.

LEMMA 3.3. *There are constants $L_{3.3}$ and $U_{3.3} < \infty$ so that when $L \geq L_{3.3}$, we have $EM_t^2 \leq U_{3.3}L^{-d}t$ for all $t \geq 0$, and hence*

$$(18) \quad E\left(\sup_{s \leq t} M_s^2\right) \leq 4U_{3.3}L^{-d}t.$$

PROOF. Using (16) and (14), we see that

$$(19) \quad |\mu(\xi_t)| \leq C_{3.3}^{(1)} = \theta(\beta + \omega + \mu + \nu)U_{(14)}.$$

To calculate EM_t^2 , let $t_i^n = it/n$.

$$\begin{aligned} EM_t^2 &= \sum_{i=0}^{n-1} E(M_{t_{i+1}^n} - M_{t_i^n})^2 \\ (20) \quad &= \sum_{i=0}^{n-1} E\left[Q(\xi_{t_{i+1}^n}) - Q(\xi_{t_i^n}) - \int_{t_i^n}^{t_{i+1}^n} \mu(\xi_s) ds\right]^2. \end{aligned}$$

The path of $M_s, s \in [0, t]$ is always a right continuous function with left limit. To control the limit of the sum, we first consider the total variation of $M_s, s \in [0, t]$. For each n , let

$$V_t^{(n)} = \sum_{i=0}^{n-1} |M_{t_{i+1}^n} - M_{\xi_{t_i^n}}|.$$

By definition,

$$\begin{aligned}
 V_t^{(n)} &\leq \sum_{i=0}^{n-1} |Q(\xi_{t_{i+1}^n}) - Q(\xi_{t_i^n})| + \sum_{i=0}^{n-1} \left| \int_{t_i^n}^{t_{i+1}^n} \mu(\xi_s) ds \right| \\
 &\leq V_t + \int_0^t |\mu(\xi_s)| ds \leq V_t + C_{3.3}^{(1)} t,
 \end{aligned}$$

where

$$V_t = \sum_{s \in \Pi_t} |Q(\xi_s) - Q(\xi_{s-})|$$

to be the total variation of $Q(\xi_s)$ in $[0, t]$ and Π_t be the set of jump times of ξ_s in $[0, t]$, which is by definition a countable set. To control V_t , write $\Pi_t = \bigcup_{k=0}^\infty \Pi_t^{(k)}$, where for each k , $\Pi_t^{(k)}$ is the set of times in which ξ has a transition at a vertex contained in $H_k = B_0(kL) \setminus B_0((k-1)L)$. Then according to (14), there is some $L_{3.3} < \infty$ and $C_{3.3}^{(2)}, C_{3.3}^{(3)} < \infty$, such that for all $L \geq L_{3.3}$,

$$\begin{aligned}
 (21) \quad EV_t &\leq \sum_{k=0}^\infty \left[E(|\Pi_t^{(k)}|) \cdot \sup_{x \in H_k} \sup_{\substack{\xi, \xi' \in \{0,1,2\}^{Z^d} \\ \xi(y) = \xi'(y) \ \forall y \neq x}} |Q(\xi_s) - Q(\xi_{s-})| \right] \\
 &\leq C_{3.3}^{(2)} L^{-d} \sum_{k=0}^\infty k^{d-1} e^{-\lambda k} t \leq C_{3.3}^{(3)} t
 \end{aligned}$$

which implies that $V_t < \infty$ almost surely, and that M_t is a process with finite variation and definitely bounded. Using Proposition 3.4 on page 67 of [8] and the fact that M_t is a bounded right-continuous martingale, we have

$$(22) \quad \sum_{i=0}^{n-1} E(M_{t_{i+1}^n} - M_{t_i^n})^2 \xrightarrow{L^1} [M]_t,$$

where $[M]_t$ is the quadratic variation of M_t . Noting that for any n

$$\sum_{i=0}^{n-1} E(M_{t_{i+1}^n} - M_{t_i^n})^2 \equiv EM_t^2,$$

combining this with the L^1 convergence in (22), we have

$$EM_t^2 = E[M]_t.$$

Since M_t is a martingale of finite variation, Exercise 3.8.12 of [2] implies

$$(23) \quad [M]_t = \sum_{s \in \Pi_t} (Q(\xi_s) - Q(\xi_{s-}))^2.$$

So for $E[M]_t$, similar as in (21), there is some $U_{3.3} < \infty$, such that when $L \geq L_{3.3}$

$$\begin{aligned}
 E[M]_t &\leq \sum_{k=0}^{\infty} \left[E(|\Pi_t^{(k)}|) \cdot \sup_{x \in H_k} \sup_{\substack{\xi, \xi' \in \{0,1,2\}^{\mathbb{Z}^d} \\ \xi(y) = \xi'(y) \ \forall y \neq x}} (Q(\xi_s) - Q(\xi_{s-}))^2 \right] \\
 (24) \qquad &\leq C_{3.3}^{(2)} L^{-2d} \sum_{k=0}^{\infty} k^{d-1} e^{-2\lambda k} t \leq U_{3.3} L^{-d} t.
 \end{aligned}$$

Equation (24) immediately implies that

$$EM_t^2 = E[M]_t \leq U_{3.3} L^{-d} t$$

which completes the proof. \square

At this point, we have all the tools needed in the proof of Lemma 3.1. If $\xi_0 \in \Xi_0$, there is a $u_{3.1} > 0$ such that for all ξ_0 in Lemma 3.1:

$$u_{3.1} L^{-d+\alpha} \leq Q(\xi_0).$$

Using (18) now

$$E\left(\sup_{s \leq t_0 \log L} M_s^2\right) \leq 4U_{3.3} L^{-d} t_0 \log L$$

so by Chebyshev’s inequality and the fact that $\alpha > d/2$:

$$\begin{aligned}
 P\left(\sup_{s \leq t_0 \log L} |M_s| \geq u_{3.1} L^{-d+\alpha} / 2\right) &\leq \frac{8U_{3.3} L^{-d} t_0 \log L}{u_{3.1}^2 L^{-2(d-\alpha)}} \\
 &= O(L^{-2\alpha+d} \log L) \\
 &= o(L^{d/2-\alpha}) \rightarrow 0.
 \end{aligned}$$

Consider the event $\{\tau > t_0 \log L\}$. For any $s \leq t_0 \log L$, $n_1(x, \xi_s) + n_2(x, \xi_s) < a_0 |\hat{B}_0|$, for all $x \in \mathbb{Z}^d$, so by Lemma 3.2, $\mu(\xi_s) \geq \rho Q(\xi_s)$. Consider the set

$$A = \left\{ \sup_{s \leq t_0 \log L} |M_s| < u_{3.1} L^{-d+\alpha} / 2 \right\} \cap \{\tau > t_0 \log L\}.$$

On A , we will have that for all $t \in [0, t_0 \log L]$,

$$Q(\xi_t) \geq u_{3.1} L^{-d+\alpha} / 2 + \rho \int_0^t Q(\xi_s) ds.$$

If we let $f(t) = e^{\rho t} u_{3.1} L^{-d+\alpha} / 2$, then

$$f(t) = u_{3.1} L^{-d+\alpha} / 2 + \rho \int_0^t f(s) ds.$$

Reasoning as in the proof of Gronwall’s inequality:

LEMMA 3.4. *On the event A , $Q(\xi_t) \geq f(t)$ for all $t \in [0, t_0 \log L]$.*

PROOF. Suppose the lemma does not hold. Let $t_1 = \inf\{t \in [0, t_0 \log L] : Q(\xi_t) < f(t)\}$. By right-continuity of $Q(\xi_t)$, $Q(\xi_{t_1}) \leq f(t_1)$ and $t_1 > 0$. However, by definition of t_1 , we have $Q(\xi_t) \geq f(t)$ on $[0, t_1)$, and by right-continuity of $Q(\xi_t)$ near $t = 0$, the inequality is strict in a neighborhood of 0. Thus, we have

$$\begin{aligned} Q(\xi_{t_1}) &\geq \frac{u_{3.1}L^{-d+\alpha}}{2} + \rho \int_0^{t_1} Q(\xi_s) ds \\ &> \frac{u_{3.1}L^{-d+\alpha}}{2} + \rho \int_0^{t_1} f(s) ds = f(t_1) \end{aligned}$$

which is a contradiction to the definition of t_1 . \square

Recalling that $t_0 = 2d/\rho$

$$f(t_0 \log L) = e^{\rho t_0 \log L} u_{3.1} L^{-d+\alpha} / 2 = u_{3.1} L^{d+\alpha} / 2.$$

When L is large, this will be $\geq \theta U_{(14)}$, the largest possible value of $Q(\xi_t)$. Thus, the assumption that $P(A > 0)$ has lead to a contradiction, and we have completed the proof of Lemma 3.1. \square

4. Proof of Theorem 2. As in the proof of Lemma 3.2, we are able to prove the extinction result in Theorem 2, which does not require the assumption of long range.

PROOF. When $\mu v \geq \omega(\beta - v)$, if $\beta \leq v$, the system dies out since η_t can be bounded by a subcritical contact process with birth rate β and death rate v (the special case of η_t when $\omega = \infty$). Otherwise, we can find a θ' such that

$$\frac{\mu + \omega}{\omega} \geq \theta' \geq \frac{\beta}{v} > 1.$$

For η_t starting from η_0 with a finite number of nonzero sites, consider

$$S(\eta_t) = \sum_{x \in \mathbb{Z}^d} 1_{\eta_t(x)=1} + \theta' 1_{\eta_t(x)=2}.$$

Similarly, let $\mu(\eta_t)$ be the infinitesimal mean of $S(\eta_t)$. Repeating the calculation in the proof of Lemma 3.2, we have

$$\mu(\eta_t) = \sum_{x \in \mathbb{Z}^d} [\omega(\theta' - 1) - \mu] 1_{\eta_t(x)=1} + [-\theta' v + f_0(x, \eta_t)\beta] 1_{\eta_t(x)=2}.$$

Noting that $\omega(\theta' - 1) - \mu \leq 0$ and that

$$-\theta' v + f_0(x, \eta_t)\beta \leq -\theta' v + \beta \leq 0$$

we have shown that $\mu(\eta_t) \leq 0$ for all $t \geq 0$. Thus, $S(\eta_t)$ is a nonnegative supermartingale. By the martingale convergence theorem, $S(\eta_t)$ converge to some limit as $t \rightarrow \infty$. Note that each jump in η_t will change $S(\eta_t)$ by $1, \theta'$ or $\theta' - 1 > 0$. Thus, to have convergence of $S(\eta_t)$, with probability one there must be only finite jumps in each path of η_t , which implies that with probability one η_t will end up at configuration of all 0's, which is the absorbing state.

For the second part of the theorem, there is no nontrivial stationary distribution when $\beta \leq \nu$. When $\beta > \nu$, note that when $\mu\nu > \omega(\beta - \nu)$, there is a θ' such that

$$\frac{\mu + \omega}{\omega} > \theta' > \frac{\beta}{\nu} > 1.$$

We again use the

$$Q'(\eta_t) = \sum_{x \in \mathbb{Z}^d} e^{-\lambda' \|x\|} w'[\eta_t(x)]$$

similar to the Q introduced at the beginning of Lemma 3.1, with $\lambda' > 0$ and

$$w'[\eta(x)] = \begin{cases} 0, & \text{if } \eta(x) = 0, \\ 1, & \text{if } \eta(x) = 1, \\ \theta', & \text{if } \eta(x) = 2. \end{cases}$$

Consider the infinitesimal mean of $Q'(\eta_t)$. Using exactly the same argument as in Lemma 3.2, we have for any η ,

$$\mu'(\eta) \leq \sum_{\eta(x)=1} [(\theta' - 1)\omega - \mu]e^{-\lambda \|x\|} + \sum_{\eta(x)=2} (\beta e^{\lambda L} - \theta' \nu)e^{-\lambda \|x\|}.$$

Thus, when λ is small enough, $\mu'(\eta) \leq 0$ for all $\eta \in \{0, 1, 2\}^{\mathbb{Z}^d}$ and $Q'(\eta_t)$ is a nonnegative supermartingale, and thus has to converge a.s. to a limit. Then for any $x \in \mathbb{Z}^d$, a flip at point x will contribute at least

$$e^{-\lambda \|x\|} \min\{1, \theta - 1\}$$

to the total value of Q' . So with probability one there is a $t < \infty$ such that there is no flip at site x after time t , which can only correspond to the case where $\eta_s(x) \equiv 0$ for all $s \in [t, \infty)$. \square

5. Spatial location of the positive density box. The argument in the previous section proves the existence of a small box \hat{B}_x with positive density, but this is not useful if we do not have control over its location. To do this, we note that the graphical representation in Section 2 shows that box process ξ_t can be stochastically bounded by Krone's model η_t starting from the same initial configuration. Krone's model can in turn be bounded by a branching random walk γ_t in which there are no deaths, 2's give birth to 2's at rate β and births are not suppressed even if the site is occupied.

LEMMA 5.1. *Suppose we start from γ_0 such that $\gamma_0(x) = 2$ for all $x \in \hat{B}_0$, $\gamma_0(x) = 0$ otherwise. Let $M_k(t)$ be the largest of the absolute values of the k th coordinate among the occupied sites at time t . If L is large enough then for any $m > 0$ we have*

$$(25) \quad P(M_k(t) \geq 1 + (2\beta + m)Lt) \leq 2e^{-mt} |\hat{B}_0|.$$

From this, it follows that there is a $C_{5.1} < \infty$ so,

$$(26) \quad E([M_k(t_0 \log L)]^2) \leq C_{5.1}(L \log L)^2.$$

PROOF. First, we will start from the case where γ_0 has only one particle at 0. Rescale space by dividing by L . In the limit as $L \rightarrow \infty$, we have a branching random walk $\bar{\gamma}_t$ with births displaced by an amount uniform on $[-1, 1]^d$. We begin by showing that the corresponding maximum has $E\bar{M}_k^2(t_0 \log L) \leq C(\log L)^2$. To this, we note that mean number of particles in A at time t

$$E(\bar{\gamma}_t(A)) = e^{\beta t} P(\bar{S}(t) \in A),$$

where $\bar{S}(t)$ is a random walk that makes jumps uniform on $[-1, 1]^d$ at rate β . Let $\bar{S}_k(t)$ be the k th coordinate of $\bar{S}(t)$. We have

$$E \exp(\theta \bar{S}_k(t)) = \exp(\beta t [\bar{\phi}(\theta) - 1]) \quad \text{with } \bar{\phi}(\theta) = (e^\theta + e^{-\theta})/2.$$

Large deviations implies that for any $\theta > 0$

$$P(\bar{S}_k(t) \geq x) \leq e^{-\theta x} \exp(\beta t [\bar{\phi}(\theta) - 1]).$$

By symmetry, we have that

$$P(|\bar{S}_k(t)| \geq x) \leq 2e^{-\theta x} \exp(\beta t [\bar{\phi}(\theta) - 1])$$

and hence that

$$(27) \quad P(\bar{M}_k(t) \geq x) \leq 2e^{-\theta x} e^{\beta t} \exp(\beta t [\phi(\theta) - 1]).$$

Since the right-hand side gives the expected number of particles with k th component $\geq x$.

To prove the lemma, now we return to the case $L < \infty$. Let $\phi(\theta) = E \exp(\theta S_k(t))$ where $S(t)$ is a random walk that makes jumps uniform on $[-1, 1]^d \cap \mathbb{Z}^d/L$ at rate β . When $\theta = 1$, $\bar{\phi}(1) - 1 = 0.543$, so if L is large $\phi(1) - 1 \leq 1$, and by the argument that led to (27)

$$P(M_k(t) \geq (2\beta + m)t) \leq 2e^{-mt}.$$

The last result is for starting for one particle at the origin. If we start with $|\hat{B}_0|$ particles in $\hat{B}_0/L \subset [-1, 1]^d$ in the initial configuration $\bar{\gamma}_0$ then

$$P(M_k(t) \geq 1 + (2\beta + m)t) \leq 2|\hat{B}_0|e^{-mt}.$$

Taking $t = t_0 \log L$, and noting that $t_0 = 2d/\rho > 1$, gives the desired result. \square

LEMMA 5.2. For any $a > 0$, let $M_k^j(t_0 \log L)$ $1 \leq j \leq L^a$ be the maximum of the absolute value of k th coordinates in the j th copy of a family of independent and identically distributed branching random walk in Lemma 5.1. There is a $C_{5.2} < \infty$ so that for large L

$$P\left(\max_{1 \leq j \leq L^a} \{M_k^j(t_0 \log L)\} \geq C_{5.2} L \log L\right) \leq 1/L.$$

PROOF. Taking $t = t_0 \log L$ in (25) and recalling $|\hat{B}_0| = O(L^d)$, the right-hand side is $\leq L^d \exp(-mt_0 \log L)$ for each copy. So the probability on the left-hand side in the lemma $\leq L^{a+d} \exp(-mt_0 \log L)$. Taking the constant m to be large enough gives the desired result. \square

6. Moving particles in η_t . Let $H_{t,x}$ be the set of nonzero sites of η_t in \hat{B}_x at time t . In this section, we will use the graphical representation in Section 2 and an argument from Durrett and Lanchier [5] to show that

LEMMA 6.1. There are constants $\delta_{6.1} > 0$ and an $L_{6.1} < \infty$ such that for all $L > L_{6.1}$ and any initial configuration η_0 with $|H_{0,0}| \geq L^{d/2}$

$$P(|H_{1,v}| \geq \delta_{6.1} |H_{0,0}|) > 1 - e^{-L^{d/4}}$$

for any $v \in \{0, \pm e_1, \dots, \pm e_d\}$.

PROOF. We begin with the case $v = 0$ which is easy. Define G_0^0 to be the set of points $x \in \hat{B}_0$, with (a) $\eta_0(x) \geq 1$, and (b) no death marks \times 's occur in $\{x\} \times [0, 1]$. We have $\xi_t(x) \geq 1$ on $S_0 = H_{0,0} \cap G_0^0$, and $|S_0| \sim \text{Binomial}(|H_{0,0}|, e^{-\mu})$, so the desired result follows from large deviations for the Binomial.

For $v \neq 0$, define G_0 to be the set of points in G_0^0 for which (c) there exists a (\bullet) , which produces growth from type 1 to type 2, in $\{x\} \times [0, 1/2]$. We define G_v to be the set of points y in \hat{B}_v so that there are no \times 's in $\{y\} \times [0, 1]$. For any $x \in \hat{B}_0$ and $y \in \hat{B}_v$ we say that x and y are connected (and write $x \rightarrow y$) if there is an arrow from x to y in $(1/2, 1)$. By definition of our process $\eta_1(y) \geq 1$ for all y in

$$S = \{y : y \in G_v, \text{ there exists an } x \in G_0 \text{ so that } x \rightarrow y\}.$$

It is easy to see that

$$(28) \quad |G_0| \sim \text{Binomial}[|H_{0,0}|, e^{-\mu}(1 - e^{-\omega/2})].$$

Conditional on $|G_0|$:

$$(29) \quad |S| \sim \text{Binomial}(|\hat{B}_v|, e^{-\mu}[1 - e^{-\beta|G_0|/2|B_0|}]),$$

by Poisson thinning since the events of being the recipient of a birth from \hat{B}_0 are independent for different sites in G_v .

Since the binomial distribution decays exponentially fast away from the mean, there is some constant $c > 0$ such that

$$(30) \quad P(|G_0| > |H_{0,0}|e^{-\mu}(1 - e^{-\omega/2})/2) \geq 1 - e^{-cL^{d/2}}.$$

To simplify the next computation, we note that $1 - e^{-\beta r} \sim \beta r$ as $r \rightarrow 0$ so if the ε_0 in the definition of the small box is small enough

$$1 - e^{-\beta|G_v|/2|B_0|} \geq \beta|G_v|/4|B_0|.$$

Let $p = e^{-\mu}\beta|G_0|/4|B_0|$. A standard large deviations result, see, for example, Lemma 2.8.5 in [4] shows that if $X = \text{Binomial}(N, p)$ then

$$P(X \leq Np/2) \leq \exp(-Np/8)$$

from which the desired result follows. \square

Let $\|\cdot\|_1$ be the L^1 -norm on \mathbb{Z}^d . Our next step is to use Lemma 6.1 $O(\log L)$ times to prove:

LEMMA 6.2. *For any $\alpha \in (d/2, d)$, let $C_{6.2}$ be a constant such that $C_{6.2} \log \delta_{6.1} > \alpha - d$. There is a finite $L_{6.2} > L_{6.1}$ such that for all $L > L_{6.2}$, any initial configuration η_0 with $|H_{0,0}| \geq a_0|\hat{B}_0|$, and any $x \in \mathbb{Z}^d$ such that $\|x\|_1 \leq C_{6.2} \log L$, we have*

$$P(|H_{x,[C_{6.2} \log L]}| \geq L^\alpha) \geq 1 - e^{-L^{d/4}/2}.$$

PROOF. Let $n = [C_{6.2} \log L]$. We can find a sequence $x_0 = 0, x_1, \dots, x_n = x$ such that for all $i = 0, \dots, n - 1, x_{i-1} - x_i \in \{0, \pm e_1, \dots, \pm e_d\}$. For any $i = 1, \dots, n$ define the event

$$A_i = \{|H_{x_i,i}| \geq \delta_{6.1}^i |H_{0,0}|\}.$$

By the definition of $C_{6.2}, |H_{x,[C_{6.2} \log L]}| \geq L^\alpha$ on A_n . To estimate $P(A_n)$ note that by Lemma 6.1

$$\begin{aligned} P(A_n) &\geq 1 - \sum_{i=1}^n P(A_i^c) \geq 1 - \sum_{i=1}^n P(A_i^c | A_{i-1}) \\ &\geq 1 - C_{6.2}(\log L)e^{-L^{d/4}} \geq 1 - e^{-L^{d/4}/2} \end{aligned}$$

when L is large. \square

7. Block construction and the proof of Theorem 1. At this point, we have all the tools to construct the block event and complete the proof of Theorem 1. Let $0 < a < \alpha/2 - d/4, K = L^{1+2a/3}, \Gamma_m = 2mKe_1 + [-K, K]^d$, and $\Gamma'_m = 2mKe_1 + [-K/2, K/2]^d$. If $m + n$ is even, we say that (m, n) is wet if there is a positive

density small box, that is, a box with size ℓ and densities of nonzero sites $\geq a_0$, in Γ'_m at some time in $[nL^a, nL^a + C_6 \log L]$, where $C_6 = C_{6.2} + t_0$. Our goal is to show

LEMMA 7.1. *If (m, n) is wet then with high probability so is $(m + 1, n + 1)$, and the events which produce this are measurable with respect to the graphical representation in $(\Gamma_m \cup \Gamma_{m+1}) \times [nL^a, (n + 1)L^a + C_6 \log L]$.*

Once this is done, Theorem 1 follows. See [3] for more details.

PROOF OF LEMMA 7.1. To prove Lemma 7.1, we will alternate two steps, starting from the location of the initial positive density box \hat{B}_{y_0} at time T_0 . Let $A_0 = \{T_0 < \infty\}$ which is the whole space. Assume given a deterministic sequence δ_i with $\|\delta_i\| \leq C_{6.2} \log L$. If we never meet a failure, the construction will terminate at the first time that $T_i > (n + 1)L^a$. The actual number steps will be random but the number is $\leq N = \lceil L^a / [C_{6.2} \log L] \rceil$. We will estimate the probability of success supposing that N steps are required. This lower bounds the probability of success when we stop at the first time $T_i \geq (n + 1)L^a$. Suppose $i \geq 1$.

Deterministic moving. If at the stopping time $T_{i-1} < \infty$, we have a positive density small box $\hat{B}_{y_{i-1}}$, then we use results in Section 6 to produce a small box $\hat{B}_{y_{i-1}+\delta_i}$ with at least L^α nonzero sites at time $S_i = T_{i-1} + [C_{6.2} \log L]$. If we fail, we let $S_i = \infty$ and the construction terminates. Let $A_i^+ = \{S_i < \infty\}$.

Random recovery. If at the stopping time $S_i < \infty$, we have a small box $\hat{B}_{y_{i-1}+\delta_i}$ with at least L^α nonzero sites then we set all of the sites outside the box to 0, and we use the recovery lemma to produce a positive density small box \hat{B}_{y_i} at time $S_i \leq T_i \leq S_i + t_0 \log L$. Again if we fail, we let $T_i = \infty$ and the construction terminates. Let $A_i = \{T_i < \infty\}$. Let $\Delta_i(\omega) = y_i - (y_{i-1} + \delta_i)$ on A_i , and $= 0$ on A_i^c .

If we define the partial sums $\bar{y}_i = y_0 + \sum_{j=1}^i \delta_j$ and $\Sigma_i = \sum_{j=1}^i \Delta_j$, then we have $y_i = \bar{y}_i + \Sigma_i$. We think of \bar{y}_i as the mean of the location of the positive density box and Σ_i as the random fluctuations in its location. We make no attempt to adjust the deterministic movements δ_i to compensate for the fluctuations. Let $y_{\text{end}} = (y_{\text{end}}^1, 0, \dots, 0) \in \mathbb{Z}^d$ be such that $2K(m + 1)e_1 \in \hat{B}_{y_{\text{end}}}$. We define the δ_i to reduce the coordinates $y_0^k, k = 2, \dots, d$ to 0 and then increase y_0^1 to y_{end}^1 , in all cases using steps of size $\leq C_{6.2} \log L$. Note that

$$\|y_0 - y_{\text{end}}\|_1 = O(K)/\ell = O(L^{2a/3}) = o(N)$$

so we can finish the movements well before N steps. And once this is done we set the remaining δ_i to 0. Moreover, note that each successful step in our iteration takes a time at most $C_6 \log L$. Thus, we will get to y_{end} by $t = nL^a + O(L^{2a/3})C_6 \log L < (n + 1)L^a$. At the first time $T_i \geq (n + 1)L^a$, we already have

$$(31) \quad \delta_i = 0, \quad \bar{y}_i = y_{\text{end}}.$$

At this point, we are ready to state the main lemma of this section that controls the spatial movement in our iteration.

LEMMA 7.2. *For any initial configuration $\eta(T_0)$ so that there is a small box $\hat{B}_{y_0} \subset \Gamma'_m$, and any sequence δ_i , $i \leq N$ with $\|\delta_i\| \leq C_{6.2} \log L$ and any $\varepsilon > 0$, there is a good event G_N with $G_N \rightarrow 1$ as $L \rightarrow \infty$ so that (a) $G_N \subset A_N$, (b) on G_N , $\|y_i - \bar{y}_i\| < \varepsilon L^{2a/3}$ for $1 \leq i \leq N$, (c) G_N depends only on the gadgets of graphical representation in $\Gamma_m \cup \Gamma_{m+1}$.*

PROOF. The first step is to show that $P(A_N) \rightarrow 1$ as $L \rightarrow \infty$. For the i th deterministic moving step, using the strong Markov property and Lemma 6.2, we have

$$P(A_i^+ | A_{i-1}) > 1 - e^{-L^{d/4}/2}.$$

Then for the random recovery phase, according to Lemma 3.1, we have the conditional probability of success:

$$P(A_i | A_i^+) = P(\tau_i < t_0 \log L) > 1 - L^{d/2-\alpha}.$$

Combining the two observations, we have

$$(32) \quad P(A_i | A_{i-1}) > (1 - e^{-L^{d/4}/2})(1 - L^{d/2-\alpha}) > 1 - e^{-L^{d/4}/2} - L^{d/2-\alpha}$$

which implies

$$\begin{aligned} P(A_N) &\geq 71 - \sum_{i=1}^N P(A_i^c) \geq 1 - \sum_{i=1}^N P(A_i^c | A_{i-1}) \\ &\geq 1 - L^a (e^{-L^{d/4}/2} + L^{d/2-\alpha}) \geq 1 - 2L^{d/4-\alpha/2} \rightarrow 1. \end{aligned}$$

The next step is to control the fluctuations in the movement of our box.

LEMMA 7.3. *Let $\mathcal{F}(T_i)$ be the filtration generated by events in the graphical representation up to stopping time T_i . For any $1 \leq k \leq d$, $\{\Sigma_i^k\}_{i=1}^N$ is a martingale with respect to $\mathcal{F}(T_i)$. $E(\Sigma_i^k) = 0$, and $\text{var}(\Sigma_N^k) \leq C_{7.3} L^a \log L$ so for any ε we have*

$$P\left(\max_{i \leq N} \|\Sigma_i\| > \varepsilon L^{2a/3}\right) \rightarrow 0.$$

PROOF. Consider the conditional expectation of Δ_i^k under $\mathcal{F}(S_i)$. According to the discussions about the truncated process right before Section 2.1, we have $E(\Delta_i^k | \mathcal{F}(S_i)) = 0$. Noting that $2\ell|\Delta_i^k|$ can be bounded by the largest k th coordinate among the occupies sites of the corresponding branching random walk at time $t_0 \log L$, Lemma 5.1 implies that $E((\Delta_i^k)^2 | \mathcal{F}(S_i)) \leq C(\log L)^2$. By orthogonality of martingale increments $\text{var}(\Sigma_N^k) \leq NC(\log L)^2$. Since $N \leq L^a / [C_{5.2} \log L] + 1$, we have the desired bound on the variances and the desired result follows from L^2 maximal inequality for martingales. \square

To check (c), now note that under A_i the success of A_{i+1}^+ depends only on gadgets in

$$(2\ell y_i + [-\ell C_{6.2} \log L, \ell C_{6.2} \log L]^d) \times [T_i, S_{i+1}]$$

and that when the i th copy of the truncated process never wanders outside

$$D_i = 2\ell(y_{i-1} + \delta_i) + [-C_{5.2}(L \log L), C_{5.2}(L \log L)]^d$$

the success of A_{i+1} under A_{i+1}^+ depends only on gadgets in D_i . According to Lemma 7.2 and the fact that $N < L^a$, with probability $\geq 1 - L^{-1} = 1 - o(1)$, our construction only depends on gadgets in the box:

$$\bigcup_{i=0}^{N-1} [(2\ell y_i + [-\ell C_{6.2} \log L, \ell C_{6.2} \log L]^d) \times [T_i, S_{i+1}]] \cup (2\ell(y_i + \delta_{i+1}) + [-C_{5.2}(L \log L), C_{5.2}(L \log L)]^d) \times [S_{i+1}, T_{i+1}].$$

The locations of the y_i are controlled by Lemma 7.3 so that it is easy to see that the box defined above is a subset of $\Gamma_n \cup \Gamma_{n+1}$, and proof of Lemma 7.2 is complete. \square

Back to the proof of Lemma 7.1, on G_N , it follows from (31) and Lemma 7.2, when we stop at the first time $T_i \geq (n + 1)L^a$:

$$\|y_i - 2K(m + 1)e_1\| \leq \ell(1 + 2\|y_i - \bar{y}_i\|) \leq 4\epsilon K$$

which implies

$$\hat{B}_{y_i} = 2\ell y_i + (-\ell, \ell]^d \subset \Gamma'_{n+1}.$$

Noting that the success of G_N only depends on gadgets in $\Gamma_n \cup \Gamma_{n+1}$, we have proved that G_N is measurable with respect to the space–time box in the statement of Lemma 7.1, which completes the proof of Lemma 7.1 and Theorem 1. \square

8. Proof of Theorem 3. Our *first step* is to construct the test functions for S and T and show that, under (6), they have positive derivatives for all sites in the region of interest. According to (7), we can choose $\Sigma_0 \in (0, 1)$ such that

$$(33) \quad \frac{\nu}{\omega_0} < \frac{\beta(1 - \Sigma_0)}{2^d(\mu + \omega_0)}.$$

Let

$$(34) \quad \gamma_0 \in \left(\frac{\nu}{\omega_0}, \frac{\beta(1 - \Sigma_0)}{2^d(\mu + \omega_0)} \right)$$

and let

$$(35) \quad T_0 = \frac{\Sigma_0}{1 + \gamma_0}, \quad S_0 = \frac{\gamma_0 \Sigma_0}{1 + \gamma_0}.$$

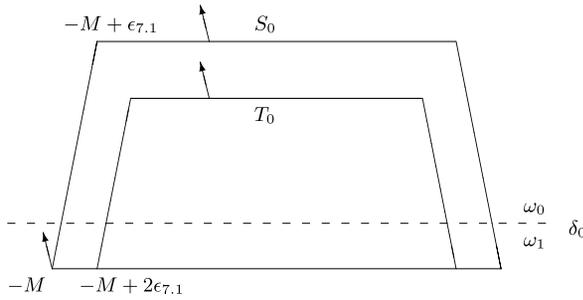


FIG. 1. Test functions for $d = 1$.

Note that

$$(36) \quad S_0 + T_0 = \Sigma_0, \quad S_0/T_0 = \gamma_0.$$

Recall that for any $x \in \mathbb{Z}^d$ and $r \geq 0$, $B(x, r)$ is defined in Section 2 to be the L^∞ neighborhood of x with range r . With S_0, T_0 defined as above and $\epsilon_{8.1}$ to be specified later, define the test functions $S_{\text{test}}(x, 0)$ and $T_{\text{test}}(x, 0)$ as follows (Figure 1 shows those test functions when $d = 1$): let $S_{\text{test}}(x, 0) = S_0$ on $B(0, M - \epsilon_{8.1})$, $S_{\text{test}}(x, 0) = 0$ on $B(0, M)^c$, and

$$(37) \quad S_{\text{test}}(x, 0) = \frac{d[x, B(0, M)^c]S_0}{d[x, B(0, M)^c] + d[x, B(0, M - \epsilon_{8.1})]}$$

for $x \in B(0, M - \epsilon_{8.1})^c \cap B(0, M)$. Similarly, let $T_{\text{test}}(x, 0) = T_0$ on $B(0, M - 3\epsilon_{8.1})$, $T_{\text{test}}(x, 0) = 0$ on $B(0, M - 2\epsilon_{8.1})^c$, and

$$(38) \quad T_{\text{test}}(x, 0) = \frac{d[x, B(0, M - 2\epsilon_{8.1})^c]T_0}{d[x, B(0, M - 2\epsilon_{8.1})^c] + d[x, B(0, M - 3\epsilon_{8.1})]}$$

for $x \in B(0, M - 3\epsilon_{8.1})^c \cap B(0, M - 2\epsilon_{8.1})$. In the definitions above, $M = \max\{4, 4\kappa\}$, $d(x, A)$ be the distance between $x \in R^d$ and $A \subset R^d$ under L^∞ -norm, $\epsilon_{8.1} = \epsilon_{8.1}(\beta, \omega_0, \mu, \nu, \kappa)$ is some positive constant that will be specified later in the proof of Lemma 8.1. The following lemma shows that the test functions have positive derivatives under IDE (6).

REMARK 2. Throughout the discussion in this section, all the ϵ 's, δ 's, t 's and c 's introduced are constants independent to the choice of L .

LEMMA 8.1. Under the conditions in Theorem 3, there are $\epsilon_{8.1}$ and $\epsilon_1 > 0$ so that under IDE (6):

$$(39) \quad \begin{aligned} \frac{dS_{\text{test}}(x, 0)}{dt} &\geq 4\epsilon_1 && \text{for all } x \in B(0, M) = \overline{\{S_{\text{test}} > 0\}}, \\ \frac{dT_{\text{test}}(x, 0)}{dt} &\geq 4\epsilon_1 && \text{for all } x \in B(0, M - 2\epsilon_{8.1}) = \overline{\{T_{\text{test}} > 0\}}. \end{aligned}$$

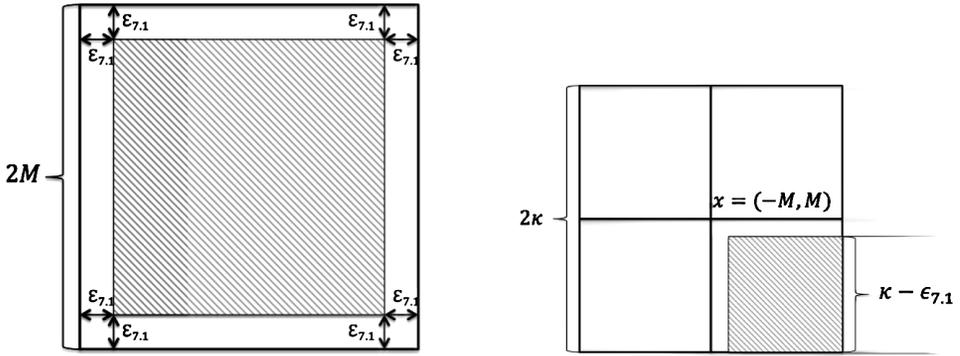


FIG. 2. S_{test} with $d = 2$. Left: S_{test} with $d = 2$. The big box of size $2M$ is region $\{S_{\text{test}} > 0\}$. The shadow area is the region where $S_{\text{test}} = S_0$. Right: Worst case for S_{test} where $x = (-M, M)$. The big box is the local grass environment. The shadow area is the region where $S_{\text{test}} = S_0$.

PROOF. With T_{test} and S_{test} defined as above, for any $x \in B(0, M)$, the local grass density can be upper bounded as follows (see Figure 2 for the case when $d = 2$):

$$D_{\kappa}^G(x, 0) \leq 1 - \left(\frac{\kappa - \varepsilon_{8.1}}{2\kappa}\right)^d S_0 \leq 1 - 2^{-d} \left(1 - d \frac{\varepsilon_{8.1}}{\kappa}\right) S_0.$$

Noting that $\delta_0 < 2^{-d} S_0$, let

$$(40) \quad \varepsilon_{(40)} = \frac{\kappa(1 - 2^d \delta_0 / S_0)}{4d}.$$

It is easy to check that when $\varepsilon_{8.1} \leq \varepsilon_{(40)}$

$$D_{\kappa}^G(x, 0) < 1 - \delta_0,$$

which implies that

$$(41) \quad \omega[D_{\kappa}^G(x, 0)] \equiv \omega_0$$

for all $x \in B(0, M)$. That is, all sites in the region of test function live in an environment with a higher growth rate ω_0 . It is easy to see that the derivative of T_{test} can be lower bounded by its derivative on the top, that is, for any $x \in B(0, M - 2\varepsilon_{8.1})$

$$(42) \quad \frac{dT_{\text{test}}(x, 0)}{dt} \geq \frac{dT_{\text{test}}(0, 0)}{dt} \geq \omega_0 S_0 - \nu T_0$$

and this holds for all $\kappa > 0$. Note that $\gamma_0 = S_0 / T_0$ according to (35). Combining this observation with the definition of γ_0 in (34)

$$(43) \quad \omega_0 S_0 - \nu T_0 = \omega_0 T_0 \left(\frac{S_0}{T_0} - \frac{\nu}{\omega_0}\right) = \omega_0 T_0 \left(\gamma_0 - \frac{\nu}{\omega_0}\right) > 0.$$

Thus, we have the derivative of T_{test} is always positive for all $x \in B(0, M - 2\varepsilon_{8.1})$.

Similarly, we can control the lower bound of derivative for test function S_{test} as follows: for any $x \in B(x, M)$

$$(44) \quad \begin{aligned} \frac{dS_{\text{test}}(x, 0)}{dt} &\geq \frac{\beta T_0(1 - 3\varepsilon_{8.1})^d}{2^d} (1 - S_0 - T_0) - (\mu + \omega_0)S_0 \\ &\geq \frac{\beta T_0(1 - 3d\varepsilon_{8.1})}{2^d} (1 - S_0 - T_0) - (\mu + \omega_0)S_0. \end{aligned}$$

For the right-hand side of (44), according to (36)

$$\begin{aligned} \frac{dS_{\text{test}}(x, 0)}{dt} &\geq \frac{\beta T_0(1 - 3d\varepsilon_{8.1})}{2^d} (1 - S_0 - T_0) - (\mu + \omega_0)S_0 \\ &= T_0(\mu + \omega_0) \left[(1 - 3d\varepsilon_{8.1}) \frac{\beta(1 - S_0 - T_0)}{2^d(\mu + \omega_0)} - \frac{S_0}{T_0} \right] \\ &= T_0(\mu + \omega_0) \left[(1 - 3d\varepsilon_{8.1}) \frac{\beta(1 - \Sigma_0)}{2^d(\mu + \omega_0)} - \gamma_0 \right]. \end{aligned}$$

Again recalling the definition in (34) that

$$\frac{\beta(1 - \Sigma_0)}{2^d(\mu + \omega_0)} > \gamma_0,$$

we let

$$(45) \quad \varepsilon_{(45)} = \left[1 - \frac{2^d(\mu + \omega_0)\gamma_0}{\beta(1 - \Sigma_0)} \right] / (6d).$$

So for any $\varepsilon_{8.1} \leq \varepsilon_{(45)}$ and $x \in B(0, M)$

$$(46) \quad \frac{dS_{\text{test}}(x, 0)}{dt} \geq T_0(\mu + \omega_0) \left[(1 - 3d\varepsilon_{(45)}) \frac{\beta(1 - \Sigma_0)}{2^d(\mu + \omega_0)} - \gamma_0 \right] > 0.$$

Thus, let

$$(47) \quad \varepsilon_{8.1} = \min\{\varepsilon_{(40)}, \varepsilon_{(45)}\} > 0$$

and

$$\varepsilon_1 = \frac{1}{4} \min \left\{ \omega_0 T_0 \left(\gamma_0 - \frac{\nu}{\omega_0} \right), T_0(\mu + \omega_0) \left[(1 - 3d\varepsilon_{(45)}) \frac{\beta(1 - \Sigma_0)}{2^d(\mu + \omega_0)} - \gamma_0 \right] \right\} > 0.$$

Combining (43) and (46) and the proof is complete. \square

With the test functions constructed, our *second step* is similar to the proof of Theorem 1. We introduce the truncated version of the Staver–Levin model, and as before, denote the process by $\tilde{\xi}_t$. For $\ell = \varepsilon L$, where the exact value of ε is specified later in Lemma 8.4, $\tilde{\xi}_t$ has birth rate $\beta|\mathcal{N}(x, L)|/(2L + 1)^d$, where $\mathcal{N}(x, L)$ in the

truncated neighborhood defined in Section 2. Type 1's and 2's in $\bar{\xi}_t$ die at the same rates as in the original χ_t , while a growth of a sapling into a tree occurs at rate:

$$\bar{\omega}[\bar{G}(x, \xi)] = \begin{cases} \omega_0, & \bar{G}(x, \xi) \in [0, 1 - \delta), \\ \omega_1, & \bar{G}(x, \xi) \in [1 - \delta, 1], \end{cases}$$

where

$$\bar{G}(x, \xi) = \frac{\# \text{ of } 0\text{'s in } \mathcal{N}(x, K)}{|\mathcal{N}(x, K)|},$$

$K = \kappa L$, and $\delta = \delta_0 + 4d\varepsilon$. First of all, with the same argument as in Section 2, we immediately have that the number of different types in each small box forms a Markov process $\bar{\zeta}_t$. According to (8):

$$\begin{aligned} \bar{G}(x, \xi) &\leq 1 - \frac{\# \text{ of } (1 + 2)\text{'s in } \mathcal{N}(x, K)}{|B(x, K)|} \\ &\leq G(x, \xi) + 1 - (1 - 4\varepsilon)^d \leq G(x, \xi) + 4d\varepsilon, \end{aligned}$$

combining this with the definition of δ , we have for any $\xi' \geq \xi$, $\omega[G(x, \xi')] \geq \bar{\omega}[\bar{G}(x, \xi)]$, which implies that the truncated $\bar{\xi}_t$ once again is dominated by the original χ_t . Thus, in order to prove Theorem 3, it suffices to show that $\bar{\xi}_t$ survives.

The *third step* is to construct an initial configuration of $\bar{\xi}_0$ according to the test functions defined in (37) and (38). For any $x \in \mathbb{Z}^d$: if $2\ell x \notin B(0, ML)$ there is no saplings or trees in \hat{B}_x under $\bar{\xi}_0$. If $2\ell x \in B(0, ML)$, $n_1(x, \bar{\xi}_0) = |\hat{B}_0|S_{\text{test}}(2\ell x/L)$, $n_2(x, \bar{\xi}_0) = |\hat{B}_0|T_{\text{test}}(2\ell x/L)$. As noted earlier in the box process $\bar{\xi}_t$, the locations of the 1's and 2's inside each small box makes no difference.

We then look at $f_i(x, \bar{\xi}_t)$, the densities of type i in each small box. For any x , the infinitesimal means of f_1 and f_2 can be written as follows:

$$\begin{aligned} \mu_1(x, \bar{\xi}) &= (2L + 1)^{-d} |\hat{B}_0| \left(\sum_{y: \hat{B}_y \subset \mathcal{N}(x, L)} f_2(y, \bar{\xi}) \right) f_0(x, \bar{\xi}) \beta \\ (48) \quad &\quad - [\bar{\omega}(\bar{G}(x, \bar{\xi})) + \mu] f_1(x, \bar{\xi}), \\ \mu_2(x, \bar{\xi}) &= \bar{\omega}(\bar{G}(x, \bar{\xi})) f_1(x, \bar{\xi}) - \nu f_2(x, \bar{\xi}). \end{aligned}$$

We prove the following.

LEMMA 8.2. *There is a $\varepsilon_{8.2} > 0$ such that for any $\ell = \varepsilon L \leq \varepsilon_{8.2} L$ and the $\bar{\xi}_0$ defined above, we have:*

- $\mu_1(x, \bar{\xi}_0) \geq 2\varepsilon_1$ for all $2\ell x \in B(0, ML + 4\ell)$.
- $\mu_2(x, \bar{\xi}_0) \geq 2\varepsilon_1$ for all $2\ell x \in B(0, ML - 2\varepsilon_{8.1} L + 4\ell)$.

PROOF. First noting that $\delta = \delta_0 + 4d\varepsilon$, $\delta_0 < 2^{-d} S_0$, for

$$(49) \quad \varepsilon_{(49)} = (2^{-d} S_0 - \delta_0)/(8d)$$

$\delta = \delta_0 + 4d\varepsilon < 2^{-d}S_0$ for $\varepsilon \leq \varepsilon_{(49)}$. Moreover, for all x such that $2\ell x \in B(0, ML + 4\ell)$,

$$\begin{aligned} \bar{G}(x, \bar{\xi}_0) &= \frac{\# \text{ of } 0\text{'s in } \mathcal{N}(x, K)}{|\mathcal{N}(x, K)|} \\ (50) \quad &\leq 1 - \left(\frac{\kappa - \varepsilon_{8.1} - 10\varepsilon}{2\kappa}\right)^d S_0 \\ &\leq 1 - 2^{-d}S_0(1 - d\varepsilon_{8.1} - 10d\varepsilon). \end{aligned}$$

Let

$$(51) \quad \varepsilon_{(51)} = \frac{S_0(1 - d\varepsilon_{8.1}) - 2^d\delta_0}{20dS_0 + 2^{d+2}d} > 0.$$

It is easy to see that $\bar{G}(x, \bar{\xi}_0) < 1 - \delta_0 - 4d\varepsilon = 1 - \delta$ for all $\varepsilon \leq \varepsilon_{(51)}$, which implies $\bar{\omega}(\bar{G}(x, \bar{\xi}_0)) \equiv \omega_0$ for all $2\ell x \in B(0, ML + 4\ell)$. Thus,

$$\mu_2(x, \bar{\xi}_0) = \bar{\omega}(\bar{G}(x, \bar{\xi}_0))f_1(x, \bar{\xi}_0) - \nu f_2(x, \bar{\xi}_0) \geq \omega_0 S_0 - \nu T_0 > 4\varepsilon_1$$

for all $2\ell x \in B(0, ML - 2\varepsilon_{8.1}L + 4\ell)$. Then for the infinitesimal mean of type 1:

$$\begin{aligned} \mu_1(x, \bar{\xi}_0) &= (2L + 1)^{-d} |\hat{B}_0| \left(\sum_{y: \hat{B}_y \subset \mathcal{N}(x, L)} f_2(y, \bar{\xi}_0) \right) f_0(x, \bar{\xi}_0) \beta \\ &\quad - [\omega_0 + \mu] f_1(x, \bar{\xi}_0). \end{aligned}$$

Note that

$$\begin{aligned} &\inf_{2\ell x \in B(0, ML + 4\ell)} (2L + 1)^{-d} |\hat{B}_0| \left(\sum_{y: \hat{B}_y \subset \mathcal{N}(x, L)} f_2(y, \bar{\xi}_0) \right) f_0(x, \bar{\xi}_0) \beta \\ &\geq 2^{-d}(1 - 3\varepsilon_{8.1} - 10\varepsilon)^d T_0(1 - S_0 - T_0)\beta \\ &\geq 2^{-d}(1 - 3d\varepsilon_{8.1} - 10d\varepsilon)T_0(1 - S_0 - T_0)\beta. \end{aligned}$$

Recalling (46), (47) and the definition of ε_1 , let

$$(52) \quad \varepsilon_{(52)} = \frac{2^{d-1}\varepsilon_1}{5dT_0(1 - S_0 - T_0)\beta}.$$

For all $\varepsilon \leq \varepsilon_{(52)}$ and any $2\ell x \in B(0, ML + 4\ell)$,

$$\mu_2(x, \bar{\xi}_0) \geq 2^{-d}(1 - 3d\varepsilon_{8.1})T_0(1 - S_0 - T_0)\beta - (\omega_0 + \mu)S_0 - 2\varepsilon_1 \geq 2\varepsilon_1.$$

Overall, let

$$\varepsilon_{8.2} = \min\{\varepsilon_{(49)}, \varepsilon_{(51)}, \varepsilon_{(52)}\}.$$

It satisfies the condition of this lemma by definition. \square

Moreover, since that the inequalities for \bar{G} 's in the proof above are strict and that all other terms in the infinitesimal mean are continuous, we have:

LEMMA 8.3. *There is some $\delta_{8.3} > 0$ so that for any configuration $\bar{\xi}'_0$ with*

$$|f_i(x, \bar{\xi}_0) - f_i(x, \bar{\xi}'_0)| \leq \delta_{8.3}, \quad i = 1, 2; 2\ell x \in B(0, 2ML)$$

we have:

- $\mu_1(x, \bar{\xi}'_0) \geq \varepsilon_1$ for all $2\ell x \in B(0, ML + 4\ell)$,
- $\mu_2(x, \bar{\xi}'_0) \geq \varepsilon_1$ for all $2\ell x \in B(0, ML - 2\varepsilon_{8.1}L + 4\ell)$

for all $\ell = \varepsilon L \leq \varepsilon_{8.2}L$.

PROOF. First note that for any x such that $2\ell x \in B(0, ML + 4\ell)$,

$$|\bar{G}(x, \bar{\xi}_0) - \bar{G}(x, \bar{\xi}'_0)| \leq 2\delta_{8.3}.$$

Let

$$(53) \quad \delta_{(53)} = [2^{-d}S_0(1 - d\varepsilon_{8.1} - 10d\varepsilon_{8.2}) - \delta_0 - 4d\varepsilon_{8.2}]/4 > 0.$$

For any $\delta_{8.3} \leq \delta_{(53)}$, recalling (50) and (51), we have

$$\begin{aligned} \bar{G}(x, \bar{\xi}'_0) &\leq \bar{G}(x, \bar{\xi}_0) + 2\delta_{8.3} \\ &\leq 1 - 2^{-d}S_0(1 - d\varepsilon_{8.1} - 10d\varepsilon_{8.2}) + 2\delta_{(53)} < 1 - \delta_0 - 4d\varepsilon_{8.2} \\ &\leq 1 - \delta_0 - 4d\varepsilon = 1 - \delta \end{aligned}$$

which implies that

$$(54) \quad \bar{\omega}(G(x, \bar{\xi}'_0)) \equiv \omega_0$$

for all x such that $2\ell x \in B(0, ML + 4\ell)$. Furthermore, under (54), (48) implies that

$$\begin{aligned} (55) \quad |\mu_2(x, \bar{\xi}_0) - \mu_2(x, \bar{\xi}'_0)| &\leq \omega_0\delta_{8.3} + \nu\delta_{8.3} = (\omega_0 + \nu)\delta_{8.3}, \\ |\mu_1(x, \bar{\xi}_0) - \mu_1(x, \bar{\xi}'_0)| &\leq (2L + 1)^{-d}|\hat{B}_0| \left(\sum_{y: \hat{B}_y \subset \mathcal{N}(x, L)} f_2(y, \bar{\xi}_0) \right) 2\beta\delta_{8.3} \\ (56) \quad &+ (2L + 1)^{-d}|\hat{B}_0| \left(\sum_{y: \hat{B}_y \subset \mathcal{N}(x, L)} \delta_{8.3} \right) f_0(x, \bar{\xi}_0)\beta \\ &+ 2\delta_{8.3}^2 + (\omega_0 + \mu)\delta_{8.3} \end{aligned}$$

for all x such that $2\ell x \in B(0, ML + 4\ell)$. Noting that $f_i \leq 1$, $\delta_{8.3} \leq 1$, (56) can be simplified as

$$|\mu_1(x, \bar{\xi}_0) - \mu_1(x, \bar{\xi}'_0)| \leq (2 + 3\beta + \omega_0 + \mu)\delta_{8.3}.$$

Thus, let

$$\delta_{8.3} = \min \left\{ \delta_{(53)}, \frac{\varepsilon_1}{2(\omega_0 + \nu)}, \frac{\varepsilon_1}{2(2 + 3\beta + \omega_0 + \mu)} \right\}.$$

Equations (54)–(56) show that $\delta_{8.3}$ satisfies the conditions in our lemma. \square

Both S_{test} and T_{test} are Lipchitz with constants $S_0\varepsilon_{8.1}^{-1}$ and $T_0\varepsilon_{8.1}^{-1}$. Let C_{lip} be the max of these two constants. At this point, we are ready to specify the size of our small box and have the lemma as follows.

LEMMA 8.4. For $\varepsilon = \varepsilon_{8.4}$, where

$$(57) \quad \varepsilon_{8.4} = \min \left\{ \frac{\delta_{8.3}\varepsilon_1}{16(\beta + \omega_0 + \mu)C_{\text{lip}}}, \frac{\varepsilon_{8.2}}{2} \right\},$$

$\bar{\xi}_t$ be the truncated process starting from $\bar{\xi}_0$. At time

$$t_{8.4} = \frac{\delta_{8.3}}{2(\beta + \omega_0 + \mu)}$$

there is some $C_{8.4} < \infty$ such that the probability that:

- $f_1(x, \bar{\xi}_{t_{8.4}}) \geq f_1(x, \bar{\xi}_0) + c_{8.4}$, when $2\ell x \in B(0, ML + 4\ell)$,
- $f_2(x, \bar{\xi}_{t_{8.4}}) \geq f_2(x, \bar{\xi}_0) + c_{8.4}$, when $2\ell x \in B(0, ML - 2\varepsilon_{8.1}L + 4\ell)$

is greater than $1 - C_{8.4}L^{-d}$ when L is large, where

$$c_{8.4} = \frac{\delta_{8.3}\varepsilon_1}{8(\beta + \omega_0 + \mu)}.$$

PROOF. Consider the stopping time

$$\bar{\tau} = \min \{ t : \exists x : 2\ell x \in B(0, 2ML), i = 1 \text{ or } 2, |f_i(x, \bar{\xi}_0) - f_i(x, \bar{\xi}_t)| > \delta_{8.3} \}.$$

Note that each site in our system flip at a rate no larger than $\beta + \omega_0 + \mu$. According to standard large deviations result as we used in Lemma 5.1, there is some $c_{6.1}, C_{6.1} \in (0, \infty)$ independent to L such that

$$P(\bar{\tau} \leq t_{8.4}) \leq C_{6.1} \exp(-c_{6.1}L^d) < C_{6.1}L^{-d}$$

when L is large. Now consider, $\sigma_i^2(x, \bar{\xi}_t)$, the infinitesimal variances of the local densities. According to exactly the same calculation as we did in Lemma 3.3, there is a $C_{3.3} < \infty$ such that

$$(58) \quad \sigma_i^2(x, \bar{\xi}) \leq C_{3.3}L^{-d}$$

for all $x \in \mathbb{Z}^d$, $i = 1, 2$ and all configurations $\bar{\xi}$. Thus, we can again define Dynkin’s martingale:

$$(59) \quad \bar{M}_i(x, t) = f_i(x, \bar{\xi}_t) - f_i(x, \bar{\xi}_0) - \int_0^t \mu_i(x, \bar{\xi}_t) dt$$

and Lemma 3.3 implies that there is a $C_{3.3}$ so that

$$P\left(\sup_{t \leq t_{8.4}} |\bar{M}_i(x, t)| > c_{8.4} \right) < C_{3.3}L^{-d}.$$

Consider the event

$$(60) \quad A_i(x) = \{\tau > t_{8.4}\} \cap \left\{ \sup_{t \leq t_{8.4}} |\bar{M}_i(x, t)| < c_{8.4} \right\}.$$

By definition, there is some $U_{8.4} < \infty$, independent to L , such that

$$P(A_i(x)) > 1 - U_{8.4}L^{-d} \quad \forall x \text{ s.t. } 2\ell x \in B(0, ML + 4\ell).$$

For any x such that $2\ell x \in B(0, ML + 4\ell)$, when $A_1(x)$ holds, Lemma 8.3 implies that for any x

$$(61) \quad f_1(x, \bar{\xi}_{t_{8.4}}) \geq \int_0^{t_{8.4}} \varepsilon_1 dt - c_{8.4} > c_{8.4}.$$

Similarly, for any x such that $2\ell x \in B(0, ML - 2\varepsilon_{8.1}L + 4\ell)$, when $A_2(x)$ holds

$$(62) \quad f_2(x, \bar{\xi}_{t_{8.4}}) \geq \int_0^{t_{8.4}} \varepsilon_1 dt - c_{8.4} > c_{8.4}.$$

So let

$$(63) \quad A = \left(\bigcap_{x: 2\ell x \in B(0, ML+4\ell)} A_1(x) \right) \cap \left(\bigcap_{x: 2\ell x \in B(0, ML-2\varepsilon_{8.1}L+4\ell)} A_2(x) \right).$$

The conditions in our lemma are satisfied on the event A . Noting that

$$P(A) \geq 1 - \sum_{x: 2\ell x \in B(0, ML+4\ell)} [P(A_1^c(x)) + P(A_2^c(x))] \geq 1 - \frac{4M^d}{\varepsilon_{8.4}^d} U_{8.4}L^{-d}$$

let $C_{8.4} = 4M^d U_{8.4} / \varepsilon_{8.4}^d$ and the proof is complete. \square

For any $x \in \mathbb{Z}^d$ and any $\xi \in \{0, 1, 2\}^{\mathbb{Z}^d}$, define $\text{shift}(\xi, x)$ to be the configuration that for any $y \in \mathbb{Z}^d$:

$$\text{shift}(\xi, x)(y) = \xi(y - x).$$

Recalling the definition of C_{lip} , on the event A , for any $i = 1, \dots, d$,

$$\bar{\xi}_{t_{8.4}} \geq \text{shift}(\bar{\xi}_0, \pm 2\ell e_i).$$

Monotonicity enables us to restart the construction above from anyone among the shifts. Note that the success probability of such a construction is of $1 - O(L^{-d})$. So when L is large, with high probability we can do it for $2d \log L$ times without a failure. This will give us a “copy” of $\bar{\xi}_0$ at $\pm 2\ell(\log L)e_i$ for each i and will take time $T = (\log L)t_{8.4}$.

Thus, we can have out block construction as follows: let

$$\Gamma_x = 2\ell(\log L)x + [-\ell \log L, \ell \log L]^d \quad \forall x \in \mathbb{Z}^d$$

and $T_n = nT, n \geq 0$. We say (x, n) is wet if

$$\bar{\xi}_{T_n} \geq \text{shift}(\bar{\xi}_0, 2\ell(\log L)x).$$

From the construction above, we immediately have that (x, n) is wet then with high probability $(x \pm e_i, n + 1)$ are all wet for $i = 1, \dots, d$.

To check that with high-probability, the block events are finite-dependent, note that $\bar{\xi}_t$ is dominated by a branching random walk with birth rate β and initial configuration $\bar{\xi}_0$. Lemma 5.1 shows that for any $m > 0$

$$P(M_k(T_1) \geq (2\beta + m)LT_1) \leq e^{-mT_1} |B(0, (M + 1)L)|$$

when L is large enough, where $M_k(t)$ is the largest k th coordinate among the occupied sites at time t . Noting that $T_1 = (\log L)t_{8.4}$ and that the choice of $t_{8.4}$ is independent to the choice of L , let

$$(64) \quad m = (d + 1)/t_{8.4}.$$

We can control the probability that $\bar{\xi}_t$ wanders too far as follows:

$$P_{\bar{\xi}_0} \left(\max_{t \in [0, T_0]} \{ \|x\| : \bar{\xi}_t(x) \neq 0 \} \geq (2\beta + m)LT_1 \right) \leq 2de^{-mT_1} |B(0, (M + 1)L)|.$$

Noting that

$$e^{-mT_1} = e^{-(\log L)t_{8.4}(d+1)/t_{8.4}} = e^{-(d+1)\log L} = L^{-d-1}$$

we have

$$(65) \quad P_{\bar{\xi}_0} \left(\max_{t \in [0, T_0]} \{ \|x\| : \bar{\xi}_t(x) \neq 0 \} \geq (2\beta + m)LT_1 \right) = L^{-d-1} O(L^d) \rightarrow 0$$

as $L \rightarrow \infty$. Thus, noting that $\ell = \varepsilon_{8.4}L$, let

$$R = \frac{(2\beta + m)LT_1}{2\ell \log L} = \frac{(2\beta + m)t_{8.4}}{2\varepsilon_{8.4}}$$

which is a finite constant independent to the choice of L . As L goes large, we have that with high probability $\bar{\xi}_t$ cannot exit the following finite union of blocks by time T_1 :

$$\bigcup_{x : \|x\| \leq R} \Gamma_x$$

which implies that the block events we constructed has finite range of dependence. Then again according to standard block argument in [3] and [11], we complete the proof of survival for $\bar{\xi}_t$ and this implies Theorem 3.

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