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An estimation of the stability and the localisability functions of multistable processes

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Abstract: Multistable processes are tangent at each point to a stable process, but where the index of stability and the index of localisability varies along the path. In this work, we give two estimators of the stability and the localisability functions, and we prove the consistency of those two estimators. We illustrate these convergences with two examples, the Lévy multistable process and the Linear Multifractional Multistable Motion.

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1. Introduction

Multifractional multistable processes have been recently introduced as models for phenomena where the regularity and the intensity of jumps are non con-

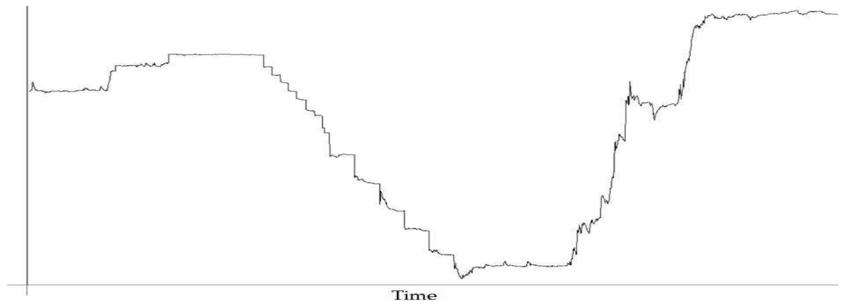


Fig 1. *Financial data where the increments do not appear to be stationary: the intensity of jumps is varying over time.*

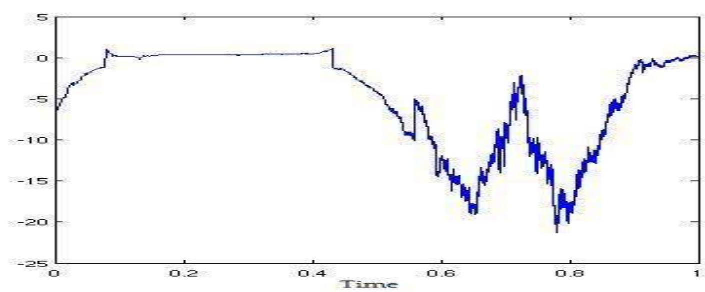


Fig 2. *Realization of a simulated multistable process. The sample size is* $n = 20000$.

stant, and particularly when the increments of the observed trajectories are not stationary. In Figure 1, we display a path of a financial data from federal funds, where the frequency of the jumps seems to vary with time. The multistable processes extend the stable models in order to take into account this additional variability (see Figure 2 for an example of a realization of such a process, computed with the simulation method explained in [4]). We describe then some events with a low intensity of jumps at some times, which may be very erratic at other times. We provide another example of application in Figure 12 of Section 6.3, where we consider a path coming from electrocardiogram.

Multistable processes are stochastic processes which are locally stable, but where the index of stability α varies with "time", and therefore is a function. They were constructed in [4, 5, 6, 8] using respectively moving averages, sums over Poisson processes, multistable measures, and the Ferguson-Klass-LePage series representation, this last definition being the representation used hereafter. These processes are, under general assumptions locally self-similar, with an index of self-similarity H which is also a function. In the remaining of this work, given one trajectory of a multistable process, we provide an estimator for each function.

The aim of this work is then to introduce, for a large class of multistable processes, an estimator of the local index of stability α. We prove in the sequel the consistency of this estimator with a convergence in all the L^r spaces. This class includes two examples considered in [5, 8], the Lévy multistable motion and linear multifractional multistable motion. We then estimate the local self-similarity function H. For the same class of multistable processes, we obtain a consistent estimator of H. In the case of the Lévy multistable motion, we are able to ascertain the asymptotic distribution of this estimator through a central limit theorem.

The remainder if this article is organized as follows: in the next section, we recall the definition of multistable processes and our two examples of interest. We present the two estimators in Section 3. Our main results on the convergence of the estimators are described in Section 4. Subsection 4.1 present the case of the index of stability α. In subsection 4.2, we state the result giving the convergence of the estimator of the local self-similarity function H, with a central limit theorem in the case of the Lévy multistable motion. In Section 5, we give intermediate results which are used in the proofs of the main theorems. Section 6 contains applications of our results to two examples and real electrocardiographic data. We give in Section 7 a list of technical conditions on the kernel of multistable processes that involve the consistency of the estimators. Finally we gather all the proofs of the statements of this article in Section 8 and Section 9.

2. Model

Let us recall the definition of a *localisable process* [2, 3]: $Y = \{Y(t) : t \in \mathbf{R}\}$ is said to be localisable at u if there exists an $H(u) \in \mathbf{R}$ and a non-trivial limiting process Y'_u such that

$$\lim_{r \to 0} \frac{Y(u+rt) - Y(u)}{r^{H(u)}} = Y'_u(t), \tag{2.1}$$

where the convergence is in finite dimensional distributions. When the limit exists, $Y'_u = \{Y'_u(t) : t \in \mathbf{R}\}$ is termed the *local form* or tangent process of Y at u. The local form Y'_u, when it exists, must be $H(u)$-self-similar, that is $Y'_u(rt) \stackrel{d}{=} r^{H(u)} Y'_u(t)$, for all $r > 0$. Under quite general conditions, Y'_u has also stationary increments, for almost all u at which (2.1) occurs in distribution. We refer to Proposition 3.7 of [2] or [3] to obtain the specific conditions. A process Y, if it is H-self similar with stationary increments, satisfy (2.1). The local form is then Y itself.

The examples of multistable processes considered in this article generalize stable processes that are self-similar with stationary increments.

Ferguson-Klass-LePage series representation

We define now the multistable processes using the Ferguson-Klass-LePage series representation, that are defined as "diagonals" of random fields that we described below. In the sequel, $(E, \mathcal{E}, m)$ will be a measure space, and U an open interval of the real line $\mathbf{R}$. We consider

$$\mathcal{F}_\alpha(E, \mathcal{E}, m) = \{f : f \text{ is measurable and } \|f\|_\alpha < \infty\},$$

where $\| \, \|_\alpha$ is the quasinorm (or norm if $1 < \alpha \leq 2$) given by

$$\|f\|_\alpha = \left(\int_E |f(x)|^\alpha m(dx) \right)^{1/\alpha}.$$

We will assume that m is either a finite or a σ-finite measure, depending on the circumstances.

Let α be a C^1 function defined on U and ranging in $[c, d] \subset (0, 2)$. Let $f(t, u, .)$ be a family of functions such that, for all $(t, u) \in U^2$, $f(t, u, .) \in \mathcal{F}_{\alpha(u)}(E, \mathcal{E}, m)$. We define also $r : E \to \mathbb{R}_+$ such that $\hat{m}(dx) = \frac{1}{r(x)} m(dx)$ is a probability measure. $(\Gamma_i)_{i \geq 1}$ will be a sequence of arrival times of a standard Poisson process and $(\gamma_i)_{i \geq 1}$ a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Let $(V_i)_{i \geq 1}$ a sequence of i.i.d. random variables with distribution $\hat{m}$ on E and we assume that the three sequences $(\Gamma_i)_{i \geq 1}$, $(V_i)_{i \geq 1}$, and $(\gamma_i)_{i \geq 1}$ are mutually independent. As in [8], we will consider the following random field:

$$X(t, u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} r(V_i)^{1/\alpha(u)} f(t, u, V_i), \tag{2.2}$$

where $C_\eta = \left(\int_0^\infty x^{-\eta} \sin(x) dx \right)^{-1}$.

Note that when the function α is constant, then (2.2) is just the Ferguson - Klass - LePage series representation of a stable random variable, and $X(., u)$ is an $\alpha(u)$-stable process. Taking m as the control measure, this define an $\alpha(u)$-stable random measure $M_{\alpha(u)}$ on E. For $f \in \mathcal{F}_{\alpha(u)}(E, \mathcal{E}, m)$, the stochastic integral of f with respect to $M_{\alpha(u)}$ exists ([16]): the scale parameter of $\int_E f(x) M_{\alpha(u)}(dx)$ is then $\|f\|_{\alpha(u)}$. Proposition 3.5.5 and Theorem 3.10.1 of [16] then imply that

$$\int_E f(t, u, x) M_{\alpha(u)}(dx) \stackrel{d}{=} C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} r(V_i)^{1/\alpha(u)} f(t, u, V_i),$$

(see [1, 7, 10, 11, 15] and [16, Theorem 3.10.1] for specific properties of this representation).

Multistable processes

Multistable processes are obtained by taking diagonals on X defined in (2.2), *i.e.*

$$Y(t) = X(t, t). \tag{2.3}$$

For a fixed t, $Y(t)$ is an $\alpha(t)$-stable random variable. It is well known that such a variable does not possess high-order moments, including a second-order moment. An explicit formula for $\mathsf{E}\left[|Y(t)|^p\right]$ is given when $0 < p < \alpha(t)$:

$$\mathsf{E}\left[|Y(t)|^p\right] = c(\alpha(t), p) \left(\int_E |f(t,t,x)|^{\alpha(t)} m(dx) \right)^{p/\alpha(t)}$$

where $c(\alpha(t), p) < \infty$, see [16], Property 1.2.17.

The process Y is not a stable process, but, as shown in Theorems 3.3 and 4.5 of [8], provided some conditions are satisfied both by X and by the function f, Y will be a localisable process whose local form is a stable process. These conditions are listed in Section 7. More precisely, it is the conditions **(R1)**, **(M1)**, **(M2)** and **(M3)**. We will always assume that $X(t,u)$ (as a process in t) is localisable at u with exponent $H(u) \in (H_-, H_+) \subset (0,1)$, with local form $X'_u(t,u)$, and $u \mapsto H(u)$ is a C^1 function. We also assume that $\alpha(u) \in [c,d] \subset (0,2)$ where $c = \min_{u \in U} \alpha(u)$ and $d = \max_{u \in U} \alpha(u)$. The definition of Y is based on the field X with equation (2.3). When the conditions are satisfied, the process Y is localisable with local form $X'_u(.,u)$. It is then necessary to assume that $X(.,u)$ is localisable. A simple way to construct Y is to consider a field X such that $X(.,u)$ is a $H(u)$-self-similar process with stationary increments, which is the case for our two examples described below.

We take as examples of multistable processes the “multistable versions” of some classical processes: the α-stable Lévy motion and the Linear Fractional Stable Motion. In the sequel, $\mathcal{L}$ will denote the Lebesgue measure and M a symmetric α-stable ($0 < \alpha < 2$) random measure on $\mathbf{R}$, with control measure $m = \mathcal{L}$. m is already a probability measure on $[0,1]$, so for the first example, we define $\hat{m} = m$ with $r(x) = 1$ for all $x \in [0,1]$. We will write

$$L_\alpha(t) := \int_0^t M(dz)$$

for α-stable Lévy motion, and we will use the Ferguson-Klass-LePage representation,

$$\forall t \in (0,1), \quad L_\alpha(t) = C_\alpha^{1/\alpha} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha} \mathbf{1}_{[0,t]}(V_i),$$

where $(V_i)_{i \geq 1}$ is a sequence of i.i.d. uniform random variables on $(0,1)$.

Let $\alpha : [0,1] \to (0,2)$ be continuously differentiable. Define

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \mathbf{1}_{[0,t]}(V_i)$$

and the symmetric multistable Lévy motion

$$Y(t) = X(t,t) = C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} \mathbf{1}_{[0,t]}(V_i).$$

We know from [8], Theorem 5.1, that the localisability function of Y is $H(t) = \frac{1}{\alpha(t)}$.

The second example is a multistable version of the *well-balanced linear fractional α-stable motion*:

$$L_{\alpha,H}(t) = \int_{-\infty}^{\infty} f_{\alpha,H}(t,x)M(dx)$$

where $t \in \mathbb{R}$, $H \in (0,1)$, and

$$f_{\alpha,H}(t,x) = |t-x|^{H-1/\alpha} - |x|^{H-1/\alpha}.$$

Let $\alpha : \mathbf{R} \to (0,2)$ and $H : \mathbf{R} \to (0,1)$ be continuously differentiable. Define $\kappa = H - \frac{1}{\alpha}$ and

$$X(t,u) = C_{\alpha(u)}^{\frac{1}{\alpha(u)}} \sum_{i,j=1}^{\infty} \gamma_i (|t-V_i|^{\kappa(u)} - |V_i|^{\kappa(u)}) (\frac{\pi^2 j^2}{3\Gamma_i})^{\frac{1}{\alpha(u)}} \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i). \tag{2.4}$$

We take for m the Lebesgue measure on $\mathbf{R}$, and we define $\hat{m}$, the distribution of each V_i, taking $r(x) = \frac{\pi^2}{3}\sum_{j=1}^{\infty} j^2 \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)$. For a fixed $u \in \mathbf{R}$, $t \mapsto X(t,u)$ is the Ferguson-Klass-LePage representation of $t \mapsto L_{\alpha(u),H(u)}(t)$. Define the linear multistable multifractional motion

$$Y(t) = X(t,t). \tag{2.5}$$

The localisability of Lévy motion and linear fractional α-stable motion simply stems from the fact that they are self-similar with stationary increments [3]. We will apply our results to these processes, that were defined in [4, 5], in Section 6.

3. Construction of the estimators

Let Y be a multistable process defined in (2.3). The estimation of the localisability function H and the stability function α is based on the increments $(Y_{k,N})$ of Y. Define the sequence $(Y_{k,N})_{k\in\mathbb{Z},N\in\mathbb{N}}$ by

$$Y_{k,N} = Y(\frac{k+1}{N}) - Y(\frac{k}{N}).$$

Let $t_0 \in \mathbb{R}$ be fixed. We introduce an estimator of $H(t_0)$ with

$$\hat{H}_N(t_0) = -\frac{1}{n(N)\log N} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \log|Y_{k,N}|$$

where $(n(N))_{N\in\mathbb{N}}$ is a sequence taking even integer values. We expect the sequence $(\hat{H}_N(t_0))_N$ to converge to $H(t_0)$ thanks to the localisability of the process Y. For the integers k and N such that $\frac{k}{N}$ is close to t_0, $\frac{Y_{k,N}}{(\frac{1}{N})^{H(t_0)}}$ is asymptotically distributed as $Y'_{t_0}(1)$. More precisely $-\frac{\log|Y_{k,N}|}{\log N} = H(t_0) + \frac{Z_{k,N}}{\log N}$ where $(Z_{k,N})_{k,N}$ converge weakly to $-\log|Y'_{t_0}(1)|$ when N tends to infinity and $\frac{k}{N}$ tends to t_0. We regulate the sequence $(Z_{k,N})$ near t_0 using the mean $\frac{1}{n(N)}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} Z_{k,N}$ and we can expect this sum will be bounded in the L^r spaces to obtain the convergence with a rate $\frac{1}{\log N}$. The convergence is proved in Theorem 4.2.

Let $p_0 \in (0, c)$ and $\gamma \in (0, 1)$, where $c = \min_{u\in U} \alpha(u)$. With the increments of the process, we define the sample moments $S_N(p)$ by

$$S_N(p) = \left(\frac{1}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} |Y_{k,N}|^p\right)^{\frac{1}{p}}.$$

Let

$$R_{\exp}^{(N)}(p) = \frac{S_N(p_0)}{S_N(p)} \text{ and } R_\alpha(p) = \frac{(\mathsf{E}|Z|^{p_0})^{1/p_0}}{(\mathsf{E}|Z|^p)^{1/p}}\mathbf{1}_{p<\alpha}$$

where Z is a standard symmetric α-stable random variable (written $Z \sim S_\alpha(1, 0, 0)$ as in [16]), i.e $\mathsf{E}|Z|^p = \frac{2^{p-1}\Gamma(1-\frac{p}{\alpha})}{p\int_0^{+\infty} u^{-p-1}\sin^2(u)du}$.

Consider the set $A_N =: \arg\min_{\alpha\in[0,2]}\left(\int_{p_0}^2 |R_{\exp}^{(N)}(p) - R_\alpha(p)|^\gamma dp\right)^{1/\gamma}$. Since the function $\alpha \to \left(\int_{p_0}^2 |R_{\exp}^{(N)}(p) - R_\alpha(p)|^\gamma dp\right)^{1/\gamma}$ is a continuous function, A_N is a non empty closed set. We define then an estimator of $\alpha(t_0)$ by

$$\hat{\alpha}_N(t_0) = \min\left(\arg\min_{\alpha\in[0,2]}\left(\int_{p_0}^2 |R_{\exp}^{(N)}(p) - R_\alpha(p)|^\gamma dp\right)^{1/\gamma}\right).$$

Under the conditions of Theorem 5.4, Y is $H(t_0)$-localisable and $Y'_{t_0}(1) \sim S_{\alpha(t_0)}(1, 0, 0)$ so $\frac{|Y_{k,N}|^p}{(\frac{1}{N})^{pH(t_0)}}$ converge weakly to $|Y'_{t_0}(1)|^p$ and taking the sample mean, $N^{H(t_0)}S_N(p)$ tends to $(\mathsf{E}|Y'_{t_0}(1)|^p)^{1/p}$ in probability, which is the result of Theorem 5.4. Following this, $\int_{p_0}^2 |R_{\exp}^{(N)}(p) - R_\alpha(p)|^\gamma dp$ tends to $\int_{p_0}^2 |R_{\alpha(t_0)}(p) - R_\alpha(p)|^\gamma dp$. Naturally, $\alpha(t_0)$ is the only solution of $\arg\min_{\alpha\in[0,2]}\int_{p_0}^2 |R_{\alpha(t_0)}(p) - R_\alpha(p)|^\gamma dp$ and this leads to the definition of $\hat{\alpha}_N(t_0)$. The consistency of $\hat{\alpha}_N(t_0)$ is proved in Theorem 4.1.

4. Main results

The following theorems apply to a diagonal process Y defined from the field X given by (2.2). For convenience, the conditions required on X and the function f that appears in (2.2) are gathered in Section 7. The two parameters of interest in the model are the stability function α and the localisability function H. Theorem 4.1 leads to the convergence in the L^r spaces of the estimator of the stability function α, while Theorem 4.2 yield the convergence of the estimator of the localisability function H. We obtain also the convergence speed in the specific case of the symmetric multistable Lévy motion.

Almost all hypotheses listed in Section 7 required for the main theorems are technical conditions. They ensure that the considered multistable processes are localisable. With previous results, we may control the various marginal distributions of the processes.

For the estimation of α, we also assume that the correlations of the increments of the underlying stable process $X(., t_0)$ decrease with time. The fact that $X(., t_0)$ is $H(t_0)$-self-similar with stationary increments is the most restrictive hypothesis. This is not a restriction for the function r because r plays no role in the distribution of the process. Indeed, the characteristic function Y does not depend on r (see [8]). A criterion of self-similarity with stationary increments is that $\int_E \frac{|f(t+h,t_0,x)-f(t,t_0,x)|^{\alpha(t_0)}}{h^{H(t_0)\alpha(t_0)}} m(dx)$ does not depend on t and h. The two main examples of such a kernel f are our two examples.

However, to obtain the same conclusions of the theorem, we think that it is enough to assume that $X(., t_0)$ is $H(t_0)$-localisable. We could then apply our results to Ornstein-Uhlenbeck processes.

4.1. Approximation of the stability function

Theorem 4.1. *Let Y be a multistable process and $t_0 \in U$. Assume the conditions (R1), (M1), (M2) and (M3). Assume in addition that:*

- $\lim_{N\to+\infty} n(N) = +\infty$ *and* $\lim_{N\to+\infty} \frac{N}{n(N)} = +\infty$.
- *The process $X(., t_0)$ is $H(t_0)$-self-similar with stationary increments and $H(t_0) < 1$.*
- $\lim_{j\to+\infty} \int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) = 0$, *where* $h_{j,t_0}(x) = f(j+1, t_0, x) - f(j, t_0, x)$.

Then for all $r > 0$,

$$\lim_{N\to+\infty} \mathsf{E} \left|\hat{\alpha}_N(t_0) - \alpha(t_0)\right|^r = 0.$$

If, in addition, the conditions hold for all $t_0 \in U$, then for all $p > 0$,

$$\lim_{N\to+\infty} \mathsf{E}\left[\int_U |\hat{\alpha}_N(t) - \alpha(t)|^p dt\right] = 0.$$

Proof. See Section 9. □

4.2. Approximation of the localisability function

Theorem 4.2. *Let Y be a multistable process. Assume that the localisability function H and the function α satisfy all the conditions (R1), (M1)-(M7) and (H1)-(H5) for an open interval U, and that* $\lim_{N\to+\infty} \frac{n(N)}{N} = 0$.

Then, for all $t_0 \in U$ and all $r > 0$,

$$\lim_{N\to+\infty} \mathsf{E}\left|\hat{H}_N(t_0) - H(t_0)\right|^r = 0.$$

Moreover, for all $[a,b] \subset U$ and all $p > 0$,

$$\lim_{N\to+\infty} \mathsf{E}\left[\int_a^b |\hat{H}_N(t) - H(t)|^p dt\right] = 0.$$

Proof. See Section 9. □

Remark. Under the conditions (R1), (M1), (M2) and (M3) listed in the theorem, Theorems 3.3 and 4.5 of [8] imply that Y is $H(t_0)-$localisable at t_0.

We obtain for the symmetric multistable Lévy motion the convergence in distribution of the estimator $\hat{H}_N(t_0)$ in the following theorem. We expect the same result holds for a more general class of processes, in particular when the conditions of Theorem 5.6 are satisfied. For Z a standard $\alpha(t_0)$-stable random variable, we define $\mu_{t_0} = \mathsf{E}[\log|Z|]$ and $\sigma^2_{t_0} = Var(\log|Z|)$. Since Z has bounded density and $\lim_{\lambda\to+\infty} \lambda^\alpha \mathsf{P}(Z > \lambda) = C_\alpha$ (see [16], Property 1.2.15), for all $p > 0$, $\mathsf{E}[|\log|Z||^p] < +\infty$. Thus, μ_{t_0} and $\sigma^2_{t_0}$ are both finite.

Theorem 4.3. *Let Y be a symmetric multistable Lévy motion with $\alpha : [0,1] \to (1,2)$ continuously differentiable, and $t_0 \in (0,1)$. Assume that $n(N) = O(N^\delta)$ with $\delta \in \left(0, \frac{2\alpha(t_0)-2}{3\alpha(t_0)+2}\right)$. Then*

$$\sqrt{n(N)}\left(\log N\left(\hat{H}_N(t_0) - H(t_0)\right) + \mu_{t_0}\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2_{t_0})$$

as $N \to +\infty$.

A simple way to estimate μ_{t_0} and $\sigma^2_{t_0}$ is to use Theorem 4.1. For Z a standard α-stable random variable ($Z \sim S_\alpha(1,0,0)$), $\mu : \alpha \mapsto \mathsf{E}[\log|Z|]$ and $\sigma^2 : \alpha \mapsto Var(\log|Z|)$ are two continuous functions, so $\mu(\hat{\alpha}_N(t_0))$ and $\sigma^2(\hat{\alpha}_N(t_0))$ are two estimators of μ_{t_0} and $\sigma^2_{t_0}$ that converge in probability. However, we can not use this to obtain confidence intervals for H because we don't have a central theorem for $\hat{\alpha}_N(t_0)$.

Nevertheless, we obtain confidence intervals for $H(t_0)$ using the relationship $H = \frac{1}{\alpha}$, available for a multistable Lévy motion (see [8]). Indeed, for $Z \sim S_\alpha(1,0,0)$, since $\mathsf{E}[|Z|^t] < +\infty$ for $0 < t < \alpha$, $\mathsf{E}[\log|Z|] = \lim_{t\to0}\left(\frac{E[|Z|^t]-1}{t}\right)$. We use the formula of Property 1.2.17 and 1.2.15 of [16] to compute:

$$\mathsf{E}[|Z|^t] = \frac{2^{t-1}\Gamma(1-t/\alpha)}{t\int_0^{+\infty} \frac{\sin^2(u)}{u^{t+1}} du} = \frac{2^{t-1}\Gamma(1-t/\alpha)}{\int_0^{+\infty} \frac{\sin(2u)}{u^t} du} = \Gamma(1-t/\alpha)C_t = \frac{\Gamma(1-t/\alpha)}{\Gamma(1-t)\cos(\frac{\pi t}{2})}.$$

Then $\mathsf{E}[\log|Z|] = \Gamma'(1)\left[1 - \frac{1}{\alpha}\right]$. Finally,

$$\mu_{t_0} = \Gamma'(1)\left[1 - \frac{1}{\alpha(t_0)}\right] = \Gamma'(1)\left[1 - H(t_0)\right].$$

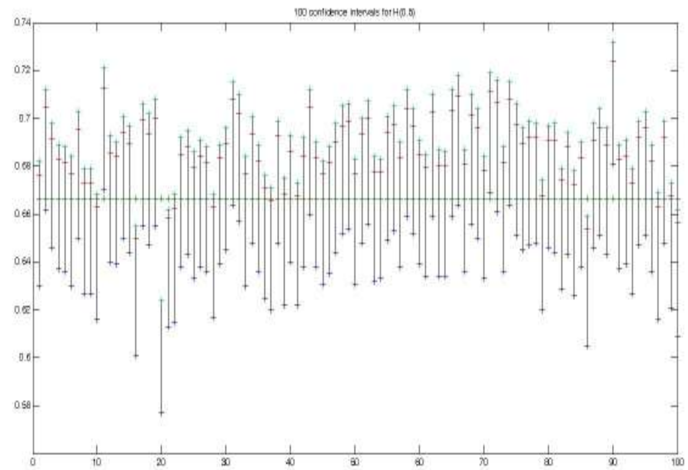


FIG 3. *100 confidence intervals for H.*

Similarly, $\mathsf{E}[(\log|Z|)^2] = \frac{d^2}{dt^2}\left(\Gamma(1-t/\alpha)C_t\right)(t=0)$ so one can compute that

$$\sigma_{t_0}^2 = \left(\Gamma''(1) - \Gamma'(1)^2\right) H(t_0)^2 + 4\Gamma'(1)^2 H(t_0) + \frac{\pi^2}{4} - \Gamma''(1) - 3\Gamma'(1)^2.$$

If a is the real number such that $\mathsf{P}(|X| \le a) = 0,95$ with $X \sim \mathcal{N}(0,1)$, then when N tends to $+\infty$,

$$\mathsf{P}\left(\frac{\sqrt{n(N)}}{\sqrt{\sigma^2(\hat{\alpha}_N(t_0))}}\left|(\log N)\hat{H}_N(t_0) + \Gamma'(1) - H(t_0)(\Gamma'(1) + \log N)\right| \le a\right) \to 0,95$$

i.e.

$$\lim_{N\to+\infty} \mathsf{P}\left(H(t_0) \in \left[\frac{(\log N)\hat{H}_N(t_0) + \Gamma'(1)}{\Gamma'(1) + \log N} \pm \frac{a\sqrt{\sigma^2(\hat{\alpha}_N(t_0))}}{(\Gamma'(1) + \log N)\sqrt{n(N)}}\right]\right) = 0,95.$$

An asymptotic 95% confidence interval is given by

$$\left[\frac{\hat{H}_N(t_0)\log N + \Gamma'(1)}{\Gamma'(1) + \log N} - \frac{a\sqrt{\sigma^2(\hat{\alpha}_N(t_0))}}{(\Gamma'(1) + \log N)\sqrt{n(N)}}; \frac{\hat{H}_N(t_0)\log N + \Gamma'(1)}{\Gamma'(1) + \log N} + \frac{a\sqrt{\sigma^2(\hat{\alpha}_N(t_0))}}{(\Gamma'(1) + \log N)\sqrt{n(N)}}\right].$$

We also notice that in order to estimate $H(t_0)$ with a weaker bias, one can use, instead of $\hat{H}_N(t_0)$, $\hat{H}_{2,N}(t_0) = \frac{(\log N)\hat{H}_N(t_0) + \Gamma'(1)}{\Gamma'(1) + \log N}$.

We present now the result of 3000 independent estimations of these confidence intervals, with $N = 20000$, $n(N) = 102$, $\alpha(t) = 1.5 + 0.48\sin(2\pi t)$, and $t_0 = \frac{1}{2}$ $(H(t_0) = \frac{2}{3})$. For the 3000 simulations, 2792 intervals contain the value $\frac{2}{3}$, that is $93,07\%$.

Another interest of Theorem 4.3 is to obtain a way to construct a statistical test in order to determine if α is actually a function, instead of a constant. Results in this direction will be presented in a forthcoming work.

5. Intermediate results

All the proofs of the intermediate results are stated in Section 8. We first give conditions for the convergence in probability of $S_N(p)$ in Theorem 5.4, which is useful to establish the consistency of the estimator $\hat{\alpha}_N(t_0)$.

Theorem 5.4. *Let Y be a multistable process. Assume the conditions (R1), (M1), (M2) and (M3). Assume in addition that:*

- $\lim_{N\to+\infty} n(N) = +\infty$ *and* $\lim_{N\to+\infty} \frac{N}{n(N)} = +\infty$.
- *The process $X(.,t_0)$ is $H(t_0)$-self-similar with stationary increments and $H(t_0) < 1$.*
- $\lim_{j\to+\infty} \int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) = 0$, *where* $h_{j,t_0}(x) = f(j+1, t_0, x) - f(j, t_0, x)$.

Then, for all $p \in [p_0, \alpha(t_0))$,

$$N^{H(t_0)} S_N(p) \underset{N\to+\infty}{\longrightarrow} (\mathsf{E}|X(1,t_0)|^p)^{1/p}$$

where the convergence is in probability.

We establish under several assumptions that the sequence $(\hat{H}_N(t))_N$ is almost surely uniformly bounded on every compact $[a,b] \subset U$.

Lemma 5.5. *Assume that the localisability function H and the function α satisfy all the conditions (R1), (M1)-(M7) and (H1)-(H5) for an open interval U, and that $\lim_{N\to+\infty} \frac{n(N)}{N} = 0$. Then there exists $B \in \mathbf{R}$ such that for all $[a,b] \subset U$,*

$$\mathsf{P}\left(\liminf_{N\to+\infty} \{ \sup_{t\in[a,b]} |\hat{H}_N(t)| \le B\}\right) = 1.$$

We state then a theorem implying the central Theorem 4.3.

Theorem 5.6. *Let Y be a multistable process and $t_0 \in U$. Assume the conditions (R1), (M1), (M2), (M3). Assume in addition that:*

- $n(N) = O(N^\delta)$ *with* $\delta \in \left(0, \frac{2\alpha(t_0)(1-H(t_0))}{2+3\alpha(t_0)}\right)$,
- *The process $X(.,u)$ is $H(u)$-self-similar with stationary increments and $H(u) < 1$, for all $u \in U$.*

Then

$$\lim_{N\to+\infty} \frac{1}{\sqrt{n(N)}} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \log \left| \frac{Y(\frac{k+1}{N}) - Y(\frac{k}{N})}{X(\frac{k+1}{N}, \frac{k}{N}) - X(\frac{k}{N}, \frac{k}{N})} \right| = 0$$

where the convergence is in probability.

Finally, we set up a technical lemma, which is useful for Theorem 5.6.

Lemma 5.7. *Assume the conditions (R1), (M1), (M2) and (M3). Let* $t_0 \in U$*. If* $X(.,u)$ *is* $H(u)$*-self-similar with stationary increments and* $H(u) < 1$*, for all* $u \in U$*, then there exists* $K_U > 0$ *such that for all* $\lambda \in (0, 1/e)$*, for all* $(k, N) \in \mathbf{Z} \times \mathbf{N}$ *with* $k \in \left[[Nt_0] - \frac{n(N)}{2}, [Nt_0] + \frac{n(N)}{2} - 1\right]$*,*

$$\mathsf{P}\left(\frac{|X(\frac{k+1}{N}, \frac{k+1}{N}) - X(\frac{k+1}{N}, \frac{k}{N})|}{|X(\frac{k+1}{N}, \frac{k}{N}) - X(\frac{k}{N}, \frac{k}{N})|} > \lambda\right) \leq K_U \frac{|\log N|^d |\log \lambda|^d}{N^{\frac{d(1-H_-)}{1+c}} \lambda^{\frac{d}{1+c}}}.$$

6. Examples and simulations

In this section, we apply the results to our two examples: the Linear multifractional multistable motion and the multistable Lévy motion. We provide then an example of application with ECG data.

6.1. Linear multistable multifractional motion

We consider first the Linear multistable multifractional motion (Lmmm) defined by (2.5).

Proposition 6.8. *Assume that* $H - \frac{1}{\alpha}$ *is a non-negative function,* $\lim_{N\to+\infty} n(N) = +\infty$ *and* $\lim_{N\to+\infty} \frac{N}{n(N)} = +\infty$*. Then for all* $r > 0$ *and all* $[a, b] \subset \mathbf{R}$*,*

$$\lim_{N\to+\infty} \mathsf{E}\left[\int_a^b |\hat{\alpha}_N(t) - \alpha(t)|^r dt\right] = 0,$$

and for all $t_0 \in \mathbf{R}$*,*

$$\lim_{N\to+\infty} \mathsf{E}\left|\hat{H}_N(t_0) - H(t_0)\right|^r = 0.$$

Proof. Let $t_0 \in [a, b] \subset \mathbf{R}$ and $r > 0$.

We know from [9] that the conditions (R1), (M1), (M2) and (M3) are satisfied. Since the process $X(., t_0)$ is a $(H(t_0), \alpha(t_0))$ linear fractional stable motion, $X(., t_0)$ is $H(t_0)$-self-similar with stationary increments [16]. Let us show that $\lim_{j\to+\infty} \int_{\mathbf{R}} |h_{0,t_0}(x) h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} dx = 0$.

Let $\varepsilon > 0$. Let $c_0 > 0$ such that $\int_{|x|>c_0} |h_{0,t_0}(x)|^{\alpha(t_0)} dx \leq \frac{\varepsilon}{2}$. By the Cauchy-Schwartz inequality, we have that

$$\int_{|x|>c_0} |h_{0,t_0}(x) h_{j,t_0}(x)|^{\alpha(t_0)/2} dx \leq (\frac{\varepsilon}{2})^{1/2} \|h_{j,t_0}\|_{\alpha(t_0)}^{\alpha(t_0)/2} = (\frac{\varepsilon}{2})^{1/2} \|h_{0,t_0}\|_{\alpha(t_0)}^{\alpha(t_0)/2}.$$

This implies the desired convergence since $\forall x \in [-c_0, c_0]$, we get the pointwise convergence $\lim_{j\to+\infty} |h_{0,t_0}(x) h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} = 0$ and $(h_{j,t_0}(x))_j$ is uniformly bounded on $[-c_0, c_0]$, and therefore

$$\lim_{j\to+\infty} \int_{|x|\leq c_0} |h_{0,t_0}(x) h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} dx = 0.$$

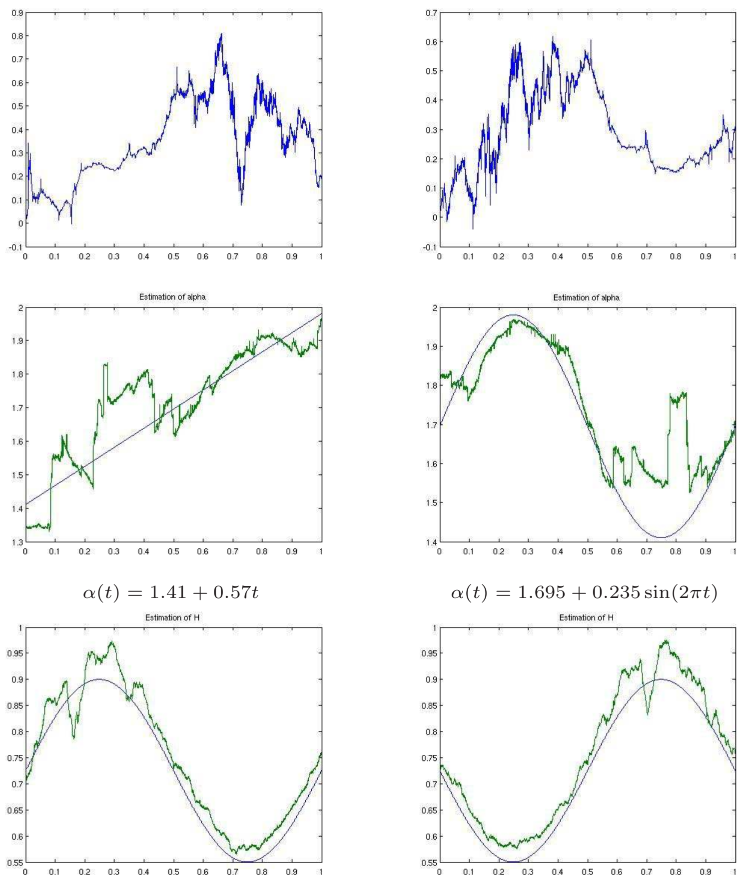
We deduce from Theorem 4.1 that for all $t_0 \in [a, b]$, $\lim_{N\to+\infty} \mathsf{E}\,|\hat{\alpha}_N(t_0) - \alpha(t_0)|^r = 0$. Since $\hat{\alpha}$ and α are bounded by 2, $\lim_{N\to+\infty} \mathsf{E}\big[\int_a^b |\hat{\alpha}_N(t) - \alpha(t)|^r dt\big] = 0$.

Let $t_0 \in \mathbf{R}$. We know from [9] that there exists U an open interval such that $t_0 \in U$ and (M3), (M4), (M5), (M6), (M7), (H1)-(H5) hold. We deduce from Theorem 4.2 that

$$\lim_{N\to+\infty} \mathsf{E}\left|\hat{H}_N(t_0) - H(t_0)\right|^r = 0.$$

□

We show on Figure 4, Figure 5 and Figure 6 some paths of Lmmm, with the two corresponding estimations of α and H. To simulate the trajectories, we have used the field (2.4). All the increments of $X(., u)$ are $(H(u), \alpha(u))$-linear



$\alpha(t) = 1.41 + 0.57t$ | $\alpha(t) = 1.695 + 0.235 \sin(2\pi t)$

$H(t) = 0.725 + 0.175 \sin(2\pi t)$ | $H(t) = 0.725 - 0.175 \sin(2\pi t)$

FIG 4. *Trajectories with $N = 20000$ in the first line, the estimations of α with $n(N) = 3000$ points in the second line, and in the last one, the estimations of H with $n(N) = 500$ points.*

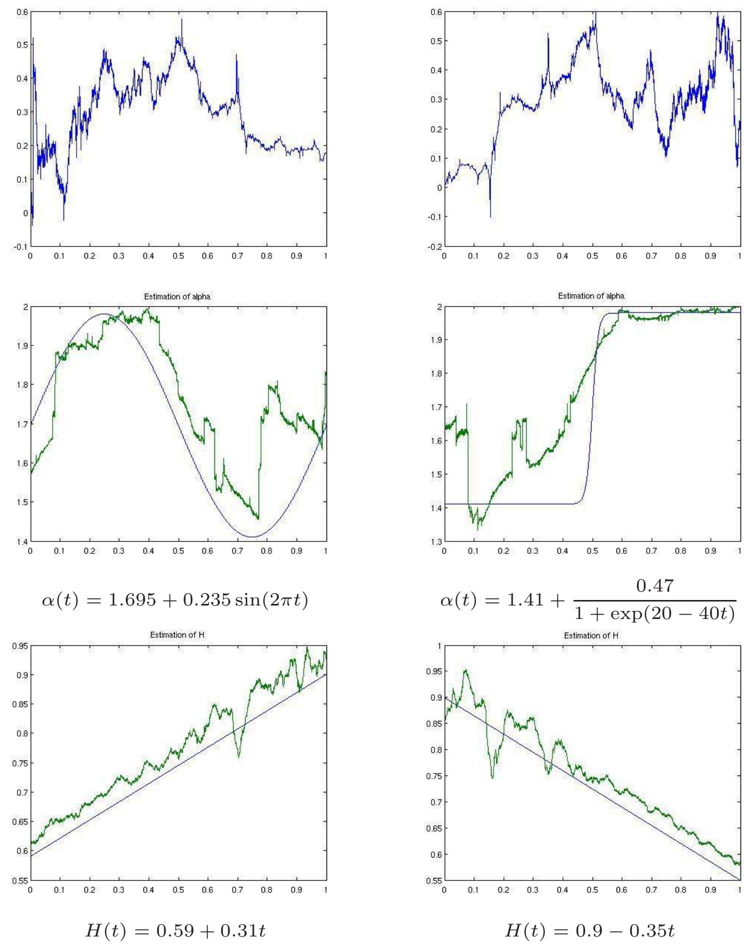


Fig 5. *Trajectories with* $N = 20000$ *in the first line, the estimations of* α *with* $n(N) = 3000$ *points in the second line, and in the last one, the estimations of* H *with* $n(N) = 500$ *points.*

fractional stable motions, generated using the LFSN program of [17]. After we have taken the diagonal process $X(t,t)$.

These estimates are overall further than the estimates in the case of the Levy process, because of greater correlations between the increments of the process. However, the estimation of H does not seem to be disturbed by this dependance. The shape of the function H is kept. For α, we notice some disruptions when the function is close to 1. We finally show an example where the estimation of α is not good enough in Figure 6. The trajectory, Figure 6.a), seems to have a big jump, which leads to decrease the estimator $\hat{\alpha}$, represented on Figure 6.b), while the jump is taken account in the $n(N)$ points. The estimation of H, represented on Figure 6.c), does not seem to be affected by this phenomenon.

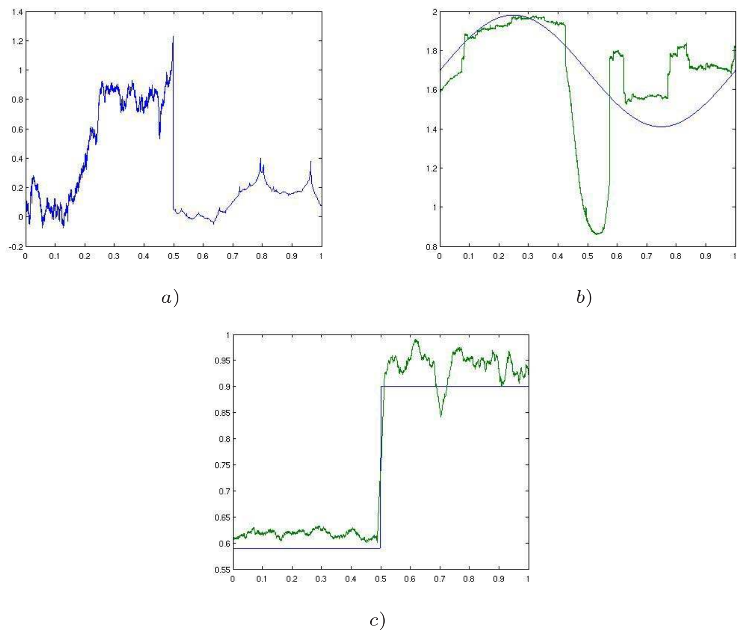


FIG 6. *Trajectories with $N = 20000$ in a), the estimations of α with $n(N) = 3000$ points in b), and in c), the estimations of H with $n(N) = 500$ points.*

6.2. Symmetric multistable Lévy motion

Let $\alpha : [0,1] \to (1,2)$ be continuously differentiable. Define

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \mathbf{1}_{[0,t]}(V_i) \tag{6.1}$$

and the symmetric multistable Lévy motion

$$Y(t) = X(t,t) = C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} \mathbf{1}_{[0,t]}(V_i).$$

Proposition 6.9. *If $\lim_{N\to+\infty} n(N) = +\infty$ and $\lim_{N\to+\infty} \frac{N}{n(N)} = +\infty$, then for all $r > 0$,*

$$\lim_{N\to+\infty} \mathsf{E}\left[\int_0^1 |\hat{\alpha}_N(t) - \alpha(t)|^r dt\right] = 0.$$

For all $[a,b] \subset (0,1)$,

$$\lim_{N\to+\infty} \mathsf{E}\left[\int_a^b |\hat{H}_N(t) - \frac{1}{\alpha(t)}|^r dt\right] = 0.$$

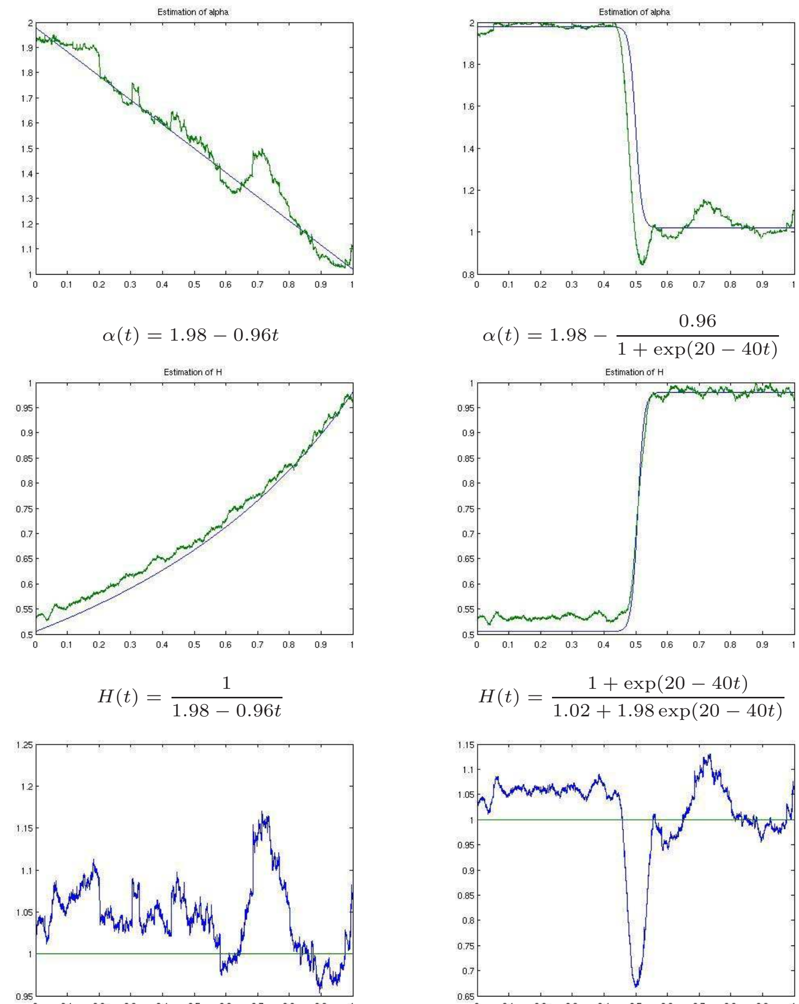


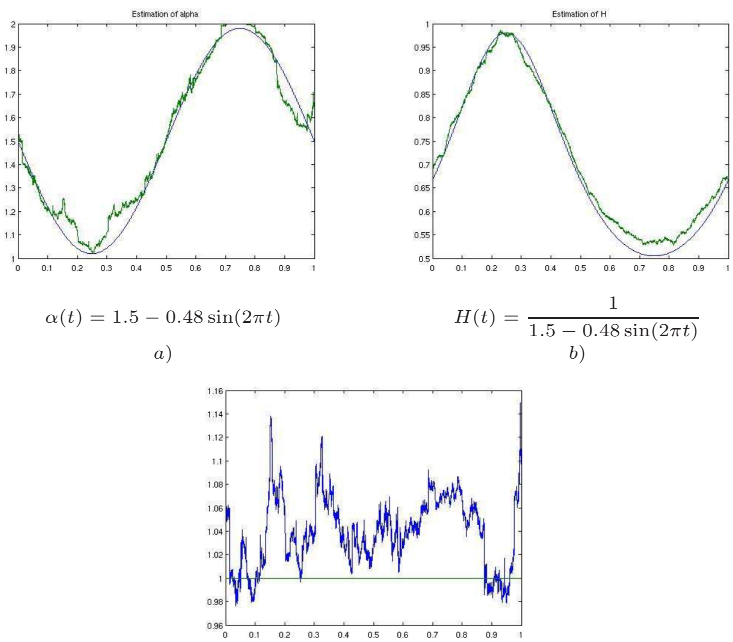
Fig 7. *Trajectories on* $(0,1)$ *with* $N = 20000$ *points,* $n(N) = 2042$ *points for the estimator* $\hat{\alpha}$*, and* $n(N) = 500$ *for* $\hat{H}$*.* α *and* $\hat{\alpha}$ *are represented in the first line,* H *and* $\hat{H}$ *in the second line, and in the last line, we have drawn the product* $\hat{\alpha}\hat{H}$*.*

Let $t_0 \in (0,1)$*. If we assume in addition that* $n(N) = O(N^{\delta})$ *with a parameter* $\delta \in \left(0, \frac{2\alpha(t_0)-2}{3\alpha(t_0)+2}\right)$*, then*

$$\sqrt{n(N)}\left(\log N\left(\hat{H}_N(t_0) - H(t_0)\right) + \mu_{t_0}\right) \xrightarrow{d} \mathcal{N}(0, \sigma_{t_0}^2)$$

as $N \to +\infty$*.*

Proof. We know from [9] that the conditions (R1), (M1), (M2) and (M3) are satisfied with $U = (0,1)$. Since the process $X(.,t_0)$ is a Lévy motion $\alpha(t_0)$-stable,



$\alpha(t) = 1.5 - 0.48\sin(2\pi t)$

a)

$H(t) = \dfrac{1}{1.5 - 0.48\sin(2\pi t)}$

b)

c)

FIG 8. *Trajectories on* $(0,1)$ *with* $N = 20000$ *points,* $n(N) = 2042$ *points for the estimator* $\hat{\alpha}$*, and* $n(N) = 500$ *for* $\hat{H}$*.* α *and* $\hat{\alpha}$ *are represented in a),* H *and* $\hat{H}$ *in b), and in c), we have drawn the product* $\hat{\alpha}\hat{H}$*.*

$X(.,t_0)$ is $\frac{1}{\alpha(t_0)}$-self-similar with stationary increments [16]. $h_{j,t_0}(x) = \mathbf{1}_{[j,j+1[}(x)$ so for $j \geq 1$,

$$\int_{\mathbf{R}} |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} dx = 0.$$

We conclude with Theorem 4.1 that $\lim_{N\to+\infty} \mathsf{E}\big[\int_0^1 |\hat{\alpha}_N(t) - \alpha(t)|^r dt\big] = 0$.

Let $[a,b] \subset (0,1)$. We easily check that the nine conditions (M4)-(M7) and (H1)-(H5) are satisfied with $U = (a,b)$ and $H(t) = \frac{1}{\alpha(t)}$. We conclude with Theorem 4.2 that $\lim_{N\to+\infty} \mathsf{E}\big[\int_a^b |\hat{H}_N(t) - \frac{1}{\alpha(t)}|^r dt\big] = 0$. The end of Proposition 6.9 is a reminder of Theorem 4.3. □

We display on Figure 7 and Figure 8 some examples of estimations for various functions α, the function H satisfying the relation $H(t) = \frac{1}{\alpha(t)}$. The trajectories have been simulated using the field (6.1). For each $u \in (0,1)$, $X(.,u)$ is a $\alpha(u)$-stable Lévy Motion. It is then an $\alpha(u)$-stable process with independent increments. We have generated these increments using the RSTAB program available in [17] or in [16], and then taken the diagonal $X(t,t)$.

Each function is pretty well-evaluated. We are able to recreate with the estimators the shape of the functions. However, we notice a significant bias on

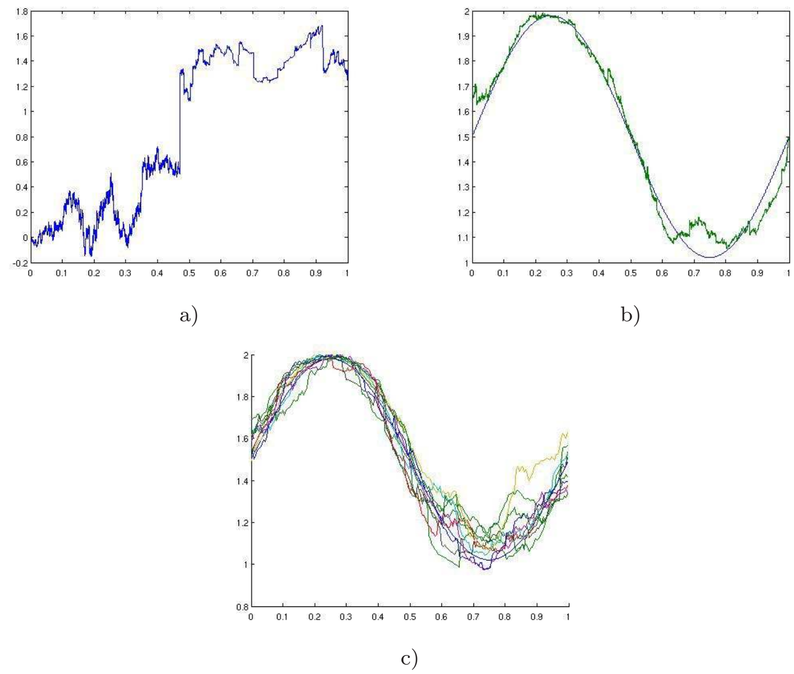


Fig 9. *Trajectory of a Levy process with $\alpha(t) = 1.5 + 0.48\sin(2\pi t)$ in figure a), and the corresponding estimation of α in figure b) with $n(N) = 2042$. The figure c) represents various estimations of α for the same function $\alpha(t) = 1.5 + 0.48\sin(2\pi t)$, with different trajectories.*

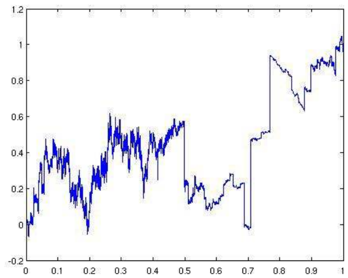
Figure 7 in the estimation of H. It seems to decrease when H is getting values close to 1. We observe this phenomenon with most trajectories, while the estimator $\hat{\alpha}$ seems to be unbiased. We have displayed the product $\hat{\alpha}\hat{H}$ in order to show the link between the estimators. We actually find again the asymptotic relationship $H(t) = \frac{1}{\alpha(t)}$.

We observe on Figure 9 c) an evolution of the variance in the estimation of α. It seems to increase when the function α is decreasing, and we conjecture that the variance at the point t_0 depends on the value $\alpha(t_0)$ in this way. In fact, the increments $Y_{k,N}$ are asymptotically distributed as an $\alpha(t_0)$-stable variable, so we expect that S_N and $R_{\exp}^{(N)}$ have a variance increasing when α is decreasing.

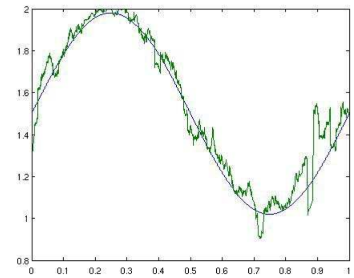
We have increased the resolution on Figure 10, taking more points for the discretization. The distance observed on Figure 9.b) for α near 1 is then corrected.

In order to represent the sampling variability of $\hat{H}$, and illustrate the central theorem of Theorem 4.3, we represent on Figure 11 the distribution of $\sqrt{n(N)}\big(\log N\big(\hat{H}_N(t_0) - H(t_0)\big) + \mu_{t_0}\big)$ with three different values of t_0, compared with the normal distribution.

We have used 3000 independent realizations of Y with $N = 20000$ and $\alpha(t) = 1.5 + 4.8\sin(2\pi t)$. The parameters are $t_0 = 0.25$, $H(t_0) = 0.5051$, $\mu_{t_0} = -0.2857$, $\sigma_{t_0}^2 = 1.2607$ and $n(N) = 102$ in Figure 11.a), $t_0 = 0.5$, $H(t_0) = 2/3$, $\mu_{t_0} =$

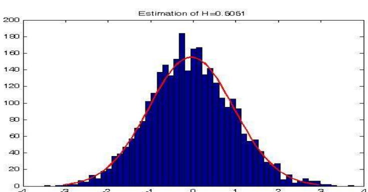


d)

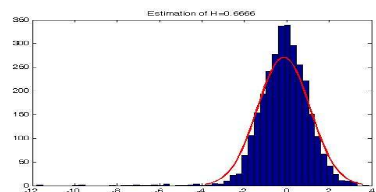


e)

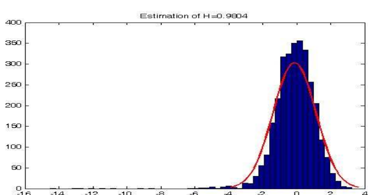
FIG 10. *Trajectory with* $N = 200000$ *in figure d), and the estimation with* $n(N) = 3546$ *in figure e).*



a)



b)



c)

FIG 11. *Simulations for Theorem 4.3.*

-0.1924, $\sigma_{t_0}^2 = 1.6149$ and $n(N) = 32$ in Figure 11.b), and $t_0 = 0.75$, $H(t_0) = 0.9804$, $\mu_{t_0} = -0.0113$, $\sigma_{t_0}^2 = 2.4128$ and $n(N) = 52$ in Figure 11.c).

6.3. Simulations with electrocardiogram

We consider an example of trajectory with a varying index of stability and a varying index of localisability. The dataset comes from [12].

We denote Z the process corresponding to an electrocardiogram. Its length is $N = 1000000$ points. We consider then the process Y defined by

$$Y(j) = \sum_{i=1}^{j} \left(Z(i) - \frac{1}{N} \sum_{k=1}^{N} Z(k) \right).$$

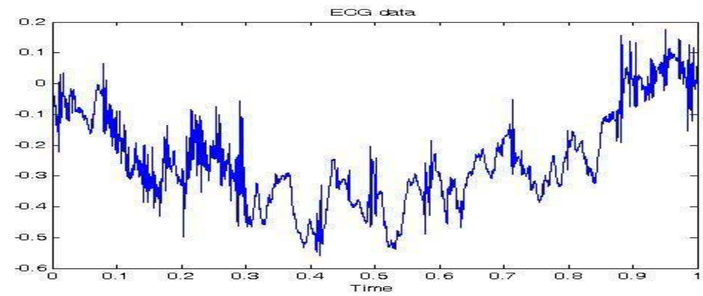


FIG 12. *Trajectory of the process Y associated to ECG series with* $N = 1000000$ *points.*

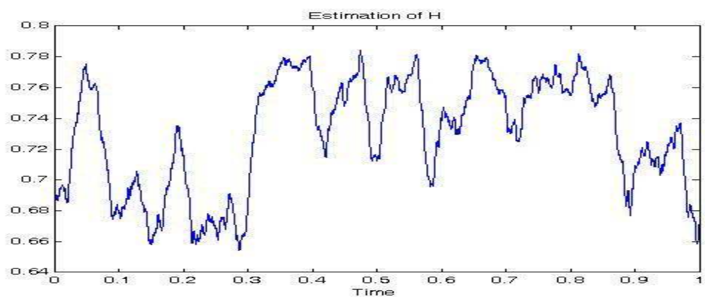


FIG 13. *Estimation of H calculated for the process represented in Figure 12.*

The realization of process Y associated to EGC series is represented in Figure 12. The increments of this process can not be regarded as stationary. We see in this example that the smoothness, as the intensity of significant jumps, is actually varying with time.

We have done an estimation of the localisability function H for this process Y. Figure 13 represents an estimation of H as function of t. The estimate of H is calculated by taking $n(N) = 25000$ points.

We notice a correlation between the noisy areas of the trajectory and the times when the exponent H is small, and also a greatest exponent when the trajectory seems to be smoother. For the estimation of the function α, we have taken $n(N) = 25000$ too. The result is presented in Figure 14. We observe also here a link between the noise and the function α. When the intensity of the significant jumps of the trajectory is high, the stability function is close to 2. A lower stability index matches to a period with a lower intensity of significant jumps.

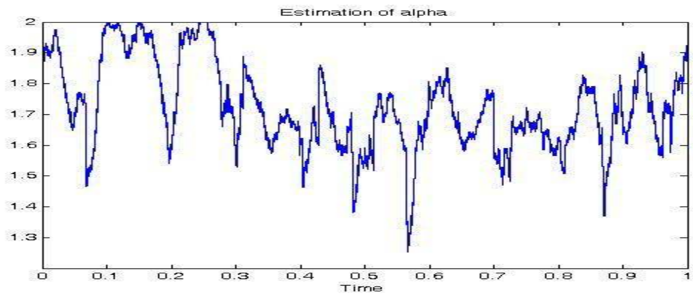


FIG 14. *Estimation of α calculated for the process of Figure 12.*

7. Assumptions

This section gathers the various conditions required on the considered processes so that our results hold. These asumptions are of three kinds: regularity condition that entail localisability, moment conditions related to the fact that we work in certain functional spaces and finally, Hölder conditions which enable to transfer the behaviour of f to the one of Y. In all the conditions, we put $c = \min_{u \in U} \alpha(u)$ and $d = \max_{u \in U} \alpha(u)$.

Regularity

- (R1) The family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in U^2 and almost all x in E. The derivatives of f with respect to v are denoted by f'_v.

Moments conditions

- (M1) There exists $\delta > \frac{d}{c} - 1$ such that:

$$\sup_{t \in U} \int_{\mathbf{R}} \left[\sup_{w \in U} (|f(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^{\delta}\, m(dx) < \infty.$$

- (M2) There exists $\delta > \frac{d}{c} - 1$ such that:

$$\sup_{t \in U} \int_{\mathbf{R}} \left[\sup_{w \in U} (|f'_v(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^{\delta}\, m(dx) < \infty.$$

- (M3) There exists $\delta > \frac{d}{c} - 1$ such that:

$$\sup_{t \in U} \int_{\mathbf{R}} \left[\sup_{w \in U} \left[|f(t,w,x)\log(r(x))|^{\alpha(w)} \right] \right]^{1+\delta} r(x)^{\delta}\, m(dx) < \infty.$$

- (M4) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbf{R}$,

$$|f(v,u,x)| \leq K_U.$$

- (M5) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbf{R}$,

$$|f'_v(v,u,x)| \leq K_U.$$

- (M6) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$,

$$\int_{\mathbf{R}} |f(v,u,x)|^2 \, m(dx) \leq K_U.$$

- (M7)

$$\inf_{v \in U} \int_{\mathbf{R}} f(v,v,x)^2 m(dx) > 0.$$

Hölder conditions

- (H1) There exists $K_U > 0$ such that $\forall (u,v) \in U^2$, $\forall x \in \mathbf{R}$,

$$\frac{1}{|v-u|^{H(u)-1/\alpha(u)}} \, |f(v,u,x) - f(u,u,x)| \leq K_U.$$

- (H2) There exists $K_U > 0$ such that $\forall (u,v) \in U^2$,

$$\frac{1}{|v-u|^{H(u)\alpha(u)}} \int_{\mathbf{R}} |f(v,u,x) - f(u,u,x)|^{\alpha(u)} m(dx) \leq K_U.$$

- (H3) There exists $p \in (d,2)$, $p \geq 1$ and $K_U > 0$ such that $\forall (u,v) \in U^2$,

$$\frac{1}{|v-u|^{1+p(H(u)-\frac{1}{\alpha(u)})}} \int_{\mathbf{R}} |f(v,u,x) - f(u,u,x)|^p \, m(dx) \leq K_U.$$

- (H4) There exists a positive function g defined on U such that

$$\lim_{r \to 0} \sup_{t \in U} \left| \frac{1}{r^{1+2(H(t)-\frac{1}{\alpha(t)})}} \int_{\mathbf{R}} \left(f(t+r,t,x) - f(t,t,x)\right)^2 m(dx) - g(t) \right| = 0.$$

- (H5) There exists $K_U > 0$ such that $\forall (u,v) \in U^2$,

$$\frac{1}{|v-u|^2} \int_{\mathbf{R}} |f(v,v,x) - f(v,u,x)|^2 \, m(dx) \leq K_U.$$

8. Proofs of Intermediate results

In all the proofs, K_U denotes a generic constant which depends on the interval U and may vary from line to line.

Proof of Lemma 5.5. Let $B \in \mathbf{R}$, $B \geq \max(5, \frac{6}{c})$. Let $[a,b] \subset U$. We denote $E_N = \{k \in \mathbf{N} \cap [0, N-1], \frac{k}{N} \in [a,b] \text{ or } \frac{k+1}{N} \in [a,b]\}$. For N large enough, since $\lim_{N \to +\infty} \frac{n(N)}{N} = 0$, for all $k \in E_N$ and $j \in \mathbf{N}$ such that $k - \frac{n(N)}{2} \leq j \leq$

$k+\frac{n(N)}{2}-1$, $\frac{j}{N}\in U$ and $\frac{j+1}{N}\in U$. The function $t\mapsto \hat{H}_N(t)$ is a step function so

$$\begin{aligned}\mathsf{P}\left(\sup_{t\in[a,b]}|\hat{H}_N(t)|>B\right) &\leq \mathsf{P}\left(\bigcup_{k\in E_N}\{|\sum_{j=k-\frac{n(N)}{2}}^{k+\frac{n(N)}{2}-1}\log|Y_{j,N}||>Bn(N)\log N\}\right)\\ &\leq \sum_{k\in E_N}\sum_{j=k-\frac{n(N)}{2}}^{k+\frac{n(N)}{2}-1}\mathsf{P}\left(|\log|Y_{j,N}||>B\log N\right)\end{aligned}$$

and

$$\begin{aligned}\mathsf{P}\left(|\log|Y_{j,N}||>B\log N\right) &\leq \mathsf{P}\left(|X(\frac{j+1}{N},\frac{j+1}{N})-X(\frac{j+1}{N},\frac{j}{N})|\geq\frac{N^B}{2}\right)\\ &\quad+\mathsf{P}\left(|X(\frac{j+1}{N},\frac{j}{N})-X(\frac{j}{N},\frac{j}{N})|\geq\frac{N^B}{2}\right)\\ &\quad+\mathsf{P}\left(|Y(\frac{j+1}{N})-Y(\frac{j}{N})|\leq\frac{1}{N^B}\right).\end{aligned}$$

We control each probability of the right term. With the conditions **(R1)**, **(M1)**, **(M2)** and **(M3)**, we can apply Proposition 4.9 of [9]: there exists $K_U>0$ such that for all $(u,v)\in U^2$ and $x>0$,

$$\begin{aligned}\mathsf{P}\left(|X(v,v)-X(v,u)|>x\right) &\leq K_U\left(\frac{|v-u|^d}{x^d}(1+|\log\frac{|v-u|}{x}|^d)\right.\\ &\quad\left.+\frac{|v-u|^c}{x^c}(1+|\log\frac{|v-u|}{x}|^c)\right). \qquad (8.1)\end{aligned}$$

We obtain the existence of a constant $K>0$ which depends on U, B, c and d such that

$$\mathsf{P}\left(|X(\frac{j+1}{N},\frac{j+1}{N})-X(\frac{j+1}{N},\frac{j}{N})|\geq\frac{N^B}{2}\right)\leq K\frac{|\log N|^d}{N^{Bc}}.$$

The process $X(.,\frac{j}{N})$ is an $\alpha(\frac{j}{N})$-stable process, so

$$\mathsf{P}\left(|X(\frac{j+1}{N},\frac{j}{N})-X(\frac{j}{N},\frac{j}{N})|\geq\frac{N^B}{2}\right)\leq\frac{2^{c/2}}{N^{\frac{Bc}{2}}}\mathsf{E}\left[|X(\frac{j+1}{N},\frac{j}{N})-X(\frac{j}{N},\frac{j}{N})|^{c/2}\right]$$

and

$$\mathsf{E}\left[|X(\tfrac{j+1}{N},\tfrac{j}{N})-X(\tfrac{j}{N},\tfrac{j}{N})|^{\frac{c}{2}}\right]=K_1\left[\int_E|f(\tfrac{j+1}{N},\tfrac{j}{N},x)-f(\tfrac{j}{N},\tfrac{j}{N},x)|^{\alpha(\frac{j}{N})}m(dx)\right]^{\frac{c}{2\alpha(\frac{j}{N})}}$$

where $K_1=\frac{2^{c/2}\Gamma(1-\frac{c}{2\alpha(\frac{j}{N})})}{c\int_0^{+\infty}u^{-\frac{c}{2}-1}\sin^2(u)du}$. With the condition **(H2)**, we obtain a constant $K_U>0$ such that $\mathsf{P}\big(|X(\frac{j+1}{N},\frac{j}{N})-X(\frac{j}{N},\frac{j}{N})|\geq\frac{N^B}{2}\big)\leq\frac{K_U}{N^{\frac{Bc}{2}}}$. With the conditions **(R1)**, **(M4)**, **(M5)**, **(M6)**, **(M7)**, **(H1)**, **(H3)**, **(H4)** and **(H5)**, we use for the third term Propositions 4.10 and 4.8 of [9]: there exists $K > 0$ and $N_0 \in \mathbf{N}$ such that for all $t \in U$, for all $N \geq N_0$ and all $x > 0$,

$$\mathsf{P}\left(|Y(t+\frac{1}{N}) - Y(t)| < x\right) \leq KN^{H(t)}x. \tag{8.2}$$

Then

$$\mathsf{P}\left(|Y(\frac{j+1}{N}) - Y(\frac{j}{N})| \leq \frac{1}{N^B}\right) \leq \frac{K}{N^B}N^{H(\frac{j}{N})}. \tag{8.3}$$

We get then

$$\mathsf{P}\left(\sup_{t\in[a,b]} |\hat{H}_N(t)| > B\right) \leq K_U N n(N)\left(\frac{|\log N|^d}{N^{Bc}} + \frac{1}{N^{\frac{Bc}{2}}} + \frac{1}{N^{B-H_+}}\right),$$

and we conclude with the Borel Cantelli lemma. □

Proof of Theorem 5.4. First, note that the condition

$$\lim_{j\to+\infty} \int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) = 0$$

implies the following condition:

- (C*) There exists $\varepsilon_1 > 0$ and $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) \leq (1-\varepsilon_1)\|h_{0,t_0}\|_{\alpha(t_0)}^{\alpha(t_0)}.$$

Let $p \in [p_0, \alpha(t_0))$. We define

$$A_N(p) = \frac{N^{pH(t_0)}}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \left|X(\frac{k+1}{N}, \frac{k+1}{N}) - X(\frac{k+1}{N}, t_0)\right|^p,$$

$$B_N(p) = \frac{N^{pH(t_0)}}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \left|X(\frac{k}{N}, \frac{k}{N}) - X(\frac{k}{N}, t_0)\right|^p$$

and

$$C_N(p) = \frac{N^{pH(t_0)}}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \left|X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)\right|^p.$$

Let $Z = X(1, t_0)$. We have, for $p \leq 1$,

$$\begin{aligned}
\mathsf{P}\left(|N^{pH(t_0)}S_N^p(p) - \mathsf{E}|Z|^p| > x\right) &\leq \mathsf{P}\left(|N^{pH(t_0)}S_N^p(p) - C_N(p)| \geq \frac{x}{2}\right) \\
&\quad + \mathsf{P}\left(|\mathsf{E}|Z|^p - C_N(p)| \geq \frac{x}{2}\right) \\
&\leq \mathsf{P}\left(|\mathsf{E}|Z|^p - C_N(p)| \geq \frac{x}{2}\right) \\
&\quad + \mathsf{P}\left(A_N(p) + B_N(p) \geq \frac{x}{2}\right)
\end{aligned}$$

and for $p \geq 1$,

$$\begin{aligned}\mathsf{P}\left(|N^{H(t_0)}S_N(p) - (\mathsf{E}|Z|^p)^{\frac{1}{p}}| > x\right) &\leq \mathsf{P}\left(|N^{H(t_0)}S_N(p) - C_N^{\frac{1}{p}}(p)| \geq \frac{x}{2}\right)\\ &\quad + \mathsf{P}\left(|C_N^{\frac{1}{p}}(p) - (\mathsf{E}|Z|^p)^{\frac{1}{p}}| \geq \frac{x}{2}\right)\\ &\leq \mathsf{P}\left(|(\mathsf{E}|Z|^p)^{\frac{1}{p}} - C_N^{\frac{1}{p}}(p)| \geq \frac{x}{2}\right)\\ &\quad + \mathsf{P}\left(A_N^{\frac{1}{p}}(p) + B_N^{\frac{1}{p}}(p) \geq \frac{x}{2}\right).\end{aligned}$$

To prove Theorem 5.4, it is enough to show that $A_N(p) \xrightarrow{\mathsf{P}} 0$, $B_N(p) \xrightarrow{\mathsf{P}} 0$ and $C_N(p) \xrightarrow{\mathsf{P}} \mathsf{E}|Z|^p$.

We consider first $A_N(p) \xrightarrow{\mathsf{P}} 0$. Let

$$\delta_N(dt) = \frac{N}{n(N)}\mathbf{1}_{\{\frac{[Nt_0]}{N} - \frac{n(N)}{2N} \leq t < \frac{[Nt_0]}{N} + \frac{n(N)}{2N}\}}dt.$$

Let U be an open interval satisfying the conditions of the theorem and $t_0 \in U$. We can fix $N_0 \in \mathbb{N}$ and $V \subset U$ an open interval depending on t_0 such that for all $N \geq N_0$ and all $t \in V$, $\frac{[Nt]+1}{N} \in U$, $\frac{[Nt]}{N} \in U$, $\int_0^1 \delta_N(dt) = \int_V \delta_N(dt)$, and such that the inequality (8.1) holds.

$$\begin{aligned}\mathsf{P}\left(A_N(p) > x\right) &= \mathsf{P}\left(\int_0^1 \left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right|^p \delta_N(dt) > x\right)\\ &\leq \frac{1}{x}\int_V \mathsf{E}\left[\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right|^p\right]\delta_N(dt).\end{aligned}$$

Let $t \in V$.

$$\mathsf{E}\Big[|\tfrac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}|^p\Big] = \int_0^\infty \mathsf{P}\Big(|\tfrac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}| > u^{\frac{1}{p}}\Big)du.$$

Let $u > 0$. We know from (8.1) that there exists $K_U > 0$ such that for all $t \in V$,

$$\begin{aligned}\mathsf{P}\Big(|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}| > u^{1/p}\Big) &\leq K_U\frac{((\log N)^c + |\log u|^c)}{N^{c(1-H(t_0))}u^{c/p}}\\ &\quad + K_U\frac{((\log N)^d + |\log u|^d)}{N^{d(1-H(t_0))}u^{d/p}},\end{aligned}$$

so, with the assumption $H(t_0) < 1$,

$$\lim_{N\to+\infty}\mathsf{P}\left(\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right| > u^{1/p}\right) = 0.$$

There exists $K_{U,p} > 0$ such that

$$\mathsf{P}\left(|\frac{X(\frac{[Nt]+1}{N},\frac{[Nt]+1}{N})-X(\frac{[Nt]+1}{N},t_0)}{(1/N)^{H(t_0)}}| > u^{\frac{1}{p}}\right) \leq \mathbf{1}_{u<1}+K_{U,p}\left(\frac{|\log u|^d}{u^{\frac{d}{p}}}+\frac{|\log u|^c}{u^{\frac{c}{p}}}\right)\mathbf{1}_{u\geq 1}. \tag{8.4}$$

Since α is a continuous function, we can fix U small enough such that $c = \inf_{t\in U}\alpha(t) > p$. We deduce from the dominated convergence theorem that for all $t \in U$,

$$\lim_{N\to+\infty} \mathsf{E}\left[\left|\frac{X(\frac{[Nt]+1}{N},\frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p\right] = 0.$$

With the inequality (8.4),

$$\mathsf{E}\left[\left|\frac{X(\frac{[Nt]+1}{N},\frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p\right] \leq 1+\int_1^{+\infty} K_{U,p}\left(\frac{|\log u|^d}{u^{d/p}}+\frac{|\log u|^c}{u^{c/p}}\right)du$$

and again with the dominated convergence theorem,

$$\lim_{N\to+\infty} \mathsf{P}\left(A_N(p) > x\right) = 0.$$

The same inequalities hold with $B_N(p)$ so we obtain $B_N(p) \xrightarrow{\mathsf{P}} 0$. We conclude proving $C_N(p) \xrightarrow{\mathsf{P}} \mathsf{E}|Z|^p$. Let $c_0 > 0$ and

$$I_N(t_0) = \{k \in \mathbb{N} \text{ such that } [Nt_0] - \frac{n(N)}{2} \leq k \leq [Nt_0] + \frac{n(N)}{2} - 1\}.$$

We use the decomposition

$$C_N(p) - \mathsf{E}|Z|^p = \frac{1}{n(N)}\sum_{k\in I_N(t_0)}\left|\frac{X(\frac{k+1}{N},t_0) - X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p \mathbf{1}_{|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}|>c_0}$$

$$+\frac{1}{n(N)}\sum_{k\in I_N(t_0)}\left|\frac{X(\frac{k+1}{N},t_0) - X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p \mathbf{1}_{|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}|\leq c_0} - \mathsf{E}|Z|^p.$$

Let $\varepsilon > 0$ and $x > 0$. By Markov's inequality, we have

$$\begin{aligned}\mathsf{P}_1 &= \mathsf{P}\left(\frac{1}{n(N)}\sum_{k\in I_N(t_0)}\left|\frac{X(\frac{k+1}{N},t_0) - X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p \mathbf{1}_{|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}|>c_0} > \frac{x}{4}\right)\\ &\leq \frac{4}{xn(N)}\sum_{k\in I_N(t_0)}\mathsf{E}\left[\left|\frac{X(\frac{k+1}{N},t_0) - X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p \mathbf{1}_{\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|>c_0}\right].\end{aligned}$$

Since $X(.,t_0)$ is $H(t_0)$-self-similar with stationary increments,

$$\mathsf{P}_1 \leq \frac{4}{x}\mathsf{E}\left[|X(1,t_0)|^p\,\mathbf{1}_{|X(1,t_0)|>c_0}\right]$$

and

$$\mathsf{E}|Z|^p \mathbf{1}_{|Z|\le c_0} = \frac{1}{n(N)} \sum_{k\in I_N(t_0)} \mathsf{E}\left[\left| \frac{X(\frac{k+1}{N},t_0) - X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}} \right|^p \mathbf{1}_{\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right| \le c_0} \right].$$

We fix c_0 large enough such that for all $N \in \mathbb{N}$, $\mathsf{P}_1 \le \frac{\varepsilon}{2}$ and $\mathsf{E}|Z|^p \mathbf{1}_{|Z|>c_0} < \frac{x}{4}$. Writing $K(x) = |x|^p \mathbf{1}_{|x|\le c_0}$ and $\Delta X_{k,t_0} = X(k+1,t_0) - X(k,t_0)$, using Chebyshev's inequality, we get

$$\begin{aligned}
\mathsf{P}\left(|C_N(p) - \mathsf{E}|Z|^p| > x\right) &\le \frac{\varepsilon}{2} + \frac{4}{x^2 n(N)^2} \sum_{k\in I_N(t_0)} \mathrm{Cov}\left(K(\Delta X_{k,t_0}), K(\Delta X_{j,t_0})\right) \\
&\le \frac{\varepsilon}{2} + \frac{4}{x^2} \frac{\mathrm{Var}\left(K(\Delta X_{0,t_0})\right)}{n(N)} \\
&\quad + \frac{4}{x^2} \frac{1}{n(N)} \sum_{j=1}^{n(N)-1} \mathrm{Cov}\left(K(\Delta X_{0,t_0}), K(\Delta X_{j,t_0})\right).
\end{aligned}$$

Under the condition **(C*)**, we can apply Theorem 2.1 of [14]: there exists a positive constant C such that

$$|Cov\left(K(\Delta X_{0,t_0}), K(\Delta X_{j,t_0})\right)| \le C\|K\|_1^2 \int_E |h_{0,t_0}(v) h_{j,t_0}(v)|^{\frac{\alpha(t_0)}{2}} m(dv).$$

Since the process $X(.,t_0)$ is $H(t_0)$-self-similar with stationary increments, the constant C does not depend on k, j. We then obtain the existence of a positive constant C_{p,c_0} depending on p, c_0 and x such that

$$\begin{aligned}
\mathsf{P}\left(|C_N(p) - \mathsf{E}|Z|^p| > x\right) &\le \frac{\varepsilon}{2} + \frac{C_{p,c_0}}{n(N)} \int_E |h_{0,t_0}(v)|^{\alpha(t_0)} m(dv) \\
&\quad + \frac{C_{p,c_0}}{n(N)} \sum_{j=1}^{n(N)-1} \int_E |h_{0,t_0}(v) h_{j,t_0}(v)|^{\frac{\alpha(t_0)}{2}} m(dv).
\end{aligned}$$

Since $\lim_{N\to+\infty} n(N) = +\infty$ and $\lim_{j\to+\infty} \int_E |h_{0,t_0}(v)h_{j,t_0}(v)|^{\frac{\alpha(t_0)}{2}} m(dv) = 0$, we conclude with Cesaro's theorem that there exists $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$,

$$\frac{C_{p,c_0}}{n(N)} \int_E |h_{0,t_0}(v)|^{\alpha(t_0)} m(dv) + \frac{C_{p,c_0}}{n(N)} \sum_{j=1}^{n(N)-1} \int_E |h_{0,t_0}(v) h_{j,t_0}(v)|^{\frac{\alpha(t_0)}{2}} m(dv) \le \frac{\varepsilon}{2}$$

and

$$\mathsf{P}\left(|C_N(p) - \mathsf{E}|Z|^p| > x\right) \le \varepsilon.$$

□

Proof of Lemma 5.7. Let $\mu > 0$ and $\lambda \in (0, 1/e)$. Since $X(., \frac{k}{N})$ is $H(\frac{k}{N})$-self-similar with stationary increments, $N^{H(\frac{k}{N})}(X(\frac{k+1}{N}, \frac{k}{N}) - X(\frac{k}{N}, \frac{k}{N}))$ is distributed as the $\alpha(\frac{k}{N})$-stable variable $X(1, \frac{k}{N})$. We deduce that there exists $K_U > 0$ such that

$$\mathsf{P}\left(\frac{|X(\frac{k+1}{N}, \frac{k}{N}) - X(\frac{k}{N}, \frac{k}{N})|}{(1/N)^{H(\frac{k}{N})}} \leq \mu\right) \leq K_U \mu.$$

Then

$$\mathsf{P}\left(\frac{|X(\frac{k+1}{N}, \frac{k+1}{N}) - X(\frac{k+1}{N}, \frac{k}{N})|}{|X(\frac{k+1}{N}, \frac{k}{N}) - X(\frac{k}{N}, \frac{k}{N})|} > \lambda\right) \leq K_U \mu + \mathsf{P}\left(\frac{|X(\frac{k+1}{N}, \frac{k+1}{N}) - X(\frac{k+1}{N}, \frac{k}{N})|}{(1/N)^{H(\frac{k}{N})}} > \lambda\mu\right).$$

With the conditions **(R1)**, **(M1)**, **(M2)** and **(M3)**, we use the inequality (8.1) to obtain (with K_U which may change from line to line)

$$\begin{aligned}\mathsf{P}\left(\frac{|X(\frac{k+1}{N}, \frac{k+1}{N}) - X(\frac{k+1}{N}, \frac{k}{N})|}{|X(\frac{k+1}{N}, \frac{k}{N}) - X(\frac{k}{N}, \frac{k}{N})|} > \lambda\right) &\leq K_U \mu + K_U \frac{N^{dH(\frac{k}{N})}}{|N\lambda\mu|^d}(1 + |\log|N\lambda\mu||^d) \\ &\quad + K_U \frac{N^{cH(\frac{k}{N})}}{|N\lambda\mu|^c}(1 + |\log|N\lambda\mu||^c).\end{aligned}$$

We choose $\mu = \frac{1}{\lambda^{\frac{\alpha(t_0)}{1+\alpha(t_0)}} N^{\frac{\alpha(t_0)(1-H(\frac{k}{N}))}{1+\alpha(t_0)}}}$ to obtain

$$\begin{aligned}\mathsf{P}\left(\frac{|X(\frac{k+1}{N}, \frac{k+1}{N}) - X(\frac{k+1}{N}, \frac{k}{N})|}{|X(\frac{k+1}{N}, \frac{k}{N}) - X(\frac{k}{N}, \frac{k}{N})|} > \lambda\right) &\leq K_U \mu + K_U \left(\frac{|\log N|^d |\log \lambda|^d}{N^{\frac{d(1-H(\frac{k}{N}))}{1+\alpha(t_0)}} \lambda^{\frac{d}{1+\alpha(t_0)}}}\right) \\ &\quad + K_U \left(\frac{|\log N|^c |\log \lambda|^c}{N^{\frac{c(1-H(\frac{k}{N}))}{1+\alpha(t_0)}} \lambda^{\frac{c}{1+\alpha(t_0)}}}\right) \\ &\leq K_U \frac{|\log N|^d |\log \lambda|^d}{N^{\frac{d(1-H_-)}{1+c}} \lambda^{\frac{d}{1+c}}}.\end{aligned}$$

□

Proof of Theorem 5.6. Let $x > 0$ and $\delta \in (0, \frac{2\alpha(t_0)(1-H(t_0))}{2+3\alpha(t_0)})$. Put $\xi_{k,N} = \frac{X(\frac{k+1}{N}, \frac{k+1}{N}) - X(\frac{k+1}{N}, \frac{k}{N})}{X(\frac{k+1}{N}, \frac{k}{N}) - X(\frac{k}{N}, \frac{k}{N})}$.

Let us show that $\frac{1}{\sqrt{n(N)}} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \log|1+\xi_{k,N}|$ tends to 0 in probability. Since α and H are continuous, we can choose $\mu > \frac{1}{2}$ and U small enough in order to have $\delta < \frac{d(1-H_-)}{1+c+\mu d}$.

Let $\lambda_N = 1 - \frac{1}{n(N)^\mu}$, $\mu_N = \frac{1}{n(N)^\mu}$, $A_N = \cup_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \{|1+\xi_{k,N}| < \lambda_N\}$ and $B_N = \cup_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \{|\xi_{k,N}| > \mu_N\}$. Since $A_N \subset B_N$, we will only show that $\mathsf{P}(B_N) = 0$. We use Lemma 5.7: there exists $K_U > 0$ such that

$$\mathsf{P}(|\xi_{k,N}| > \mu_N) \quad \leq \quad K_U \frac{|\log N|^d n(N)^{\frac{d\mu}{1+c}}}{N^{\frac{d(1-H_-)}{1+c}}}.$$

Then

$$\begin{aligned}\mathsf{P}(B_N) &\leq n(N)K_U\frac{|\log N|^d n(N)^{\frac{d\mu}{1+c}}}{N^{\frac{d(1-H_-)}{1+c}}}\\ &\leq K_U|\log N|^d N^{\delta(1+\frac{d\mu}{1+c})-\frac{d(1-H_-)}{1+c}}\end{aligned}$$

so $\lim_{N\to+\infty}\mathsf{P}(B_N)=0$.

We obtain then

$$\begin{aligned}\mathsf{P}\left(\sum_{k\in I_N(t_0)}\frac{\log|1+\xi_{k,N}|}{\sqrt{n(N)}}<-x\right) &\leq \mathsf{P}(A_N)+\mathsf{P}\left(\{\sum_{k\in I_N(t_0)}\frac{\log|1+\xi_{k,N}|}{\sqrt{n(N)}}<-x\}\cap\bar{A_N}\right)\\ &\leq \mathsf{P}(A_N)+\mathsf{P}\left(n(N)\frac{\log\lambda_N}{\sqrt{n(N)}}<-x\right).\end{aligned}$$

Since $\mu>\frac{1}{2}$, $\lim_{N\to+\infty}\mathsf{P}\big(\sqrt{n(N)}\log\lambda_N<-x\big)=0$. We obtain in the same way

$$\begin{aligned}\mathsf{P}\left(\sum_{k\in I_N(t_0)}\frac{\log|1+\xi_{k,N}|}{\sqrt{n(N)}}>x\right) &\leq \mathsf{P}(B_N)+\mathsf{P}\left(\{\sum_{k\in I_N(t_0)}\frac{\log|1+\xi_{k,N}|}{\sqrt{n(N)}}>x\}\cap\bar{B_N}\right)\\ &\leq \mathsf{P}(B_N)+\mathsf{P}\left(n(N)\frac{\log|1+\mu_N|}{\sqrt{n(N)}}>x\right).\end{aligned}$$

Since $\mu>\frac{1}{2}$, $\lim_{N\to+\infty}\mathsf{P}\big(\sqrt{n(N)}\log|1+\mu_N|>x\big)=0$. □

9. Proofs of Main results

Proof of Theorem 4.1. Since $x\to x^\gamma$ is an increasing function on $\mathbf{R}_+$ (we take $\gamma\in(0,1)$),

$$\hat{\alpha}_N(t_0)=\min\left(\arg\min_{\alpha\in[0,2]}\int_{p_0}^2|R_{\exp}^{(N)}(p)-R_\alpha(p)|^\gamma dp\right).$$

Let $g_N(\alpha)=\int_{p_0}^2|R_{\exp}^{(N)}(p)-R_\alpha(p)|^\gamma dp$ and $g(\alpha)=\int_{p_0}^2|R_{\alpha(t_0)}(p)-R_\alpha(p)|^\gamma dp$.

g is a continuous function on $(0,2]$, with $g(0)>0$, $g(2)>0$. The only solution of the equation $g(\alpha)=0$ is $\alpha(t_0)$. Moreover, $\lim_{\alpha\to\alpha(t_0)}\frac{|g(\alpha)-g(\alpha(t_0))|}{|\alpha-\alpha(t_0)|^\gamma}>0$. The function $f:\alpha\mapsto\frac{|g(\alpha)|}{|\alpha-\alpha(t_0)|}$ is then strictly positive, continuous for $\alpha\neq\alpha(t_0)$, and satisfy $\lim_{\alpha\to\alpha(t_0)}f(\alpha)=+\infty$: there exists $K_{\alpha(t_0)}$ a positive constant depending only on $\alpha(t_0)$ such that

$$\forall\alpha\in(0,2),\quad|g(\alpha)|\geq K_{\alpha(t_0)}|\alpha-\alpha(t_0)|. \tag{9.1}$$

We estimate now $|g(\hat{\alpha}_N(t_0))|$.

$$\begin{aligned}|g(\hat{\alpha}_N(t_0))| &\leq |g(\hat{\alpha}_N(t_0)) - g_N(\hat{\alpha}_N(t_0))| + |g_N(\hat{\alpha}_N(t_0))| \\ &\leq |g(\hat{\alpha}_N(t_0)) - g_N(\hat{\alpha}_N(t_0))| + g_N(\alpha(t_0)),\end{aligned}$$

and

$$\begin{aligned}& |g(\hat{\alpha}_N(t_0)) - g_N(\hat{\alpha}_N(t_0))| \\ = & \left|\int_{p_0}^2 (\left|R_{\alpha(t_0)}(p) - R_{\hat{\alpha}_N(t_0)}(p)\right|^\gamma - \left|R_{\exp}^{(N)}(p) - R_{\hat{\alpha}_N(t_0)}(p)\right|^\gamma) dp\right| \\ \leq & \int_{p_0}^2 \left|R_{\alpha(t_0)}(p) - R_{\exp}^{(N)}(p)\right|^\gamma dp \\ = & g_N(\alpha(t_0)).\end{aligned}$$

From (9.1),

$$\begin{aligned}|\hat{\alpha}_N(t_0) - \alpha(t_0)| &\leq \frac{1}{K_{\alpha(t_0)}} g(\hat{\alpha}_N(t_0)) \\ &\leq \frac{2}{K_{\alpha(t_0)}} g_N(\alpha(t_0)).\end{aligned}$$

Let us show that $\lim_{N\to+\infty} \mathsf{E}\,|g_N(\alpha(t_0))|^r = 0$ for any $r > 0$. One has, using the inequality $S_N(p) \leq S_N(q)$ for $p \leq q$,

$$\begin{aligned}g_N(\alpha(t_0)) &= \int_{p_0}^{\alpha(t_0)} |R_{\exp}^{(N)}(p) - R_{\alpha(t_0)}(p)|^\gamma dp + \int_{\alpha(t_0)}^{2} |R_{\exp}^{(N)}(p)|^\gamma dp \\ &\leq \int_{p_0}^{\alpha(t_0)} |R_{\exp}^{(N)}(p) - R_{\alpha(t_0)}(p)|^\gamma dp + (2 - \alpha(t_0)) \left|\frac{S_N(p_0)}{S_N(\alpha(t_0))}\right|^\gamma.\end{aligned}$$

For the first term, we use Theorem 5.4: for all $p \in [p_0, \alpha(t_0))$,

$$N^{H(t_0)} S_N(p) \xrightarrow{\mathsf{P}} (\mathsf{E}|X(1,t_0)|^p)^{1/p} \tag{9.2}$$

It is clear that $\forall p \in [p_0, \alpha(t_0))$,

$$\left(N^{H(t_0)} S_N(p_0), N^{H(t_0)} S_N(p)\right) \xrightarrow{\mathsf{P}} \left((\mathsf{E}|X(1,t_0)|^{p_0})^{1/p_0}, (\mathsf{E}|X(1,t_0)|^p)^{1/p}\right),$$

and

$$R_{\exp}^{(N)}(p) = \frac{S_N(p_0)}{S_N(p)} \xrightarrow{\mathsf{P}} R_{\alpha(t_0)}(p). \tag{9.3}$$

Note that $\forall N \in \mathbb{N}$, $\forall p \in [p_0, \alpha(t_0))$, $|R_{\exp}^{(N)}(p)| \leq 1$ so there exists a positive constant K depending on γr, $\alpha(t_0)$ and p such that

$$\mathsf{E}|R_{\exp}^{(N)}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} = \int_0^K \mathsf{P}\left(|R_{\exp}^{(N)}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} > x\right) dx.$$

Finally, with (9.3), $\forall p \in [p_0, \alpha(t_0))$, $\mathsf{E}|R_{\exp}^{(N)}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} \underset{N\to+\infty}{\longrightarrow} 0$. With the inequality $\mathsf{E}|R_{\exp}^{(N)}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} \leq 2C_{\gamma r}$ where $C_{\gamma r}$ is a positive constant depending on γr, by the dominating convergence theorem,

$$\lim_{N\to+\infty} \int_{p_0}^{\alpha(t_0)} \mathsf{E}|R_{\exp}^{(N)}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} dp = 0.$$

To conclude we show that $\left|\frac{S_N(p_0)}{S_N(\alpha(t_0))}\right|^{\gamma} \xrightarrow{L^r} 0$. Since $\forall N \in \mathbb{N}$, $\left|\frac{S_N(p_0)}{S_N(\alpha(t_0))}\right|^{\gamma} \leq 1$, it is enough to show $\frac{S_N(p_0)}{S_N(\alpha(t_0))} \xrightarrow{\mathsf{P}} 0$. Let $p < \alpha(t_0)$.

$$\mathsf{P}\left(\frac{1}{N^{H(t_0)} S_N(\alpha(t_0))} > x\right) \leq \mathsf{P}\left(\frac{1}{N^{H(t_0)} S_N(p)} > x\right).$$

So,

$$\begin{aligned} \limsup_{N\to+\infty} \mathsf{P}\left(\frac{1}{N^{H(t_0)} S_N(\alpha(t_0))} > x\right) &\leq \limsup_{N\to+\infty} \mathsf{P}\left(\frac{1}{N^{H(t_0)} S_N(p)} > x\right) \\ &= \lim_{N\to+\infty} \mathsf{P}\left(\frac{1}{N^{H(t_0)} S_N(p)} > x\right) \\ &= \mathsf{P}\left(\frac{1}{(\mathsf{E}|X(1,t_0)|^p)^{1/p}} > x\right), \end{aligned}$$

with (9.2). Since $\lim_{p\to\alpha(t_0)} \mathsf{P}\left(\frac{1}{(\mathsf{E}|X(1,t_0)|^p)^{1/p}} > x\right) = 0$, we have also for the left side $\limsup_{N\to+\infty} \mathsf{P}\left(\frac{1}{N^{H(t_0)} S_N(\alpha(t_0))} > x\right) = 0$ and $\frac{1}{N^{H(t_0)} S_N(\alpha(t_0))} \xrightarrow{\mathsf{P}} 0$. Using the convergence $N^{H(t_0)} S_N(p_0) \xrightarrow{\mathsf{P}} (\mathsf{E}|X(1,t_0)|^{p_0})^{1/p_0}$, we obtain $\frac{S_N(p_0)}{S_N(\alpha(t_0))} \xrightarrow{\mathsf{P}} 0$. This entails the first part of Theorem 4.1, that is

$$\lim_{N\to+\infty} \mathsf{E}\left|\hat{\alpha}_N(t_0) - \alpha(t_0)\right|^r = 0.$$

If in addition, we assume that the conditions hold for all $t_0 \in U$, we obtain for all $r > 0$ and all $t \in U$

$$\lim_{N\to+\infty} \mathsf{E}\left|\hat{\alpha}_N(t) - \alpha(t)\right|^r = 0.$$

$\hat{\alpha}_N$ and α are two bounded functions on U so for all $r > 0$,

$$\lim_{N\to+\infty} \int_U \mathsf{E}\left[\left|\hat{\alpha}_N(t) - \alpha(t)\right|^r\right] dt = 0.$$

□

Proof of Theorem 4.2. Note that it is sufficient to prove the result of Theorem 4.2 for $r \geq 1$ since the convergence in L^p implies the convergence in L^q for all

$q < p$. Let $r \geq 1$. We write

$$\begin{aligned} \hat{H}_N(t_0) - H(t_0) &= -\frac{1}{n(N)\log N} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \log\left|\frac{Y_{k,N}}{(\frac{1}{N})^{H(t_0)}}\right| \\ &= -\frac{N}{n(N)\log N} \int_{\frac{[Nt_0]}{N}-\frac{n(N)}{2N}}^{\frac{[Nt_0]}{N}+\frac{n(N)}{2N}} \log\left|\frac{Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})}{(\frac{1}{N})^{H(t_0)}}\right| dt. \end{aligned}$$

We write again $\delta_N(dt) = \frac{N}{n(N)} \mathbf{1}_{\{\frac{[Nt_0]}{N}-\frac{n(N)}{2N} \leq t < \frac{[Nt_0]}{N}+\frac{n(N)}{2N}\}} dt$ and we define the function $f_N(t) = \log|\frac{Y(\frac{[Nt]+1}{N})-Y(\frac{[Nt]}{N})}{(\frac{1}{N})^{H(t)}}|$. Since $\int_0^1 \delta_N(dt) = 1$, we obtain

$$\hat{H}_N(t_0) - H(t_0) = -\frac{1}{\log N}\int_0^1 f_N(t)\delta_N(dt) + \int_0^1 (H(t) - H(t_0))\,\delta_N(dt).$$

Then, there exists a constant $K_r > 0$ depending on r such that

$$\begin{aligned} \mathsf{E}\left[|\hat{H}_N(t_0) - H(t_0)|^r\right] &\leq K_r \frac{\mathsf{E}\left(|\int_0^1 f_N(t)\delta_N(dt)|^r\right)}{|\log N|^r} \\ &\quad + K_r \left|\int_0^1 (H(t) - H(t_0))\,\delta_N(dt)\right|^r. \end{aligned}$$

H is continuously differentiable and $\lim_{N\to+\infty} \frac{N}{n(N)} = +\infty$ so

$$\lim_{N\to+\infty} \int_0^1 (H(t) - H(t_0))\,\delta_N(dt) = 0.$$

To conclude, it is sufficient to show that there exists a constant $K > 0$ depending on t_0 and r such that for all $N \in \mathbb{N}$, $\mathsf{E}\big(|\int_0^1 f_N(t)\delta_N(dt)|^r\big) \leq K$. Let U an open interval satisfying all the conditions **(R-)**, **(M-)** and **(H-)**, and $t_0 \in U$. We can fix $N_0 \in \mathbb{N}$ and $V \subset U$ an open interval depending on t_0 such that for all $N \geq N_0$ and all $t \in V$, $\frac{[Nt]+1}{N} \in U$, $\frac{[Nt]}{N} \in U$ and $\int_0^1 f_N(t)\delta_N(dt) = \int_V f_N(t)\delta_N(dt)$. With the Jensen inequality,

$$\mathsf{E}\left(|\int_0^1 f_N(t)\delta_N(dt)|^r\right) \leq \int_V \mathsf{E}|f_N(t)|^r \delta_N(dt).$$

We consider $\mathsf{E}|f_N(t)|^r = \int_0^{+\infty} \mathsf{P}\left(|f_N(t)|^r > x\right) dx$.

$$\begin{aligned} \mathsf{E}|f_N(t)|^r &= \int_0^{+\infty} \mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})\right| < \frac{e^{-x^{1/r}}}{N^{H(t)}}\right) dx. \\ &\quad + \int_0^{+\infty} \mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})\right| > \frac{e^{x^{1/r}}}{N^{H(t)}}\right) dx. \end{aligned}$$

Thanks to the conditions **(R1)**, **(M4)**, **(M5)**, **(M6)**, **(M7)**, **(H1)**, **(H3)**, **(H4)** and **(H5)**, we use the equality (8.2) to control the first term: there exists $K_U > 0$ (that may change from line to line) and $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$ and all $t \in V$,

$$\mathsf{P}\left(|Y(\frac{[Nt]}{N}+\frac{1}{N})-Y(\frac{[Nt]}{N})| < \frac{e^{-x^{1/r}}}{N^{H(t)}}\right) \leq K_U N^{H(\frac{[Nt]}{N})}\frac{e^{-x^{1/r}}}{N^{H(t)}}.$$

We get then

$$\begin{aligned}\int_0^{+\infty}\mathsf{P}\left(|Y(\tfrac{[Nt]+1}{N})-Y(\tfrac{[Nt]}{N})| < \frac{e^{-x^{1/r}}}{N^{H(t)}}\right)dx \;&\leq\; K_U\left(\int_0^{+\infty} e^{-x^{1/r}}dx\right)N^{H(\frac{[Nt]}{N})-H(t)}\\ &\leq\; K_U.\end{aligned}$$

For the second term, we write

$$\begin{aligned}&\mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N})-Y(\frac{[Nt]}{N})\right| > \frac{e^{x^{1/r}}}{N^{H(t)}}\right)\\ \leq\;&\mathsf{P}\left(\left|X(\frac{[Nt]+1}{N},\frac{[Nt]+1}{N})-X(\frac{[Nt]+1}{N},\frac{[Nt]}{N})\right| > \frac{e^{x^{1/r}}}{N^{H(t)}}\right)\\ &+\mathsf{P}\left(\left|X(\frac{[Nt]+1}{N},\frac{[Nt]}{N})-X(\frac{[Nt]}{N},\frac{[Nt]}{N})\right| > \frac{e^{x^{1/r}}}{N^{H(t)}}\right).\end{aligned}$$

With the conditions **(R1)**, **(M1)**, **(M2)** and **(M3)**, we use the equality (8.1) to obtain a positive constant $K_U > 0$ such that:

$$\begin{aligned}&\mathsf{P}\left(\left|X(\frac{[Nt]+1}{N},\frac{[Nt]+1}{N})-X(\frac{[Nt]+1}{N},\frac{[Nt]}{N})\right| > \frac{e^{x^{1/r}}}{N^{H(t)}}\right)\\ &\leq K_U\left(\frac{(\log N)^c}{N^{c(1-H(t))}e^{cx^{1/r}}}\right)+K_U\left(\frac{x^{c/r}}{N^{c(1-H(t))}e^{cx^{1/r}}}\right)\\ &+K_U\left(\frac{(\log N)^d}{N^{d(1-H(t))}e^{dx^{1/r}}}\right)+K_U\left(\frac{x^{d/r}}{N^{d(1-H(t))}e^{dx^{1/r}}}\right).\end{aligned}$$

Since $H_+ = \max_{u\in U} H(u) < 1$, we conclude that

$$\lim_{N\to+\infty}\int_0^{+\infty}\sup_{t\in U}\mathsf{P}(|X(\frac{[Nt]+1}{N},\frac{[Nt]+1}{N})-X(\frac{[Nt]+1}{N},\frac{[Nt]}{N})| \geq \frac{e^{x^{1/r}}}{N^{H(t)}})dx = 0.$$

Let $\eta < c$. The Markov inequality gives

$$\begin{aligned}&\mathsf{P}\left(\left|X(\frac{[Nt]+1}{N},\frac{[Nt]}{N})-X(\frac{[Nt]}{N},\frac{[Nt]}{N})\right| \geq \frac{e^{x^{1/r}}}{N^{H(t)}}\right)\\ \leq\;&\frac{N^{\eta H(t)}}{e^{\eta x^{1/r}}}\mathsf{E}\left[|X(\frac{[Nt]+1}{N},\frac{[Nt]}{N})-X(\frac{[Nt]}{N},\frac{[Nt]}{N})|^{\eta}\right]\end{aligned}$$

and Property 1.2.17 of [16], since $X(.,t_N)$ is an $\alpha(t_N)$-stable process,

$$\begin{aligned} & \mathsf{E}\left[|X(\frac{[Nt]+1}{N},t_N)-X(t_N,t_N)|^\eta\right] \\ = \quad & c_{\alpha(t_N),0}(\eta)^\eta\left(\int_E(|f(\frac{[Nt]+1}{N},t_N,x)-f(t_N,t_N,x)|^{\alpha(t_N)}\, m(dx)\right)^{\eta/\alpha(t_N)}. \end{aligned}$$

where $t_N=\frac{[Nt]}{N}$ and $c_{\alpha(t_N),0}(\eta)^\eta=\Gamma(1-\eta/\alpha(t_N))C_{\alpha(t_N)}$. With the condition **(H2)**, there exists $K_U>0$ such that for all $N\geq N_0$ and all $t\in V$,

$$\int\limits_0^{+\infty}\mathsf{P}\left(\left|X(\frac{[Nt]+1}{N},\frac{[Nt]}{N})-X(\frac{[Nt]}{N},\frac{[Nt]}{N})\right|\geq\frac{e^{x^{1/r}}}{N^{H(t)}}\right)dx\leq K_U.$$

The conclusion is that for all $t_0\in U$,

$$\lim_{N\to+\infty}\mathsf{E}\left|\hat{H}_N(t_0)-H(t_0)\right|^r=0. \tag{9.4}$$

Let $[a,b]\subset U$, $p>0$ and $\eta\in(0,1)$. We denote $A=\liminf_{N\to+\infty}\{\sup_{t\in[a,b]}|\hat{H}_N(t)-H(t)|\leq B\}$. Thanks to Lemma 5.5, there exists $B\in\mathbf{R}$ such that $\mathsf{P}(A)=1$. Then

$$\begin{aligned} \mathsf{E}\left[\int\limits_a^b|\hat{H}_N(t)-H(t)|^p dt\right] &= \int_a^b\mathsf{E}\left[|\hat{H}_N(t)-H(t)|^p\right]dt \\ &= \int_a^b\int_0^{+\infty}\mathsf{P}\left(\{|\hat{H}_N(t)-H(t)|^p>x\}\cap A\right)dx\,dt \\ &= \int_a^b\int_0^{B^p}\mathsf{P}\left(\{|\hat{H}_N(t)-H(t)|^p>x\}\cap A\right)dx\,dt \\ &\leq \int_a^b\int_0^{B^p}\mathsf{P}\left(\{|\hat{H}_N(t)-H(t)|^p>x\}\right)dx\,dt. \end{aligned}$$

The equality (9.4) available for all $r>0$ easily leads to

$$\lim_{N\to+\infty}\mathsf{E}\left[\int\limits_a^b|\hat{H}_N(t)-H(t)|^p dt\right]=0.$$

□

Proof of Theorem 4.3. We write

$$H_N^s(t_0)=-\frac{1}{n(N)\log N}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\log\left|X(\frac{k+1}{N},\frac{k}{N})-X(\frac{k}{N},\frac{k}{N})\right|$$

and the following decomposition

$$\log N(\hat{H}_N(t_0) - H(t_0)) = \log N(\hat{H}_N(t_0) - H_N^s(t_0)) + \log N(H_N^s(t_0) - H(t_0)).$$

We know from [9] that Y is satisfying the conditions (**R1**), (**M1**), (**M2**) and (**M3**). For all $u \in (0,1)$, $X(.,u)$ is a $\alpha(u)$-stable Lévy motion, so is $\frac{1}{\alpha(u)}$-self-similar with stationary increments. We apply Theorem 5.6 to obtain the convergence in probability to 0 of $\sqrt{n(N)}\log N(\hat{H}_N(t_0) - H_N^s(t_0))$. For the second term, notice that

$$\begin{aligned} H_N^s(t_0) &= -\frac{1}{n(N)\log N}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \log\left| X(\frac{k+1}{N},\frac{k}{N}) - X(\frac{k}{N},\frac{k}{N})\right| \\ &= -\frac{1}{n(N)\log N}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \log\left| \frac{X(\frac{k+1}{N},\frac{k}{N}) - X(\frac{k}{N},\frac{k}{N})}{(1/N)^{H(k/N)}}\right| \\ &\quad + \frac{1}{n(N)}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} H(\frac{k}{N}). \end{aligned}$$

Put $z_{k,N} = \log\left|\frac{X(\frac{k+1}{N},\frac{k}{N}) - X(\frac{k}{N},\frac{k}{N})}{(1/N)^{H(k/N)}}\right|$. Then

$$\begin{aligned} \sqrt{n(N)}(\log N(H_N^s(t_0) - H(t_0)) + \mu_{t_0}) &= \frac{\log N}{\sqrt{n(N)}}\sum_{k\in I_N(t_0)}(H(\frac{k}{N}) - H(t_0)) \\ &\quad + \frac{1}{\sqrt{n(N)}}\sum_{k\in I_N(t_0)}(\mu_{t_0} - \mu_{\frac{k}{N}}) \\ &\quad + \frac{1}{\sqrt{n(N)}}\sum_{k\in I_N(t_0)}(\mu_{\frac{k}{N}} - z_{k,N}). \end{aligned}$$

$H = \frac{1}{\alpha}$ is a $\mathcal{C}^1$ function, so there exists a positive constant $K > 0$ such that $|H(\frac{k}{N}) - H(t_0)| \leq K|\frac{k}{N} - t_0|$, and

$$\begin{aligned} \left|\frac{\log N}{\sqrt{n(N)}}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}(H(\frac{k}{N}) - H(t_0))\right| &\leq K\frac{\log N}{\sqrt{n(N)}}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}|\frac{k}{N} - t_0| \\ &\leq K\frac{n(N)^{\frac{3}{2}}}{N}\log N. \end{aligned}$$

Since $\delta < \frac{2\alpha(t_0)-2}{3\alpha(t_0)+2}$, $\frac{3}{2}\delta < 1$ and $\lim_{N\to+\infty}\frac{\log N}{\sqrt{n(N)}}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}(H(\frac{k}{N}) - H(t_0)) = 0$.

With $Z \sim S_\eta(1,0,0)$, $\eta \mapsto \mathsf{E}[\log|Z|] = \Gamma'(1)\big[1 - \frac{1}{\eta}\big]$ is continuously differentiable. With the hypothesis on the function α, the function $t \mapsto \mu_t$ is a $\mathcal{C}^1$ function. We get then, as for H, $\lim_{N\to+\infty}\frac{1}{\sqrt{n(N)}}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}(\mu_{\frac{k}{N}} - \mu_{t_0}) = 0$.

To finish the proof, let us show the convergence $\frac{1}{\sqrt{n(N)}}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}(\mu_{\frac{k}{N}} - z_{k,N}) \xrightarrow{d} \mathcal{N}(0,\sigma_{t_0}^2)$. Let $X_{k,N} = \frac{\mu_{\frac{k}{N}} - z_{k,N}}{\sqrt{n(N)}}$, $\varepsilon > 0$ and $c = \inf_{t\in U}\alpha(t)$.

$$\begin{aligned}\mathsf{P}\left(|X_{k,N}| > \varepsilon\right) &= \mathsf{P}\left(\left|\frac{X(\frac{k+1}{N},\frac{k}{N}) - X(\frac{k}{N},\frac{k}{N})}{(1/N)^{H(k/N)}}\right| \le e^{\mu_{\frac{k}{N}}} e^{-\varepsilon\sqrt{n(N)}}\right)\\ &\quad + \mathsf{P}\left(\left|\frac{X(\frac{k+1}{N},\frac{k}{N}) - X(\frac{k}{N},\frac{k}{N})}{(1/N)^{H(k/N)}}\right|^{\frac{c}{2}} > e^{\frac{c}{2}\mu_{\frac{k}{N}}} e^{\frac{c\varepsilon\sqrt{n(N)}}{2}}\right).\end{aligned}$$

$\frac{X(\frac{k+1}{N},\frac{k}{N}) - X(\frac{k}{N},\frac{k}{N})}{(1/N)^{H(k/N)}}$ is a standard $\alpha(\frac{k}{N})$-stable random variable, then there exists $K > 0$ such that

$$\mathsf{P}\left(|X_{k,N}| > \varepsilon\right) \le K(e^{-\frac{c\varepsilon\sqrt{n(N)}}{2}} + e^{-\varepsilon\sqrt{n(N)}}). \tag{9.5}$$

$(X_{k,N})_k$ is thus satisfying $\lim_{N\to+\infty}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\mathsf{P}(|X_{k,N}| > \varepsilon) = 0$.

$\mu_{\frac{k}{N}} = \mathsf{E}[z_{k,N}]$ so

$$\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\mathsf{E}[X_{k,N}\mathbf{1}_{|X_{k,N}|\le\varepsilon}] = -\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\mathsf{E}[X_{k,N}\mathbf{1}_{|X_{k,N}|>\varepsilon}].$$

With the inequality (9.5), we obtain $\lim_{N\to+\infty}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\mathsf{E}[X_{k,N}\mathbf{1}_{|X_{k,N}|\le\varepsilon}] = 0$.

Finally, we have

$$\sum_{k\in I_N(t_0)}\mathsf{Var}(X_{k,N}\mathbf{1}_{|X_{k,N}|\le\varepsilon}) = \int_0^1 \mathsf{Var}\left((\mu_{\frac{[Nt]}{N}} - z_{[Nt],N})\mathbf{1}_{|X_{[Nt],N}|\le\varepsilon}\right)\delta_N(dt),$$

where we have put $\delta_N(dt) = \frac{N}{n(N)}\mathbf{1}_{\{\frac{[Nt_0]}{N}-\frac{n(N)}{2N}\le t<\frac{[Nt_0]}{N}+\frac{n(N)}{2N}\}}dt$. Use again the fact that $e^{z_{[Nt],N}}$ is a standard $\alpha(t_N)$-stable random variable and the inequality (9.5) to obtain the convergence $\mathsf{Var}\big((\mu_{\frac{[Nt]}{N}} - z_{[Nt],N})\mathbf{1}_{|X_{[Nt],N}|\le\varepsilon}\big) \to \sigma_t^2$ and

$$\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\mathsf{Var}(X_{k,N}\mathbf{1}_{|X_{k,N}|\le\varepsilon}) \to \sigma_{t_0}^2.$$

We conclude the proof using Theorem 4.1 of [13]. □

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