

Sub-Gaussian bound for the one-dimensional Bouchaud trap model

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Abstract. We establish a sub-Gaussian lower bound for the transition kernel of the one-dimensional, symmetric Bouchaud trap model, which provides a positive answer to the behavior predicted by Bertin and Bouchaud in (*Phys. Rev. E* (3) **67** (2013) 026128). The proof rests on the Ray–Knight description of the local time of a one-dimensional Brownian motion. Using the same ideas, we also prove the corresponding result for the FIN singular diffusion.

1 Introduction

The *Bouchaud trap model* (BTM) is a random walk in a random medium. The medium represents a landscape composed of *traps* which retain the walk for some amount of time. In this paper, we will consider the BTM taking values on the line \mathbb{Z} . Each $z \in \mathbb{Z}$ represents a trap of some depth $\tau_z > 0$. For every fixed realization of the traps $\tau = (\tau_z)_{z \in \mathbb{Z}}$, let $(X_t)_{t \geq 0}$ be a continuous-time Markov chain with $X_0 = 0$ and whose jump rates are given by

$$c(x, y) := \begin{cases} (2\tau_x)^{-1} & \text{if } |x - y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $(X_t)_{t \geq 0}$ is a random walk on \mathbb{Z} which, each time it visits a site $z \in \mathbb{Z}$, it waits there an exponentially distributed time with mean τ_z and then jumps to $z - 1, z + 1$ with probability $1/2$ each. Hence, τ_z should be regarded as the depth of the trap at z . The environment τ is chosen at random, more precisely, $(\tau_z)_{z \in \mathbb{Z}}$ is an i.i.d. sequence of positive random variables. The random walk $(X_t)_{t \geq 0}$ defined in this way is the one-dimensional Bouchaud trap model.

When the environment is highly inhomogeneous (i.e., when the distribution of τ_0 has heavy tails), the trapping mechanism becomes relevant even in the large scale behavior of the system, and the BTM displays some striking features, such as *localization* (see Fontes et al. (1999)), *subdiffusivity* (see Fontes et al. (2002)) and *aging* (see Fontes et al. (2002), Černý (2006), Ben Arous and Černý (2005)). More precisely, *localization* means that, for \mathbb{P} -a.e. realization of the environment τ , we have that

$$\limsup_{t \rightarrow \infty} \max_{z \in \mathbb{Z}} \mathbb{P}(X_t = z | \tau) > 0,$$

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that is, for arbitrarily large times, there is a site z which carries a positive proportion of the distribution of X_t . Localization reflects the fact that, at all time scales, the depth of the deepest trap found by the BTM is of the same order of magnitude as the sum of the depths of all the traps visited by the BTM up to that time.

Hence, the evolution of the BTM is characterized by the effect of the deepest traps visited, which dominate the dynamics at all time scales. In agreement with this picture, the BTM has an anomalously slow evolution. In fact, the *subdiffusivity* displayed by the BTM means that the typical displacement of X at time t is $o(t^{1/2})$.

Other interesting properties of the BTM can be found by analyzing some *two-time* functions as $f(t_w, t) := \mathbb{P}(X_{t_w+t} = X_{t_w})$. We should think that we are letting that the system *ages* for a time t_w , and then we measure the probability that the system is at the “initial position” X_{t_w} after a time t . In equilibrium, one would expect that $f(t_w, t)$ depends only on t (i.e., the aging has no effect on the measurement). Nevertheless, for the BTM on a strongly inhomogeneous environment, $f(t_w, t)$ depends on the ratio t/t_w rather than on t . More precisely, as shown in Fontes et al. (2002), we have that

$$\lim_{\substack{t_w \rightarrow \infty \\ t/t_w \rightarrow \theta}} f(t_w, t) = F(\theta),$$

where F is a non-trivial function of θ . This phenomenon is usually referred as *aging* and is yet another consequence of the ever slower dynamics of the BTM.

The standard assumption to ensure an inhomogeneous environment is that the law of τ_0 is in the domain of attraction of an α -stable law, for some $\alpha \in (0, 1)$. For the sake of simplicity, in this article we will make the stronger assumption that there exists $\alpha \in (0, 1)$ such that

$$\lim_{u \rightarrow \infty} u^\alpha \mathbb{P}(\tau_x \geq u) = 1.$$

A basic question of the model is to describe its transition kernel. In Bertin and Bouchaud (2003), it is predicted that the anomalously slow dynamics should be reflected on a non-Gaussian diffusion front. Moreover, they conjectured that the transition kernel has a decay given by a stretched exponential. More precisely, they claimed that $\mathbb{P}(|X_t| \geq x)$ should behave as $C \exp(-c(\frac{x}{t^\gamma})^{1+\alpha})$ for some positive constants C, c , where $\gamma = \alpha/(1+\alpha)$. Their prediction was supported by numerical simulations and non-rigorous arguments. In this article we confirm the predicted behavior by providing the sub-Gaussian lower bound for the transition kernel. The corresponding upper bound was previously obtained in Černý (2006).

The main result of this article is the following theorem.

Theorem 1. *There exist positive constants C_1, c_1, C_2, c_2 and ε_1 such that*

$$C_1 \exp\left(-c_1 \left(\frac{x}{t^\gamma}\right)^{1+\alpha}\right) \leq \mathbb{P}(|X_t| \geq x) \leq C_2 \exp\left(-c_2 \left(\frac{x}{t^\gamma}\right)^{1+\alpha}\right)$$

for all $t \geq 0$ and $x \geq 0$ such that $\frac{x}{\varepsilon_1} \leq t$.

As we have previously stated, the upper bound in Theorem 1 has been already obtained in Černý (2006), Lemma 3.2.

The proof that we will present for the lower bound relies heavily on the fact that $(X_t)_{t \geq 0}$ has a clearly identified scaling limit. This scaling limit is the *Fontes–Isopi–Newman singular diffusion* (FIN) $(Z_t)_{t \geq 0}$. It was discovered by Fontes, Isopi and Newman in Fontes et al. (2002) and it should be thought of as a Brownian motion moving among a random environment of traps. More accurately, the FIN singular diffusion Z is obtained as a speed-measure change of a Brownian motion through a random, purely atomic measure ρ , where ρ is the Stieltjes measure associated to an α -stable subordinator.

It is a known fact that α -stable subordinators are pure-jump processes. Hence, we can write

$$\rho = \sum_{i \in \mathbb{N}} v_i \delta_{x_i}. \quad (1.1)$$

Each atom $v_i \delta_{x_i}$ of ρ plays the role of a trap located at x_i and whose *depth* is v_i . It can be shown that $\{x_i : i \in \mathbb{N}\}$ is dense in \mathbb{R} . Hence, the FIN diffusion is a Brownian motion moving among a random, dense set of traps $v_i \delta_{x_i}$, $i \in \mathbb{N}$.

As we have said, the FIN singular diffusion is the scaling limit of the one-dimensional Bouchaud trap model, more precisely, as proved in Fontes et al. (2002), Theorem 4.1, we have that

$$(\varepsilon^{-1} X_{\varepsilon^{1/\gamma} t})_{t \geq 0} \xrightarrow{d} (Z_t)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0,$$

where \xrightarrow{d} denotes convergence in distribution. Note that the scaling in the display above is sub-diffusive, in accordance with the subdiffusivity of the BTM.

The techniques used to prove Theorem 1 can also be applied to establish the sub-Gaussian behavior of the transition kernel for the FIN singular diffusion, that is, we will prove the following theorem.

Theorem 2. *There exist positive constants C_3, c_3, C_4 and c_4 such that*

$$C_3 \exp\left(-c_3 \left(\frac{x}{t^\gamma}\right)^{1+\alpha}\right) \leq \mathbb{P}(|Z_t| \geq x) \leq C_4 \exp\left(-c_4 \left(\frac{x}{t^\gamma}\right)^{1+\alpha}\right)$$

for all $t \geq 0$ and $x \geq 0$.

Again, the upper bound of Theorem 2 was obtained in Černý (2006), Corollary 3.3.

At this point we would like to give some references concerning the BTM. The model was introduced in Bouchaud (1992) as a toy model to study metastability on some complex system such as spin glasses. In this case the state space of the walk was a large complete graph. Each vertex x of the complete graph represented a metastable state of the system, and the depths were chosen as $\tau_x := \exp(-\beta E_x)$,

where E_x represented the energy barrier that the system must overcome to leave that metastable state and β is the inverse temperature. When the state space is \mathbb{Z}^d , $d \geq 2$, the BTM has a behavior completely different from the one-dimensional case, as shown in Ben Arous and Černý (2007), and by Ben Arous, Černý and Mountford in Ben Arous et al. (2006) (see also Mourrat (2011)). In these papers, it is shown that the scaling limit of the BTM on \mathbb{Z}^d ($d \geq 2$) is the *fractional kinetics process* (FK), which is a time-change of a d -dimensional Brownian motion through the inverse of an α -stable subordinator. In Ben Arous et al. (2003a) and Ben Arous et al. (2003b) Ben Arous, Bovier and Gayraud obtained aging properties of the model on the complete graph. For an spectral characterization of aging see Bovier and Faggionato (2005). More on the one-dimensional BTM can be found in Ben Arous and Černý (2005), where it was shown that the model displays several aging regimes. The hydrodynamic behavior of one-dimensional BTM can be found in Jara et al. (2011). The response of the one-dimensional BTM to a drift is the subject of Zindy (2009). In Gantert et al. (2010) a drift which decays to 0 is introduced, and a phase transition in terms of the speed of decay of the drift is identified (see also Cabezas (2010)). A study of the BTM on a wider class of graphs can be found on Ben Arous and Černý (2008). For a general account on the mathematical study of the Bouchaud trap model and the FIN singular diffusion, we refer to Ben Arous and Černý (2006).

The organization of the paper is as follows: In Section 2, we give some preliminaries and we briefly sketch the main ideas of the proofs. The proofs of the main results are the subject of Section 3. Section 3.1 contains the proof of Theorem 2 and Section 3.2 deals with the proof of Theorem 1.

In this article, C and c will represent positive constants which might change their values from line to line.

2 Preliminaries and strategy of the proof

2.1 Preliminaries

Here we will give the precise definition of the FIN singular diffusion and we will state the Ray–Knight Theorem.

To define the FIN singular diffusion, first we recall the definition of a speed-measure changed Brownian motion. Let $(B_t)_{t \geq 0}$ be a standard one-dimensional Brownian motion starting at zero and $l(t, x)$ be a jointly continuous version of its local time. Given any locally finite measure μ on \mathbb{R} , denote

$$\phi_\mu(s) := \int_{\mathbb{R}} l(s, y) \mu(dy),$$

and its right-continuous generalized inverse by

$$\psi_\mu(t) := \inf\{s > 0 : \phi_\mu(s) > t\}.$$

The speed-measure change of B with speed-measure μ , $(B[\mu]_t)_{t \geq 0}$, is defined as

$$B[\mu]_t := B_{\psi_{\mu}(t)}. \tag{2.1}$$

Now, we proceed to define the random measure appearing on the definition of the FIN singular diffusion. Let $(V_x)_{x \in \mathbb{R}}$ be a two sided, α -stable subordinator independent of B . Let ρ be the Lebesgue–Stieltjes measure associated with V , that is $\rho(a, b] := V_b - V_a$. The process $(Z_t)_{t \geq 0}$ defined as $Z_s := B[\rho]_s$ is the FIN singular diffusion.

Observation 3. The measure ρ has scale invariance in the sense that $\lambda^{-1/\alpha} \rho(0, \lambda)$ is distributed as $\rho(0, 1)$ for all $\lambda > 0$. Also, the Brownian motion B is scale invariant in the sense that $(\lambda^{-1/2} B_{\lambda t})_{t \geq 0}$ is distributed as $(B_t)_{t \geq 0}$. Those two facts imply that Z is scale invariant in the sense that $(\lambda^{-\gamma} Z_{\lambda t})_{t \geq 0}$ is distributed as $(Z_t)_{t \geq 0}$ for all $\lambda > 0$. This fact reflects that the FIN singular diffusion has a subdiffusive behavior.

Next, we will recall the main tool of our proof, that is, the Ray–Knight description of the local time of the Brownian motion. Recall that B is a standard one-dimensional Brownian motion started at the origin and $l(t, x)$ is its local time. For any $b \in \mathbb{R}$ let $\tau_b := \inf\{t \geq 0 : B_t = b\}$. The Ray–Knight Theorem (Ray (1963) and Knight (1963)) states the following.

Proposition 4 (Ray–Knight). *For each $a > 0$, the process $(l(\tau_{-a}, x) : x \geq -a)$ is Markovian. Moreover, $(l(\tau_{-a}, x) : -a \leq x \leq 0)$ is distributed as a squared Bessel process of dimension $d = 2$ started at 0 ($l(\tau_{-a}, -a) = 0$). Further, $(l(\tau_{-a}, x) : x \geq 0)$ is distributed as a squared Bessel process of dimension $d = 0$ started at $l(\tau_{-a}, 0)$ and killed at 0.*

2.2 Strategy of the proof

In what follows, we will sketch the proof of Theorem 2. The proof of Theorem 1 follows the same line of reasoning but the technical details are slightly more complicated.

The main step in the proof of Theorem 2 is to find a lower bound for $\mathbb{P}(\min\{Z_s : s \in [0, t]\} \leq -x)$. That is equivalent to find a lower bound for $\mathbb{P}(H_{-x} \leq t)$, where, for any $b \in \mathbb{R}$, $H_b := \inf\{t \geq 0 : Z_t = b\}$. The idea of the proof is to split the interval $[-x, 0)$ into n sub-intervals $[-x \frac{i}{n}, -x \frac{(i-1)}{n})$, $i = 1, \dots, n$. Let Δ_i be the time spent in $[-x \frac{i}{n}, -x \frac{(i-1)}{n})$ up to H_{-x} . Then we consider the events:

$$O_i := \left\{ \Delta_i \leq \frac{t}{n} \right\}, \quad i = 1, \dots, n. \tag{2.2}$$

Clearly, we have that the intersection of those events implies that the time spent by Z in the negative axis before hitting $-x$ is less than t . If we ignore for the moment the problem of controlling the time spent in the positive axis, we just have

to control the probability of the intersection $\bigcap_{i=1}^n O_i$. At this point, we would like to use the product rule to compute the probability of such intersection. Unfortunately, we are dealing with events which are not independent, and we cannot use the product rule. Nevertheless, one can proceed by using a similar strategy, and obtain independence by virtue of the Ray–Knight Theorem.

First, note that, recalling the definition of the FIN singular diffusion, we have that

$$\Delta_i = \int_{[-xi/n, -x(i-1)/n]} l(\tau_{-x}, u) \rho(du), \quad (2.3)$$

where ρ is the measure appearing in the definition of the FIN singular diffusion and $l(\cdot, \cdot)$ is the Brownian local time.

We start by dealing with the dependence between O_1 and $(O_i)_{i=2, \dots, n}$. Since ρ is defined as the Lebesgue–Stieltjes measure associated to a Lévy process (more precisely, an α -stable subordinator), we have that ρ has independent increments, in the sense that, for any sequence $(J_i)_{i \in \mathbb{N}}$ of disjoint Borel subsets of \mathbb{R} , we have that the family of random variables $(\rho(J_i))_{i \in \mathbb{N}}$ is independent. Moreover, by the Ray–Knight Theorem, we know that $(l(\tau_{-x}, u))_{u \geq 0}$ is a Markov process. The key step in our reasoning is that, by virtue of the independence of increments of ρ and the Markovianity of $l(\tau_{-x}, u)$, we have that Δ_1 depends of Δ_i , $i = 2, \dots, n$ only through the value of $l(\tau_{-x}, -x/n)$. Hence, O_1 depends of O_i , $i = 1, \dots, n$ only through $l(\tau_{-x}, -x/n)$. As we will see in Section 3, it is not hard to deal with that kind dependence.

The same argument can be applied to deal with the dependence between O_j and O_i , $i = j + 1, \dots, n$, for all $j = 1, \dots, n - 1$. In this way we will obtain a formula for the probability of the intersection $\bigcap_{i=1}^n O_i$ which will play the role of the product rule.

Finally, thanks to the scale invariance of the FIN singular diffusion one can choose the number of sub-intervals n in such a way that the probability of the events O_i is a constant c which does not depend on x , t or n . The proper choice of n will be $n = (\frac{x}{t^\gamma})^{1+\alpha}$. Hence, by using our product rule-type formula, we will get that the probability of $\{\min\{Z_s : s \in [0, t]\} \leq -x\}$ will be bounded below by $c^{(x/t^\gamma)^{1+\alpha}}$ and that is the type of result that we want.

3 Proofs of Theorems 1 and 2

3.1 Proof of Theorem 2

We will prove only the lower bound, since the upper bound is proved in Černý (2006), Lemma 3.2. Thanks to the scale invariance of Z (see Observation 3), to prove the lower bound in Theorem 2 it is enough to show that there are positive constants C , $c > 0$ such that

$$\mathbb{P}(|Z_1| \geq x) \geq C \exp(-cx^{1+\alpha}) \quad \text{for all } x \geq 0.$$

Moreover, both $\mathbb{P}(|Z_1| \geq x)$ and $C \exp(-cx^{1+\alpha})$ are decreasing in x . Hence it will suffice to show that there are positive constants $C, c > 0$ such that

$$\mathbb{P}(|Z_1| \geq a_n) \geq C \exp(-cn) \quad \text{for all } n \in \mathbb{N},$$

where $a_n := n^{1/(1+\alpha)}$. The main step in the proof of Theorem 2 is the following lemma:

Lemma 5. *There exist positive constants $C, c > 0$ such that*

$$\mathbb{P}(\min\{Z_s : s \in [0, 1]\} \leq -a_n) \geq C \exp(-cn) \quad \text{for all } n \in \mathbb{N}.$$

Proof. Note that the event $\{\min\{Z_s : s \in [0, 1]\} \leq -a_n\}$ is equivalent to $\{H_{-a_n} \leq 1\}$. We recall that ρ is the random measure appearing in display (1.1), B is the Brownian motion used in the construction of Z and $l(\cdot, \cdot)$ is its local time. We have that

$$H_{-a_n} = \int_{-a_n}^{\infty} l(\tau_{-a_n}, u) \rho(du).$$

We split $[-a_n, 0)$ into n subintervals $I_i := [-in^{-\gamma}, -(i-1)n^{-\gamma})$, $i = 1, \dots, n$ (recall that $\gamma = \alpha/(1+\alpha)$). Let us define the events A_i , $i = 1, \dots, n-1$ as

$$A_i := \left\{ \int_{I_i} l(\tau_{-a_n}, u) \rho(du) \leq \frac{1}{n} \right\}.$$

Also define

$$A_n := \left\{ \int_{I_n} l(\tau_{-a_n}, u) \rho(du) \leq \frac{1}{2n} \right\}$$

and

$$A_0 := \left\{ \int_{[0, \infty)} l(\tau_{-a_n}, u) \rho(du) \leq \frac{1}{2n} \right\}.$$

Clearly

$$\bigcap_{i=0}^n A_i \subset \{H_{-a_n} \leq 1\}. \quad (3.1)$$

At this point we would like to use the product rule on the display above to obtain a lower bound for $\mathbb{P}(H_{-a_n} \leq 1)$, but the events A_i , $i = 0, \dots, n$ are not independent. Nevertheless, we will harness the Markovianity of $l(\tau_{-a_n}, u)$ (guaranteed by the Ray–Knight Theorem) and the independence of increments of ρ to obtain a formula for $\mathbb{P}(\bigcap_{i=0}^n A_i)$ which will play the role of a product rule. We will start by dealing with the dependence between A_0 and $\bigcap_{i=1}^n A_i$.

As we have explained in Section 2.2, the measure ρ has independent increments. The key observation here is that, by virtue of the independence of increments of

ρ and by the Markovianity of $(l(\tau_{-a_n}, u))_{u \geq -a_n}$ we have that the random variable $\int_{[0, \infty)} l(\tau_{-a_n}, u) \rho(du)$ depends of the random variables $\int_{I_i} l(\tau_{-a_n}, u) \rho(du)$, $i = 1, \dots, n$ only through the value of $l(\tau_{-a_n}, 0)$. In particular, we have that A_0 depends of A_i , $i = 1, \dots, n$ only through the value of $l(\tau_{-a_n}, 0)$.

To get rid of that dependence we will condition on the event that $l(\tau_{-a_n}, 0)$ is below a certain fixed level $n^{-\gamma}$. That is, we introduce the event

$$L_1 := \{l(\tau_{-a_n}, 0) \leq n^{-\gamma}\}.$$

The strategy will be that, when conditioned on the event L_1 , the event A_0 is stochastically dominated by $\{\int_{[0, \infty)} Y_u \rho(du) \leq \frac{1}{2n}\}$, where $(Y_u)_{u \geq 0}$ is a squared Bessel process of dimension $d = 0$ started at $n^{-\gamma}$ ($Y_0 = n^{-\gamma}$) and independent of ρ . In this way, we will overcome the dependence between A_0 and $l(\tau_{-a_n}, 0)$.

We start by writing

$$\mathbb{P}\left(\bigcap_{i=0}^n A_i\right) \geq \mathbb{P}\left(A_0 \cap L_1 \bigcap_{i=1}^n A_i\right) = \mathbb{P}\left(A_0 \mid L_1 \bigcap_{i=1}^n A_i\right) \cdot \mathbb{P}\left(L_1 \bigcap_{i=1}^n A_i\right). \quad (3.2)$$

On the other hand, as we have said, the Markovianity of $l(\tau_{-a_n}, \cdot)$ and the independence of increments of ρ imply that A_0 depends of $L_1 \bigcap_{i=1}^n A_i$ only through the value of $l(\tau_{-a_n}, 0)$. Hence, we can write

$$\mathbb{P}\left(A_0 \mid L_1 \bigcap_{i=1}^n A_i\right) = \int_0^{n^{-\gamma}} \mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = y) \nu^0(dy), \quad (3.3)$$

where ν^0 is the distribution of $l(\tau_{-a_n}, 0)$ conditioned on $L_1 \bigcap_{i=1}^n A_i$.

Let us make a technical comment about the meaning of the conditional probabilities $\mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = y)$ appearing in (3.3). One can use any regular version of the conditional probabilities of $(l(\tau_{-a_n}, u))_{u \geq 0}$ given $l(\tau_{-a_n}, 0)$ to define the function $y \mapsto \mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = y)$ up to a set of ν^0 -measure zero. Nevertheless, it will be convenient to define $\mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = y)$ for all $y \geq 0$. To do so, we use the fact that $(l(\tau_{-a_n}, u))_{u \geq 0}$ is a squared Bessel process, and Bessel processes can be defined as starting from any $y \geq 0$. Hence, for all $y \geq 0$ we define $\mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = y)$ using the law of a 0-dimensional squared Bessel process started at y .

Now, we will show that, for all $y \in [0, n^{-\gamma}]$, we have that

$$\mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = y) \geq \mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = n^{-\gamma}). \quad (3.4)$$

For all $y \in [0, n^{-\gamma}]$, we can couple two 0-dimensional squared Bessel processes, $(X_t^1)_{t \geq 0}$, $(X_t^2)_{t \geq 0}$ with $X_0^1 = y$, $X_0^2 = n^{-\gamma}$ and such that $\mathbb{P}(X_s^1 \leq X_s^2 \text{ for all } s \geq 0) = 1$. In particular, we have that for any measure μ on \mathbb{R}

$$\int_0^\infty X_u^1 \mu(du) \leq \int_0^\infty X_u^2 \mu(du).$$

Recalling the definition of the event A_0 , we see that the previous display implies (3.4). On the other hand, displays (3.4) plus (3.3) imply that

$$\mathbb{P}\left(A_0 \mid L_1 \bigcap_{i=1}^n A_i\right) \geq \mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = n^{-\gamma}).$$

Hence, using (3.2) we get that

$$\mathbb{P}\left(\bigcap_{i=0}^n A_i\right) \geq \mathbb{P}\left(L_1 \bigcap_{i=1}^n A_i\right) \mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = n^{-\gamma}). \quad (3.5)$$

In this way we have been able to deal with the dependence between A_0 and A_i , $i = 1, \dots, n$.

The next step is to get rid of the dependence between A_1 and A_i , $i = 2, \dots, n$. We will proceed analogously. Let

$$L_2 := \{l(\tau_{-a_n}, -n^{-\gamma}) \leq n^{-\gamma}\}.$$

We have that

$$\mathbb{P}\left(L_1 \bigcap_{i=1}^n A_i\right) \geq \mathbb{P}\left(A_1 \cap L_1 \cap L_2 \bigcap_{i=2}^n A_i\right),$$

which, in turn, equals

$$\mathbb{P}\left(A_1 \cap L_1 \mid L_2 \bigcap_{i=2}^n A_i\right) \cdot \mathbb{P}\left(L_2 \bigcap_{i=2}^n A_i\right).$$

Again, by the Markovianity of $l(\tau_{-a_n}, \cdot)$ and the independence of increments of ρ , we have that $A_1 \cap L_1$ depends on $L_2 \bigcap_{i=2}^n A_i$ only through the value of $l(\tau_{-a_n}, -n^{-\gamma})$. That is

$$\mathbb{P}\left(A_1 \cap L_1 \mid L_2 \bigcap_{i=2}^n A_i\right) = \int_0^{n^{-\gamma}} \mathbb{P}(A_1 \cap L_1 \mid l(\tau_{-a_n}, -n^{-\gamma}) = y) \nu^1(dy),$$

where ν^1 is the distribution of $l(\tau_{-a_n}, -n^{-\gamma})$ conditioned on $L_2 \bigcap_{i=2}^n A_i$. We can use the same type of coupling that we have used to obtain (3.4) to see that

$$\mathbb{P}(A_1 \cap L_1 \mid l(\tau_{-a_n}, -n^{-\gamma}) = y) \geq \mathbb{P}(A_1 \cap L_1 \mid l(\tau_{-a_n}, -n^{-\gamma}) = n^{-\gamma})$$

for all $y \leq n^{-\gamma}$. Hence, the same reasoning used to obtain (3.5) gives us

$$\mathbb{P}\left(L_1 \bigcap_{i=1}^n A_i\right) \geq \mathbb{P}\left(L_2 \bigcap_{i=2}^n A_i\right) \mathbb{P}(A_1 \cap L_1 \mid l(\tau_{-a_n}, -n^{-\gamma}) = n^{-\gamma}), \quad (3.6)$$

which allows us to deal with the dependence between A_1 and A_i , $i = 2, \dots, n$.

By analogous arguments, we get that, for all $j = 1, \dots, n-1$

$$\mathbb{P}\left(L_j \bigcap_{i=j}^n A_i\right) \geq \mathbb{P}\left(L_{j+1} \bigcap_{i=j+1}^n A_i\right) \mathbb{P}(A_j \cap L_j | l(\tau_{-a_n}, -jn^{-\gamma}) = n^{-\gamma}), \quad (3.7)$$

where

$$L_j := \{l(\tau_{-a_n}, -(j-1)n^{-\gamma}) \leq n^{-\gamma}\}.$$

Putting together displays (3.5), (3.6) and (3.7), we get that

$$\begin{aligned} \mathbb{P}\left(\bigcap_{i=0}^n A_i\right) &\geq \mathbb{P}(A_n \cap L_n) \mathbb{P}(A_0 | l(\tau_{-a_n}, 0) = n^{-\gamma}) \\ &\times \prod_{i=1}^{n-1} \mathbb{P}(A_i \cap L_i | l(\tau_{-a_n}, -in^{-\gamma}) = n^{-\gamma}). \end{aligned} \quad (3.8)$$

The formula above is the sort of product rule we were after.

Next, note that using the spatial homogeneity of the random measure ρ , display (3.8) yields

$$\begin{aligned} \mathbb{P}\left(\bigcap_{i=0}^n A_i\right) &\geq \mathbb{P}(A_n \cap L_n) \mathbb{P}(A_0 | l(\tau_{-a_n}, 0) = n^{-\gamma}) \\ &\times (\mathbb{P}(A_1 \cap L_1 | l(\tau_{-a_n}, -n^{-\gamma}) = n^{-\gamma}))^{n-1}. \end{aligned} \quad (3.9)$$

Lemma 5 will follow after we have showed that the quantities appearing in the display above are independent of n and positive. The key at this point is the scale invariance of the FIN diffusion together with our choice of a_n .

First, we will deal with $\mathbb{P}(A_1 \cap L_1 | l(\tau_{-a_n}, -n^{-\gamma}) = n^{-\gamma})$. By definition, we have that $\mathbb{P}(A_1 \cap L_1 | l(\tau_{-a_n}, -n^{-\gamma}) = n^{-\gamma})$ equals

$$\mathbb{P}\left(\int_{-n^{-\gamma}}^0 l(\tau_{-a_n}, u) \rho(du) \leq \frac{1}{n}; l(\tau_{-a_n}, 0) \leq n^{-\gamma} \mid l(\tau_{-a_n}, -n^{-\gamma}) = n^{-\gamma}\right).$$

Recall that, if $(Y_t)_{t \geq 0}$ is a squared Bessel process started at some $a \geq 0$, then, for each $\lambda \geq 0$, $(\lambda Y_{\lambda^{-1}t})_{t \geq 0}$ is a squared Bessel process started at λa . Hence, by choosing $\lambda = n^{-\gamma}$, we have that the last display equals

$$\mathbb{P}\left(\int_{-n^{-\gamma}}^0 n^{-\gamma} l(\tau_{-a_n}, un^\gamma) \rho(du) \leq \frac{1}{n}; l(\tau_{-a_n}, 0) \leq 1 \mid l(\tau_{-a_n}, -1) = 1\right).$$

Using the scale invariance of the measure ρ we obtain that the previous display equals

$$\mathbb{P}\left(\int_{-n^{-\gamma}}^0 l(\tau_{-a_n}, un^\gamma) \rho(n^{-\gamma} du) \leq 1; l(\tau_{-a_n}, 0) \leq 1 \mid l(\tau_{-a_n}, -1) = 1\right).$$

Finally, let us perform a change of variables inside the integral in the display above to get

$$\mathbb{P}\left(\int_{-1}^0 l(\tau_{-a_n}, u)\rho(du) \leq 1; l(\tau_{-a_n}, 0) \leq 1 \mid l(\tau_{-a_n}, -1) = 1\right). \quad (3.10)$$

Hence $\mathbb{P}(A_1 \cap L_1 \mid l(\tau_{-a_n}, -n^{-\gamma}) = n^{-\gamma})$ is equal to (3.10), which clearly does not depend on n .

Similar arguments can be used to show that

$$\mathbb{P}(A_n \cap L_n) = \mathbb{P}\left(\int_{-1}^0 l(\tau_{-1}, u)\rho(du) \leq \frac{1}{2}; l(\tau_{-1}, 0) \leq 1\right) \quad (3.11)$$

and

$$\mathbb{P}(A_0 \mid l(\tau_{-a_n}, 0) = n^{-\gamma}) = \mathbb{P}\left(\int_0^\infty l(\tau_{-a_n}, u)\rho(du) \leq \frac{1}{2} \mid l(\tau_{-a_n}, 0) = 1\right), \quad (3.12)$$

which shows that the last two expressions are independent of n .

The core of the proof has been completed. It only remains to show the trivial fact that displays (3.10), (3.11) and (3.12) are non-zero. Using the independence between the Brownian motion B and the random measure ρ , we have that display (3.10) is bigger or equal than

$$\mathbb{P}\left(\sup_{u \in [-1, 0]} l(\tau_{-a_n}, u) \leq \delta^{-1}; l(\tau_{-a_n}, 0) \leq 1 \mid l(\tau_{-a_n}, -1) = 1\right) \cdot \mathbb{P}(\rho(-1, 0] \leq \delta)$$

for any $\delta \geq 0$. We can see that the above display is positive for $\delta < 1$ by noticing that the first factor in the product above is positive for $\delta < 1$ (this follows from standard properties of Bessel processes) and, on the other hand, we have that $\mathbb{P}(\rho[-1, 0] \leq \delta) > 0$ for each $\delta > 0$. An analogous argument gives that the quantity appearing in (3.11) is positive. To see the positivity of the quantity appearing in (3.12), we need also to use the fact that, for any $t > 0$, a 0-dimensional Bessel process hits the origin before time t with positive probability. \square

We have managed to show Lemma 5, which provides the sub-Gaussian lower bound for $\mathbb{P}(\min\{Z_s : s \in [0, 1]\} \geq -a_n)$. To prove Lemma 8, we need to obtain the same type of bound for Z_1 instead of $\min\{Z_s : s \in [0, 1]\}$. The standard argument of applying symmetry of the process after the stopping time H_{-a_n} to obtain $\mathbb{P}(Z_1 \leq -a_n) = \frac{1}{2}\mathbb{P}(\min\{Z_s : s \in [0, 1]\} \leq -a_n)$ cannot be applied in our case, because the annealed FIN singular diffusion is not Markovian (in particular, the event $\{\min\{Z_s : s \in [0, 1]\} \leq -a_n\}$ should be positively correlated with the absence of very deep traps on $[-a_n, 0]$).

Instead, we will proceed as follows, Lemma 5 guarantees that the probability that Z reaches $-a_n$ before time 1 is at least c^{-n} . From Lemma 6 below, it will follow that, after hitting $-a_n$, the probability that Z remains “to the left” of $-a_n$ for a time larger than 1 is also larger than c^{-n} . In the argument we follow the reasoning used in Černý (2006), Lemma 3.2.

Lemma 6. *There exist positive constants $C, c > 0$ such that*

$$\mathbb{P}\left(\int_{-a_n}^0 l(\tau_{-a_n}, u)\rho(du) \geq 1; Z_s \leq n^{-\gamma} \text{ for all } s \leq H_{-a_n}\right) \geq C \exp(-cn)$$

for all $n \in \mathbb{N}$.

Proof. We divide the interval $[-a_n, 0)$ into n subintervals

$$I_i := [-in^{-\gamma}, -(i-1)n^{-\gamma}), \quad i = 1, \dots, n$$

(recall that $\gamma = \alpha/(1 + \alpha)$). For $i = 1, \dots, n$ we define

$$T_i := \int_{I_i} (l(\tau_{-in^{-\gamma}}, u) - l(\tau_{-(i-1)n^{-\gamma}}, u))\rho(du).$$

That is, T_i is the time spent by Z on the interval I_i between times $H_{-(i-1)n^{-\gamma}}$ and $H_{-in^{-\gamma}}$. Also define, for $i = 1, \dots, n$

$$\theta_i := \min\{s \geq H_{-(i-1)n^{-\gamma}} : Z_s \in \{-in^{-\gamma}, -(i-2)n^{-\gamma}\}\}$$

and

$$D_i := \left\{T_i \geq \frac{1}{n}; Z_{\theta_i} = -in^{-\gamma}\right\}.$$

That is, D_i is the event that the time spent by Z on the interval I_i between times $H_{-(i-1)n^{-\gamma}}$ and $H_{-in^{-\gamma}}$ is greater than $1/n$ and that, after reaching $-(i-1)n^{-\gamma}$, Z exits the interval I_i through $-in^{-\gamma}$. We have that

$$\bigcap_{i=1}^n D_i \subset \left\{\int_{-a_n}^0 l(\tau_{-a_n}, u)\rho(du) \geq 1; Z_s \leq n^{-\gamma} \text{ for all } s \leq H_{-a_n}\right\}.$$

Observe that the event D_i depends on the realization of ρ only in the interval I_i . Also, the family of intervals $I_i, i = 1, \dots, n$ is disjoint. That, plus the independence of increments of the measure ρ and the strong Markov property of the Brownian motion (applied at the stopping times $\tau_{-in^{-\gamma}}, i = 1, \dots, n-1$) implies that the events $D_i, i = 1, \dots, n$ are independent. Hence, we have that

$$\prod_{i=1}^n \mathbb{P}(D_i) \leq \mathbb{P}\left(\int_{-a_n}^0 l(\tau_{-a_n}, u)\rho(du) \geq 1; Z_s \leq n^{-\gamma} \forall s \leq H_{-a_n}\right). \quad (3.13)$$

On the other hand, using the scale invariance of the measure ρ and the scale invariance of the Brownian motion B , we can show that

$$\mathbb{P}(D_i) = \mathbb{P}\left(\int_{-1}^0 l(\tau_{-1}, u)\rho(du) \geq 1; H_1 \geq H_{-1}\right).$$

That is, the probability of D_i does not depend on n (and is greater than 0). That, plus display (3.13), proves Lemma 6. \square

Now we use Lemmas 5 and 6 to prove Theorem 2. We define

$$\begin{aligned} G_1 &:= \{H_{-(n+1)n^{-\gamma}} \leq 1\}, \\ I^* &:= [-(2n+1)n^{-\gamma}, -(n+1)n^{-\gamma}), \\ T &:= \int_{I^*} (l(\tau_{-(2n+1)n^{-\gamma}}, u) - l(\tau_{-(n+1)n^{-\gamma}}, u)) \rho(du) \quad \text{and} \\ G_2 &:= \{T \geq 1; Z_s \leq -a_n \text{ for all } s \in [H_{-(n+1)n^{-\gamma}}, H_{-(2n+1)n^{-\gamma}}]\}. \end{aligned}$$

In the event G_1 , we have that Z reaches $-(n+1)n^{-\gamma}$ before time $t = 1$. On the other hand, in the event G_2 we have that, after reaching $-(n+1)n^{-\gamma}$, the process Z remains inside the interval $[-(2n+1)n^{-\gamma}, -n^{\alpha/(1+\alpha)}]$ for at least a unit of time. Hence, in the event $G_1 \cap G_2$, we have that $Z_1 \in [-(2n+1)n^{-\gamma}, -n^{\alpha/(1+\alpha)}]$, in particular

$$G_1 \cap G_2 \subset \{Z_1 \leq -a_n\}.$$

We know, by Lemma 5, that $\mathbb{P}(H_{-a_n} \leq 1) \geq C_1 \exp(-c_1 n)$, for suitable $C_1, c_1 > 0$. From that, it follows that there exist constants $C, c \geq 0$ such that $\mathbb{P}(G_1) \geq C \exp(-cn)$ (to show that, take $M \in \mathbb{N}$ such that $(Mn)^{1/(1+\alpha)} \geq (n+1)n^{-\gamma}$ for all $n \in \mathbb{N}$ and then apply Lemma 5 at Mn). Also, Lemma 6 (plus the strong Markov property of the Brownian motion B applied at time $\tau_{-(n+1)n^{-\gamma}}$) implies that $\mathbb{P}(G_2) \geq C_2 \exp(-c_2)$ for suitable $C_2, c_2 > 0$. Now, observe that the event G_1 depends on the realization of ρ only on the interval $[-(n+1)n^{-\gamma}, \infty)$, whereas G_2 depends on the realization of ρ only on the interval I^* . Those intervals are disjoint, hence, by virtue of the independence of increments of the measure ρ and the strong Markov property of the Brownian motion B applied at the stopping time $\tau_{-(n+1)n^{-\gamma}}$, we deduce that the events G_1 and G_2 are independent. We have showed that

$$\mathbb{P}(|Z_1| \geq a_n) \geq CC_2 \exp(-(c+c_2)n),$$

and that proves Theorem 2.

3.2 Proof of Theorem 1

The strategy to prove Theorem 1 will be to mimic the arguments leading to Theorem 2 using the fact that the scaling limit of the one-dimensional BTM is the FIN singular diffusion. The main tool used in Fontes et al. (2002) to show that the BTM scales to the FIN singular diffusion is a coupling between different time scales of the BTM. We will make use of this coupling for the proof of Theorem 1, so we proceed to recall it.

To better understand the idea of the coupling, first we have to stress the fact that some discrete-space processes can be expressed as speed-measure changed Brownian motions (see Stone (1963)). In particular, we will express the scalings of the BTM, $(\varepsilon X_{\varepsilon^{-(1+\alpha)/\alpha_t}})_{t \geq 0}$, $\varepsilon > 0$ as speed-measure changed Brownian

motions, using a family of random, discrete measures ρ^ε , $\varepsilon > 0$ as their speed-measures. In fact, we will have that for each $\varepsilon > 0$, $(B[\rho^\varepsilon]_t)_{t \geq 0}$ will be distributed as $(\varepsilon X_{\varepsilon^{-(1+\alpha)/\alpha}t})_{t \geq 0}$, where B is a Brownian motion independent of the ρ^ε , $\varepsilon > 0$ and $B[\rho^\varepsilon]$ is as in (2.1).

On the other hand, the random measures ρ^ε , $\varepsilon > 0$ will be coupled in such a way that $\rho^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \rho$ almost surely, where ρ is the random measure appearing in the definition of the FIN singular diffusion. At this point, we recall that convergence of speed measure changed Brownian motions follows from the convergence of their respective speed measures. More precisely Stone (1963), Theorem 1, states that, if μ^ε is a family of measures on \mathbb{R} which converges to some measure μ as $\varepsilon \rightarrow 0$, then we have that $(B[\mu^\varepsilon]_t)_{t \geq 0}$ converges to $(B[\mu]_t)_{t \geq 0}$ as $\varepsilon \rightarrow 0$ in the Skorohod topology. Hence, we will have that the convergence of ρ^ε to ρ will imply the convergence of $B[\rho^\varepsilon]$ to $B[\rho]$, i.e., we will have that the FIN diffusion is the scaling limit of the BTM.

Next, we present the construction of such measures. Let $G : [0, \infty) \rightarrow [0, \infty)$ be the function defined by the relation

$$\mathbb{P}(V_1 > G(u)) := \mathbb{P}(\tau_0 > u),$$

where we recall from the introduction that $(V_x)_{x \in \mathbb{R}}$ is a two sided α -stable subordinator and τ_0 is the depth of the trap at $x = 0$. The function G is well defined since V_1 has a continuous distribution function. Moreover, G is non-decreasing and right-continuous. Thus, G has a right-continuous generalized inverse $G^{-1}(s) := \inf\{t : G(t) > s\}$. Now, for all $\varepsilon > 0$ and $z \in \mathbb{Z}$, we define the random variables τ_z^ε as

$$\tau_z^\varepsilon := G^{-1}(\varepsilon^{-1/\alpha} \rho(\varepsilon z, \varepsilon(z + 1))).$$

As V has independent and stationary increments, we have that $(\tau_z^\varepsilon)_{z \in \mathbb{Z}}$ is i.i.d. Moreover, the function G was defined in order to have that τ_0^ε is distributed according to τ_0 . For a proof of that fact, we refer to Fontes et al. (2002), Proposition 3.1.

Finally, we define the coupled family of random measures as

$$\rho^\varepsilon := \sum_{z \in \mathbb{Z}} \varepsilon^{1/\alpha} \tau_z^\varepsilon \delta_{\varepsilon z}$$

for all $\varepsilon > 0$.

Observation 7. Define, for each measure μ over \mathbb{R} and $r \in \mathbb{R}_+$, the following rescaling

$$\mu_r(\cdot) := r^{1/\alpha} \mu(r^{-1}\cdot). \tag{3.14}$$

Then $\rho_r^1 = \sum_{z \in \mathbb{Z}} r^{1/\alpha} \tau_z^1 \delta_{rz}$ and, since $(\tau_z^1)_{z \in \mathbb{Z}}$ is distributed as $(\tau_z^r)_{z \in \mathbb{Z}}$, we have that ρ_r^1 is distributed as ρ^r .

The next proposition states the two key properties of the coupled measures ρ^ε , $\varepsilon > 0$.

Proposition 8 (Fontes, Isopi, Newman). *For all $\varepsilon > 0$ the stochastic process $(\varepsilon X_{t\varepsilon^{-(1+\alpha)/\alpha}})_{t \geq 0}$ has the same distribution as $(B[\rho^\varepsilon]_t)_{t \geq 0}$. Moreover, we have that*

$$\rho^\varepsilon \xrightarrow{v} \rho \quad \mathbb{P}\text{-a.s. as } \varepsilon \rightarrow 0, \tag{3.15}$$

where \xrightarrow{v} denotes vague convergence of measures.

For the proof of this statement, we refer to [Fontes et al. \(2002\)](#), Proposition 3.1. Having recalled the coupling between different time scales of the BTM, we turn our attention to the proof of Theorem 1.

First, note that Proposition 8 implies, in particular, that $(X_t)_{t \geq 0}$ is distributed as $(B[\rho^1]_t)_{t \geq 0}$. Hence, we have that

$$\mathbb{P}(|X_t| \geq x) = \mathbb{P}(|B[\rho^1]_t| \geq x)$$

for all $x \geq 0$ and $t \geq 0$.

For any $b \in \mathbb{Z}$, we define $H_b^0 := \inf\{t \geq 0 : B[\rho^1]_t = b\}$. Theorem 1 will be deduced from the two following lemmas, which are the analogous of Lemmas 5 and 6, respectively.

Lemma 9. *There exist positive constants $C, c, \varepsilon > 0$ such that*

$$\mathbb{P}\left(\min\left\{X_s : s \in \left[0, \frac{m^{1/\gamma}}{n^{1/\alpha}}\right]\right\} \leq -m\right) \geq C \exp(-cn),$$

for all $m, n \in \mathbb{N}$ such that $n/m \leq \varepsilon$.

Lemma 10. *There exist positive constants $C, c, \varepsilon > 0$ such that*

$$\mathbb{P}\left(\int_{-m}^0 l(\tau_{-m}, u)\rho^1(du) \geq \frac{m^{1/\gamma}}{n^{1/\alpha}}; Z_s \leq \frac{m}{n} \text{ for all } s \leq H_{-m}^0\right) \geq C \exp(-cn),$$

for all $m, n \in \mathbb{N}$ such that $n/m \leq \varepsilon$.

First, we will prove Lemma 9 by imitating the proof of Lemma 5. Note that, since X is distributed as $B[\rho^1]$, we have that $\{\min\{X_s : s \in [0, \frac{m^{1/\gamma}}{n^{1/\alpha}}]\} \leq -m\}$ is equivalent to $\{H_{-m}^0 \leq \frac{m^{1/\gamma}}{n^{1/\alpha}}\}$. Next, define the events $A_i^0, i = 1, \dots, n - 1$ as

$$A_i^0 := \left\{ \int_{-mi/n}^{-m(i-1)/n} l(\tau_{-m}, u)\rho^1(du) \leq \left(\frac{m}{n}\right)^{1/\gamma} \right\}.$$

Also define

$$A_n^0 := \left\{ \int_{-m}^{-m(n-1)/n} l(\tau_{-m}, u)\rho^1(du) \leq \frac{1}{2} \left(\frac{m}{n}\right)^{1/\gamma} \right\}$$

and

$$A_0^0 := \left\{ \int_{\mathbb{R}_+} l(\tau_{-m}, u) \rho^1(du) \leq \frac{1}{2} \left(\frac{m}{n} \right)^{1/\gamma} \right\}.$$

Clearly, we have that

$$\bigcap_{i=0}^n A_i^0 \subset \left\{ H_{-m}^0 \leq \frac{m^{1/\gamma}}{n^{1/\alpha}} \right\}.$$

As in the proof of Lemma 5, we would like to apply the product rule to the intersection of events above, but those events are not independent. We will deal with the dependence by using the same reasoning as in the proof of Lemma 5. For $i = 1, \dots, n$, we define

$$L_i^0 := \{l(\tau_{-m}, -m(i-1)/n) \leq m/n\}.$$

The same argument leading to display (3.8) gives that

$$\begin{aligned} \mathbb{P}\left(H_{-m}^0 \leq \frac{m^{1/\gamma}}{n^{1/\alpha}}\right) &\geq \mathbb{P}(A_n^0 \cap L_n^0) \mathbb{P}(A_0^0 | l(\tau_{-m}, 0) = m/n) \\ &\quad \times \prod_{i=1}^{n-1} \mathbb{P}(A_i^0 \cap L_i^0 | l(\tau_{-m}, -im/n) = m/n). \end{aligned} \tag{3.16}$$

Having dealt with the dependence of the events A_i^0 , $i = 0, \dots, n$, it remains to show that the quantities above can be bounded from below, uniformly on i , n and m . In the proof of Lemma 5, the uniform lower bound for the probability of A_i , $i = 0, \dots, n$ follows from the scale invariance of the FIN singular diffusion. A technicality arises at this point, since the BTM is not scale invariant. Instead, we will use that, under appropriate scaling, the BTM converges to the FIN diffusion. Specifically, will rely on display (3.15).

Note that, for $i = 1, \dots, n-1$, $\mathbb{P}(A_i^0 \cap L_i^0 | l(\tau_{-m}, -im/n) = m/n)$ equals

$$\begin{aligned} \mathbb{P}\left(\int_{-m/n}^0 l(\tau_{-m}, u) \rho^1(du - m(i-1)/n) \leq \left(\frac{m}{n}\right)^{1/\gamma} ; l(\tau_{-m}, 0) \leq \frac{m}{n} \mid \right. \\ \left. l(\tau_{-m}, -m/n) = \frac{m}{n}\right). \end{aligned} \tag{3.17}$$

The measure ρ^1 is translation invariant in the sense that $\rho^1(\cdot - k)$ is distributed as ρ^1 for any $k \in \mathbb{Z}$. Hence, we can replace $\rho^1(du - m\frac{i-1}{n})$ by $\rho^{(i)}(du) := \rho^1(du - (m\frac{i-1}{n} - \lfloor m\frac{i-1}{n} \rfloor))$ on display (3.17). Moreover, performing a change of variables inside the integral (and recalling the notation introduced in (3.14)), we obtain that display (3.17) equals

$$\begin{aligned} \mathbb{P}\left(\int_{-1}^0 l(\tau_{-m}, um/n) \left(\frac{m}{n}\right)^{1/\alpha} \rho_{n/m}^{(i)}(du) \leq \left(\frac{m}{n}\right)^{1/\gamma} ; l(\tau_{-m}, 0) \leq \frac{m}{n} \mid \right. \\ \left. l(\tau_{-m}, -m/n) = \frac{m}{n}\right). \end{aligned}$$

Using the scale invariance of the local time, we obtain that the display above is equal to

$$\mathbb{P}\left(\int_{-1}^0 l(\tau_{-m}, u) \rho_{n/m}^{(i)}(du) \leq 1; l(\tau_{-m}, 0) \leq 1 \mid l(\tau_{-m}, -1) = 1\right).$$

By the independence between the random measure ρ^1 and the Brownian motion B , we can see that the previous display is at least

$$\begin{aligned} &\mathbb{P}\left(\sup_{u \in [-1, 0]} l(\tau_{-m}, u) \leq \delta^{-1}; l(\tau_{-m}, 0) \leq 1 \mid l(\tau_{-m}, -1) = 1\right) \\ &\cdot \mathbb{P}(\rho_{n/m}^{(i)}(-1, 0] \leq \delta) \end{aligned} \tag{3.18}$$

for any $\delta > 0$.

We will show that the display above is uniformly bounded away from zero for all $\delta < 1$ and $m, n \in \mathbb{N}$ with n/m is small enough. It follows from standard considerations about Bessel processes that the first factor in (3.18) is positive for $\delta < 1$. Then, it suffices to bound from below $\mathbb{P}(\rho_{n/m}^{(i)}(-1, 0] \leq \delta)$.

From Observation 7, it follows that $\rho_{n/m}^{(i)}$ is distributed as

$$\rho^{n/m}\left(\cdot - \frac{n}{m}\left(m\frac{i-1}{n} - \left\lfloor m\frac{i-1}{n} \right\rfloor\right)\right). \tag{3.19}$$

On the other hand, using display (3.15) in Lemma 8 and the fact that $m\frac{i-1}{n} - \lfloor m\frac{i-1}{n} \rfloor \leq 1$ we obtain that (3.19) converges to ρ as $\frac{n}{m}$ goes to 0. Moreover, we see that $\mathbb{P}(\rho_{n/m}^{(i)}(-1, 0] \leq \delta)$ converges to $\mathbb{P}(\rho(-1, 0] \leq \delta)$ uniformly on i as $\frac{n}{m}$ goes to 0, for each $\delta > 0$. Thus, there exists ε small enough such that, for $n, m \in \mathbb{N}$ such that $n/m \leq \varepsilon$ we have that the quantity on display (3.17) is bounded away from zero uniformly on n, m and i .

Following analogous arguments, we can find uniform lower bounds for $\mathbb{P}(A_n^0 \cap L_n^0)$ and $\mathbb{P}(A_0^0 \mid l(\tau_{-x}, 0) = m/n)$. That, plus display (3.16), proves Lemma 9.

In a similar way, we can apply the vague convergence of ρ^ε to ρ to obtain Lemma 10 by imitating the argument leading to Lemma 6.

Now we are ready to prove Theorem 1. The strategy is analogous to the proof of Theorem 2. Since both, $\mathbb{P}(|X_t| \geq x)$ and $C \exp(-c(\frac{x}{t^\gamma})^{1+\alpha})$ are decreasing on x , it will suffice to show that there exist constants $C, c, \varepsilon_1 > 0$ such that

$$\mathbb{P}(|X_t| \geq m) \geq C \exp\left(-c\left(\frac{m}{t^\gamma}\right)^{1+\alpha}\right)$$

for all $m \in \mathbb{N}$ and t such that $\frac{m}{\varepsilon} \leq t$. Moreover, we can restrict ourselves to $t \leq m^{1/\gamma}$, because for $t \geq m^{1/\gamma}$, we have that $\mathbb{P}(|X_t| \geq m) \geq \mathbb{P}(|X_t| \geq t^\gamma)$, and the convergence of $(\varepsilon^{-1} X_{\varepsilon^{1/\gamma} t})_{t \in \mathbb{R}_+}$ to the FIN singular diffusion (proved in Fontes et al. (2002), Theorem 4.1) implies that $\mathbb{P}(|X_t| \geq t^\gamma)$ converges to $\mathbb{P}(|Z_1| \geq 1)$ as $t \rightarrow \infty$.

Let $n \geq 2$ be a natural number such that $t \in [\frac{m^{1/\gamma}}{n^{1/\alpha}}, \frac{m^{1/\gamma}}{(n-1)^{1/\alpha}}]$ (which exists because $t \leq m^{1/\gamma}$) and $K \in \mathbb{N}$ be a fixed constant satisfying $K^\alpha > 2$. Hence, since $t \leq \frac{m^{1/\gamma}}{(n-1)^{1/\alpha}}$, we have

$$t \leq \frac{(Km)^{1/\gamma}}{(Kn)^{1/\alpha}}. \quad (3.20)$$

Let us define

$$G_1^0 := \left\{ \min \left\{ X_s : s \in \left[0, \frac{m^{1/\gamma}}{n^{1/\alpha}} \right] \right\} \leq -\frac{n+1}{n}m \right\};$$

$$T^0 := \int_{-((K+1)n+1)/nm}^{-(n+1)/nm} (l(\tau_{-(((K+1)n+1)/n)m}, u) - l(\tau_{-((n+1)/n)m}, u)) \rho^1(du),$$

and

$$G_2^0 := \left\{ T^0 \geq \frac{(Km)^{1/\gamma}}{(Kn)^{1/\alpha}}; \right. \\ \left. X_s \leq -m \text{ for all } s \in [H_{-((n+1)/n)m}^0, H_{-(((K+1)n+1)/n)m}^0] \right\}.$$

In the event G_1^0 , X reaches $-\frac{n+1}{n}m$ before time $\frac{m^{1/\gamma}}{n^{1/\alpha}}$ and, since $t \geq \frac{m^{1/\gamma}}{n^{1/\alpha}}$, this implies that X reaches $-\frac{n+1}{n}m$ before time t . On the other hand, in the event G_2^0 we have that, after reaching $-\frac{n+1}{n}m$, X remains inside the interval $(-\infty, -m]$ for a time larger than $\frac{(Km)^{1/\gamma}}{(Kn)^{1/\alpha}}$ which, by (3.20), is in turn larger than t . Hence, we have that

$$G_1^0 \cap G_2^0 \subset \{X_t \leq -m\}.$$

As in the proof of Theorem 2, we have that G_1^0 and G_2^0 are independent. Thus, we can apply Lemma 9 to find that $\mathbb{P}(G_1) \geq C_1 \exp(-c_1 n)$. We can also apply Lemma 10 (at Km and Kn) to find that $\mathbb{P}(G_2) \geq C_2 \mathbb{P}(c_2 Kn)$. Thus we have that

$$\mathbb{P}(X_t \leq -m) \geq C_1 C_2 \exp(-(c_1 + Kc_2)n).$$

As we have chosen n such that $t \leq \frac{m^{1/\gamma}}{(n-1)^{1/\alpha}}$, we have that $n \leq (\frac{m}{t^\gamma})^{1+\alpha} + 1$ and we obtain

$$\mathbb{P}(X_t \leq -m) \geq C_1 C_2 \exp(-(c_1 + Kc_2)n) \exp\left(- (c_1 + Kc_2) \left(\frac{m}{t^\gamma}\right)^{1+\alpha}\right).$$

The last display provides the lower desired lower bound with $c = c_1 + Kc_2$ and $C = C_1 C_2 \exp(-(c_1 + Kc_2)n)$.

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