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# Slope influence diagnostics in conditional heteroscedastic time series models

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**Abstract.** In this paper, we provide useful and simple expressions for slope influence diagnostics of several conditional heteroscedastic time series models under innovative model perturbations. These expressions are obtained by establishing a connection between the local influence and residual diagnostics. Monte Carlo experiments provided good results in terms of the size and power of the proposed statistics. To illustrate the results, we analyze the financial time series returns of the S&P500 and DJIA indexes.

## 1 Introduction

Local influence is a valuable device for model diagnostics. It permits the identification of atypical observations that cannot be accommodated by the model. Local influence analyses were initially proposed by Cook (1986) to assess the effects (influence) of minor perturbations in the model through the curvature of the influence graph. However, because curvature is not invariant to reparameterizations of the model, Billor and Loynes (1993) suggested the use of the slope of a modified influence graph.

Conditional heteroscedastic time series models have been applied successfully to the analysis of financial time series (see Engle, 2002). Influence diagnostics in this class of models has been studied Liu (2004) on GARCH models with elliptical errors (but without statistical analyses), by Zhang and King (2005) on GARCH models with Gaussian errors and by Zevallos and Hotta (2012) on GARCH models with Gaussian or Student's-$t$ errors.

The major challenge in time series diagnostics using local influence is to find the distribution of the statistics, such as the slope or curvature, which is required to characterize an observation as influential. Because general theoretical expressions were not available, an alternative was proposed by Zhang and King (2005), which was to simulate the distribution of the statistics. This approach was applied by Zhang and King (2005) and by Zevallos and Hotta (2012) for GARCH models.

Motivated by the works of Schwarzmann (1991) and Billor and Loynes (1993), both in the regression context, this paper derives useful and simple expressions for the slope influence diagnostics of several conditional heteroscedastic time series models under innovative model perturbations. These expressions are obtained

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by establishing a connection between local influence and residual diagnostics and allow us to find the analytical asymptotic distribution of the influence statistics.

It can be argued that the slope is not enough for influence diagnostics and that curvature diagnostics is necessary. However, Zevallos et al. (2012) showed that for AR(1) models with innovative perturbation scheme the slope and Cook's curvature are the same. That the same occurs for other time series models deserves more investigation and we leave this as a further research topic.

The remainder of this paper is organized as follows. In Section 2, we briefly discuss the slope influence analysis and present the overall and individual statistics. The distributions of these statistics are derived in Section 3 for a broad class of conditional heteroscedastic time series models. Monte Carlo experiments are presented in Section 4 to assess the size and power of the proposed statistics and to compare this method with other methods from the literature. Section 5 is devoted to illustrating the methodology by detecting influential observations in two real financial time series. Finally, the conclusions are provided in Section 6, and the technical proofs are sketched in the Appendix.

## 2 Slope influence diagnostics

Let $\mathbf{y} = (y_1, \ldots, y_n)'$ be a time series generated by a postulated model with log-likelihood $L(\boldsymbol{\omega})$ and where $\boldsymbol{\theta}$ is a vector of unknown parameters. Suppose $\mathbf{y}$ is perturbed according to a perturbation scheme with perturbation vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)'$. As a result, we obtain an observation vector which has a perturbed log-likelihood $L(\boldsymbol{\theta}|\boldsymbol{\omega})$. Let $\boldsymbol{\omega}_0$ be the point of null perturbation, the point which satisfies $L(\boldsymbol{\theta}|\boldsymbol{\omega}_0) = L(\boldsymbol{\theta})$. Let $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}}$ be the maximum likelihood estimates under $L(\boldsymbol{\theta})$ and $L(\boldsymbol{\theta}|\boldsymbol{\omega})$, respectively. Note that $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}_0}$.

To assess the influence of minor perturbations $\boldsymbol{\omega}$ on the postulated model, Billor and Loynes (1993) suggested using the modified likelihood displacement, $\mathrm{MLD}(\boldsymbol{\omega}) = -2[L(\hat{\boldsymbol{\theta}}) - L(\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}}|\boldsymbol{\omega})]$. They proposed the analysis of the direction vector associated with the maximum slope of MLD

$$S = 2\frac{\partial L(\boldsymbol{\theta}|\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} \tag{2.1}$$

evaluated at $\boldsymbol{\omega}_0$ and $\hat{\boldsymbol{\theta}}$. Thus, $S = (s_1, \ldots, s_n)'$ denotes the slope vector with elements $s_i = 2\,\partial L(\boldsymbol{\theta}|\boldsymbol{\omega})/\partial\omega_i$.

Slope influence diagnostics are based on the vector $S = (s_1, \ldots, s_n)$. To perform the diagnostics, the following two criteria are considered. The first is an *overall* criterion that indicates whether the time series has at least one influential point. The second criterion, named the *individual*, serves to identify the specific influential points. These measures could be defined as follows:

**Overall criterion.** Given by

$$O_{ve} = \frac{1}{n}\|S\|^2 = \frac{1}{n}\sum_{i=1}^{n} s_i^2. \tag{2.2}$$

The time series under study has at least one influential point if $O_{ve}$ is large.

**Individual criterion.** The observation $y_i$ is considered influential if $s_i$ is large.

After these statistics are calculated, the remaining problem is how to assess whether $O_{ve}$ or the $s_i$ values are statistically large enough to consider the $i$th observation as influential. We deal with this issue in the next section.

## 3 Main results

Let $\{y_t\}$ be a stochastic process, and let $\mathcal{F}_{t-1} = \{y_{t-1}, y_{t-2}, \ldots\}$ be the past information. In addition, assume that $\varphi$ is a generic function. Conditional heteroscedastic time series models are defined as follows:

$$y_t = \sigma_t \varepsilon_t, \tag{3.1}$$

$$\sigma_t = \varphi(\mathcal{F}_{t-1}), \tag{3.2}$$

$$\varepsilon_t \sim \text{IID}(0, 1), \tag{3.3}$$

where IID(0, 1) means that the sequence of errors $\{\varepsilon_t\}$ is independent and identically distributed with a mean of zero and unit variance. In finance, $\sigma_t$ is known as the *volatility* at time $t$ and corresponds to the conditional standard deviation of $y_t$ given the past.

Several volatility specifications have been proposed in the literature. Parametric specifications include the following: GARCH models (Bollerslev, 1986), EGARCH models (Nelson, 1991), PGARCH models (Ding et al., 1993), and TGARCH models (Glosten et al., 1993). These models are defined as

$$\text{GARCH}(p,q): \quad \sigma_t^2 = \delta + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{3.4}$$

$$\text{EGARCH}(p,q): \quad \ln(\sigma_t^2) = \delta + \sum_{i=1}^{p} \alpha_i \left\{ \frac{|y_{t-i}|}{\sigma_{t-i}} + \gamma_i \frac{y_{t-i}}{\sigma_{t-i}} \right\} + \sum_{j=1}^{q} \beta_j \ln(\sigma_{t-j}^2), \tag{3.5}$$

$$\text{PGARCH}(p,q): \quad \sigma_t^d = \delta + \sum_{i=1}^{p} \alpha_i \{|y_{t-i}| + \gamma_i y_{t-i}\}^d + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^d, \tag{3.6}$$

$$\text{TGARCH}(p,q):\quad \sigma_t^2 = \delta + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{i=1}^{p} \gamma_i U_{t-i} y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{3.7}$$

where $U_t = 0$ if $y_t < 0$ and $U_t = 1$ if $y_t \geq 0$; see Zivot and Wang (2006). For an account of other volatility specifications, see Franses (2000).

On the other hand, in empirical applications, besides the Gaussian distribution two families of distribution for the errors $\varepsilon_t$ have been employed, the standardized Student' $t$-distribution,

$$f(\varepsilon_t) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\pi(\nu-2)}}\left(1 + \frac{\varepsilon_t^2}{\nu-2}\right)^{-(\nu+1)/2}, \qquad \nu > 2, \tag{3.8}$$

which is denoted henceforth by $\mathrm{St}(\nu, 0, 1)$, and the standardized Generalized Exponential Distribution (GED),

$$f(\varepsilon_t) = \frac{\nu}{\lambda\Gamma(1/\nu)2^{1+1/\nu}} \exp\left\{-\frac{1}{2}\lambda^{-\nu}|\varepsilon_t|^{\nu}\right\}, \tag{3.9}$$

where $\lambda^2 = 2^{-2/\nu}\Gamma(1/\nu)/\Gamma(3/\nu)$. Note that when $\nu \to \infty$ in (3.8) or $\nu = 2$ in (3.9) we obtain the standard Gaussian density.

We are interested in the slope diagnostics of model (3.1)–(3.3) where the errors assume Gaussian, Student-$t$ or GED distributions. The chosen scheme of perturbation is the innovative model perturbation that was discussed by Zhang and King (2005) and Liu (2004). Here, the perturbation $\omega_t$ is introduced into the model (3.1)–(3.3) via the conditional variance, and its effect is carried over to both the future observations and the future conditional variances. Thus, the innovations in (3.3) are given by

$$\varepsilon_t \sim \mathrm{IID}(0, \omega_t^{-1}), \tag{3.10}$$

and the point of the null perturbation is $\boldsymbol{\omega}_0 = (1, \dots, 1)^\top$, that is, $\omega_t = 1$ for $i = 1, \dots, n$. This perturbation scheme is very useful when a perturbation in the economy increases the volatility, but the impact will eventually die out.

As described in the previous section, we have to calculate (2.1), which is evaluated at $\boldsymbol{\omega}_0$ and $\hat{\theta}$, where the perturbed log-likelihood $L(\boldsymbol{\theta}|\boldsymbol{\omega})$ is

$$L(\boldsymbol{\theta}|\boldsymbol{\omega}) = \sum_{t=1}^{n} l_t, \qquad l_t = \ln f_t(\boldsymbol{\theta}|\boldsymbol{\omega}), \tag{3.11}$$

and $f_t(\boldsymbol{\theta}|\boldsymbol{\omega})$ is the conditional density of $y_t$ given its past.

It is convenient to write the slope as a function of the standardized residuals, $e_t$, which are defined as

$$e_t = y_t/\hat{\sigma}_t, \qquad t = 1, \dots, n, \tag{3.12}$$

where $\hat{\sigma}_t$ is the estimated conditional standard deviation of the unperturbed model. Next, we present the results concerning the overall and individual criteria discussed in Section 2.

**Theorem 3.1.** *Consider the innovative model perturbation* (3.10) *in model* (3.1)–(3.3). *Let* (3.12) *be the standardized residuals. Then, we have the following expressions for the* $i$*th element of the vector slope*:

(a) *If the errors follow a standard Gaussian distribution*

$$s_i = 1 - e_i^2. \tag{3.13}$$

(b) *When the errors follow a GED distribution*

$$s_i = 1 - \frac{\nu}{2}\lambda^{-\nu}|e_i|^{\nu}. \tag{3.14}$$

(c) *If the errors follow an* $\mathrm{St}(\nu, 0, 1)$ *distribution*

$$s_i = 1 - \frac{(\nu+1)e_i^2}{(\nu-2)+e_i^2}. \tag{3.15}$$

Please refer to the Appendix for the proof.

Note that when the errors are Gaussian, observations with absolute large residuals are considered influential. To determine whether they are statistically significant, we have to find the distribution of the $s_i$ statistics which depends on the distribution of residuals. As far as we know, there are not general results for the distribution of the maximum likelihood residuals in heteroscedastic time series models. Then, to derive the properties of the influence statistics we assume that the residuals have the same probabilistic behavior of the errors. Monte Carlo experiments of Section 4 evidence that the critical values obtained under this assumption are very close to the true theoretical values.

Theorem 3.1 make it clear that instead of working with $s_i$, it is more convenient to consider $1 - s_i$ to construct the individual criteria. Thus, the Gaussian errors $\{e_t^2\}$ are asymptotically I.I.D. with a $\chi^2_{(1)}$ distribution. As a consequence, $y_i$ is considered influential at the 5% (1%) level, if $1 - s_i = e_i^2 > 3.84$ (6.63). For the GED and Student's $t$ cases, the critical values are easily calculated using well known distributions. Thus, it is straightforward to prove that $1 - s_i \sim \mathrm{Gamma}(1/\nu, \nu)$ for the GED errors and that $(1 - s_i)/(\nu + 1) \sim \mathrm{Beta}(1/2, \nu/2)$ for the Student's $t$ errors (refer to the Appendix for the proof of Theorem 3.1). However, for the Student's case, we obtain better results in terms of power by working with the statistic $e^2\nu/(\nu - 2)$ instead of $1 - s$. This statistic has the Snedecor's $F$ distribution with 1 and $\nu$ degrees of freedom, $F_{1,\nu}$.

It is worth stressing that the critical values discussed in the last paragraph are valid for a fixed position $i$. Let $\mathrm{Inf}_i$ be the statistic at time $i$, for example,

$\text{Inf}_i = 1 - s_i$ for Gaussian errors. Because the position of the influential observation is not known, we use $\max_i$ ($\text{Inf}_i$) as, what is henceforth called, the *global individual statistic*. Let $\alpha_g$ be the global significance level, i.e., the probability of incorrectly detecting at least one observation as influential in a time series of size $n$. Considering that the statistics are asymptotically independent, we have $1 - \alpha_g = (1 - \alpha_i)^n$ where $\alpha_i$ is the $i$th significance level. Thus, if we want $\alpha_g = 0.05$ (0.10), then $\alpha_i = 1 - 0.9999591$ $(1 - 0.999916)$ for $n = 1255$. If we assume Gaussian errors, the 0.9999591-quantile for a chi-square of one degree of freedom is 16.83. Therefore, the observation $y_i$ is considered influential at the 5% (global significance) level if $e_i^2 > 16.83$. For the GED and Student's $t$ errors, the benchmarks at the 5% level are the 0.9999591-quantile for the Gamma$(1/\nu, \nu)$ density and the 0.9999591-quantile for the $F_{1,\nu}$ density, respectively. For example, Table 1 shows the critical values or benchmarks obtained from the asymptotic distribution, which is denoted by $AB$, for three different model specifications and several sample sizes, including $n = 1255$ which corresponds to the sample size of the empirical application of Section 5.1.

Now, we present the results for the overall criterion. These are obtained using the central limit theorem. The result (a) was derived by Billor and Loynes (1993) in the regression context.

**Table 1** *The estimated size (S) of the asymptotic global individual benchmarks (AB). The true distribution is estimated using a simulation with* 2000 *replications for the* GARCH(1, 1) *models with Gaussian errors, the* EGARCH(1, 1) *models with a GED distribution using the parameter* $\nu = 1.736$ *and the* GARCH(1, 1) *models with the Student's t-distribution using* $\nu = 7.87$ *degrees of freedom. SB is the simulated benchmark*

| | | GARCH(1, 1) Gaussian | | | EGARCH(1, 1) GED | | | GARCH(1, 1) Student's $t$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | Level | SB | AB | S (%) | SB | AB | S (%) | SB | AB | S (%) |
| 500 | 10% | 13.41 | 13.73 | 8.4 | 13.13 | 12.40 | 15.5 | 42.52 | 41.73 | 10.7 |
| | 5% | 14.79 | 15.09 | 4.3 | 14.68 | 13.59 | 8.1 | 52.46 | 51.51 | 5.3 |
| | 1% | 17.52 | 18.18 | 0.7 | 18.99 | 16.30 | 2.5 | 78.81 | 81.53 | 0.9 |
| 1000 | 10% | 14.83 | 15.04 | 9.1 | 14.37 | 13.55 | 15.3 | 51.02 | 51.10 | 10.0 |
| | 5% | 16.19 | 16.40 | 4.4 | 15.72 | 14.74 | 8.3 | 63.75 | 62.77 | 5.4 |
| | 1% | 19.08 | 19.50 | 0.7 | 18.62 | 17.46 | 1.9 | 99.6 | 98.58 | 1.1 |
| 1255 | 10% | 15.16 | 15.47 | 9.0 | 14.66 | 13.92 | 14.0 | 53.27 | 54.56 | 9.3 |
| | 5% | 16.39 | 16.83 | 3.8 | 16.24 | 15.12 | 8.4 | 66.02 | 66.92 | 4.8 |
| | 1% | 19.59 | 19.94 | 0.7 | 19.80 | 17.84 | 2.3 | 104.01 | 104.86 | 1.0 |
| 5000 | 10% | 17.88 | 18.09 | 8.9 | 16.64 | 16.22 | 12.4 | 80.48 | 80.47 | 10.1 |
| | 5% | 19.44 | 19.46 | 4.9 | 17.98 | 17.42 | 6.7 | 99.96 | 98.04 | 5.5 |
| | 1% | 22.41 | 22.59 | 0.9 | 20.63 | 20.15 | 1.4 | 162.49 | 151.96 | 1.2 |

**Theorem 3.2.** *Consider the innovative model perturbation* (3.10) *in model* (3.1)–(3.3) *and let* (3.12) *be the standardized residuals. Assuming that the residuals have the same probabilistic behavior of the errors, then we have the following expressions for the asymptotic distribution of the overall statistic* $O_{ve}$ *in* (2.2):

(a) *If the errors follow a standard Gaussian distribution*

$$\sqrt{n}(O_{ve} - 2) \to_{\mathcal{L}} N(0, 56). \tag{3.16}$$

(b) *When the errors follow a GED distribution*

$$\sqrt{n}(O_{ve} - \nu) \to_{\mathcal{L}} N\big(0, 2\nu^2(1 + 3\nu)\big). \tag{3.17}$$

(c) *If the errors follow an* $\mathrm{St}(\nu, 0, 1)$ *distribution*

$$\sqrt{n}\left(O_{ve} - \frac{2\nu}{(\nu + 3)}\right) \to_{\mathcal{L}} N\big(0, \varphi(\nu)\big), \tag{3.18}$$

*where*

$$\varphi(\nu) = \frac{8\nu(7\nu^3 + 12\nu^2 - 25\nu + 18)}{(\nu + 3)^2(\nu + 5)(\nu + 7)}.$$

Please refer to the Appendix for the proof.

## 4 Monte Carlo experiments

In practice, the benchmarks for the proposed statistics are calculated based on estimated parameters instead of the true (unknown) parameters of the proposed model. Besides, the estimated innovations are not independent. Therefore, it is important to evaluate the robustness of the benchmarks face to estimation, that is, to check whether the true size of the estimated benchmarks are close to the nominal values. In addition, it is worthwhile to assess whether the methodology allows us to identify simulated influential observations (power). In this section, both aspects are discussed by mean of simulations.

First, we assess the effects of estimation on the benchmarks. Thus, 2000 time series of sizes $n = 500$, 1000, 1255 and 5000 were simulated for each of the following three models: GARCH(1, 1) models with Gaussian errors and parameters

$$\delta = 12.6 \times 10^{-6}, \qquad \alpha = 0.1025, \qquad \beta = 0.8211, \tag{4.1}$$

GARCH(1, 1) models with Student's $t$ errors and parameters

$$\delta = 8.5 \times 10^{-6}, \qquad \alpha = 0.0713, \qquad \beta = 0.8738, \qquad \nu = 7.87, \tag{4.2}$$

and EGARCH(1, 1) models with GED errors and parameters

$$\begin{aligned} &\delta = -0.8568, \qquad \alpha = 0.0610, \qquad \beta = 0.9096, \\ &\gamma = -1, \qquad \nu = 1.736. \end{aligned} \tag{4.3}$$

These models and parameters correspond to the estimated models in Section 5.1. Then, for each time series, the parameters are estimated, and based on

**Table 2** *The estimated size* ($S$) *of the asymptotic overall benchmarks* ($AB$). *The true distribution is estimated using a simulation with* 2000 *replications for the* GARCH(1, 1) *models with Gaussian errors*, *the* EGARCH(1, 1) *models with a GED distribution using the parameter* $\nu = 1.736$ *and the* GARCH(1, 1) *models with the Student's* $t$*-distribution using* $\nu = 7.87$ *degrees of freedom*. *SB is the simulated benchmark*.

| | | GARCH(1, 1) Gaussian | | | EGARCH(1, 1) GED | | | GARCH(1, 1) Student's $t$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | Level | SB | AB | S (%) | SB | AB | S (%) | SB | AB | S (%) |
| 500 | 10% | 2.25 | 2.43 | 3.1 | 2.30 | 2.09 | 20.8 | 1.69 | 1.64 | 15.2 |
| | 5% | 2.35 | 2.55 | 1.2 | 2.47 | 2.19 | 15.2 | 1.77 | 1.69 | 10.1 |
| | 1% | 2.57 | 2.78 | 0.2 | 3.08 | 2.37 | 7.5 | 1.93 | 1.79 | 4.1 |
| 1000 | 10% | 2.18 | 2.30 | 2.6 | 2.12 | 1.98 | 19.6 | 1.61 | 1.58 | 14.3 |
| | 5% | 2.24 | 2.39 | 0.9 | 2.25 | 2.05 | 13.5 | 1.64 | 1.62 | 7.9 |
| | 1% | 2.37 | 2.55 | 0.1 | 2.59 | 2.19 | 7.6 | 1.75 | 1.69 | 2.5 |
| 1255 | 10% | 2.17 | 2.27 | 2.9 | 2.07 | 1.96 | 20.7 | 1.57 | 1.57 | 11.4 |
| | 5% | 2.23 | 2.35 | 1.3 | 2.19 | 2.02 | 14.0 | 1.62 | 1.60 | 6.0 |
| | 1% | 2.37 | 2.49 | 0.1 | 2.45 | 2.14 | 6.7 | 1.74 | 1.67 | 2.4 |
| 5000 | 10% | 2.09 | 2.14 | 2.5 | 1.91 | 1.85 | 23.9 | 1.51 | 1.51 | 8.9 |
| | 5% | 2.11 | 2.17 | 0.6 | 1.95 | 1.88 | 16.3 | 1.53 | 1.53 | 5.1 |
| | 1% | 2.16 | 2.25 | 0.0 | 2.02 | 1.94 | 6.4 | 1.55 | 1.56 | 0.9 |

the 2000 simulated time series, the global individual and overall benchmarks are calculated. These benchmarks, which are called *simulated benchmarks* (*SB*), are presented in Tables 1 and 2, for global individual and overall statistics, respectively. Moreover, these values are compared with the *asymptotic benchmarks* (*AB*) that are calculated from the asymptotic distributions given in Section 3. In addition, we reported the size of the asymptotic benchmarks, in percentages ($S$), which were estimated by the tail probability for the empirical simulated distribution.

From Table 1, we observe that the sizes of the simulated global individual benchmarks are very close to the asymptotic ones for the Gaussian and Student's cases. Therefore, we can conclude that the estimation process almost does not affect the individual benchmarks found by the asymptotic distribution. This also occurs for the GED errors when $n = 5000$; however, for smaller sample sizes, the asymptotic benchmarks are smaller than the simulated benchmarks.

With respect to the overall benchmarks, the results presented in Table 2 evidence the effects of the estimation for Gaussian and GED errors, even for $n = 5000$. Thus, the simulated overall benchmarks are smaller than the asymptotic benchmarks. However, for Student's $t$ errors when $n \geq 1255$, the results are good, and the values, obtained for the simulated benchmarks are close to the corresponding asymptotic values.

We also assess the performance of the influence statistics is terms of power. For instance, we simulate 1000 perturbed time series of models (4.1), (4.2) and (4.3).

**Table 3** *The estimated power (%) of influential statistics is based on 1000 replications of time series of size 1255 generated by GARCH(1, 1) models with Gaussian errors, the EGARCH(1, 1) models with GED errors using the parameter $\nu = 1.736$ and the GARCH(1, 1) models with Student's t-errors using $\nu = 7.87$ degrees of freedom. In each simulated time series, two innovative perturbations are included at positions 206 and 418 with the values $\omega_{206}^{-1/2} = 7$ and $\omega_{418}^{-1/2} = 5$, respectively. Each entry corresponds to the percentage of detection using the asymptotic benchmarks ($AB_1$). The benchmarks $AB_2$, are calculated using $\nu = 1.736 + 0.4$ for the GED errors and $\nu = 7.87 + 3.7$ for the Student's t-errors. For the Gaussian distribution, the power of Charles and Darné (2005) test (CD) for IO is presented at the 5% level based on 2000 replications*

| | GARCH(1, 1) Gaussian | | EGARCH(1, 1) GED | | GARCH(1, 1) Student's $t$ | |
|---|---|---|---|---|---|---|
| | $AB_1$ | CD | $AB_1$ | $AB_2$ | $AB_1$ | $AB_2$ |
| Position 206 | 57.4 | 45.5 | 46.3 | 54.3 | 21.5 | 34.6 |
| Position 418 | 42.4 | 37.8 | 33.6 | 41.1 | 7.8 | 18.8 |
| Positions 206 and 418 | 25.0 | 17.5 | 14.2 | 21.9 | 0.6 | 5.8 |
| Overall | 68.0 | 10.9 | 48.5 | 97.1 | 0 | 0 |

In each simulated time series with size of $n = 1255$, two innovative perturbations are considered at positions 206 and 418 with the values $\omega_{206}^{-1/2} = 7$ and $\omega_{418}^{-1/2} = 5$. Then, we calculate the frequency of influence detection (in percentage) using the asymptotic global individual influential statistics, which are denoted by $AB_1$, and the overall influential statistics at the 5% level. The results are summarized in Table 3. Here, we observe that for the Gaussian case the overall statistics has reasonably power. In addition, the power of the individual statistic is quite high. Note that the two perturbed observations are detected simultaneously approximately 25% of the time. For the GED errors, the results are also good but the test has small power compared to the Gaussian case. On the contrary, for the Student's $t$ errors, the influential statistics are almost incapable of detecting the outliers.

The small power that was observed for the Student's $t$ case can be explained by the strong influence of the perturbed observations on the estimation of the degrees of freedom. This also occurs, to a lesser extent, for the parameter $\nu$ in the GED distribution. Because benchmarks are constructed using estimated parameters, the influence performance is affected. A comparison between the estimated parameters of the unperturbed and perturbed time series is reported in Table 4. As shown, based on the median values, the perturbation does not affect the estimates of $\alpha$, $\beta$, $\gamma$, but the estimates of $\delta$ may change dramatically for the Gaussian and Student's $t$ cases. In addition, the effects of the perturbation are severe on the degrees of freedom ($\nu$) estimates for the Student's $t$ errors. Thus, to accommodate the perturbations, the degrees of freedom are subestimated, which increases the benchmarks. This also occurs to a lesser extent for $\nu$ in the GED case.

We assessed the effects of the subestimation of $\nu$ on the GED and Student's distributions in terms of power. Thus, for the GED errors, we calculated the asymp-

**Table 4** *The estimation robustness assessment. The entries correspond to the percentiles of the differences between the parameter estimates of the perturbed minus the unperturbed models based on* 1000 *simulations of time series with a size of* 1255. *Unperturbed models include the* GARCH(1, 1) *models with Gaussian errors, the* EGARCH(1, 1) *models with GED errors using the parameter* $\nu = 1.736$, *and the* GARCH(1, 1) *models with the Student's t-errors using* $\nu = 7.87$ *degrees of freedom. The perturbed models are obtained by including the innovative perturbations at positions* 206 *and* 418, *with values* $\omega_{206}^{-1/2} = 7$ *and* $\omega_{418}^{-1/2} = 5$ *in each simulated time series*

| Model | Perc. | $\delta$ | $\alpha$ | $\beta$ | $\nu$ | $\gamma$ |
|---|---|---|---|---|---|---|
| GARCH(1, 1)- | 5% | $-6.8 \times 10^{-6}$ | −0.0314 | −0.0991 | | |
| Gaussian | 50% | $0.5 \times 10^{-6}$ | 0.0027 | −0.0032 | | |
| | 95% | $13.7 \times 10^{-6}$ | 0.0853 | 0.0745 | | |
| GARCH(1, 1)- | 5% | $-4 \times 10^{-6}$ | −0.0083 | −0.0265 | −3.7252 | |
| Student's *t* | 50% | $0.2 \times 10^{-6}$ | 0.0014 | −0.0002 | −0.8989 | |
| | 95% | $3.4 \times 10^{-6}$ | 0.0138 | 0.0400 | 0.0858 | |
| EGARCH(1, 1)- | 5% | −1.2147 | −0.0466 | −0.1343 | −0.4054 | −0.2895 |
| GED | 50% | 0.0401 | 0.0000 | 0.0039 | −0.0993 | 0.0000 |
| | 95% | 1.6043 | 0.0562 | 0.1767 | 0.0129 | 0.4385 |

totic benchmarks, which are denoted by $AB_2$ in Table 3, using $\nu$ as the estimated $\nu$ plus 0.4. The value 0.4 is the 95% percentile of the difference between the parameter estimates of the unperturbed and perturbed time series; see Table 4. The same calculation was made for the Student's errors, where the degrees of freedom were computed as the estimated $\nu$ plus 3.7. Table 3 shows an improvement in the power of the individual statistics, especially for the GED errors. However, the power of the overall statistic in the Student's case can not be improved.

### 4.1 Comparison with other outliers tests

A very well-known outlier test for financial time series was proposed in 1998 by Hotta and Tsay (2012). They only worked with GARCH models with Gaussian errors and defined the volatility outlier as disturbing the volatility additively, whereas in our case, the perturbation is multiplicative; see (3.10). However Hotta and Tsay's statistic test is given by $e_i^2$, that is, the same as the proposed test.

On the other hand, Abraham and Yatawara (1988) suggested that the asymptotic distribution of the maximum in a $n$-dependent stationary process that is based on a lemma by Leadbetter (1983) be used. The critical value is given by

$$x_c = F^{-1}\left(1 + \frac{\log(1-\alpha)}{n\gamma}\right), \tag{4.4}$$

where $\gamma$ is the extremal index and $F$ is the marginal distribution. Because $\gamma$ is difficult to evaluate, they suggested that $\gamma = 0.8$ be used. When the model is in-

**Table 5** *The estimated size and critical values of the individual influence statistics calculated by the asymptotic benchmarks* (*AB*) *and calculated by Abraham and Yatawara* (1988) *using extremal index equal to* 0.8 (*AY*-0.8) *and* 0.999 (*AY*-0.999). *The true distribution is estimated by a simulation with* 2000 *replications of time series with a size of* 1255. *Data generating processes*: GARCH(1, 1) *models with Gaussian errors*, *the* EGARCH(1, 1) *models with GED errors using the parameter* $\nu = 1.736$, *and the* GARCH(1, 1) *models with Student's t-distribution using* $\nu = 7.87$ *degrees of freedom*

| Model | Level | Critical values | | | Size | | |
|---|---|---|---|---|---|---|---|
| | | AB | AY-0.80 | AY-0.999 | AB | AY-0.80 | AY-0.999 |
| GARCH(1, 1)- | 10% | 15.47 | 15.05 | 15.47 | 9.0 | 10.9 | 9.0 |
| Gaussian | 5% | 16.83 | 16.41 | 16.83 | 3.8 | 4.9 | 3.8 |
| | 1% | 19.94 | 19.51 | 19.93 | 0.7 | 1.1 | 0.7 |
| EGARCH(1, 1)- | 10% | 13.92 | 13.55 | 13.92 | 14.0 | 16.5 | 14.0 |
| GED | 5% | 15.12 | 14.75 | 15.12 | 8.4 | 9.7 | 8.4 |
| | 1% | 17.84 | 17.46 | 17.83 | 2.3 | 2.8 | 2.3 |
| GARCH(1, 1)- | 10% | 54.56 | 51.17 | 54.55 | 9.3 | 11.3 | 9.3 |
| Student's *t* | 5% | 66.92 | 62.84 | 66.90 | 4.8 | 6.1 | 4.8 |
| | 1% | 104.86 | 98.69 | 104.83 | 1.0 | 1.1 | 1.0 |

dependent, $\gamma = 1$. We suggest that independence be assumed when evaluating the critical value. In Table 5, we present the critical values (4.4) given by our suggestion ($\gamma = 0.999$) and by the Abraham and Yatawara suggestion ($\gamma = 0.8$). The true distribution of the statistic is estimated by a simulation with 2000 replications of the time series with a size of 1255 from models (4.1)–(4.3). The size of the critical values is also estimated using this simulated distribution. The results show that for the Abraham–Yatawara (AY) method, the critical values calculated using independence are the best values. In fact, the AY benchmarks are very close to the influence statistics benchmarks.

We also compared the performance of our test with the IO test from Charles and Darné (2005) for Gaussian errors. They considered that the GARCH(1, 1) model can be written as the ARMA(1, 1) model

$$y_t^2 = \delta + (\alpha_1 + \beta_1) y_{t-1}^2 + \nu_t - \beta_1 \nu_{t-1}, \tag{4.5}$$

where $\nu_t = y_t^2 - \sigma_t^2$. In their model, in the presence of an innovation outlier of size $\omega$ at the $\tau$th observation, the squared observed value, $x_t^2$ is given by:

$$x_t^2 = y_t^2 + \frac{1 - \beta_1 B}{1 - (\alpha_1 + \beta_1) B} \omega I_\tau(t), \tag{4.6}$$

where $B$ is the backshift operator, $\omega$ is the size of the innovation outlier in $x_t^2$, and $I_\tau(t)$ is the indicator function, which is equal to one when $t = \tau$, and zero elsewhere. The statistic proposed by Charles and Darné to test for the presence of an IO at the $\tau$th observation is given by $\hat{\tau}(\tau) = \hat{\eta}_\tau/\hat{\sigma}_\tau$, where $\hat{\eta}_\tau$ is the estimated error when the GARCH(1, 1) model is fitted to the observed series and $\hat{\sigma}_\tau$ is the sample variance of $\hat{\eta}_j$, $j = 1, \ldots, \tau - 1, \tau + 1, \ldots, n$. Their global test statistic is given by $\hat{\tau}_{\max} = \max_{1 \leq \tau \leq n} |\hat{\tau}(\tau)|$. To compare the results with our test, we estimated the power of this test using the following steps.

1. Generate a sample of size 1255 with the two IOs.
2. Fit a GARCH(1, 1) model.
3. For the estimated model, estimate the critical values for the individual test $\hat{\tau}(\tau)$ for $\tau = 206{,}418$ and for the global statistics $\hat{\tau}_{\max}$. To control the size of the test in 5%, the critical values were estimated based on 2000 replications.
4. Verify whether the statistics $\hat{\tau}(206)$, $\hat{\tau}(418)$ and $\hat{\tau}_{\max}$ are larger than their estimated 5% critical values.
5. Repeat steps 1–4 1000 times.

The results are presented in Table 3. We did not use an iterative procedure because the power of the proposed statistics was estimated without using the iterative procedure. The power of our proposed test is always larger than that of Charles and Darné (2005), especially for the overall test.

## 5 Illustrations

The proposed methodology is applied to identify influential observations in the following three well-known financial time series: the daily S&P500 returns and the monthly and daily returns of the Dow Jones Industrial Average.

### 5.1 Standard and Poor's 500

The first illustration intends to identify influential observations in continuously compounded daily returns on the S&P500s closest composite index from January 3, 1997 to December 31, 2001 ($n = 1255$ returns).

The following models were fitted: GARCH(1, 1) models with Gaussian and Student's $t$ errors, EGARCH(1, 1) models with Gaussian and GED errors, TGARCH(1, 1) models with GED errors and PGARCH(1, 1) models with the Student's $t$ errors [see Equations (3.4)–(3.9)]. With the exception of the EGARCH-GED case, the estimated parameters are highly significant, and the $p$-values of the Box–Ljung (BL) statistics for the standardized residuals and squared standardized residuals indicate that all the fitted models explain the correlation structure of the level and the volatility. For example, for the EGARCH-GED case, the $p$-values of the BL statistic that correspond to the squared standardized residuals are 0.0548 and 0.0415 for 12 and 24 lags, respectively.

To characterize the points as influential, we used the asymptotic benchmarks. The estimated degrees of freedom for GARCH-$t$ and PGARCH-$t$ are 7.87 and

**Table 6** *Slope Influence Diagnostics of the S&P500 time series. The influential points at 5% level are marked in bold. The significant overall statistics ($O_{ve}$) at 5% level are marked in bold*

| Date *yy\|mm\|dd* | Point | GARCH Gaussian | GARCH Student | EGARCH Gaussian | EGARCH GED | TGARCH GED | PGARCH Student |
|---|---|---|---|---|---|---|---|
| 97\|10\|27 | 206 | **34.31** | 50.32 | **34.66** | **23.72** | **19.45** | 44.84 |
| 98\|08\|31 | 418 | **19.14** | 29.17 | **22.95** | **17.44** | 8.83 | 30.16 |
| 00\|01\|04 | 757 | 16.08 | 21.94 | **16.92** | 11.66 | 11.83 | 24.18 |
| 00\|04\|14 | 828 | **20.23** | 28.41 | **20.88** | **15.67** | 11.01 | 26.92 |
| 01\|09\|17 | 1182 | 15.13 | 21.83 | 14.80 | 11.35 | 7.38 | 21.10 |
| $O_{ve}$ | | **3.75** | 1.44 | **4.70** | **2.81** | 1.85 | 1.45 |
| *Benchmarks* | | | | | | | |
| Individual 5% | | 16.83 | 66.92 | 16.83 | 15.12 | 14.52 | 65.80 |
| Overall 5% | | 2.35 | 1.60 | 2.35 | 2.02 | 1.91 | 1.61 |

7.953, respectively. The estimated parameter of $\nu$ for the EGARCH-GED and TGARCH-GED models are 1.736 and 1.646, respectively. The points that were detected as influential at the 5% global level are marked in bold in Table 6. Thus, in the GARCH-Gaussian fit we identified three influential points: 206, 418 and 828, and points 757 and 1182 were influential at 10% and 11.8% levels, respectively. For the EGARCH-Gaussian case, we identified the points 206, 418, 757 and 828 as influential at the 5% level. For the EGARCH-GED model we found points 206, 418 and 828 to be influential at the 5% level. For the TGARCH-GED model just one point, 206, was determined to be influential, and no points were found to be influential for the GARCH-$t$ and PGARCH-$t$ models, even at the 10% level. Therefore, we identified more influential points with Gaussian errors, and, as expected, fewer influential points when GED errors were used. Furthermore, if the errors follow a Student's $t$ distribution, the model was able to reproduce heavy tails in such a way that it accommodated all extremal points.

The identified influential points, except for the 757th, are associated with the following important historical events: the Asian Flu in October of 1997 (point 206), the Russian cold in August of 1998 (point 418), the NASDAQ fall in April of 2000 (point 828) and the World Trade Center attack in September of 2001 (point 1182).

On the other hand, the overall test at 5% reported that there are influential observations in the GARCH-Gaussian, EGARCH-Gaussian and EGARCH-GED cases, with $z$ values of 8.29, 12.80 and 6.25, respectively. For the TGARCH-GED case, the $z$ value is 1.26, which reaches significance at 10%. For the other models we did not obtain any significance. In the GARCH-$t$ and PGARCH-$t$ models the corresponding $p$-values are 0.53 and 0.51, respectively.

Figure 1 shows the S&P500 time series returns with the individual influence statistics that correspond to the GARCH-Gaussian, EGARCH-GED and

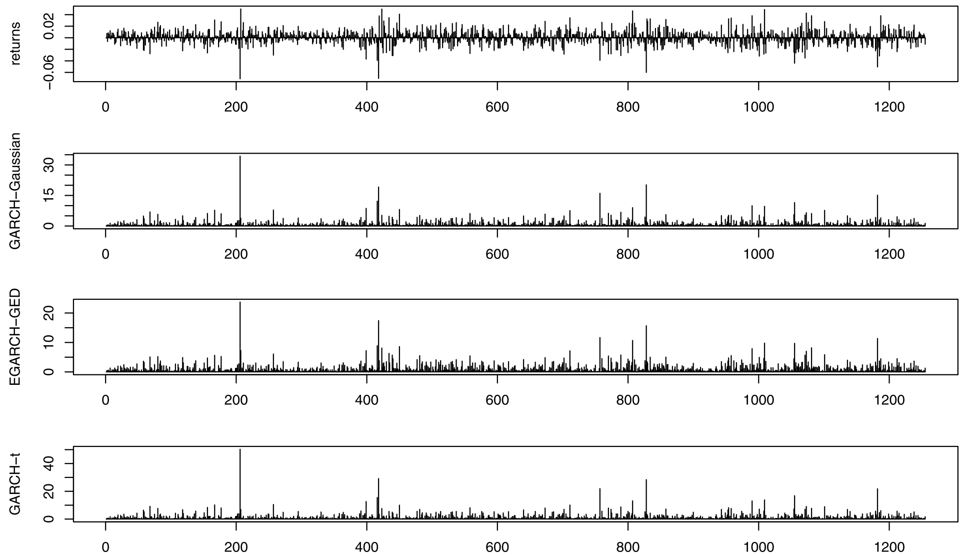


**Figure 1** *Slope Influential Diagnostics of SP500. From top to bottom: returns and slopes for GARCH-Gaussian, EGARCH-GED and GARCH-t models.*

GARCH-$t$ cases. All cases contained some peaks that were associated with the detected influential observations in the S&P500 time series.

In addition, we performed de Charles and Darné test for the Gaussian errors case. Two observations were identified as IO outliers: point 206 (test statistic 15.86 and $p$-value 0.049) and point 418 (test statistic 15.04 and $p$-value 0.069). The remaining points exhibit $p$-values greater than 0.30.

## 5.2 Dow Jones Industrial Average

As a second application, we identify the influential observations for the daily and monthly returns of the Dow Jones Industrial Average index. The data set was obtained from www.djindexes.com. Because these time series were also used by Doornik and Ooms (2005) in their study of outlier detection, we can compare the performance of our tests with theirs. Doornik and Ooms (2005) reported that their test compared favorably against tests from Hotta and Tsay (2012) and Franses (1999). Our proposed tests are much more simple than all of the aforementioned tests.

The annualized daily returns are calculated as $r_t = 276\Delta\log(p_t)$, where $p_t$, $t = 1, \ldots, n$, are the closest daily price indexes for the period from May 26, 1896, to December 5, 2001 ($n = 26{,}422$ returns). For the monthly returns, we use $r_t = 12\Delta\log(p_t)$, where $p_t$ is the last daily trading index in each month ($n = 1263$ returns).

The daily returns were fitted using the GARCH(1, 1) models with Gaussian, GED and Student's $t$ errors. The benchmarks use to characterize the individual

**Table 7** *The influential observations at 1% and 5% (in italics) levels for the daily DJIA time series returns*

| Date *yy\|mm\|dd* | Point | GARCH Gaussian | EGARCH GED | Date *yy\|mm\|dd* | Point | GARCH Gaussian | EGARCH GED |
|---|---|---|---|---|---|---|---|
| 1898\|03\|28 | 464 | *23.33* | | 1939\|01\|23 | 10,589 | 42.41 | |
| 1899\|02\|20 | 690 | 45.02 | | 1939\|09\|05 | 10,744 | 42.42 | |
| 1899\|12\|08 | 889 | *25.46* | | 1940\|05\|13 | 10,915 | 32.59 | |
| 1899\|12\|18 | 895 | 33.99 | | 1946\|09\|03 | 12,491 | 27.47 | |
| 1904\|12\|07 | 2141 | 26.40 | | 1947\|04\|14 | 12,643 | 28.28 | |
| 1907\|03\|14 | 2709 | 28.61 | | 1948\|11\|03 | 13,033 | 30.22 | |
| 1909\|02\|23 | 3201 | 54.59 | | 1950\|06\|26 | 13,444 | 46.41 | *16.42* |
| 1913\|01\|20 | 4182 | *25.45* | | 1955\|09\|26 | 14,760 | 162.93 | 45.49 |
| 1914\|07\|28 | 4565 | 29.92 | | 1962\|05\|28 | 16,439 | 27.09 | |
| 1914\|07\|30 | 4567 | 34.85 | | 1982\|08\|17 | 21,513 | 27.13 | |
| 1914\|12\|14 | 4568 | 87.63 | 31.11 | 1986\|09\|11 | 22,542 | 26.05 | |
| 1917\|02\|01 | 5105 | 38.15 | | 1987\|10\|19 | 22,821 | 130.38 | 31.92 |
| 1924\|02\|15 | 6856 | *23.22* | | 1989\|10\|13 | 23,324 | 122.86 | 32.48 |
| 1927\|10\|10 | 7773 | 30.52 | | 1991\|01\|17 | 23,642 | *24.30* | |
| 1929\|10\|28 | 8286 | 28.58 | | 1991\|11\|15 | 23,853 | 48.99 | |
| 1931\|06\|22 | 8697 | 27.73 | | 1997\|10\|27 | 25,370 | 36.41 | |
| 1933\|03\|15 | 9123 | 26.11 | | 2001\|09\|17 | 26,366 | 33.78 | |

observations as influential are 25.79 (22.65) at 1% (5%) for the Gaussian case and 18.54 (16.40) at 1% (5%) for the GED case using $\hat{\nu} = 1.333$. The results of the influential diagnostics are summarized in Table 7. Thus, with Gaussian errors, we detect 29 and 34 influential observations at 1% and 5%, respectively. However, using the GED errors, we detect only 4 and 5 points at 1% and 5%, respectively. In addition, the overall statistic and its standardized value are equal to 6.60 and 99.8, respectively, for Gaussian errors, and 1.57 and 9.08 for GED errors. All these values are highly significant.

The identified influential points can be associated with historical events such as World War I and II, the *Crash* of 1929, *Black Monday* in October 1987, the Asian Flu in October 1997, and the World Trade Center attack in September 2001, to name a few.

However, when the Student's $t$ errors are used, no points are detected as influential even at 10%, and the overall statistic $O_{ve} = 1.326$ is not significant ($p$-value $= 0.5$). However, as evidenced in the last section, influential points also affect the parameter estimation, especially the degree of freedom, which is estimated as $\hat{\nu} = 5.904$. Thus, we recalculate the influence statistics using $\hat{\nu} = 7$, which is a plausible value considering the 50th percentile in Table 4. As a result, the points 14,760 (with statistic equal to 226.20) and 22,821 (with statistic equal to 204.16) are detected as influential at 5% (the benchmark at 5% is 204.61), and point 23,324 with statistic 172.75 is influential at 10% (the benchmark at 10% is 165.42). The

**Table 8** *The influential observations at* 1% *and* 5% (*in italics*) *levels for the monthly DJIA time series returns. The significant overall statistics* ($O_{ve}$) *at* 5% *level are marked in bold.*

| Date *yy\|mm* | Point | GARCH Gaussian | EGARCH GED |
|---|---|---|---|
| 1914\|12 | 219 | 24.99 | |
| 1938\|03 | 498 | *16.88* | |
| 1940\|05 | 524 | 22.01 | |
| 1987\|10 | 1093 | 31.44 | *14.18* |
| $O_{ve}$ | | **4.21** | 1.52 |
| *Benchmarks* | | | |
| Individual 1% | | 19.95 | 14.79 |
| Individual 5% | | 16.84 | 12.62 |

first two points, which correspond to September 26, 1955, and October 19, 1987, are considered by financial analysts to be the most critical days for the Dow Jones Industrial Average index in the 20th century. In addition, the overall statistic $O_{ve}$ equals 1.409, which is non significant and has a $p$-value of 0.32.

The monthly DJIA returns were estimated using GARCH(1, 1) models with Gaussian, GED and Student's $t$ errors. In all cases, the estimated parameters are highly significant. For GED errors, $\hat{\nu} = 1.361$, and for the Student's $t$ errors, the estimated degree of freedom is $\hat{\nu} = 6.376$. The standardized overall statistic for the Gaussian errors (equal to 10.5) is highly significant, but is not significant for the GED errors (equal to 1.28). For the Student's $t$ case, the overall statistic $O_{ve} = 1.36$ is not significant and has a $p$-value of 0.51. In Table 8, we present the influential points that were identified using the Gaussian and GED errors. We did not find any influential points when the Student's $t$ errors were used.

We also compared our findings with the findings of Doornik and Ooms (2005). For daily returns, assuming Gaussian errors, they detect 34 outliers (at 5%) and we also detected 34 influential points, 27 of them at the same date. For monthly returns, assuming Gaussian errors, both studies detected 4 influential points, and 3 of them were common between the two studies. As reported by Doornik and Ooms (2005), these three points were also detected using the Hotta and Tsay (2012) and the Franses (1999) outlier detection tests. Furthermore, for the Student's $t$ errors case without the correction of the degree of freedom, we did not detect any influential observations in the monthly or daily series. The same result was obtained by Doornik and Ooms (2005).

## 6 Conclusions

This paper examines the slope local influence diagnostics for heteroscedastic time series models under innovative model perturbations. We derive some simple and useful statistics for characterizing influential observations. We stress that these statistics are useful for several parameterizations of the conditional standard deviations ($\sigma_t$) and not only for the models discussed in this paper. The asymptotic distribution of the proposed statistics where first derived under the assumption that the innovations are known. Simulations showed that the these benchmarks are close to the real values when the innovations are estimated. The simulation study reveals that for Gaussian and GED errors the power of the influence statistics is quite high, but for the Student's $t$, it is low, especially for the overall statistic. As illustrated by the empirical applications, the proposed statistics are useful for identifying influential observations and compare favorably with existing methods. In practice, we recommend using individual influence statistics even if the overall criterion may indicate insignificance.

## Appendix: Proof of Theorems

### Proof of Theorem 3.1

The perturbed density conditional on $\mathcal{F}_{t-1}$ is $f_z(z_t) = f(z_t\omega_t^{1/2}\sigma_t^{-1})\omega_t^{1/2}\sigma_t^{-1}$, where $f$ is the density of errors. Part (a) is a consequence of either (b) when $\nu = 2$, or (c) when $\nu \to \infty$. In part (b), the $t$th component of the perturbed log-likelihood in (3.11) is

$$l_t = A(\nu) - \frac{1}{2}\ln(\sigma_t^2) + \frac{1}{2}\ln(\omega_t) - \frac{1}{2}\lambda^{-\nu}\sigma_t^{-\nu}|z_t\omega_t^{1/2}|^{\nu},$$

where $A(\nu) = \ln(\nu/2) - (3/2)\ln\Gamma(1/\nu) + (1/2)\ln\Gamma(3/\nu)$. Because $\frac{\partial|x|^\nu}{\partial x} = \nu x|x|^{\nu-2}$ then

$$\frac{\partial l_t}{\partial\omega_i} = \frac{1}{2}\left[\frac{\partial\omega_t}{\partial\omega_i}\right]\left\{\frac{1}{\omega_t} - \frac{\nu}{2}\lambda^{-\nu}\left|\frac{z_t}{\sigma_t}\right|^{\nu}\omega_t|\omega_t|^{\nu/2-2}\right\}.$$

Because $\frac{\partial\omega_t}{\partial\omega_i} = 1$ for $t = i$ and zero elsewhere, and evaluating at $\hat{\theta}$,

$$2\frac{\partial L(\theta|\omega)}{\partial\omega_i} = 2\sum_{t=1}^{n}\frac{\partial l_t}{\partial\omega_i} = \frac{1}{\omega_i} - \frac{\nu}{2}\lambda^{-\nu}\left|\frac{z_i}{\hat{\sigma}_i}\right|^{\nu}\omega_i|\omega_i|^{\nu/2-2}.$$

When evaluating this expression at $\omega_0$ and substituting (3.12), we obtain (3.14). Finally, part (c) is proved in the same manner as (b) when considering that the $t$th component of the perturbed log-likelihood in (3.11) is

$$l_t = B(\nu) - \frac{1}{2}\ln(\sigma_t^2) + \frac{1}{2}\ln(\omega_t) - \frac{1}{2}(\nu+1)\ln\left(1 + \frac{\omega_t z_t^2}{(\nu-2)\sigma_t^2}\right),$$

where $B(\nu) = \ln\Gamma((\nu+1)/2) - \ln\Gamma(\nu/2) - \frac{1}{2}\ln\pi - \frac{1}{2}\ln(\nu-2)$. Then,

$$\frac{\partial l_t}{\partial\omega_i} = \frac{1}{2}\left[\frac{\partial\omega_t}{\partial\omega_i}\right]\left\{\frac{1}{\omega_t} - \frac{(\nu+1)z_t^2}{(\nu-2)\sigma_t^2 + z_t^2\omega_t}\right\}.$$

**Proof of Theorem 3.2**

Part (a) follows from part (b) when $\nu = 2$. Parts (b) and (c) follow from a direct application of the Central Limit Theorem on the $\{s_i^2\}$ sequence, assuming that the asymptotic distribution of $e_i$ is the same as the asymptotic distribution of $\varepsilon_i$. Therefore, we have to calculate the first and second moments of $\{s_i^2\}$.

For part (b), $s_i^2 = (1 - x_i)^2$ where $x_i = \frac{\nu}{2}\lambda^{-\nu}|\varepsilon_i|^\nu$. Because $x_i$ has Gamma$(1/\nu, \nu)$ density, then by a known property, $E(x_i) = 1$, $E(x_i^2) = 1 + \nu$, $E(x_i^3) = (1+\nu)(1+2\nu)$, and $E(x_i^4) = (1+\nu)(1+2\nu)(1+3\nu)$. Therefore, $E(s_i^2) = \nu$ and $E(s_i^4) = 3(\nu^2 + 2\nu^3)$, which implies that $\mathrm{Var}(s_i^2) = E(s_i^4) - E^2(s_i^2) = 2\nu^2(1+3\nu)$.

The distribution of $x_i$ can be calculated from

$$F_{x_i}(x) = P\big[|\varepsilon| \le (2/\nu)^{1/\nu}\lambda x^{1/\nu}\big] = F_\varepsilon\big((2/\nu)^{1/\nu}\lambda x^{1/\nu}\big) - F_\varepsilon\big(-(2/\nu)^{1/\nu}\lambda x^{1/\nu}\big),$$

and expression (3.9), which results in $f(x) = cx^{1/\nu - 1}e^{-x/\nu}$.

For part (c) from (3.15), we can write $1 - s_i$ as $1 - s_i = (\nu+1)z_i/(\nu + z_i)$ with $z_i = e_i^2\nu/(\nu - 2)$. Then, $z_i \sim F_{1,\nu}$, and therefore, $x_i = z_i/(\nu + z_i) \sim$ Beta$(1/2, \nu/2)$; see Johnson et al. (1995, page 327). Using $E(x_i^k) = E(x_i^{k-1}) \times (1/2 + k - 1)/(1/2 + \nu/2 + k - 1)$ for $k = 1, 2, \dots,$ we calculated the first fourth moments of $x_i$. Then, we substitute these values into $E(s_i^2) = E[1 - (\nu+1)x_i]^2 = 1 + (\nu+1)^2E(x_i^2) - 2(\nu+1)E(x_i)$, $E(s_i^4) = E[1 - (\nu+1)x_i]^4$ and $\mathrm{Var}(s_i^2) = E(s_i^4) - E^2(s_i^2)$, and the result follows.

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