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Domains of operator semi-attraction of probability measures on Banach spaces

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Abstract. The paper deals with operator (semi-) stability and domains of operator (semi-) attraction of probability measures on infinite dimensional Banach spaces: characterizations of operator (semi-) stability and of domains of (normal) operator (semi-) attraction are given; it is shown that the set of operator stable probability measures is a closed subset under weak topology; the domain of operator semi-attraction of a given stable probability measure coincides with its domain of operator attraction; and a probability measure is (semi-) stable iff its finite-dimensional projections are (semi-) stable.

1 Introduction and notation

Central limit theorem and Gaussian distributions play a crucial role in classical statistics. The development of applied statistics in various sciences implied the creation of new tools based on the broader class of stable probability measures containing the set of all Gaussian distributions (see Hougaard (1986); Kozubowski and Rachev (1994); Kuruoglu et al. (1998); McCulloch (1996); Nikias and Shao (1995); Palmer et al. (2008)). Further, the larger family of semi-stable probability measures can be also considered. In the last century, the concepts of semistability and domain of semi-attraction of probability measures were studied intensively on real line, on finite dimensional spaces, as well as on abstract spaces. Recently, the topic still attracts attention of many authors (Becker-Kern (2002, 2003, 2004, 2007); Bouzar and Jayakumar (2008); Chuprunov and Terekhova (2009); Csörgö (2007); Csörgö and Megyesi (2002); Divanji (2004); Fazekas and Chuprunov (2007); Fedosenko (2005); Hazod and Shah (2001); Ho Dang (2009); Kevei (2009); Kevei and Csörgö (2009); Maejima and Riddhi (2006); Maejima and Riddhi (2007); Meerschaert and Scheffler (2002); Rajput and Rama-Murthy (2004); Sato and Watanabe (2005); Shah (2007)). That motivates our study on (semi-) stability and domain of (semi-) attraction of probability measures on Banach spaces.

Section 2 of the paper studies limit behaviour of operator semistability of measures. It is found that, if $(\boldsymbol{\mu}_n)$ is a sequence of operator semistable measures convergent weakly to a probability measure $\boldsymbol{\mu}$, then $\boldsymbol{\mu}$ is also operator semistable. As a

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consequence, given a full measure $\boldsymbol{\mu}$, the set $H(\boldsymbol{\mu})$ of all real numbers r, such that $\boldsymbol{\mu}$ is r-operator semistable, is a closed multiplicative subgroup of R^+. Besides, on Banach spaces with countable Schauder basis, a measure is (semi-) stable if and only if its finite-dimensional projections are (semi-) stable.

Section 3 takes up the problem of domains of operator-semi-attraction, attempting to generalize some results given by Ho Dang (1987, 2009). Specifically, it shows that the domain of operator attraction coincides with the domain of operator r-semi-attraction for every $r \in (0, 1)$.

Section 4 is devoted to characterizing the normal domains of semi-attraction and the normal domains of attraction. Some results of Jurek (1980) have been generalized in this section.

Throughout the paper, the following notation will be used: $\mathbf{E}$ denotes an infinite dimensional separable real Banach space, $\mathbf{E}'$—its dual space, $\langle\cdot,\cdot\rangle$—the dual pairing between $\mathbf{E}$ and $\mathbf{E}'$, $\partial\mathbf{U}$—the boundary and $\mathbf{U}^c$—the complement of a set $\mathbf{U}$, $\mathbf{E}_r = \{\mathbf{x} \in \mathbf{E} : \|\mathbf{x}\| \le r\}$ $(r > 0)$. Further, $B(\mathbf{E})$ will denote the algebra of continuous linear operators on $\mathbf{E}$ with norm topology, $\mathrm{Aut}(\mathbf{E})$—the sub-algebra of $B(\mathbf{E})$ containing all invertible bounded linear operators, $\mathbf{I}$ and $\boldsymbol{\Theta}$—the unit and zero operators, respectively,

$$t^{\mathbf{B}} = \sum_{k=0}^{\infty} \ln t^k \mathbf{B}^k / k!, \qquad t > 0, \mathbf{B} \in B(\mathbf{E}).$$

By a *measure on* $\mathbf{E}$, we mean a nonnegative measure defined on σ-algebra of Borel subsets of $\mathbf{E}$. Given a σ-finite measure $\boldsymbol{\mu}$ on $\mathbf{E}$, $C(\boldsymbol{\mu}) = \{r > 0 : \boldsymbol{\mu}(\partial\mathbf{E}_r) = 0\}$, it is observed that $C(\boldsymbol{\mu})^c \cap R^+$ is countable. For a linear continuous map $\mathbf{B}$ from $\mathbf{E}$ to another real Banach space $\mathbf{E}_1$, the measure $\mathbf{B}\boldsymbol{\mu}$ is defined by $\mathbf{B}\boldsymbol{\mu}(\mathbf{F}) = \boldsymbol{\mu}(\mathbf{B}^{-1}\mathbf{F})$ for every Borel subset $\mathbf{F}$ of $\mathbf{E}_1$, $\bar{\boldsymbol{\mu}} = -\mathbf{I}\boldsymbol{\mu}$ and $|\boldsymbol{\mu}|^2 = \bar{\boldsymbol{\mu}} * \boldsymbol{\mu}$ is the symmetrization of $\boldsymbol{\mu}$, where $*$ denotes the convolution of measures. In the case when $\mathbf{B} = c\mathbf{I}$, $c > 0$, we write $T_c\boldsymbol{\mu}$ instead of $c\mathbf{I}\boldsymbol{\mu}$. A Borel subset $\mathbf{G}$ of $\mathbf{E}$ is called a $\boldsymbol{\mu}$*-continuity* set if $\boldsymbol{\mu}(\partial\mathbf{G}) = 0$, $\boldsymbol{\mu}|_{\mathbf{G}}$ is the measure defined by $\boldsymbol{\mu}|_{\mathbf{G}}(\mathbf{F}) = \boldsymbol{\mu}(\mathbf{G} \cap \mathbf{F})$. A measure on $\mathbf{E}$ is said to be *full* if its support is not contained in any proper hyperplane of $\mathbf{E}$. If $(\boldsymbol{\mu}_n)$ converges weakly to $\boldsymbol{\mu}$ we write $\boldsymbol{\mu}_n \to_w \boldsymbol{\mu}$.

Let $P(\mathbf{E})$ denote the class of all *probability measures (p.m.'s)* on $\mathbf{E}$. A sequence $(\boldsymbol{\mu}_n)$ of p.m.'s is called *shift convergent* if there exists a sequence $(\mathbf{x}_n) \subset \mathbf{E}$ such that $(\boldsymbol{\mu}_n) * \delta(\mathbf{x}_n)$ is weakly convergent, where $\delta(\mathbf{x})$ is the p.m. concentrated at the point $\mathbf{x} \in \mathbf{E}$. Similarly, $(\boldsymbol{\mu}_n)$ is said to be *relatively shift compact* if $(\boldsymbol{\mu}_n * \delta(\mathbf{x}_n))$ is relatively weak compact. It is well known that $P(\mathbf{E})$ with the topology of weak convergent is a separable metric space (see Parthasarathy (1967), Theorem II.6.2). Moreover, one can find in this space a *shift-invariant metric* ρ (e.g. the Lévy–Prokhorov metric) such that

$$\rho\big(\boldsymbol{\nu} * \delta(\mathbf{x}), \boldsymbol{\mu} * \delta(\mathbf{x})\big) = \rho(\boldsymbol{\nu}, \boldsymbol{\mu})$$

for all $\boldsymbol{\nu}, \boldsymbol{\mu} \in P(\mathbf{E})$ and $\mathbf{x} \in \mathbf{E}$.

A p.m. $\boldsymbol{\mu}$ is said to be *infinitely divisible (inf. div.)* whenever for each positive integer n there exists a p.m. $\boldsymbol{\mu}_n$ such that $\boldsymbol{\mu} = \boldsymbol{\mu}_n^n$, where the power is taken in the sense of convolution. Then the power $\boldsymbol{\mu}^c$ is well defined for every $c \in R^+$ (see Tortrat (1967) for example). For any bounded measure $\mathbf{M}$ on $\mathbf{E}$, the Poisson measure $\mathrm{Pois}(\mathbf{M})$ is defined as

$$\mathrm{Pois}(\mathbf{M}) = \exp(-\mathbf{M}(\mathbf{E})) \sum_{k=0}^{\infty} \mathbf{M}^{*k}/k!,$$

where $\mathbf{M}^{*0} = \delta(\mathbf{0})$. Let $\mathbf{M}$ be an arbitrary measure on $\mathbf{E}$ vanishing at $\mathbf{0}$. Then $\mathbf{M}$ is called a *Lévy measure* if there exists a representation $\mathbf{M} = \sup \mathbf{M}_n$, where $\mathbf{M}_n$ are bounded and the sequence $(\mathrm{Pois}(\mathbf{M}_n))$ of associated Poisson measures is relatively shift compact. Then each cluster point of the relatively compact sequence $(\mathrm{Pois}(\mathbf{M}_n) * \delta(\mathbf{x}_n))$, $\mathbf{x}_n \in \mathbf{E}$, is called a *generalized Poisson measure* and is denoted by $e(\mathbf{M})$. According to Tortrat (1967), $e(\mathbf{M})$ is uniquely determined up to translation, that is, for two cluster points, say $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, of $(\mathrm{Pois}(\mathbf{M}_n) * \delta(\mathbf{x}_n))$ and $(\mathrm{Pois}(\mathbf{M}_n) * \delta(\mathbf{y}_n))$, respectively, we have $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 * \delta(\mathbf{x})$ for certain $\mathbf{x} \in \mathbf{E}$. Let $L(\mathbf{E})$ denote the set of all Lévy measures on $\mathbf{E}$. Tortrat (1967) (see also Dettweiler (1978)) pointed out that each inf. div. p.m. $\boldsymbol{\mu}$ on $\mathbf{E}$ has an unique representation $\boldsymbol{\mu} = \boldsymbol{\gamma} * e(\mathbf{M})$, where $\boldsymbol{\gamma}$ is a symmetric Gaussian measure on $\mathbf{E}$ and $\mathbf{M} \in L(\mathbf{E})$.

2 Semistability in limit

A probability measure $\boldsymbol{\mu}$ is called an *operator r-semistable measure*, $r \in R^+$, if it is inf. div. and there are an operator $\mathbf{B} \in B(\mathbf{E})$ and an element $\mathbf{x} \in \mathbf{E}$ such that $\boldsymbol{\mu}^r = \mathbf{B}\boldsymbol{\mu} * \delta(\mathbf{x})$. More precisely, we say that $\boldsymbol{\mu}$ is $(\mathbf{B}, r)$-*semistable*. Further, if $\mathbf{B}$ is of the form $c\mathbf{I}$, $c \in R$, we can omit the "operator" and say that $\boldsymbol{\mu}$ is (c, r)-*semistable* instead of $(c\mathbf{I}, r)$-semistable. In the latter case, it is well known that, for a non degenerated measure $\boldsymbol{\mu}$, the unique solution p of the equation $|c|^p = r$ lies in the interval $(0, 2]$, $p = 2$ if and only if $\boldsymbol{\mu}$ is a Gaussian measure, and $\boldsymbol{\mu}$ has no Gaussian component if $p \neq 2$. The number p is called the *exponent* of the measure $\boldsymbol{\mu}$.

In addition, we say that $\boldsymbol{\mu}$ is $\mathbf{B}$-*stable* (or *operator stable*) if it is inf. div. and for all $t \in (0, 1]$ there exists an element $\mathbf{y}_t \in \mathbf{E}$ such that

$$\boldsymbol{\mu}^t = t^{\mathbf{B}}\boldsymbol{\mu} * \delta(\mathbf{y}_t).$$

Moreover, if the operator $\mathbf{B}$ in the above formula is of the form $c\mathbf{I}$ then $\boldsymbol{\mu}$ is said to be a *stable* measure.

Let $H(\boldsymbol{\mu})$ be the set of all positive numbers r such that $\boldsymbol{\mu}$ is operator $(\mathbf{B}, r)$-semistable with $\mathbf{B} \in \mathrm{Aut}(\mathbf{E})$. In the case of finite dimensional spaces, Luczak (1984) pointed out that $H(\boldsymbol{\mu})$ is a closed multiplicative subgroup of R^+, that is, $H(\boldsymbol{\mu}) = R^+$ or $H(\boldsymbol{\mu})$ is a discrete group generated by some element $s \in (0, 1)$.

Using the terminology given in Ho Dang (2009), if $\boldsymbol{\mu}$ is $(\mathbf{B}, r)$-semistable and $H(\boldsymbol{\mu})$ is the discrete multiplicative subgroup generated by s then we say that $\boldsymbol{\mu}$ is operator (s)-*semistable*, or $(\mathbf{B}, (s))$-*semistable*. Below we show that for the case of full measure $\boldsymbol{\mu}$ on Banach spaces, $H(\boldsymbol{\mu})$ is also a closed multiplicative subgroup of R^+. At first, we formulate auxiliary lemmas:

Lemma 1. *Let $\boldsymbol{\mu}$, $\boldsymbol{\mu}_n$, $n = 1, 2, \ldots$ be inf. div. measures and t, $t_n \in [0, \infty)$. Suppose that $\boldsymbol{\mu}_n \to_w \boldsymbol{\mu}$ and $t_n \to t$, then there exists a sequence $(\mathbf{z}_n) \subset \mathbf{E}$ such that*

$$\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{z}_n) \to_w \boldsymbol{\mu}^t.$$

Proof. Let N be a natural number such that $t \le N$ and $t_n \le N$ for all n, then $\boldsymbol{\mu}_n^N \to_w \boldsymbol{\mu}^N$. Plus, because $\boldsymbol{\mu}_n^{t_n}$ is a factor of $\boldsymbol{\mu}_n^N$, Theorem III.5.1 of Parthasarathy (1967) implies the relative shift compactness of $(\boldsymbol{\mu}_n^{t_n})$, that is, there is a sequence $(\mathbf{x}_n) \subset \mathbf{E}$ such that the sequence $(\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{x}_n))$ is relatively weak compact. Meanwhile, it is clear that for every $\mathbf{y}' \in \mathbf{E}'$ we have

$$(\mathbf{y}'\boldsymbol{\mu}_n)^{t_n} \to_w (\mathbf{y}'\boldsymbol{\mu})^t.$$

Therefore, if $\boldsymbol{\nu}$ is any cluster point of the sequence $(\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{x}_n))$, meaning $\boldsymbol{\mu}_{n_k}^{t_{n_k}} * \delta(\mathbf{x}_{n_k}) \to_w \boldsymbol{\nu}$ for some subsequence (n_k) of natural numbers, then the Convergence of Type theorem implies $\mathbf{y}'\boldsymbol{\nu} = (\mathbf{y}'\boldsymbol{\mu})^t * \delta(\mathbf{x}_{\mathbf{y}'})$ for some real $\mathbf{x}_{\mathbf{y}'}$.

Hence, with $\boldsymbol{\lambda}^\wedge$ denoted for the characteristic functional of a p.m. $\boldsymbol{\lambda}$, we see that $[\mathbf{y}'(\boldsymbol{\mu}_{n_k}^{t_{n_k}} * \delta(\mathbf{x}_{n_k}))]^\wedge(\mathbf{u}) \to [\mathbf{y}'\boldsymbol{\nu}]^\wedge(\mathbf{u})$ for any real number $\mathbf{u}$, and

$$\left([\mathbf{y}'\boldsymbol{\mu}_{n_k}]^\wedge(\mathbf{u})\right)^{t_{n_k}}.e^{i\mathbf{u}\langle \mathbf{y}';\mathbf{x}_{n_k}\rangle} \to \left([\mathbf{y}'\boldsymbol{\mu}]^\wedge(\mathbf{u})\right)^t.e^{i\mathbf{u}\mathbf{x}_{\mathbf{y}'}},$$

that yields $\mathbf{u}\langle \mathbf{y}'; \mathbf{x}_{n_k}\rangle \to \mathbf{u}\mathbf{x}_{\mathbf{y}'}$. Consequently, $(\mathbf{x}_{n_k})$ as a sequence of elements in the second dual $\mathbf{E}''$ is $*$-weakly convergent to some $\mathbf{x}_0'' \in \mathbf{E}''$. With $\boldsymbol{\mu}$, $\boldsymbol{\mu}_{n_k}$ and $\boldsymbol{\nu}$ treated as p.m.'s on $\mathbf{E}''$ and passing to the limit, we obtain

$$\boldsymbol{\nu}^\wedge(\mathbf{y}') = [\boldsymbol{\mu}^\wedge(\mathbf{y}')]^t.e^{i\langle \mathbf{y}';\mathbf{x}_0''\rangle} = [\boldsymbol{\mu}^t * \delta(\mathbf{x}_0'')]^\wedge(\mathbf{y}'),$$

which means $\boldsymbol{\nu} = \boldsymbol{\mu}^t * \delta(\mathbf{x}_0'')$. However, both $\boldsymbol{\nu}$ and $\boldsymbol{\mu}^t$ are concentrated on $\mathbf{E}$, $\mathbf{x}_0''$ must be an element of $\mathbf{E}$ (renamed as $\mathbf{x}_0$), and $\boldsymbol{\nu} = \boldsymbol{\mu}^t * \delta(\mathbf{x}_0)$.

Let $\mathbf{L} = \mathrm{LIM}(\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{x}_n))$ be the set of all cluster points of the sequence $(\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{x}_n))$. Then, since $(\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{x}_n))$ is a relatively weak compact sequence, $\mathbf{L}$ is a compact subset of $P(\mathbf{E})$ and $\mathbf{L} = \{\boldsymbol{\nu} = \boldsymbol{\mu}^t * \delta(\mathbf{x}) : \mathbf{x} \in \mathbf{D}\}$ with compact subset $\mathbf{D} \subset \mathbf{E}$.

Let d denote the distance between a point and a subset in $P(\mathbf{E})$, i.e. $d(\boldsymbol{\lambda}, \mathbf{L}) = \inf_{\boldsymbol{\nu} \in \mathbf{L}} \rho(\boldsymbol{\lambda}, \boldsymbol{\nu})$ for any $\lambda \in P(\mathbf{E})$, where ρ is the Lévy–Prokhorov metric in $P(\mathbf{E})$. Then $d(\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{x}_n), \mathbf{L}) \to 0$. In such case, for every n we can choose an element $\mathbf{x}_n^o \in \mathbf{D}$ such that

$$\rho\left(\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{x}_n), \boldsymbol{\mu}^t * \delta(\mathbf{x}_n^o)\right) < d\left(\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{x}_n), \mathbf{L}\right) + 1/n.$$

In consequence, $\rho(\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{x}_n - \mathbf{x}_n^o), \boldsymbol{\mu}^t) \to 0$, which means $\boldsymbol{\mu}_n^{t_n} * \delta(\mathbf{z}_n) \to_w \boldsymbol{\mu}^t$ with $\mathbf{z}_n = \mathbf{x}_n - \mathbf{x}_n^o$, the lemma is proved. $\square$

Lemma 2. *Let $\boldsymbol{\mu} \neq \delta(\mathbf{0})$ be an inf. div. p.m. and $\mathbf{B}_n \in B(\mathbf{E})$, $\mathbf{x}_n \in \mathbf{E}$, $t_n \in R^+$,*

$$\boldsymbol{\mu}^{t_n} = \mathbf{B}_n \boldsymbol{\mu} * \delta(\mathbf{x}_n),$$

for $n = 1, 2, \ldots$.

(a) *If* $\sup \|\mathbf{B}_n\| < \infty$ *and $t_n \to t$, $t > 0$, then $(\mathbf{B}_n)$, $(\mathbf{x}_n)$ are compact sequences and $\boldsymbol{\mu}$ is an operator t-semistable measure.*

(b) *If $\mathbf{B}_n \to \mathbf{B} \in B(\mathbf{E})$, $\mathbf{x}_n \to \mathbf{x} \in \mathbf{E}$ and $c_1 \le t_n \le c_2$, $0 < c_1 < c_2 < \infty$, $n = 1, 2, \ldots$, then there exists a number $t \in [c_1, c_2]$ such that $t_n \to t$ and $\boldsymbol{\mu}$ is $(\mathbf{B}, t)$-semistable.*

Proof. (a) Let $t_n \to t$, from Theorem 5 in Chung Dong et al. (1982),

$$\mathbf{B}_n \boldsymbol{\mu} * \delta(\mathbf{x}_n) = \boldsymbol{\mu}^{t_n} \to_w \boldsymbol{\mu}^t.$$

Hence, because $\sup \|\mathbf{B}_n\| < \infty$, Theorem 4.11 of Linde and Siegel (1990) implies the relative compactness (in the sense of point wise convergence) of the sequences $(\mathbf{B}_n)$ and $(\mathbf{x}_n)$. Given $\mathbf{B}_{n'} \to \mathbf{B} \in B(\mathbf{E})$, $\mathbf{x}_{n'} \to \mathbf{x} \in \mathbf{E}$ for some subsequence (n') of natural numbers, taking into account Theorem 3.1 of Linde and Siegel (1990) we get

$$\mathbf{B}_{n'} \boldsymbol{\mu} * \delta(\mathbf{x}_{n'}) \to_w \mathbf{B}\boldsymbol{\mu} * \delta(\mathbf{x}),$$

and therefore $\boldsymbol{\mu}^t = \mathbf{B}\boldsymbol{\mu} * \delta(\mathbf{x})$, i.e. $\boldsymbol{\mu}$ is $(\mathbf{B}, t)$-semistable.

(b) Let $\mathbf{B}_n \to \mathbf{B} \in B(\mathbf{E})$ and $\mathbf{x}_n \to \mathbf{x} \in \mathbf{E}$, then $\sup \|\mathbf{B}_n\| < \infty$. With the same argument as the above, we see that

$$\boldsymbol{\mu}^{t_n} = \mathbf{B}_n \boldsymbol{\mu} * \delta(\mathbf{x}_n) \to_w \mathbf{B}\boldsymbol{\mu} * \delta(\mathbf{x}). \tag{2.1}$$

On the other hand, (t_n) is relatively compact and if t, s are two cluster points of this sequence then $t \neq 0 \neq s$. Therefore, $\boldsymbol{\mu}^t \neq \delta(\mathbf{0}) \neq \boldsymbol{\mu}^s$. Meanwhile, in view of Theorem 3.1 in Linde and Siegel (1990), Lemma 1 and (2.1), there exist elements $\mathbf{z}_t, \mathbf{z}_s \in \mathbf{E}$ such that

$$\boldsymbol{\mu}^t = \mathbf{B}\boldsymbol{\mu} * \delta(\mathbf{x} + \mathbf{z}_t);$$
$$\boldsymbol{\mu}^s = \mathbf{B}\boldsymbol{\mu} * \delta(\mathbf{x} + \mathbf{z}_s).$$

Hence $t = s$ by virtue of Proposition I.4.7 in Vakhaniya et al. (1985), the proof is complete. □

Proposition 1. *Let $\boldsymbol{\mu}$ be a full inf. div. p.m. on a Banach space. Then $H(\boldsymbol{\mu})$ a closed multiplicative subgroup of R^+.*

Proof. It is clear that if $\boldsymbol{\mu}$ is full $(\mathbf{B}, t)$-semistable and $\mathbf{B}$ is invertible then $\boldsymbol{\mu}$ is $(\mathbf{B}^{-1}, t^{-1})$-semistable. Besides, it is quite trivial that $H(\boldsymbol{\mu})$ is a multiplicative subgroup of R^+.

Let now $(t_n) \subset H(\boldsymbol{\mu})$ and $t_n \to t \in R^+$. Then for $n = 1, 2, \ldots$ we have

$$\boldsymbol{\mu}^{t_n} = \mathbf{B}_n \boldsymbol{\mu} * \delta(\mathbf{x}_n) \tag{2.2}$$

with invertible $\mathbf{B}_n \in \mathrm{Aut}(\mathbf{E})$ and $\mathbf{x}_n \in \mathbf{E}$. We claim that

$$\sup \|\mathbf{B}_n\| < \infty. \tag{2.3}$$

Indeed, if $\|\mathbf{B}_{n'}\| \to \infty$ for some subsequence (n') of natural numbers, then

$$\|\mathbf{B}_{n'}\|^{-1}\mathbf{I}\boldsymbol{\mu}^{t_{n'}} = \|\mathbf{B}_{n'}\|^{-1}\mathbf{B}_{n'}\boldsymbol{\mu} * \delta(\mathbf{x}_{n'}).$$

It is clear that the left-hand side of the above equation tends to $\delta(\mathbf{0})$. On the other hand, because $\|\|\mathbf{B}_{n'}\|^{-1}\mathbf{B}_{n'}\| = 1$, Theorem 4.11 of Linde and Siegel (1990) implies the existence of cluster points $\mathbf{x}_0 \in \mathbf{E}$ of the sequence $(\mathbf{x}_{n'})$ and $\mathbf{A} \in B(\mathbf{E})$ of the sequence $(\|\mathbf{B}_{n'}\|^{-1}\mathbf{B}_{n'})$, $\|\mathbf{A}\| = 1$, such that $\mathbf{A}\boldsymbol{\mu} * \delta(\mathbf{x}_0)$ is a cluster point of the right hand side of the equation. Consequently, $\delta(\mathbf{0}) = \mathbf{A}\boldsymbol{\mu} * \delta(\mathbf{x}_0)$, which yields $\delta(-\mathbf{x}_0) = \mathbf{A}\boldsymbol{\mu}$, meaning $\boldsymbol{\mu}$ is supported on the proper hyperplane $\mathbf{A}^{-1}(\{-\mathbf{x}_0\})$, as $\mathbf{A} \neq \boldsymbol{\Theta}$. That indicates $\boldsymbol{\mu}$ is not a full measure and it contradicts the initial assumption. Thus, (2.3) is valid.

Consequently, Lemma 2 together with (2.2) and (2.3) confirms that $\boldsymbol{\mu}$ is $(\mathbf{B}, t)$-semistable with some cluster point $\mathbf{B} \in B(\mathbf{E})$ of the sequence $(\mathbf{B}_n)$. To complete the proof we need only to point out that $\mathbf{B}$ is convertible, that implies $t \in H(\boldsymbol{\mu})$.

Indeed, let (n_k) be a subsequence of natural numbers such that $\mathbf{B}_{n_k} \to \mathbf{B}$. Then from (2.2), we have

$$\boldsymbol{\mu}^{1/t_{n_k}} = \mathbf{B}_{n_k}^{-1}\boldsymbol{\mu} * \delta\big(-(1/t_{n_k})\mathbf{B}_{n_k}^{-1}\mathbf{x}_{n_k}\big). \tag{2.4}$$

With the same argument as the above, using (2.4) instead of (2.2) and $(\mathbf{B}_{n_k}^{-1})$ in place of $(\mathbf{B}_n)$, we conclude that $\sup \|\mathbf{B}_{n_k}^{-1}\| < \infty$ and there exists a cluster point $\mathbf{B}_0 \in B(\mathbf{E})$ of the sequence $(\mathbf{B}_{n_k}^{-1})$.

Let $(n_{k'})$ be another subsequence of the sequence (n_k) such that $\mathbf{B}_{n_{k'}}^{-1} \to \mathbf{B}_0$. Then $\mathbf{B}_{n_{k'}} \to \mathbf{B}$ and $\mathbf{B}_{n_{k'}}\mathbf{B}_{n_{k'}}^{-1} = \mathbf{I}$, $\mathbf{B}_{n_{k'}}^{-1}\mathbf{B}_{n_{k'}} = \mathbf{I}$, for every $n_{k'} \in (n_{k'})$. Therefore, $\mathbf{B}\mathbf{B}_0 = \mathbf{I}$ and $\mathbf{B}_0\mathbf{B} = \mathbf{I}$, or $\mathbf{B}_0 = \mathbf{B}^{-1}$ and $\mathbf{B}$ is invertible. The proposition is just proved. $\square$

From the proposition, we can see that the concept of $(\mathbf{B}, (s))$-semistability is well defined for full measures on infinite dimensional Banach spaces. Namely, a full measure $\boldsymbol{\mu}$ is said to be $(\mathbf{B}, (s))$-*semistable* with an invertible operator $\mathbf{B}$ if it is operator semistable and $H(\boldsymbol{\mu})$ is the discrete multiplicative subgroup generated by s.

Lemma 3. *Given* $\boldsymbol{\mu}, \boldsymbol{\mu}_n \in P(\mathbf{E})$, $\mathbf{B}_n \in B(\mathbf{E})$, $0 < c_1 < t_n < c_2 < \infty$, *suppose that* $\boldsymbol{\mu}_n$ *are* $(\mathbf{B}_n, t_n)$*-semistable*, $\sup \|\mathbf{B}_n\| < \infty$ *with* $n = 1, 2, \ldots$ *and* $\boldsymbol{\mu}_n \to_w \boldsymbol{\mu}$. *Then there exist a number* $t \in [c_1, c_2]$ *and an operator* $\mathbf{B} \in B(\mathbf{E})$ *such that* $\boldsymbol{\mu}$ *is* $(\mathbf{B}, t)$*-semistable.*

Proof. From the assumption, we get $\boldsymbol{\mu}^{t_n} = \mathbf{B}_n\boldsymbol{\mu}_n * \delta(\mathbf{x}_n)$ for certain $\mathbf{x}_n \in \mathbf{E}$, $n = 1, 2, \ldots$. On the other hand, by virtue of Lemma 1, we can find a subsequence (n') of natural numbers, $(\mathbf{z}_n) \subset \mathbf{E}$ and a number $t \in [c_1, c_2]$ such that $\boldsymbol{\mu}_{n'}^{t_{n'}} * \delta(\mathbf{z}_{n'}) \to_w \boldsymbol{\mu}^t$. Therefore, $\mathbf{B}_{n'}\boldsymbol{\mu}_{n'} * \delta(\mathbf{x}_{n'} + \mathbf{z}_{n'}) \to_w \boldsymbol{\mu}^t$. Hence, by virtue of Theorem 4.11 in Linde and Siegel (1990), there exists an operator $\mathbf{B} \in B(\mathbf{E})$ such that $\boldsymbol{\mu}$ is $(\mathbf{B}, t)$-semistable. □

It is well known that a p.m. on a Banach space is Gaussian if and only if all its one-dimensional projections are Gaussian. A similar feature of (semi-) stability can be described as follows.

Proposition 2. *Let* $\mathbf{E}$ *be a real Banach space with countable Schauder basis and* $\boldsymbol{\mu}$ *be an inf. div. p.m. on* $\mathbf{E}$. *Then*

(a) *For given* $t \in R^+$, $\boldsymbol{\mu}$ *is* t*-semistable if and only if all its projections on finite-dimensional subspace of* $\mathbf{E}$ *is* t*-semistable,*

(b) $\boldsymbol{\mu}$ *is stable if and only if all its projections on finite-dimensional subspace of* $\mathbf{E}$ *is stable.*

Proof. Let $\mathbf{\Pi}$ be any linear projector from $\mathbf{E}$ into a given subspace. Then $\mathbf{\Pi}(\boldsymbol{\lambda}_1 * \boldsymbol{\lambda}_2) = \mathbf{\Pi}\boldsymbol{\lambda}_1 * \mathbf{\Pi}\boldsymbol{\lambda}_2$ for $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in P(\mathbf{E})$. Therefore, $\mathbf{\Pi}\boldsymbol{\mu}$ is inf. div. and $(\mathbf{\Pi}\boldsymbol{\mu})^s = \mathbf{\Pi}(\boldsymbol{\mu}^s)$ for each positive $s \in R^+$ by virtue of the infinite divisibility of $\boldsymbol{\mu}$. Suppose $\boldsymbol{\mu}$ is t-semistable for given $t \in R^+$, i.e. $\boldsymbol{\mu}^t = T_c\boldsymbol{\mu} * \delta(\mathbf{x})$ for some real c. Then

$$(\mathbf{\Pi}\boldsymbol{\mu})^t = \mathbf{\Pi}(\boldsymbol{\mu}^t) = \mathbf{\Pi}\big(T_c\boldsymbol{\mu} * \delta(\mathbf{x})\big) = \mathbf{\Pi}(c\mathbf{I})\boldsymbol{\mu} * \delta(\mathbf{\Pi}\mathbf{x}) = T_c(\mathbf{\Pi}\boldsymbol{\mu}) * \delta(\mathbf{\Pi}\mathbf{x}),$$

that means the t-semistability of $(\mathbf{\Pi}\boldsymbol{\mu})$ and the "only if" part of (a) is shown.

To prove the sufficient condition of (a), let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$ be a countable Schauder basis of $\mathbf{E}$. For each natural n, by $\mathbf{E}_n$ we denote the subspace of $\mathbf{E}$ generated by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, that is, $\mathbf{E}_n = \mathrm{lin}(\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\})$, by $\mathbf{\Pi}_n$—the natural linear projector from $\mathbf{E}$ into $\mathbf{E}_n$. Then it is evident that $\mathbf{\Pi}_n \to \mathbf{I}$ and $\mathbf{\Pi}_n\boldsymbol{\mu} \to_w \boldsymbol{\mu}$.

Assume now $\mathbf{\Pi}_n\boldsymbol{\mu}$ are t-semistable, that is,

$$\mathbf{\Pi}_n\boldsymbol{\mu}^t = T_{c_n}(\mathbf{\Pi}_n\boldsymbol{\mu}) * \delta(\mathbf{x}_n) \tag{2.5}$$

with $c_n \in R$, $\mathbf{x}_n \in \mathbf{E}_n$, $n = 1, 2, \ldots$, we have $\mathbf{\Pi}_n\mathbf{\Pi}_m = \mathbf{\Pi}_m\mathbf{\Pi}_n = \mathbf{\Pi}_n$, $\mathbf{\Pi}_n\mathbf{x}_m = \mathbf{x}_n$, and $\mathbf{\Pi}_n\boldsymbol{\mu}^t = T_{c_m}(\mathbf{\Pi}_n\boldsymbol{\mu}) * \delta(\mathbf{x}_n)$, for all natural numbers $n < m$. The last equation together with (2.5) implies

$$T_{c_n}(\mathbf{\Pi}_n\boldsymbol{\mu}) = T_{c_m}(\mathbf{\Pi}_n\boldsymbol{\mu}). \tag{2.6}$$

First, we assume that $\boldsymbol{\mu}$ is a full measure on $\mathbf{E}$. In that case $\boldsymbol{\mu}_n = \mathbf{\Pi}_n\boldsymbol{\mu}$ is a full measure on $\mathbf{E}_n$, therefore from (2.6) we confirm that $c_n = c_m = c$ with some constant $c > 0$, independent of n. Hence, the condition of Lemma 3 is satisfied for $\boldsymbol{\mu}_n = \mathbf{\Pi}_n\boldsymbol{\mu}$ and $\mathbf{B}_n = c\mathbf{\Pi}_n$. In consequence, $\boldsymbol{\mu}$ is $(\mathbf{B}, t)$-semistable for some operator $\mathbf{B} \in B(\mathbf{E})$. Then, from (2.5) we can conclude $\mathbf{B} = c\mathbf{I}$, implying $\boldsymbol{\mu}$ is t-semistable.

In the case when the fullness of the measure $\boldsymbol{\mu}$ is not true, we can treat $\boldsymbol{\mu}$ as a full measure on the closed subspace $\mathbf{E}_{\mu}$ generated by its support and all finite projections of that full measure in $\mathbf{E}_{\mu}$ are t-semistable. Then by the similar to the above argument, $\boldsymbol{\mu}$ is a t-semistable measure on $\mathbf{E}_{\mu}$ and then $\boldsymbol{\mu}$ is also a t-semistable measure on $\mathbf{E}$.

To complete the proof, we use the fact that (b) is an immediate consequence of (a), because a measure is stable when and only when it is t-semistable for every positive number t. $\square$

3 Domains of operator semi-attraction

Let $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \in P(\mathbf{E})$, $(\mathbf{A}_k) \subset B(\mathbf{E})$, $(\mathbf{x}_k) \subset \mathbf{E}$ and (n_k) be a subsequence of natural numbers. In this section, we study the convergence

$$\mathbf{A}_k \boldsymbol{\lambda}^{n_k} * \delta(\mathbf{x}_k) \to_w \boldsymbol{\mu} \tag{3.1}$$

endowed with one of the following conditions:

$$n_k/n_{k+1} \to r \in (0, 1), \tag{$*$}$$

$$n_k/n_{k+1} \geq c > 0, \qquad k = 1, 2, \ldots, \tag{$**$}$$

$$\mathrm{LIM}(n_k/n_{k+1}) \cap (0, 1) \neq \varnothing, \tag{$***$}$$

where $\mathrm{LIM}(s_k)$ denotes the set of all cluster points of the sequence (s_k).

Now, by the same argument of Theorem 3.1 in Siegel (1989), we have the following lemma.

Lemma 4. *Suppose that $\boldsymbol{\lambda}$ is non degenerated. If (3.1) is satisfied, then*

$$\mathbf{A}_k |\boldsymbol{\lambda}|^2 \to_w \delta(\mathbf{0})$$

and $\boldsymbol{\mu}$ is inf. div.

The following lemma is also necessary for further exploration.

Lemma 5. *Let $\boldsymbol{\mu}, \boldsymbol{\lambda}_k \in P(\mathbf{E})$, $k = 1, 2, \ldots$, (n_k) and (m_k) be two subsequences of natural numbers. Suppose that $\boldsymbol{\lambda}_k^{n_k} \to_w \boldsymbol{\mu}$ and $m_k/n_k \to t \in R$. Then there exists a sequence $(\mathbf{y}_k) \subset \mathbf{E}$ such that*

$$\boldsymbol{\lambda}_k^{m_k} * \delta(\mathbf{y}_k) \to_w \boldsymbol{\mu}^t.$$

Proof. In the same way as in the proof of Theorem 3.1 given by Siegel (1989), we can show that $\boldsymbol{\mu}$ is inf. div. and then $\boldsymbol{\mu}^t$ is well defined. For a natural number N such that $m_k/n_k \leq N$ for all k, the lemma is proved by an argument similar to that used in the proof of Lemma 1. □

We say that $\boldsymbol{\lambda}$ belongs to the *domain of operator r-semi-attraction* of $\boldsymbol{\mu}$, i.e. $\boldsymbol{\lambda} \in \mathrm{DOSA}(r, \boldsymbol{\mu})$ if (3.1) and $(*)$ are true. In the finite dimensional case, Jajte (1977) has shown that a full p.m. $\boldsymbol{\mu}$ is an operator semistable measure if and only if its DOSA is not empty. Moreover, according to Theorem 4 in Ho Dang (2009), a p.m. $\boldsymbol{\lambda}$ on a finite dimensional space is an operator semistable measure if (3.1) and $(***)$ hold. The infinite dimensional version of that is the following.

Proposition 3. *Given a full measure $\boldsymbol{\mu} \in P(\mathbf{E})$, the following conditions are equivalent*:

(i) *There exist an invertible operator* $\mathbf{B}$ *and a number* $r \in (0, 1)$ *such that* $\boldsymbol{\mu}$ *is* $(\mathbf{B}, r)$*-semistable.*

(ii) *There exist p.m.* $\boldsymbol{\lambda} \in P(\mathbf{E})$, *a subsequence* (n_k) *of natural numbers, a sequence of invertible operators* $\mathbf{A_k} \in B(\mathbf{E})$ *and a sequence* $(\mathbf{x}_k) \subset \mathbf{E}$ *such that* (3.1) *and* $(***)$ *hold.*

Proof. If $\boldsymbol{\mu}$ is a full $(\mathbf{B}, r)$-semistable measure with invertible $\mathbf{B}$, that is, $\boldsymbol{\mu}$ is inf. div. and

$$\boldsymbol{\mu}^r = \mathbf{B}\boldsymbol{\mu} * \delta(\mathbf{x})$$

for some $\mathbf{x} \in \mathbf{E}$, then

$$\boldsymbol{\mu} = \mathbf{B}^k \boldsymbol{\mu}^{r^{-k}} * \delta(\mathbf{z}_k) \tag{3.2}$$

with $\mathbf{z}_k \in \mathbf{E}$, $k = 1, 2, \ldots$.

Thus, by virtue of Theorem III.2.2 in Parthasarathy (1967), the sequence $(\mathbf{B}^k \boldsymbol{\mu}^{[r^{-k}]})$ is shift compact, where $[s]$ denotes the integer part of a number $s \in R$. Hence, Lemma 4 implies $\mathbf{B}^k |\boldsymbol{\mu}|^2 \to_w \delta(\mathbf{0})$. Then, the sequence $(\mathbf{B}^k \boldsymbol{\mu}^{r^{-k}-[r^{-k}]})$ is shift convergent to $\delta(\mathbf{0})$. This together with (3.2) implies the existence of a sequence $(\mathbf{x}_k) \subset \mathbf{E}$, such that

$$\mathbf{B}^k \boldsymbol{\mu}^{[r^{-k}]} * \delta(\mathbf{x}_k) \to_w \boldsymbol{\mu},$$

i.e. (ii) is satisfied with invertible $\mathbf{A_k} = \mathbf{B}^k$ for $k = 1, 2, \ldots$.

Conversely, let (ii) be true. Then, in view of Lemma 4, $\boldsymbol{\mu}$ is inf. div. and

$$\left(\mathbf{A}_k \boldsymbol{\lambda} * \delta\left(\frac{1}{n_k}\mathbf{x}_k\right)\right)^{n_k} = \mathbf{A}_k \boldsymbol{\lambda}^{n_k} * \delta(\mathbf{x}_k) \to_w \boldsymbol{\mu}, \tag{3.3}$$

$$\begin{aligned} &(\mathbf{A}_{k+1}\mathbf{A}_k^{-1})\left(\mathbf{A}_k \boldsymbol{\lambda} * \delta\left(\frac{1}{n_k}\mathbf{x}_k\right)\right)^{n_{k+1}} * \delta(\mathbf{z}_k) \\ &= \mathbf{A}_{k+1}\boldsymbol{\lambda}^{n_{k+1}} * \delta(\mathbf{x}_{k+1}) \to_w \boldsymbol{\mu} \end{aligned} \tag{3.4}$$

with $\mathbf{z}_k = \mathbf{x}_{k+1} - \mathbf{A}_{k+1}\mathbf{A}_k^{-1}(\frac{n_{k+1}}{n_k}\mathbf{x}_k)$.

We contend at first that

$$\sup\|\mathbf{A}_{k+1}\mathbf{A}_k^{-1}\| \leq \infty. \tag{3.5}$$

Indeed, if (3.5) fails, then there always exists a subsequence (k') of natural numbers such that $\|\mathbf{A}_{k'+1}\mathbf{A}_{k'}^{-1}\| \to \infty$ and (3.4) leads to

$$\begin{aligned}&\frac{\mathbf{A}_{k'}\mathbf{A}_{k'+1}^{-1}}{\|\mathbf{A}_{k'+1}\mathbf{A}_{k'}^{-1}\|}\left(\mathbf{A}_{k'+1}\boldsymbol{\lambda} * \delta\left(\frac{1}{n_{k'+1}}\mathbf{x}_{k'+1}\right)\right)^{n_{k'}} * \delta(\mathbf{y}_{k'})\\ &= \frac{\mathbf{I}}{\|\mathbf{A}_{k'+1}\mathbf{A}_{k'}^{-1}\|}(\mathbf{A}_k\boldsymbol{\lambda}^{n_{k'}} * \delta(\mathbf{x}_{k'})) \to_w \delta(\mathbf{0})\end{aligned}$$

with $\mathbf{y}_{k'} = \mathbf{z}_{k'+1}/\|\mathbf{A}_{k'+1}\mathbf{A}_{k'}^{-1}\|$ and $\|\frac{\mathbf{A}_{k'}\mathbf{A}_{k'+1}^{-1}}{\|\mathbf{A}_{k'+1}\mathbf{A}_{k'}^{-1}\|}\| = 1$ for all $k' \in (k')$.

Combining the above evidence together with (3.3), Lemma 5 and Theorem 4.11 of Linde and Siegel (1990) we conclude the existence of an operator $\mathbf{A} \in B(\mathbf{E})$ with $\|\mathbf{A}\| = 1$ and an element $\mathbf{x}_0 \in \mathbf{E}$ such that $\mathbf{A}\boldsymbol{\mu}^r * \delta(\mathbf{x}_0) = \delta(\mathbf{0})$. Hence, by the same reason as in the proof of Proposition 1, $\boldsymbol{\mu}$ is not full, that contradicts the assumption, therefore (3.5) must be satisfied.

Taking $(***)$ into account, we get $\frac{n_{k'}}{n_{k'+1}} \to r$ for some subsequence (k') of natural numbers and for some $r \in (0, 1)$. Then we can invoke (3.3), (3.4), (3.5), Theorem 4.11 of Linde and Siegel (1990) and Lemma 5 to deduce that

$$\boldsymbol{\mu} = \mathbf{B}\boldsymbol{\mu}^{-r} * \delta(\mathbf{x})$$

with certain element $\mathbf{x} \in \mathbf{E}$ and certain operator $\mathbf{B} \in B(\mathbf{E})$, being a cluster point of the sequence of invertible operators $(\mathbf{A}_{k'+1}\mathbf{A}_{k'}^{-1})$ in the strong operator topology. Next, arguing as in the proof of Proposition 1 we can infer that the operator $\mathbf{B}$ is invertible. The proof is complete. □

Note. If (3.1) holds for some sequences (n_k) then under power of Lemma 2 we see that

$$\mathbf{A}_k\boldsymbol{\lambda}^{n_k+1} * \delta(\mathbf{x}_k) \to_w \boldsymbol{\mu} \tag{3.1$'$}$$

and $(n_k/n_k + 1) \to 1$. Then from (3.1) and (3.1′) we can build new sequences $(\mathbf{A}_k^1) \subset B(\mathbf{E})$, $(\mathbf{x}_k^1) \subset \mathbf{E}$ and (n_k^1), such that (3.1) holds for them and $1 \in \mathrm{LIM}(n_k^1/n_{k+1}^1)$ for arbitrary inf. div. measure $\boldsymbol{\mu}$. Therefore, the condition $(***)$ in the above theorem can not be replaced by $(**)$.

Proposition 4. *Let* $\boldsymbol{\mu}, \boldsymbol{\lambda} \in P(\mathbf{E})$, $\mathbf{B} \in \mathrm{Aut}(\mathbf{E})$ *and* $r \in (0, 1)$. *Suppose that* $\boldsymbol{\mu}$ *is a full* $(\mathbf{B}, (r))$*-semistable measure and there exists a sequence of invertible operators* $(\mathbf{A_k}) \subset \mathrm{Aut}(\mathbf{E})$ *such that* (3.1) *and* $(**)$ *hold. Then* $\boldsymbol{\lambda} \in \mathrm{DOSA}(r, \boldsymbol{\mu})$.

Proof. Let N be a natural number determined by $r^N \geq c > r^{N+1}$. We define $n_{i,k} = [\frac{n_k}{r^{i-1}}]$; $\mathbf{A}_{i,k} = \mathbf{B}^{i-1}\mathbf{A}_k$ for $i = 1, 2, \dots, N$; $k = 1, 2, \dots$. Then, by virtue of (3.1) and Lemma 5, for every $i = 1, 2, \dots, N$, there exists a sequence $(\mathbf{y}_{i,k}) \subset \mathbf{E}$ such that $\mathbf{A}_{i,k}\boldsymbol{\lambda}^{n_{i,k}} * \delta(\mathbf{y}_{i,k}) \to_w \mathbf{B}^{i-1}\boldsymbol{\mu}^{r^{-(i-1)}}$ when $k \to \infty$. Meanwhile, from the $(\mathbf{B}, r)$-semistability of $\boldsymbol{\mu}$ we see that $\mathbf{B}^{i-1}\boldsymbol{\mu}^{r^{-(i-1)}} = \boldsymbol{\mu} * \delta(\mathbf{y}_i)$ for some $\mathbf{y}_i \in \mathbf{E}$, $i = 1, 2, \dots, N$. Hence,

$$\mathbf{A}_{i,k}\boldsymbol{\lambda}^{n_{i,k}} * \delta(\mathbf{y}_{i,k} - \mathbf{y}_i) \to_w \boldsymbol{\mu} \tag{3.6}$$

when $k \to \infty$, $i = 1, 2, \dots, N$.

From ($**$) we have $\mathrm{LIM}(n_k/n_{k+1}) \subset [c, 1]$. If $t \in \mathrm{LIM}(n_k/n_{k+1})$ then by an argument analogous to that used for the proof of Proposition 3, we see that $\boldsymbol{\mu}$ is $(\mathbf{B}_t, t)$-semistable for some operator $\mathbf{B}_t \in \mathrm{Aut}(\mathbf{E})$. Hence, by the $(\mathbf{B}, (r))$-semistability of $\boldsymbol{\mu}$, we get $t = r^m$ for some $m \in \{1, 2, \dots, N\}$, and

$$\mathrm{LIM}(n_k/n_{k+1}) = \{r^{h_M}, r^{h_{M-1}}, \dots, r^{h_1}\} \subset \{r^N, r^{N-1}, \dots, r\} \subset [c, 1] \tag{3.7}$$

for certain natural number $M \leq N$.

Let the sets of indices K_m, $m = 1, 2, \dots, N$, be determined by

$$\begin{aligned}
K_1 &= \{k : r^{h_2 - 1} \leq n_k/n_{k+1} < 1\}, \\
K_2 &= \{k : r^{h_3 - 1} \leq n_k/n_{k+1} < r^{h_2 - 1}\}, \\
&\dots \\
K_{M-1} &= \{k : r^{h_M - 1} \leq n_k/n_{k+1} < r^{h_{M-1} - 1}\}, \\
K_M &= \{k : c \leq n_k/n_{k+1} < r^{h_M - 1}\}.
\end{aligned}$$

Then, in view of (3.7) and ($**$), it is clear that $\{K_1, K_2, \dots, K_M\}$ forms a disjoint partition of natural numbers, each K_m is a countable infinite set and

$$\lim_{k\to\infty, k\in K_m} (n_k/n_{k+1}) = r^{h_m}$$

for $m = 1, 2, \dots, M$. Consequently,

$$\lim_{k\to\infty, k\in K_m} \left([n_k/r^{h_m - j}]/n_{k+1}\right) = r^j \tag{3.8}$$

for $m = 1, 2, \dots, M$ and every natural j. We extend the subsequence (n_k) to a new subsequence (n_k^*) by adding suitable natural numbers between n_k and (n_{k+1}) to make the segment $(n_k, [n_k/r], \dots, [n_k/(r^{h_m - 1})], n_{k+1})$ for $k \in K_m$. Then (3.8) implies $\lim_{k\to\infty}(n_k^*/n_{k+1}^*) = r$.

The sequences $(\mathbf{A}_k)$ and $(\mathbf{x}_k)$ are also extended to new sequences $(\mathbf{A}_k^*)$ and $(\mathbf{x}_k^*)$ by merging suitably the segments in the form of

$$\left(\mathbf{A}_k, \mathbf{B}\mathbf{A}_k, \dots, \mathbf{B}^{h_m - 1}\mathbf{A}_k, \mathbf{A}_{k+1}\right), \qquad k \in K_m; m = 1, 2, \dots, M$$

and the segments in the form of

$$\big(\mathbf{x}_k, (\mathbf{y}_{2,k} - \mathbf{y}_2), \ldots, (\mathbf{y}_{h_m,k} - \mathbf{y}_{h_m}), \mathbf{x}_{k+1}\big), \qquad k \in K_m; m = 1, 2, \ldots, M.$$

Then (3.6) and (3.8) lead to

$$\mathbf{A}_k^* \boldsymbol{\lambda}^{n_k^*} * \delta\big(\mathbf{x}_k^*\big) \to_w \boldsymbol{\mu}$$

and $(n_k^*/n_{k+1}^*) \to r$ as $k \to \infty$, that is, $\boldsymbol{\lambda} \in \mathrm{DOSA}(r, \boldsymbol{\mu})$. The proposition is proved. □

Under some conditions set for the operator $\mathbf{B}$ and the operator $\mathbf{A}_k$ in (3.1), Krakowiak (1979) has shown that a full measure is $\mathbf{B}$-stable if and only if its DOA is not empty. The next theorem characterizes the domains of operator attraction.

Theorem 1. *Let* $\boldsymbol{\mu}, \boldsymbol{\lambda} \in P(\mathbf{E})$, $\mathbf{B} \in B(\mathbf{E})$. *Suppose that* $\boldsymbol{\mu}$ *is* $\mathbf{B}$*-stable and there exists a sequences* $(\mathbf{A_k})$, $(\mathbf{x}_k)$ *and* (n_k) *such that* (3.1) *and* $(**)$ *hold. Then* $\boldsymbol{\lambda} \in \mathrm{DOA}(\boldsymbol{\mu})$.

Proof. For all $t \in (0, 1]$, the $\mathbf{B}$-stability of $\boldsymbol{\mu}$ implies the existence of an element $\mathbf{y}_t \in \mathbf{E}$ such that

$$\boldsymbol{\mu}^t = t^{\mathbf{B}} \boldsymbol{\mu} * \delta(\mathbf{y}_t).$$

Let the sequences $(\mathbf{A}_m^1)$ and $(\mathbf{x}_m^1)$ be defined by:

$$\begin{aligned} \mathbf{A}_m^1 &= (n_j/m)^{\mathbf{B}} \mathbf{A}_j, \\ \mathbf{x}_m^1 &= (m/n_j).(n_j/m)^{\mathbf{B}} \mathbf{x}_j + \mathbf{y}_{n_j/m} \end{aligned}$$

for $n_j \le m < n_{j+1}$, $j = 1, 2, \ldots$. We infer that

$$\mathbf{A}_m^1 \boldsymbol{\lambda}^m * \delta\big(\mathbf{x}_m^1\big) \to_w \boldsymbol{\mu} \tag{3.9}$$

as $m \to \infty$. Namely, let (m') be any subsequence of natural numbers, then for each $m' \in (m')$ one can find a natural number $j_{m'}$ such that

$$n_{j_{m'}} \le m' < n_{j_{m'}+1}.$$

Hence, from $(**)$ we have

$$1 \ge n_{j_{m'}}/m' > n_{j_{m'}}/n_{j_{m'}+1} \ge c.$$

Then one can pick from (m') another subsequence (m'') such that

$$n_{j_{m''}}/m'' \to u \in [c, 1].$$

Now, taking into account the fact that

$$t^{\mathbf{B}} \to t_0^{\mathbf{B}}, \qquad y_t \to y_{t_0} \qquad \text{as } t \to t_0,$$

for $t, t_0 \in (0, 1]$, and applying Theorem 3.1 of Linde and Siegel (1990), we get

$$\begin{aligned}&\mathbf{A}^1_{m''}\boldsymbol{\lambda}^{m''} * \delta(\mathbf{x}^1_{m''})\\&= \left(\frac{n_{j_{m''}}}{m''}\right)^{\mathbf{B}} (\mathbf{A}_{j_{m''}}\boldsymbol{\lambda}^{n_{j_{m''}}} * \delta(\mathbf{x}_{j_{m''}}))^{m''/n_{j_{m''}}} * \delta\left(\frac{m''}{n_{j_{m''}}}\mathbf{y}_{j_{m''}/m''}\right)\\&\to_w u^{\mathbf{B}}\boldsymbol{\mu}^{1/u} * \delta((1/u)\mathbf{y}_u) = (u^{\mathbf{B}}\boldsymbol{\mu} * \delta(\mathbf{y}_u))^{1/u} = \boldsymbol{\mu}.\end{aligned}$$

Thus, (3.9) is true by virtue of Theorem 2.3 in Billingsley (1968), that means $\boldsymbol{\lambda} \in \mathrm{DOA}(\boldsymbol{\mu})$, which concludes the theorem. □

By definitions, it seems DOSA's would be broader than DOA's. However, DOSA's and DOA of a given stable measure in fact are equal. Namely the above theorem implies the following.

Corollary 1. *If $\boldsymbol{\mu}$ is a $\mathbf{B}$-stable measure, $\mathbf{B} \in B(\mathbf{E})$, then*

$$\mathrm{DOA}(\boldsymbol{\mu}) = \mathrm{DOSA}(r, \boldsymbol{\mu})$$

for every number r such that $0 < r < 1$.

4 Normal domains of operator semi-attraction and normal domains of operator attraction

Let $\boldsymbol{\mu}, \boldsymbol{\lambda} \in P(\mathbf{E})$, $\mathbf{B} \in B(\mathbf{E})$. If there are an operator $\mathbf{B}$, a number $r \in (0, 1)$ and a sequence $(\mathbf{x}_k)$ of elements from $\mathbf{E}$ such that $\boldsymbol{\mu}$ is $\mathbf{B}$-semistable and

$$\mathbf{B}^k\boldsymbol{\lambda}^{[r^{-k}]} * \delta(\mathbf{x}_k) \to_w \boldsymbol{\mu} \tag{4.1}$$

then we say that $\boldsymbol{\lambda}$ belongs to the *normal domain of operator semi-attraction* of $\boldsymbol{\mu}$ ($\boldsymbol{\lambda} \in \mathrm{NDOSA}(\boldsymbol{\mu})$).

Here, the term "normal" is used to emphasize the fact that in (3.1) the norming operators $\mathbf{A}_k$, in the form of $\mathbf{B}^k$, and the natural numbers in the subsequence (n_k), in the form of $[r^{-k}]$, are closely related to the $(\mathbf{B}, r)$-semistability of the measure $\boldsymbol{\mu}$.

While normal domains of operator semi-attraction were studied by Jurek (1981, 1980) and Hudson et al. (1983) in finite dimensional spaces, this section of the paper examine the problem of infinite dimensional spaces.

Theorem 2. *Let $r \in (0, 1)$, $\mathbf{M} \in L(\mathbf{E})$ and $\boldsymbol{\mu} = e(\mathbf{M})$ be $(\mathbf{B}, r)$-semistable with invertible $\mathbf{B} \in \mathrm{Aut}(\mathbf{E})$ and*

$$\|\mathbf{B}\| < r^{1/2}. \tag{4.2}$$

Then (4.1) fulfils when and only when the following two conditions hold:

(a) *For every* $u \in C(\boldsymbol{\mu})$,

$$[r^{-k}](\mathbf{B}^k\boldsymbol{\lambda})|_{\mathbf{E}_u^c} \to_w \mathbf{M}|_{\mathbf{E}_u^c}. \tag{4.3}$$

(b) *There exist* $u > 0$, $q > 0$ *and a sequence* $(\mathbf{F}_n)$ *of finite dimensional subspaces of* $\mathbf{E}$ *such that*

$$\lim_n \sup_k \int_{\mathbf{E}_u} d^q(\mathbf{x}, \mathbf{F}_n)\mathbf{B}^k\big(|\boldsymbol{\lambda}|^2\big)^{[r^{-k}]}(d\mathbf{x}) = 0, \tag{4.4}$$

where d *refers to the distance in the Banach space* $\mathbf{E}$.

Proof. Let (4.1) hold. Then Theorems 3.4 and 3.11 of Acosta et al. (1978) imply the existence of a sequence $(\mathbf{x}_k^1) \subset \mathbf{E}$, such that

$$\mathrm{Pois}\big([r^{-k}]\mathbf{B}^k\boldsymbol{\lambda}\big) * \delta\big(\mathbf{x}_k^1\big) \to_w e(\mathbf{M})$$

which yields (4.3) by virtue of Theorems 1.6 and 1.10 of Acosta et al. (1978). On the other hand, from (4.1) we have

$$\mathbf{B}^k\big(|\boldsymbol{\lambda}|^2\big)^{[r^{-k}]} \to_w |\boldsymbol{\mu}|^2.$$

This together with Theorem 2.3 of Acosta (1970) implies (4.4).

Conversely, suppose that (4.3) and (4.4) are true. Let $b > 0$ be any number from $C(\boldsymbol{\mu})$ and $\mathbf{U}_n = \mathbf{B}^n\mathbf{E}_b \cap (\mathbf{B}^{n+1}\mathbf{E}_b)^c$ for $n = 0, 1, 2, \ldots$. Then from (4.2) we see that $\mathbf{B}^n\mathbf{E}_b \downarrow \{\mathbf{0}\}$ as $n \to \infty$ and $\bigcup_{n=0}^{\infty} \mathbf{U}_n = \mathbf{E}_b \cap \{\mathbf{0}\}^c$.

Let $a = \|\mathbf{B}\| < r^{1/2}$. Then for any $\mathbf{x} \in \mathbf{B}^k\mathbf{E}_b$ we have $\mathbf{x} = \mathbf{B}^k\mathbf{y}$ with $\mathbf{y} \in \mathbf{E}_b$ and $\|\mathbf{x}\| \le \|\mathbf{B}^k\|^2\|\mathbf{y}\|^2$. Hence,

$$0 \le \int_{\mathbf{B}^k\mathbf{E}_b} \|\mathbf{x}\|^2[r^{-k}]\big(\mathbf{B}^k\boldsymbol{\lambda}\big)(d\mathbf{x}) \le a^{2k}b^2r^{-k} = \big(a^2/r\big)^k b^2 \to 0 \tag{4.5}$$

as $k \to \infty$.

Let (b_n) be a sequence of positive numbers such that $b_n \downarrow 0$. Since $\mathbf{B}^{-1} \in B(\mathbf{E})$, for every n there exists a natural number m_n such that

$$\mathbf{E}_{b_n} \subset \mathbf{B}^{m_n}\mathbf{E}_b; \qquad \mathbf{E}_{b_n} \cap \big(\mathbf{B}^{m_n+1}\mathbf{E}_{b_n}\big)^c \neq \varnothing.$$

Because $m_n \to \infty$ as $n \to \infty$ and $\mathbf{M}(\partial\mathbf{U}_0) = 0$, (4.3) leads to

$$\int_{\mathbf{U}_0} \|\mathbf{x}\|^2[r^{-k}]\big(\mathbf{B}^k\boldsymbol{\lambda}\big)(d\mathbf{x}) \to \int_{\mathbf{U}_0} \|\mathbf{x}\|^2\mathbf{M}(d\mathbf{x}).$$

Hence, one can find a number $D > 0$ such that

$$0 \le \int_{\mathbf{U}_0} \|\mathbf{x}\|^2[r^{-k}]\big(\mathbf{B}^k\boldsymbol{\lambda}\big)(d\mathbf{x}) \le D$$

for all $k \geq 0$. Then for $n < k$ we have

$$0 \leq \int_{\mathbf{U}_n} \|\mathbf{x}\|^2 [r^{-k}](\mathbf{B}^k \boldsymbol{\lambda})(d\mathbf{x}) = \int_{\mathbf{U}_0} \|\mathbf{B}^n \mathbf{y}\|^2 [r^{-k}](\mathbf{B}^{k-n} \boldsymbol{\lambda})(d\mathbf{y})$$
$$\leq \int_{\mathbf{U}_0} a^{2n} \|\mathbf{y}\|^2 [r^{-k}](\mathbf{B}^{k-n} \boldsymbol{\lambda})(d\mathbf{y})$$
$$\leq (a^2/r)^n \int_{\mathbf{U}_0} \|\mathbf{y}\|^2 [r^{-(k-n)}](\mathbf{B}^{k-n} \boldsymbol{\lambda})(d\mathbf{y}) \leq (a^2/r)^n D.$$

Therefore, in view of (4.5), for fixed $\varepsilon > 0$ and $k > m_n$ such that

$$\int_{\mathbf{B}^k \mathbf{E}_b} \|\mathbf{x}\|^2 [r^{-k}](\mathbf{B}^k \boldsymbol{\lambda})(d\mathbf{x}) < \varepsilon/2$$

we have

$$0 \leq \int_{\mathbf{E}_{b_n}} \|\mathbf{x}\|^2 [r^{-k}](\mathbf{B}^k \boldsymbol{\lambda})(d\mathbf{x}) \leq \int_{\mathbf{B}^{m_n} \mathbf{E}_b} \|\mathbf{x}\|^2 [r^{-k}](\mathbf{B}^k \boldsymbol{\lambda})(d\mathbf{x})$$
$$\leq \int_{\mathbf{B}^k \mathbf{E}_b} \|\mathbf{x}\|^2 [r^{-k}](\mathbf{B}^k \boldsymbol{\lambda})(d\mathbf{x}) + \sum_{j=m_n}^{k-1} \left(\int_{\mathbf{U}_j} \|\mathbf{x}\|^2 [r^{-j}](\mathbf{B}^j \boldsymbol{\lambda})(d\mathbf{x}) \right)$$
$$< \varepsilon/2 + \sum_{j=m_n}^{k-1} (a^2/r)^j D < \varepsilon/2 + (a^2/r)^{m_n} D/(1 - (a^2/r)) < \varepsilon$$

for n sufficiently large.

Consequently,

$$\lim_n \lim_k \int_{\mathbf{E}_{b_n}} \|\mathbf{x}\|^2 [r^{-k}](\mathbf{B}^k \boldsymbol{\lambda})(d\mathbf{x}) = 0. \tag{4.6}$$

Besides, due to (4.3) we have

$$[r^{-k}](\mathbf{B}^k |\boldsymbol{\lambda}|^2)|_{\mathbf{E}_b^c} \to_w |\mathbf{M}|^2|_{\mathbf{E}_b^c} \tag{4.7}$$

for every $b \in C(\boldsymbol{\mu})$. Therefore, repeating the above reasoning we get

$$\lim_n \lim_k \int_{\mathbf{E}_{b_n}} \|\mathbf{x}\|^2 [r^{-k}](\mathbf{B}^k |\boldsymbol{\lambda}|^2)(d\mathbf{x}) = 0 \tag{4.8}$$

for each sequence $b_n \downarrow 0$.

Using the same technique as in Theorem 2.14 of Acosta et al. (1978), by virtue of (4.4), (4.7) and (4.8), we can infer that the sequence $((\mathbf{B}^k |\boldsymbol{\lambda}|^2)^{r^{-k}})$ is relatively compact. Hence, taking into account of the conditions (4.3), (4.6) and Theorem 2.10 of Acosta et al. (1978), by the same argument in the proof of Theorem 2.14 given by Acosta et al. (1978), we can conclude that (4.1) is true. □

Remark. In the above theorem, the condition (b) can be replaced by the following one:

(b′) *There exist* $u > 0$, $q > 0$, *a sequence* $(\mathbf{x}_k) \subset \mathbf{E}$ *and a sequence* $(\mathbf{F}_n)$ *of finite dimensional subspaces of* $\mathbf{E}$ *such that*

$$\lim_n \sup_k \int_{\mathbf{E}_u} d^q(x, \mathbf{F}_n - \mathbf{x}_k)\mathbf{B}^k \boldsymbol{\lambda}^{[r^{-k}]}(d\mathbf{x}) = 0.$$

Lemma 6. *Let* $\mathbf{B} \in B(\mathbf{E})$ *be invertible and* $\|\mathbf{B}\| < 1$. *For* $t > 0$ *let denote*

$$\mathbf{U}_t = \{\mathbf{x} \in \mathbf{E} : \|\mathbf{x}\| \geq t, \|\mathbf{B}\mathbf{x}\| < t\}.$$

Then

(a) $\mathbf{B}^n\mathbf{U}_t \cap \mathbf{B}^k\mathbf{U}_t = \varnothing$ *for all integers* $n \neq k$;
(b) $\mathbf{E} \cap \{\mathbf{0}\}^c = \bigcup_{n=-\infty}^{\infty} \mathbf{B}^n\mathbf{U}_t$;
(c) *An inf. div. measure* $\boldsymbol{\mu} = e(\mathbf{M})$, $\mathbf{M} \in L(\mathbf{E})$, *is* $(\mathbf{B}, r)$*-semistable*, $0 < r < 1$, *if and only if there exists a finite measure* $\boldsymbol{\gamma}_t$ *on* $\mathbf{U}_t$ *such that*

$$\mathbf{M}(\mathbf{F}) = \sum_{n=-\infty}^{\infty} r^{-n} \int_{\mathbf{U}_t} \mathbf{I}_{\mathbf{F}}(\mathbf{B}^n\mathbf{x})\boldsymbol{\gamma}_t(d\mathbf{x}), \tag{4.9}$$

where $\mathbf{I}_{\mathbf{F}}$ *denotes the indicator of a Borel subset* $\mathbf{F}$ *of* $\mathbf{E} \cap \{\mathbf{0}\}^c$.

Proof. (a) Assume that $\mathbf{B}^k\mathbf{x} = \mathbf{B}^n\mathbf{x}$ for certain $\mathbf{x} \in \mathbf{U}_t$ and for certain integers $k < n$. Then $\mathbf{x} = \mathbf{B}^{-k}\mathbf{B}^n\mathbf{x} = \mathbf{B}^{n-k}\mathbf{x}$ and because $\|\mathbf{B}\| < 1$ and $n - k > 0$, we get

$$t \leq \|\mathbf{x}\| \leq \left\|\mathbf{B}^{n-k-1}\right\|.\|\mathbf{B}\mathbf{x}\| < t,$$

showing an obvious contradiction.

(b) Let $\mathbf{x} \in \mathbf{E} \cap \{\mathbf{0}\}^c$. If $\|\mathbf{x}\| = t$ then $\mathbf{x} \in \mathbf{B}^0\mathbf{U}_t$. In the case when $\|\mathbf{x}\| > t$, we see that $\mathbf{B}^m\mathbf{x} \to \mathbf{0}$ as $m \to \infty$. Hence, if n is the smallest natural number such that $\|\mathbf{B}^n\mathbf{x}\| \geq t$, then $\mathbf{x} \in \mathbf{B}^n\mathbf{U}_t$. Finally, if $\|\mathbf{x}\| < t$ then from $\|\mathbf{B}^k\| \to 0$ as $k \to \infty$, resulting in $\|\mathbf{B}^{-k}\mathbf{x}\| \geq \|\mathbf{x}\|/\|\mathbf{B}^k\| \to \infty$ and $\mathbf{x} \in \mathbf{B}^{-n}\mathbf{U}_t$, where n is the smallest natural number such that $\|\mathbf{B}^{-n+1}\mathbf{x}\| < t$. Consequently, we always have $\mathbf{x} \in \mathbf{B}^m\mathbf{U}_t$ for certain integer m, which means (b) is true.

For the final assertion of the lemma, it is clear that (c) follows directly from (a) and (b), by repeating the main argument of Theorem 3 in Krakowiak (1980). Hence, the lemma is proved. □

Proposition 5. *Let* $0 < r < 1$, $\mathbf{M} \in L(\mathbf{E})$ *and* $\mathbf{B} \in B(\mathbf{E})$ *be invertible. Suppose that* (4.2) *fulfils*, $\boldsymbol{\mu} = e(\mathbf{M})$ *is* $(\mathbf{B}, r)$*-semistable and* t *is a positive number such that* $\mathbf{U}_t$ *is an* $\mathbf{M}$*-continuity subset. Then* (4.1) *is confirmed if and only if* (4.4) *is true and*

$$\lim_n [r^{-n}]\mathbf{B}^n\boldsymbol{\lambda}(\mathbf{G}) = \boldsymbol{\gamma}_t(\mathbf{G}) \tag{4.3a}$$

for all $\boldsymbol{\gamma}_t$*-continuity subset* $\mathbf{G}$ *of* $\mathbf{U}_t$, *where* $\boldsymbol{\gamma}_t$ *is defined by Lemma* 6.

Proof. The number t, such that $\mathbf{U}_t$ is an $\mathbf{M}$-continuity set, always exists. Therefore, by Theorem 2, we need only to prove that (4.3) and (4.3a) are equivalent.

From the equation $\boldsymbol{\gamma}_t = \mathbf{M}|_{\mathbf{U}_t}$, (4.3) directly implies (4.3a). Conversely, assuming (4.3a) is true, then for every $\mathbf{M}$-continuity subset $\mathbf{G}$ of $\mathbf{B}^k\mathbf{U}_t$, $k = 0, \pm 1, \pm 2, \ldots$, the set $\mathbf{B}^{-k}\mathbf{G}$ is a $\boldsymbol{\gamma}_t$-continuity subset of $\mathbf{U}_t$. Therefore, in view of Lemma 6, we have

$$\begin{aligned}[r^{-n}]\mathbf{B}^n\boldsymbol{\lambda}(\mathbf{G}) &= [r^{-n}]\mathbf{B}^n\boldsymbol{\lambda}(\mathbf{B}^k\mathbf{B}^{-k}\mathbf{G}) = ([r^{-n}]/[r^{-n+k}]).[r^{-n+k}]\mathbf{B}^{n-k}\boldsymbol{\lambda}(\mathbf{B}^{-k}\mathbf{G}) \\ &\to r^{-k}\boldsymbol{\gamma}_t(\mathbf{B}^{-k}\mathbf{G}) = r^{-k}\mathbf{M}(\mathbf{B}^{-k}\mathbf{G}) = \mathbf{M}(\mathbf{G}),\end{aligned}$$

i.e.

$$[r^{-n}]\mathbf{B}^n\boldsymbol{\lambda}(\mathbf{G}) \to \mathbf{M}(\mathbf{G})$$

for every $\mathbf{M}$-continuity subset $\mathbf{G}$ of $\mathbf{B}^k\mathbf{U}_t$, $k = 0, \pm 1, \pm 2, \ldots$. On the other hand, because $\mathbf{U}_t$ is an $\mathbf{M}$-continuity set and $\mathbf{B}$ is an invertible continuous operator, $\mathbf{B}^k\mathbf{U}_t$ is an $\mathbf{M}$-continuity set for every integer k. This certifies the fact that (4.9) holds for every $\mathbf{M}$-continuity subset $\mathbf{G}$ of $\mathbf{E} \cap \{\mathbf{0}\}^c$ such that

$$\mathbf{G} \subset \bigcup_{k=k_1}^{k_2} \mathbf{B}^k\mathbf{U}_t, \qquad -\infty < k_1 \le k_2 < \infty.$$

Hence, based on Theorem 1.2.2, Billingsley (1968), we get (4.3), the proposition is proved. □

Proposition 6. *With the assumption of Proposition 5, the condition (4.1) fulfils if and only if (4.4) is true and*

$$\lim_n [r^{-n}]\boldsymbol{\lambda}(\{\mathbf{B}^k\mathbf{x} : \mathbf{x} \in \mathbf{G}, k \ge n\}) = \boldsymbol{\gamma}_t(\mathbf{G})/(1-r) \tag{4.3b}$$

for every $\boldsymbol{\gamma}_t$-continuity subset $\mathbf{G}$ of $\mathbf{U}_t$, where $\boldsymbol{\gamma}_t$ is as in Lemma 6.

Proof. In view of Theorem 2, it is sufficient to verify that (4.3) and (4.3b) are equivalent. Let (4.3) hold and $\mathbf{G}$ be a $\boldsymbol{\gamma}_t$-continuity subset of $\mathbf{U}_t$. Then

$$\mathbf{F} = \{\mathbf{B}^{-k}\mathbf{x} : \mathbf{x} \in \mathbf{G}, k = 0, 1, 2, \ldots\} = \bigcup_{k=0}^{\infty} \mathbf{B}^{-k}\mathbf{G}$$

is an $\mathbf{M}$-continuity set. Moreover $\mathbf{F}$ is separated from $\mathbf{0}$. Then (4.3) together with Lemma 6 yields

$$\lim_n [r^{-n}](\mathbf{B}^n\boldsymbol{\lambda})(\mathbf{F}) = \mathbf{M}(\mathbf{F}) = \sum_{k=0}^{\infty} r^{-k}\boldsymbol{\gamma}_t(\mathbf{G}) = \boldsymbol{\gamma}_t(\mathbf{G})/(1-r).$$

Consequently we have (4.3b).

Now, let us suppose that (4.3b) is true. Then for a $\boldsymbol{\gamma}_t$-continuity set $\mathbf{G}$ of $\mathbf{U}_t$ we have

$$[r^{-n}]\boldsymbol{\lambda}(\{\mathbf{B}^k\mathbf{x}:\mathbf{x}\in\mathbf{G}, k\geq n+1\})$$
$$=([r^{-n}]/[r^{-n-1}])[r^{-n-1}]\boldsymbol{\lambda}(\{\mathbf{B}^k\mathbf{x}:\mathbf{x}\in\mathbf{G}, k\geq n+1\})\to r\boldsymbol{\gamma}_t(\mathbf{G})/(1-r).$$

Therefore,

$$\begin{aligned}
&[r^{-n}]\mathbf{B}^n\boldsymbol{\lambda}(\mathbf{G})\\
&\quad=[r^{-n}]\boldsymbol{\lambda}(\{\mathbf{B}^{-k}\mathbf{x}:\mathbf{x}\in\mathbf{G}, k\geq n\}\setminus\{\mathbf{B}^{-k}\mathbf{x}:\mathbf{x}\in\mathbf{G}, k\geq n+1\})\\
&\quad=[r^{-n}]\boldsymbol{\lambda}(\{\mathbf{B}^{-k}\mathbf{x}:\mathbf{x}\in\mathbf{G}, k\geq n\})-[r^{-n}]\boldsymbol{\lambda}(\{\mathbf{B}^{-k}\mathbf{x}:\mathbf{x}\in\mathbf{G}, k\geq n+1\})\\
&\quad\to\boldsymbol{\gamma}_t(\mathbf{G})/(1-r)-r\boldsymbol{\gamma}_t(\mathbf{G})/(1-r)=\boldsymbol{\gamma}_t(\mathbf{G}),
\end{aligned}$$

that is, (4.3a) holds, which implies (4.3) by virtue of Proposition 5. The proof is complete. □

When $\mathbf{B}$ is of the form $c\mathbf{I}$, $c\neq 0$, and $\boldsymbol{\mu}$ is a non-Gaussian (c,r)-semistable measure, that is, the exponent p of $\boldsymbol{\mu}$ satisfies $0<p<2$, then $\boldsymbol{\mu}$ is of the form $\boldsymbol{\mu}=e(\mathbf{M})$, $\mathbf{M}\in L(\mathbf{E})$ and $\|\mathbf{B}\|=|c|<r^{1/2}$. Further, (4.1) is reduced to

$$T_{c^k}\boldsymbol{\lambda}^{[r^{-k}]}*\delta(\mathbf{x}_k)\to_w\mu.$$

In this case, we say that $\boldsymbol{\lambda}$ belongs to the *normal domain of semi-attraction* of $\boldsymbol{\mu}$ ($\boldsymbol{\lambda}\in\mathrm{NDSA}(\boldsymbol{\mu})$). Meantime, (4.4) is reduced to

(4.4b) *There exist $u>0$, $q>0$ and a sequence $(\mathbf{F}_n)$ of finite dimensional subspaces of $\mathbf{E}$ such that*

$$\lim_n\sup_k\int_{\mathbf{E}_u}d^q(x,\mathbf{F}_n)T_{c^k}\left(|\boldsymbol{\lambda}|^2\right)^{[r^{-k}]}(dx)=0.$$

Moreover, the set $\mathbf{U}_t$ in Lemma 6 is reduced to

$$\mathbf{V}_t=\{\mathbf{x}\in\mathbf{E}:t\leq\|\mathbf{x}\|<t/|c|\}.$$

Hence, we have the following characterizations of normal domains of semi-attraction of non-Gaussian semistable measures.

Corollary 2. *Let $\boldsymbol{\mu}$ be (c,r)-semistable with exponent $p\in(0,2)$, $0<r<1$. Then $\boldsymbol{\lambda}\in\mathrm{NDSA}(\boldsymbol{\mu})$ if and only if* (4.4b) *holds and*

$$[r^{-n}]T_{c^k}\boldsymbol{\lambda}(\mathbf{G})\to\mathbf{M}(\mathbf{G})$$

as $n\to\infty$ for all $\mathbf{M}$-continuity subset $\mathbf{G}$ of $\mathbf{V}_t$, where $\mathbf{M}$ is the Lévy measure of $\boldsymbol{\mu}$ and t is a positive number such that $\mathbf{M}(\partial\mathbf{V}_t)=0$.

Corollary 3. *With the assumption as in Corollary 2*, $\boldsymbol{\lambda} \in \mathrm{NDSA}(\boldsymbol{\mu})$ *if and only if* (4.4b) *is true and*

$$[r^{-n}]\boldsymbol{\lambda}\big(\{c^{-k}\mathbf{x} : \mathbf{x} \in \mathbf{G}, k \geq n\}\big) \to \mathbf{M}(\mathbf{G})/(1-r)$$

as $n \to \infty$ *for every* $\mathbf{M}$*-continuity subset* $\mathbf{G}$ *of* $\mathbf{V}_t$, *where* $\mathbf{M}(\partial \mathbf{V}_t) = 0$.

In the following part, we examine the concept of normal domains of operator attraction of operator stable measures. Let $\boldsymbol{\mu}, \boldsymbol{\lambda} \in P(\mathbf{E})$, $\mathbf{B} \in B(\mathbf{E})$ and $\boldsymbol{\mu}$ be $\mathbf{B}$-stable. Then we say that $\boldsymbol{\lambda}$ belongs to the *normal domain of operator attraction* of $\boldsymbol{\mu}$ (in symbols $\boldsymbol{\lambda} \in \mathrm{NDOA}(\boldsymbol{\mu})$) if

$$n^{-\mathbf{B}}\boldsymbol{\lambda}^{n} * \delta(\mathbf{y}_n) \to_w \boldsymbol{\mu} \tag{4.10}$$

as $n \to \infty$ for some sequence $(\mathbf{y}_n)$ of elements from $\mathbf{E}$.

Lemma 7. *Let* $\boldsymbol{\mu}$ *be a* $\mathbf{B}$*-stable measure. Then the condition* (4.10) *holds if and only if there exist a number* $r \in (0, 1)$ *and sequence* $(\mathbf{x}_k) \subset B(\mathbf{E})$ *such that*

$$r^{k\mathbf{B}}\boldsymbol{\lambda}^{[r^{-k}]} * \delta(\mathbf{x}_k) \to_w \boldsymbol{\mu} \tag{4.11}$$

as $k \to \infty$.

Proof. It is obvious that (4.11) comes from (4.10), taking $r = 1/2$ for example. Conversely, by an argument analogous to that used for proving Theorem 1, (4.11) implies (4.10). The lemma is proved. □

Combining Lemma 7 and Theorem 2, we conclude the following proposition.

Proposition 7. *Let* $\mathbf{M} \in L(\mathbf{E})$ *and* $\boldsymbol{\mu} = e(\mathbf{M})$ *be* $\mathbf{B}$*-stable for some* $\mathbf{B} \in B(\mathbf{E})$. *Suppose that there exists a number* $r \in (0, 1)$ *such that* $\|r^{\mathbf{B}}\| < r^{1/2}$. *Then* (4.10) *holds if and only if the following two conditions are true*:

(a) *For every* $u \in C(\boldsymbol{\mu})$,

$$[r^{-k}]r^{k\mathbf{B}}\boldsymbol{\lambda}|_{\mathbf{E}_u^c} \to_w \mathbf{M}|_{\mathbf{E}_u^c}. \tag{4.12}$$

(b) *There exist* $u > 0$, $q > 0$ *and a sequence* $(\mathbf{F}_n)$ *of finite dimensional subspaces of* $\mathbf{E}$ *such that*

$$\lim_n \sup_k \int_{\mathbf{E}_u} d^q(\mathbf{x}, \mathbf{F}_n) r^{k\mathbf{B}}(|\boldsymbol{\lambda}|^2)^{[r^{-k}]}(d\mathbf{x}) = 0, \tag{4.13}$$

where d *refers to the distance in the Banach space* $\mathbf{E}$.

Given an operator $\mathbf{B}$, we denote

$$\mathbf{V}_r(t) = \{\mathbf{x} \in \mathbf{E} : \|\mathbf{x}\| \geq t;\ \|r^{\mathbf{B}}\mathbf{x}\| < t\}.$$

Then Proposition 5 together with Lemma 6 yields the following proposition.

Proposition 8. *Let r, $\boldsymbol{\mu}$, $\mathbf{B}$ be as in Proposition 7 and t be a positive number such that*

$$\mathbf{M}\big(\partial \mathbf{V}_r(t)\big) = 0. \tag{4.14}$$

Then (4.10) fulfils if and only if (4.13) holds and

$$\big[r^{-n}\big] r^{n\mathbf{B}} \boldsymbol{\lambda}(\mathbf{G}) \to \boldsymbol{\gamma}_t(\mathbf{G})$$

as $n \to \infty$ for every $\boldsymbol{\gamma}_t$-continuity subset $\mathbf{G}$ of $\mathbf{V}_r(t)$, where $\boldsymbol{\gamma}_t = \mathbf{M}|_{\mathbf{V}_r(t)}$.

As a corollary of Theorems 3.2 and 4.2 in Krakowiak (1979), we have the following lemma.

Lemma 8. *Let $\boldsymbol{\mu}$ be a full p.m. on $\mathbf{E}$, $\boldsymbol{\mu} = e(\mathbf{M})$, $\mathbf{M} \in L(\mathbf{E})$ and $\mathbf{B} \in B(\mathbf{E})$ satisfy*

$$t^{\mathbf{B}} \to \mathbf{0} \tag{4.15}$$

as $t \to 0$. Then $\boldsymbol{\mu}$ is $\mathbf{B}$-stable if only if there exist a finite measure $\boldsymbol{\gamma}$ on the unit sphere $\mathbf{S}$ of $\mathbf{E}$ such that

$$\mathbf{M}(\mathbf{F}) = \int_{\mathbf{S}} \int_0^\infty \mathbf{I}_{\mathbf{F}}(t^{\mathbf{B}}\mathbf{x}) t^{-2}\, dt\, \boldsymbol{\gamma}(d\mathbf{x}), \tag{4.16}$$

where $\mathbf{I}_{\mathbf{F}}$ denotes the indicator of a Borel subset $\mathbf{F}$ of $\mathbf{E} \setminus \{\mathbf{0}\}$.

The measure $\boldsymbol{\gamma}$ defined in Lemma 8 is called a *spectral measure* of $\boldsymbol{\mu}$. In view of the lemma we see that if (4.15) fulfils and $\boldsymbol{\mu} = e(\mathbf{M})$ is a full $\mathbf{B}$-stable measure then (4.14) is always true. Moreover, the following Proposition 9 is the generalization of Theorem 1.1 of Jurek (1980).

Proposition 9. *Let $r \in (0, 1)$ and $\mathbf{B} \in B(\mathbf{E})$ satisfy*

$$\|r^{\mathbf{B}}\| < r^{1/2}.$$

Assume that $\boldsymbol{\mu} = e(\mathbf{M})$, $\mathbf{M} \in L(\mathbf{E})$, is full $\mathbf{B}$-stable on $\mathbf{E}$. Then (4.10) is true if and only if (4.13) holds and

$$\lim_{\alpha \to \infty} \alpha \boldsymbol{\lambda}\big(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \geq \alpha\}\big) = \boldsymbol{\gamma}(\mathbf{G}) \tag{4.12b}$$

for every $\boldsymbol{\gamma}$-continuity subset $\mathbf{G}$ of the unit sphere $\mathbf{S}$.

Proof. We infer that (4.15) holds and consequently (4.16) is true. Namely, let (t_n) be any sequence of positive numbers tending to 0. Then there exists a sequence (k_n) of natural numbers such that $k_n \to \infty$ as $n \to \infty$ and

$$t_n = r^{k_n}.s_n, \qquad r \leq s_n < 1.$$

Therefore

$$0 \leq \|t_n^{\mathbf{B}}\| < \|s_n^{\mathbf{B}}\|.\|r^{k_n\mathbf{B}}\| \to 0,$$

due to

$$\|r^{k_n\mathbf{B}}\| \le (r^{1/2})^{k_n} \to 0$$

and of the inequalities $\|s_n^{\mathbf{B}}\| \le (1/s_n)^{\|\mathbf{B}\|} \le (1/r)^{\|\mathbf{B}\|}$ are obtained straightforwardly from the power series expansion. Thus, (4.15) is satisfied.

When (4.10) holds, (4.13) comes from Proposition 7. Besides, arguing as in Theorem 2, by virtue of Theorems 1.6, 1.10, 3.4 and 3.11 in Acosta et al. (1978), we can infer that

$$n(n^{-\mathbf{B}}\boldsymbol{\lambda})|_{\mathbf{E}_u^c} \to_w \mathbf{M}|_{\mathbf{E}_u^c} \tag{4.17}$$

for every $u \in C(\boldsymbol{\mu})$.

Let $\mathbf{G} \subset \mathbf{S}$ be a $\boldsymbol{\gamma}$-continuity set. Then in view of Lemma 6, it is easy to see that $\mathbf{F} = \{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge 1\}$ is an $\mathbf{M}$-continuity set. Moreover, $\mathbf{F}$ is separated from $\mathbf{0}$. Hence, (4.16) together with (4.17) implies

$$n(n^{-\mathbf{B}}\boldsymbol{\lambda})(\mathbf{F}) \to \mathbf{M}(\mathbf{F}) = \boldsymbol{\gamma}(\mathbf{G}) \int_1^{\infty} t^{-2}\,dt = \boldsymbol{\gamma}(\mathbf{G}).$$

In addition,

$$n(n^{-\mathbf{B}}\boldsymbol{\lambda})(\mathbf{F}) = n\boldsymbol{\lambda}(\{(nu)^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge 1\}) = n\boldsymbol{\lambda}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge n\}).$$

Consequently,

$$n\boldsymbol{\lambda}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge n\}) \to \boldsymbol{\gamma}(\mathbf{G}) \tag{4.18}$$

as $n \to \infty$ for every $\boldsymbol{\gamma}$-continuity subset $\mathbf{G}$ of $\mathbf{S}$. Meanwhile,

$$\begin{aligned}[a]\boldsymbol{\lambda}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge [a]+1\}) &\le a\boldsymbol{\lambda}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge a\}) \\ &\le ([a]+1)\boldsymbol{\lambda}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge [a]\}).\end{aligned}$$

This, together with (4.18), implies $a\boldsymbol{\lambda}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge a\}) \to \boldsymbol{\gamma}(\mathbf{G})$ as $a \to \infty$. Then (4.12b) is proved.

Conversely, if (4.12b) and (4.13) hold, then for each $\boldsymbol{\gamma}$-continuity subset $\mathbf{G}$ of $\mathbf{S}$ and for every positive number b, by virtue of (4.16) we have

$$\begin{aligned}&n.\boldsymbol{\lambda}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge nb\}) \\ &\quad = (1/b).nb.\boldsymbol{\lambda}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge nb\}) \to (1/b)\boldsymbol{\gamma}(\mathbf{G})\end{aligned}$$

as $n \to \infty$ and

$$(1/b)\boldsymbol{\gamma}(\mathbf{G}) = \boldsymbol{\gamma}(\mathbf{G}) \int_b^{\infty} t^{-2}\,dt = \mathbf{M}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, u \ge b\}).$$

Hence, for $0 < b < c \le \infty$ we get

$$\begin{aligned}&n.(n^{-\mathbf{B}}\boldsymbol{\lambda})(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, b \le u < c\}) \\ &\quad = n.\boldsymbol{\lambda}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, nb \le u < nc\}) \to \mathbf{M}(\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, b \le u < c\})\end{aligned}$$

as $n \to \infty$, and

$$n(n^{-\mathbf{B}}\boldsymbol{\lambda})|_{\mathbf{E}_u^c} \to_w \mathbf{M}|_{\mathbf{E}_u^c}$$

for every $u \in C(\boldsymbol{\mu})$, because the sets of the form $\{u^{\mathbf{B}}\mathbf{x} : \mathbf{x} \in \mathbf{G}, b \le u < c\}$, where $\boldsymbol{\gamma}(\partial\mathbf{G}) = 0$, $0 < b < c \le \infty$, satisfy the condition of Theorem 1.2.2, Billingsley (1968). In particular, we get

$$[r^{-k}]([r^{-k}]^{-\mathbf{B}}\boldsymbol{\lambda})|_{\mathbf{E}_u^c} \to_w \mathbf{M}|_{\mathbf{E}_u^c}$$

for every $u \in C(\boldsymbol{\mu})$. This implies (4.12), therefore Proposition 7 yields (4.10). The proof is obtained. $\square$

Let us consider a $\mathbf{B}$-stable measure $\boldsymbol{\mu}$, where $\mathbf{B}$ is of the form $(1/p).\mathbf{I}$ and p is a positive number. Then $\boldsymbol{\mu}$ is a stable measure on $\mathbf{E}$ and $0 < p \le 2$. Besides, (4.10) is reduced to

$$T_{n^{-1/p}}(\boldsymbol{\lambda}^n) * \delta(\mathbf{y}_n) \to_w \boldsymbol{\mu}.$$

In this case, $\boldsymbol{\lambda}$ is said to belong to the *normal domain of attraction* of $\boldsymbol{\mu}$. Besides, (4.13) is reduced to

(4.13a) *There exist* $s > 0$, $q > 0$ *and a sequence* $(\mathbf{F}_n)$ *of finite dimensional subspaces of* $\mathbf{E}$ *such that*

$$\lim_n \sup_k \int_{\mathbf{E}_u} d^q(\mathbf{x}, \mathbf{F}_n) T_{r^{k/p}}(|\boldsymbol{\lambda}|^2)^{[r^{-k}]}(d\mathbf{x}) = 0.$$

Thus, Proposition 9 gives the following characterization of normal domain of attraction of full stable measure with exponent $p \in (0, 2)$:

Corollary 4. *Let $\boldsymbol{\mu}$ be a full stable with exponent $p \in (0, 2)$. Then a p.m. $\boldsymbol{\lambda}$ belongs to the normal domain of attraction of $\boldsymbol{\mu}$ if and only if there exists a number $r \in (0, 1)$ such that* (4.13a) *holds and*

$$t^p \boldsymbol{\lambda}(\{\mathbf{x} : \mathbf{x}/\|\mathbf{x}\| \in \mathbf{G}, \|\mathbf{x}\| \ge t\}) \to \boldsymbol{\gamma}(\mathbf{G})$$

as $t \to \infty$ for all $\boldsymbol{\gamma}$-continuity subset $\mathbf{G}$ of $\mathbf{S}$, where $\boldsymbol{\gamma}$ is the spectral measure of $\boldsymbol{\mu}$.

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References

Acosta, A. (1970). Existence and convergence of probability measures in Banach spaces. *Transaction of the American Mathematical Society* **152**, 273–298. MR0267614

Acosta, A., Araujo, A. and Giné, E. (1978). On Poisson measures, Gaussian measures and the Central Limit Theorem in Banach spaces. In *Probability on Banach Spaces*. *Advances in Probability and Related Topics* **4**, 1–66. New York: Dekker. MR0515429

Becker-Kern, P. (2002). Limit theorems with random sample size for generalized domains of semistable attraction. *Journal of Mathematical Sciences (New York)* **111**, 3820–3829. MR1945210

Becker-Kern, P. (2003). Stable and semistable hemigroups: Domains of attraction and self-decomposability. *Journal of Theoretical Probability* **16**, 573–598. MR2009194

Becker-Kern, P. (2004). Stochastic summability methods for domains of normal attraction of semistable laws. *Journal of Mathematical Sciences (New York)* **121**, 2603–2612. MR2087736

Becker-Kern, P. (2007). Almost sure limit theorems of mantissa type for semistable domains of attraction. *Acta Mathematica Hungarica* **114**, 301–336. MR2305821

Billingsley, P. (1968). *Convergence of Probability Measures*. New York: Wiley. MR0233396

Bouzar, N. and Jayakumar, K. (2008). Time series with discrete semistable marginals. *Statistics Papers* **49**, 619–635. MR2439010

Chung Dong, M., Rajput, B. S. and Tortrat, A. (1982). Semistable laws on topological vector spaces. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **60**, 209–218. MR0663902

Chuprunov, A. N. and Terekhova, L. P. (2009). An almost sure limit theorem for random sums of independent random variables in the domain of attraction of a semistable law (Russian). *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika* **11**, 85–88. MR2657245

Csörgö, S. (2007). Fourier analysis of semistable distributions. *Acta Applicandae Mathematicae* **96**, 159–174. MR2327532

Csörgö, S. and Megyesi, Z. (2002). Merging to semistable laws. *Teorija Veroyatnost i Primeneniya* **47**, 90–109. MR1978698

Dettweiler, E. (1978). Grenzweztsatze für Wahrscheinlichkeitsmasse auf Badrikianschen Raumen. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **34**, 285–311. MR0402849

Divanji, G. (2004). Law of the iterated logarithm for subsequences of partial sums which are in the domain of partial attraction of a semistable law. *Probability and Mathematical Statistics* **24**, 433–442. MR2157216

Fazekas, I. and Chuprunov, A. (2007). An almost sure functional limit theorem for the domain of geometric partial attraction of semistable laws. *Journal of Theoretical Probability* **20**, 339–353. MR2324535

Fedosenko, A. G. (2005). On the convergence of limit distributions of sums of dependent random variables to a semistable law (Russian). *Vestnik Belorusskogo Gosudarstvennogo Universiteta. Seriya 1. Fizika, Matematika, Informatika* **2**, 57–61, 118. MR2209854

Hazod, W. and Shah, R. (2001). Semistable selfdecomposable laws on groups. *Journal of Applied Analysis* **7**, 1–22. MR1836762

Ho Dang, P. (1987). Domains of semi-attraction of semistable laws on topological vector spaces. *Acta Mathematica Vietnamica* **12**, 39–50. MR0970591

Ho Dang, P. (2009). Domains of operator semi-attraction of operator semistable probability measures. *Acta Mathematica Vietnamica* **34**, 259–269. MR2547005

Hougaard, P. (1986). Survival models for heterogeneous populations derived from stable distributions. *Biometrika* **73**, 387–396. MR0855898

Hudson, W. N., Mason, J. D. and Veeh, A. J. (1983). The domain of normal attraction of an operator stable law. *The Annals of Probability* **11**, 178–184. MR0682808

Jajte, R. (1977). Semistable probability measures on R^N. *Studia Mathematica* **61**, 29–39. MR0436263

Jurek, Z. J. (1980). Domain of normal attraction of operator stable measures on Euclidean spaces. *Bulletin of the Polish Academy of Sciences. Mathematics, Astronomy, Physics* **28**, 397–401. MR0628057

Jurek, Z. J. (1981). On Gaussian measures on R^d. In *Proceedings of the Sixth Conference on Probability Theory (Brasov, 1979)* 313–316. Bucharest: Ed. Acad. S.R. Romania. MR0633938

Kevei, P. (2009). Merging asymptotic expansions for semistable random variables. *Lithuanian Mathematical Journal* **49**, 40–54. MR2491709

Kevei, P. and Csörgö, S. (2009). Merging of linear combinations to semistable laws. *Journal of Theoretical Probability* **22**, 772–790. MR2530112

Kozubowski, T. J. and Rachev, S. T. (1994). The theory of geometric stable distributions and its use in modeling financial data. *European Journal of Operational Research* **74**, 310–324.

Krakowiak, W. (1979). Operator-stable probability measures on Banach spaces. *Colloquium Mathematicum* **41**, 313–326. MR0591939

Krakowiak, W. (1980). Operator semistable probability measures on Banach spaces. *Colloquium Mathematicum* **43**, 351–363. MR0628190

Kuruoglu, E., Fitzgerald, W. and Rayner, P. (1998). Near optimal detection of signals in impulsive noise modeled with a symmetric α-stable distribution. *IEEE Communications Letters* **2**, 282–284.

Linde, W. and Siegel, G. (1990). On the convergence of types for Radon probability measures in Banach spaces. In *Probability in Banach Spaces 6 (Sandbjerg, 1986). Progr. Probab.* **20**, 234–251. Boston, MA: Birkhauser Boston. MR1056713

Luczak, A. (1984). Elliptical symmetry and characterization of operator stable and operator semistable probability measures. *The Annals of Probability* **12**, 1217–1223. MR0757779

Maejima, M. and Riddhi, S. (2006). Moments and projections of semistable probability measures on p-adic vector spaces. *Journal of Theoretical Probability* **19**, 447–459. MR2283385

Maejima, M. and Riddhi, S. (2007). Operator-semistable, operator semi-self-decomposable probability measures and related nested classes on p-adic vector spaces. *Monatshefte für Mathematik* **151**, 293–318. MR2329090

McCulloch, J. (1996). Financial applications of stable distributions. In *Handbook of Statistics, Vol. 14* (G. Maddala and C. Rao, eds.) 393–425. North-Holland: Elsevier Science Publishers. MR1602156

Meerschaert, M. M. and Scheffler, H. P. (2002). Semistable Lévy motion. *Fractional Calculus and Applied Analysis* **5**, 27–54. MR1890124

Nikias, C. L. and Shao, M. (1995). *Signal Processing with α-Stable Distributions and Applications*. New York: Wiley.

Palmer, K. J., Ridout, M. S. and Morgan, B. J. T. (2008). Modelling cell generation times using the tempered stable distribution. *Journal of the Royal Statistical Society, Series C* **57**, 379–397. MR2526124

Parthasarathy, K. R. (1967). *Probability Measures on Metric Spaces*. New York: Academic Press. MR0226684

Rajput, B. S. and Rama-Murthy, K. (2004). Comparison of tail probabilities of strictly semistable/stable random vectors and their symmetrized counterparts with application. *Probability and Mathematical Statistics* **24**, 367–379. MR2157212

Sato, K.-i. and Watanabe, T. (2005). Last exit times for transient semistable processes. *Annales de l'Institut Henri Poincare. Probabilites et Statistiques* **41**, 929–951. MR2165258

Shah, R. (2007). On the uniqueness of semistable embedding and domain of semistable attraction for probability measures on p-adic groups. *Tokyo Journal of Mathematics* **30**, 383–396. MR2376516

Siegel, G. (1989). Operator-stable distributions in separable Banach spaces. *Teorija Veroyatnost i Primeneniya* **34**, 552–560. MR1022638

Tortrat, A. (1967). Structure des indefiniment divisibles dans un E.V.T. In *Symposium on Probability Methods in Analysis*. *Lecture Notes in Mathematics* **31**, 229–238. Berlin: Springer. MR0226692

Vakhaniya, N. N., Tarieladze, V. and Tchobanyan, S. A. (1985). *Probability Distributions in Banach Spaces* (in Russian). Moscow: Nauka. MR0787803

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