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RELATIVE ERRORS FOR BOOTSTRAP APPROXIMATIONS OF THE SERIAL CORRELATION COEFFICIENT

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We consider the first serial correlation coefficient under an AR(1) model where errors are not assumed to be Gaussian. In this case it is necessary to consider bootstrap approximations for tests based on the statistic since the distribution of errors is unknown. We obtain saddle-point approximations for tail probabilities of the statistic and its bootstrap version and use these to show that the bootstrap tail probabilities approximate the true values with given relative errors, thus extending the classical results of Daniels [*Biometrika* **43** (1956) 169–185] for the Gaussian case. The methods require conditioning on the set of odd numbered observations and suggest a conditional bootstrap which we show has similar relative error properties.

1. Introduction. A central limit theorem for the first-order serial correlation for an autoregression with general errors was obtained by Anderson (1959), and Edgeworth expansions were obtained by Bose (1988) who used this to prove the validity of the bootstrap approximation. There have been several papers which consider saddle-point approximations for autoregressive processes [Daniels (1956), Phillips (1978), Lieberman (1994b)] under the assumption of normal errors and more generally for a ratio of quadratic forms of normal variables [Lieberman (1994a)]. Our results, in contrast, give relative errors, valid for nonnormal errors and are used to show that the bootstrap has better than first-order relative accuracy in a moderately large region.

Let $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$ be independent and identically distributed random variables with distribution function F and density f, assume that $E\varepsilon_0 = 0$, define $X_i = \rho X_{i-1} + \varepsilon_i$, $i = 2, \dots, n$ and take X_1 to be distributed as $\varepsilon_0/\sqrt{1-\rho^2}$, which, although not of the correct form of the stationary distribution when we do not assume normal errors, has a variance in common with that case. We consider approximating the distribution of the first serial correlation coefficient,

$$R = \frac{\sum_{i=2}^{n} X_i X_{i-1}}{X_1^2/2 + \sum_{i=2}^{n-1} X_i^2 + X_n^2/2}, \tag{1}$$

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following Section 6 of Daniels (1956) who obtained a saddle-point approximation for this when f was the density of a normal variable. Note that without loss of generality we can assume $E\varepsilon_0^2 = 1$. We wish to consider testing the hypothesis $\rho \le \rho_0$ using R.

When F is unknown we will consider a bootstrap approximation to the test, generating a bootstrap sample, $X_1^*, \dots, X_n^*$, under the hypothesis using methods described later. Then we can obtain R^* by replacing $X_1, \dots, X_n$ by $X_1^*, \dots, X_n^*$ in the definition of R. We use a test based on R^*, so we need to know the accuracy of the approximations $P^*(R^* > u)$ to $P(R > u)$, where P^* refers to probabilities under the bootstrap sampling given the original sample.

We are unable to obtain a saddle-point approximation to this tail area directly. Instead we will consider conditioning over a subset of the random variables and obtain an approximation to the conditional tail area. In order to get the unconditional tail area, we take the expected value over the conditioning variables. We will show that we can approximate the conditional distribution with a saddle-point approximation where the conditioning is on $\mathbf{C}$, the odd numbered observations. The approximation is

$$P(R \ge u|\mathbf{C}) = \bar{\Phi}\big(\sqrt{m}W^+(u)\big)\big(1 + O_P(1/m)\big), \tag{2}$$

where m is the number of even numbered observations, $\bar{\Phi}(z) = P(Z \ge z)$ for Z a standard normal variable, and $W^+(u)$ is defined later. We obtain a similar approximation for $P^*(R^* \ge u|\mathbf{C}^*)$.

We want the relative error of the unconditional bootstrap tail area under ρ_0 as an approximation of the true tail area. We use the saddle-point approximation as a device to enable this comparison. Since we cannot get a saddle-point for the unconditional probability, we need to work from the conditional approximations. Now $P(R \ge u) = EP(R \ge u|\mathbf{C})$ and $P^*(R^* \ge u) = E^*P^*(R^* \ge u|\mathbf{C}^*)$, where E^* is expectation under the bootstrap resampling given the original sample. Then the relative error is

$$\frac{P(R \ge u) - P^*(R^* \ge u)}{P(R \ge u)}. \tag{3}$$

The above conditioning suggests a different conditional bootstrap, in which we condition on the odd numbered observations $\mathbf{C}$ and obtain conditional bootstrap samples for the even observations. This permits a direct comparison of the conditional distributions of the ratios R and a bootstrap counterpart given the same odd numbered observations, $\mathbf{C}$. We describe this conditional bootstrap and compare tests based on it to tests based on the unconditional bootstrap. We introduce this conditional bootstrap and obtain a saddle-point approximation for it.

The next section provides the details of the conditioning and is followed by a section giving results for the Gaussian case for both conditional and unconditional cases, then by sections giving the derivation of the main result. A final section provides some numerical results illustrating the accuracy of the approximations and comparing the power of the conditional and unconditional bootstraps.

2. Conditioning. Assume that $n = 2m + 1$. Let

$$S = \sum_{i=2}^{n} X_i X_{i-1} - u\left(X_1^2/2 + \sum_{i=2}^{n-1} X_i^2 + X_n^2/2\right),$$

then $P(R > u) = P(S > 0)$. Let $A_i = X_{2i-1} + X_{2i+1}$, $B_i = (X_{2i-1}^2 + X_{2i+1}^2)/2$ for $i = 1, \dots, m$, and $\mathbf{C} = (X_1, X_3, \dots, X_n)$, and write

$$\begin{aligned} S &= \sum_{i=1}^{m} \bigl(A_i X_{2i} - u(X_{2i}^2 + B_i)\bigr) \\ &= -u \sum_{i=1}^{m} (X_{2i} - A_i/2u)^2 + m \frac{\bar{A}^2 - 4u^2 \bar{B}}{4u}, \end{aligned} \tag{4}$$

where $m\bar{A}^2 = \sum_{i=1}^{m} A_i^2$ and $m\bar{B} = \sum_{i=1}^{m} B_i$. So for $u > 0$, $P(S > 0|\mathbf{C}) = 0$ if $\bar{A}^2 - 4u^2\bar{B} < 0$.

It is clear that when $\rho_0 = 0$, conditional on $\mathbf{C}$, the terms in the sums in S are independent random variables. If $\rho_0 \neq 0$ the first step is to show that the X_{2i}'s are independent conditional on $\mathbf{C}$. This follows since we can factor the joint density of $\mathbf{D} = (X_2, X_4, \dots, X_{n-1})$ conditional on $\mathbf{C} = (X_1, X_3, \dots, X_n)$.

3. The Gaussian case. We will first give a brief account of the saddle-point approximations for the Gaussian case where both an unconditional and conditional approach are possible with explicit forms for the approximations.

Consider the unconditional normal case. If $\varepsilon_1, \dots, \varepsilon_n$ are independent standard normal, $X_1 = \varepsilon_1/\sqrt{1-\rho^2}$ and $X_i = \rho X_{i-1} + \varepsilon_i$ for $i = 2, \dots, n$, and

$$S = \sum_{i=2}^{n} X_i X_{i-1} - u\left(X_1^2/2 + \sum_{i=2}^{n-1} X_i^2 + X_n^2/2\right) = x^T(A - uB)x,$$

with A and B symmetric. We find the saddle-point approximation to $P(S \geq 0)$ following the method of Lieberman (1994b). The cumulative generating function of S is

$$\begin{aligned} \kappa(t) &= \log\bigl((2\pi)^{n/2}|\Sigma|^{1/2}\bigr)^{-1} \int e^{tx^T(A-uB)x - x^T\Sigma^{-1}x/2}\,dx \\ &= \log\bigl|I - 2tU(A - uB)U^T\bigr|^{-1/2} \\ &= -\frac{1}{2}\sum_{i=1}^{n} \log(1 - 2t\lambda_i), \end{aligned}$$

where $\sigma_{ij} = \rho^{|i-j|}$, $\Sigma = U^T U$, U is upper triangular and $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of $U(A - uB)U^T$. So the Barndorff–Nielsen approximation [see Section 1.2 of Field and Robinson (2013)] is

$$P(S \geq 0) = \bar{\Phi}\bigl(\sqrt{m}w^\dagger\bigr)\bigl(1 + O(1/n)\bigr),$$

where $w^\dagger = w - \log\psi(w)/nw$ for $w = (-2\kappa(\hat{t}))^{1/2}$, where $\hat{t}$ is the solution to $\kappa'(\hat{t}) = 0$ and $\psi(w) = w/\hat{t}(\kappa''(\hat{t}))^{1/2}$. Note that $\kappa(t)$, $\hat{t}$, w and so $w^\dagger$ all are functions of u, but this dependence is suppressed to simplify notation.

To consider the power of the test $H_0\colon \rho = \rho_0$ versus the alternative $H_1\colon \rho = \rho_1 > \rho_0$, we can find the critical values from the saddle-point approximation under H_0 for a fixed level and then the power directly under H_1.

Now consider the conditional test. If the observations are as above and $A_1, \dots, A_m$ and $B_1, \dots, B_m$ are defined as in Section 2, then we need to find $P(S \geq 0|\mathbf{C})$. Recall that

$$S = \sum_{i=1}^{m} \big(X_{2i} A_i - u\big(X_{2i}^2 + B_i\big)\big),$$

and in this case, given A_i and B_i, X_{2i} are conditionally independent with conditional distribution normal with mean $\rho A_i/(1+\rho^2)$ and variance $1/(1+\rho^2)$. The test of H_0 will be performed by considering the conditional distribution of S given $\mathbf{C}$ obtained when X_{2i} are assumed to be conditionally independent normal variables with mean $\rho_0 A_i/(1+\rho_0^2)$ and variance $1/(1+\rho_0^2)$. So the critical value at a fixed level can be calculated from this distribution. Then the power can be calculated using the conditional distribution of S given $\mathbf{C}$ using X_{2i} conditionally independent normal variables with mean $\rho_1 A_i/(1+\rho_1^2)$ and variance $1/(1+\rho_1^2)$. These conditional distributions can be approximated by a saddle-point method as in the unconditional case, by using the conditional cumulative generating function of S, given by

$$\begin{aligned} \kappa(t) &= \frac{1}{m}\sum_{i=1}^{m} \log \sqrt{\frac{1+\rho^2}{2\pi}} \int e^{-tu(z-A_i/2u)^2-(1+\rho^2)(z-\rho A_i/(1+\rho^2))^2/2}\,dz \\ &\qquad + t\frac{\bar{A}^2 - 4u^2\bar{B}}{4u} \qquad (5) \\ &= -\frac{1}{2}\log\left(1+\frac{2tu}{1+\rho^2}\right) - tu\bar{B} + \frac{\bar{A}^2(\rho+t)^2}{2(1+\rho^2+2tu)} - \frac{\bar{A}^2\rho^2}{2(1+\rho^2)}. \end{aligned}$$

From (5), $\kappa(0) = 0$, and differentiating (5) shows that for $u > 0$, $\kappa'(0) < 0$ and that $\kappa'(t) < 0$ for all $t > 0$ if $\bar{A}^2 - 4u^2\bar{B} < 0$ and that $\kappa'(t) \to (\bar{A}^2 - 4u^2\bar{B})/4u$ as $t \to \infty$. So $\kappa'(t) = 0$ has a solution, if and only if $\bar{A}^2 - 4u^2\bar{B} > 0$. Then the Barndorff–Nielsen approximation for the conditional distribution can be obtained as before.

4. The general case. We can get a general bootstrap sample by considering the residuals $\varepsilon_i = X_i - \rho_0 X_{i-1}$, $i = 2, \dots, n$ and drawing bootstrap replicates by sampling $\varepsilon_1^*, \dots, \varepsilon_n^*$ from $F_n(x) = \sum_{i=2}^n I((\varepsilon_i - \bar{\varepsilon})/\sigma_n \leq x)/(n-1)$, where $\bar{\varepsilon} = \sum_{i=2}^n \varepsilon_i/(n-1)$ and $\sigma_n^2 = \sum_{i=2}^n (\varepsilon_i - \bar{\varepsilon})^2/(n-1)$, then generating bootstrap versions of the sample as $X_1^* = \varepsilon_1^*/\sqrt{1-\rho_0^2}$, $X_i^* = \rho_0 X_{i-1}^* + \varepsilon_i^*$ for $i = 2, \dots, n$. From this bootstrap sample we can calculate R^* unconditionally.

We consider saddle-point approximations to the conditional distribution of S given $\mathbf{C}$ then get the approximation to the unconditional distribution by considering the expectation of these. For the bootstrap no density exists, so we consider a smoothed bootstrap by adding independent normal variables with zero mean and small standard deviation τ to each bootstrap value $\varepsilon_1^*, \dots, \varepsilon_n^*$ obtaining $\varepsilon_1^\dagger, \dots, \varepsilon_n^\dagger$. Then we can proceed in the same way to approximate the bootstrap distribution as the expectation of the approximation to the conditional distribution. Finally we show that for a suitable choice of τ the smoothed bootstrap approximates the unconditional bootstrap with appropriate relative error.

We also consider a conditional bootstrap where we condition on $\mathbf{C}$, the same conditioning variables used for the true distribution. Here we are able to obtain relative errors for the approximation to the conditional distribution of S given $\mathbf{C}$.

4.1. *Approximations under conditioning.* From the factorization of the joint density of $\mathbf{D} = (X_2, X_4, \dots, X_{n-1})$ conditional on $\mathbf{C} = (X_1, X_3, \dots, X_n)$, we get the conditional density of X_{2i} given X_{2i-1} and X_{2i+1} is

$$\begin{aligned} g(z|X_{2i-1}, X_{2i+1}) &\\ &= f(z|X_{2i-1}) f(X_{2i+1}|z)/f(X_{2i+1}|X_{2i-1}) \\ &= \frac{f_\varepsilon(z - \rho_0 X_{2i-1}) f_\varepsilon(X_{2i+1} - \rho_0 z)}{\int f_\varepsilon(z - \rho_0 X_{2i-1}) f_\varepsilon(X_{2i+1} - \rho_0 z)\, dz}, \end{aligned}$$

where f_ε is the density of the errors $\varepsilon_2, \dots, \varepsilon_n$. Define S as in (4). Then we can get approximations to the distribution of S given $\mathbf{C}$ using this density.

The conditional cumulant generating function for S given $\mathbf{C}$ is

$$\begin{aligned} mK(t,u) &= \sum_{i=1}^m \log \int e^{\{t(A_i z - u(z^2 + B_i))\}} g(z|X_{2i-1}, X_{2i+1})\, dz \\ &= \sum_{i=1}^m \log \int e^{-tu(z - A_i/2u)^2} g(z|X_{2i-1}, X_{2i+1})\, dz \\ &\quad + m\frac{t(\bar{A}^2 - 4u^2\bar{B})}{4u}. \end{aligned} \tag{6}$$

Note that this will exist whenever $tu > 0$. We use the notation $K_{ij}(t,u) = \partial^{i+j} K(t,u)/\partial t^i \partial u^j$. Then differentiating (6) with respect to t gives

$$K_{10}(t,u) = -\frac{1}{m}\sum_{i=1}^m K_i(t,u) + \frac{(\bar{A}^2 - 4u^2\bar{B})}{4u} \tag{7}$$

and

$$K_{20}(t,u) = \frac{1}{m}\sum_{i=1}^{m} \frac{\int u^2(z - A_i/2u)^4 e^{-tu(z-A_i/2u)^2} g(z|X_{2i-1}, X_{2i+1})\,dz}{\int e^{-tu(z-A_i/2u)^2} g(z|X_{2i-1}, X_{2i+1})\,dz} - \frac{1}{m}\sum_{i=1}^{m} K_i(t,u)^2, \tag{8}$$

where

$$K_i(t,u) = \frac{\int u(z - A_i/2u)^2 e^{-tu(z-A_i/2u)^2} g(z|X_{2i-1}, X_{2i+1})\,dz}{\int e^{-tu(z-A_i/2u)^2} g(z|X_{2i-1}, X_{2i+1})\,dz}. \tag{9}$$

Note from (6) that $K(0,u) = 0$ and from (7) that if $\bar{A}^2 - 4u^2\bar{B} < 0$, then $K_{10}(t,u)$ is always negative, so there is no solution to the saddle-point equation $K_{10}(t,u) = 0$. For $\bar{A}^2 - 4u^2\bar{B} > 0$ we first find a value of u such that $K_{10}(0,u) = 0$. Now

$$K_{10}(0,u) = \frac{1}{m}\sum_{i=1}^{m} \int (zA_i - uz^2) g(z|X_{2i-1}, X_{2i+1})\,dz - u\bar{B}.$$

Let u_0 be such that $K_{10}(0,u_0) = 0$, then

$$u_0 = \frac{\sum_{i=1}^{m} \int zg(x|X_{2i-1}, X_{2i+1})\,dz A_i}{\sum_{i=1}^{m} \int z^2 g(z|X_{2i-1}, X_{2i+1})\,dz + m\bar{B}}. \tag{10}$$

So for $u > u_0$,

$$K_{10}(0,u) = (u_0 - u)\left(\frac{1}{m}\sum_{i=1}^{m} \int z^2 g(z|X_{2i-1}, X_{2i+1})\,dz + \bar{B}\right) < 0$$

and $K_{20}(t,u) > 0$. So for $u > u_0$, $K_{10}(t,u)$ is increasing in t, is negative for $t = 0$ and as $t \to \infty$,

$$K_{10}(t,u) \to \frac{\bar{A}^2 - 4u^2\bar{B}}{4u},$$

since the first term in (7) tends to 0 as $t \to \infty$. Thus the saddle-point equation $K_{10}(t,u) = 0$, has a finite solution, $t(u)$ for $u > u_0$, if and only if $\bar{A}^2 - 4u^2\bar{B} > 0$. Further, $K(t(u),u)$ exists and is finite if $\bar{A}^2 - 4u^2\bar{B} > 0$. If $\bar{A}^2 - 4u^2\bar{B} < 0$, $K(t,u) \to -\infty$ as $t \to \infty$.

If $\bar{A}^2 - 4u^2\bar{B} > 0$, the Barndorff–Nielsen form of the saddle-point approximation is

$$P(S \geq 0|\mathbf{C} = \mathbf{c}) = \bar{\Phi}(\sqrt{m}W^+)(1 + O_P(m^{-1})), \tag{11}$$

where

$$W^+ = W - \log(\Psi(W))/(mW), \tag{12}$$

with

$$W=\sqrt{-2K\big(t(u),u\big)} \quad \text{and} \quad \Psi(W)=W/\big(t(u)\sqrt{K_{20}\big(t(u),u\big)}\big). \tag{13}$$

The proof of this result is given in Section 1 of the supplementary material of Field and Robinson (2013).

The bootstrap distribution of $\varepsilon_1^*,\dots,\varepsilon_n^*$ does not have a density, but we can approximate the distribution by a smoothed version which is continuous. Let

$$f_n(z)=\frac{1}{n-1}\sum_{k=2}^{n}\frac{e^{-(z-\eta_k)^2/2\tau^2}}{\sqrt{2\pi\tau^2}}, \tag{14}$$

where $\eta_k=(\varepsilon_k-\bar{\varepsilon})/\sigma_n$. If we draw a sample $\varepsilon_1^\dagger,\dots,\varepsilon_n^\dagger$ from this distribution and obtain $X_1^\dagger=\varepsilon_1^\dagger/(1-\rho_0^2)$ and $X_i^\dagger=\rho_0X_{i-1}^\dagger+\varepsilon_i^\dagger$, then choosing τ small enough, we can approximate the bootstrap distribution of R^* by the bootstrap version of $R^\dagger$. With this new smoothed bootstrap we can proceed to get the saddle-point approximation to its distribution by using the expectation of the conditional bootstrap as we do for the saddle-point approximation of the distribution of R.

The conditional density of $X_{2i}^\dagger$ given $X_{2i-1}^\dagger$ and $X_{2i+1}^\dagger$ is

$$g^\dagger\big(z|X_{2i-1}^\dagger,X_{2i+1}^\dagger\big)=\frac{f_n(z-\rho_0X_{2i-1}^\dagger)f_n(X_{2i+1}^\dagger-\rho_0z)}{\int f_n(z-\rho_0X_{2i-1}^\dagger)f_n(X_{2i+1}^\dagger-\rho_0z)\,dz}, \tag{15}$$

where

$$g^\dagger\big(z|X_{2i-1}^\dagger,X_{2i+1}^\dagger\big)=\frac{1}{(n-1)^2}\sum_k\sum_l g_{ikl}^\dagger(z) \tag{16}$$

for

$$g_{ikl}^\dagger(z)=\frac{(n-1)^2e^{-(z-\rho_0X_{2i-1}^\dagger-\eta_k)^2/2\tau^2-(X_{2i+1}^\dagger-\rho_0z-\eta_l)^2/2\tau^2}}{\sum_k\sum_l\int e^{-(z-\rho_0X_{2i-1}^\dagger-\eta_k)^2/2\tau^2-(X_{2i+1}^\dagger-\rho_0z-\eta_l)^2/2\tau^2}\,dz}. \tag{17}$$

Now

$$\begin{aligned}&\big[\big(z-\rho_0X_{2i-1}^\dagger-\eta_k\big)^2+\big(X_{2i+1}^\dagger-\rho_0z-\eta_l\big)^2\big]\\&\quad=(1+\rho_0^2)\left(z'-\frac{\eta_k-\rho_0\eta_l}{1+\rho_0^2}\right)^2\\&\qquad+\frac{(X_{2i+1}^\dagger-\rho_0^2X_{2i-1}^\dagger-\rho_0\eta_k-\eta_l)^2}{(1+\rho_0^2)},\end{aligned}$$

where $z'=z-\rho_0(X_{2i-1}^\dagger+X_{2i+1}^\dagger)/(1+\rho_0^2)$. So, integrating with respect to z in the denominator of $g_{ikl}^\dagger(z)$ we have

$$g_{ikl}^\dagger(z)=\frac{e^{-(1+\rho_0^2)(z'-(\eta_k-\rho_0\eta_l)/(1+\rho_0^2))^2/2\tau^2-(X_{2i+1}^\dagger-\rho_0^2X_{2i-1}^\dagger-\rho_0\eta_k-\eta_l)^2/2\tau^2(1+\rho_0^2)}}{\sqrt{2\pi\tau^2}\sum_k\sum_l e^{-(X_{2i+1}^\dagger-\rho_0^2X_{2i-1}^\dagger-\rho_0\eta_k-\eta_l)^2/2\tau^2(1+\rho_0^2)}}.$$

Define $S^\dagger$ as in (4) using $X^\dagger$ in place of X, with analogous definitions for $A_i^\dagger$, $B_i^\dagger$, $R^\dagger$ and $C^\dagger$. Then the conditional cumulant generating function of $S^\dagger$ given $\mathbf{C}^\dagger$ is

$$\begin{aligned} mK^\dagger(t,u) &= \sum_{i=1}^{m} \log \int e^{-tu(z-A_i^\dagger/2u)^2} g^\dagger\big(z|X_{2i-1}^\dagger, X_{2i+1}^\dagger\big)\,dz \\ &\quad + m\frac{t(\bar{A}^{\dagger 2} - 4u^2\bar{B}^\dagger)}{4u}, \end{aligned} \tag{18}$$

which is of the same form as the formula for $K(t,u)$ with $g^\dagger(z|X_{2i-1}^\dagger, X_{2i+1}^\dagger)$ replacing $g(z|X_{2i-1}, X_{2i+1})$. So we can obtain analogous results to those of (7)–(10) and to the argument following these, to show that, when $\bar{A}^{\dagger 2} - 4u^2\bar{B}^\dagger > 0$, if $t^\dagger(u)$ is the solution of $K_{10}^\dagger(t,u) = 0$, then the saddle-point approximation is

$$P^\dagger\big(S^\dagger \geq 0|\mathbf{C}^\dagger\big) = \bar{\Phi}\big(\sqrt{m}W^{\dagger +}\big)\big(1 + O_P(m^{-1})\big),$$

where

$$W^{\dagger +} = W^* - \log\big(\Psi^\dagger(W)\big)/\big(mW^\dagger\big),$$

with

$$W^\dagger = \sqrt{-2K^\dagger\big(t^\dagger(u), u\big)} \quad \text{and} \quad \Psi^\dagger(W) = W^\dagger/\Big(t^*(u)\sqrt{K_{20}^\dagger\big(t^\dagger(u), u\big)}\Big).$$

We can summarize these results in the following theorem:

THEOREM 1. *For $u \geq 0$, $P(S > 0|\mathbf{C}) = 0$ if $\bar{A}^2 - 4u^2\bar{B} < 0$ and $P(S^\dagger > 0|\mathbf{C}^\dagger) = 0$ if $\bar{A}^{\dagger 2} - 4u^2\bar{B}^\dagger < 0$. If $\bar{A}^2 - 4u^2\bar{B} > 0$ and $\bar{A}^{\dagger 2} - 4u^2\bar{B}^\dagger > 0$, for $u > u_0$ from (10) and $u > u_0^\dagger$ defined analogously, $t(u)$ and $t^\dagger(u)$, solutions of $K_{10}(t,u) = 0$ and $K_{10}^\dagger(t,u) = 0$, exist and are both finite and positive, and if EX_1^8 is bounded,*

$$P(R > u|\mathbf{C}) = \bar{\Phi}\big(\sqrt{m}W^+\big)\big[1 + O_P(1/m)\big]$$

and

$$P^\dagger\big(R^\dagger > u|\mathbf{C}^\dagger\big) = \bar{\Phi}\big(\sqrt{m}W^{\dagger +}\big)\big[1 + O_P(1/m)\big],$$

where $W(u)$, W^+ and $\Psi(W^\dagger)$ are defined as in (12) and (13) and

$$W^{\dagger +} = W^\dagger - \log\big(\Psi^\dagger\big)/\big(mW^\dagger\big),$$

with

$$W^\dagger = \sqrt{-2K^\dagger\big(t^\dagger(u), u\big)} \quad \textit{and} \quad \Psi^\dagger\big(W^\dagger\big) = W^\dagger/\Big(t^\dagger(u)\sqrt{K_{20}^\dagger\big(t(u), u\big)}\Big).$$

REMARK. If R' has the denominator in R replaced by $\sum_{i=1}^n X_i^2$, then $P(R' > u|\mathbf{C}) = P(S > u(X_1^2 + X_n^2)/2|\mathbf{C})$. So we can proceed with the saddle-point approximation obtaining results with the relative error unchanged, since throughout the errors will be affected by a term of $O_P(u/m)$. A similar argument gives results for n even.

4.2. *The relative error of the bootstrap.* Assume throughout this section that the conditions of Theorem 1 hold. Let $\mathcal{A} = \{\mathbf{C} : \bar{A}^2 - 4u^2\bar{B} > 0\}$. Now $E(\bar{A}^2 - 4u^2\bar{B}) = (2(1-2u^2) + 2\rho_0^2)/(1-\rho_0^2)$ and $\operatorname{var}(\bar{A}^2 - 4u^2\bar{B}) = O(1/m)$, so for $1 - 2u^2 + \rho_0^2 > \delta > 0$, it follows from the Chebychev inequality that $P(\mathcal{A}^c) = P(\bar{A}^2 - 4u^2\bar{B} < 0) = O(1/m)$. So, since $P(S > 0|\mathbf{C})I(\mathcal{A}^c) = 0$,

$$\begin{aligned} P(S>0) &= E\big[P(S>0|\mathbf{C})I(\mathcal{A})\big] + E\big[P(S>0|\mathbf{C})I(\mathcal{A}^c)\big] \\ &= E\big[\bar{\Phi}(\sqrt{m}W^+)I(\mathcal{A})\big(1 + O_P(1/m)\big)\big]. \end{aligned} \tag{19}$$

Restrict attention to $\mathcal{A}$, so with u_0 given in (10), $K_{10}(0, u_0) = 0$ and thus $t(u_0) = 0$ and

$$t(u) = t'(u_o)(u - u_0) + \tfrac{1}{2}t''(u_0)(u - u_o)^2 + O_P\big((u - u_0)^3\big).$$

Further, since $K_{10}(t(u), u) = 0$, $t'(u_0) = -K_{11}/K_{20}$, where we write $K_{ij} = K_{ij}(0, u_0)$. Then expanding $K(t(u), u)$ about u_0 we obtain,

$$K\big(t(u), u\big) = -D_1(u - u_0)^2 - D_2(u - u_0)^3 + O_P\big((u - u_0)^4\big),$$

where $D_1 = K_{11}^2/2K_{20}$ and

$$D_2 = \tfrac{1}{2}\big[t_0''K_{11} + t_0'K_{12} + t_0'^2K_{21} + \tfrac{1}{3}t_0'^3K_{30}\big]. \tag{20}$$

So

$$W = (u - u_0)\sqrt{2D_1}\big(1 + (u - u_0)D_2/2D_1\big) + O_P\big((u - u_0)^3\big). \tag{21}$$

Note that u_0 is given in (10), so

$$u_0 = \frac{E[E(X_2|X_1, X_3)(X_1 + X_3)]}{E[E(X_2^2|X_1, X_3) + X_1^2]} + J_u/\sqrt{m} + O_P(1/m), \tag{22}$$

where, here and in the sequel, values of J denote zero mean random variables with finite variances. Further, since $X_2 = \rho_0 X_1 + \varepsilon_2$, $X_3 = \rho_0^2 X_1 + \rho_0\varepsilon_2 + \varepsilon_3$ and X_1 is independent of ε_2 and ε_3, the numerator in (22) is

$$\rho_0 E X_1(X_1 + X_3) + E\big[E(\varepsilon_2|\rho_0\varepsilon_2 + \varepsilon_3)\big(\rho_0^2 X_1 + \rho_0\varepsilon_2 + \varepsilon_3\big)\big],$$

and since $\varepsilon_2 = ((\varepsilon_2 - \rho_0\varepsilon_3) + \rho_0(\rho_0\varepsilon_2 + \varepsilon_3))/(1 + \rho_0^2)$, the numerator is

$$\frac{\rho_0}{1-\rho_0^2} + \frac{\rho_0^3}{1-\rho_0^2} + \rho_0 = \frac{2\rho_0}{1-\rho_0^2}.$$

The denominator of (22) is

$$E\big[E\big(X_2^2|X_1, X_3\big) + X_1^2\big] = E\big(X_2^2\big) + E\big(X_1^2\big) = \frac{2}{1-\rho_0^2}.$$

So

$$u_0 = \rho_0 + J_u/\sqrt{m} + O(1/m). \tag{23}$$

From (7) and (9)

$$K_{11} = -\frac{1}{m}\sum_{i=1}^{m}\int z^2 g(z|X_{2i-1}, X_{2i+1})\,dz - \bar{B}$$

so

$$EK_{11} = -E(X_2^2 + X_1^2) = -\frac{2}{1-\rho_0^2}$$

and

$$K_{11} = -\frac{2}{1-\rho_0^2} + J_{11}/\sqrt{m} + O_P(1/m). \tag{24}$$

From (8), and using (23), we can write

$$\begin{aligned} K_{20} &= \frac{1}{m}\sum_{i=1}^{m}\biggl\{\int (\rho_0 z^2 - A_i z)^2 g(z|X_{2i-1}, X_{2i+1})\,dz \\ &\qquad - \biggl[\int (\rho_0 z^2 - A_i z) g(z|X_{2i-1}, X_{2i+1})\,dz\biggr]^2\biggr\} \\ &\quad + J_{20}/\sqrt{m} + O_P(1/m) \\ &= \frac{1}{m}\sum_{i=1}^{m}\gamma(X_{2i-1}, X_{2i+1}) + J_{20}/\sqrt{m} + O_P(1/m), \end{aligned} \tag{25}$$

so

$$K_{20} = E_{20} + J'_{20}/\sqrt{m} + O_P(1/m), \tag{26}$$

where

$$E_{20} = \frac{1}{m}\sum_{i=1}^{m} E\gamma(X_{2i-1}, X_{2i+1}).$$

Now, recalling that $D_1 = K_{11}^2/2K_{20}$, and using (24) and (26), we have

$$D_1 = \frac{2}{(1-\rho_0^2)^2 E_{20}} + J_D/\sqrt{m} + O_P(1/m), \tag{27}$$

$t(u) = -(u-u_0)K_{11}/K_{20} + O_P((u-u_0)^2)$, $\Psi(u) = W/t(u)\sqrt{K_{20}} = 1 + O_P(u - u_0)$, so $\log \Psi(u)/mW = O_P(1/m)$, and, from (12), (21), (23) and (27),

$$\begin{aligned} &W^+ - EW^+ \\ &\quad = (u-\rho_0)\biggl(\frac{J_W}{\sqrt{m}} + (u-\rho_0)\frac{H}{\sqrt{m}}\biggr) + O_P\biggl((u-\rho_o)^3 + \frac{1}{m}\biggr), \end{aligned} \tag{28}$$

where $H = \sqrt{m}(D_2/2D_1 - ED_2/2ED_1)$.

We can consider the smoothed bootstrap introduced in Section 4.1 in the same way. Let $W^\dagger$, $W^{\dagger+}$ be defined as in the statement of Theorem 1, and let $\mathcal{A}^\dagger = \{\mathbf{C}^\dagger : \bar{A}^{\dagger 2} - 4u^2\bar{B}^\dagger > 0\}$ and $E^\dagger_+(\cdot) = E^\dagger(\cdot|\mathcal{A}^\dagger)$. Then restricting attention to $\mathcal{A}^\dagger$, $K^\dagger_{10}(0, u^\dagger_0) = 0$, so $t^\dagger(u^\dagger_0) = 0$ and

$$t^\dagger(u) = t^{\dagger\prime}(u^\dagger_0)(u - u^\dagger_0) + \tfrac{1}{2}t^{\dagger\prime\prime}(u^\dagger_0)(u - u^\dagger_0)^2 + O_P\big((u - u^\dagger_0)^3\big),$$

with

$$t^{\dagger\prime}(u^\dagger_0) = -K^\dagger_{11}/K^\dagger_{20},$$

where $K^\dagger_{ij} = K^\dagger_{ij}(0, u_0)$. Now we proceed as above with $X^\dagger_i$, $g^\dagger_i(\cdot|X^\dagger_{2i-1}, X^\dagger_{2i+1}))$, $E^\dagger(\cdot)$ and $E^\dagger(\cdot|\cdot)$ replacing X_i, $g(z|X_{2i-1}, X_{2i+1})$, $E(\cdot)$ and $E(\cdot|\cdot)$. So

$$u^\dagger_0 = \rho_0 + J^\dagger_u/\sqrt{m} + O_P\left(\frac{\rho_0}{\sqrt{m}}\right),$$

$$K^\dagger_{11} = -\frac{2}{1-\rho_0^2} + J^\dagger_{11}/\sqrt{m} + O_P(1/\sqrt{m}) \tag{29}$$

and

$$\begin{aligned} K^\dagger_{20} = \frac{1}{m}\sum_{i=1}^m \bigg\{ &\int (\rho_0 z^2 - A^\dagger_i z)^2 g^\dagger(z|X^\dagger_{2i-1}, X^\dagger_{2i+1})\,dz \\ &- \left[\int (\rho_0 z^2 - A^\dagger_i z) g^\dagger(z|X^\dagger_{2i-1}, X^\dagger_{2i+1})\,dz\right]^2\bigg\} \\ &+ J^\dagger_{20}/\sqrt{m} + O_P(1/m) \\ = \frac{1}{m}\sum_{i=1}^m &\gamma^\dagger(X^\dagger_{2i-1}, X^\dagger_{2i+1}) + J^\dagger_{20}/\sqrt{m} + O_P(1/m). \end{aligned} \tag{30}$$

In order to compare the first terms of (25) and (30), we need first to replace $\gamma^\dagger(\cdot)$ in this first term by $\gamma(\cdot)$ appearing in E_{20}. The following lemma, the proof of which is given in Section 2 of the supplementary material of Field and Robinson (2013), accomplishes this.

LEMMA 1. *For* $\tau = O(1/\sqrt{m})$,

$$\int h(z)g^\dagger(z|X^\dagger_{2i-1}, X^\dagger_{2i+1})\,dz = \int h(z)g(z|X^\dagger_{2i-1}, X^\dagger_{2i+1})\,dz + \frac{J_h}{\sqrt{m}} + O_P\left(\frac{1}{m}\right).$$

Using Lemma 1,

$$\frac{1}{m}\sum_{i=1}^m \gamma^\dagger(X^\dagger_{2i-1}, X^\dagger_{2i+1}) = \frac{1}{m}\sum_{i=1}^m \gamma(X^\dagger_{2i-1}, X^\dagger_{2i+1}) + \frac{J_h}{\sqrt{m}} + O_P\left(\frac{1}{m}\right),$$

so

$$E_{20}^{\dagger} = \frac{1}{m} \sum_{i=1}^{m} E\gamma^{\dagger}(X_{2i-1}^{\dagger}, X_{2i+1}^{\dagger}) = E_{20} + J_{20}^{\ddagger}/\sqrt{m} + O_P(1/m). \tag{31}$$

Now, as before $D_1^{\dagger} = K_{11}^{\dagger 2}/2K_{20}^{\dagger}$, so using (29) and (31), we have

$$D_1^{\dagger} = \frac{2}{(1-\rho_0^2)^2 E_{20}} + J_D^{\dagger}/\sqrt{m} + O_P(1/m),$$

and an equation equivalent to (28) holds for $W^{\dagger +} - E^{\dagger} W^{\dagger +}$.

For some $0 < c < C < \infty$, let

$$\mathcal{E} = \left\{ \mathbf{C} : \frac{1}{m+1} \sum_{i=0}^{m} X_{2i+1}^8 < C, \frac{1}{m+1} \sum_{i=0}^{m} X_{2i+1}^2 > c \right\}. \tag{32}$$

In Theorem 1, the $O_P(1/m)$, can be replaced by θM_m, where $|\theta| < C$ and

$$M_m = m \sum_{i=1}^{m} EY_i^4 \Big/ \left[\sum_{i=1}^{m} EY_i^2 \right]^2$$

as shown in Section 1 of the supplementary material of Field and Robinson (2013), and for $\mathbf{C} \in \mathcal{E}$, M_m is bounded. So

$$P(R > u|\mathcal{E}) = E[P(S > 0|\mathbf{C})|\mathcal{E}] = E[\bar{\Phi}(\sqrt{m} W^+)|\mathcal{E}](1 + O_P(1/m)).$$

Using this and the equivalent term for $P^{\dagger}(R^{\dagger} > u|\mathcal{E})$, we have

$$\begin{aligned} &\frac{|P(R > u|\mathcal{E}) - P^{\dagger}(R^{\dagger} > u|\mathcal{E})|}{P(R > u|\mathcal{E})} \\ &\quad = \frac{|E^{\dagger}[\bar{\Phi}(\sqrt{m} W^{\dagger +})|\mathcal{E}] - E[\bar{\Phi}(\sqrt{m} W^+)|\mathcal{E}]|}{E[\bar{\Phi}(\sqrt{m} W^+)|\mathcal{E}]} \\ &\quad \le \frac{I_1 + I_2 + I_3}{\bar{\Phi}(\sqrt{m} E(W^+|\mathcal{E}))}, \end{aligned} \tag{33}$$

where we have used Jensen's inequality in the denominator and

$$I_1 = |\bar{\Phi}(\sqrt{m} E^{\dagger}(W^{\dagger +}|\mathcal{E})) - \bar{\Phi}(\sqrt{m} E(W^+|\mathcal{E}))|, \tag{34}$$

$$I_2 = |E^{\dagger}[\bar{\Phi}(\sqrt{m} W^{\dagger +})|\mathcal{E}] - \bar{\Phi}(\sqrt{m} E^{\dagger}(W^{\dagger +}|\mathcal{E}))| \tag{35}$$

and

$$I_3 = |E[\bar{\Phi}(\sqrt{m} W^+)|\mathcal{E}] - \bar{\Phi}(\sqrt{m} E(W^+|\mathcal{E}))|. \tag{36}$$

Noting that, for $\varphi(x) = -\bar{\Phi}'(x)$, $\varphi'(x) = -x\varphi(x)$ and $x < \varphi(x)/\bar{\Phi}(x) < 1 + x$, we have

$$\bar{\Phi}(\sqrt{m} E(W^+|\mathcal{E})) > \varphi(\sqrt{m} E(W^+|\mathcal{E}))/(1 + \sqrt{m} E(W^+|\mathcal{E})).$$

Then

$$\frac{I_3}{\bar{\Phi}(\sqrt{m}E(W^+|\mathcal{E}))} \le \frac{m}{2}\frac{E[(W^+ - E(W^+|\mathcal{E}))^2\varphi(\sqrt{m}W^{\ddagger})|\mathcal{E}]}{\varphi(\sqrt{m}E(W^+|\mathcal{E}))/(1+\sqrt{m}E(W^+|\mathcal{E}))},$$

where $W^{\ddagger}$ lies between W^+ and $E(W^+|\mathcal{E})$. Now, for $\mathbf{C} \in \mathcal{E}$, noting (21) and (23),

$$\frac{\varphi(\sqrt{m}W^{\ddagger})}{\varphi(\sqrt{m}E(W^+|\mathcal{E}))} = O_P\bigl(e^{\sqrt{m}(u-\rho_0)^2}\bigr) = O_P(1)$$

for $u = O(m^{-1/4})$, and using (21) and (28), we have

$$\frac{I_3}{\bar{\Phi}(\sqrt{m}E(W^+|\mathcal{E}))} = O_P\bigl(m(u-\rho_0)^4 + 1/m\bigr).$$

An equivalent result holds for I_2. Also, using the same results gives

$$\begin{aligned}\frac{I_1}{\bar{\Phi}(\sqrt{m}E(W^+|\mathcal{E}))} &= \frac{\sqrt{m}|E^{\dagger}(W^{\dagger+}|\mathcal{E}) - E(W^+|\mathcal{E})|\varphi(\sqrt{m}W^*)}{\varphi(\sqrt{m}E(W^+|\mathcal{E}))/(1+\sqrt{m}E(W^+|\mathcal{E}))} \\ &= O_P\bigl(\sqrt{m}(u-\rho_0)^3 + 1/m\bigr),\end{aligned}$$

where W^* lies between $E(W^+|\mathcal{E})$ and $E^{\dagger}(W^{\dagger+}|\mathcal{E})$.

Finally, we need to consider the relative errors of the bootstrap and the smoothed bootstrap.

LEMMA 2. *For $\tau = O(1/\sqrt{m})$ and $u - \rho_0 = O(n^{-1/4})$,*

$$P^{\dagger}\bigl(R^{\dagger} \ge u|\mathcal{E}\bigr)/P^*\bigl(R^* \ge u|\mathcal{E}\bigr) = 1 + O_P\bigl(m(u-\rho_0)^4 + 1/m\bigr).$$

The proof of Lemma 2 is given in Section 2 of the supplementary material of Field and Robinson (2013). Thus we have the following theorem:

THEOREM 2. *For $\mathcal{E}$ defined in (32), $u \ge \rho_0$, $u - \rho_0 = O(m^{-1/4})$ and $1 - 2u^2 + \rho_0^2 > \delta > 0$,*

$$\frac{P(R > u|\mathcal{E}) - P^*(R^* > u|\mathcal{E})}{P(R > u|\mathcal{E})} = O_P\bigl(m(u-\rho_0)^4 + 1/m\bigr).$$

Further, if $E\varepsilon_1^8$ exists, then $P(\mathcal{E}) = 1 - o(1)$, if $E\varepsilon_1^{16}$ exists, then $P(\mathcal{E}) = 1 - O(1/m)$ and if ε_1 is bounded, then $P(\mathcal{E}) = 1$, in which case the conditional probabilities can be replaced by their expectations over $\mathcal{E}$.

4.3. *The conditional bootstrap.* Consider obtaining a smoothed conditional bootstrap given $\mathbf{C}$. Let

$$f_n(z) = \frac{1}{n-1}\sum_{k=2}^{n}\frac{e^{-(z-\varepsilon_k)^2/2\tau^2}}{\sqrt{2\pi\tau^2}},$$

where $\varepsilon_i = X_i - \rho_0 X_{i-1}$, for $i = 2, \ldots, n$. Note that this differs from f_n of (14) in that the unstandardized errors are used. Then the conditional density of $X_{2i}^{\#}$, the smoothed bootstrap values of the even subscripted variable, given X_{2i-1} and X_{2i+1} is

$$g^{\#}(z|X_{2i-1}, X_{2i+1}) = \frac{f_n(z - \rho_0 X_{2i-1}) f_n(X_{2i+1} - \rho_0 z)}{\int f_n(z - \rho_0 X_{2i-1}) f_n(X_{2i+1} - \rho_0 z)\, dz},$$

where

$$g^{\#}(z|X_{2i-1}, X_{2i+1}) = \frac{1}{(n-1)^2} \sum_k \sum_l g_{ikl}^{\#}(z),$$

and, as in Section 4.1, this can be reduced to

$$g_{ikl}^{\#}(z) = \left(2\pi\tau^2/(1+\rho_0^2)\right)^{-1/2} e^{-(1+\rho_0^2)(z'-(\varepsilon_k - \rho_0\varepsilon_l)/(1+\rho_0^2))^2/2\tau^2} w_{ikl}^{\#},$$

where

$$w_{ikl}^{\#} = \frac{e^{-(X_{2i+1} - \rho_0^2 X_{2i-1} - \rho_0\varepsilon_k - \varepsilon_l)^2/2\tau^2(1+\rho_0^2)}}{\sum_k \sum_l e^{-(X_{2i+1} - \rho_0^2 X_{2i-1} - \rho_0\varepsilon_k - \varepsilon_l)^2/2\tau^2(1+\rho_0^2)}}$$

and $z' = z - \rho_0(X_{2i-1} + X_{2i+1})/(1+\rho_0^2)$.

For each i we sample from this distribution by first choosing $\varepsilon_k, \varepsilon_l$ with probabilities $w_{ikl}^{\#}$, then obtaining a random normal variable Z_i' with mean $(\varepsilon_k - \rho_0\varepsilon_l)/(1+\rho_0^2)$ and variance $\tau^2/(1+\rho_0^2)$, then taking $X_{2i}^{\#} = Z_i' + \rho_0(X_{2i-1} + X_{2i+1})/(1+\rho_0^2)$.

Then the conditional cumulant generating function of $S^{\#}$ given $\mathbf{C}$ is

$$\begin{aligned} mK^{\#}(t,u) &= \sum_{i=1}^{m} \log \int e^{\{t(A_i z - u(z^2 + B_i))\}} g^{\#}(z|X_{2i-1}, X_{2i+1})\, dz \\ &= \sum_{i=1}^{m} \log \int e^{-tu(z - A_i/2u)^2} g^{\#}(z|X_{2i-1}, X_{2i+1})\, dz + m\frac{t(\bar{A}^2 - 4u^2\bar{B})}{4u}. \end{aligned}$$

Proceeding as in Section 4.1 we have

$$K_{10}^{\#}(0,u) = \frac{1}{m} \sum_{i=1}^{m} \int (zA_i - uz^2) g^{\#}(z|X_{2i-1}, X_{2i+1})\, dz - u\bar{B}.$$

Let $u_0^{\#}$ be such that $K_{10}^{\#}(0, u_0^{\#}) = 0$, then

$$u_0^{\#} = \frac{\sum_{i=1}^{m} \int z g^{\#}(z|X_{2i-1}, X_{2i+1})\, dz A_i}{\sum_{i=1}^{m} \int z^2 g^{\#}(z|X_{2i-1}, X_{2i+1})\, dz + m\bar{B}}. \qquad (37)$$

So for $u > u_0^{\#}$,

$$K_{10}^{\#}(0,u) = (u_0^{\#} - u)\left(\frac{1}{m} \sum_{i=1}^{m} \int z^2 g(z|X_{2i-1}, X_{2i+1})\, dz + \bar{B}\right) < 0$$

and $K_{20}^{\#}(t,u) > 0$. So for $u > u_0^{\#}$, $K_{10}^{\#}(t,u)$ is increasing in t, is negative for $t = 0$ and as $t \to \infty$,

$$K_{10}^{\#}(t,u) \to \frac{\bar{A}^2 - 4u^2\bar{B}}{4u}.$$

Thus the saddle-point equation $K_{10}^{\#}(t,u) = 0$ has a finite solution $t^{\#}(u)$ for $u > u_0^{\#}$, if and only if $\bar{A}^2 - 4u^2\bar{B} > 0$. Further, $K^{\#}(t^{\#}(u), u)$ exists and is finite if $\bar{A}^2 - 4u^2\bar{B} > 0$. If $\bar{A}^2 - 4u^2\bar{B} < 0$, $K^{\#}(t,u) \to -\infty$ as $t \to \infty$.

Let $W^{\#}$, $W^{\#+}$ be defined in the same way as in the statement of Theorem 1, then

$$P^{\#}(R^{\#} > u) = \bar{\Phi}(\sqrt{m}W^{\#+})(1 + O_P(1/m)). \tag{38}$$

Now $K_{10}^{\#}(0, u_0^{\#}) = 0$, so $t^{\#}(u_0^{\#}) = 0$ and

$$t^{\#}(u) = t^{\#\prime}(u_0^{\#})(u - u_0^{\#}) + \tfrac{1}{2}t^{\#\prime\prime}(u_0^{\#})(u - u_0^{\#})^2 + O_P((u - u_0^{\#})^3),$$

with

$$t^{\#\prime}(u_0^{\#}) = -K_{11}^{\#}/K_{20}^{\#},$$

where $K_{ij}^{\#} = K_{ij}^{\#}(0, u_0)$. Then

$$K_{11}^{\#} = \frac{1}{m}\sum_{i=1}^{m}\int z^2 g^{\#}(z|X_{2i-1}, X_{2i+1})\,dz - \bar{B}$$

and

$$K_{20}^{\#} = \frac{1}{m}\sum_{i=1}^{m}\biggl\{\int (u_0^{\#}z^2 - A_i z)^2 g^{\#}(z|X_{2i-1}, X_{2i+1})\,dz - \biggl[\int (u_0^{\#}z^2 - A_i z) g^{\#}(z|X_{2i-1}, X_{2i+1})\,dz\biggr]^2\biggr\}.$$

Now, as before, $D_1^{\#} = K_{11}^{\#2}/2K_{20}^{\#}$. To compare $D_1^{\#}$ and D_1 we need the following lemma, the proof of which is given in Section 2 of the supplementary material Field and Robinson (2013).

LEMMA 3.

$$\int h(z)g^{\#}(z|X_1, X_3)\,dz = \int h(z)g(z|X_1, X_3)\,dz + O_P\biggl(\frac{1}{m}\biggr).$$

So, applying the lemma to $u_0^{\#}$, $K_{11}^{\#}$ and $K_{20}^{\#}$,

$$D_1^{\#} = D_1 + O_P(1/m).$$

Now using (12) and an analogous term for $W^{\#}$ and noting that $D_2 - D_2^{\#} = O_P(1/\sqrt{m})$, we have

$$\sqrt{m}(W^+ - W^{\#+})(1 + \sqrt{m}W^+) = O(\sqrt{m}(u - \rho_0)^3 + 1/m).$$

Summarizing these results we have the following theorem:

THEOREM 3. *For* $u \geq 0$, $P(S > 0|\mathbf{C}) = 0$ *and* $P(S^{\#} > 0|\mathbf{C}) = 0$ *if* $\bar{A}^2 - 4u^2\bar{B} < 0$ *and if* $\bar{A}^2 - 4u^2\bar{B} > 0$ $t(u)$ *and* $t^{\#}(u)$, *solutions of* $K_{10}(t,u) = 0$ *and* $K_{10}^{\#}(t,u) = 0$, *exist and are both finite and positive*, *and if* EX_1^8 *is bounded*, (38) *holds and*

$$P(R > u|\mathbf{C}) = P^{\#}(R^{\#} > u|\mathbf{C})\big[1 + O_P\big(\sqrt{m}(u - \rho_0)^3 + 1/m\big)\big].$$

5. Numerical results. Monte Carlo simulations, bootstraps and tail area approximations both unconditionally and conditionally are used to illustrate accuracy of results and to compare the power of the unconditional and the conditional bootstrap.

First we describe the computational methods. The true distribution of $\hat{\rho}$ is approximated by Monte Carlo simulations of 1,000,000. For the bootstrap, we consider testing $H_0 : \rho = \rho_0$. The unconditional bootstrap is straightforward in that we compute $n - 1$ residuals, $\varepsilon_i = x_i - \rho_0 x_{i-1}$, center them and sample these with replacement. Then $x_i^* = \rho_0 x_{i-1}^* + \varepsilon_i^*$ with $x_1^* = \varepsilon_1^*/(1 - \rho^2)$, and we compute R^* and obtain an estimate of $P^*(R^* > u)$ from repetitions. For the conditional bootstrap of Section 4.2, we draw samples $\varepsilon_i^{\dagger}$'s from f_n in (14) with τ equal to $1/m$. We first generate $X_i^{\dagger}$'s from the $\varepsilon_i^{\dagger}$'s. Then $X_{2i}^{\dagger}$ are replaced by generating an observation from the normal mixture given in (15)–(17), $R^{\dagger}$ is computed and repetitions give an estimate of $P^{\dagger}(R^{\dagger} > u|\mathbf{C}^*)$. Now repeating this entire process from sampling $\varepsilon_i^{\dagger}$'s and averaging the conditional probabilities gives an estimate of $P^{\dagger}(R^{\dagger} > u)$. For the conditional bootstrap of Section 4.3, we replace X_{2i} by $X_{2i}^{\#}$ drawn from (15), calculate $R^{\#}$ and repeat this process to get an estimate of $P^{\#}(R^{\#} > u|\mathbf{C})$.

The results for the approximations of Section 3 for the Gaussian case are given in the upper part of Table 1 for the unconditional results (U) and the lower part for the conditional case (C). As can be seen, the agreements between the simulation results and the saddle-point, computed as in Section 3 for normal data, are excellent with very accurate results, even for $n = 9$. The accuracy for values of $\rho < 0.5$ is even better.

In Table 2, we use a single sample from a t_{10} distribution to compare the unconditional bootstrap and the smoothed bootstrap averaged over $\mathbf{C}^{\dagger}$'s for $\rho_0 = 0.5$, to demonstrate the results of Lemma 2, and we obtain an estimate of $E_+^{\dagger}\bar{\Phi}(\sqrt{m}W^{\dagger +})$, the expected value of the saddle-point approximation given in Theorem 1, by averaging over 100 values of $\mathbf{C}^{\dagger}$, comparing this to the Monte Carlo estimates. These results, which would vary from sample to sample from the t_{10} distribution, illustrate excellent relative accuracy, and we note that better results are obtained for $0 \leq \rho_0 < 0.5$.

In Table 3, to illustrate the main results of Theorem 2, we compare the simulated distribution, when sampling from the t_{10}-distribution and the exponential distribution shifted to have mean 0, with the bootstrap averages over 40 samples. The average bootstrap is quite accurate, while the standard deviation shows that

TABLE 1
Comparison of saddle-point and simulated tail areas for normal distribution from Section 3 with the unconditional case (U) and the conditional case (C) at both $n = 39$ and $n = 9$

		Tail prob. exceeds				
		$\rho + 0.05$	$\rho + 0.10$	$\rho + 0.15$	$\rho + 0.20$	$\rho + 0.25$
$n = 39$	$\rho = 0.5$					
UC	saddle-point	0.3210	0.1923	0.0946	0.0353	0.0088
	simulations	0.3223	0.1922	0.0946	0.0352	0.0094
$n = 9$	$\rho = 0.5$					
UC	saddle-point	0.3629	0.2937	0.2261	0.1624	0.1066
	simulations	0.3695	0.2994	0.2310	0.1660	0.1081
$n = 39$	$\rho = 0.5$					
C	saddle-point	0.3133	0.1888	0.0983	0.0412	0.0118
	simulation	0.3136	0.1884	0.0983	0.0410	0.0118
$n = 9$	$\rho = 0.5$					
C	saddle-point	0.4077	0.3413	0.2713	0.1972	0.1267
	simulation	0.4094	0.3432	0.2722	0.1999	0.1280

the relative error of the bootstrap becomes larger in the tails, as expected since this is shown to be of order $m(u - \rho_0)^4$ in Theorem 2. For $0 \leq \rho_0 < 0.5$, there is even better accuracy.

Table 4 illustrates the accuracy of the results of Theorem 3 using random samples for ρ_0 equal to 0 and 0.5 for centered exponential errors. The saddle-point approximation has the relative accuracy property. In this case, there is considerable variation in tail areas as different random samples are taken, but similar accuracy is achieved with other samples. Similar results are obtained for the t_{10} distribution and for $0 \leq \rho_0 < 0.5$.

TABLE 2
Unconditional bootstrap (BS: 100,000 replicates) and expected conditional bootstrap averages over $\mathbf{C}^{\dagger}$ *(ECBS: using 500 sets of the conditional bootstrap with 10,000 replicates) and average of conditional saddle-point approximation (ECSP: over 500 replicates), from the same original sample from* t_{10}

		Tail prob. exceeds				
		$\rho + 0.05$	$\rho + 0.10$	$\rho + 0.15$	$\rho + 0.20$	$\rho + 0.25$
$n = 39$	$\rho = 0.5$					
	BS	0.3206	0.1921	0.0943	0.0350	0.0086
	ECBS	0.3160	0.1833	0.0860	0.0309	0.0075
	ECSP	0.3131	0.1823	0.0861	0.0308	0.0075

TABLE 3
Simulated tail probabilities (SIM: 1,000,000 samples), estimates of expected bootstrap tail probabilities and standard deviations of bootstrap tail probabilities based on means and standard deviations of 40 samples (EBS and SDBS: 100,000 bootstrap replications) from t_{10} and centered exponential distributions

		Tail prob. exceeds				
		$\rho + 0.05$	$\rho + 0.10$	$\rho + 0.15$	$\rho + 0.20$	$\rho + 0.25$
$n = 39$	$\rho = 0.5$					
t_{10}	SIM	0.3171	0.1885	0.0916	0.0340	0.0083
	EBS	0.3215	0.1932	0.0957	0.0361	0.0094
	SDBS	0.0016	0.0017	0.0019	0.0015	0.0009
exp	SIM	0.3174	0.1937	0.1020	0.0442	0.0154
	EBS	0.3223	0.1991	0.1059	0.0473	0.0173
	SDBS	0.0044	0.0088	0.0123	0.0121	0.0089

TABLE 4
Comparison of tail areas for conditional bootstrap (CBS) and conditional saddle-point tail area (CSP) for one sample from a centered exponential with $\rho_0 = 0.0$ and $\rho_0 = 0.5$, as in Section 4.3

		Tail prob. exceeds				
	ρ	$\rho + 0.05$	$\rho + 0.10$	$\rho + 0.15$	$\rho + 0.20$	$\rho + 0.25$
$n = 39$						
CSP	0.0	0.4300	0.3103	0.2074	0.1268	0.0697
CBS	0.0	0.4378	0.3147	0.2080	0.1274	0.0688
CSP	0.5	0.2499	0.0863	0.0145	0.0004	0.0000
CBS	0.5	0.2456	0.0843	0.0132	0.0002	0.0000

Finally, we compare the power of the two tests based on the unconditional bootstrap and the conditional bootstrap in Table 5 for the Gaussian case of Section 3 and for the general case from Sections 4.2 and 4.3. We note that the tests have

TABLE 5
Power under unconditional (U) and conditional (C) tests for the Gaussian case in the left half of the table and the general case from t_{10} in the right half

	U	C	U	C	U	C	U	C
$\rho_0 =$	0	0	0.4	0.4	0	0	0.4	0.4
$\rho_1 = \rho_0 + 0.1$	0.15	0.15	0.18	0.12	0.15	0.13	0.18	0.11
$\rho_1 = \rho_0 + 0.3$	0.58	0.59	0.73	0.42	0.58	0.53	0.73	0.38
$\rho_1 = \rho_0 + 0.5$	0.92	0.90	0.98	0.89	0.93	0.90	0.98	0.78

equal power up to computational accuracy when $\rho_0 = 0$, as might be expected since there is no loss of information due to conditioning in this case, but there is some loss of power in the case of $\rho_0 = 0.2$ and a considerable loss for $\rho_0 = 0.4$.

SUPPLEMENTARY MATERIAL

Supplement to "Relative errors for bootstrap approximations of the serial correlation coefficient" (DOI: 10.1214/13-AOS1111SUPP; .pdf). We provide details and proofs needed for a number of results in the paper.

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