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ITÔ ISOMORPHISMS FOR L^p-VALUED POISSON STOCHASTIC INTEGRALS[1]

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Motivated by the study of existence, uniqueness and regularity of solutions to stochastic partial differential equations driven by jump noise, we prove Itô isomorphisms for L^p-valued stochastic integrals with respect to a compensated Poisson random measure. The principal ingredients for the proof are novel Rosenthal type inequalities for independent random variables taking values in a (noncommutative) L^p-space, which may be of independent interest. As a by-product of our proof, we observe some moment estimates for the operator norm of a sum of independent random matrices.

1. Introduction. In the functional analytic approaches to stochastic partial differential equations (SPDEs), one studies an SPDE by reformulating it as a stochastic ordinary differential equation in a suitable infinite-dimensional state space X. A particularly popular method, known as the semigroup approach, has proven very effective in obtaining existence, uniqueness and regularity results for large classes of SPDEs with Gaussian noise. A demonstration of this approach for SPDEs driven by Gaussian noise in Hilbert spaces can be found in the monograph of Da Prato and Zabczyk [6]. In the last decade, there has been increased interest in SPDEs driven by Poisson-type noise; see, for instance, [2, 10, 23, 24] and the recent monograph [29]. To obtain existence, uniqueness and regularity results for such equations, one requires as a basic tool L^p-estimates for vector-valued Poisson stochastic integrals. Concretely, one needs to answer the following fundamental question. Suppose that we are given a compensated Poisson random measure $\tilde{N}$ on $\mathbb{R}_+ \times J$, where J is a σ-finite measure space, and a simple, adapted X-valued process F. Can one find a suitable Banach space $\mathcal{I}_{p,X}$ such that

$$c_{p,X} \|\!| F \|\!|_{\mathcal{I}_{p,X}} \leq \left(\mathbb{E} \left\| \int_{\mathbb{R}_+ \times J} F \, d\tilde{N} \right\|_X^p \right)^{1/p} \leq C_{p,X} \|\!| F \|\!|_{\mathcal{I}_{p,X}} \tag{1.1}$$

for constants $c_{p,X}, C_{p,X}$ depending only on p and X? In the SPDE literature, the right-hand side inequality is often referred to as a Bichteler–Jacod inequality. This estimate allows one to define an Itô-type stochastic integral, sometimes

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called a strong or L^p-stochastic integral in the literature [1, 35], for all elements in the closure of the simple adapted processes in $\mathcal{I}_{p,X}$. If both inequalities in (1.1) hold simultaneously, then we shall speak of an Itô isomorphism. In this situation, the choice of the space $\mathcal{I}_{p,X}$ is optimal and therefore $\mathcal{I}_{p,X}$ provides the proper framework to study well-posedness and regularity questions. We will call the corresponding Bichteler–Jacod inequality optimal in this case, even though the constants $c_{p,X}, C_{p,X}$ in (1.1) are not required to be optimal. In the case of Gaussian noise, Itô isomorphisms in UMD Banach spaces were obtained in [28]. The optimality of these estimates proved crucial in obtaining maximal regularity results for stochastic parabolic evolution equations driven by Gaussian noise [27]. One can expect optimal Bichteler–Jacod inequalities to be similarly useful in the investigation of maximal regularity for equations driven by Poisson or, more generally, Lévy noise.

Although Bichteler–Jacod inequalities are fundamental to the study of SPDEs driven by jump noise and have been investigated by many authors (see [1, 2, 11, 23, 24, 29, 35] and the references therein), a general Itô isomorphism as available in the Gaussian case is still missing. In fact, it seems that the optimality of Bichteler–Jacod inequalities has not yet been investigated, not even in the scalar-valued case. The main aim of this paper is to provide optimal estimates of the form (1.1) in the important case where X is an L^q-space. On the one hand, this result can serve as a stepping stone in the development of Itô isomorphisms in more general Banach spaces needed in the study of SPDEs. On the other hand, our estimates are in itself valuable for existence, uniqueness and regularity questions that can be addressed in the setting of L^q-spaces; see, for example, [24] for interesting examples.

With some additional effort, our estimates can be extended to the situation where X is a noncommutative L^q-space associated with a semifinite von Neumann algebra $\mathcal{M}$, for any $1 < q < \infty$. To keep our exposition accessible to readers who have little familiarity with noncommutative analysis, we choose to focus on classical L^q-spaces and only later indicate the modifications needed to prove our results in full generality.

To formulate our main result for classical L^q-spaces, Theorem 1.1, we introduce the following spaces. Let (S, Σ, σ) be any measure space. We consider the completions $\mathcal{S}_q^p$, $\mathcal{D}_{q,q}^p$ and $\mathcal{D}_{p,q}^p$ of the space of all simple functions in the respective norms

$$\begin{aligned}
&\|F\|_{\mathcal{S}_q^p} = \left(\mathbb{E}\left\|\left(\int_{\mathbb{R}_+\times J} |F|^2\,dt \times d\nu\right)^{1/2}\right\|_{L^q(S)}^p\right)^{1/p},\\
&\|F\|_{\mathcal{D}_{q,q}^p} = \left(\mathbb{E}\left(\int_{\mathbb{R}_+\times J} \|F\|_{L^q(S)}^q\,dt \times d\nu\right)^{p/q}\right)^{1/p}, \qquad (1.2)\\
&\|F\|_{\mathcal{D}_{p,q}^p} = \left(\int_{\mathbb{R}_+\times J} \mathbb{E}\|F\|_{L^q(S)}^p\,dt \times d\nu\right)^{1/p}.
\end{aligned}$$

We use the following notation. If A, B are quantities depending on a parameter α, then we write $A \lesssim_\alpha B$ if there is a constant $c_\alpha > 0$ depending only on α such that $A \le c_\alpha B$. We write $A \simeq_\alpha B$ if both $A \lesssim_\alpha B$ and $B \lesssim_\alpha A$ hold. Also, we use χ_A to denote the indicator function of a set A. Finally, to avoid ambiguity, let us mention that we always take the notation $a < p, q < b$ to mean that both $a < p < b$ and $a < q < b$ hold.

THEOREM 1.1 (Itô isomorphism). *Let $1 < p, q < \infty$. For any $B \in \mathcal{J}$, any $t > 0$ and any simple, adapted $L^q(S)$-valued process F,*

$$\left(\mathbb{E} \sup_{0<s\le t} \left\| \int_{(0,s]\times B} F \, d\tilde{N} \right\|_{L^q(S)}^p \right)^{1/p} \simeq_{p,q} \|F\chi_{(0,t]\times B}\|_{\mathcal{I}_{p,q}}, \tag{1.3}$$

where $\mathcal{I}_{p,q}$ is given by

$$\begin{array}{ll} \mathcal{S}_q^p \cap \mathcal{D}_{q,q}^p \cap \mathcal{D}_{p,q}^p & \text{if } 2 \le q \le p < \infty, \\ \mathcal{S}_q^p \cap (\mathcal{D}_{q,q}^p + \mathcal{D}_{p,q}^p) & \text{if } 2 \le p \le q < \infty, \\ (\mathcal{S}_q^p \cap \mathcal{D}_{q,q}^p) + \mathcal{D}_{p,q}^p & \text{if } 1 < p < 2 \le q < \infty, \\ (\mathcal{S}_q^p + \mathcal{D}_{q,q}^p) \cap \mathcal{D}_{p,q}^p & \text{if } 1 < q < 2 \le p < \infty, \\ \mathcal{S}_q^p + (\mathcal{D}_{q,q}^p \cap \mathcal{D}_{p,q}^p) & \text{if } 1 < q \le p \le 2, \\ \mathcal{S}_q^p + \mathcal{D}_{q,q}^p + \mathcal{D}_{p,q}^p & \text{if } 1 < p \le q \le 2. \end{array}$$

Moreover, the estimate $\lesssim_{p,q}$ in (1.3) remains valid if $q = 1$.

To understand the estimates in (1.3), recall that if X and Y are two Banach spaces which are continuously embedded in some Hausdorff topological vector space, then their intersection $X \cap Y$ and sum $X + Y$ are Banach spaces under the norms

$$\|z\|_{X\cap Y} = \max\{\|z\|_X, \|z\|_Y\}$$

and

$$\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y : z = x + y, x \in X, y \in Y\}.$$

So, for example, if $2 \le p \le q < \infty$ then $\|F\chi_{(0,t]\times B}\|_{\mathcal{I}_{p,q}}$ is equal to

$$\begin{aligned} \max\Bigg[& \left(\mathbb{E}\left\| \left(\int_{(0,t]\times B} |F|^2 \, dt \times d\nu\right)^{1/2} \right\|_{L^q(S)}^p\right)^{1/p}, \\ & \inf\Bigg\{ \left(\mathbb{E}\left(\int_{(0,t]\times B} \|F_1\|_{L^q(S)}^q \, dt \times d\nu\right)^{p/q}\right)^{1/p} \\ & \qquad + \left(\int_{(0,t]\times B} \mathbb{E}\|F_2\|_{L^q(S)}^p \, dt \times d\nu\right)^{1/p} \Bigg\}\Bigg], \end{aligned}$$

where the infimum is taken over all decompositions $F = F_1 + F_2$ with $F_1 \in \mathcal{D}^p_{q,q}$ and $F_2 \in \mathcal{D}^p_{p,q}$.

In comparison, recall that if W is a Gaussian random measure on $\mathbb{R}_+ \times J$, then for any $1 < p, q < \infty$,

$$\left(\mathbb{E} \sup_{0<s\leq t}\left\|\int_{(0,s]\times B} F\,dW\right\|^p_{L^q(S)}\right)^{1/p} \simeq_{p,q} \|F\chi_{(0,t]\times B}\|_{\mathcal{S}^p_q}.$$

In the proof of the latter inequalities, as well as the more general results in [28], crucial use is made of the fact that any mean-zero, real-valued Gaussian random variable has a standard normal distribution once we divide it by its standard deviation. It is the lack of this type of stability of Poisson random variables that accounts for the more involved isomorphisms in Theorem 1.1.

The result in Theorem 1.1 improves and extends all the known estimates for L^q-valued Poisson stochastic integrals. In fact, it seems that only the estimate "$\lesssim_{p,q}$" in (1.3) was obtained earlier in [11] for $q = 2$, $p = 2^n$ for some $n \in \mathbb{N}$ (see also [23] for a near-optimal estimate for $q = 2$, $2 \leq p < \infty$). As it turns out, this estimate is optimal. In all other cases, our optimal estimates improve the results in the literature. We make a detailed comparison with existing results at the end of Section 7.

The proof of Theorem 1.1 relies on the following decoupling inequalities. Let $\tilde{N}^c$ be a copy of $\tilde{N}$ defined on a different probability space $(\Omega_c, \mathcal{F}_c, \mathbb{P}_c)$, so that $\tilde{N}^c$ is independent of both $\tilde{N}$ and the simple, adapted process F. If X is a UMD Banach space, then for any $1 < p < \infty$,

$$\left(\mathbb{E}\left\|\int_{(0,t]\times B} F\,d\tilde{N}\right\|^p_X\right)^{1/p} \simeq_{p,X} \left(\mathbb{E}\mathbb{E}_c\left\|\int_{(0,t]\times B} F\,d\tilde{N}^c\right\|^p_X\right)^{1/p}. \tag{1.4}$$

These inequalities are a special case of the decoupling inequalities for martingale difference sequences in UMD Banach spaces due to McConnell [25] and Hitczenko [12]. A relatively simple direct proof of (1.4) can be found in, for example [38], Theorem 2.4.1. For completeness, we reproduce this argument in Appendix A. Observe that for a simple process F, the decoupled stochastic integral on the right-hand side can be written as a sum of conditionally independent, mean-zero random variables. Thus, the key to obtaining an Itô isomorphism as in (1.1) lies in answering the following question: given $1 \leq p < \infty$ and a Banach space X, can we find constants $c_{p,X}, C_{p,X}$ depending only on p and X such that for any sequence of independent, mean-zero X-valued random variables (ξ_i)

$$c_{p,X}|||(\xi_i)|||_{p,X} \leq \left(\mathbb{E}\left\|\sum_i \xi_i\right\|^p_X\right)^{1/p} \leq C_{p,X}|||(\xi_i)|||_{p,X} \tag{1.5}$$

for a suitable norm $|||\cdot|||_{p,X}$ which can be computed explicitly in terms of the (moments of the) individual summands ξ_i? These kind of inequalities can be termed vector-valued Rosenthal inequalities, since in the case $X = \mathbb{C}$ the well-known answer to this question is due to Rosenthal [34]: For $2 \le p < \infty$, there exists an absolute constant c such that

$$
\begin{aligned}
\left(\mathbb{E}\left|\sum_i \xi_i\right|^p\right)^{1/p} &\le c\frac{p}{\log p}\max\left\{\left(\sum_i \mathbb{E}|\xi_i|^p\right)^{1/p}, \left(\sum_i \mathbb{E}|\xi_i|^2\right)^{1/2}\right\},\\
\left(\mathbb{E}\left|\sum_i \xi_i\right|^p\right)^{1/p} &\ge \frac{1}{2}\max\left\{\left(\sum_i \mathbb{E}|\xi_i|^p\right)^{1/p}, \left(\sum_i \mathbb{E}|\xi_i|^2\right)^{1/2}\right\}.
\end{aligned} \tag{1.6}
$$

A version of (1.6) for noncommutative random variables, as well as a version for $1 < p \le 2$, was recently obtained by Junge and Xu [16]. Their main results yield two-sided bounds of the form (1.5) if X is a (noncommutative) L^q-space and $p = q$. Various upper bounds for the moments of a martingale with values in a uniformly 2-smooth Banach space were obtained by Pinelis [30]. However, these results lead to a two-sided estimate of the form (1.5) only if X is a Hilbert space (see [30], Theorem 5.2).

Our main result in this direction provides Rosenthal-type inequalities for independent random variables taking values in a noncommutative L^q-space. We state the version for classical L^q-spaces. We consider the following norms on the linear space of all finite sequences (f_i) of random variables in $L^\infty(\Omega; L^q(S))$. For $1 \le p, q < \infty$, we set

$$
\begin{aligned}
\|(f_i)\|_{S_q} &= \left\|\left(\sum_i \mathbb{E}|f_i|^2\right)^{1/2}\right\|_{L^q(S)},\\
\|(f_i)\|_{D_{p,q}} &= \left(\sum_i \mathbb{E}\|f_i\|^p_{L^q(S)}\right)^{1/p}.
\end{aligned} \tag{1.7}
$$

THEOREM 1.2. *Let $1 < p, q < \infty$ and let (S, Σ, σ) be a measure space. If (ξ_i) is a sequence of independent, mean-zero random variables taking values in $L^q(S)$, then*

$$
\left(\mathbb{E}\left\|\sum_i \xi_i\right\|^p_{L^q(S)}\right)^{1/p} \simeq_{p,q} \|(\xi_i)\|_{s_{p,q}}, \tag{1.8}
$$

where $s_{p,q}$ is given by

$$
\begin{aligned}
&S_q \cap D_{q,q} \cap D_{p,q} && \text{if } 2 \le q \le p < \infty,\\
&S_q \cap (D_{q,q} + D_{p,q}) && \text{if } 2 \le p \le q < \infty,\\
&(S_q \cap D_{q,q}) + D_{p,q} && \text{if } 1 < p < 2 \le q < \infty,\\
&(S_q + D_{q,q}) \cap D_{p,q} && \text{if } 1 < q < 2 \le p < \infty,\\
&S_q + (D_{q,q} \cap D_{p,q}) && \text{if } 1 < q \le p \le 2,\\
&S_q + D_{q,q} + D_{p,q} && \text{if } 1 < p \le q \le 2.
\end{aligned}
$$

Moreover, the estimate $\lesssim_{p,q}$ *in* (1.8) *remains valid if* $p=1$, $q=1$ *or both.*

The notational similarity between the spaces introduced in (1.2) and (1.7) is intentional. Indeed, when applying Theorem 1.2 to the decoupled Poisson stochastic integral on the right-hand side of (1.4), the spaces S_q, $D_{q,q}$, and $D_{p,q}$ give rise to $\mathcal{S}_q^p$, $\mathcal{D}_{q,q}^p$ and $\mathcal{D}_{p,q}^p$, respectively.

If $p=q$, then the result in Theorem 1.2 (as well as its generalization in Theorem 5.1) is a special case of the noncommutative Rosenthal inequalities in [16] and the only novelty here is a new proof. However, in applications of Theorem 1.1, and hence of Theorem 1.2, one is typically also interested in the case $p \neq q$.

As said before, we can even prove an extension of the Itô isomorphism in Theorem 1.1 in which $L^q(S)$ is replaced by a general noncommutative L^q-space associated with a semifinite von Neumann algebra $\mathcal{M}$. This result is stated and proved in Theorem 7.1 below. The proof proceeds along the same lines as the result for classical L^q-spaces and in particular requires a version of the Rosenthal-type inequalities stated above for random variables taking values in a noncommutative L^q-space, which we prove in Theorem 5.1. As a by-product of the proof of Theorem 5.1, we take the opportunity to observe the following estimates for the moments of the operator norm of a sum of independent, mean-zero $d_1 \times d_2$ random matrices (x_i), which may be of independent interest. If $2 \leq p < \infty$ and $d = \min\{d_1, d_2\}$, then

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_i x_i\right\|^p\right)^{1/p} \leq C_{p,d} \max\Biggl\{&\left\|\left(\sum_i \mathbb{E}|x_i|^2\right)^{1/2}\right\|, \left\|\left(\sum_i \mathbb{E}|x_i^*|^2\right)^{1/2}\right\|,\\ &C_{p/2,d}\left(\mathbb{E}\max_i \|x_i\|^p\right)^{1/p}\Biggr\},\end{aligned}$$

where $C_{p,d}$ is of order $\max\{\sqrt{p}, \sqrt{\log d}\}$. In Section 6, we compare this result to known estimates for random matrices.

An application of Theorem 1.1 is discussed in [8].

2. L^q-valued Rosenthal inequalities. We start by proving Theorem 1.2. Throughout, we fix a measure space (S, Σ, σ). Let us collect some tools that we will use in the proof. First recall the Khintchine inequalities for $L^q(S)$. Let (r_i) be a Rademacher sequence, that is, a sequence of independent, identically distributed random variables satisfying $\mathbb{P}(r_i = 1) = \mathbb{P}(r_i = -1) = 1/2$. Then, for any $0 < p, q < \infty$ and any finite sequence (x_i) in $L^q(S)$ we have

$$\left(\mathbb{E}\left\|\sum_i r_i x_i\right\|_{L^q(S)}^p\right)^{1/p} \simeq_{p,q} \left\|\left(\sum_i |x_i|^2\right)^{1/2}\right\|_{L^q(S)}. \tag{2.1}$$

We will frequently use this result in combination with the following well-known symmetrization inequalities (see, e.g., [20], Lemma 6.3). Let $1 \leq p < \infty$, let X

be a Banach space and (ξ_i) a sequence of independent, mean-zero X-valued random variables. If (r_i) is a Rademacher sequence defined on a probability space $(\Omega_r, \mathcal{F}_r, \mathbb{P}_r)$, then

$$\frac{1}{2}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{1/p} \leq \Big(\mathbb{E}_r\mathbb{E}\Big\|\sum_i r_i\xi_i\Big\|_X^p\Big)^{1/p} \leq 2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{1/p}. \tag{2.2}$$

As a first consequence, we find the following useful estimates.

LEMMA 2.1. *Suppose that $1 \leq p, q \leq 2$. Let (ξ_i) be a finite sequence of independent, mean-zero $L^q(S)$-valued random variables. Then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{1/p} \lesssim_{p,q} \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(S)}.$$

On the other hand, if $2 \leq p, q < \infty$ then

$$\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(S)} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{1/p}.$$

PROOF. Let $1 \leq p, q \leq 2$. Combining (2.2) and (2.1) yields

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{1/p} &\simeq_{p,q}\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p} \\ &= \Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2\Big\|_{L^{q/2}(S)}^{p/2}\Big)^{1/p} \\ &\leq \Big\|\sum_i \mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(S)}^{1/2} = \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(S)}.\end{aligned}$$

Note that in the final inequality we apply Jensen's inequality, using that $\frac{p}{2}, \frac{q}{2} < 1$. If we assume $2 \leq p, q < \infty$, then this inequality is reversed. $\square$

We recall the notions of type and cotype. A Banach space X is said to have *type s* for some $1 \leq s \leq 2$ if for any finite sequence (x_i) in X

$$\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_X^2\Big)^{1/2} \lesssim_{s,X} \Big(\sum_i \|x_i\|_X^s\Big)^{1/s}.$$

A Banach space X is said to have *cotype s* for some $2 \leq s < \infty$ if for any finite sequence (x_i) in X

$$\Big(\sum_i \|x_i\|_X^s\Big)^{1/s} \lesssim_{s,X} \Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_X^2\Big)^{1/2}.$$

It is well known that any L^q-space with $1 \leq q < \infty$ has type $\min\{q, 2\}$ and cotype $\max\{q, 2\}$. The following observation is well known, we include a proof for the convenience of the reader. The main ingredients are Kahane's inequalities (see, e.g., [20], Theorem 4.7): for any $0 < p, q < \infty$ there exists a constant $\kappa_{p,q}$ such that for any Banach space X and $x_1, \dots, x_n \in X$,

$$\left(\mathbb{E}\left\|\sum_{i=1}^n r_i x_i\right\|_X^p\right)^{1/p} \leq \kappa_{p,q}\left(\mathbb{E}\left\|\sum_{i=1}^n r_i x_i\right\|_X^q\right)^{1/q}. \tag{2.3}$$

LEMMA 2.2. *Fix $1 \leq p < \infty$. Let X be a Banach space and (ξ_i) be a finite sequence of independent, mean-zero X-valued random variables. If X has type $1 \leq s \leq 2$, then*

$$\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_X^p\right)^{1/p} \lesssim_{p,s,X} \left(\mathbb{E}\left(\sum_i \|\xi_i\|_X^s\right)^{p/s}\right)^{1/p}.$$

On the other hand, if X has cotype $2 \leq s < \infty$, then

$$\left(\mathbb{E}\left(\sum_i \|\xi_i\|_X^s\right)^{p/s}\right)^{1/p} \lesssim_{p,s,X} \left(\mathbb{E}\left\|\sum_i \xi_i\right\|_X^p\right)^{1/p}.$$

PROOF. Suppose X has type s. By symmetrization, Kahane's inequalities and the type s inequality we obtain

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_X^p\right)^{1/p} &\simeq \left(\mathbb{E}\mathbb{E}_r\left\|\sum_i r_i\xi_i\right\|_X^p\right)^{1/p} \\ &\simeq_p \left(\mathbb{E}\left(\mathbb{E}_r\left\|\sum_i r_i\xi_i\right\|_X^2\right)^{p/2}\right)^{1/p} \lesssim_{s,X} \left(\mathbb{E}\left(\sum_i \|\xi_i\|_X^s\right)^{p/s}\right)^{1/p}.\end{aligned}$$

The second assertion is proved similarly. $\square$

The following result is the key to the Rosenthal-type inequalities in the cases where $2 \leq p, q < \infty$.

THEOREM 2.3. *Suppose that $2 \leq p, q < \infty$. If (ξ_i) is a finite sequence of independent, mean-zero $L^q(S)$-valued random variables, then*

$$\begin{aligned}&\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_{L^q(S)}^p\right)^{1/p} \\ &\quad\simeq_{p,q} \max\left\{\left\|\left(\sum_i \mathbb{E}|\xi_i|^2\right)^{1/2}\right\|_{L^q(S)}, \left(\mathbb{E}\left(\sum_i \|\xi_i\|_{L^q(S)}^q\right)^{p/q}\right)^{1/p}\right\}.\end{aligned} \tag{2.4}$$

PROOF. We first prove the estimate $\gtrsim_{p,q}$. By Lemma 2.1,

$$\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(S)} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{1/p}.$$

Moreover, since $L^q(S)$ has cotype q Lemma 2.2 implies

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum\nolimits_i \xi_i\Big\|_{L^q(S)}^p\Big)^{1/p}.$$

We now prove the reverse inequality in (2.4). By symmetrization and the Khintchine inequalities (2.1),

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{1/p} \simeq_{p,q} \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p}. \tag{2.5}$$

By the triangle inequality, we obtain

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad = \Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2\Big\|_{L^{q/2}(S)}^{p/2}\Big)^{1/p} \qquad (2.6)\\
&\quad \le \bigg(\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2 - \mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(S)}^{p/2}\Big)^{2/p} + \Big\|\sum_i \mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(S)}\bigg)^{1/2}.
\end{aligned}$$

Suppose first that $q \le 4$. Then $L^{q/2}(S)$ has type $\frac{q}{2}$, so by Lemma 2.2,

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2 - \mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(S)}^{p/2}\Big)^{2/p} \\
&\quad \lesssim_{p,q} \Big(\mathbb{E}\Big(\sum_i \big\||\xi_i|^2 - \mathbb{E}|\xi_i|^2\big\|_{L^{q/2}(S)}^{q/2}\Big)^{p/q}\Big)^{2/p} \\
&\quad \le \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{p/q}\Big)^{2/p} + \Big(\sum_i \big\|\mathbb{E}|\xi_i|^2\big\|_{L^{q/2}(S)}^{q/2}\Big)^{2/q} \\
&\quad \le \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{p/q}\Big)^{2/p} + \mathbb{E}\Big(\sum_i \big\||\xi_i|^2\big\|_{L^{q/2}(S)}^{q/2}\Big)^{2/q} \\
&\quad \le 2\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{p/q}\Big)^{2/p},
\end{aligned}$$

where in the final two steps we apply Jensen's inequality, using that the $\ell^{q/2}(L^{q/2}(S))$-norm is convex, and subsequently use Hölder's inequality.

Suppose now that $q > 4$. By applying symmetrization and the Khintchine inequalities (2.1), we find

$$
\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2 - \mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(S)}^{p/2}\Big)^{2/p} \\
&\quad \simeq_{p,q} \Big(\mathbb{E}\Big\|\Big(\sum_i ||\xi_i|^2 - \mathbb{E}|\xi_i|^2|^2\Big)^{1/2}\Big\|_{L^{q/2}(S)}^{p/2}\Big)^{2/p} \\
&\quad \leq \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^4\Big)^{1/2}\Big\|_{L^{q/2}(S)}^{p/2}\Big)^{2/p} + \Big\|\Big(\sum_i |\mathbb{E}|\xi_i|^2|^2\Big)^{1/2}\Big\|_{L^{q/2}(S)} \\
&\quad \leq \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^4\Big)^{1/4}\Big\|_{L^{q}(S)}^{p}\Big)^{2/p} + \Big\|\sum_i \mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(S)}.
\end{aligned}
\tag{2.7}
$$

Since $q > 4$, there is some $0 < \theta < \frac{1}{2}$ such that $\frac{1}{4} = \frac{\theta}{2} + \frac{1-\theta}{q}$. By applying Hölder's inequality three times (the second and third time with parameters $\frac{1}{q} = \frac{\theta}{q} + \frac{1-\theta}{q}$ and $\frac{1}{p} = \frac{\theta}{p} + \frac{1-\theta}{p}$, resp.), we obtain

$$
\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^4\Big)^{1/4}\Big\|_{L^q(S)}^{p}\Big)^{2/p} \\
&\quad \leq \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\theta/2}\Big(\sum_i |\xi_i|^q\Big)^{(1-\theta)/q}\Big\|_{L^q(S)}^{p}\Big)^{2/p} \\
&\quad \leq \Big(\mathbb{E}\Big(\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\theta/2}\Big\|_{L^{q/\theta}(S)}\Big\|\Big(\sum_i |\xi_i|^q\Big)^{(1-\theta)/q}\Big\|_{L^{q/(1-\theta)}(S)}\Big)^{p}\Big)^{2/p} \\
&\quad \leq \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\theta/2}\Big\|_{L^{q/\theta}(S)}^{p/\theta}\Big)^{2\theta/p} \\
&\qquad \times \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^q\Big)^{(1-\theta)/q}\Big\|_{L^{q/(1-\theta)}(S)}^{p/(1-\theta)}\Big)^{2(1-\theta)/p} \\
&\quad = \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{1/2}\Big\|_{L^q(S)}^{p}\Big)^{2\theta/p}\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^q\Big)^{1/q}\Big\|_{L^q(S)}^{p}\Big)^{2(1-\theta)/p}.
\end{aligned}
\tag{2.8}
$$

Combining (2.6), (2.7) and (2.8), we arrive at the inequality

$$
a^2 \lesssim_{p,q} a^{2\theta} b^{2(1-\theta)} + c^2,
$$

where we set $a = (\mathbb{E}\|(\sum_i |\xi_i|^2)^{1/2}\|_{L^q(S)}^p)^{1/p}$, $b = (\mathbb{E}(\sum_i \|\xi_i\|_{L^q(S)}^q)^{p/q})^{1/p}$ and $c = \|(\sum_i \mathbb{E}|\xi_i|^2)^{1/2}\|_{L^q(S)}$. Notice that if $a \leq b$ then the claim immediately follows from (2.5). Hence, we may assume $a > b$. Since $0 < 2\theta < 1$ we then have

$$a^{2\theta} b^{2(1-\theta)} = b^2 \left(\frac{a}{b}\right)^{2\theta} \leq ab.$$

Thus, we obtain the inequality

$$a^2 \lesssim_{p,q} ab + c^2.$$

Solving this quadratic inequality, we find that $a \lesssim_{p,q} \max\{b, c\}$. That is,

$$\begin{aligned}&\left(\mathbb{E}\left\|\left(\sum_i |\xi_i|^2\right)^{1/2}\right\|_{L^q(S)}^p\right)^{1/p} \\ &\quad\lesssim_{p,q} \max\left\{\left(\mathbb{E}\left(\sum_i \|\xi_i\|_{L^q(S)}^q\right)^{p/q}\right)^{1/p}, \left\|\left(\sum_i \mathbb{E}|\xi_i|^2\right)^{1/2}\right\|_{L^q(S)}\right\}.\end{aligned}$$

The result now follows from (2.5). This completes the proof. $\square$

Recall the spaces $s_{p,q}$ defined in the statement of Theorem 1.2. In the proof of this result, we shall make use of the fact that for any $1 < p, q < \infty$

$$(s_{p,q})^* = s_{p',q'}, \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1\right) \tag{2.9}$$

holds isometrically. This follows from the following general principle. Suppose that X and Y are two Banach spaces which are continuously embedded in some Hausdorff topological vector space and assume moreover that $X \cap Y$ is dense in both X and Y. Then we have

$$(X \cap Y)^* = X^* + Y^*, \qquad (X + Y)^* = X^* \cap Y^* \tag{2.10}$$

isometrically. The duality brackets under these identifications are given by

$$\langle x, x^*\rangle = \langle x, x^*|_{X\cap Y}\rangle \qquad (x^* \in X^* + Y^*),$$

where $x^*|_{X\cap Y}$ denotes the restriction of x^* to $X \cap Y$, and

$$\langle x, x^*\rangle = \langle y, x^*\rangle + \langle z, x^*\rangle \qquad (x^* \in X^* \cap Y^*, x = y + z \in X + Y),$$

respectively; see, for example, [17], Theorem I.3.1. In our case of interest, the spaces S_q, $D_{p,q}$ and $D_{q,q}$ have dense intersection and, therefore, the duality of these individual spaces imply together with (2.10) that (2.9) holds, with associated duality bracket

$$\langle (f_i), (g_i)\rangle = \sum_i \mathbb{E}\int f_i g_i \, d\sigma.$$

We need two more ingredients for the proof of Theorem 1.2. The first are the hypercontractive-type inequalities due to Hoffmann–Jørgensen [13] (see also [18, 20] for a proof yielding a constant of optimal order)

$$\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_X^p\right)^{1/p} \lesssim \frac{p}{\log 2p}\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_X + \left(\mathbb{E}\max_i \|\xi_i\|_X^p\right)^{1/p}\right), \tag{2.11}$$

valid for any $1 \le p < \infty$ and any sequence (ξ_i) of independent, mean-zero random variables taking values in a Banach space X. Finally, let us recall the Rosenthal inequalities for a sequence (f_i) of positive scalar-valued random variables: if $1 \le p < \infty$, then

$$\left(\mathbb{E}\left|\sum_i f_i\right|^p\right)^{1/p} \lesssim_p \max\left\{\left(\sum_i \mathbb{E}|f_i|^p\right)^{1/p}, \sum_i \mathbb{E}|f_i|\right\}. \tag{2.12}$$

We are now ready to prove our first main result.

PROOF OF THEOREM 1.2. Let us note that the inequalities "$\gtrsim_{p,q}$" in (1.8) follow by duality once the reverse inequalities have been established. Indeed, if (η_i) is a finite sequence in $s_{p',q'}$ of norm 1, then

$$\begin{aligned}
\langle(\xi_i),(\eta_i)\rangle &= \sum_i \mathbb{E}\int(\xi_i\eta_i)\,d\sigma \\
&= \sum_i \mathbb{E}\int\big(\xi_i\big(\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)\big)\big)\,d\sigma \\
&= \sum_{i,j} \mathbb{E}\int\big(\xi_i\big(\mathbb{E}(\eta_j|\xi_j) - \mathbb{E}(\eta_j)\big)\big)\,d\sigma \qquad (2.13)\\
&= \mathbb{E}\int\Big(\sum_i \xi_i\Big)\Big(\sum_j \mathbb{E}(\eta_j|\xi_j) - \mathbb{E}(\eta_j)\Big)\,d\sigma \\
&\le \left(\mathbb{E}\left\|\sum_i \xi_i\right\|_{L^q(S)}^p\right)^{1/p}\left(\mathbb{E}\left\|\sum_j \mathbb{E}(\eta_j|\xi_j) - \mathbb{E}(\eta_j)\right\|_{L^{q'}(S)}^{p'}\right)^{1/p'}.
\end{aligned}$$

Since the elements $\mathbb{E}(\eta_j|\xi_j) - \mathbb{E}(\eta_j)$ are independent and mean-zero,

$$\begin{aligned}
\langle(\xi_i),(\eta_i)\rangle &\lesssim_{p',q'} \left(\mathbb{E}\left\|\sum_i \xi_i\right\|_{L^q(S)}^p\right)^{1/p} \big\|\big(\mathbb{E}(\eta_j|\xi_j) - \mathbb{E}(\eta_j)\big)\big\|_{s_{p',q'}} \qquad (2.14)\\
&\le 2\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_{L^q(S)}^p\right)^{1/p}.
\end{aligned}$$

By (2.9), the claim follows by taking the supremum over all (η_i) as above. We now prove the estimates $\lesssim_{p,q}$ case by case.

Case $2 \le q \le p < \infty$: Recall that Theorem 2.3 says that

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{1/p} \\ \lesssim_{p,q} \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(S)}, \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p}\Big\}.\end{aligned}$$

Since $q \le p$, applying (2.12) with $f_i = \|\xi\|_{L^q(S)}^q$ yields

$$\begin{aligned}\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p} \\ \lesssim_{p,q} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^p\Big)^{1/p}, \Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^q\Big)^{1/q}\Big\}.\end{aligned}$$

Case $2 \le p \le q < \infty$: If $p \le q$, the contractive embeddings $L^q(\Omega) \subset L^p(\Omega)$ and $\ell^p \subset \ell^q$ imply

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p} \le \Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^q\Big)^{1/q} \tag{2.15}$$

and

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p} \le \Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^p\Big)^{1/p}. \tag{2.16}$$

By the triangle inequality,

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p} \le \|(\xi_i)\|_{D_{p,q}+D_{q,q}}.$$

The asserted estimate now follows from Theorem 2.3.

Case $1 \le p \le q \le 2$: Let $(\eta_i) \in S_q$, $(\theta_i) \in D_{p,q}$ and $(\kappa_i) \in D_{q,q}$ be such that $\xi_i = \eta_i + \theta_i + \kappa_i$. Then

$$\xi_i = \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i) + \mathbb{E}(\kappa_i|\xi_i) - \mathbb{E}(\kappa_i).$$

By Lemma 2.1,

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\sum_i \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)\Big\|_{L^q(S)}^p\Big)^{1/p} \\ &\quad\lesssim_{p,q} \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\Big)^{1/2}\Big\|_{L^q(S)} \\ &\quad\le 2\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{1/2}\Big\|_{L^q(S)},\end{aligned} \tag{2.17}$$

where the final step follows from the triangle inequality and Jensen's inequality. Now apply Lemma 2.2 [using that $L^q(S)$ has type q], (2.16) and Jensen's inequality to find

$$\begin{aligned}
&\left(\mathbb{E}\left\|\sum_i \mathbb{E}(\theta_i|\xi_i)-\mathbb{E}(\theta_i)\right\|_{L^q(S)}^p\right)^{1/p} \\
&\quad\lesssim_{p,q}\left(\mathbb{E}\left(\sum_i \|\mathbb{E}(\theta_i|\xi_i)-\mathbb{E}(\theta_i)\|_{L^q(S)}^q\right)^{p/q}\right)^{1/p} \\
&\quad\leq\left(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i)-\mathbb{E}(\theta_i)\|_{L^q(S)}^p\right)^{1/p} \\
&\quad\leq 2\left(\sum_i \mathbb{E}\|\theta_i\|_{L^q(S)}^p\right)^{1/p}.
\end{aligned}$$

Similarly, Lemma 2.2, (2.15) and Jensen's inequality yield

$$\left(\mathbb{E}\left\|\sum_i \mathbb{E}(\kappa_i|\xi_i)-\mathbb{E}(\kappa_i)\right\|_{L^q(S)}^p\right)^{1/p}\lesssim_{p,q}\left(\sum_i \mathbb{E}\|\kappa_i\|_{L^q(S)}^q\right)^{1/q}.$$

The asserted estimate now follows by the triangle inequality.

Case $1\leq q\leq p\leq 2$: The proof is very similar to the previous case. Let $(\eta_i)\in S_q$ and $(\theta_i)\in D_{p,q}\cap D_{q,q}$ be such that $\xi_i=\eta_i+\theta_i$, then

$$\xi_i=\mathbb{E}(\eta_i|\xi_i)-\mathbb{E}(\eta_i)+\mathbb{E}(\theta_i|\xi_i)-\mathbb{E}(\theta_i).$$

By the same argument as in (2.17),

$$\left(\mathbb{E}\left\|\sum_i \mathbb{E}(\eta_i|\xi_i)-\mathbb{E}(\eta_i)\right\|_{L^q(S)}^p\right)^{1/p}\lesssim_{p,q}\left\|\left(\sum_i \mathbb{E}|\eta_i|^2\right)^{1/2}\right\|_{L^q(S)}.$$

Moreover, successively applying Lemma 2.2, the Rosenthal inequality (2.12) (using that $q\leq p$) and Jensen's inequality yields

$$\begin{aligned}
&\left(\mathbb{E}\left\|\sum_i \mathbb{E}(\theta_i|\xi_i)-\mathbb{E}(\theta_i)\right\|_{L^q(S)}^p\right)^{1/p} \\
&\quad\lesssim_{p,q}\left(\mathbb{E}\left(\sum_i \|\mathbb{E}(\theta_i|\xi_i)-\mathbb{E}(\theta_i)\|_{L^q(S)}^q\right)^{p/q}\right)^{1/p} \\
&\quad\lesssim_{p,q}\max\left\{\left(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i)-\mathbb{E}(\theta_i)\|_{L^q(S)}^p\right)^{1/p},\right. \\
&\qquad\qquad\left.\left(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i)-\mathbb{E}(\theta_i)\|_{L^q(S)}^q\right)^{1/q}\right\} \\
&\quad\leq 2\max\left\{\left(\sum_i \mathbb{E}\|\theta_i\|_{L^q(S)}^p\right)^{1/p},\left(\sum_i \mathbb{E}\|\theta_i\|_{L^q(S)}^q\right)^{1/q}\right\}.
\end{aligned}$$

The result now follows by the triangle inequality.

Case $1 \le q \le 2 \le p < \infty$: By Hoffmann–Jørgensen's inequality (2.11), we have

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{1/p} \lesssim_p \max\Big\{\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^q\Big)^{1/q}, \Big(\mathbb{E}\max_i \|\xi_i\|_{L^q(S)}^p\Big)^{1/p}\Big\}.$$

By the previous case (with $p = q$), we have

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^q\Big)^{1/q} \simeq_{p,q} \|(\xi_i)\|_{S_q + D_{q,q}}$$

and obviously

$$\Big(\mathbb{E}\max_i \|\xi_i\|_{L^q(S)}^p\Big)^{1/p} \le \Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^p\Big)^{1/p}.$$

Case $1 \le p \le 2 \le q < \infty$: Let $\xi_i = \eta_i + \theta_i$ with $(\eta_i) \in S_q \cap D_{q,q}$ and $(\theta_i) \in D_{p,q}$. Then, $\xi_i = \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)$. Since the $\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)$ are independent and mean-zero, we can subsequently use Hölder's inequality and the already established estimate in the case $p = q \ge 2$ to find

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad\le \Big(\mathbb{E}\Big\|\sum_i \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)\Big\|_{L^q(S)}^q\Big)^{1/q} \\
&\quad\lesssim_{p,q} \max\Big\{\Big(\sum_i \mathbb{E}\|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)\|_{L^q(S)}^q\Big)^{1/q}, \\
&\qquad\qquad \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\Big)^{1/2}\Big\|_{L^q(S)}\Big\} \\
&\quad\le 2\max\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|_{L^q(S)}^q\Big)^{1/q}, \Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{1/2}\Big\|_{L^q(S)}\Big\}.
\end{aligned}$$

On the other hand, as $L^q(S)$ has type 2, it has type p and therefore Lemma 2.2 implies

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad\lesssim_{p,q} \Big(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad\le 2\Big(\sum_i \mathbb{E}\|\theta_i\|_{L^q(S)}^p\Big)^{1/p}.
\end{aligned}$$

The claimed inequality now follows by the triangle inequality. This completes the proof. □

3. Itô-isomorphisms: Classical L^q-spaces. In this section, we present a proof of the Itô isomorphism stated in Theorem 1.1. Let us first define the Poisson stochastic integral.

DEFINITION 3.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(E, \mathcal{E}, \mu)$ be a measure space. We say that a random measure N on E is a *Poisson random measure* if the following conditions hold:

(i) For disjoint $A_1, \dots, A_n \in \mathcal{E}$ the random variables $N(A_1), \dots, N(A_n)$ are independent and

$$N\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} N(A_i),$$

(ii) For any $A \in \mathcal{E}$ with $\mu(A) < \infty$ the random variable $N(A)$ is Poisson distributed with parameter $\mu(A)$.

Let $\mathcal{E}_\mu = \{A \in \mathcal{E} : \mu(A) < \infty\}$. Then the random measure $\tilde{N}$ on $(E, \mathcal{E}_\mu, \mu)$ defined by

$$\tilde{N}(A) := N(A) - \mu(A) \qquad (A \in \mathcal{E}_\mu),$$

is called the *compensated Poisson random measure* associated with N.

As is well known, one can always construct a Poisson random measure on any given σ-finite measure space $(E, \mathcal{E}, \mu)$; see, for example, [36].

Throughout, we let $(J, \mathcal{J}, \nu)$ be a σ-finite measure space and we fix a Poisson random measure N on $\mathbb{R}_+ \times J$. To arrive at a satisfactory stochastic integration theory with respect to the associated compensated Poisson random measure, we need to impose the following standard compatibility assumption.

ASSUMPTION 3.2. Throughout we fix a filtration $(\mathcal{F}_t)_{t>0}$ such that for any $0 \le s < t < \infty$ and any $A \in \mathcal{J}$ the random variable $\tilde{N}((s,t] \times A)$ is $\mathcal{F}_t$-measurable and independent of $\mathcal{F}_s$.

DEFINITION 3.3. Fix a Banach space X and let $F : \Omega \times \mathbb{R}_+ \times J \to X$. We say that F is a *simple, adapted X-valued process* if there is a finite partition $\pi = \{0 = t_1 < \cdots < t_{l+1} < \infty\}$ of $\mathbb{R}_+$, $F_{i,j,k} \in L^\infty(\mathcal{F}_{t_i})$, $x_{i,j,k} \in X$ and disjoint sets $A_1, \dots A_m$ in $\mathcal{J}$ satisfying $\nu(A_j) < \infty$ for $i = 1, \dots, l$, $j = 1, \dots, m$ and $k = 1, \dots, n$ such that

$$F = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} F_{i,j,k} \chi_{(t_i, t_{i+1}]} \chi_{A_j} x_{i,j,k}. \tag{3.1}$$

Given $t > 0$ and $B \in \mathcal{J}$, we define the (*compensated*) *Poisson stochastic integral of* F on $(0,t] \times B$ with respect to $\tilde{N}$ by

$$\int_{(0,t]\times B} F\, d\tilde{N} = \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} F_{i,j,k}\tilde{N}\big((t_i \wedge t, t_{i+1} \wedge t] \times (A_j \cap B)\big)x_{i,j,k},$$

where $s \wedge t := \min\{s,t\}$.

The following elementary observation will be important for our proof. The upper estimate in (3.2) in the case $1 \le p \le 2$ was noted earlier in [2], Lemma C.3.

LEMMA 3.4. *Let N be a Poisson distributed random variable with parameter $0 \le \lambda \le 1$. Then for every $1 \le p < \infty$ there exist constants $b_p, c_p > 0$ such that*

$$b_p\lambda \le \mathbb{E}|N-\lambda|^p \le c_p\lambda. \tag{3.2}$$

PROOF. The inequalities are trivial if $\lambda = 0$, so we may assume $\lambda > 0$. Suppose first that $2 \le p < \infty$. We begin by proving the inequality on the left-hand side of (3.2). We have

$$\begin{aligned}\mathbb{E}|N-\lambda|^p &= \sum_{k=0}^{\infty}|k-\lambda|^p\frac{\lambda^k e^{-\lambda}}{k!}\\ &\ge \sum_{k=2}^{\infty}|k-\lambda|^2\frac{\lambda^k e^{-\lambda}}{k!} + |\lambda|^p e^{-\lambda} + |1-\lambda|^p\lambda e^{-\lambda}.\end{aligned} \tag{3.3}$$

Hence,

$$\begin{aligned}&\mathbb{E}|N-\lambda|^p\\ &\quad\ge \mathbb{E}|N-\lambda|^2 - |\lambda|^2e^{-\lambda} - |1-\lambda|^2\lambda e^{-\lambda} + |\lambda|^p e^{-\lambda} + |1-\lambda|^p\lambda e^{-\lambda}\\ &\quad= \lambda + \lambda e^{-\lambda}\big(-\lambda - (1-\lambda)^2 + \lambda^{p-1} + (1-\lambda)^p\big)\\ &\quad= \lambda\big(1 + e^{-\lambda}f_p(\lambda)\big),\end{aligned} \tag{3.4}$$

where

$$f_p(\lambda) = \lambda^{p-1} - \lambda^2 + \lambda - 1 + (1-\lambda)^p. \tag{3.5}$$

One easily sees that $\min_{0\le\lambda\le1}(1 + e^{-\lambda}f_p(\lambda)) = b_p > 0$. Indeed,

$$1 + e^{-\lambda}f_p(\lambda) > 1 + e^{-\lambda}\big(-\lambda^2 + \lambda - 1\big) + e^{-\lambda}(1-\lambda)^p.$$

Now,

$$1 + e^{-\lambda}\big(-\lambda^2 + \lambda - 1\big) + e^{-\lambda}(1-\lambda)^p > 0$$

if and only if

$$(1-\lambda)^p > -e^{\lambda} + \lambda^2 - \lambda + 1 = -2\lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{6} - \frac{\lambda^4}{24} - \cdots.$$

Clearly, this holds if $0 \le \lambda \le 1$. This proves the left-hand side inequality of (3.2) if $2 \le p < \infty$. We now consider the right-hand side inequality. It suffices to prove this in the case where p is an even integer n. Since the moment generating function of $N - \lambda$ is given by

$$\mathbb{E}(e^{t(N-\lambda)}) = e^{\lambda(e^t - 1 - t)} = \exp\left(\lambda \sum_{n=2}^{\infty} \frac{t^n}{n!}\right),$$

it is easy to see that the nth moment of $N - \lambda$ can be written as $\lambda p_n(\lambda)$ for some polynomial p_n with positive coefficients. Since $\max_{0 \le \lambda \le 1} |p_n(\lambda)| \le c_n$ for some constant $c_n > 0$, our proof for the case $2 \le p < \infty$ is complete.

Suppose now that $1 \le p < 2$. Then, by the Cauchy–Schwartz inequality,

$$\begin{aligned}\lambda = \mathbb{E}|N-\lambda|^2 &= \mathbb{E}|N-\lambda|^{p/2}|N-\lambda|^{2-p/2} \\ &\le \left(\mathbb{E}|N-\lambda|^p\right)^{1/2}\left(\mathbb{E}|N-\lambda|^{4-p}\right)^{1/2}.\end{aligned}$$

Since $4 - p \ge 2$, we find by the above that

$$\lambda^2 \le \mathbb{E}|N-\lambda|^p \mathbb{E}|N-\lambda|^{4-p} \le \mathbb{E}|N-\lambda|^p c_{4-p}\lambda.$$

To prove the right-hand side inequality in (3.2), note that if $1 \le p < 2$ the inequalities in (3.3) and (3.4) reverse and, therefore,

$$\mathbb{E}|N-\lambda|^p \le \lambda \max_{0 \le \lambda \le 1} \left(1 + e^{-\lambda} f_p(\lambda)\right),$$

where f_p is the continuous function defined in (3.5). $\square$

REMARK 3.5. By refining the partition π in Definition 3.3 if necessary, we can and will always assume that $(t_{i+1} - t_i)\nu(A_j) \le 1$ for all $i = 1, \dots, l$, $j = 1, \dots, m$. This will allow us to apply Lemma 3.4 to the compensated Poisson random variables $\tilde{N}((t_i \wedge t, t_{i+1} \wedge t] \times (A_j \cap B))$.

Let us finally record the following easy observation for further reference.

LEMMA 3.6. *Suppose that $(E, \mathcal{E}, \mu)$ is a σ-finite measure space and let X be a Banach space. Let $A_1, \dots, A_n$ be disjoint sets in Σ satisfying $\mu(A_i) < \infty$ and let $\mathcal{A}$ be the σ-algebra generated by $A_1, \dots, A_n$. Then, for any $G \in L^1(E; X)$ supported on $\bigcup_{i=1}^n A_i$,*

$$\mathbb{E}(G|\mathcal{A}) = \sum_{i=1}^{n} \chi_{A_i} y_i$$

for certain $y_i \in X$.

PROOF. Let A_{n+1} be the complement of $\bigcup_{i=1}^n A_i$ in E. Since $A_1, \dots, A_{n+1}$ are disjoint, $\mathcal{A}$ is actually a finite algebra consisting of $A_1, \dots, A_{n+1}$ and all their possible unions. Moreover, for any $1 \le i \le n$,

$$\int_{A_i} G\,d\mu = \int_{A_i} \sum_{\{1\le j\le n:\,\mu(A_j)\neq 0\}} (\mu(A_j))^{-1}\Big(\int_{A_j} G\,d\mu\Big)\chi_{A_j}\,d\mu.$$

Since $\int_{A_{n+1}} G\,d\mu = 0$ by assumption, we conclude that

$$\mathbb{E}(G|\mathcal{A}) = \sum_{\{1\le j\le n:\,\mu(A_j)\neq 0\}} (\mu(A_j))^{-1}\chi_{A_j}\int_{A_j} G\,d\mu. \qquad \square$$

We are now ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Using Assumption 3.2, it is not difficult to show that the process $(\int_{(0,s]\times B} F\,d\tilde{N})_{s>0}$ is a martingale. Therefore, the map

$$s \mapsto \Big\|\int_{(0,s]\times B} F\,d\tilde{N}\Big\|_{L^q(S)}$$

defines a positive submartingale in $L^p(\Omega)$ and by Doob's maximal inequality (see, e.g., [33], Theorem 1.7) we have for any $p > 1$,

$$\Big(\mathbb{E}\sup_{0<s\le t}\Big\|\int_{(0,s]\times B} F\,d\tilde{N}\Big\|_{L^q(S)}^p\Big)^{1/p} \le p'\Big(\mathbb{E}\Big\|\int_{(0,t]\times B} F\,d\tilde{N}\Big\|_{L^q(S)}^p\Big)^{1/p},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, $L^q(S)$ has the UMD property if $1 < q < \infty$, so in view of the decoupling inequalities (1.4) it suffices to prove

$$\Big(\mathbb{E}\mathbb{E}_c\Big\|\int_{(0,t]\times B} F\,d\tilde{N}^c\Big\|_{L^q(S)}^p\Big)^{1/p} \simeq_{p,q} \|F\chi_{(0,t]\times B}\|_{\mathcal{I}_{p,q}}, \tag{3.6}$$

where $\tilde{N}^c$ is an independent copy of $\tilde{N}$ on a probability space $(\Omega_c, \mathcal{F}_c, \mathbb{P}_c)$. We show this in the cases $2 \le q \le p < \infty$ and $1 < p \le q \le 2$ in detail. All the main technical difficulties occur in these two cases. For the similar proof in the other cases, we refer the reader to Appendix B. Let F be the simple adapted process given in (3.1), taking Remark 3.5 into account. We may assume that $t = t_{l+1}$ and $B = \bigcup_{j=1}^m A_j$. We write $\tilde{N}_{i,j}^c := \tilde{N}^c((t_i, t_{i+1}] \times A_j)$ for brevity.

Case $2 \le q \le p < \infty$: Set $y_{i,j} = \sum_{k=1}^n F_{i,j,k}x_{i,j,k}$, then the doubly indexed sequence $d_{i,j} = y_{i,j}\tilde{N}_{i,j}^c$ satisfies

$$\int_{(0,t]\times B} F\,d\tilde{N} = \sum_{i,j} d_{i,j}.$$

Moreover, for any fixed $\omega \in \Omega$ the sequence $(d_{i,j}(\omega))_{i,j}$ consists of independent, mean-zero random variables. By applying Theorem 1.2 pointwise in Ω, we find

$$\begin{aligned}
&\Big(\mathbb{E}_c\Big\|\sum_{i,j} d_{i,j}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad\simeq_{p,q} \max\Big\{\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j}|^2\Big)^{1/2}\Big\|_{L^q(S)}, \Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j}\|_{L^q(S)}^q\Big)^{1/q}, \\
&\qquad\qquad \Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j}\|_{L^q(S)}^p\Big)^{1/p}\Big\}
\end{aligned}$$

and by taking the $L^p(\Omega)$-norm on both sides we arrive at

$$\begin{aligned}
&\Big(\mathbb{E}\mathbb{E}_c\Big\|\sum_{i,j} d_{i,j}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad\simeq_{p,q} \max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j}|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p}, \\
&\qquad\qquad \Big(\mathbb{E}\Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j}\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p}, \Big(\sum_{i,j}\mathbb{E}\mathbb{E}_c\|d_{i,j}\|_{L^q(S)}^p\Big)^{1/p}\Big\}.
\end{aligned}$$

Using Lemma 3.4 and Remark 3.5, we compute

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j}|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad= \Big(\mathbb{E}\Big\|\Big(\sum_{i,j}|y_{i,j}|^2\mathbb{E}_c|\tilde{N}_{i,j}^c|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p} \qquad (3.7)\\
&\quad= \Big(\mathbb{E}\Big\|\Big(\sum_{i,j}|y_{i,j}|^2(t_{i+1}-t_i)\nu(A_j)\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p} = \|F\|_{\mathcal{S}_q^p}
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\mathbb{E}\Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j}\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p} \\
&\quad= \Big(\mathbb{E}\Big(\sum_{i,j}\|y_{i,j}\|_{L^q(S)}^q\mathbb{E}_c|\tilde{N}_{i,j}^c|^q\Big)^{p/q}\Big)^{1/p} \qquad (3.8)\\
&\quad\simeq_q \Big(\mathbb{E}\Big(\sum_{i,j}\|y_{i,j}\|_{L^q(S)}^q(t_{i+1}-t_i)\nu(A_j)\Big)^{p/q}\Big)^{1/p} = \|F\|_{\mathcal{D}_{q,q}^p}.
\end{aligned}$$

Finally,

$$
\begin{aligned}
&\Big(\sum_{i,j} \mathbb{E}\mathbb{E}_c \|d_{i,j}\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad = \Big(\sum_{i,j} \mathbb{E}\|y_{i,j}\|_{L^q(S)}^p \mathbb{E}_c|\tilde{N}_{i,j}^c|^p\Big)^{1/p} \\
&\quad \simeq_p \Big(\sum_{i,j} \mathbb{E}\|y_{i,j}\|_{L^q(S)}^p (t_{i+1}-t_i)\nu(A_j)\Big)^{1/p} = \|F\|_{\mathcal{D}_{p,q}^p}.
\end{aligned} \tag{3.9}
$$

We conclude that (3.6) holds.

Case $1 < p \le q \le 2$: Let $\mathcal{I}_{\mathrm{elem}}$ denote the linear space of all simple functions on $\Omega \times \mathbb{R}_+ \times J \times S$ with support of finite measure. Note that $\mathcal{I}_{\mathrm{elem}}$ is dense in $\mathcal{S}_q^p$, $\mathcal{D}_{p,q}^p$ and $\mathcal{D}_{q,q}^p$. Hence, if we fix $\varepsilon > 0$, we can find a decomposition $F = F_1 + F_2 + F_3$ with $F_\alpha \in \mathcal{I}_{\mathrm{elem}}$ for $\alpha = 1, 2, 3$ such that

$$
\|F_1\|_{\mathcal{S}_q^p} + \|F_2\|_{\mathcal{D}_{p,q}^p} + \|F_3\|_{\mathcal{D}_{q,q}^p} \le \|F\|_{\mathcal{I}_{p,q}} + \varepsilon.
$$

Clearly, we may assume that F_1, F_2 and F_3 have the same support in $\mathbb{R}_+ \times J$ as F. Let $\mathcal{A}$ be the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+) \times \mathcal{J}$ generated by the sets $(t_i, t_{i+1}] \times A_j$. The associated conditional expectation $\mathbb{E}(\cdot|\mathcal{A})$ is well defined, as $\mathcal{J}$ is σ-finite. By Lemma 3.6, $\mathbb{E}(F_\alpha|\mathcal{A})$ is of the form

$$
\mathbb{E}(F_\alpha|\mathcal{A}) = \sum_{i,j,k} F_{i,j,k,\alpha} \chi_{(t_i,t_{i+1}]} \chi_{A_j} x_{i,j,k,\alpha} \qquad (\alpha = 1, 2, 3).
$$

Let $y_{i,j,\alpha} = \sum_{k=1}^n F_{i,j,k,\alpha} x_{i,j,k,\alpha}$ and set $d_{i,j,\alpha} = y_{i,j,\alpha} N_{i,j}^c$, so that $d_{i,j} = y_{i,j} N_{i,j}^c$ satisfies

$$
d_{i,j} = d_{i,j,1} + d_{i,j,2} + d_{i,j,3}.
$$

We apply Theorem 1.2 pointwise in Ω to find

$$
\begin{aligned}
&\Big(\mathbb{E}_c \Big\|\sum_{i,j} d_{i,j}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad \lesssim_{p,q} \Big\|\Big(\sum_{i,j} \mathbb{E}_c |d_{i,j,1}|^2\Big)^{1/2}\Big\|_{L^q(S)} \\
&\qquad + \Big(\sum_{i,j} \mathbb{E}_c \|d_{i,j,2}\|_{L^q(S)}^p\Big)^{1/p} + \Big(\sum_{i,j} \mathbb{E}_c \|d_{i,j,3}\|_{L^q(S)}^q\Big)^{1/q}.
\end{aligned}
$$

By taking $L^p(\Omega)$-norms on both sides and using the triangle inequality, we obtain

$$\begin{aligned}
&\Big(\mathbb{E}\mathbb{E}_c\Big\|\sum_{i,j} d_{i,j}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad \lesssim_{p,q} \Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j,1}|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\qquad + \Big(\sum_{i,j}\mathbb{E}\mathbb{E}_c\|d_{i,j,2}\|_{L^q(S)}^p\Big)^{1/p} + \Big(\mathbb{E}\Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j,3}\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p}.
\end{aligned}$$

By the computations in (3.7), (3.8) and (3.9),

$$\begin{aligned}
\Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j,1}|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p} &= \|\mathbb{E}(F_1|\mathcal{A})\|_{\mathcal{S}_q^p} \le \|F_1\|_{\mathcal{S}_q^p}, \\
\Big(\sum_{i,j}\mathbb{E}\mathbb{E}_c\|d_{i,j,2}\|_{L^q(S)}^p\Big)^{1/p} &\simeq_p \|\mathbb{E}(F_2|\mathcal{A})\|_{\mathcal{D}_{p,q}^p} \le \|F_2\|_{\mathcal{D}_{p,q}^p}, \\
\Big(\mathbb{E}\Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j,3}\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p} &\simeq_q \|\mathbb{E}(F_3|\mathcal{A})\|_{\mathcal{D}_{q,q}^p} \le \|F_3\|_{\mathcal{D}_{q,q}^p}.
\end{aligned} \tag{3.10}$$

We conclude that

$$\begin{aligned}
\Big(\mathbb{E}\mathbb{E}_c\Big\|\int_{(0,t]\times B} F\,d\tilde{N}^c\Big\|_{L^q(S)}^p\Big)^{1/p} &\lesssim_{p,q} \|F_1\|_{\mathcal{S}_q^p} + \|F_2\|_{\mathcal{D}_{p,q}^p} + \|F_3\|_{\mathcal{D}_{q,q}^p} \\
&\le \|F\|_{\mathcal{I}_{p,q}} + \varepsilon.
\end{aligned}$$

We deduce the reverse inequality by duality. If p', q' are the Hölder conjugates of p and q, then $(\mathcal{S}_q^p)^* = \mathcal{S}_{q'}^{p'}$, $(\mathcal{D}_{q,q}^p)^* = \mathcal{D}_{q',q'}^{p'}$ and $(\mathcal{D}_{p,q}^p)^* = \mathcal{D}_{p',q'}^{p'}$. Therefore, it follows from (2.10) that $\mathcal{I}_{p,q}^* = \mathcal{I}_{p',q'}$. We let

$$\langle F, G\rangle = \int_{\Omega\times\mathbb{R}_+\times J\times S} FG\,d\mathbb{P}\,dt\,d\nu\,d\sigma$$

denote the associated duality bracket. If $G \in \mathcal{I}_{\mathrm{elem}}$ has the same support as F, then $\mathbb{E}(G|\mathcal{A})$ is of the form

$$\mathbb{E}(G|\mathcal{A}) = \sum_{i,j,k} G_{i,j,k}\chi_{(t_i,t_{i+1}]}\chi_{A_j}x_{i,j,k}^*,$$

where $G_{i,j,k} \in L^\infty(\Omega)$. Now,

$$\begin{aligned}
\langle F, G\rangle &= \langle F, \mathbb{E}(G|\mathcal{A})\rangle \\
&= \sum_{i,j,k}\mathbb{E}(F_{i,j,k}G_{i,j,k})\,dt \times d\nu\big((t_i,t_{i+1}]\times A_j\big)\langle x_{i,j,k}, x_{i,j,k}^*\rangle
\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&= \sum_{i,j,k} \mathbb{E}(F_{i,j,k} G_{i,j,k})\, dt \times d\nu\big((t_i, t_{i+1}] \times A_j\big) \langle x_{i,j,k}, x^*_{i,j,k} \rangle \\
&= \sum_{i,j,k,l,m,n} \mathbb{E}(F_{i,j,k} G_{l,m,n}) \mathbb{E}_c\big(\tilde{N}^c_{i,j} \tilde{N}^c_{l,m}\big) \langle x_{i,j,k}, x^*_{l,m,n} \rangle \\
&= \sum_{i,j,k,l,m,n} \mathbb{E}\mathbb{E}_c\big(F_{i,j,k} \tilde{N}^c_{i,j} G_{l,m,n} \tilde{N}^c_{l,m} \langle x_{i,j,k}, x^*_{l,m,n} \rangle\big) \\
&= \Big\langle \sum_{i,j,k} F_{i,j,k} \tilde{N}^c_{i,j} x_{i,j,k}, \sum_{l,m,n} G_{l,m,n} \tilde{N}^c_{l,m} x^*_{l,m,n} \Big\rangle \\
&\le \Big\| \int_{(0,t]\times B} F\, d\tilde{N}^c \Big\|_{L^p(\Omega\times\Omega_c; L^q(S))} \\
&\quad \times \Big\| \sum_{l,m,n} G_{l,m,n} \tilde{N}^c_{l,m} x^*_{l,m,n} \Big\|_{L^{p'}(\Omega\times\Omega_c; L^{q'}(S))}.
\end{aligned}
$$

Since $2 \le q' \le p' < \infty$, our previously established case implies that

$$
\Big\| \sum_{l,m,n} G_{l,m,n} \tilde{N}^c_{l,m} x^*_{l,m,n} \Big\|_{L^{p'}(\Omega\times\Omega_c; L^{q'}(S))} \lesssim_{p,q} \|\mathbb{E}(G|\mathcal{A})\|_{\mathcal{I}_{p',q'}} \le \|G\|_{\mathcal{I}_{p',q'}}.
$$

Summarizing, we find

$$
\langle F, G \rangle \lesssim_{p,q} \Big\| \int_{(0,t]\times B} F\, d\tilde{N}^c \Big\|_{L^p(\Omega; L^q(S))} \|G\|_{\mathcal{I}_{p',q'}}.
$$

Taking the supremum over all $G \in \mathcal{I}_{\mathrm{elem}}$ yields the result.

For the proof of the final assertion, note that $L^1(S)$ is not a UMD space. However, for any $1 \le p < \infty$, the one-sided decoupling inequality

$$
\Big(\mathbb{E}\Big\| \int_{(0,t]\times B} F\, d\tilde{N} \Big\|^p_{L^1(S)}\Big)^{1/p} \lesssim_p \Big(\mathbb{E}\mathbb{E}_c\Big\| \int_{(0,t]\times B} F\, d\tilde{N}^c \Big\|^p_{L^1(S)}\Big)^{1/p}
$$

still holds, see [5]. The remainder of the proof is the same as in the case $q > 1$. $\square$

REMARK 3.7. It is clear from the proof that the inequality

$$
\mathbb{E}\Big\| \int_{(0,t]\times B} F\, d\tilde{N} \Big\|_{L^q(S)} \lesssim_q \|F\|_{\mathcal{I}_{1,q}}
$$

is valid if $1 \le q < \infty$.

4. Preliminaries on noncommutative L^q-spaces. We now turn to the extension of the Itô isomorphism in Theorem 1.1 to integrands taking values in a noncommutative L^q-space. We begin by reviewing some facts on noncommutative L^q-spaces. References for proofs of the results presented below can be found in the survey [32]. Let $\mathcal{M}$ be a von Neumann algebra acting on a complex Hilbert space H, which is equipped with a normal, semi-finite faithful trace τ. We say that a closed, densely defined linear operator x on H is *affiliated* with the von Neumann algebra $\mathcal{M}$ if $ux = xu$ for any unitary element u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. For such an operator, we define its *distribution function* by

$$d(v; x) = \tau\big(e^{|x|}(v,\infty)\big) \qquad (v \geq 0),$$

where $e^{|x|}$ is the spectral measure of $|x|$. The *decreasing rearrangement* of x is defined by

$$\mu_t(x) = \inf\{v > 0 : d(v; x) \leq t\} \qquad (t \geq 0).$$

We call x τ*-measurable* if $d(v; x) < \infty$ for some $v > 0$. We let $S(\tau)$ denote the linear space of all τ-measurable operators. One can show that $S(\tau)$ is a metrizable, complete topological $*$-algebra with respect to the measure topology. Moreover, the trace τ extends to a trace (again denoted by τ) on the set $S(\tau)_+$ of positive τ-measurable operators by setting

$$\tau(x) = \int_0^\infty \mu_t(x)\, dt \qquad \big(x \in S(\tau)_+\big). \tag{4.1}$$

For $0 < q < \infty$, we define

$$\|x\|_{L^q(\mathcal{M})} = \big(\tau\big(|x|^q\big)\big)^{1/q} \qquad \big(x \in S(\tau)\big). \tag{4.2}$$

The linear space $L^q(\mathcal{M}, \tau)$ of all $x \in S(\tau)$ satisfying $\|x\|_{L^q(\mathcal{M})} < \infty$ is called the *noncommutative* L^q*-space* associated with the pair $(\mathcal{M}, \tau)$. We usually denote $L^q(\mathcal{M}, \tau)$ by $L^q(\mathcal{M})$ for brevity. The map $\|\cdot\|_{L^q(\mathcal{M})}$ in (4.2) defines a norm (or q-norm if $0 < q < 1$) on the space $L^q(\mathcal{M})$ under which it becomes a Banach space (resp., quasi-Banach space). It can alternatively be viewed as the completion of $\mathcal{M}$ in the (quasi-)norm $\|\cdot\|_{L^q(\mathcal{M})}$. We use the expression $L^\infty(\mathcal{M})$ to denote $\mathcal{M}$ equipped with its operator norm. By (4.1) and using that $\mu(|x|^q) = \mu(x)^q$, the noncommutative L^q-(quasi-)norm can alternatively be computed as

$$\|x\|_{L^q(\mathcal{M})} = \bigg(\int_0^\infty \mu_t(x)^q\, dt\bigg)^{1/q} \qquad \big(x \in L^q(\mathcal{M})\big).$$

If (S, Σ, σ) is a Maharam measure space, then $\mathcal{M} = L^\infty(S)$ is a von Neumann algebra, which can be equipped with the normal, semifinite faithful trace $\tau(f) = \int f\, d\sigma$. In this case, $L^q(\mathcal{M})$ coincides with the usual Bochner space $L^q(S)$. Another familiar example is obtained by taking $\mathcal{M} = B(H)$, for a Hilbert space H. If $B(H)$ is equipped with its standard trace, then the associated noncommutative L^q-spaces are the usual Schatten spaces.

Below we shall use the following facts. First, recall Hölder's inequality: if $0 < q, r, s \leq \infty$ are such that $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$ and $x \in L^r(\mathcal{M})$, $y \in L^s(\mathcal{M})$, then $xy \in L^q(\mathcal{M})$ and

$$\|xy\|_{L^q(\mathcal{M})} \leq \|x\|_{L^r(\mathcal{M})} \|y\|_{L^s(\mathcal{M})}. \tag{4.3}$$

For $1 \leq q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$, the familiar duality $L^q(\mathcal{M})^* = L^{q'}(\mathcal{M})$ holds isometrically, with the duality bracket given by $\langle x, y \rangle = \tau(xy)$. In particular, $L^q(\mathcal{M})$ is reflexive if and only if $1 < q < \infty$ and $L^1(\mathcal{M}) = \mathcal{M}_*$ isometrically, where $\mathcal{M}_*$ is the predual of $\mathcal{M}$. We recall that $L^q(\mathcal{M})$ is a UMD Banach space if and only if $1 < q < \infty$. If $1 \leq q < \infty$, then $L^q(\mathcal{M})$ has type $\min\{q, 2\}$ and cotype $\max\{q, 2\}$.

We conclude this section by describing the column and row spaces and their conditional versions. Let $1 \leq q < \infty$. For a finite sequence (x_i) in $L^q(\mathcal{M})$, we define

$$\tag{4.4} \begin{aligned} \|(x_i)\|_{L^q(\mathcal{M};\ell^2_c)} &= \left\| \left(\sum_i x_i^* x_i \right)^{1/2} \right\|_{L^q(\mathcal{M})}, \\ \|(x_i)\|_{L^q(\mathcal{M};\ell^2_r)} &= \left\| \left(\sum_i x_i x_i^* \right)^{1/2} \right\|_{L^q(\mathcal{M})}. \end{aligned}$$

Given $x_1, \ldots, x_n$, we let $\operatorname{diag}(x_i)$, $\operatorname{row}(x_i)$ and $\operatorname{col}(x_i)$ denote the matrix with the x_i on its diagonal, first row and first column, respectively, and zeroes elsewhere. Let $\mathcal{M} \otimes B(\ell^2)$ be the von Neumann tensor product equipped with its product trace $\tau \otimes \operatorname{Tr}$. By noting that

$$\begin{aligned} \left\| \left(\sum_{i=1}^n x_i^* x_i \right)^{1/2} \right\|_{L^q(\mathcal{M})} &= \|\operatorname{col}(x_i)\|_{L^q(\mathcal{M}\otimes B(\ell^2))}, \\ \left\| \left(\sum_{i=1}^n x_i x_i^* \right)^{1/2} \right\|_{L^q(\mathcal{M})} &= \|\operatorname{row}(x_i)\|_{L^q(\mathcal{M}\otimes B(\ell^2))}, \end{aligned}$$

one sees that the expressions in (4.4) define two norms on the linear space of all finitely nonzero sequences in $L^q(\mathcal{M})$. The completions of this space in these norms are called the *column* and *row space*, respectively.

We shall need a conditional version of these two spaces. Suppose that $\mathcal{N}$ is a von Neumann algebra equipped with a normal, semifinite faithful trace σ and let $\mathcal{K}$ be a von Neumann subalgebra such that $\sigma|_{\mathcal{K}}$ is again semifinite. Let $\mathcal{E} : \mathcal{N} \to \mathcal{K}$ be the conditional expectation with respect to $\mathcal{K}$. For a finite sequence (x_i) in $\mathcal{N}$, we define

$$\tag{4.5} \begin{aligned} \|(x_i)\|_{L^q(\mathcal{N};\mathcal{E},\ell^2_c)} &= \left\| \left(\sum_i \mathcal{E}|x_i|^2 \right)^{1/2} \right\|_{L^q(\mathcal{N})}, \\ \|(x_i)\|_{L^q(\mathcal{N};\mathcal{E},\ell^2_r)} &= \left\| \left(\sum_i \mathcal{E}|x_i^*|^2 \right)^{1/2} \right\|_{L^q(\mathcal{N})}. \end{aligned}$$

Using techniques from Hilbert C^*-modules, it was shown by M. Junge [14] that

$$\begin{aligned} &\left\{ (x_i)_{i=1}^n : x_i \in \mathcal{N}, n \geq 1, \|(x_i)\|_{L^q(\mathcal{N};\mathcal{E},\ell^2_c)} < \infty \right\} \quad \text{and} \\ &\left\{ (x_i)_{i=1}^n : x_i \in \mathcal{N}, n \geq 1, \|(x_i)\|_{L^q(\mathcal{N};\mathcal{E},\ell^2_r)} < \infty \right\} \end{aligned}$$

are normed linear spaces. By taking the completion of these spaces, we obtain the *conditional column* and *row space*, respectively. Moreover, one can identify these spaces with complemented subspaces of $L^q(\mathcal{K};\ell_c^2)$ and $L^q(\mathcal{K};\ell_r^2)$ and in this way show that for any $1<q<\infty$ and $\frac{1}{q}+\frac{1}{q'}=1$

$$(L^q(\mathcal{N};\mathcal{E},\ell_c^2))^* = L^{q'}(\mathcal{N};\mathcal{E},\ell_r^2), \qquad (L^q(\mathcal{N};\mathcal{E},\ell_r^2))^* = L^{q'}(\mathcal{N};\mathcal{E},\ell_c^2), \tag{4.6}$$

isometrically, with duality bracket given by

$$\langle (x_i),(y_i)\rangle = \sum_i \tau(x_i y_i).$$

We refer to Section 2 of [14] for more information.

5. L^q-valued Rosenthal inequalities: Noncommutative case. In this section, we prove an extension of Theorem 1.2 for random variables taking values in a noncommutative L^q-space. To state our main result, we introduce the following norms on the linear space of all finite sequences (f_i) of random variables in $L^\infty(\Omega;L^q(\mathcal{M}))$, which serve as substitutes for the norms considered in (1.7). First, for $1\le p,q<\infty$ we define

$$\|(f_i)\|_{D_{p,q}} = \Bigl(\sum_i \mathbb{E}\|f_i\|_{L^q(\mathcal{M})}^p\Bigr)^{1/p} \tag{5.1}$$

and we consider a column and row version of the space S_q in considered earlier, that is, we set

$$\begin{aligned} \|(f_i)\|_{S_{q,c}} &= \Bigl\|\Bigl(\sum_i \mathbb{E}|f_i|^2\Bigr)^{1/2}\Bigr\|_{L^q(\mathcal{M})}, \\ \|(f_i)\|_{S_{q,r}} &= \Bigl\|\Bigl(\sum_i \mathbb{E}|f_i^*|^2\Bigr)^{1/2}\Bigr\|_{L^q(\mathcal{M})}. \end{aligned} \tag{5.2}$$

Here, f_i^* denotes the (pointwise) adjoint of f_i. To see that the latter two expressions define two norms, we identify them with a particular instance of the conditional row and column norms in (4.5). We let $\mathcal{N}$ be the tensor product von Neumann algebra $L^\infty(\Omega)\otimes\mathcal{M}$, equipped with the tensor product trace $\mathbb{E}\otimes\tau$. Let us recall that, for any $1\le q<\infty$, the map defined on simple functions in the Bochner space $L^q(\Omega;L^q(\mathcal{M}))$ by

$$I_q\Bigl(\sum_i \chi_{A_i}x_i\Bigr) = \sum_i \chi_{A_i}\otimes x_i$$

extends to an isometric isomorphism

$$L^q(\Omega;L^q(\mathcal{M})) = L^q(L^\infty(\Omega)\otimes\mathcal{M}). \tag{5.3}$$

Let $\mathcal{K}$ be the von Neumann subalgebra of $\mathcal{N}$ given by $\mathcal{K} = \mathbb{C}\mathbf{1} \otimes \mathcal{M}$ and let $\mathcal{E}$ be the associated conditional expectation. Under the identification (5.3), the element $\mathcal{E}(f)$ coincides with the Bochner integral $\mathbb{E}(f)$, whenever $f \in L^q(\mathcal{N})$. In particular, for any finite sequence (f_i) in $\mathcal{N}$,

$$\|(f_i)\|_{L^q(\mathcal{N};\mathcal{E},\ell_c^2)} = \|(f_i)\|_{S_{q,c}}, \qquad \|(f_i)\|_{L^q(\mathcal{N};\mathcal{E},\ell_r^2)} = \|(f_i)\|_{S_{q,r}}.$$

We denote by $D_{p,q}$, $S_{q,c}$ and $S_{q,r}$ the completion of the linear space of all finite sequences (f_i) of random variables in $L^\infty(\Omega; L^q(\mathcal{M}))$ with respect to the norms in (5.1) and (5.2). By (4.6), we have the duality

$$(S_{q,c})^* = S_{q',r}, \qquad (S_{q,r})^* = S_{q',c} \qquad \left(1 < q < \infty, \frac{1}{q} + \frac{1}{q'} = 1\right).$$

We are now ready to state the extension of Theorem 1.2.

THEOREM 5.1. *Let $1 < p, q < \infty$. If (ξ_i) is a finite sequence of independent, mean-zero $L^q(\mathcal{M})$-valued random variables, then*

$$\left(\mathbb{E}\left\|\sum_i \xi_i\right\|_{L^q(\mathcal{M})}^p\right)^{1/p} \simeq_{p,q} \|(\xi_i)\|_{s_{p,q}}, \tag{5.4}$$

where $s_{p,q}$ is given by

$$\begin{aligned}
&S_{q,c} \cap S_{q,r} \cap D_{q,q} \cap D_{p,q} && \text{if } 2 \le q \le p < \infty,\\
&S_{q,c} \cap S_{q,r} \cap (D_{q,q} + D_{p,q}) && \text{if } 2 \le p \le q < \infty,\\
&(S_{q,c} \cap S_{q,r} \cap D_{q,q}) + D_{p,q} && \text{if } 1 < p < 2 \le q < \infty,\\
&(S_{q,c} + S_{q,r} + D_{q,q}) \cap D_{p,q} && \text{if } 1 < q < 2 \le p < \infty,\\
&S_{q,c} + S_{q,r} + (D_{q,q} \cap D_{p,q}) && \text{if } 1 < q \le p \le 2,\\
&S_{q,c} + S_{q,r} + D_{q,q} + D_{p,q} && \text{if } 1 < p \le q \le 2.
\end{aligned}$$

To prove Theorem 5.1, we shall need to generalize Lemma 2.1 and Theorem 2.3. Let us first recall the noncommutative version of Khintchine's inequalities (2.1).

THEOREM 5.2 (Noncommutative Khintchine inequalities). *Let (r_i) be a Rademacher sequence and fix $1 \le p < \infty$. If $2 \le q < \infty$, then, for any finite sequence (x_i) in $L^q(\mathcal{M})$,*

$$\begin{aligned}
&\left(\mathbb{E}\left\|\sum_i r_i x_i\right\|_{L^q(\mathcal{M})}^p\right)^{1/p}\\
&\quad \le K_{p,q} \max\left\{\left\|\left(\sum_i |x_i|^2\right)^{1/2}\right\|_{L^q(\mathcal{M})}, \left\|\left(\sum_i |x_i^*|^2\right)^{1/2}\right\|_{L^q(\mathcal{M})}\right\}
\end{aligned} \tag{5.5}$$

and

$$\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{L^q(\mathcal{M})}^2\Big)^{1/2} \geq \max\Big\{\Big\|\Big(\sum_i |x_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i |x_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big\}.$$

On the other hand, *if* $1 \leq q \leq 2$, *then*

$$\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{L^q(\mathcal{M})}^2\Big)^{1/2} \leq \inf\Big\{\Big\|\Big(\sum_i |y_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i |z_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big\}$$

and

$$\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{L^q(\mathcal{M})}^q\Big)^{1/q} \gtrsim_{p,q} \inf\Big\{\Big\|\Big(\sum_i |y_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i |z_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big\},$$

where the infimum is taken over all decompositions $x_i = y_i + z_i$ *in* $L^q(\mathcal{M})$.

REMARK 5.3. Theorem 5.2 was proved for $p = q$ in [21, 22]. The general case immediately follows by applying Kahane's inequalities (2.3). It is known that the constant $\kappa_{p,q}$ in (2.3) satisfies $\kappa_{p,q} \leq (p-1)^{1/2}/(q-1)^{1/2}$ if $1 < q < p < \infty$ (see, e.g., [7], Theorem 3.1). It was proved by Buchholz that $K_{2n}^{2n} = (2n)!/(2^n n!)$ if $n \in \mathbb{N}$ ([3], Theorem 5 and the remark following it). From this, it follows that $K_{q,q} < \sqrt{q}$ if $q \geq 2$. Summarizing, if $2 \leq q < p < \infty$, then

$$K_{p,q} \leq \kappa_{p,q} K_{q,q} \leq (p-1)^{1/2}/(q-1)^{1/2} q^{1/2} \leq \sqrt{2}\sqrt{p-1}$$

and if $2 \leq p \leq q < \infty$, then $K_{p,q} \leq K_{q,q} < \sqrt{q}$.

In the proof of the next result, we use for $0 < q \leq 1$ and $\xi \in L^1(\Omega; L^q(\mathcal{M})_+)$,

$$\mathbb{E}\|\xi\|_{L^q(\mathcal{M})} \leq \|\mathbb{E}\xi\|_{L^q(\mathcal{M})}. \tag{5.6}$$

This follows by approximation by step functions using the inequality

$$\|x + y\|_{L^q(\mathcal{M})} \geq \|x\|_{L^q(\mathcal{M})} + \|y\|_{L^q(\mathcal{M})} \qquad \big(x, y \in L^q(\mathcal{M})_+\big).$$

LEMMA 5.4. *Let* (ξ_i) *be a finite sequence of independent*, *mean-zero* $L^q(\mathcal{M})$-*valued random variables*. *If* $1 \leq p, q < 2$, *then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \leq 4\inf\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\theta_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in S_{q,c}$ *and* $(\theta_i) \in S_{q,r}$ *such that* $\xi_i = \eta_i + \theta_i$. *On the other hand, if* $2 \leq p, q < \infty$, *then*

$$2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \geq \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big\}.$$

PROOF. Suppose $1 \leq p, q < 2$. Let (α_i) be a finite sequence in $S_{q,c}$ of independent, mean-zero $L^q(\mathcal{M})$-valued random variables. By symmetrization (2.2) and Theorem 5.2,

$$\begin{aligned}
\Big(\mathbb{E}\Big\|\sum_i \alpha_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} &\leq 2\Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i\alpha_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \\
&\leq 2\Big(\mathbb{E}\Big\|\Big(\sum_i |\alpha_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \\
&= 2\Big(\mathbb{E}\Big\|\sum_i |\alpha_i|^2\Big\|_{L^{q/2}(\mathcal{M})}^{p/2}\Big)^{1/p} \\
&\leq 2\Big(\mathbb{E}\Big\|\sum_i |\alpha_i|^2\Big\|_{L^{q/2}(\mathcal{M})}\Big)^{1/2} \\
&\leq 2\Big\|\sum_i \mathbb{E}|\alpha_i|^2\Big\|_{L^{q/2}(\mathcal{M})}^{1/2} = 2\Big\|\Big(\sum_i \mathbb{E}|\alpha_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}.
\end{aligned}$$

Note that in the final two inequalities we apply Jensen's inequality and (5.6), respectively, using that $\frac{p}{2}, \frac{q}{2} < 1$. Applying this for (α_i^*) yields

$$\Big(\mathbb{E}\Big\|\sum_i \alpha_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \leq \Big\|\Big(\sum_i \mathbb{E}|\alpha_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}.$$

Let (η_i) and (θ_i) be finite sequences in $S_{q,c}$ and $S_{q,r}$, respectively, such that $\xi_i = \eta_i + \theta_i$, then $\xi_i = \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)$. Since $(\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i))$ and $(\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i))$ are sequences of independent, mean-zero random variables, we obtain by the triangle inequality and the above,

$$\begin{aligned}
\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} &\leq 2\Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})} \\
&\quad + 2\Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\theta_i^*|\xi_i) - \mathbb{E}(\theta_i^*)|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}.
\end{aligned}$$

Therefore, by the triangle inequality in $S_{q,c}$ and $S_{q,r}$ we find

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \\
&\quad\le 2\Big(\Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i)|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i)|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})} \\
&\qquad\quad + \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\theta_i^*|\xi_i)|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\theta_i^*)|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big) \\
&\quad\le 4\Big(\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\theta_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big).
\end{aligned}$$

Note that the final step follows directly from Kadison's inequality for (noncommutative) conditional expectations if η_i, θ_i are, in addition, in $L^\infty \otimes \mathcal{M}$. For general η_i and θ_i as above, the asserted inequality then follows by a density argument. This proves the first statement.

Suppose now that $2 \le p, q < \infty$. By symmetrization (2.2) and Theorem 5.2,

$$\begin{aligned}
&2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \\
&\quad\ge \Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \\
&\quad\ge \max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p}, \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p}\Big\} \\
&\quad= \max\Big\{\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2\Big\|_{L^{q/2}(\mathcal{M})}^{p/2}\Big)^{1/p}, \Big(\mathbb{E}\Big\|\sum_i |\xi_i^*|^2\Big\|_{L^{q/2}(\mathcal{M})}^{p/2}\Big)^{1/p}\Big\} \\
&\quad\ge \max\Big\{\Big\|\sum_i \mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(\mathcal{M})}^{1/2}, \Big\|\sum_i \mathbb{E}|\xi_i^*|^2\Big\|_{L^{q/2}(\mathcal{M})}^{1/2}\Big\} \\
&\quad= \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big\}.
\end{aligned}$$

This completes the proof. □

For our discussion in Section 6, we will keep track of the dependence of the constants on p and q in the inequalities (5.7) and (5.8) below.

THEOREM 5.5. *Suppose that $2 \le p, q < \infty$. If (ξ_i) is a finite sequence of independent, mean-zero $L^q(\mathcal{M})$-valued random variables, then*

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{1/p} \\ \le C_{p,q}(1+\sqrt{2})\max\Big\{&\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}, \\ &C_{p/2,q/2}\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{p/q}\Big)^{1/p}\Big\},\end{aligned} \tag{5.7}$$

where $C_{p,q} = 2K_{p,q} < \max\{2\sqrt{2}\sqrt{p-1}, 2\sqrt{q}\}$ and $K_{p,q}$ is the constant in (5.5). *Moreover, if $\kappa_{p,q}$ is the constant in* (2.3) *then*

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{1/p} \\ \ge \frac{1}{2}\max\Big\{&(\kappa_{q,p})^{-1}\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{p/q}\Big)^{1/p}, \\ &\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big\}.\end{aligned} \tag{5.8}$$

PROOF. We first prove (5.8). By Lemma 5.4,

$$\max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}\Big\} \le 2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{1/p}.$$

By successively applying the cotype q inequality for $L^q(\mathcal{M})$, Kahane's inequalities (2.3) and (2.2), we see that

$$\begin{aligned}\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{p/q}\Big)^{1/p} &\le \Big(\mathbb{E}\Big(\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|^q_{L^q(\mathcal{M})}\Big)^{p/q}\Big)^{1/p} \\ &\le \kappa_{q,p}\Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{1/p} \\ &\le 2\kappa_{q,p}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{1/p}.\end{aligned} \tag{5.9}$$

We refer to [9] for a proof that (5.9) holds with constant 1.

We now prove (5.7). By (2.2) and Theorem 5.2, we have

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \\
&\quad\le 2K_{p,q}\max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p}, \\
&\qquad\qquad \Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p}\Big\}.
\end{aligned}\tag{5.10}$$

By the triangle inequality in $L^{p/2}(\Omega; L^{q/2}(\mathcal{M}))$, it follows that

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \\
&\quad=\Big(\mathbb{E}\Big\|\sum_i|\xi_i|^2\Big\|_{L^{q/2}(\mathcal{M})}^{p/2}\Big)^{1/p} \\
&\quad\le\Big(\Big(\mathbb{E}\Big\|\sum_i|\xi_i|^2-\mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(\mathcal{M})}^{p/2}\Big)^{2/p}+\Big\|\sum_i\mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(\mathcal{M})}\Big)^{1/2}.
\end{aligned}\tag{5.11}$$

We now estimate the first term on the far right-hand side. By applying (2.2) and Theorem 5.2 once again, we obtain

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i|\xi_i|^2-\mathbb{E}|\xi_i|^2\Big\|_{L^{q/2}(\mathcal{M})}^{p/2}\Big)^{2/p} \\
&\quad\le 2K_{p/2,q/2}\Big(\mathbb{E}\Big\|\Big(\sum_i\big||\xi_i|^2-\mathbb{E}|\xi_i|^2\big|^2\Big)^{1/2}\Big\|_{L^{q/2}(\mathcal{M})}^{p/2}\Big)^{2/p} \\
&\quad\le C_{p/2,q/2}\Big(\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^4\Big)^{1/2}\Big\|_{L^{q/2}(\mathcal{M})}^{p/2}\Big)^{2/p} \\
&\qquad\qquad+\Big\|\Big(\sum_i\big|\mathbb{E}|\xi_i|^2\big|^2\Big)^{1/2}\Big\|_{L^{q/2}(\mathcal{M})}\Big),
\end{aligned}\tag{5.12}$$

where the final inequality is a consequence of the triangle inequality in $L^{p/2}(\Omega; L^{q/2}(\mathcal{M}; \ell_c^2))$. Note that the second term on the right-hand side is smaller than the first one. Indeed,

$$\begin{aligned}
&\Big\|\Big(\sum_i\big|\mathbb{E}|\xi_i|^2\big|^2\Big)^{1/2}\Big\|_{L^{q/2}(\mathcal{M})} \\
&\quad=\big\|\mathrm{col}\big(\mathbb{E}|\xi_i|^2\big)\big\|_{L^{q/2}(\mathcal{M}\otimes B(\ell^2))}
\end{aligned}\tag{5.13}$$

$$
\begin{aligned}
&= \|\mathbb{E}(\mathrm{col}(|\xi_i|^2))\|_{L^{q/2}(\mathcal{M}\otimes B(\ell^2))} \\
&\leq \mathbb{E}\|\mathrm{col}(|\xi_i|^2)\|_{L^{q/2}(\mathcal{M}\otimes B(\ell^2))} \\
&\leq \big(\mathbb{E}\|\mathrm{col}(|\xi_i|^2)\|^{p/2}_{L^{q/2}(\mathcal{M}\otimes B(\ell^2))}\big)^{2/p} \\
&= \left(\mathbb{E}\left\|\left(\sum_i |\xi_i|^4\right)^{1/2}\right\|^{p/2}_{L^{q/2}(\mathcal{M})}\right)^{2/p}.
\end{aligned}
$$

Write $x = \mathrm{col}(|\xi_i|)$ and $y = \mathrm{diag}(|\xi_i|)$ for the matrices with the $|\xi_i|$ in their first column and diagonal, respectively, and zeroes elsewhere. By the noncommutative Hölder inequality (4.3),

$$
\begin{aligned}
&\left(\mathbb{E}\left\|\left(\sum_i |\xi_i|^4\right)^{1/2}\right\|^{p/2}_{L^{q/2}(\mathcal{M})}\right)^{2/p} \\
&\quad= \big(\mathbb{E}\|(x^*y^*yx)^{1/2}\|^{p/2}_{L^{q/2}(\mathcal{M}\otimes B(\ell^2))}\big)^{2/p} \\
&\quad= \big(\mathbb{E}\|yx\|^{p/2}_{L^{q/2}(\mathcal{M}\otimes B(\ell^2))}\big)^{2/p} \\
&\quad\leq \big(\mathbb{E}\big|\|y\|_{L^q(\mathcal{M}\otimes B(\ell^2))}\|x\|_{L^q(\mathcal{M}\otimes B(\ell^2))}\big|^{p/2}\big)^{2/p} \qquad (5.14)\\
&\quad\leq \big(\mathbb{E}\|y\|^p_{L^q(\mathcal{M}\otimes B(\ell^2))}\big)^{1/p}\big(\mathbb{E}\|x\|^p_{L^q(\mathcal{M}\otimes B(\ell^2))}\big)^{1/p} \\
&\quad= \left(\mathbb{E}\left(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\right)^{p/q}\right)^{1/p}\left(\mathbb{E}\left\|\left(\sum_i |\xi_i|^2\right)^{1/2}\right\|^p_{L^q(\mathcal{M})}\right)^{1/p}.
\end{aligned}
$$

Collecting our estimates (5.11), (5.12), (5.13) and (5.14), we find the quadratic inequality

$$
a^2 \leq (2C_{p/2,q/2})ab + c^2,
$$

where we set $a = (\mathbb{E}\|(\sum_i |\xi_i|^2)^{1/2}\|^p_{L^q(\mathcal{M})})^{1/p}$, $b = (\mathbb{E}(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})})^{p/q})^{1/p}$ and $c = \|(\sum_i \mathbb{E}|\xi_i|^2)^{1/2}\|_{L^q(\mathcal{M})}$. Solving this quadratic inequality, we obtain

$$
a \leq \tfrac{1}{2}\big(2C_{p/2,q/2}b + \big((2C_{p/2,q/2}b)^2 + 4c^2\big)^{1/2}\big) \leq \tfrac{1+\sqrt{2}}{2}\max\{2C_{p/2,q/2}b, 2c\},
$$

that is,

$$
\begin{aligned}
&\left(\mathbb{E}\left\|\left(\sum_i |\xi_i|^2\right)^{1/2}\right\|^p_{L^q(\mathcal{M})}\right)^{1/p} \\
&\quad\leq (1+\sqrt{2})\max\left\{\left\|\left(\sum_i \mathbb{E}|\xi_i|^2\right)^{1/2}\right\|_{L^q(\mathcal{M})},\right. \\
&\qquad\qquad\left. C_{p/2,q/2}\left(\mathbb{E}\left(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\right)^{p/q}\right)^{1/p}\right\}.
\end{aligned}
$$

Applying this to the sequence (ξ_i^*), we obtain

$$\begin{aligned}\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p}&\\ \leq (1+\sqrt{2})\max\Big\{&\Big\|\Big(\sum_i\mathbb{E}|\xi_i^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})},\\ &C_{p/2,q/2}\Big(\mathbb{E}\Big(\sum_i\|\xi_i\|_{L^q(\mathcal{M})}^q\Big)^{p/q}\Big)^{1/p}\Big\}.\end{aligned}$$

Inequality (5.7) now follows from (5.10). □

Note that even if $\mathcal{M}$ is commutative, the proof of Theorem 5.5 is different from the one presented for Theorem 2.3. We are now ready to prove Theorem 5.1.

PROOF OF THEOREM 5.1. Observe that the spaces $S_{q,c}$, $S_{q,r}$, $D_{p,q}$ and $D_{q,q}$ have dense intersection and, therefore, the duality of these individual spaces imply together with (2.10) that

$$(s_{p,q})^* = s_{p',q'}, \qquad \frac{1}{p}+\frac{1}{p'}=1, \qquad \frac{1}{q}+\frac{1}{q'}=1,$$

with associated duality bracket

$$\langle (f_i),(g_i)\rangle = \sum_i \mathbb{E}\tau(f_i g_i).$$

Thus, the lower estimates $\gtrsim_{p,q}$ in (5.4) can be deduced from the upper ones using the duality argument presented in (2.13) and (2.14).

The upper estimates $\lesssim_{p,q}$ follow essentially as in the proof of Theorem 1.2 once we replace the use of Lemma 2.1 and Theorem 2.3 by their noncommutative versions Lemma 5.4 and Theorem 5.5, respectively. The straightforward modifications are left to the reader. □

Before deducing Itô isomorphisms for Poisson stochastic integrals taking values in a noncommutative L^q-space from Theorem 5.1, we take the opportunity to observe some moment estimates for the norm of a sum of random matrices.

6. Intermezzo on random matrices. Let us recall the following noncommutative Khintchine inequality for the operator norm of a Rademacher sum of matrices. Let $d_1, d_2 \in \mathbb{N}$ and set $d = \min\{d_1, d_2\}$. If $x_1, \dots, x_n$ are $d_1 \times d_2$ random matrices, then there is a constant $C_{p,d}$ depending only on p and d such that

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n r_i x_i\Big\|^p\Big)^{1/p} \leq C_{p,d}\max\Big\{\Big\|\Big(\sum_{i=1}^n|x_i|^2\Big)^{1/2}\Big\|, \Big\|\Big(\sum_{i=1}^n|x_i^*|^2\Big)^{1/2}\Big\|\Big\}. \tag{6.1}$$

Indeed, this inequality can readily be deduced from the noncommutative Khintchine inequalities for Schatten spaces. Since $\|x\|_{\log d} \leq e\|x\| \leq e\|x\|_{\log d}$ for any $d_1 \times d_2$ matrix x,

$$\begin{aligned}
&\left(\mathbb{E}\left\|\sum_{i=1}^{n} r_i x_i\right\|^p\right)^{1/p} \\
&\quad \leq \left(\mathbb{E}\left\|\sum_{i=1}^{n} r_i x_i\right\|_{\log d}^p\right)^{1/p} \\
&\quad \leq K_{p,\log d} \max\left\{\left\|\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\right\|_{\log d}, \left\|\left(\sum_{i=1}^{n} |x_i^*|^2\right)^{1/2}\right\|_{\log d}\right\} \\
&\quad \leq eK_{p,\log d} \max\left\{\left\|\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\right\|, \left\|\left(\sum_{i=1}^{n} |x_i^*|^2\right)^{1/2}\right\|\right\}.
\end{aligned}$$

By the remark following Theorem 5.2, if $2 \leq \log d \leq p$ then

$$C_{p,d} \leq eK_{p,\log d} \leq e\sqrt{2}\sqrt{p-1}$$

and $C_{p,d} \leq e\sqrt{\log d}$ if $2 \leq p \leq \log d$.

REMARK 6.1. The Khintchine inequality (6.1) cannot hold with a constant independent of the dimensions d_1, d_2. Indeed, it was shown by Seginer ([37], Theorem 3.1) that there is an absolute constant C such that for any a_{ij}, $i = 1, \dots, d_1$ $j = 1, \dots, d_2$ in $\mathbb{C}$ and any $1 \leq p \leq 2\log\max\{d_1, d_2\}$ the rank one matrices $x_{ij} = a_{ij} \otimes e_{ij}$ satisfy

$$\begin{aligned}
&\left(\mathbb{E}\left\|\sum_{i,j} r_{ij} x_{ij}\right\|^p\right)^{1/p} \\
&\quad \leq C(\log d)^{1/4} \max\left\{\left\|\left(\sum_{i,j} |x_{ij}|^2\right)^{1/2}\right\|, \left\|\left(\sum_{i,j} |x_{ij}^*|^2\right)^{1/2}\right\|\right\}.
\end{aligned} \tag{6.2}$$

Moreover, the order of growth $(\log d)^{1/4}$ in (6.2) is optimal ([37], Theorem 3.2).

THEOREM 6.2. *Let $2 \leq p < \infty$. If (ξ_i) is a finite sequence of independent, mean-zero $d_1 \times d_2$ random matrices, then*

$$\begin{aligned}
&\left(\mathbb{E}\left\|\sum_{i} \xi_i\right\|^p\right)^{1/p} \\
&\quad \leq 2(1+\sqrt{2})C_{p,d} \max\left\{\left\|\left(\sum_{i} \mathbb{E}|\xi_i|^2\right)^{1/2}\right\|, \left\|\left(\sum_{i} \mathbb{E}|\xi_i^*|^2\right)^{1/2}\right\|,\right. \\
&\qquad\qquad \left. 2C_{p/2,d}\left(\mathbb{E}\max_i \|\xi_i\|^p\right)^{1/p}\right\},
\end{aligned}$$

where $d = \min\{d_1, d_2\}$. *The reverse inequality holds with constant* $2^{1+1/p}$.

PROOF. By repeating the proof of Theorem 5.5 using (6.1) instead of the noncommutative Khintchine inequality (5.5), we find

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p\Big)^{1/p} \\
&\quad \leq 2(1+\sqrt{2})C_{p,d}\max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\|, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{1/2}\Big\|, \\
&\qquad\qquad 2C_{p/2,d}\big(\mathbb{E}\|\operatorname{diag}(\xi_i)\|^p\big)^{1/p}\Big\}.
\end{aligned}$$

Clearly, $\|\operatorname{diag}(\xi_i)\| = \max_i \|\xi_i\|$, so the first assertion holds.

For the second assertion, let (r_i) be a Rademacher sequence on a probability space $(\Omega_r, \mathcal{F}_r, \mathbb{P}_r)$. Then

$$\begin{aligned}
\Big(\mathbb{E}\max_i \|\xi_i\|_X^p\Big)^{1/p} &= \Big(\mathbb{E}\mathbb{E}_r \max_i \|r_i\xi_i\|_X^p\Big)^{1/p} \\
&\leq 2^{1/p}\Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|_X^p\Big)^{1/p} \\
&\leq 2^{1+1/p}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{1/p},
\end{aligned}$$

where the first inequality follows by the Lévy–Octaviani inequality in [18], Proposition 1.1.1. Moreover,

$$\begin{aligned}
\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{1/2}\Big\| &= \Big\|\mathbb{E}\mathbb{E}_r \sum_{i,j} r_i r_j \xi_i^*\xi_j\Big\|^{1/2} \\
&\leq \Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_{i,j} r_i r_j \xi_i^*\xi_j\Big\|\Big)^{1/2} \\
&= \Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|^2\Big)^{1/2} \leq 2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p\Big)^{1/p},
\end{aligned}$$

where the final inequality follows from (2.2). $\square$

As a consequence, we find the following moment inequalities for the norm of a random matrix with independent, mean-zero entries.

COROLLARY 6.3. *Let* $2 \leq p < \infty$. *Suppose that* x_{ij}, $i = 1, \dots, d_1$, $j = 1, \dots, d_2$ *are independent, mean-zero random variables in* $L^p(\Omega)$. *If* x *is the*

$d_1 \times d_2$ random matrix (x_{ij}), then

$$
\begin{aligned}
&\big(\mathbb{E}\|x\|^p\big)^{1/p} \\
&\le 2(1+\sqrt{2})C_{p,d}\max\Biggl\{\max_{j=1,\dots,d_2}\Biggl(\sum_{i=1}^{d_1}\mathbb{E}x_{ij}^2\Biggr)^{1/2}, \max_{i=1,\dots,d_1}\Biggl(\sum_{j=1}^{d_2}\mathbb{E}x_{ij}^2\Biggr)^{1/2}, \\
&\qquad 2C_{p/2,d}\Bigl(\mathbb{E}\max_{i,j}|x_{ij}|^p\Bigr)^{1/p}\Biggr\},
\end{aligned} \tag{6.3}
$$

with $C_{p,d} < e\max\{\sqrt{\log d}, \sqrt{2}\sqrt{p-1}\}$ as in Theorem 6.2.

PROOF. Let e_{ij} be the $d_1 \times d_2$ matrix having 1 in entry (i, j) and zeroes elsewhere. Set $y_{ij} = x_{ij} \otimes e_{ij}$, then (y_{ij}) is a doubly indexed sequence of independent, mean-zero random matrices and $x = \sum_{i,j} y_{ij}$. Notice that

$$y_{ij}^* y_{ij} = x_{ij}^2 \otimes e_{ji}e_{ij} = x_{ij}^2 \otimes e_{jj},$$

so

$$\Biggl\|\Biggl(\sum_{i,j}\mathbb{E}|y_{ij}|^2\Biggr)^{1/2}\Biggr\| = \Biggl\|\sum_j\Biggl(\sum_i\mathbb{E}x_{ij}^2\Biggr)^{1/2}\otimes e_{jj}\Biggr\| = \max_j\Biggl(\sum_i\mathbb{E}x_{ij}^2\Biggr)^{1/2}.$$

Moreover,

$$y_{ij}y_{ij}^* = x_{ij}^2 \otimes e_{ij}e_{ji} = x_{ij}^2 \otimes e_{ii}$$

and, therefore,

$$\Biggl\|\Biggl(\sum_{i,j}\mathbb{E}|y_{ij}^*|^2\Biggr)^{1/2}\Biggr\| = \Biggl\|\sum_i\Biggl(\sum_j\mathbb{E}x_{ij}^2\Biggr)^{1/2}\otimes e_{ii}\Biggr\| = \max_i\Biggl(\sum_j\mathbb{E}x_{ij}^2\Biggr)^{1/2}.$$

Finally, it is clear that

$$\Bigl(\mathbb{E}\max_{i,j}\|y_{ij}\|^p\Bigr)^{1/p} = \Bigl(\mathbb{E}\max_{i,j}|x_{ij}|^p\Bigr)^{1/p}.$$

The result now follows from Theorem 6.2. □

In [19], Latała showed that there is a universal constant $C > 0$ such that

$$
\begin{aligned}
\mathbb{E}\|x\| \le C\Biggl(&\max_{i=1,\dots,d_1}\Biggl(\sum_{j=1}^{d_2}\mathbb{E}x_{ij}^2\Biggr)^{1/2} + \max_{j=1,\dots,d_2}\Biggl(\sum_{i=1}^{d_1}\mathbb{E}x_{ij}^2\Biggr)^{1/2} \\
&+ \Biggl(\sum_{i,j}\mathbb{E}x_{ij}^4\Biggr)^{1/4}\Biggr)
\end{aligned} \tag{6.4}
$$

for any random matrix $x=(x_{ij})$ with independent, mean-zero entries in $L^4(\Omega)$. To compare this result to Corollary 6.3, observe that (6.4) implies together with (2.11) that there is a universal constant $C>0$ such that for all $1\leq p<\infty$,

$$(6.5)\quad \begin{aligned}\big(\mathbb{E}\|x\|^p\big)^{1/p} \leq C\frac{p}{\log p}\Bigg(&\max_{i=1,\dots,d_1}\Bigg(\sum_{j=1}^{d_2}\mathbb{E}x_{ij}^2\Bigg)^{1/2}+\max_{j=1,\dots,d_2}\Bigg(\sum_{i=1}^{d_1}\mathbb{E}x_{ij}^2\Bigg)^{1/2}\\ &+\Bigg(\sum_{i,j}\mathbb{E}x_{ij}^4\Bigg)^{1/4}+\Big(\mathbb{E}\max_{i,j}|x_{ij}|^p\Big)^{1/p}\Bigg).\end{aligned}$$

The upper bound in Corollary 6.3 exhibits different growth behavior in p and does not contain the factor $(\sum_{i,j}\mathbb{E}x_{ij}^4)^{1/4}$. In particular, the bound (6.3) is applicable to random matrices having entries with infinite fourth moment. On the other hand, note that the bound in (6.5) is of order $\sqrt{d}$ for matrices with uniformly bounded entries, which is optimal for $d\to\infty$ (see the discussion in [19]). Through the use of the noncommutative Khintchine inequality in our proof, we incur an extra factor of order $\sqrt{\log d}$. As the order $(\log d)^{1/4}$ of the constant in (6.2) is optimal, this additional factor is an inevitable product of our method.

7. Itô-isomorphisms: Noncommutative L^q-spaces. We now present an extension of Theorem 1.1 for integrands taking values in a noncommutative L^q-space. In the statement of our main result, we will use the following noncommutative L^2-valued L^q-spaces, which were introduced by Pisier in [31] and treated in more detail in [15]. For any simple function on a measure space $(E,\mathcal{E},\mu)$ with values in $L^q(\mathcal{M})$, $F=\sum_i \chi_{E_i}x_i$ say, we set

$$\|F\|_{L^q(\mathcal{M};L^2(\mathbb{R}_+\times J)_c)}=\Bigg\|\Bigg(\sum_i|x_i|^2\mu(E_i)\Bigg)^{1/2}\Bigg\|_{L^q(\mathcal{M})},$$

$$\|F\|_{L^q(\mathcal{M};L^2(\mathbb{R}_+\times J)_r)}=\Bigg\|\Bigg(\sum_i|x_i^*|^2\mu(E_i)\Bigg)^{1/2}\Bigg\|_{L^q(\mathcal{M})}.$$

It can be shown that these expression define two norms on the simple functions, and we let $L^q(\mathcal{M};L^2(E)_c)$ and $L^q(\mathcal{M};L^2(E)_r)$ denote the respective completions in these norms. Alternatively, one can describe these spaces as complemented subspaces of $L^q(\mathcal{M}\otimes B(L^2(E)))$ and in this way one can show that for $1<q,q'<\infty$ with $\frac{1}{q}+\frac{1}{q'}=1$,

$$(7.1)\quad \begin{aligned}\big(L^q(\mathcal{M};L^2(E)_c)\big)^*&=L^{q'}(\mathcal{M};L^2(E)_r),\\ \big(L^q(\mathcal{M};L^2(E)_r)\big)^*&=L^{q'}(\mathcal{M};L^2(E)_c).\end{aligned}$$

We refer to Chapter 2 of [15] for details. Now, for any $1\leq p,q<\infty$ we set

$$\mathcal{S}_{q,c}^p=L^p(\Omega;L^q(\mathcal{M};L^2(\mathbb{R}_+\times J)_c)),\qquad \mathcal{S}_{q,r}^p=L^p(\Omega;L^q(\mathcal{M};L^2(\mathbb{R}_+\times J)_r)).$$

Since $L^q(\mathcal{M};L^2(\mathbb{R}_+\times J)_c)$ and $L^q(\mathcal{M};L^2(\mathbb{R}_+\times J)_r)$ can be identified with closed subspaces of $L^q(\mathcal{M}\otimes B(L^2(\mathbb{R}_+\times J)))$, they are reflexive if $1<q<\infty$. Therefore, it follows from (7.1) that for any $1<p,q<\infty$,

$$(\mathcal{S}^p_{q,c})^* = \mathcal{S}^{p'}_{q',r}, \qquad (\mathcal{S}^p_{q,r})^* = \mathcal{S}^{p'}_{q',c}, \qquad \left(\frac{1}{p}+\frac{1}{p'}=1, \frac{1}{q}+\frac{1}{q'}=1\right). \tag{7.2}$$

If $\mathcal{M}$ is commutative, then $\mathcal{S}^p_{q,c}$ and $\mathcal{S}^p_{q,r}$ coincide and are equal to the Bochner space $S^p_q = L^p(\Omega; L^q(S; L^2(\mathbb{R}_+\times J)))$ considered earlier.

We are now ready to prove our main theorem.

THEOREM 7.1. *Let $1<p,q<\infty$. For any $B\in\mathcal{J}$, any $t>0$ and any simple, adapted $L^q(\mathcal{M})$-valued process F,*

$$\left(\mathbb{E}\sup_{0<s\le t}\left\|\int_{(0,s]\times B} F\,d\tilde{N}\right\|^p_{L^q(\mathcal{M})}\right)^{1/p} \simeq_{p,q} \|F\chi_{(0,t]\times B}\|_{\mathcal{I}_{p,q}}, \tag{7.3}$$

where $\mathcal{I}_{p,q}$ is given by

$$\begin{array}{ll} \mathcal{S}^p_{q,c}\cap\mathcal{S}^p_{q,r}\cap\mathcal{D}^p_{q,q}\cap\mathcal{D}^p_{p,q} & \text{if } 2\le q\le p<\infty, \\ \mathcal{S}^p_{q,c}\cap\mathcal{S}^p_{q,r}\cap(\mathcal{D}^p_{q,q}+\mathcal{D}^p_{p,q}) & \text{if } 2\le p\le q<\infty, \\ (\mathcal{S}^p_{q,c}\cap\mathcal{S}^p_{q,r}\cap\mathcal{D}^p_{q,q})+\mathcal{D}^p_{p,q} & \text{if } 1<p<2\le q<\infty, \\ (\mathcal{S}^p_{q,c}+\mathcal{S}^p_{q,r}+\mathcal{D}^p_{q,q})\cap\mathcal{D}^p_{p,q} & \text{if } 1<q<2\le p<\infty, \\ \mathcal{S}^p_{q,c}+\mathcal{S}^p_{q,r}+(\mathcal{D}^p_{q,q}\cap\mathcal{D}^p_{p,q}) & \text{if } 1<q\le p\le 2, \\ \mathcal{S}^p_{q,c}+\mathcal{S}^p_{q,r}+\mathcal{D}^p_{q,q}+\mathcal{D}^p_{p,q} & \text{if } 1<p\le q\le 2. \end{array}$$

PROOF. The proof is similar to the one for Theorem 1.1, we sketch the main differences in the cases $2\le q\le p<\infty$ and $1<p\le q\le 2$. Since $L^q(\mathcal{M})$ is a UMD space if $1<q<\infty$, by the decoupling inequality (1.4) and Doob's maximal inequality it suffices to show that

$$\left(\mathbb{E}\mathbb{E}_c\left\|\int_{(0,t]\times B} F\,d\tilde{N}^c\right\|^p_{L^q(\mathcal{M})}\right)^{1/p} \simeq_{p,q} \|F\chi_{(0,t]\times B}\|_{\mathcal{I}_{p,q}}.$$

Let F be the simple adapted process given in (3.1), taking Remark 3.5 into account. We may assume that $t=t_{l+1}$ and $B=\bigcup_{j=1}^m A_j$. We write $\tilde{N}^c_{i,j} := \tilde{N}^c((t_i,t_{i+1}]\times A_j)$ for brevity.

Case $2\le q\le p<\infty$: Set $y_{i,j}=\sum_{k=1}^n F_{i,j,k}x_{i,j,k}$ and $d_{i,j}=y_{i,j}\tilde{N}^c_{i,j}$, then clearly

$$\int_{(0,t]\times B} F\,d\tilde{N}^c = \sum_{i,j} d_{i,j}. \tag{7.4}$$

Moreover, for every fixed $\omega \in \Omega$ the random variables $d_{i,j}(\omega)$ are independent and mean-zero. Therefore, we can apply Theorem 5.1 pointwise in Ω and subsequently take the $L^p(\Omega)$-norm on both sides to obtain

$$\begin{aligned}
&\Big(\mathbb{E}\mathbb{E}_c\Big\|\sum_{i,j} d_{i,j}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p} \\
&\quad\lesssim_{p,q} \max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j}|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p}, \\
&\qquad\qquad \Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j}^*|^2\Big)^{1/2}\Big\|_{L^q(\mathcal{M})}^p\Big)^{1/p}, \\
&\qquad\qquad \Big(\mathbb{E}\Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j}\|_{L^q(\mathcal{M})}^q\Big)^{p/q}\Big)^{1/p}, \Big(\sum_{i,j}\mathbb{E}\mathbb{E}_c\|d_{i,j}\|_{L^q(\mathcal{M})}^p\Big)^{1/p}\Big\} \\
&\quad\simeq_{p,q} \max\big\{\|F\|_{\mathcal{S}_{q,c}^p}, \|F\|_{\mathcal{S}_{q,r}^p}, \|F\|_{\mathcal{D}_{q,q}^p}, \|F\|_{\mathcal{D}_{p,q}^p}\big\},
\end{aligned}$$

where the final step follows by calculations analogous to (3.7), (3.8) and (3.9).

Case $1 < p \leq q \leq 2$: Let $\mathcal{I}_{\mathrm{elem}}$ denote the algebraic tensor product

$$\mathcal{I}_{\mathrm{elem}} = L^\infty(\Omega) \otimes L^\infty(\mathbb{R}_+) \otimes \big(L^1 \cap L^\infty\big)(\mathcal{J}) \otimes \big(L^1 \cap L^\infty\big)(\mathcal{M}).$$

Since this linear space is dense in $\mathcal{S}_{q,c}^p$, $\mathcal{S}_{q,r}^p$, $\mathcal{D}_{p,q}^p$ and $\mathcal{D}_{q,q}^p$, we can find, for any fixed $\varepsilon > 0$, a decomposition $F = F_1 + F_2 + F_3 + F_4$ with $F_\alpha \in \mathcal{I}_{\mathrm{elem}}$ such that

$$\|F_1\|_{\mathcal{S}_{q,c}^p} + \|F_2\|_{\mathcal{S}_{q,r}^p} + \|F_3\|_{\mathcal{D}_{p,q}^p} + \|F_4\|_{\mathcal{D}_{q,q}^p} \leq \|F\|_{\mathcal{I}_{p,q}} + \varepsilon.$$

We may assume that the F_α have the same support in $\mathbb{R}_+ \times J$ as F. Let $\mathcal{A}$ be the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+) \times \mathcal{J}$ generated by the sets $(t_i, t_{i+1}] \times A_j$. By Lemma 3.6 $\mathbb{E}(F_\alpha|\mathcal{A})$ is of the form

$$\mathbb{E}(F_\alpha|\mathcal{A}) = \sum_{i,j,k} F_{i,j,k,\alpha}\chi_{(t_i,t_{i+1}]}\chi_{A_j} x_{i,j,k,\alpha} \qquad (\alpha = 1, 2, 3, 4).$$

Let $y_{i,j,\alpha} = \sum_{k=1}^n F_{i,j,k,\alpha} x_{i,j,k,\alpha}$ and set $d_{i,j,\alpha} = y_{i,j,\alpha}\tilde{N}_{i,j}^c$, then (7.4) holds and

$$d_{i,j} = d_{i,j,1} + d_{i,j,2} + d_{i,j,3} + d_{i,j,4}.$$

By computations similar to (3.7), (3.8) and (3.9),

$$\begin{aligned}
\big\|(d_{i,j,1})\big\|_{S_{q,c}^p} &= \big\|\mathbb{E}(F_1|\mathcal{A})\big\|_{\mathcal{S}_{q,c}^p} \leq \|F_1\|_{\mathcal{S}_{q,c}^p}, \\
\big\|(d_{i,j,2})\big\|_{S_{q,r}^p} &= \big\|\mathbb{E}(F_2|\mathcal{A})\big\|_{\mathcal{S}_{q,r}^p} \leq \|F_2\|_{\mathcal{S}_{q,r}^p}, \\
\big\|(d_{i,j,3})\big\|_{D_{p,q}^p} &\simeq_p \big\|\mathbb{E}(F_3|\mathcal{A})\big\|_{\mathcal{D}_{p,q}^p} \leq \|F_3\|_{\mathcal{D}_{p,q}^p}, \\
\big\|(d_{i,j,4})\big\|_{D_{q,q}^p} &\simeq_q \big\|\mathbb{E}(F_4|\mathcal{A})\big\|_{\mathcal{D}_{q,q}^p} \leq \|F_4\|_{\mathcal{D}_{q,q}^p}.
\end{aligned}$$

By applying Theorem 5.1 pointwise in Ω and subsequently taking $L^p(\Omega)$-norms on both sides, we conclude that

$$\Big(\mathbb{E}\mathbb{E}_c\Big\|\int_{(0,t]\times B} F\,d\tilde{N}^c\Big\|^p_{L^q(\mathcal{M})}\Big)^{1/p} \lesssim_{p,q} \|F_1\|_{\mathcal{S}^p_{q,c}} + \|F_2\|_{\mathcal{S}^p_{q,r}} + \|F_3\|_{\mathcal{D}^p_{p,q}} + \|F_4\|_{\mathcal{D}^p_{q,q}} \leq \|F\|_{\mathcal{I}_{p,q}} + \varepsilon.$$

For the reverse estimate, observe that if p', q' are the Hölder conjugates of p and q, then in view of (7.2) and (2.10), we have $\mathcal{I}^*_{p,q} = \mathcal{I}_{p',q'}$, with associated duality bracket

$$\langle F, G\rangle = \int_{\Omega\times\mathbb{R}_+\times J} \tau(FG)\,d\mathbb{P}\,dt\,d\nu.$$

The reverse inequality can therefore be deduced using the duality argument (3.11) explained in the proof of Theorem 1.1. $\square$

Let us make a detailed comparison of our main result with the existing results in the literature. We restrict our attention to [2, 11, 23, 24] and refer to the references in these papers for earlier achievements. In [23], Marinelli, Prévôt and Röckner showed using Itô's formula that if H is a Hilbert space and $2 \leq p < \infty$, then

$$\begin{aligned}&\Big(\mathbb{E}\sup_{0<s\leq t}\Big\|\int_{(0,s]\times B} F\,d\tilde{N}\Big\|^p_H\Big)^{1/p} \\ &\quad\lesssim_{p,t} \Big(\mathbb{E}\int_{(0,t]}\Big(\int_B \|F\|^2_H\,d\nu\Big)^{p/2} dt\Big)^{1/p} + \Big(\mathbb{E}\int_{(0,t]\times B}\|F\|^p_H\,dt\,d\nu\Big)^{1/p}.\end{aligned}\tag{7.5}$$

Due to the first term on the right-hand side, this estimate is only near-optimal. Indeed, since

$$\Big(\mathbb{E}\Big(\int_{(0,t]\times B}\|F\|^2_H\,d\nu\Big)^{p/2} dt\Big)^{1/p} \leq t^{1/2-1/p}\Big(\mathbb{E}\int_{(0,t]}\Big(\int_B\|F\|^2_H\,d\nu\Big)^{p/2} dt\Big)^{1/p},$$

Theorem 7.1 implies (7.5) but not vice versa. In [24], Marinelli and Röckner proved the bound

$$\begin{aligned}&\Big(\mathbb{E}\sup_{0<s\leq t}\Big\|\int_{(0,s]\times B} F\,d\tilde{N}\Big\|^p_{L^p(S)}\Big)^{1/p} \\ &\quad\lesssim_{p,t} \Big(\mathbb{E}\int_{(0,t]}\Big(\int_B \|F\|^2_{L^p(S)}\,d\nu\Big)^{p/2} dt\Big)^{1/p} \\ &\qquad + \Big(\mathbb{E}\int_{(0,t]\times B}\|F\|^p_{L^p(S)}\,dt\,d\nu\Big)^{1/p},\end{aligned}\tag{7.6}$$

valid for any $2 \leq p < \infty$. This result is deduced by a Fubini-type argument from the estimate (7.5) for $H = \mathbb{R}$. Of course, such an argument can only work if

$p = q$ (in our notation). Observe that the optimal bound in Theorem 7.1 improves upon (7.6). Also note that the constants in (7.3) do not depend on t, in contrast to (7.5) and (7.6). Finally, let us recall the following bounds valid for a Banach space X with martingale type $1 < q \leq 2$. Brzeźniak and Hausenblas showed ([2], Corollary B.6) that if $1 < p \leq q$ then

$$\left(\mathbb{E} \sup_{0<s\leq t} \left\| \int_{(0,s]\times B} F\, d\tilde{N} \right\|_X^p \right)^{1/p} \lesssim_{p,q,X} \left(\mathbb{E}\left(\int_{(0,t]\times B} \|F\|_X^q \, dt\, d\nu\right)^{p/q}\right)^{1/p}. \tag{7.7}$$

Moreover, Hausenblas proved ([11], Proposition 2.14) that if $p = q^n$ for some $n \in \mathbb{N}$, then

$$\begin{aligned} &\left(\mathbb{E} \sup_{0<s\leq t} \left\| \int_{(0,s]\times B} F\, d\tilde{N} \right\|_X^p \right)^{1/p} \\ &\quad \lesssim_{p,q,X} \left(\mathbb{E}\left(\int_{(0,t]\times B} \|F\|_X^q \, dt\, d\nu\right)^{p/q}\right)^{1/p} + \left(\mathbb{E} \int_{(0,t]\times B} \|F\|_X^p \, dt\, d\nu\right)^{1/p}. \end{aligned} \tag{7.8}$$

If $X = L^2(\mathcal{M})$, so that $q = 2$, and $p = 2^n$ then (7.8) reproduces the optimal upper bound in Theorem 7.1. In all other cases, however, both (7.7) and (7.8) yield suboptimal bounds for L^q-spaces.

APPENDIX A: DECOUPLING

In this appendix, we give a proof of the decoupling inequality (1.4). Recall that a Banach space X is called a *UMD space* if for some (then, every) $1 < p < \infty$ there is a constant $C_{p,X} \geq 0$ such that for any X-valued martingale difference sequence $(d_n)_{n\geq 1}$, any sequence of signs $(\varepsilon_n)_{n\geq 1}$ and any $N \geq 1$ one has

$$\left(\mathbb{E}\left\| \sum_{n=1}^N \varepsilon_n d_n \right\|_X^p\right)^{1/p} \leq C_{p,X} \left(\mathbb{E}\left\| \sum_{n=1}^N d_n \right\|_X^p\right)^{1/p}. \tag{A.1}$$

It is well known that any L^q-space, classical or noncommutative, is a UMD space if and only if $1 < q < \infty$. We refer to [4] for more information on UMD spaces.

The decoupling inequality (1.4) is a direct consequence of the following observation. For the convenience of the reader, we reproduce its short proof, which appeared in [38], Theorem 2.4.1 (see also [26], Theorem 13.1).

LEMMA A.1. *Let $1 < p < \infty$ and let X be a UMD Banach space. Consider a filtration $(\mathcal{G}_i)_{i=0}^n$ in $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that for every $1 \leq i \leq n$ we are given a $\mathcal{G}_i$-measurable, mean-zero, real-valued random variable M_i which is independent of $\mathcal{G}_{i-1}$ and, moreover, a $\mathcal{G}_{i-1}$-measurable, X-valued random variable G_i. Let $(M_i^c)_{i=1}^n$ be an independent copy of $(M_i)_{i=1}^n$ on a probability space $(\Omega_c, \mathcal{F}_c, \mathbb{P}_c)$. Then*

$$\left(\mathbb{E}\left\| \sum_{i=1}^n G_i M_i \right\|_X^p\right)^{1/p} \leq C_{p,X} \left(\mathbb{E}\mathbb{E}_c\left\| \sum_{i=1}^n G_i M_i^c \right\|_X^p\right)^{1/p}. \tag{A.2}$$

PROOF. For $i = 1, \dots, n$ let $\mathcal{G}_i^c$ be the sub-σ-algebra generated by $(M_j^c)_{j=1}^i$. Define

$$d_{2i} = \tfrac{1}{2} G_i(M_i - M_i^c), \qquad d_{2i-1} = \tfrac{1}{2} G_i(M_i + M_i^c).$$

We claim that $(d_i)_{i=1}^{2n}$ is a martingale difference sequence on $\Omega \times \Omega_c$ with respect to the filtration $(\mathcal{F}_i)_{i=1}^{2n}$ defined by

$$\mathcal{F}_{2i} = \sigma(\mathcal{G}_i, \mathcal{G}_i^c), \qquad \mathcal{F}_{2i-1} = \sigma(\mathcal{G}_{i-1}, \mathcal{G}_{i-1}^c, M_i + M_i^c) \qquad (i = 1, \dots, n).$$

The result immediately follows from this claim and the UMD-property, since

$$\sum_{i=1}^{2n} d_i = \sum_{i=1}^{n} G_i M_i, \qquad \sum_{i=1}^{2n} (-1)^{i+1} d_i = \sum_{i=1}^{n} G_i M_i^c.$$

To prove the claim, note that $(d_i)_{i=1}^{2n}$ is adapted. Moreover, by our assumptions on the G_i and M_i,

$$\mathbb{E}(d_{2i-1} | \mathcal{F}_{2i-2}) = \tfrac{1}{2} G_i \mathbb{E}(M_i + M_i^c | \mathcal{G}_{i-1}, \mathcal{G}_{i-1}^c) = 0$$

and

$$\begin{aligned}\mathbb{E}(d_{2i} | \mathcal{F}_{2i-1}) &= \tfrac{1}{2} G_i \mathbb{E}(M_i - M_i^c | \mathcal{G}_{i-1}, \mathcal{G}_{i-1}^c, M_i + M_i^c) \\ &= \tfrac{1}{2} G_i \mathbb{E}(M_i - M_i^c | M_i + M_i^c) = 0,\end{aligned}$$

where the final step follows from a direct computation, using that M_i and M_i^c are independent and identically distributed. $\square$

LEMMA A.2. *Let $1 < p < \infty$ and let X be a UMD Banach space. Let N be a Poisson random measure on $\mathbb{R}_+ \times J$ and let N^c be an independent copy of N. Fix a filtration $(\mathcal{F}_t)_{t>0}$ in Ω satisfying Assumption 3.2. If F is a simple, adapted X-valued process, then for all $t > 0$ and $B \in \mathcal{J}$,*

$$\left(\mathbb{E}\left\| \int_{(0,t]\times B} F \, d\tilde{N} \right\|_X^p\right)^{1/p} \le C_{p,X} \left(\mathbb{E}\mathbb{E}_c \left\| \int_{(0,t]\times B} F \, d\tilde{N}^c \right\|_X^p\right)^{1/p}, \tag{A.3}$$

where $C_{p,X}$ is the constant in (A.1).

PROOF. Let F be the simple adapted process in (3.1). We may assume that $t = t_{l+1}$ and $B = \bigcup_{j=1}^m A_j$. For every $1 \le i \le l$ and $1 \le j \le m$, we set

$$G_{(i,j)} = \sum_{k=1}^{n} F_{i,j,k} x_{i,j,k}, \qquad M_{(i,j)} = \tilde{N}((t_i, t_{i+1}] \times A_j),$$

$$M_{(i,j)}^c = \tilde{N}^c((t_i, t_{i+1}] \times A_j).$$

Under Assumption 3.2, the subalgebras defined for $i = 1, \dots, l$ and $j = 1, \dots, m$ by

$$\mathcal{G}_{(i,j)} = \sigma\big(\mathcal{F}_{t_i}, \tilde{N}\big((t_i, t_{i+1}] \times A_k\big), k = 1, \dots, j\big) \qquad \text{if } 1 \le j \le m-1,$$
$$\mathcal{G}_{(i,m)} = \mathcal{F}_{t_{i+1}}$$

form a filtration if we equip the pairs (i, j) with the lexicographic ordering. Moreover, the sequences $(G_{(i,j)})_{(i,j)}$, $(M_{(i,j)})_{(i,j)}$, and $(M^c_{(i,j)})_{(i,j)}$ satisfy the conditions of Lemma A.1 and inequality (A.3) exactly corresponds to the estimate (A.2). □

APPENDIX B: PROOF OF THEOREM 1.1: REMAINING CASES

For completeness, we give a proof here of the remaining cases of Theorem 1.1. We continue to use the same notation, in particular $\mathcal{I}_{\text{elem}}$ is the space of all simple functions on $\Omega \times \mathbb{R}_+ \times J \times S$ with support of finite measure and $\mathcal{A}$ denotes the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+) \times \mathcal{J}$ generated by the sets $(t_i, t_{i+1}] \times A_j$. Let us note that it suffices to prove the upper estimates $\lesssim_{p,q}$ in (1.3). The reverse estimates then follow by the duality argument presented in the case $1 < p \le q \le 2$.

Case $2 \le p \le q \le 2$: Fix $\varepsilon > 0$. By density of $\mathcal{I}_{\text{elem}}$ in $\mathcal{D}^p_{p,q}$ and $\mathcal{D}^p_{q,q}$, we can find a decomposition $F = F_1 + F_2$ with $F_\alpha \in \mathcal{I}_{\text{elem}}$ for $\alpha = 1, 2$ such that

$$\|F_1\|_{\mathcal{D}^p_{p,q}} + \|F_2\|_{\mathcal{D}^p_{q,q}} \le \|F\|_{\mathcal{D}^p_{p,q} + \mathcal{D}^p_{q,q}} + \varepsilon.$$

We may assume that F_1 and F_2 have the same support in $\mathbb{R}_+ \times J$ as F. By Lemma 3.6 $\mathbb{E}(F_\alpha|\mathcal{A})$ is of the form

$$\mathbb{E}(F_\alpha|\mathcal{A}) = \sum_{i,j,k} F_{i,j,k,\alpha} \chi_{(t_i,t_{i+1}]} \chi_{A_j} x_{i,j,k,\alpha} \qquad (\alpha = 1, 2). \tag{B.1}$$

Let $y_{i,j,\alpha} = \sum_{k=1}^n F_{i,j,k,\alpha} x_{i,j,k,\alpha}$ and set $d_{i,j,\alpha} = y_{i,j,\alpha} N^c_{i,j}$, so that

$$d_{i,j} = d_{i,j,1} + d_{i,j,2}.$$

If we apply Theorem 1.2 pointwise in Ω and subsequently take $L^p(\Omega)$-norms on both sides, we find

$$\begin{aligned}
&\bigg(\mathbb{E}\mathbb{E}_c\bigg\|\int_{(0,t]\times B} F\,d\tilde{N}^c\bigg\|^p_{L^q(S)}\bigg)^{1/p} \\
&\quad = \bigg(\mathbb{E}\mathbb{E}_c\bigg\|\sum_{i,j} d_{i,j}\bigg\|^p_{L^q(S)}\bigg)^{1/p} \\
&\quad \lesssim_{p,q} \max\bigg\{\bigg(\mathbb{E}\bigg\|\bigg(\sum_{i,j} \mathbb{E}_c|d_{i,j}|^2\bigg)^{1/2}\bigg\|^p_{L^q(S)}\bigg)^{1/p}, \\
&\qquad\qquad \bigg(\sum_{i,j} \mathbb{E}\mathbb{E}_c\|d_{i,j,1}\|^p_{L^q(S)}\bigg)^{1/p} + \bigg(\mathbb{E}\bigg(\sum_{i,j} \mathbb{E}_c\|d_{i,j,2}\|^q_{L^q(S)}\bigg)^{p/q}\bigg)^{1/p}\bigg\} \\
&\quad \lesssim_{p,q} \max\big\{\|F\|_{\mathcal{S}^p_q}, \|F_1\|_{\mathcal{D}^p_{p,q}} + \|F_2\|_{\mathcal{D}^p_{q,q}}\big\} \le \|F\|_{\mathcal{I}_{p,q}} + \varepsilon,
\end{aligned}$$

where the penultimate inequality follows by the computations in (3.10).

Case $1 < p < 2 \le q < \infty$: Fix $\varepsilon > 0$. By density of $\mathcal{I}_{\mathrm{elem}}$ in $\mathcal{D}^p_{p,q}$ and $\mathcal{S}^p_q \cap \mathcal{D}^p_{q,q}$, we can find a decomposition $F = F_1 + F_2$ with $F_\alpha \in \mathcal{I}_{\mathrm{elem}}$ for $\alpha = 1, 2$ such that

$$\|F_1\|_{\mathcal{D}^p_{p,q}} + \|F_2\|_{\mathcal{S}^p_q \cap \mathcal{D}^p_{q,q}} \le \|F\|_{\mathcal{I}_{p,q}} + \varepsilon.$$

We may assume that F_1 and F_2 have the same support in $\mathbb{R}_+ \times J$ as F. By Lemma 3.6, $\mathbb{E}(F_\alpha|\mathcal{A})$ is of the form (B.1). Let $y_{i,j,\alpha} = \sum_{k=1}^n F_{i,j,k,\alpha} x_{i,j,k,\alpha}$ and set $d_{i,j,\alpha} = y_{i,j,\alpha} N^c_{i,j}$, so that

$$d_{i,j} = d_{i,j,1} + d_{i,j,2}.$$

We apply Theorem 1.2 pointwise in Ω and subsequently take $L^p(\Omega)$-norms on both sides to find

$$\begin{aligned}
&\Big(\mathbb{E}\mathbb{E}_c \Big\| \int_{(0,t]\times B} F\, d\tilde{N}^c \Big\|^p_{L^q(S)}\Big)^{1/p} \\
&\quad = \Big(\mathbb{E}\mathbb{E}_c \Big\| \sum_{i,j} d_{i,j} \Big\|^p_{L^q(S)}\Big)^{1/p} \\
&\quad \lesssim_{p,q} \Big(\sum_{i,j} \mathbb{E}\mathbb{E}_c \|d_{i,j,1}\|^p_{L^q(S)}\Big)^{1/p} \\
&\qquad\qquad + \max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j,2}|^2\Big)^{1/2}\Big\|^p_{L^q(S)}\Big)^{1/p}, \\
&\qquad\qquad\qquad \Big(\mathbb{E}\Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j,2}\|^q_{L^q(S)}\Big)^{p/q}\Big)^{1/p}\Big\} \\
&\quad \lesssim_{p,q} \|F_1\|_{\mathcal{D}^p_{p,q}} + \max\{\|F_2\|_{\mathcal{S}^p_q}, \|F_2\|_{\mathcal{D}^p_{q,q}}\} \le \|F\|_{\mathcal{I}_{p,q}} + \varepsilon,
\end{aligned}$$

where the penultimate inequality follows by (3.10).

Case $1 < q < 2 \le p < \infty$: Let $\varepsilon > 0$. By density of $\mathcal{I}_{\mathrm{elem}}$ in $\mathcal{S}^p_q$ and $\mathcal{D}^p_{q,q}$, we can find a decomposition $F = F_1 + F_2$ with $F_\alpha \in \mathcal{I}_{\mathrm{elem}}$ for $\alpha = 1, 2$ such that

$$\|F_1\|_{\mathcal{S}^p_q} + \|F_2\|_{\mathcal{D}^p_{q,q}} \le \|F\|_{\mathcal{S}^p_q + \mathcal{D}^p_{q,q}} + \varepsilon.$$

We may assume that F_1 and F_2 have the same support in $\mathbb{R}_+ \times J$ as F. By Lemma 3.6 $\mathbb{E}(F_\alpha|\mathcal{A})$ is of the form (B.1). Let $y_{i,j,\alpha} = \sum_{k=1}^n F_{i,j,k,\alpha} x_{i,j,k,\alpha}$ and set $d_{i,j,\alpha} = y_{i,j,\alpha} N^c_{i,j}$, so that

$$d_{i,j} = d_{i,j,1} + d_{i,j,2}.$$

We apply Theorem 1.2 pointwise in Ω and subsequently take $L^p(\Omega)$-norms on both sides to obtain

$$\begin{aligned}
&\Big(\mathbb{E}\mathbb{E}_c\Big\|\int_{(0,t]\times B} F\,d\tilde{N}^c\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad = \Big(\mathbb{E}\mathbb{E}_c\Big\|\sum_{i,j} d_{i,j}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad \lesssim_{p,q} \max\Big\{\Big(\sum_{i,j}\mathbb{E}\mathbb{E}_c\|d_{i,j}\|_{L^q(S)}^p\Big)^{1/p}, \\
&\qquad\qquad \Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j,1}|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\qquad\qquad + \Big(\mathbb{E}\Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j,2}\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p}\Big\} \\
&\quad \lesssim_{p,q} \max\{\|F\|_{\mathcal{D}_{p,q}^p}, \|F_1\|_{\mathcal{S}_q^p} + \|F_2\|_{\mathcal{D}_{q,q}^p}\} \le \|F\|_{\mathcal{I}_{p,q}} + \varepsilon,
\end{aligned}$$

where the penultimate inequality follows by the computations in (3.10).

Case $1 < q \le p \le 2$: Fix $\varepsilon > 0$. By density of $\mathcal{I}_{\mathrm{elem}}$ in $\mathcal{S}_q^p$ and $\mathcal{D}_{q,q}^p \cap \mathcal{D}_{p,q}^p$, we can find a decomposition $F = F_1 + F_2$ with $F_\alpha \in \mathcal{I}_{\mathrm{elem}}$ for $\alpha = 1, 2$ such that

$$\|F_1\|_{\mathcal{S}_q^p} + \|F_2\|_{\mathcal{D}_{q,q}^p \cap \mathcal{D}_{p,q}^p} \le \|F\|_{\mathcal{I}_{p,q}} + \varepsilon.$$

We may assume that F_1 and F_2 have the same support in $\mathbb{R}_+ \times J$ as F. By Lemma 3.6, $\mathbb{E}(F_\alpha|\mathcal{A})$ is of the form (B.1). Let $y_{i,j,\alpha} = \sum_{k=1}^n F_{i,j,k,\alpha} x_{i,j,k,\alpha}$ and set $d_{i,j,\alpha} = y_{i,j,\alpha} N_{i,j}^c$, so that

$$d_{i,j} = d_{i,j,1} + d_{i,j,2}.$$

We apply Theorem 1.2 pointwise in Ω and subsequently take $L^p(\Omega)$-norms on both sides to find

$$\begin{aligned}
&\Big(\mathbb{E}\mathbb{E}_c\Big\|\int_{(0,t]\times B} F\,d\tilde{N}^c\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad = \Big(\mathbb{E}\mathbb{E}_c\Big\|\sum_{i,j} d_{i,j}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\quad \lesssim_{p,q} \Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\mathbb{E}_c|d_{i,j,1}|^2\Big)^{1/2}\Big\|_{L^q(S)}^p\Big)^{1/p} \\
&\qquad\quad + \max\Big\{\Big(\mathbb{E}\Big(\sum_{i,j}\mathbb{E}_c\|d_{i,j,2}\|_{L^q(S)}^q\Big)^{p/q}\Big)^{1/p},
\end{aligned}$$

$$\left(\mathbb{E}\left\|\left(\sum_{i,j}\mathbb{E}_c|d_{i,j,2}|^2\right)^{1/2}\right\|_{L^q(S)}^p\right)^{1/p}\Bigg\}$$

$$\lesssim_{p,q} \|F_1\|_{\mathcal{S}_q^p} + \max\{\|F_2\|_{\mathcal{D}_{q,q}^p}, \|F_2\|_{\mathcal{D}_{p,q}^p}\} \le \|F\|_{\mathcal{I}_{p,q}} + \varepsilon,$$

where the penultimate inequality follows as in (3.10). This completes the proof.

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