

STEIN'S METHOD AND THE RANK DISTRIBUTION OF RANDOM MATRICES OVER FINITE FIELDS

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With $\mathcal{Q}_{q,n}$ the distribution of n minus the rank of a matrix chosen uniformly from the collection of all $n \times (n + m)$ matrices over the finite field \mathbb{F}_q of size $q \geq 2$, and \mathcal{Q}_q the distributional limit of $\mathcal{Q}_{q,n}$ as $n \rightarrow \infty$, we apply Stein's method to prove the total variation bound

$$\frac{1}{8q^{n+m+1}} \leq \|\mathcal{Q}_{q,n} - \mathcal{Q}_q\|_{\text{TV}} \leq \frac{3}{q^{n+m+1}}.$$

In addition, we obtain similar sharp results for the rank distributions of symmetric, symmetric with zero diagonal, skew symmetric, skew centrosymmetric and Hermitian matrices.

1. Introduction. We study the distribution of the rank for various ensembles of random matrices over finite fields. To give a flavor of our results, let M_n be chosen uniformly from all $n \times (n + m)$ matrices over the finite field \mathbb{F}_q of size $q \geq 2$. Letting $\mathcal{Q}_{q,n} = n - \text{rank}(M_n)$, it is known (page 38 of [3]) that for all k in $U_n = \{0, \dots, n\}$,

$$(1) \quad P(\mathcal{Q}_{q,n} = k) = p_{k,n}$$

where $p_{k,n} = \frac{1}{q^{k(m+k)}} \frac{\prod_{i=1}^{n+m} (1 - 1/q^i) \prod_{i=k+1}^n (1 - 1/q^i)}{\prod_{i=1}^{n-k} (1 - 1/q^i) \prod_{i=1}^{m+k} (1 - 1/q^i)}.$

Clearly, for any fixed $k \in \mathbb{N}_0$, the collection of nonnegative integers,

$$(2) \quad \lim_{n \rightarrow \infty} p_{k,n} = p_k \quad \text{where } p_k = \frac{1}{q^{k(m+k)}} \frac{\prod_{i=k+1}^{\infty} (1 - 1/q^i)}{\prod_{i=1}^{m+k} (1 - 1/q^i)}.$$

For readability and notational agreement with the examples that follow, we suppress m in the definition of these distributions. Throughout, we also adopt the convention that an empty product takes the value 1. One of our main results, Theorem 1.1, provides sharp upper and lower bounds on the total variation distance

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between $\mathcal{Q}_{q,n}$, the distribution of $\mathcal{Q}_{q,n}$ in (1) and its limit in (2), denoted \mathcal{Q}_q . Recall that the total variation distance between two probability distributions P_1, P_2 on a finite set S is given by

$$(3) \quad \|P_1 - P_2\|_{\text{TV}} := \frac{1}{2} \sum_{s \in S} |P_1(s) - P_2(s)| = \max_{A \subset S} |P_1(A) - P_2(A)|.$$

THEOREM 1.1. For $q \geq 2, n \geq 1$ and $m \geq 0$,

$$(4) \quad \frac{1}{8q^{n+m+1}} \leq \|\mathcal{Q}_{q,n} - \mathcal{Q}_q\|_{\text{TV}} \leq \frac{3}{q^{n+m+1}}.$$

The upper bound in Theorem 1.1 appears quite difficult to compute directly by substituting the expressions for the point probabilities given in (1) and (2) into the defining expressions for the total variation distance in (3). In particular, even when $m = 0, n = 2$, the $p_{k,n}$ are not monotonic in k . On the other hand, use of Stein's method [11, 28] makes for a quite tractable computation. In Sections 4–7, we also apply our methods to ensembles of random matrices with symmetry constraints, in particular, to symmetric, symmetric with zero diagonal, skew symmetric, skew centrosymmetric and Hermitian matrices.

Next, we give five pointers to the large literature on the rank distribution of random matrices over finite fields, demonstrating that the subject is of interest. First, one of the earliest systematic studies of ranks of random matrices from the finite classical groups is due to Rudvalis and Shinoda [26, 27]. They determine the rank distribution of random matrices from finite classical groups, and relate distributions such as \mathcal{Q}_q of (2) to identities of Euler. Second, ranks of random matrices from finite classical groups appear in works on the ‘‘Cohen–Lenstra heuristics’’ of number theory; see [32] for the finite general linear groups and [24] for the finite symplectic groups. Third, the rank distribution of random matrices over finite fields is useful in coding theory; see [4] and Chapter 15 of [23]. Fourth, the distribution of ranks of uniformly chosen random matrices over finite fields has been used to test random number generators [14], and there is interest in the rate of convergence to \mathcal{Q}_q . Fifth, there is work on ranks of random matrices over finite fields where the matrix entries are independent and identically distributed, but not necessarily uniform. For example, the paper [10] uses a combination of Möbius inversion, finite Fourier transforms and Poisson summation, to find conditions on the distribution of matrix entries under which the probability of a matrix being invertible tends to p_0 as $n \rightarrow \infty$. Further results in this direction, including rank distributions of sparse matrices, can be found in [5, 12, 13, 20]. It would be valuable (but challenging) to extend our methods to these settings.

The organization of this paper is as follows. Section 2 provides some general tools for our application of Stein's method, and useful bounds on products such as $\prod_i (1 - 1/q^i)$. The development followed here is along the lines of the ‘‘comparison of generators’’ method as in [18] and [19]. Section 3 treats the rank distribution of uniformly chosen $n \times (n + m)$ matrices over a finite field, proving

Theorem 1.1. Section 4 treats the rank distribution of random symmetric matrices over a finite field. Section 5 provides results for the rank distribution of a uniformly chosen symmetric matrix with 0 diagonal; these are called “symplectic” matrices in Chapter 15 of [23], which uses their rank distribution in the context of error correcting codes. The same formulas for the rank distribution of symmetric matrices with zero diagonal also apply to the rank distribution of random skew-symmetric matrices, when q is odd. Section 6 treats the rank distribution of random skew centrosymmetric matrices over finite fields, and Section 7 treats the rank distribution of random Hermitian matrices over finite fields. The Appendix gives an algebraic proof, for the special case $m = 0$ of square matrices, of the crucial fact (proved probabilistically in Section 3 in general) that if Q_n has distribution $Q_{q,n}$ of (1), then $E(q^{Q_n}) = 2 - 1/q^n$.

In the interest of notational simplicity, in Sections 4–7, the specific rank distributions of the $n \times n$ matrices of interest, and their limits, will apply only locally in the section or subsection that contains them, and will there be consistently denoted by $Q_{q,n}$ and Q_q , respectively.

2. Preliminaries. We begin with a general result for obtaining characterizations of discrete integer distributions. We note that a version of Lemma 2.1 can be obtained by replacing $f(x)$ by $f(x)b(x)$ in Theorem 2.1 of [21], followed by a reversal of the interval $[a, b]$, with similar remarks applying to the use of Proposition 2.1 and Corollary 2.1 of [18]. However, the following lemma and its short, simple proof contain the precise conditions used throughout this work and keep the paper self-contained.

We say a nonempty subset \mathbb{I} of the integers \mathbb{Z} is an interval if $a, b \in \mathbb{I}$ with $a \leq b$ then $[a, b] \cap \mathbb{Z} \subset \mathbb{I}$. Let $\mathcal{L}(X)$ denote the distribution of a random variable X .

LEMMA 2.1. *Let $\{r_k, k \in \mathbb{I}\}$ be the distribution of a random variable Y having support the integer interval \mathbb{I} . Then if $a(k)$ and $b(k)$ are any functions such that*

$$(5) \quad a(k)r_{k-1} = b(k)r_k \quad \text{for all } k \in \mathbb{Z},$$

then a random variable X having distribution $\mathcal{L}(Y)$ satisfies

$$(6) \quad E[a(X + 1)f(X + 1)] = E[b(X)f(X)]$$

for all functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ for which the expectations in (6) exist.

Conversely, if $a(k)$ and $b(k)$ satisfy (5) and $a(k) \neq 0$ for all $k \in \mathbb{I}$ then X has distribution $\mathcal{L}(Y)$ whenever X has support \mathbb{I} and satisfies (6) for all functions $f(x) = \mathbf{1}(x = k), k \in \mathbb{I}$.

When Y has support \mathbb{N}_0 then $k \in \mathbb{Z}$ in (5) may be replaced by $k \in \mathbb{N}_0$, while if Y has support $U_n = \{0, 1, \dots, n\}$ for some $n \in \mathbb{N}_0$, then (5) may be replaced by the condition that (5) holds for $k \in U_n$ and that $a(n + 1) = 0$.

PROOF. First suppose that (5) holds and that $\mathcal{L}(X) = \mathcal{L}(Y)$. Then for all $k \in \mathbb{Z}$,

$$\begin{aligned} E(a(X + 1)\mathbf{1}(X + 1 = k)) &= a(k)P(X = k - 1) \\ &= a(k)r_{k-1} \\ &= b(k)r_k \\ &= b(k)P(X = k) \\ &= E(b(X)\mathbf{1}(X = k)). \end{aligned}$$

Hence, (6) holds for $f(x) = \mathbf{1}(x = k), k \in \mathbb{Z}$. By linearity, (6) holds for all functions with finite support, and hence for all the claimed functions by dominated convergence.

Conversely, if (6) holds for X with $f(x) = \mathbf{1}(x = k)$ for $k \in \mathbb{I}$ then

$$a(k)P(X = k - 1) = b(k)P(X = k).$$

Hence, using that $a(k) \neq 0, r_k \neq 0$ for $k \in \mathbb{I}$ and that X has the same support as Y yields

$$\frac{P(X = k - 1)}{P(X = k)} = \frac{b(k)}{a(k)} = \frac{r_{k-1}}{r_k}.$$

If $\mathbb{I} = \{s, \dots, t\}$, then for $j \in \mathbb{I}$

$$\frac{P(X = j)}{P(X = t)} = \prod_{k=j+1}^t \frac{P(X = k - 1)}{P(X = k)} = \frac{r_j}{r_t}.$$

Summing over $j \in \mathbb{I}$ yields $P(X = t) = r_t$, and hence $P(X = j) = r_j$, showing $\mathcal{L}(X) = \mathcal{L}(Y)$. One may argue similarly for the remaining cases where \mathbb{I} is an unbounded integer interval.

Lastly, when the support of Y is a subset of \mathbb{N}_0 then (5) holds trivially for $k \notin \mathbb{N}_0$, and when Y has support $U_n = \{0, 1, \dots, n\}$ then (5) also holds trivially for $k \geq n + 2$, and at $k = n + 1$ when $a(n + 1) = 0$. \square

For example, when Y has the Poisson distribution $\mathcal{P}(\lambda)$ with parameter λ , then $r_k = e^{-\lambda}\lambda^k/k!$, and we obtain

$$\frac{r_{k-1}}{r_k} = \frac{k}{\lambda} \quad \text{for all } k \in \mathbb{N}_0.$$

Setting $b(k) = k$ and $a(k) = \lambda$ yields the standard characterization of the Poisson distribution [2],

$$E[\lambda f(Y + 1)] = E[Yf(Y)].$$

Of particular interest here is the characterization (6) of Lemma 2.1 for limiting distributions \mathcal{Q}_q with distribution $P(Q = k) = p_k$ having support \mathbb{N}_0 . In this

case, when applying Lemma 2.1 we take $a(k) > 0$ for all $k \in \mathbb{N}_0$, whence $b(0) = 0$ by (5), and let the values of $a(k)$ and $b(k)$ for $k \notin \mathbb{N}_0$ be arbitrary. For such functions $a(k)$ and $b(k)$, we consider solutions f to recursive ‘‘Stein equations’’ of the form

$$(7) \quad a(k + 1)f(k + 1) - b(k)f(k) = h(k) - \mathcal{Q}_q h \quad \text{for } k \in \mathbb{N}_0,$$

where $\mathcal{Q}_q h = Eh(Q)$.

Solving (7) for $f(k), k \in \mathbb{N}_0$, when the functions $a(k), b(k)$ satisfy only $b(0) = 0$ and $a(k) > 0$ one may take $f(0) = 0$ arbitrarily, and easily verify that the remaining values are uniquely determined and given by

$$(8) \quad f(k + 1) = \sum_{j=0}^k \left(\frac{\prod_{l=j+1}^k b(l)}{\prod_{l=j+1}^{k+1} a(l)} \right) [h(j) - \mathcal{Q}_q h] \quad \text{for } k \in \mathbb{N}_0.$$

In the case where the distribution $\{p_k, k \in \mathbb{N}_0\}$ with support \mathbb{N}_0 satisfies (5) with p_k replacing r_k , the solution (8) simplifies to

$$(9) \quad f(k + 1) = \frac{1}{a(k + 1)p_k} \sum_{j=0}^k [h(j) - \mathcal{Q}_q h] p_j$$

$$(10) \quad = \frac{E[(h(Q) - \mathcal{Q}_q h)\mathbf{1}(Q \leq k)]}{a(k + 1)p_k} \quad \text{for } k \in \mathbb{N}_0.$$

In particular, for $h_A(k) = \mathbf{1}(k \in A)$ with $A \subset \mathbb{N}_0$ and $U_k = \{0, 1, \dots, k\}$, as in Barbour et al. [2], Lemma 1.1.1, for $k \in \mathbb{N}_0$, as $\mathcal{Q}_q h_A = P(Q \in A)$, the numerator of (9) is given by

$$P(Q \in A \cap U_k) - P(Q \in A)P(Q \in U_k).$$

Now replacing $P(Q \in A \cap U_k)$ and $P(Q \in A)$ in the first and second term, respectively, by

$$P(Q \in A \cap U_k)[P(Q \in U_k) + P(Q \in U_k^c)]$$

and

$$P(Q \in A \cap U_k) + P(Q \in A \cap U_k^c),$$

canceling the resulting common factor demonstrates that the solution f_A satisfies

$$(11) \quad f_A(k + 1) = \frac{P(Q \in A \cap U_k)P(Q \in U_k^c) - P(Q \in A \cap U_k^c)P(Q \in U_k)}{a(k + 1)p_k}$$

$$\leq \frac{P(Q \in A \cap U_k)P(Q \in U_k^c)}{a(k + 1)p_k}$$

$$(12) \quad \leq \frac{P(Q \in U_k)P(Q \in U_k^c)}{a(k + 1)p_k}$$

with equality when $A = U_k$. Since $f_{A^c}(k) = -f_A(k)$ the bound (12) holds for $|f_A(k + 1)|$.

LEMMA 2.2. *Let Q have distribution $\{p_k, k \in \mathbb{N}_0\}$ with $p_k > 0$ for all $k \in \mathbb{N}_0$, and let $a(k), b(k)$ satisfy (5) with p_k replacing r_k , and for $A \subset \mathbb{N}_0$ let f_A be the solution to (7) given by (11). Then*

$$|f_A(1)| \leq \frac{P(Q \geq 1)}{a(1)}.$$

PROOF. From (11) with $k = 0$, we obtain

$$f_A(1) = \frac{P(Q \in A \cap U_0)P(Q \geq 1) - P(Q \in A \cap U_0^c)P(Q = 0)}{a(1)p_0}.$$

If $A \ni 0$, then

$$\begin{aligned} |f_A(1)| &= \left| \frac{P(Q = 0)P(Q \geq 1) - P(Q \in A \setminus \{0\})P(Q = 0)}{a(1)p_0} \right| \\ &= \frac{P(Q \geq 1) - P(Q \in A \setminus \{0\})}{a(1)} \leq \frac{P(Q \geq 1)}{a(1)}, \end{aligned}$$

while if $A \not\ni 0$ then again

$$|f_A(1)| = \frac{P(Q \in A)P(Q = 0)}{a(1)p_0} \leq \frac{P(Q \geq 1)}{a(1)}. \quad \square$$

Lemma 2.3 collects some bounds that will be useful. We first state the simple inequality

$$(13) \quad \prod_{i=1}^n (1 - a_i) \geq 1 - \sum_{i=1}^n a_i$$

valid for $a_i \in [0, 1], i = 1, \dots, n$, and easily shown by induction.

LEMMA 2.3. *Let $q \geq 2$. Then*

$$\begin{aligned} \prod_{i=1}^n (1 - 1/q^i) &\geq 1 - 1/q - 1/q^2, \\ \prod_{i \geq 1} (1 - 1/q^i) &\geq 1 - 1/q - 1/q^2 + 1/q^5 + 1/q^7 - 1/q^{12} - 1/q^{15}, \\ \prod_{\substack{i \geq 1 \\ i \text{ odd}}} (1 - 1/q^i) &\geq 1 - 1/q - 1/q^3 \end{aligned}$$

and

$$\prod_{\substack{i \geq 3 \\ i \text{ odd}}} (1 - 1/q^i) \geq 1 - 2/q^3.$$

For $0 \leq m + 1 \leq n$,

$$\prod_{i=m+1}^n (1 - 1/q^i) \geq 1 - 2/q^{m+1}.$$

PROOF. The first claim is Lemma 3.5 of [25], and arguing as there yields the second claim. Thus,

$$\begin{aligned} \prod_{\substack{i \geq 1 \\ i \text{ odd}}} (1 - 1/q^i) &\geq \frac{\prod_{i \geq 1} (1 - 1/q^i)}{1 - 1/q^2} \\ &\geq \frac{1 - 1/q - 1/q^2 + 1/q^5 + 1/q^7 - 1/q^{12} - 1/q^{15}}{1 - 1/q^2} \\ &\geq 1 - 1/q - 1/q^3, \end{aligned}$$

where the last inequality holds since

$$\begin{aligned} &(1 - 1/q - 1/q^2 + 1/q^5 + 1/q^7 - 1/q^{12} - 1/q^{15}) \\ &\quad - (1 - 1/q^2)(1 - 1/q - 1/q^3) = \frac{q^8 - q^3 - 1}{q^{15}}, \end{aligned}$$

which is positive for $q \geq 2$. The next inequality now follows by applying the one just shown to obtain

$$\prod_{\substack{i \geq 3 \\ i \text{ odd}}} (1 - 1/q^i) \geq 1 - \frac{1}{q^3(1 - 1/q)}$$

and using that $q \geq 2$.

For the final claim, using (13) yields

$$\begin{aligned} \prod_{i=m+1}^n (1 - 1/q^i) &\geq 1 - \sum_{i=m+1}^n 1/q^i \\ &\geq 1 - \sum_{i=m+1}^{\infty} 1/q^i \\ &= 1 - \frac{1}{q^{m+1}(1 - 1/q)} \\ &\geq 1 - \frac{2}{q^{m+1}}. \end{aligned}$$

□

REMARK. Since

$$\prod_{i=1}^n (1 - 1/q^i) \geq \prod_{i \geq 1} (1 - 1/q^i),$$

it is easy to see that the second claim of Lemma 2.3 implies the first.

3. Uniform matrices over finite fields. In this section, we study the rank distribution of matrices chosen uniformly from those of dimension $n \times (n + m)$ with entries from the finite field \mathbb{F}_q , and take the distributions \mathcal{Q}_q and $\mathcal{Q}_{q,n}$ as in (2) and (1), respectively; throughout this section, we take $q \geq 2$. The goal of this section is to prove Theorem 1.1.

The following lemma is our first application of the characterizations provided by Lemma 2.1.

LEMMA 3.1. *If Q has the \mathcal{Q}_q distribution then*

$$(14) \quad E[qf(Q + 1)] = E[(q^Q - 1)(q^{Q+m} - 1)f(Q)]$$

for all functions f for which these expectations exist.

If Q_n has the $\mathcal{Q}_{q,n}$ distribution, then

$$(15) \quad E[q(1 - q^{-n+Q_n})f(Q_n + 1)] = E[(q^{Q_n} - 1)(q^{Q_n+m} - 1)f(Q_n)]$$

for all functions f for which these expectations exist.

PROOF. From (2), we obtain

$$\frac{p_{k-1}}{p_k} = \frac{(q^k - 1)(q^{m+k} - 1)}{q} \quad \text{for all } k \in \mathbb{N}_0.$$

An application of Lemma 2.1 with $a(k) = q$ and $b(k) = (q^k - 1)(q^{k+m} - 1)$ yields (14). Similarly, from (1) we obtain

$$(16) \quad \frac{p_{k-1,n}}{p_{k,n}} = \frac{(q^k - 1)(q^{k+m} - 1)}{q(1 - q^{-n+k-1})} \quad \text{for all } k \in U_n.$$

An application of Lemma 2.1 with $a(k) = q(1 - q^{-n+k-1})$, $b(k) = (q^k - 1)(q^{k+m} - 1)$, noting $a(n + 1) = 0$, yields (15). \square

Here, we calculate $E(q^{Q_n})$ using the characterization (15). An algebraic proof for the case $m = 0$ of Lemma 3.2 appears in the Appendix. After reading the first version of this paper, Dennis Stanton has shown us a proof of this special case using the q -Chu–Vandermonde summation formula.

LEMMA 3.2. *If Q_n has the $\mathcal{Q}_{q,n}$ distribution on $U_n = \{0, 1, \dots, n\}$ given by (1), then*

$$E(q^{Q_n}) = 1 + q^{-m} - q^{-(n+m)}.$$

PROOF. Applying the characterization (15) with the choice $f(x) = q^{kx}$, we obtain

$$E[q(1 - q^{-n+Q_n})q^{k(Q_n+1)}] = E[(q^{Q_n} - 1)(q^{Q_n+m} - 1)q^k Q_n].$$

Letting $c_k = E q^k Q_n$ yields the recursion

$$(17) \quad q^m c_{k+2} = (1 + q^m - q^{-n+k+1})c_{k+1} + (q^{k+1} - 1)c_k.$$

Since $Q_{q,n}$ is a probability distribution, $c_0 = 1$, and setting $k = -1$ in (17) yields the claim. \square

In the remainder of this section, we consider the Stein equation (7), with

$$(18) \quad a(k) = q \quad \text{and} \quad b(k) = (q^k - 1)(q^{k+m} - 1)$$

for the target distribution Q_q , and for $A \subset \mathbb{N}_0$ we let f_A denote the solution (9) when $h(k) = \mathbf{1}(k \in A)$.

For a function $f : \mathbb{N}_0 \rightarrow \mathbb{R}$, let

$$\|f\| = \sup_{k \in \mathbb{N}_0} |f(k)|.$$

LEMMA 3.3. *The solution f_A satisfies*

$$\sup_{A \subset \mathbb{N}_0} \|f_A\| \leq \frac{2}{q^{m+2}}.$$

If $m = 0$, the bound can be improved to

$$\sup_{A \subset \mathbb{N}_0} \|f_A\| \leq \frac{1}{q^2} + \frac{1}{q^3}.$$

PROOF. As we may set $f_A(0) = 0$, it suffices to consider $f_A(k + 1)$ for $k \in \mathbb{N}_0$. By Lemma 2.2, for all $A \subset \mathbb{N}_0$

$$\begin{aligned} |f_A(1)| &\leq \frac{P(Q \geq 1)}{q} \\ &= \frac{1 - p_0}{q} \\ &= \frac{1}{q} \left(1 - \prod_{i \geq m+1} \left(1 - \frac{1}{q^i} \right) \right) \\ &\leq \frac{2}{q^{m+2}}, \end{aligned}$$

where we have applied the last part of Lemma 2.3. For $m = 0$, using the first inequality of Lemma 2.3 in the last step gives that

$$|f_A(1)| \leq \frac{1}{q^2} + \frac{1}{q^3}.$$

Now consider the case $k \geq 1$. By (12) and (18), we have

$$(19) \quad |f_A(k+1)| \leq \frac{P(Q \in U_k)P(Q \in U_k^c)}{qp_k}$$

and by neglecting the term $P(Q \in U_k)$ in (19) and applying (2) we obtain

$$\begin{aligned} & |f_A(k+1)| \\ & \leq \frac{P(Q \in U_k^c)}{qp_k} \\ & = q^{k(m+k)-1} \frac{\prod_{i=1}^{m+k}(1-1/q^i)}{\prod_{i=k+1}^{\infty}(1-1/q^i)} \sum_{l=k+1}^{\infty} \frac{1}{q^{l(m+l)}} \frac{\prod_{i=l+1}^{\infty}(1-1/q^i)}{\prod_{i=1}^{m+l}(1-1/q^i)} \\ & = \frac{q^{k(m+k)-1}}{\prod_{i=k+1}^{\infty}(1-1/q^i)} \sum_{l=k+1}^{\infty} \frac{1}{q^{l(m+l)}} \frac{\prod_{i=l+1}^{\infty}(1-1/q^i)}{\prod_{i=m+k+1}^{m+l}(1-1/q^i)} \\ & \leq \frac{q^{k(m+k)-1}}{\prod_{i=k+1}^{\infty}(1-1/q^i) \prod_{i=m+k+1}^{\infty}(1-1/q^i)} \sum_{l=k+1}^{\infty} \frac{1}{q^{l(m+l)}} \\ & \leq \frac{q^{k(m+k)-1}}{(1-\sum_{j=k+1}^{\infty}(1/q^j))(1-\sum_{j=m+k+1}^{\infty}(1/q^j))} \sum_{l=k+1}^{\infty} \frac{1}{q^{l(m+l)}} \\ & = \frac{q^{k(m+k)-1}}{(1-(q^{-(k+1)}/(1-q^{-1}))) (1-(q^{-(m+k+1)}/(1-q^{-1})))} \sum_{l=k+1}^{\infty} \frac{1}{q^{l(m+l)}} \\ & = \frac{1}{q(1-(1/q^k(q-1)))(1-(1/q^{m+k}(q-1)))} \sum_{l=1}^{\infty} \frac{1}{q^{2lk+l^2+lm}} \\ & \leq \frac{1}{q(q^k-(1/(q-1)))(q^{m+k}-(1/(q-1)))} \sum_{l=1}^{\infty} \frac{1}{q^{l^2}}, \end{aligned}$$

where for the third inequality we have applied (13).

We claim that

$$4\left(q^k - \frac{1}{q-1}\right)\left(q^{m+k} - \frac{1}{q-1}\right) \geq q^{m+2}.$$

As the left-hand side is increasing in $k \geq 1$, it suffices to prove the claim for $k = 1$.

In this case, the claim may be rewritten as

$$3q^{m+2} + \frac{4}{(q-1)^2} \geq \frac{4}{q-1}(q+q^{m+1}).$$

As $q \geq 2$, the result is a consequence of the two easily verified inequalities

$$2q^{m+2} \geq \frac{4q^{m+1}}{q-1} \quad \text{and} \quad q^{m+2} + \frac{4}{(q-1)^2} \geq \frac{4q}{q-1}.$$

Hence, for $k \geq 1$, using $q \geq 2$, we obtain

$$|f_A(k + 1)| \leq \frac{4}{q^{m+3}} \sum_{l=1}^{\infty} \frac{1}{q^{l^2}} \leq \frac{4}{q^{m+3}} \left(\frac{1}{2} + \sum_{l=2}^{\infty} \frac{1}{2^{2+l}} \right) \leq \frac{1}{q^{m+2}} + \frac{1}{q^{m+3}},$$

where the final inequality used that $2/q^{m+3} \leq 1/q^{m+2}$, and that $\sum_{l=2}^{\infty} \frac{1}{2^{2+l}} \leq 1/4$, thus completing the proof of the lemma. \square

We now present the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. We first compute the lower bound on the total variation distance by estimating the difference of the two distributions at $k = 0$. In particular, by (3), (1) and (2),

$$\begin{aligned} & \|Q_{q,n} - Q_q\|_{TV} \\ & \geq \frac{1}{2} [p_{0,n} - p_0] \\ & = \frac{1}{2} \left[\prod_{m+1 \leq i \leq m+n} (1 - 1/q^i) - \prod_{i \geq m+1} (1 - 1/q^i) \right] \\ & \geq \frac{1}{2} [(1 - 1/q^{m+1}) \cdots (1 - 1/q^{n+m}) - (1 - 1/q^{m+1}) \cdots (1 - 1/q^{n+m+1})] \\ & = \frac{1}{2q^{n+m+1}} (1 - 1/q^{m+1}) \cdots (1 - 1/q^{n+m}) \\ & \geq \frac{1}{2q^{n+m+1}} (1 - 1/q) \cdots (1 - 1/q^n) \\ & \geq \frac{1}{2q^{n+m+1}} (1 - 1/q - 1/q^2) \\ & \geq \frac{1}{8q^{n+m+1}}. \end{aligned}$$

The fourth inequality used Lemma 2.3, and the last that $q \geq 2$.

For the upper bound, with $h_A(k) = \mathbf{1}(k \in A)$ we obtain

$$\begin{aligned} & |P(Q_n \in A) - P(Q \in A)| \\ & = |E[h_A(Q_n)] - Q_q h_A| \\ & = |E[q f_A(Q_n + 1) - (q^{Q_n} - 1)(q^{Q_n+m} - 1) f_A(Q_n)]| \\ & = |E[q^{-n+Q_n+1} f_A(Q_n + 1)]| \leq \|f_A\| E q^{-n+Q_n+1}, \end{aligned}$$

where we have applied (15) in the third equality. Applying Lemmas 3.3 and 3.2 gives that for $m \geq 1$,

$$\begin{aligned} \|f_A\| E q^{-n+Q_{n+1}} &\leq \frac{2}{q^{m+2}} q^{-n+1} \left(1 + q^{-m} - \frac{1}{q^{n+m}}\right) \\ &\leq \frac{2(1 + 1/q)}{q^{n+m+1}} \leq \frac{3}{q^{n+m+1}}. \end{aligned}$$

For $m = 0$, applying Lemmas 3.3 and 3.2 gives that

$$\begin{aligned} \|f_A\| E q^{-n+Q_{n+1}} &\leq \left(\frac{1}{q^2} + \frac{1}{q^3}\right) q^{-n+1} \left(2 - \frac{1}{q^n}\right) \\ &\leq \frac{2(1 + 1/q)}{q^{n+1}} \leq \frac{3}{q^{n+1}}. \end{aligned}$$

Now taking the supremum over all $A \subset \mathbb{N}_0$ and applying definition (3) completes the proof. \square

REMARK. When $m = 0$, the limit distribution \mathcal{Q}_q also arises in the study of the dimension of the fixed space of a random element of $GL(n, q)$. More precisely, Rudvalis and Shinoda [26] prove that for k fixed, as $n \rightarrow \infty$ the probability that a random element of $GL(n, q)$ has a k dimensional fixed space tends to p_k . See [15] for another proof.

4. Symmetric matrices over finite fields. Let S be the set of symmetric matrices with entries in the finite field \mathbb{F}_q (where q is a prime power). Clearly, $|S| = q^{\binom{n+1}{2}}$. The paper [7] determines the rank distribution of a matrix chosen uniformly from S when q is odd, and the paper [22] determines this distribution for q both odd and even, given by (21).

Throughout this section $q \geq 2$, and we let \mathcal{Q}_q be the distribution on \mathbb{N}_0 with mass function

$$(20) \quad p_k = \frac{\prod_{i \geq 1, i \text{ odd}} (1 - 1/q^i)}{\prod_{i=1}^k (q^i - 1)}$$

and for $n \in \mathbb{N}_0$ we let $\mathcal{Q}_{q,n}$ be the distribution on $U_n = \{0, \dots, n\}$ with mass function

$$p_{k,n} = \frac{N(n, n - k)}{q^{\binom{n+1}{2}}}$$

$$(21) \quad \text{where } N(n, 2h) = \prod_{i=1}^h \frac{q^{2i}}{(q^{2i} - 1)} \prod_{i=0}^{2h-1} (q^{n-i} - 1) \text{ for } 2h \leq n \text{ and}$$

$$N(n, 2h + 1) = \prod_{i=1}^h \frac{q^{2i}}{(q^{2i} - 1)} \prod_{i=0}^{2h} (q^{n-i} - 1) \text{ for } 2h + 1 \leq n.$$

THEOREM 4.1. *If n is even, we have*

$$\frac{0.18}{q^{n+1}} \leq \|Q_{q,n} - Q_q\|_{TV} \leq \frac{2.25}{q^{n+1}}.$$

If n is odd, we have

$$\frac{0.18}{q^{n+2}} \leq \|Q_{q,n} - Q_q\|_{TV} \leq \frac{2}{q^{n+2}}.$$

We again begin by using Lemma 2.1 to develop characterizations for the two distributions of interest. For $n \in \mathbb{N}_0$ we let $\mathbf{1}_n = \mathbf{1}(n \text{ is even})$, the indicator function that n is even.

LEMMA 4.2. *If Q has the Q_q distribution then*

$$E[f(Q + 1)] = E[(q^Q - 1)f(Q)]$$

for all functions f for which these expectations exist.

If Q_n has the $Q_{q,n}$ distribution then

$$(22) \quad E[(1 - \mathbf{1}_{n-Q_n} q^{-(n-Q_n)})f(Q_n + 1)] = E[(q^{Q_n} - 1)f(Q_n)]$$

for all functions f for which these expectations exist.

PROOF. By taking ratios in (20), we obtain

$$\frac{p_{k-1}}{p_k} = q^k - 1.$$

Setting $a(k) = 1$ and $b(k) = q^k - 1$ applying Lemma 2.1 yields the first result.

If n and k are of the same parity, then $n - k = 2h$ for some h , and we have

$$\begin{aligned} \frac{p_{k-1,n}}{p_{k,n}} &= \frac{N(n, n - k + 1)}{N(n, n - k)} = \frac{N(n, 2h + 1)}{N(n, 2h)} \\ &= q^{n-2h} - 1 = q^k - 1, \quad k \in U_n. \end{aligned}$$

In this case, we set $a(k) = 1$ and $b(k) = q^k - 1$.

If k and n are of opposite parity, then $n - k = 2h + 1$ for some h and we obtain

$$\begin{aligned} \frac{p_{k-1,n}}{p_{k,n}} &= \frac{N(n, n - k + 1)}{N(n, n - k)} = \frac{N(n, 2(h + 1))}{N(n, 2h + 1)} \\ &= \frac{q^{2(h+1)}}{q^{2(h+1)} - 1} (q^{n-2h-1} - 1) \\ &= \frac{q^{n-k+1}}{q^{n-k+1} - 1} (q^k - 1) = \frac{q^k - 1}{1 - q^{-n+k-1}} \quad \text{for } k \in U_n. \end{aligned}$$

In this case, we set $a(k) = 1 - q^{-n+k-1}$ and $b(k) = q^k - 1$.

Writing $a(k) = 1 - \mathbf{1}_{n-k+1}q^{-n+k-1}$ and $b(k) = q^k - 1$ combines both cases. Noting that $a(n + 1) = 0$ an application of Lemma 2.1 completes the proof. \square

LEMMA 4.3. *If Q_n has distribution $Q_{q,n}$ then*

$$E\mathbf{1}_{n-Q_n}q^{Q_n} = 1.$$

PROOF. Setting $f(x) = \mathbf{1}_{n-x}$ in (22) yields

$$E[(1 - \mathbf{1}_{n-Q_n}q^{-(n-Q_n)})\mathbf{1}_{n-Q_n-1}] = E[(q^{Q_n} - 1)\mathbf{1}_{n-Q_n}].$$

Since $\mathbf{1}_{n-Q_n}\mathbf{1}_{n-Q_n-1} = 0$, we obtain

$$E[\mathbf{1}_{n-Q_n-1}] = E[(q^{Q_n} - 1)\mathbf{1}_{n-Q_n}]$$

and rearranging yields

$$E[\mathbf{1}_{n-Q_n}q^{Q_n}] = E[\mathbf{1}_{n-Q_n-1}] + E[\mathbf{1}_{n-Q_n}] = 1$$

as claimed. \square

In the remainder of this section, we consider the Stein equation (7) for the target distribution Q_q with

$$a(k) = 1 \quad \text{and} \quad b(k) = q^k - 1$$

and for $A \subset \mathbb{N}_0$ we let f_A denote the solution (9) when $h(k) = \mathbf{1}(k \in A)$.

LEMMA 4.4. *The solution f_A satisfies*

$$\sup_{A \subset \mathbb{N}_0} |f_A(1)| \leq \frac{1}{q} + \frac{1}{q^3} \quad \text{and} \quad \sup_{A \subset \mathbb{N}_0, k \geq 2} |f_A(k)| \leq \frac{2}{q^2}.$$

PROOF. By Lemma 2.2, for all $A \subset \mathbb{N}_0$,

$$\begin{aligned} |f_A(1)| &\leq P(Q \geq 1) \\ &= 1 - p_0 \\ &= 1 - \prod_{i \geq 1, i \text{ odd}} \left(1 - \frac{1}{q^i}\right) \\ &\leq \frac{1}{q} + \frac{1}{q^3}, \end{aligned}$$

where we applied the third inequality in Lemma 2.3.

For $k \geq 1$, using (12) and (20),

$$\begin{aligned}
 |f_A(k+1)| &\leq \frac{P(Q \in U_k^c)}{p_k} \\
 &= \prod_{i=1}^k (q^i - 1) \sum_{l=k+1}^{\infty} \frac{1}{\prod_{i=1}^l (q^i - 1)} \\
 &= \sum_{l=k+1}^{\infty} \frac{1}{\prod_{i=k+1}^l (q^i - 1)} \\
 &= \sum_{l=k+1}^{\infty} \frac{1}{q^{(l(l+1)-k(k+1))/2} \prod_{i=k+1}^l (1 - q^{-i})} \\
 &\leq \frac{1}{\prod_{i=k+1}^{\infty} (1 - q^{-i})} \sum_{l=k+1}^{\infty} \frac{1}{q^{(l(l+1)-k(k+1))/2}} \\
 &= \frac{1}{\prod_{i=k+1}^{\infty} (1 - q^{-i})} \left(\frac{1}{q^{k+1}} + \sum_{l=2}^{\infty} \frac{1}{q^{lk+l^2/2+l/2}} \right) \\
 &\leq \frac{1}{\prod_{i=k+1}^{\infty} (1 - q^{-i})} \left(\frac{1}{q^{k+1}} + \frac{1}{q^{2k}} \sum_{l=2}^{\infty} \frac{1}{q^{l(l+1)/2}} \right).
 \end{aligned}$$

In particular, for all $k \geq 1$ we obtain

$$|f_A(k+1)| \leq \frac{1}{q^2 \prod_{i=2}^{\infty} (1 - q^{-i})} \left(1 + \sum_{l=2}^{\infty} \frac{1}{q^{l(l+1)/2}} \right)$$

and the proof is now completed by using the fact that for all $q \geq 2$

$$\frac{1}{\prod_{i=2}^{\infty} (1 - q^{-i})} \left(1 + \sum_{l=2}^{\infty} \frac{1}{q^{l(l+1)/2}} \right) \leq (1.732)(1.142) \leq 2.$$

The upper bound on the first factor used the second assertion of Lemma 2.3. Indeed,

$$\prod_{i \geq 2} (1 - q^{-i}) \geq \frac{1 - 1/q - 1/q^2 + 1/q^5 + 1/q^7 - 1/q^{12} - 1/q^{15}}{1 - 1/q}.$$

The upper bound on the second factor used that

$$\begin{aligned}
 1 + \sum_{l=2}^{\infty} \frac{1}{q^{l(l+1)/2}} &\leq 1 + 1/2^3 + 1/2^6 + 1/2^{10} + \sum_{l=5}^{\infty} \frac{1}{2^{10+l}} \\
 &= 1 + 1/2^3 + 1/2^6 + 1/2^{10} + 2/2^{15} \leq 1.142. \quad \square
 \end{aligned}$$

We now present the proof of Theorem 4.1.

PROOF OF THEOREM 4.1. For the lower bound, one computes from the formula for $p_{0,n}$ in (21), in the case $n = 2m$ is even, that

$$\begin{aligned} p_{0,n} &= \frac{N(n, n)}{q^{\binom{n+1}{2}}} = (1 - 1/q)(1 - 1/q^2) \cdots (1 - 1/q^n) \prod_{i=1}^m \frac{1}{1 - q^{-2i}} \\ &= (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^{n-1}). \end{aligned}$$

Thus, the total variation distance between $\mathcal{Q}_{q,n}$ and \mathcal{Q}_q is at least

$$\begin{aligned} &\frac{1}{2}[p_{0,n} - p_0] \\ &\geq \frac{1}{2} \left[\left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{q^3}\right) \cdots \left(1 - \frac{1}{q^{n-1}}\right) \right. \\ &\quad \left. - \left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{q^3}\right) \cdots \left(1 - \frac{1}{q^{n+1}}\right) \right] \\ &= \frac{1}{2q^{n+1}} (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^{n-1}) \\ &\geq \frac{1}{2q^{n+1}} (1 - 1/q - 1/q^3) \\ &\geq \frac{0.18}{q^{n+1}}. \end{aligned}$$

The second inequality used Lemma 2.3, and the final inequality that $q \geq 2$.

When $n = 2m + 1$ is odd, we obtain similarly that

$$\begin{aligned} &\frac{1}{2}[p_{0,n} - p_0] \\ &\geq \frac{1}{2} \left[\left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{q^3}\right) \cdots \left(1 - \frac{1}{q^n}\right) \right. \\ &\quad \left. - \left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{q^3}\right) \cdots \left(1 - \frac{1}{q^{n+2}}\right) \right] \\ &= \frac{1}{2q^{n+2}} (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^n) \\ &\geq \frac{1}{2q^{n+2}} (1 - 1/q - 1/q^3) \\ &\geq \frac{0.18}{q^{n+2}}. \end{aligned}$$

To prove the upper bound, for any $A \subset \mathbb{N}_0$ we have

$$\begin{aligned}
 & |P(Q_n \in A) - P(Q \in A)| \\
 &= |E[h_A(Q_n)] - Q_q h_A| \\
 &= |E[f_A(Q_n + 1) - (q^{Q_n} - 1)f_A(Q_n)]| \\
 &= |E[\mathbf{1}_{n-Q_n} q^{-(n-Q_n)} f_A(Q_n + 1)]| \\
 &\leq \mathbf{1}_n q^{-n} |f_A(1)| P(Q_n = 0) \\
 &\quad + |E[\mathbf{1}_{n-Q_n} q^{-(n-Q_n)} f_A(Q_n + 1) \mathbf{1}(Q_n \geq 1)]| \\
 &\leq \mathbf{1}_n q^{-n} |f_A(1)| + E[\mathbf{1}_{n-Q_n} q^{-(n-Q_n)} \mathbf{1}(Q_n \geq 1)] \sup_{k \geq 2} |f_A(k)| \\
 &\leq \mathbf{1}_n q^{-n} |f_A(1)| + E[\mathbf{1}_{n-Q_n} q^{-(n-Q_n)}] \sup_{k \geq 2} |f_A(k)| \\
 &= \mathbf{1}_n q^{-n} |f_A(1)| + q^{-n} \sup_{k \geq 2} |f_A(k)| \\
 &\leq \mathbf{1}_n q^{-n} \left(\frac{1}{q} + \frac{1}{q^3} \right) + q^{-n} \left(\frac{2}{q^2} \right)
 \end{aligned}$$

and the result easily follows. The last two steps used Lemmas 4.3 and 4.4, respectively. \square

5. Symmetric matrices over finite fields with zero diagonal. This section treats the rank distribution (24), (27) of a random symmetric matrix with zero diagonal over a finite field \mathbb{F}_q , when q is a power of 2. Such matrices were termed “symplectic” in [23], which studied their rank distribution in the context of coding theory. We remark that by [8] and elementary manipulations, the quantity $N(n, 2h)$ defined in (24) below is also equal to the number of $n \times n$ skew-symmetric matrices of rank $2h$ (where now q is odd), so our results also apply in that context. We also mention that the two limiting distributions studied in this section arise in the work of the number theorist Swinnerton–Dyer on 2-Selmer groups [30]. We consider the cases where n is even and odd separately.

5.1. *Case of n even.* Throughout this subsection, let $n = 2m$, an even, nonnegative integer, and with $q \geq 2$, let Q_q be the distribution on \mathbb{N}_0 with mass function

$$(23) \quad p_k = \prod_{i \geq 1, i \text{ odd}} (1 - 1/q^i) \frac{q^{2k}}{\prod_{i=1}^{2k} (q^i - 1)}.$$

For $n \in \mathbb{N}_0$, let $Q_{q,n}$ be the distribution on $U_m = \{0, \dots, m\}$ with mass function

$$(24) \quad p_{k,n} = \frac{N(n, n - 2k)}{q^{\binom{n}{2}}} \quad \text{where } N(n, 2h) = \prod_{i=1}^h \frac{q^{2i-2}}{q^{2i} - 1} \prod_{i=0}^{2h-1} (q^{n-i} - 1).$$

THEOREM 5.1. *We have that*

$$\frac{0.18}{q^{n+1}} \leq \|Q_{q,n} - Q_q\|_{TV} \leq \frac{1.5}{q^{n+1}}.$$

We begin the proof of Theorem 5.1 by developing characterizations of the two distributions of interest.

LEMMA 5.2. *If Q has the Q_q distribution, then*

$$E[q^2 f(Q + 1)] = E[(q^{2Q-1} - 1)(q^{2Q} - 1)f(Q)]$$

for all functions f for which these expectations exist.

If Q_n has the $Q_{q,n}$ distribution then

$$(25) \quad \begin{aligned} E[(q^2 - q^{-2(m-Q_n-1)})f(Q_n + 1)] \\ = E[(q^{2Q_n-1} - 1)(q^{2Q_n} - 1)f(Q_n)] \end{aligned}$$

for all functions f for which these expectations exist.

PROOF. By taking ratios in (23), we obtain that for $k \in \mathbb{N}_0$

$$\frac{p_{k-1}}{p_k} = \frac{(q^{2k-1} - 1)(q^{2k} - 1)}{q^2}.$$

Setting $a(k) = q^2$ and $b(k) = (q^{2k-1} - 1)(q^{2k} - 1)$, applying Lemma 2.1 yields the first result.

Similarly, the second claim can be shown using Lemma 2.1 and (24) to yield

$$\frac{p_{k-1,n}}{p_{k,n}} = \frac{N(2m, 2(m - k + 1))}{N(2m, 2(m - k))} = \frac{(q^{2k-1} - 1)(q^{2k} - 1)}{q^2 - q^{-2(m-k)}},$$

upon setting $a(k) = q^2 - q^{-2(m-k)}$ and $b(k) = (q^{2k-1} - 1)(q^{2k} - 1)$, noting that $a(m + 1) = 0$. \square

LEMMA 5.3. *If Q_n has distribution $Q_{q,n}$, then*

$$Eq^{2Q_n} = q + 1 - q^{-n+1}.$$

PROOF. For k any integer, letting $f(x) = q^{kx}$ in (25) yields

$$E[(q^2 - q^{-2(m-Q_n-1)})q^{k(Q_n+1)}] = E[(q^{2Q_n-1} - 1)(q^{2Q_n} - 1)q^{kQ_n}].$$

Setting $c_k = Eq^{kQ_n}$, this identity yields

$$q^{-1}c_{k+4} - (1 + q^{-1} - q^{-2m+2+k})c_{k+2} + (1 - q^{k+2})c_k = 0.$$

Substituting $k = -2$ and using that $c_0 = 1$ we obtain

$$q^{-1}c_2 - (1 + q^{-1} - q^{-2m}) = 0,$$

so that

$$c_2 = q(1 + q^{-1} - q^{-2m}) = q + 1 - q^{-2m+1}. \quad \square$$

In the remainder of this subsection we consider the Stein equation (7) for the target distribution \mathcal{Q}_q with

$$a(k) = q^2 \quad \text{and} \quad b(k) = (q^{2k-1} - 1)(q^{2k} - 1)$$

and for $A \subset \mathbb{N}_0$ we let f_A denote the solution (9) when $h(k) = \mathbf{1}(k \in A)$.

LEMMA 5.4. *The function f_A satisfies*

$$\sup_{A \subset \mathbb{N}_0} |f_A(1)| \leq \frac{1}{q^3} + \frac{1}{q^5} \quad \text{and} \quad \sup_{A \subset \mathbb{N}_0, k \geq 2} |f_A(k)| \leq \frac{1.31}{q^7}.$$

PROOF. By Lemma 2.2, for all $A \subset \mathbb{N}_0$,

$$\begin{aligned} |f_A(1)| &\leq \frac{P(Q \geq 1)}{q^2} \\ &= \frac{1 - p_0}{q^2} \\ &= \frac{1}{q^2} \left(1 - \prod_{i \geq 1, i \text{ odd}} \left(1 - \frac{1}{q^i} \right) \right) \\ &\leq \frac{1}{q^2} \left(\frac{1}{q} + \frac{1}{q^3} \right) \\ &= \frac{1}{q^3} + \frac{1}{q^5}, \end{aligned}$$

where the second inequality used Lemma 2.3.

For $k \geq 1$, by (12) and (23),

$$\begin{aligned} |f_A(k+1)| &\leq \frac{P(Q \in U_k^c)}{q^2 p_k} \\ &= \frac{\prod_{i=1}^{2k} (q^i - 1)}{q^{2k+2}} \sum_{l=k+1}^{\infty} \frac{q^{2l}}{\prod_{i=1}^{2l} (q^i - 1)} \\ &= \frac{1}{q^2} \sum_{l=k+1}^{\infty} \frac{q^{2(l-k)}}{\prod_{i=2k+1}^{2l} (q^i - 1)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{q^2} \sum_{l=k+1}^{\infty} \frac{q^{l-k}}{q^{2(l^2-k^2)} \prod_{i=2k+1}^{2l} (1-q^{-i})} \\
 &\leq \frac{1}{q^2 \prod_{i=2k+1}^{\infty} (1-q^{-i})} \sum_{l=k+1}^{\infty} \frac{q^{l-k}}{q^{2(l^2-k^2)}} \\
 &= \frac{1}{q^2 \prod_{i=2k+1}^{\infty} (1-q^{-i})} \left(\frac{1}{q^{4k+1}} + \sum_{l=2}^{\infty} \frac{1}{q^{2l^2+4lk-l}} \right) \\
 &\leq \frac{1}{q^2 \prod_{i=2k+1}^{\infty} (1-q^{-i})} \left(\frac{1}{q^{4k+1}} + \frac{1}{q^{8k}} \sum_{l=2}^{\infty} \frac{1}{q^{2l^2-l}} \right).
 \end{aligned}$$

Hence, for all $k \geq 1$ we obtain

$$|f_A(k+1)| \leq \frac{1}{q^7 \prod_{i=3}^{\infty} (1-q^{-i})} \left(1 + \frac{1}{q^3} \sum_{l=2}^{\infty} \frac{1}{q^{2l^2-l}} \right)$$

and the proof is now completed using the fact that for all $q \geq 2$

$$\frac{1}{\prod_{i=3}^{\infty} (1-q^{-i})} \left(1 + \frac{1}{q^3} \sum_{l=2}^{\infty} \frac{1}{q^{2l^2-l}} \right) \leq (1.29854)(1.002) \leq 1.31.$$

The upper bound on the first factor used part 2 of Lemma 2.3. The upper bound on the second factor used that

$$\begin{aligned}
 1 + \frac{1}{q^3} \sum_{l=2}^{\infty} \frac{1}{q^{2l^2-l}} &\leq 1 + \frac{1}{2^3} \left(\frac{1}{2^6} + \sum_{l=3}^{\infty} \frac{1}{2^{12+l}} \right) \\
 &= 1 + \frac{1}{8} \left(\frac{1}{2^6} + \frac{2}{2^{15}} \right) \leq 1.002. \quad \square
 \end{aligned}$$

We now present the proof of Theorem 5.1.

PROOF OF THEOREM 5.1. From the formula for $p_{0,n}$, one has that

$$p_{0,n} = \frac{N(n,n)}{q^{\binom{n}{2}}} = (1-1/q)(1-1/q^3) \cdots (1-1/q^{n-1}).$$

The argument in the proof of Theorem 4.1 now shows that total variation distance between $\mathcal{Q}_{q,n}$ and \mathcal{Q}_q is at least $0.18/q^{n+1}$.

For the upper bound, arguing as in the proof of Theorem 4.1 we obtain

$$\begin{aligned}
 &|P(Q_n \in A) - P(Q \in A)| \\
 &= |E[h_A(Q_n)] - \mathcal{Q}_q h_A| \\
 &= |E[q^2 f_A(Q_n + 1) - (q^{2Q_n-1} - 1)(q^{2Q_n} - 1) f_A(Q_n)]|
 \end{aligned}$$

$$\begin{aligned}
 &= |E[q^{-2(m-Q_n-1)} f_A(Q_n + 1)]| \\
 &\leq |q^{-2(m-1)} f_A(1)| P(Q_n = 0) \\
 &\quad + |E[q^{-2(m-Q_n-1)} f_A(Q_n + 1) \mathbf{1}(Q_n \geq 1)]| \\
 &\leq q^{-2(m-1)} |f_A(1)| + E[q^{-2(m-Q_n-1)} \mathbf{1}(Q_n \geq 1)] \sup_{k \geq 2} |f_A(k)| \\
 &\leq q^{-2(m-1)} |f_A(1)| + E[q^{-2(m-Q_n-1)}] \sup_{k \geq 2} |f_A(k)| \\
 &= q^{-2(m-1)} |f_A(1)| + q^{-2(m-1)} (q + 1 - q^{-2m+1}) \sup_{k \geq 2} |f_A(k)| \\
 &\leq q^{-2(m-1)} |f_A(1)| + q^{-2(m-1)} (q + 1) \sup_{k \geq 2} |f_A(k)| \\
 &\leq q^{-n+2} \left(\frac{1}{q^3} + \frac{1}{q^5} \right) + 1.31 (q^{-n+3} + q^{-n+2}) \frac{1}{q^7} \\
 &= q^{-(n+1)} + q^{-(n+3)} + 1.31 q^{-(n+4)} + 1.31 q^{-(n+5)} \\
 &\leq 1.5 q^{-(n+1)}
 \end{aligned}$$

as claimed. Note that Lemma 5.3 was used in the fourth equality, and Lemma 5.4 in the second to last inequality. \square

5.2. *Case of n odd.* Throughout this subsection, let $n = 2m + 1$, a positive, odd integer, and with $q \geq 2$, let \mathcal{Q}_q be the distribution on \mathbb{N}_0 with mass function

$$(26) \quad p_k = \prod_{i \geq 1, i \text{ odd}} (1 - 1/q^i) \frac{q^{2k+1}}{\prod_{i=1}^{2k+1} (q^i - 1)}.$$

For $n \in \mathbb{N}_0$ let $\mathcal{Q}_{q,n}$ be the distribution on $\{0, \dots, m\}$ with mass function

$$(27) \quad p_{k,n} = \frac{N(n, n - 1 - 2k)}{q^{\binom{n}{2}}},$$

where $N(n, 2h)$ is given in (24).

Our main result is the following theorem.

THEOREM 5.5. *We have that*

$$\frac{0.37}{q^{n+2}} \leq \|\mathcal{Q}_{q,n} - \mathcal{Q}_q\|_{\text{TV}} \leq \frac{2.2}{q^{n+2}}.$$

We again begin by developing characterizing equations for the distributions under study.

LEMMA 5.6. *If Q has the \mathcal{Q}_q distribution, then*

$$E[q^2 f(Q + 1)] = E[(q^{2Q+1} - 1)(q^{2Q} - 1)f(Q)]$$

for all functions f for which these expectations exist.

If Q_n has the $\mathcal{Q}_{q,n}$ distribution then

$$(28) \quad \begin{aligned} E[(q^2 - q^{-2(m-Q_n-1)})f(Q_n + 1)] \\ = E[(q^{2Q_n+1} - 1)(q^{2Q_n} - 1)f(Q_n)] \end{aligned}$$

for all functions f for which these expectations exist.

PROOF. By taking ratios in (26), we obtain

$$\frac{p_{k-1}}{p_k} = \frac{(q^{2k+1} - 1)(q^{2k} - 1)}{q^2}.$$

Setting $a(k) = q^2$ and $b(k) = (q^{2k+1} - 1)(q^{2k} - 1)$, Lemma 2.1 yields the first claim. Similarly, the second can be shown by applying (27) to yield

$$\frac{p_{k-1,n}}{p_{k,n}} = \frac{N(n, 2(m - k + 1))}{N(n, 2(m - k))} = \frac{(q^{2k+1} - 1)(q^{2k} - 1)}{q^2 - q^{-2(m-k)}}$$

and then invoking Lemma 2.1 with $a(k) = q^2 - q^{-2(m-k)}$ and $b(k) = (q^{2k+1} - 1)(q^{2k} - 1)$, noting $a(m + 1) = 0$. \square

LEMMA 5.7. *If Q_n has distribution $\mathcal{Q}_{q,n}$ then*

$$E q^{2Q_n} = 1 + q^{-1} - q^{-n}.$$

PROOF. For k any integer, letting $f(x) = q^{kx}$ in (28) yields

$$E[(q^2 - q^{-2(m-Q_n-1)})q^{k(Q_n+1)}] = E[(q^{2Q_n+1} - 1)(q^{2Q_n} - 1)q^{kQ_n}].$$

Setting $c_k = E[q^{kQ_n}]$, this identity yields

$$q c_{k+4} - (1 + q - q^{-2m+2+k})c_{k+2} + (1 - q^{k+2})c_k = 0.$$

Substituting $k = -2$ and using that $c_0 = 1$ we obtain

$$q c_2 - (1 + q - q^{-2m}) = 0,$$

so that

$$c_2 = q^{-1}(1 + q - q^{-2m}) = 1 + q^{-1} - q^{-2m-1}. \quad \square$$

In the remainder of this subsection, we consider the Stein equation (7) for the target distribution \mathcal{Q}_q with

$$a(k) = q^2 \quad \text{and} \quad b(k) = (q^{2k+1} - 1)(q^{2k} - 1)$$

and for $A \subset \mathbb{N}_0$ we let f_A denote the solution (9) when $h(k) = \mathbf{1}(k \in A)$.

LEMMA 5.8. *The function f_A satisfies*

$$\sup_{A \subset \mathbb{N}_0} |f_A(1)| \leq \frac{2}{q^5} \quad \text{and} \quad \sup_{A \subset \mathbb{N}_0, k \geq 2} |f_A(k)| \leq \frac{1.14}{q^9}.$$

PROOF. By Lemma 2.2, for all $A \subset \mathbb{N}_0$,

$$\begin{aligned} |f_A(1)| &\leq \frac{P(Q \geq 1)}{q^2} \\ &= \frac{1 - p_0}{q^2} \\ &= \frac{1}{q^2} \left(1 - \frac{q}{q-1} \prod_{i \geq 1, i \text{ odd}} \left(1 - \frac{1}{q^i} \right) \right) \\ &= \frac{1}{q^2} \left(1 - \prod_{i \geq 3, i \text{ odd}} \left(1 - \frac{1}{q^i} \right) \right) \\ &\leq \frac{1}{q^2} \left(\frac{2}{q^3} \right) = \frac{2}{q^5}, \end{aligned}$$

where the second inequality used Lemma 2.3.

For $k \geq 1$, by (12) and (26),

$$\begin{aligned} |f_A(k+1)| &\leq \frac{P(Q \in U_k^c)}{q^2 p_k} \\ &= \frac{\prod_{i=1}^{2k+1} (q^i - 1)}{q^{2k+3}} \sum_{l=k+1}^{\infty} \frac{q^{2l+1}}{\prod_{i=1}^{2l+1} (q^i - 1)} \\ &= \frac{1}{q^2} \sum_{l=k+1}^{\infty} \frac{q^{2(l-k)}}{\prod_{i=2k+2}^{2l+1} (q^i - 1)} \\ &= \frac{1}{q^2} \sum_{l=k+1}^{\infty} \frac{1}{q^{2(l^2-k^2)+(l-k)} \prod_{i=2k+2}^{2l+1} (1 - q^{-i})} \\ &\leq \frac{1}{q^2 \prod_{i=2k+2}^{\infty} (1 - q^{-i})} \sum_{l=k+1}^{\infty} \frac{1}{q^{2(l^2-k^2)+(l-k)}} \\ &= \frac{1}{q^2 \prod_{i=2k+2}^{\infty} (1 - q^{-i})} \left(\frac{1}{q^{4k+3}} + \sum_{l=2}^{\infty} \frac{1}{q^{2l^2+4lk+l}} \right) \\ &\leq \frac{1}{q^2 \prod_{i=2k+2}^{\infty} (1 - q^{-i})} \left(\frac{1}{q^{4k+3}} + \frac{1}{q^{8k}} \sum_{l=2}^{\infty} \frac{1}{q^{2l^2+l}} \right). \end{aligned}$$

Hence, for all $k \geq 1$, we obtain

$$|f_A(k + 1)| \leq \frac{1}{q^9 \prod_{i=4}^{\infty} (1 - q^{-i})} \left(1 + \frac{1}{q} \sum_{l=2}^{\infty} \frac{1}{q^{2l^2+l}} \right)$$

and the proof is now completed by using the fact that for all $q \geq 2$,

$$\frac{1}{\prod_{i=4}^{\infty} (1 - q^{-i})} \left(1 + \frac{1}{q} \sum_{l=2}^{\infty} \frac{1}{q^{2l^2+l}} \right) \leq (1.137)(1.0005) \leq 1.14.$$

The inequality $\prod_{i=4}^{\infty} (1 - q^{-i})^{-1} \leq 1.137$ is obtained by applying part 2 of Lemma 2.3. We also used that

$$\begin{aligned} 1 + \frac{1}{q} \sum_{l=2}^{\infty} \frac{1}{q^{2l^2+l}} &\leq 1 + \frac{1}{2} \left(\frac{1}{2^{10}} + \sum_{l=3}^{\infty} \frac{1}{2^{18+l}} \right) \\ &= 1 + \frac{1}{2} \left(\frac{1}{2^{10}} + \frac{2}{2^{21}} \right) \leq 1.0005. \end{aligned} \quad \square$$

We now present the proof of Theorem 5.5.

PROOF OF THEOREM 5.5. From the formula (27) for $p_{0,n}$, we obtain

$$\begin{aligned} p_{0,n} &= \frac{N(n, n - 1)}{q^{\binom{n}{2}}} \\ &= (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^n) \frac{q}{q - 1}. \end{aligned}$$

Thus, now applying (26), the total variation distance between $\mathcal{Q}_{q,n}$ and \mathcal{Q}_q is at least

$$\begin{aligned} &\frac{1}{2} [p_{0,n} - p_0] \\ &\geq \frac{q}{2(q - 1)} \left[\left(1 - \frac{1}{q} \right) \left(1 - \frac{1}{q^3} \right) \cdots \left(1 - \frac{1}{q^n} \right) \right. \\ &\quad \left. - \left(1 - \frac{1}{q} \right) \left(1 - \frac{1}{q^3} \right) \cdots \left(1 - \frac{1}{q^{n+2}} \right) \right] \\ &= \frac{q}{2(q - 1)q^{n+2}} (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^n) \\ &= \frac{1}{2q^{n+2}} (1 - 1/q^3) \cdots (1 - 1/q^n) \\ &\geq \frac{1}{2q^{n+2}} (1 - 2/q^3) \\ &\geq \frac{0.37}{q^{n+2}}. \end{aligned}$$

The second inequality used the fourth claim of Lemma 2.3.

Arguing as for the proof of Theorem 4.1, for any $A \subset \mathbb{N}_0$ we have

$$\begin{aligned}
 & |P(Q_n \in A) - P(Q \in A)| \\
 &= |E[h_A(Q_n)] - Q_q h_A| \\
 &= |E[q^2 f_A(Q_n + 1) - (q^{2Q_n+1} - 1)(q^{2Q_n} - 1) f_A(Q_n)]| \\
 &= |E[q^{-2(m-Q_n-1)} f_A(Q_n + 1)]| \\
 &\leq q^{-2(m-1)} |f_A(1)| P(Q_n = 0) \\
 &\quad + |E[q^{-2(m-Q_n-1)} f_A(Q_n + 1) \mathbf{1}(Q_n \geq 1)]| \\
 &\leq q^{-2(m-1)} |f_A(1)| + E[q^{-2(m-Q_n-1)} \mathbf{1}(Q_n \geq 1)] \sup_{k \geq 2} |f_A(k)| \\
 &\leq q^{-2(m-1)} |f_A(1)| + E[q^{-2(m-Q_n-1)}] \sup_{k \geq 2} |f_A(k)| \\
 &= q^{-2(m-1)} |f_A(1)| + q^{-2(m-1)} (1 + q^{-1} - q^{-n}) \sup_{k \geq 2} |f_A(k)| \\
 &\leq q^{-2(m-1)} |f_A(1)| + q^{-2(m-1)} (1 + q^{-1}) \sup_{k \geq 2} |f_A(k)| \\
 &\leq q^{-n+3} \frac{2}{q^5} + 1.14(q^{-n+3} + q^{-n+2}) \frac{1}{q^9} \\
 &= 2q^{-(n+2)} + 1.14q^{-(n+6)} + 1.14q^{-(n+7)} \\
 &\leq 2.2q^{-(n+2)}
 \end{aligned}$$

as claimed, where we have applied Lemmas 5.7 and 5.8 in the second to last equality, and inequality, respectively. \square

6. Skew centrosymmetric matrices over finite fields. An $n \times n$ matrix A is called skew centrosymmetric if $A_{ij} = -A_{ji}$ and $A_{ij} = A_{n+1-j, n+1-i}$. This section studies the rank distributions (29) and (30) of a randomly chosen skew centrosymmetric matrix with entries in \mathbb{F}_q for q odd.

Suppose that n is even. Waterhouse [33] shows that the total number of skew centrosymmetric matrices is $q^{(n/2)^2}$, that all such matrices have even rank, and that the proportion of $n \times n$ skew centrosymmetric matrices of rank $n - 2k$ is equal to

$$(29) \quad P_{k,n} = \frac{N(n, n - 2k)}{q^{(n/2)^2}},$$

$$\text{where } N(n, 2h) = \prod_{j=0}^{n/2-h-1} \frac{q^{n/2} - q^j}{q^{n/2-h} - q^j} \prod_{i=0}^{h-1} (q^{n/2} - q^i).$$

We claim that $p_{k,n}$ in (29) is exactly equal to the probability that a uniformly chosen $n/2 \times n/2$ random matrix with entries from \mathbb{F}_q has rank $n/2 - k$. Indeed, pulling out factors of q , one can write (29) as

$$\frac{1}{q^{k^2}} \prod_{j=0}^{k-1} \left(\frac{1 - q^{j-n/2}}{1 - q^{j-k}} \right) \prod_{j=k+1}^{n/2} (1 - q^{-j}).$$

Comparing this expression with (1) for the case $m = 0$ with n replaced by $n/2$ shows that it is sufficient to prove that

$$\prod_{j=0}^{k-1} \left(\frac{1 - q^{j-n/2}}{1 - q^{j-k}} \right) = \frac{\prod_{j=k+1}^{n/2} (1 - q^{-j})}{\prod_{j=1}^{n/2-k} (1 - q^{-j})}.$$

This identity holds since both

$$\prod_{j=0}^{k-1} (1 - q^{j-n/2}) \prod_{j=1}^{n/2-k} (1 - q^{-j})$$

and

$$\prod_{j=0}^{k-1} (1 - q^{j-k}) \prod_{j=k+1}^{n/2} (1 - q^{-j})$$

are equal to $\prod_{i=1}^{n/2} (1 - 1/q^i)$. Hence, the following corollary is immediate from Theorem 1.1.

COROLLARY 6.1. *For $q \geq 2$, let \mathcal{Q}_q be the distribution (2) on \mathbb{N}_0 , specialized to $m = 0$. For n even in \mathbb{N}_0 , let $\mathcal{Q}_{q,n}$ be the distribution on $\{0, \dots, n/2\}$ with mass function (29). Then*

$$\frac{1}{8q^{n/2+1}} \leq \|\mathcal{Q}_{q,n} - \mathcal{Q}_q\|_{\text{TV}} \leq \frac{3}{q^{n/2+1}}.$$

Now suppose that n is odd. Waterhouse [33] shows that the total number of skew centrosymmetric matrices is $q^{(n-1)^2/4+(n-1)/2}$, that all such matrices have even rank and that the number of $n \times n$ skew centrosymmetric matrices of rank $2h$ is equal to

$$N(n, 2h) = \prod_{j=0}^{(n-1)/2-h} \frac{q^{(n-1)/2+1} - q^j}{q^{(n-1)/2+1-h} - q^j} \prod_{i=0}^{h-1} (q^{(n-1)/2} - q^i).$$

Hence,

$$(30) \quad p_{k,n} = \frac{N(n, n - 2k - 1)}{q^{(n-1)^2/4+(n-1)/2}}, \quad k \in U_{(n-1)/2} = \{0, 1, \dots, (n-1)/2\}$$

is the proportion of skew centrosymmetric matrices of rank $n - 2k - 1$. The main result in this section is Theorem 6.2, which provides bounds on the total variation distance between $\mathcal{Q}_{q,n}$, the distribution given in (30), and \mathcal{Q}_q , given by

$$(31) \quad p_k = \frac{\prod_{i \geq 1} (1 - 1/q^i)}{q^{k^2+k} (1 - 1/q^{k+1}) \prod_{i=1}^k (1 - 1/q^i)^2}, \quad k \in \mathbb{N}_0.$$

THEOREM 6.2. *For $n \geq 1$ odd, and $q \geq 2$, we have that*

$$\frac{1}{4q^{(n+3)/2}} \leq \|\mathcal{Q}_{q,n} - \mathcal{Q}_q\|_{TV} \leq \frac{3}{q^{(n+3)/2}}.$$

We begin with the following characterization lemma.

LEMMA 6.3. *If Q has the \mathcal{Q}_q distribution, then*

$$E[qf(Q + 1)] = E[(q^Q - 1)(q^{Q+1} - 1)f(Q)]$$

for all functions f for which these expectations exist.

If Q_n has the $\mathcal{Q}_{q,n}$ distribution, then

$$(32) \quad E[(q - q^{Q_n+1-(n-1)/2})f(Q_n + 1)] = E[(q^{Q_n} - 1)(q^{Q_n+1} - 1)f(Q_n)]$$

for all functions f for which these expectations exist.

PROOF. For the first assertion, one calculates that

$$\frac{p_{k-1}}{p_k} = \frac{(q^k - 1)(q^{k+1} - 1)}{q}, \quad k \in \mathbb{N}_0.$$

Taking $a(k) = q$ and $b(k) = (q^k - 1)(q^{k+1} - 1)$ in Lemma 2.1, the first assertion follows.

For the second assertion, one calculates that

$$\frac{p_{k-1,n}}{p_{k,n}} = \frac{N(n, n - 2k + 1)}{N(n, n - 2k - 1)} = \frac{(q^k - 1)(q^{k+1} - 1)}{q - q^{k-(n-1)/2}}, \quad k \in U_{(n-1)/2}.$$

Taking $a(k) = q - q^{k-(n-1)/2}$ and $b(k) = (q^k - 1)(q^{k+1} - 1)$, noting that $a((n - 1)/2 + 1) = 0$, the second assertion follows by Lemma 2.1. \square

Lemma 6.4 calculates the expected value of q^{Q_n} .

LEMMA 6.4. *If Q_n has distribution $\mathcal{Q}_{q,n}$, then*

$$E[q^{Q_n}] = 1 + \frac{1}{q} - \frac{1}{q^{(n+1)/2}}.$$

PROOF. Let $c_k = E[q^k Q_n]$, and set $f(x) = q^{kx}$ in (32). Elementary manipulations yield the recurrence

$$q c_{k+2} = (q + 1 - q^{k+1-(n-1)/2})c_{k+1} + (q^{k+1} - 1)c_k.$$

The result now follows by setting $k = -1$ and using that $c_0 = 1$. \square

In the remainder of this section, we consider the Stein equation (7) for the target distribution Q_q with

$$a(k) = q \quad \text{and} \quad b(k) = (q^k - 1)(q^{k+1} - 1)$$

and for $A \subset \mathbb{N}_0$ we let f_A denote the solution (9) when $h(k) = \mathbf{1}(k \in A)$.

LEMMA 6.5. *The function f_A satisfies*

$$\sup_{A \subset \mathbb{N}_0} |f_A(k)| \leq \frac{2}{q^3}.$$

PROOF. By Lemma 2.2 and (31),

$$\begin{aligned} |f_A(1)| &\leq \frac{P(Q \geq 1)}{q} \\ &= \frac{1 - p_0}{q} \\ &= \frac{1 - \prod_{i \geq 2} (1 - 1/q^i)}{q} \\ &\leq \frac{1 - (1 - \sum_{i \geq 2} 1/q^i)}{q} \\ &= \frac{1}{q^3(1 - 1/q)} \\ &\leq 2/q^3, \end{aligned}$$

where we have applied (13) in the second inequality, and used that $q \geq 2$.

For $k \geq 1$, by (12),

$$\begin{aligned} |f_A(k + 1)| &\leq \frac{P(Q \in U_k^c)}{q p_k} \\ &= q^{k^2+k-1} (1 - 1/q^{k+1}) \\ &\quad \times \prod_{i=1}^k (1 - 1/q^i)^2 \sum_{l=k+1}^{\infty} \frac{1}{q^{l^2+l} (1 - 1/q^{l+1}) \prod_{i=1}^l (1 - 1/q^i)^2} \end{aligned}$$

$$\begin{aligned}
 &= q^{k^2+k-1} \sum_{l=k+1}^{\infty} \frac{1}{q^{l^2+l}(1-1/q^{k+1})(1-1/q^{l+1}) \prod_{j=k+2}^l (1-1/q^j)^2} \\
 &\leq q^{k^2+k-1} \sum_{l=k+1}^{\infty} \frac{1}{q^{l^2+l} \prod_{j=k+1}^l (1-1/q^j)^2} \\
 &\leq \frac{q^{k^2+k-1}}{\prod_{j=k+1}^{\infty} (1-1/q^j)^2} \sum_{l=k+1}^{\infty} \frac{1}{q^{l^2+l}} \\
 &\leq \frac{q^{k^2+k-1}}{(1-\sum_{j=k+1}^{\infty} 1/q^j)^2} \sum_{l=k+1}^{\infty} \frac{1}{q^{l^2+l}} \\
 &= \frac{q^{k^2+k-1}}{(1-(1/(q^k(q-1))))^2} \sum_{l=1}^{\infty} \frac{1}{q^{(k+l)^2+k+l}} \\
 &\leq \frac{4}{q} \sum_{l=1}^{\infty} \frac{1}{q^{l^2+l+2kl}} \\
 &\leq \frac{4}{q^3} \sum_{l=1}^{\infty} \frac{1}{q^{l^2+l}} \\
 &\leq \frac{4}{q^3} \sum_{l=1}^{\infty} \frac{1}{2^{l+1}} \\
 &= \frac{2}{q^3},
 \end{aligned}$$

where (13) was applied in the fourth inequality. \square

We now present the proof of Theorem 6.2.

PROOF OF THEOREM 6.2. From the formula (30) for $p_{0,n}$, one computes that

$$p_{0,n} = \frac{N(n, n-1)}{q^{(n-1)^2/4+(n-1)/2}} = (1-1/q^2)(1-1/q^3) \cdots (1-1/q^{(n+1)/2}).$$

Thus, using (31), the total variation distance between $\mathcal{Q}_{q,n}$ and \mathcal{Q}_q is at least

$$\begin{aligned}
 \frac{1}{2}[p_{0,n} - p_0] &\geq \frac{1}{2}[(1-1/q^2)(1-1/q^3) \cdots (1-1/q^{(n+1)/2})] \\
 &\quad - \frac{1}{2}[(1-1/q^2)(1-1/q^3) \cdots (1-1/q^{(n+3)/2})] \\
 &= \frac{1}{2q^{(n+3)/2}}(1-1/q^2)(1-1/q^3) \cdots (1-1/q^{(n+1)/2}).
 \end{aligned}$$

By part 1 of Lemma 2.3,

$$\begin{aligned} & (1 - 1/q^2)(1 - 1/q^3) \cdots (1 - 1/q^{(n+1)/2}) \\ &= \frac{(1 - 1/q)(1 - 1/q^2) \cdots (1 - 1/q^{(n+1)/2})}{(1 - 1/q)} \\ &\geq \frac{1 - 1/q - 1/q^2}{1 - 1/q} \\ &\geq 1/2. \end{aligned}$$

It follows that the total variation distance between $\mathcal{Q}_{q,n}$ and \mathcal{Q}_q is at least $1/(4q^{(n+3)/2})$.

For the upper bound, arguing as in Theorem 1.1,

$$\begin{aligned} & |P(Q_n \in A) - P(Q \in A)| \\ &= |E[h_A(Q_n)] - \mathcal{Q}_q h_A| \\ &= |E[qf_A(Q_n + 1) - (q^{Q_n} - 1)(q^{Q_n+1} - 1)f_A(Q_n)]| \\ &= |E[q^{Q_n+1-(n-1)/2} f_A(Q_n + 1)]| \\ &\leq \|f_A\| E[q^{Q_n+1-(n-1)/2}]. \end{aligned}$$

By Lemmas 6.4 and 6.5, this quantity is at most

$$\frac{2}{q^3} q^{1-(n-1)/2} (1 + 1/q) \leq \frac{3}{q^{(n+3)/2}}. \quad \square$$

7. Hermitian matrices over finite fields. Let q be odd. Suppose that $\theta \in \mathbb{F}_{q^2}, \theta^2 \in \mathbb{F}_q$, but $\theta \notin \mathbb{F}_q$. Then any $\alpha \in \mathbb{F}_{q^2}$ can be written $\alpha = a + b\theta$ with $a, b \in \mathbb{F}_q$. By the conjugate of α , we mean $\bar{\alpha} = a - b\theta$. If $A = (\alpha_{ij})$ is a square matrix, $\alpha_{ij} \in \mathbb{F}_{q^2}$, let $A^* = \bar{A}' = (\bar{\alpha}_{ij})'$, where the prime denotes transpose. Then A is said to be Hermitian if and only if $A^* = A$.

By [9], for q odd the total number of $n \times n$ Hermitian matrices over \mathbb{F}_q is q^{n^2} , and the total number of such matrices with rank r is

$$N(n, r) = q^{\binom{n}{2}} \prod_{i=1}^r \frac{q^{2n-2(r-i)} - 1}{q^i - (-1)^i}.$$

Hence, the proportion of such matrices with rank $n - k$ is given by

$$(33) \quad p_{k,n} = \frac{N(n, n - k)}{q^{n^2}}, \quad k \in U_n = \{0, \dots, n\}.$$

In this section we compute total variation bounds between the distribution (33), denoted $\mathcal{Q}_{q,n}$, and the distribution

$$(34) \quad p_k = \prod_{i \text{ odd}} \frac{1}{1 + 1/q^i} \cdot \frac{1}{q^{k^2} \prod_{i=1}^k (1 - 1/q^{2i})},$$

which we denote here by \mathcal{Q}_q .

REMARK. The distribution (34) also arises as a limiting law in the study of the dimension of the fixed space of a random element of the finite unitary group $U(n, q)$. More precisely, the paper [26] proves that for k fixed, the chance that a uniformly chosen random element of $U(n, q)$ has a k dimensional fixed space tends to p_k as $n \rightarrow \infty$. See [15] for another proof.

The main theorem of this section is the following result.

THEOREM 7.1. *For all $n \geq 1$ and $q \geq 2$, we have*

$$\frac{0.07}{q^{n+1}} \leq \|\mathcal{Q}_{q,n} - \mathcal{Q}_q\|_{\text{TV}} \leq \frac{2.3}{q^{n+1}}.$$

The following lemma characterizes the two distributions of interest in this section.

LEMMA 7.2. *If Q has the \mathcal{Q}_q distribution, then*

$$E[qf(Q + 1)] = E[(q^{2Q} - 1)f(Q)]$$

for all functions f for which these expectations exist.

If Q_n has the $\mathcal{Q}_{q,n}$ distribution, then

$$(35) \quad E[(q - (-1)^{n-Q_n} q^{Q_n - n + 1})f(Q_n + 1)] = E[(q^{2Q_n} - 1)f(Q_n)]$$

for all functions f for which these expectations exist.

PROOF. For the first assertion, one calculates from (34) that

$$\frac{p_{k-1}}{p_k} = \frac{q^{2k} - 1}{q} \quad \text{for all } k \in \mathbb{N}_0.$$

Taking $a(k) = q$ and $b(k) = q^{2k} - 1$ in Lemma 2.1, the first assertion follows.

For the second assertion, one calculates that

$$\frac{p_{k-1,n}}{p_{k,n}} = \frac{N(n, n - k + 1)}{N(n, n - k)} = \frac{q^{2k} - 1}{q - (-1)^{n-k+1} q^{k-n}}.$$

Taking $a(k) = q - (-1)^{n-k+1} q^{k-n}$ and $b(k) = q^{2k} - 1$ in Lemma 2.1, and noting $a(n + 1) = 0$, the second assertion follows. \square

Next, we handle the moment $E[q^{Q_n}]$. Unlike all our other moment computations where we obtain equality, here we derive an upper bound.

LEMMA 7.3. *If Q_n has the $\mathcal{Q}_{q,n}$ distribution, then*

$$E(q^{Q_n}) \leq 2 + q^{-n}.$$

PROOF. Setting $f(x) = q^{-x}$ in (35) implies that

$$E[q^{-Q_n} - (-1)^{n-Q_n}q^{-n}] = E[q^{Q_n} - q^{-Q_n}].$$

Thus,

$$E[q^{Q_n}] = E[2q^{-Q_n} - (-1)^{n-Q_n}q^{-n}] \leq 2 + q^{-n}. \quad \square$$

In the remainder of this section, we consider the Stein equation (7) for the target distribution Q_q with

$$a(k) = q \quad \text{and} \quad b(k) = q^{2k} - 1$$

and for $A \subset \mathbb{N}_0$ we let f_A denote the solution (9) when $h(k) = \mathbf{1}(k \in A)$. Our next task is to provide a bound on f_A . In the following, we will apply the identity

$$(36) \quad \prod_{i \text{ odd}} (1 - 1/q^i) \prod_{i \text{ even}} (1 + 1/q^i) = \prod_{i \text{ odd}} \frac{1}{(1 + 1/q^i)},$$

which holds since

$$\prod_{i \text{ odd}} (1 - 1/q^i) = \frac{\prod_i (1 - 1/q^i)}{\prod_i (1 - 1/q^{2i})} = \prod_i \frac{1}{(1 + 1/q^i)}.$$

LEMMA 7.4. *The function f_A satisfies*

$$\sup_{A \subset \mathbb{N}_0} |f_A(1)| \leq \frac{1.1}{q^2} \quad \text{and} \quad \sup_{A \subset \mathbb{N}_0, k \geq 2} |f_A(k)| \leq \frac{1.8}{q^4} \quad \text{for all } q \geq 2.$$

PROOF. By Lemma 2.2,

$$|f_A(1)| \leq \frac{P(Q \geq 1)}{q} = \frac{1 - p_0}{q}.$$

By (34), (36) and the third claim of Lemma 2.3,

$$\begin{aligned} p_0 &= \prod_{i \text{ odd}} (1 - 1/q^i) \prod_{i \text{ even}} (1 + 1/q^i) \\ &\geq (1 - 1/q - 1/q^3)(1 + 1/q^2) \\ &\geq 1 - 1/q - 1/q^5. \end{aligned}$$

Thus, $1 - p_0 \leq 1/q + 1/q^5 \leq 1.1/q$, and hence $|f_A(1)| \leq 1.1/q^2$, for all $q \geq 2$.

For $k \geq 1$, by (12),

$$\begin{aligned} |f_A(k+1)| &\leq \frac{P(Q \geq k+1)}{qp_k} \\ (37) \quad &= q^{k^2-1} \prod_{i=1}^k (1 - 1/q^{2i}) \sum_{l=k+1}^{\infty} \frac{1}{q^{l^2} \prod_{j=1}^l (1 - 1/q^{2j})} \end{aligned}$$

$$\begin{aligned}
 &= q^{k^2-1} \sum_{l=k+1}^{\infty} \frac{1}{q^{l^2} \prod_{j=k+1}^l (1 - 1/q^{2j})} \\
 &\leq \frac{q^{k^2-1}}{\prod_{j=k+1}^{\infty} (1 - 1/q^{2j})} \sum_{l=k+1}^{\infty} \frac{1}{q^{l^2}}.
 \end{aligned}$$

Since $k \geq 1$, using (13) we have that

$$\begin{aligned}
 (38) \quad \frac{1}{\prod_{j=k+1}^{\infty} (1 - 1/q^{2j})} &\leq \frac{1}{\prod_{j=2}^{\infty} (1 - 1/q^{2j})} \leq \frac{1}{1 - \sum_{j=2}^{\infty} q^{-2j}} = d_q \\
 &\text{where } d_q = \frac{1}{1 - (1/(q^4 - q^2))}.
 \end{aligned}$$

Thus, from (37),

$$\begin{aligned}
 (39) \quad |f_A(k + 1)| &\leq d_q \cdot q^{k^2-1} \sum_{l=k+1}^{\infty} \frac{1}{q^{l^2}} \\
 &= d_q \cdot q^{k^2-1} \sum_{l=0}^{\infty} \frac{1}{q^{(k+1+l)^2}} \\
 &\leq d_q \frac{q^{k^2-1}}{q^{(k+1)^2}} \sum_{l=0}^{\infty} \frac{1}{q^{l^2}} \\
 &= \frac{d_q}{q^{2k+2}} \sum_{l=0}^{\infty} \frac{1}{q^{l^2}} \\
 &\leq \frac{d_q s_q}{q^4} \quad \text{where } s_q = \sum_{l=0}^{\infty} \frac{1}{q^{l^2}},
 \end{aligned}$$

using $k \geq 1$ in the final inequality. Now using that d_q and s_q are decreasing for $q \geq 2$, and that

$$\sum_{l=0}^{\infty} \frac{1}{q^{l^2}} \leq 1 + \frac{1}{2} + \sum_{l=2}^{\infty} \frac{1}{2^{l+2}} = 1.625$$

we obtain the second claim of the lemma. \square

Now we present the proof of the main result of this section, Theorem 7.1.

PROOF OF THEOREM 7.1. We first compute a lower bound for the case where n is odd. From (33), we have

$$\begin{aligned}
 p_{0,n} &= (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^n) \\
 &\quad \times (1 + 1/q^2)(1 + 1/q^4) \cdots (1 + 1/q^{n-1}).
 \end{aligned}$$

By (34) and (36),

$$\begin{aligned}
 p_0 &\geq (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^{n+2}) \\
 &\quad \times (1 + 1/q^2)(1 + 1/q^4) \cdots (1 + 1/q^{n+1}) \\
 &= (1 + 1/q^{n+1})(1 - 1/q^{n+2})p_{0,n}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (40) \quad p_0 - p_{0,n} &\geq (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^n) \\
 &\quad \times (1 + 1/q^2)(1 + 1/q^4) \cdots (1 + 1/q^{n-1}) \\
 &\quad \times [(1 + 1/q^{n+1})(1 - 1/q^{n+2}) - 1] \\
 &\geq (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^n) \\
 &\quad \times [(1 + 1/q^{n+1})(1 - 1/q^{n+2}) - 1] \\
 &\geq (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^n) \left[\frac{(1 - 1/q - 1/q^3)}{q^{n+1}} \right] \\
 &\geq (1 - 1/q - 1/q^3)^2 / q^{n+1} \\
 &\geq 0.14/q^{n+1},
 \end{aligned}$$

where the fourth inequality used the third claim of Lemma 2.3. Thus, the total variation distance between $\mathcal{Q}_{q,n}$ and \mathcal{Q}_q is at least $\frac{1}{2}[p_0 - p_{0,n}] \geq 0.07/q^{n+1}$.

Now we compute a lower bound for n even. From (33),

$$\begin{aligned}
 p_{0,n} &= (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^{n-1}) \\
 &\quad \times (1 + 1/q^2)(1 + 1/q^4) \cdots (1 + 1/q^n)
 \end{aligned}$$

and by (34) and (36),

$$\begin{aligned}
 p_0 &\leq (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^{n+1}) \\
 &\quad \times (1 + 1/q^2)(1 + 1/q^4) \cdots (1 + 1/q^{n+2}) \\
 &= p_{0,n}(1 - 1/q^{n+1})(1 + 1/q^{n+2}).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (41) \quad p_{0,n} - p_0 &\geq (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^{n-1}) \\
 &\quad \times (1 + 1/q^2)(1 + 1/q^4) \cdots (1 + 1/q^n) \\
 &\quad \times [1 - (1 - 1/q^{n+1})(1 + 1/q^{n+2})] \\
 &\geq (1 - 1/q)(1 - 1/q^3) \cdots (1 - 1/q^{n-1}) \\
 &\quad \times [1 - (1 - 1/q^{n+1})(1 + 1/q^{n+2})]
 \end{aligned}$$

$$\begin{aligned} &\geq \frac{(1 - 1/q)}{q^{n+1}} \prod_{i \text{ odd}} (1 - 1/q^i) \\ &\geq \frac{(1 - 1/q)(1 - 1/q - 1/q^3)}{q^{n+1}} \\ &\geq 0.18/q^{n+1}, \end{aligned}$$

where the fourth inequality used the third claim of Lemma 2.3. Thus, the total variation distance between $\mathcal{Q}_{q,n}$ and \mathcal{Q}_q is at least $\frac{1}{2}[p_{0,n} - p_0] \geq 0.09/q^{n+1}$.

For the upper bound, arguing as in the proof of Theorem 4.1,

$$\begin{aligned} &|P(Q_n \in A) - P(Q \in A)| \\ &= |E[h_A(Q_n)] - Q_q h_A| \\ &= |E[qf_A(Q_n + 1) - (q^{2Q_n} - 1)f_A(Q_n)]| \\ &= |E[(-1)^{n-Q_n} q^{-n+Q_n+1} f_A(Q_n + 1)]| \\ (42) \quad &\leq q^{-n+1} |f_A(1)| P(Q_n = 0) \\ &\quad + E[q^{-n+Q_n+1} |f_A(Q_n + 1)| \mathbf{1}(Q_n \geq 1)] \\ &\leq q^{-n+1} |f_A(1)| + q^{-n+1} E[q^{Q_n}] \sup_{k \geq 2} |f_A(k)| \\ &\leq q^{-n+1} \frac{1.1}{q^2} + q^{-n+1} (2 + q^{-n}) \frac{1.8}{q^4} \\ &\leq q^{-(n+1)} (1.1 + 3.6q^{-2} + 1.8q^{-3}) \\ &\leq 2.3/q^{n+1} \end{aligned}$$

for $n \geq 1$. The third inequality used Lemmas 7.3 and 7.4. \square

REMARK 7.5. The distribution $p_{k,n}$ of (33) holds for $q \geq 3$. Over this range, the bounds of Theorem 7.1 may be slightly improved by applying (38) and (39) to replace 1.8 in Lemma 7.4 by 1.4, and then using this value in (42). One may similarly improve the lower bound by replacing 0.14 by 0.38 in (40), and 0.18 by 0.41 in (41), resulting in

$$\frac{0.19}{q^{n+1}} \leq \|\mathcal{Q}_{q,n} - \mathcal{Q}_q\|_{\text{TV}} \leq \frac{1.5}{q^{n+1}} \quad \text{for all } q \geq 3.$$

APPENDIX

The main purpose of this appendix is to give an algebraic proof of Lemma 3.2 in the special case that $m = 0$. The proof assumes familiarity with rational canonical

forms of matrices (i.e., the theory of Jordan forms over finite fields), and with cycle index generating functions. Background on these topics can be found in [16] or [29], or in the survey [17].

PROOF OF LEMMA 3.2 WHEN $m = 0$. The sought equation is

$$(43) \quad \sum_{k=0}^n q^k p_{k,n} = 2 - 1/q^n.$$

From the expression for $p_{k,n}$ in (1) specialized to the case $m = 0$, it is clear that if one multiplies (43) by $q^t(1 - 1/q) \cdots (1 - 1/q^n)$ where t is sufficiently large as a function of n , then both sides become polynomials in q . Since polynomials in q agreeing for infinitely many values of q are equal, it is enough to prove the result for infinitely many values of q , so we demonstrate it for q a prime power.

Let $\text{Mat}(n, q)$ be the collection of all $n \times n$ matrices with entries in \mathbb{F}_q and $M \in \text{Mat}(n, q)$. Then n minus the rank of M is equal to $l(\lambda_z(M))$, the number of parts in the partition corresponding to the degree one polynomial z in the rational canonical form of M .

$$(44) \quad E(q^{Q_n}) = \frac{1}{q^{n^2}} \sum_{M \in \text{Mat}(n, q)} q^{l(\lambda_z(M))},$$

where $\text{Mat}(n, q)$ denotes the set of $n \times n$ matrices over the finite field \mathbb{F}_q .

From the cycle index for $\text{Mat}(n, q)$ (Lemma 1 of [29]), it follows that

$$(45) \quad \begin{aligned} 1 + \sum_{n \geq 1} \frac{u^n}{|\text{GL}(n, q)|} \sum_{M \in \text{Mat}(n, q)} q^{l(\lambda_z(M))} \\ = \left[\sum_{\lambda} \frac{q^{l(\lambda)} u^{|\lambda|}}{c_{\text{GL}, z}(\lambda)} \right] \prod_{\phi \neq z} \sum_{\lambda} \frac{u^{|\lambda| \deg(\phi)}}{c_{\text{GL}, \phi}(\lambda)}. \end{aligned}$$

Here, λ ranges over all partitions of all natural numbers, and $l(\lambda)$ is the number of parts of λ . The quantity $c_{\text{GL}, \phi}(\lambda)$ is a certain function of λ, ϕ which depends on the polynomial ϕ only through its degree. The product is over all monic, irreducible polynomials ϕ over \mathbb{F}_q other than $\phi = z$.

From the cycle index for $\text{GL}(n, q)$ (Lemma 1 of [29]), it follows that

$$(46) \quad \frac{1}{1 - u} = 1 + \sum_{n \geq 1} \frac{u^n}{|\text{GL}(n, q)|} \sum_{\alpha \in \text{GL}(n, q)} 1 = \prod_{\phi \neq z} \sum_{\lambda} \frac{u^{|\lambda| \deg(\phi)}}{c_{\text{GL}, \phi}(\lambda)}.$$

Summarizing, it follows from (45) and (46) that

$$(47) \quad 1 + \sum_{n \geq 1} \frac{u^n}{|\text{GL}(n, q)|} \sum_{M \in \text{Mat}(n, q)} q^{l(\lambda_z(M))} = \frac{1}{1 - u} \sum_{\lambda} \frac{q^{l(\lambda)} u^{|\lambda|}}{c_{\text{GL}, z}(\lambda)}.$$

The next step is to compute

$$\sum_{\lambda} \frac{q^{l(\lambda)} u^{|\lambda|}}{c_{\text{GL},z}(\lambda)} = \sum_{\lambda} \frac{q^{l(\lambda)} u^{|\lambda|}}{c_{\text{GL},z-1}(\lambda)}.$$

This equality holds because $c_{\text{GL},\phi}(\lambda)$ depends on the polynomial ϕ only through its degree. From the cycle index of $\text{GL}(n, q)$, it follows that

$$\begin{aligned} & 1 + \sum_{n \geq 1} \frac{u^n}{|\text{GL}(n, q)|} \sum_{\alpha \in \text{GL}(n, q)} q^{l(\lambda_{z-1}(\alpha))} \\ &= \sum_{\lambda} \frac{q^{l(\lambda)} u^{|\lambda|}}{c_{\text{GL},z-1}(\lambda)} \prod_{\phi \neq z, z-1} \sum_{\lambda} \frac{u^{|\lambda| \deg(\phi)}}{c_{\text{GL},\phi}(\lambda)} \\ &= \frac{\sum_{\lambda} ((q^{l(\lambda)} u^{|\lambda|}) / (c_{\text{GL},z-1}(\lambda)))}{\sum_{\lambda} (u^{|\lambda|} / (c_{\text{GL},z-1}(\lambda)))} \prod_{\phi \neq z} \sum_{\lambda} \frac{u^{|\lambda| \deg(\phi)}}{c_{\text{GL},\phi}(\lambda)} \\ &= \frac{1}{1-u} \frac{\sum_{\lambda} ((q^{l(\lambda)} u^{|\lambda|}) / (c_{\text{GL},z-1}(\lambda)))}{\sum_{\lambda} (u^{|\lambda|} / (c_{\text{GL},z-1}(\lambda)))} \\ &= \frac{\prod_{i \geq 1} (1 - u/q^i)}{1-u} \sum_{\lambda} \frac{q^{l(\lambda)} u^{|\lambda|}}{c_{\text{GL},z-1}(\lambda)}. \end{aligned}$$

The third equality used (46) and the final equality is from Lemma 6 of [29] and page 19 of [1].

Next, we can use group theory to find an alternate expression for

$$1 + \sum_{n \geq 1} \frac{u^n}{|\text{GL}(n, q)|} \sum_{\alpha \in \text{GL}(n, q)} q^{l(\lambda_{z-1}(\alpha))}.$$

Indeed, by the theory of rational canonical forms, $q^{l(\lambda_{z-1}(\alpha))}$ is the number of fixed points of α in its action on the underlying n dimensional vector space V . By Burnside’s lemma (page 95 of [31]), the average number of fixed points of a finite group acting on a finite set is the number of orbits of the action on the set. For $\text{GL}(n, q)$ acting on V , there are two such orbits, consisting of the zero vector and the set of nonzero vectors. Thus,

$$1 + \sum_{n \geq 1} \frac{u^n}{|\text{GL}(n, q)|} \sum_{\alpha \in \text{GL}(n, q)} q^{l(\lambda_{z-1}(\alpha))} = 1 + \sum_{n \geq 1} 2u^n = \frac{1+u}{1-u}.$$

Comparing the final equations of the previous two paragraphs gives that

$$(48) \quad \sum_{\lambda} \frac{q^{l(\lambda)} u^{|\lambda|}}{c_{\text{GL},z}(\lambda)} = \frac{1+u}{\prod_{i \geq 1} (1 - u/q^i)}.$$

It follows from (47) and (48) that

$$1 + \sum_{n \geq 1} \frac{u^n}{|\text{GL}(n, q)|} \sum_{M \in \text{Mat}(n, q)} q^{l(\lambda_z(M))} = \frac{1+u}{1-u} \prod_{i \geq 1} \frac{1}{1-u/q^i}.$$

Thus, by (44), $E(q^{Q_n})$ is $\frac{|\text{GL}(n, q)|}{q^{n^2}}$ multiplied by the coefficient of u^n in

$$\frac{1+u}{1-u} \prod_{i \geq 1} \frac{1}{1-u/q^i}.$$

From page 19 of [1], the coefficient of u^n in

$$\frac{1}{1-u} \prod_{i \geq 1} \frac{1}{1-u/q^i}$$

is equal to $[(1 - 1/q)(1 - 1/q^2) \cdots (1 - 1/q^n)]^{-1}$. Thus,

$$\begin{aligned} E(q^{Q_n}) &= \frac{|\text{GL}(n, q)|}{q^{n^2}} \left[\frac{1}{(1 - 1/q) \cdots (1 - 1/q^n)} + \frac{1}{(1 - 1/q) \cdots (1 - 1/q^{n-1})} \right] \\ &= 2 - \frac{1}{q^n}, \end{aligned}$$

where the last equality used that $|\text{GL}(n, q)| = q^{n^2} (1 - 1/q) \cdots (1 - 1/q^n)$. \square

We close this section with two remarks about the distribution $Q_{q,n}$ in (1) (for general m) from the [Introduction](#).

- From [3], there is a natural Markov chain on $\{0, 1, \dots, n\}$ which has $Q_{q,n}$ as its stationary distribution. This chain has transition probabilities

$$\begin{aligned} M(i, i + 1) &= \frac{q^{n-i-1}(q^{n-i} - 1)}{(q^n - 1)(q^{n+m} - 1)}, \\ M(i, i - 1) &= \frac{(q^n - q^{n-i})(q^{n+m} - q^{n-i})}{(q^n - 1)(q^{n+m} - 1)}, \\ M(i, i) &= 1 - M(i, i - 1) - M(i, i + 1). \end{aligned}$$

This Markov chain describes how the rank of a matrix evolves by adding a uniformly chosen rank one matrix at each step.

- The following known lemma gives a formula for the chance that a random $k \times n$ matrix with entries from \mathbb{F}_q has rank r . For its statement, we let

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-m+1} - 1)}{(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)}$$

be the q -binomial coefficient.

LEMMA A.1 ([31], page 338). *The chance that a random $k \times n$ matrix with entries from \mathbb{F}_q has rank r is equal to*

$$(49) \quad \frac{1}{q^{kn}} \begin{bmatrix} n \\ r \end{bmatrix}_q \sum_{l=0}^r (-1)^{r-l} \begin{bmatrix} r \\ l \end{bmatrix}_q q^{kl + \binom{r-l}{2}}.$$

Following a suggestion of Dennis Stanton, we indicate how Lemma A.1 can be used to derive the product formula for $p_{k,n}$ in the Introduction. By replacing k by n , n by $n + m$, and r by $n - k$ in (49), we get that the probability that a random $n \times (n + m)$ matrix has rank $n - k$ is equal to

$$\begin{aligned} & \frac{1}{q^{n(n+m)}} \begin{bmatrix} n+m \\ n-k \end{bmatrix}_q \sum_{l=0}^{n-k} (-1)^{n-k-l} \begin{bmatrix} n-k \\ l \end{bmatrix}_q q^{nl + \binom{n-k-l}{2}} \\ &= \frac{1}{q^{n(n+m)}} \begin{bmatrix} n+m \\ n-k \end{bmatrix}_q \sum_{l=0}^{n-k} (-1)^l \begin{bmatrix} n-k \\ l \end{bmatrix}_q q^{n(n-k-l) + \binom{l}{2}} \\ &= \frac{1}{q^{n(m+k)}} \begin{bmatrix} n+m \\ n-k \end{bmatrix}_q \sum_{l=0}^{n-k} \left[\frac{-1}{q^{n+1}} \right]^l \begin{bmatrix} n-k \\ l \end{bmatrix}_q q^{\binom{l+1}{2}}. \end{aligned}$$

Plugging into the q -binomial theorem (page 78 of [6])

$$(1+xq)(1+xq^2) \cdots (1+xq^r) = \sum_{l=0}^r \begin{bmatrix} r \\ l \end{bmatrix}_q q^{l(l+1)/2} x^l$$

with $r = n - k$ and $x = -1/q^{n+1}$ gives that the probability that a random $n \times (n + m)$ matrix over \mathbb{F}_q has rank $n - k$ is equal to

$$\frac{1}{q^{n(m+k)}} \begin{bmatrix} n+m \\ n-k \end{bmatrix}_q (1 - 1/q^n) \cdots (1 - 1/q^{k+1}).$$

It follows from elementary manipulations that this is equal to

$$\frac{1}{q^{k(m+k)}} \frac{\prod_{i=1}^{n+m} (1 - 1/q^i) \prod_{i=k+1}^n (1 - 1/q^i)}{\prod_{i=1}^{n-k} (1 - 1/q^i) \prod_{i=1}^{m+k} (1 - 1/q^i)}.$$

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