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LOCALISATION AND AGEING IN THE PARABOLIC ANDERSON MODEL WITH WEIBULL POTENTIAL

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The parabolic Anderson model is the Cauchy problem for the heat equation on the integer lattice with a random potential ξ. We consider the case when $\{\xi(z): z \in \mathbb{Z}^d\}$ is a collection of independent identically distributed random variables with Weibull distribution with parameter $0 < \gamma < 2$, and we assume that the solution is initially localised in the origin. We prove that, as time goes to infinity, the solution completely localises at just one point with high probability, and we identify the asymptotic behaviour of the localisation site. We also show that the intervals between the times when the solution relocalises from one site to another increase linearly over time, a phenomenon known as ageing.

1. Introduction and main results.

1.1. *Parabolic Anderson model.* We consider the heat equation with random potential on the integer lattice $\mathbb{Z}^d$ and study the Cauchy problem with localised initial condition,

$$
(1)\qquad \begin{aligned} \partial_t u(t,z) &= \Delta u(t,z) + \xi(z)u(t,z), \qquad (t,z) \in (0,\infty) \times \mathbb{Z}^d, \\ u(0,z) &= \mathbb{1}_{\{0\}}(z), \qquad z \in \mathbb{Z}^d, \end{aligned}
$$

where

$$
(\Delta f)(z) = \sum_{y \sim z} [f(y) - f(z)], \qquad z \in \mathbb{Z}^d, f: \mathbb{Z}^d \to \mathbb{R}
$$

is the discrete Laplacian, and the potential $\{\xi(z): z \in \mathbb{Z}^d\}$ is a collection of independent identically distributed random variables. The problem (1) and its variants are often called the *parabolic Anderson model*.

The model originates from the seminal work [1] of the Nobel laureate P. W. Anderson, who used the Hamiltonian $\Delta + \xi$ to describe electron localisation inside a semiconductor, a phenomenon now known as Anderson localisation. The parabolic version of the model appears naturally in the context of reaction–diffusion equations; see [5, 14], describing a system of noninteracting particles diffusing in space

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according to the Laplacian Δ and branching at rate $\xi(z)\,dt$ at any given point z. It turns out that the solution $u(t,z)$ gives the average number of such particles at time t at location z.

1.2. *Intermittency and localisation.* A lot of mathematical attention to the parabolic Anderson model over the last 30 years has been due to the fact that it exhibits the *intermittency effect*. In general, a random model is said to be intermittent if its long-term behaviour cannot be described using an averaging principle; see [18]. In the context of the parabolic Anderson model, this means that, for large times t, the solution $u(t,z)$ is mainly concentrated on a small number of remote random islands; see [7] for a survey.

The long-term behaviour of the parabolic Anderson model is determined by the upper tail of the underlying distribution of the potential ξ, and it is believed that the intermittency is more pronounced for heavier tails. However, an initial approach to understanding intermittency was proposed for light-tailed potentials (those with finite exponential moments). It was suggested to study large time asymptotics of the moments of the total mass of the solution

$$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z),$$

which are finite for such potentials. The model was defined as intermittent if higher moments exhibited a faster growth rate, and it was proved in [9] that the parabolic Anderson model is intermittent in this sense. This method, however, does not work for heavy-tailed potentials (those with infinite exponential moments), as for them the moments of $U(t)$ are infinite. Such distributions include the exponential distribution and all heavier-tailed distributions.

In order to understand the intermittent picture in more detail, it proved to be useful to study various large-time asymptotics of the total mass $U(t)$, as they provided some insight into the geometry of the intermittent islands. It was shown in [16] that there are four types of behaviour the parabolic Anderson model can exhibit depending on the tail of the underlying distribution. The prime examples from each class are the following distributions:

(1) Weibull distribution with parameter $\gamma > 1$, that is, $F(x) = 1 - e^{-x^\gamma}$.
(2) Double-exponential distribution with parameter $\rho > 0$, that is, $F(x) = 1 - e^{-e^{x/\rho}}$.
(3) "Almost bounded" distributions, including some unbounded distributions with tails lighter than double-exponential and some bounded distributions.
(4) Other bounded distributions.

The asymptotics of the total mass $U(t)$ was studied in [10] for cases (1) and (2), in [16] for case (3) and in [4] for case (4). Heuristics based on the asymptotics of $U(t)$ suggests that the intermittent islands will be single lattice points in case (1), bounded regions in case (2) and of size growing to infinity in cases (3) and (4).

However, a rigorous geometric picture of intermittency has not been well understood. In particular, it is not clear how many intermittent islands are needed to carry the total mass of the solution, and where those islands are located.

Moreover, the four classes above only cover light-tailed potential, and the class of all heavy-tailed distributions should be included to complete the picture. The prime examples of such distributions are

(0a) Pareto distributions, that is, $F(x) = 1 - x^{-\alpha}$, $\alpha > d$;
(0b) Weibull potentials with parameter $\gamma \le 1$.

Heavy-tailed potentials were first studied in [17], and it turned out that the asymptotics of $U(t)$ in this case becomes nondeterministic and difficult to control. It was suggested to study the nondeterministic nature of $U(t)$ using extreme value theory and point processes techniques. This approach was further developed in [12], where the intermittency was fully described in its original geometric sense for Pareto potentials (0a). Polynomial tails are the heaviest tails for which the solution of the parabolic Anderson model still exists (see [9]), and one expected the localisation islands to be small and not numerous. It was proved that the extreme form of this conjecture is true, namely, that there is only one localisation island consisting of only one site. In other words, at any time the solution is localised at just one point with high probability, a phenomenon called *complete localisation*.

It is a challenging problem to describe geometric intermittency for lighter tails. In [8], intermittent islands were described for potentials from classes (1) and (2), but the question about the number of islands remained open. Case (0b) was studied in [13], and it was shown that the solution is localised on an island of size $o(\frac{t(\log t)^{1/\gamma-1}}{\log\log t})$. However, it was believed that a much smaller region should actually contribute to the solution.

In this paper, we assume that the potential has *Weibull distribution* with parameter $\gamma > 0$, that is, the distribution function of each $\xi(z)$ is

$$F(x) = \mathrm{Prob}\{\xi(z) < x\} = 1 - e^{-x^\gamma}, \qquad x \ge 0. \tag{2}$$

We focus on $0 < \gamma < 2$, which covers case (0b) and partly case (1). We prove that for such potentials the solution of the parabolic Anderson model completely localises at just one single site, exhibiting the strongest form of intermittency similar to the Pareto case (0a). This was plausible for $0 < \gamma < 1$ as in this case the spectral gap of the Anderson Hamiltonian $\Delta + \xi$ in a relevant t-dependent large box tends to infinity, but is quite surprising for the exponential distribution ($\gamma = 1$) where the spectral gap is bounded, and even more so for $1 < \gamma < 2$ where the spectral gap tends to zero. We identify the localisation site explicitly in terms of the potential ξ and describe its scaling limit.

For all sufficiently large t (so that $\log\log t$ is well defined), denote

$$\Psi_t(z) = \xi(z) - \frac{|z|}{\gamma t}\log\log t, \qquad z \in \mathbb{Z}^d, \tag{3}$$

and let $Z_t^{(1)}$ be such that

$$\Psi_t(Z_t^{(1)}) = \max_{z \in \mathbb{Z}^d} \Psi_t(z).$$

The existence of $Z_t^{(1)}$ will be proved in Lemma 2.2.

Denote by $|x|$ the ℓ^1-norm of $x \in \mathbb{R}^d$, and denote by $\Longrightarrow$ weak convergence.

THEOREM 1.1 (Complete localisation). *Let $0 < \gamma < 2$. As $t \to \infty$,*

$$\lim_{t \to \infty} \frac{u(t, Z_t^{(1)})}{U(t)} = 1 \qquad \textit{in probability.}$$

REMARK 1. It is easy to see that the solution cannot be localised at one point for all large times t since occasionally it has to relocalise continuously from one site to another, and at those periods the solution will be concentrated at more than one point. It was shown in [12] that for Pareto potentials the solution in fact remains localised at just two points at all large times t almost surely. We conjecture that the same is true for Weibull potentials with $0 < \gamma < 2$.

REMARK 2. There is a chance that our proof could be adjusted to the case $\gamma = 2$. However, new ideas are required to deal with $\gamma > 2$, and there is a high chance that complete localisation will simply fail in that case. The technical reasons why our proof breaks down for $\gamma \geq 2$ are explained in Remark 8 and Remark 9 in Sections 4 and 5, respectively.

THEOREM 1.2 (Scaling limit for the localisation site). *Let $\gamma > 0$. Then*

$$\frac{Z_t^{(1)}}{r_t} \Longrightarrow X^{(1)},$$

as $t \to \infty$ where

$$r_t = \frac{t (\log t)^{1/\gamma - 1}}{\log \log t} \tag{4}$$

and $X^{(1)}$ is an $\mathbb{R}^d$-valued random variable with independent exponentially distributed coordinates with parameter $d^{1-1/\gamma}$ and uniform random signs, that is, with density

$$p^{(1)}(x) = \frac{d^{d(1-1/\gamma)}}{2^d} \exp\{-d^{1-1/\gamma} |x|\}, \qquad x \in \mathbb{R}^d.$$

REMARK 3. Although we prove Theorem 1.2 for all $\gamma > 0$, it only describes the scaling limit for the concentration site for $0 < \gamma < 2$ as otherwise the solution may not be localised at $Z_t^{(1)}$.

REMARK 4. This scaling limit agrees with the scaling limit for the centre of the intermittent island obtained in [13] for $0 < \gamma \le 1$. However, according to Theorem 1.1, this island is now of radius zero (being a single point) rather than $o(r_t)$, and the result holds for the wider range $0 < \gamma < 2$.

1.3. *Ageing*. The notion of *ageing* is a key paradigm in studying the long-term dynamics of large disordered systems. A system exhibits ageing if, being in a certain state at time t, it is likely to remain in this state for some time $s(t)$ which depends increasingly, and often linearly, on the time t. Roughly speaking, the system becomes increasingly more conservative and reluctant to change.

The ageing phenomenon has been extensively studied for disordered systems such as trap models and spin glasses; see [3] and references therein. In the context of the parabolic Anderson model, a certain form of ageing based on correlations was studied for some time-dependent potentials in [2, 6], and it was shown that such systems exhibit no ageing. The recent paper [11] dealt with potentials from class (1) and studied the correlation ageing (which gives only indirect information about the evolution of localisation) and more explicit annealed ageing (which, in contrast to the quenched setting, is based on the evolution of the islands contributing to the solution averaged over the environment). It was shown that these two forms of ageing are similar, and somewhat surprisingly, ageing was observed for Weibull potentials with parameter $\gamma > 2$ but not for heavier-tailed Weibull potentials with parameter $1 < \gamma \le 2$.

The explicit ageing in the quenched setting has so far only been observed for Pareto potentials; see [15]. In that case, the solution completely localises at just one point and ageing of the parabolic Anderson model is equivalent to ageing of the concentration site process. In this paper, we use a similar approach to show that the parabolic Anderson model with Weibull potential with parameter $0 < \gamma < 2$ exhibits ageing as well. Notice that, remarkably, this is in sharp contrast to the absence of annealed and correlation ageing observed for $\gamma > 1$ in [11].

For each $t > 0$, denote

$$T_t = \inf\{s > 0 : Z_{t+s}^{(1)} \neq Z_t^{(1)}\}.$$

THEOREM 1.3 (Ageing). *Let $\gamma > 0$. As $t \to \infty$*

$$\frac{T_t}{t} \Longrightarrow \Theta,$$

where Θ is a nondegenerate almost surely positive random variable.

REMARK 5. In the proof of Theorem 1.3, we identify the distribution function of Θ as a certain integral over $\mathbb{R}^d \times \mathbb{R}$.

REMARK 6. Although we prove Theorem 1.3 for all $\gamma > 0$, it only characterises the ageing behaviour of the parabolic Anderson model for $0 < \gamma < 2$ as otherwise the solution may not be localised at $Z_t^{(1)}$.

1.4. *Outline of the proofs*. It follows from [9], Theorem 2.1, that the parabolic Anderson model with Weibull potential possesses a unique nonnegative solution $u : (0,\infty) \times \mathbb{Z}^d \to [0,\infty)$, which has a *Feynman–Kac representation*

$$u(t,z) = \mathbb{E}_0\left[\exp\left\{\int_0^t \xi(X_s)\,ds\right\}\mathbb{1}\{X_t = z\}\right], \qquad (t,z) \in (0,\infty) \times \mathbb{Z}^d,$$

where $(X_s : s \geq 0)$ is a continuous-time simple random walk on the lattice $\mathbb{Z}^d$ with generator Δ, and $\mathbb{P}_z$ and $\mathbb{E}_z$ denote the corresponding probability and expectation given that the random walk starts at $z \in \mathbb{Z}^d$.

The Feynman–Kac formula suggests that the main contribution to the solution u at time t comes from paths (X_s) spending a lot of time at sites z where the value $\xi(z)$ of the potential is high but which are reasonably close to the origin so that the random walk would have a fair chance of reaching them in time t. It turns out that the functional Ψ_t defined in (3) captures this trade-off, being the difference of the energetic term $\xi(z)$ and an entropic term responsible for the cost of going to a point z in time t and staying there. Furthermore, the maximiser $Z_t^{(1)}$ of Ψ_t turns out to be the site where the solution u is localised at time t.

In order to prove this, we decompose the solution u into the sum

$$u(t,z) = u_1(t,z) + u_2(t,z)$$

according to two groups of paths ending at z:

(I) paths visiting $Z_t^{(1)}$ before time t and staying in the ball B_t centred in the origin with radius $|Z_t^{(1)}|(1+\rho_t)$, where ρ_t is a certain function tending to zero;
(II) all other paths.

We show that u_1 localises around $Z_t^{(1)}$ and that the total mass of u_2 is negligible.

To prove the localisation of u_1, we use spectral analysis of the Anderson Hamiltonian $\Delta + \xi$ in the ball B_t. In order to do so, we show that, although the spectral gap tends to zero for $\gamma > 1$, it is still reasonably large. We suggest a new technique which allows us to show that the principal eigenfunction just manages to localise at $Z_t^{(1)}$. Then we use a result from [8] to show that this is sufficient for the localisation of u_1.

In order to prove that the total mass of u_2 is negligible, we notice that the paths from the second group fall into one of the following three subgroups:

(1) paths having the maximum of the potential at the point $Z_t^{(1)}$ but making more than $|Z_t^{(1)}|(1+\rho_t)$ steps;
(2) paths having the maximum of the potential not at the point $Z_t^{(1)}$, with the maximum being reasonably large;
(3) paths missing all high values of the potential.

In Section 4, we show that the total mass of the paths corresponding to each group is negligible. In all cases, this is due to an imbalance between the energetic forces (which do not contribute enough if the site $Z_t^{(1)}$ is not visited) and entropic forces (as the probabilistic cost is too high if a path is too long), as well as to the fact that the gap between $\Psi_t(Z_t^{(1)})$ and the second largest value of Ψ_t is too large.

Denote by $Z_t^{(2)}$ a point where the second largest value of Ψ_t is attained, that is,

$$\Psi_t(Z_t^{(2)}) = \max\{\Psi_t(z) : z \in \mathbb{Z}^d, z \neq Z_t^{(1)}\}.$$

In order to find the scale of growth of $\Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)})$ as well as of $Z_t^{(1)}$ and $Z_t^{(2)}$ we extend the point processes techniques developed in [17] and [12]. For sufficiently large t, we denote

$$a_t = (d \log t)^{1/\gamma} \quad \text{and} \quad d_t = (d \log t)^{1/\gamma - 1}.$$

Further, for all $z \in \mathbb{Z}^d$ and all sufficiently large t, we denote

$$Y_{t,z} = \frac{\Psi_t(z) - a_{r_t}}{d_{r_t}}, \tag{5}$$

where r_t is defined by (4), and define a point process

$$\Pi_t = \sum_{z \in \mathbb{Z}^d} \varepsilon_{(z r_t^{-1}, Y_{t,z})}, \tag{6}$$

where we write ε_x for the Dirac measure in x. In Section 3, we show that the point processes Π_t are well defined on a carefully chosen domain, and that they converge in law to a Poisson point process with certain density. This allows us to analyse the joint distribution of the random variables $Z_t^{(1)}$, $Z_t^{(2)}$, $\Psi_t(Z_t^{(1)})$, $\Psi_t(Z_t^{(2)})$ and, in particular, prove Theorem 1.2.

Finally, to prove ageing, we argue that due to the form of the functional Ψ_t the probability of $\{Z_{t+wt}^{(1)} = Z_t^{(1)}\}$, for each $w > 0$, is roughly equal to

$$\int_{\mathbb{R}^d \times \mathbb{R}} \mathrm{Prob}\{\Pi_t(dx \times dy) = 1, \Pi_t(D_w(x, y)) = 0\}, \tag{7}$$

where

$$\begin{aligned} D_w(x, y) = {} & \left\{(\bar{x}, \bar{y}) \in \mathbb{R}^d \times \mathbb{R} : y + \frac{w\theta|x|}{1+w} \le \bar{y} + \frac{w\theta|\bar{x}|}{1+w}\right\} \\ & \cup \left(\mathbb{R}^d \times [y, \infty)\right), \end{aligned} \tag{8}$$

and

$$\theta = \gamma^{-1} d^{1-1/\gamma}. \tag{9}$$

In particular, the integral in (7) converges to the corresponding finite integral with respect to the Poisson point process Π as $t \to \infty$. This proves Theorem 1.3 since that integral is a continuous function of w decreasing from one to zero as w varies from zero to infinity and so it is the tail of a distribution function.

The paper is organised as follows. In Section 2, we introduce notation and prove some preliminary results. In Section 3, we develop a point processes approach, analyse the joint distribution of $Z_t^{(1)}$, $Z_t^{(2)}$, $\Psi_t(Z_t^{(1)})$, $\Psi_t(Z_t^{(2)})$ and prove Theorem 1.2. In Section 4, we deal with the total mass corresponding to the paths from groups (1)–(3) and show that it is negligible. In Section 5, we discuss the localisation of u_1 and prove Theorem 1.1. Finally, in Section 6, we study ageing and prove Theorem 1.3.

2. Preliminaries. We focus on potentials with Weibull distribution (2) with parameter $0 < \gamma < 2$. However, most of our point processes results can be obtained for all $\gamma > 0$ at no additional cost. Therefore, we will assume $\gamma > 0$ in Sections 2, 3 and 6, and restrict ourselves to the case $0 < \gamma < 2$ in Sections 4 and 5.

2.1. *Extreme value notation and preliminary results.* We denote the upper order statistics of the potential ξ in the centred ball of radius $r > 0$ by

$$\xi_r^{(1)} = \max_{|z|\le r} \xi(z)$$

and

$$\xi_r^{(i)} = \max\{\xi(z) : |z| \le r, \xi(z) < \xi_r^{(i-1)}\}$$

for $2 \le i \le \ell_r$, where ℓ_r is the number of points in the ball. Observe that throughout the paper we use the ℓ^1-norm.

Let $0 < \rho < \sigma < 1/2$ and for all sufficiently large r let

$$F_r = \{z \in \mathbb{Z}^d : |z| \le r, \exists i \le r^\rho \text{ such that } \xi(z) = \xi_r^{(i)}\},$$
$$G_r = \{z \in \mathbb{Z}^d : |z| \le r, \exists i \le r^\sigma \text{ such that } \xi(z) = \xi_r^{(i)}\}.$$

The sets F_r and G_r contain the sites in the centred ball of radius r where the highest $\lfloor r^\rho \rfloor$ and $\lfloor r^\sigma \rfloor$ values of the potential ξ are achieved, respectively.

LEMMA 2.1. *Almost surely*

$$\xi_r^{(1)} \sim (d \log r)^{1/\gamma} \qquad \text{as } r \to \infty.$$

PROOF. This result was proved in [17] for the case $0 < \gamma \le 1$ but it can be easily extended to all $\gamma > 0$ by observing that $\zeta(z) = \xi(z)^\gamma$, $z \in \mathbb{Z}$, are exponential identically distributed random variables. Denote the maximum of the potential ζ by

$$\zeta_r^{(1)} = \max_{|z|\le r} \zeta(z).$$

Since $\xi_r^{(1)} = (\zeta_r^{(1)})^{1/\gamma}$ and $\zeta_r^{(1)} \sim d \log r$ by [17], Lemma 4.1, with $\gamma = 1$, we obtain the required asymptotics. □

For all $c \in \mathbb{R}$, $z \in \mathbb{Z}^d$, and all sufficiently large t define

$$\Psi_{t,c}(z) = \Psi_t(z) + \frac{c|z|}{t}.$$

Denote by $Z_t^{(1,c)}$ and $Z_t^{(2,c)}$ points where the first and second largest values of the functional $\Psi_{t,c}$ are achieved, that is,

$$\begin{aligned}\Psi_{t,c}(Z_t^{(1,c)}) &= \max\{\Psi_{t,c}(z) : z \in \mathbb{Z}^d\},\\ \Psi_{t,c}(Z_t^{(2,c)}) &= \max\{\Psi_{t,c}(z) : z \in \mathbb{Z}^d, z \neq Z_t^{(1,c)}\}.\end{aligned} \tag{10}$$

Observe that $\Psi_t = \Psi_{t,0}$ and so $Z_t^{(1)} = Z_t^{(1,0)}$ and $Z_t^{(2)} = Z_t^{(2,0)}$. We are mostly interested in the case $c = 0$, but some understanding of the general case is needed for Lemma 4.5. This is explained more carefully in Remark 7 in Section 3.

LEMMA 2.2. *For each c, the maximisers $Z_t^{(1,c)}$ and $Z_t^{(2,c)}$ (and, in particular, $Z_t^{(1)}$ and $Z_t^{(2)}$) are well defined for all sufficiently large t almost surely.*

PROOF. Observe that $\Psi_{t,c}(0) > 0$ and $\Psi_{t,c}(1) > 0$ almost surely if t is large enough. On the other hand, by Lemma 2.1 for all sufficiently large t there exists a random radius $\rho(t) > 0$ such that, almost surely,

$$\xi(z) \le \xi_{|z|}^{(1)} \le \big(2d \log|z|\big)^{1/\gamma} \le \frac{|z|}{\gamma t} \log\log t - \frac{c|z|}{t} \qquad \text{for all } |z| > \rho(t).$$

Hence, $\Psi_{t,c}(z) \le 0$ for all $|z| > \rho(t)$ and so $\Psi_{t,c}$ takes only finitely many positive values. This implies that the maxima in (10) exist for all c. The existence of $Z_t^{(1)}$ and $Z_t^{(2)}$ follows as a particular case when $c = 0$. $\square$

Choose

$$\begin{cases}\beta \in (1 - 1/\gamma, 1/\gamma) & \text{if } 1 \le \gamma < 2,\\ \beta = 0 & \text{if } 0 < \gamma < 1.\end{cases}$$

Observe that $\beta \ge 0$ and define

$$\mu_r = (\log r)^{-\beta} \tag{11}$$

for all r large enough. For $0 < \gamma < 1$, the gaps between higher order statistics of the potential get larger (as $r \to \infty$) and the auxiliary scaling function μ_r is not needed (so that we can simply set $\mu_r = 1$ as above). For $\gamma = 1$, the gaps are of finite order, and for $\gamma > 1$ they tend to zero, and an extra effort is required to control this effect. This is done by the correction term μ_r. It is essential for the choice of μ_r that, on the one hand, it is negligible with respect to d_r and so with respect to the gap $\Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)})$ (which is achieved by the condition $\beta > 1 - 1/\gamma$) and on the other hand $-\log \mu_r$ must be smaller than $\log \xi_r^{(1)}$ (which is guaranteed by $\beta < 1/\gamma$). However, this method only works for $\gamma < 2$ as the interval $(-1/\gamma + 1, 1/\gamma)$ is empty otherwise. This is explained in more detail in Remark 8 in Section 4.

We introduce four auxiliary positive scaling functions $f_t \to 0$, $g_t \to \infty$, $\lambda_t \to 0$, $\rho_t \to 0$ satisfying the following conditions as $t \to \infty$:

$$\text{(a)} \quad f_t^{-1}, g_t, \lambda_t^{-1}, \rho_t^{-1} \text{ are } o(\log\log t), \tag{12}$$

$$\text{(b)} \quad g_t \rho_t \lambda_t^{-1} \to 0. \tag{13}$$

Further, we define

$$k_t = \lfloor (r_t g_t)^{\rho} \rfloor \quad \text{and} \quad m_t = \lfloor (r_t g_t)^{\sigma} \rfloor.$$

For any $c \in \mathbb{R}$, we introduce the event

$$\begin{aligned} \mathcal{E}_c(t) = \{ & r_t f_t < |Z_t^{(1)}| < r_t g_t, \Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)}) > d_t \lambda_t, \\ & \Psi_t(Z_t^{(1)}) > a_{r_t} - d_t g_t, \Psi_t(Z_t^{(2)}) > a_{r_t} - d_t g_t, \\ & |Z_t^{(1,c)}| < r_t g_t, |Z_t^{(2,c)}| < r_t g_t \}. \end{aligned} \tag{14}$$

For any $x, y \in \mathbb{R}$, we denote by $x \wedge y$ and $x \vee y$ the minimum and the maximum of x and y, respectively, and we denote $x_- = -x \vee 0$.

2.2. *Geometric paths on the lattice.* For each $n \in \mathbb{N} \cup \{0\}$ denote by

$$\mathcal{P}_n = \{ y = (y_0, \dots, y_n) \in (\mathbb{Z}^d)^{n+1} : |y_i - y_{i-1}| = 1 \text{ for all } 1 \le i \le n \}$$

the set of all geometric paths in $\mathbb{Z}^d$. Define

$$q(y) = \max_{0 \le i \le n} \xi(y_i) \quad \text{and} \quad p(y) = \max_{0 \le i \le n} |y_i - y_0|,$$

and denote by $z(y)$ a point y_i of the path y such that $\xi(y_i) = q(y)$.

Let (τ_i), $i \ge 0$, be waiting times of the random walk (X_s), which are independent exponentially distributed random variables with parameter $2d$. Denote by E the expectation with respect to (τ_i). For each $y \in \mathcal{P}_n$, denote by

$$\begin{aligned} P(t, y) = \{ & X_0 = y_0, X_{\tau_0 + \cdots + \tau_{i-1}} = y_i \text{ for all } 1 \le i \le n, \\ & \text{and } t - \tau_n \le \tau_0 + \cdots + \tau_{n-1} < t \} \end{aligned}$$

the event that the random walk has the trajectory y up to time t. Here, we assume that the random walk is continuous from the right. Denote by

$$U(t, y) = \mathbb{E}_0 \left[\exp\left\{ \int_0^t \xi(X_s)\, ds \right\} \mathbb{1}_{P(t,y)} \right] \tag{15}$$

the contribution of the event $P(t, y)$ to the total mass of the solution u of the parabolic Anderson model.

For any set $A \subset \mathbb{Z}^d$ and any geometric path $y \in \mathcal{P}_n$ denote

$$n_+(y, A) = |\{0 \le i \le n : y_i \in A\}| \quad \text{and} \quad n_-(y, A) = |\{0 \le i \le n : y_i \notin A\}|.$$

We call a set $A \subset \mathbb{Z}^d$ totally disconnected if $|x - y| \ne 1$ whenever $x, y \in A$.

LEMMA 2.3. *Let A be a totally disconnected finite subset of $\mathbb{Z}^d$, and $y \in \mathcal{P}_n$ for some n. Then*

$$n_+(y, A) \le \frac{n - p(y)}{2} + |A| \wedge \left\lceil \frac{p(y)+1}{2} \right\rceil.$$

PROOF. Let $i(y) = \min\{i : |y_i - y_0| = p(y)\}$ and denote $z = y_{i(y)}$. Similarly to [12], page 371, we first erase loops that the path y may have made before reaching z for the first time and extract from $(y_0, \dots, y_{i(y)})$ a self-avoiding path $(y_{i_0}, \dots, y_{i_{p(y)}})$ starting at y_0 of length $p(y)$, where we take $i_0 = 0$ and

$$i_{j+1} = \min\{i : y_l \ne y_{i_j} \ \forall l \in [i, i(y)]\}.$$

Since this path is self-avoiding and has length $p(y)$, at most $|A| \wedge \lceil \frac{p(y)+1}{2} \rceil$ of its points belong to A. Next, for each $0 \le j \le p(y) - 1$, we consider the path $(y_{i_j+1}, \dots, y_{i_{j+1}-1})$, which was removed during erasing the jth loop. It contains an even number $i_{j+1} - i_j - 1$ of steps and at most half of them belong to A since A is totally disconnected. Finally, the remaining piece $(y_{i_{p(y)}+1}, \dots, y_n)$ consists of $n - i_{p(y)}$ points, and at most half of them lie in A for the same reason. We obtain

$$\begin{aligned} n_+(y, A) &\le |A| \wedge \left\lceil \frac{p(y)+1}{2} \right\rceil + \sum_{j=0}^{p(y)-1} \frac{i_{j+1} - i_j - 1}{2} + \frac{n - i_{p(y)}}{2} \\ &= |A| \wedge \left\lceil \frac{p(y)+1}{2} \right\rceil + \frac{n - p(y)}{2} \end{aligned}$$

as required. □

3. A point processes approach. In this section, we use point processes techniques to understand the joint scaling limit of the random variables $Z_t^{(1,c)}$, $Z_t^{(2,c)}$, $\Psi_{t,c}(Z_t^{(1,c)})$, $\Psi_{t,c}(Z_t^{(2,c)})$ for each c and, in particular, that of $Z_t^{(1)}$, $Z_t^{(2)}$, $\Psi_t(Z_t^{(1)})$, $\Psi_t(Z_t^{(2)})$. We show that $Z_t^{(1,c)}$ and $Z_t^{(2,c)}$ grow at scale r_t and that $\Psi_{t,c}(Z_t^{(1,c)}) - a_{r_t}$ and $\Psi_{t,c}(Z_t^{(2,c)}) - a_{r_t}$ grow or decay at scale d_t (which goes to infinity for $\gamma < 1$, is a constant for $\gamma = 1$, and tends to zero for $\gamma > 1$), and we find their joint scaling limit in Proposition 3.2. In particular, we show that the probability of the event $\mathcal{E}_c(t)$ defined in (14) tends to one for any c and so it suffices to prove complete localisation and ageing on the event $\mathcal{E}_c(t)$ for a sufficiently large constant c. This constant will be identified later in Proposition 4.3 in Section 4. Finally, in the end of this section we prove Theorem 1.2.

For all $z \in \mathbb{Z}^d$ and all sufficiently large r, denote

$$X_{r,z} = \frac{\xi(z) - a_r}{d_r}$$

and define

$$\Sigma_r = \sum_{z\in\mathbb{Z}^d} \varepsilon_{(zr^{-1}, X_{r,z})},$$

where ε_x denotes the Dirac measure in x. For each $\tau \in \mathbb{R}$ and $q > 0$, let

$$H^q_\tau = \{(x, y) \in \dot{\mathbb{R}}^d \times (-\infty, \infty] : y \geq q|x| + \tau\},$$

where $\dot{\mathbb{R}}^d$ denotes the one-point compactification of the Euclidean space. It was proved in [17], Lemma 4.3, that for $0 < \gamma \leq 1$ the restriction of each Σ_r to H^q_τ is a point process and, as $r \to \infty$, $\Sigma_r|_{H^q_\tau}$ converges in law to a Poisson point process Σ on H^q_τ with intensity measure

$$\eta(dx, dy) = dx \otimes \gamma e^{-\gamma y}\, dy.$$

However, it is easy to check that the same proof works for all $\gamma > 0$.

Observe that we need to restrict Σ_r from $\mathbb{R}^d \times \mathbb{R}$ to H^q_τ in order to ensure that there are only finitely many points of Σ_r in every relatively compact set. This is achieved with the help of q, and τ makes it possible for the spaces H^q_τ to capture the behaviour of Σ_r on the whole space $\mathbb{R}^d \times \mathbb{R}$ as it can be chosen arbitrarily small.

For each $\tau \in \mathbb{R}$ and $\alpha > -\theta$, let

$$\hat{H}^\alpha_\tau = \{(x, y) \in \dot{\mathbb{R}}^{d+1} : y \geq \alpha|x| + \tau\},$$

where the hat over H reflects the fact that the spaces $\dot{\mathbb{R}}^d \times (-\infty, \infty]$ and $\dot{\mathbb{R}}^{d+1}$ have different topology.

For all $c \in \mathbb{R}$, $z \in \mathbb{Z}^d$, and all sufficiently large t define

$$Y_{t,z,c} = \frac{\Psi_{t,c}(z) - a_{r_t}}{d_{r_t}} \quad \text{and} \quad \Pi_{t,c} = \sum_{z\in\mathbb{Z}^d} \varepsilon_{(zr_t^{-1}, Y_{t,z,c})}.$$

Recall the definitions of $Y_{t,z}$ and Π_t from (5) and (6) and observe that $Y_{t,z,c} = Y_{t,z,0}$ and $\Pi_t = \Pi_{t,0}$.

LEMMA 3.1. *Let $c \in \mathbb{R}$. For all sufficiently large t, $\Pi_{t,c}$ is a point process on $\hat{H}^\alpha_\tau$. As $t \to \infty$, $\Pi_{t,c}$ converges in law to a Poisson point process Π on $\hat{H}^\alpha_\tau$ with intensity measure*

$$\nu(dx, dy) = dx \otimes \gamma \exp\{-\gamma(y + \theta|x|)\}\, dy.$$

PROOF. Observe that

$$Y_{t,z,c} = \frac{\xi(z) - a_{r_t}}{d_{r_t}} - \frac{|z|}{\gamma t d_{r_t}} \log\log t + \frac{c|z|}{t d_{r_t}} = \frac{\xi(z) - a_{r_t}}{d_{r_t}} - (\theta + o(1))\frac{|z|}{r_t}.$$

Choose α' and q so that $-\theta < \alpha' < \alpha$ and $\alpha' + \theta < q < \alpha + \theta$. Then

$$\Pi_{t,c}|_{\hat{H}^\alpha_\tau} = (\Sigma_{r_t}|_{H^q_\tau} \circ T^{-1}_{t,c})|_{\hat{H}^\alpha_\tau}, \tag{16}$$

where $T_{t,c}: H^q_\tau \to \hat{H}^{\alpha'}_\tau$ is such that

$$T_{t,c}:(x,y)\mapsto \begin{cases}\big(x, y-(\theta+o(1))|x|\big), & \text{if } x\neq\infty \text{ and } y\neq\infty,\\ \infty, & \text{otherwise.}\end{cases}$$

We define $T: H^q_\tau \to \hat{H}^{\alpha'}_\tau$ by

$$T:(x,y)\mapsto \begin{cases}\big(x, y-\theta|x|\big), & \text{if } x\neq\infty \text{ and } y\neq\infty,\\ \infty, & \text{otherwise.}\end{cases}$$

It was proved in [17], Lemma 2.5, that one can pass to the limit in (16) as $t\to\infty$ simultaneously in the mapping $T_{t,c}$ and the point process Σ_{r_t} to get

$$\Pi_{t,c}|_{\hat{H}^\alpha_\tau} \Longrightarrow \big(\Sigma|_{H^q_\tau}\circ T^{-1}\big)\big|_{\hat{H}^\alpha_\tau}.$$

Observe that the conditions of that lemma are satisfied as T is continuous, H^q_τ is compact, $T_{t,c}\to T$ uniformly on $\{(x,y)\in H^q_\tau : |x|\geq n\}$ as $t\to\infty$ for each $n\in\mathbb{N}$, and

$$\eta\big\{(x,y)\in H^q_\tau : |x|\geq n\big\}\to 0 \qquad \text{as } n\to\infty$$

since $\eta(H^q_\tau)$ is finite. Finally, it remains to notice that $(\Sigma|_{H^q_\tau}\circ T^{-1})|_{\hat{H}^\alpha_\tau}$ is a Poisson process with intensity measure $\eta\circ T^{-1}=\nu$ restricted on $\hat{H}^\alpha_\tau$. $\square$

PROPOSITION 3.2. *Let $c\in\mathbb{R}$.*

(a) *As $t\to\infty$,*

$$\left(\frac{Z_t^{(1,c)}}{r_t}, \frac{\Psi_{t,c}(Z_t^{(1,c)})-a_{r_t}}{d_{r_t}}, \frac{Z_t^{(2,c)}}{r_t}, \frac{\Psi_{t,c}(Z_t^{(2,c)})-a_{r_t}}{d_{r_t}}\right) \Longrightarrow \big(X^{(1)}, Y^{(1)}, X^{(2)}, Y^{(2)}\big),$$

where the limit random variable has density

$$p(x_1,y_1,x_2,y_2) = \gamma^2\exp\big\{-\gamma\big(y_1+y_2+\theta|x_1|+\theta|x_2|\big)-2^d(\gamma\theta)^{-d}e^{-\gamma y_2}\big\}\mathbb{1}_{\{y_1>y_2\}}.$$

(b) $\mathrm{Prob}\{\mathcal{E}_c(t)\}\to 1$ *as $t\to\infty$.*

PROOF. (a) Let $A\subset \hat{H}^0_\tau\times\hat{H}^0_\tau$ for some τ, and assume that $\mathrm{Leb}(\partial A)=0$. Since H^0_τ is compact, we have by Lemma 3.1

$$\begin{aligned}\mathrm{Prob}\left\{\left(\frac{Z_t^{(1,c)}}{r_t}, \frac{\Psi_{t,c}(Z_t^{(1,c)})-a_{r_t}}{d_{r_t}}, \frac{Z_t^{(2,c)}}{r_t}, \frac{\Psi_{t,c}(Z_t^{(2,c)})-a_{r_t}}{d_{r_t}}\right)\in A\right\}\\ = \int_A \mathbb{1}_{\{y_1>y_2\}}\mathrm{Prob}\big\{\Pi_{t,c}(dx_1\times dy_1)=\Pi_{t,c}(dx_2\times dy_2)=1,\\ \Pi_{t,c}\big(\mathbb{R}^d\times(y_1,\infty)\big)=\Pi_{t,c}\big(\mathbb{R}^d\times(y_2,y_1)\big)=0\big\}\end{aligned} \tag{17}$$

$$\to \int_A \mathbb{1}_{\{y_1>y_2\}} \operatorname{Prob}\{\Pi(dx_1\times dy_1)=1\}\operatorname{Prob}\{\Pi(dx_2\times dy_2)=1\}$$
$$\times \operatorname{Prob}\{\Pi(\mathbb{R}^d\times(y_1,\infty))=0\}\operatorname{Prob}\{\Pi(\mathbb{R}^d\times(y_2,y_1))=0\}$$
$$= \int_A \mathbb{1}_{\{y_1>y_2\}}\nu(dx_1,dy_1)\nu(dx_2,dy_2)\exp\{-\nu(\mathbb{R}^d\times(y_2,\infty))\}.$$

Integrating we obtain

$$\nu(\mathbb{R}^d\times(y_2,\infty)) = \gamma\int_{\mathbb{R}^d}\int_{y_2}^{\infty}\exp\{-\gamma y-\gamma\theta|x|\}\,dy\,dx$$
$$= 2^d(\gamma\theta)^{-d}e^{-\gamma y_2}. \tag{18}$$

Substituting this, as well as the expressions for $\nu(dx_1,dy_1)$ and $\nu(dx_2,dy_2)$ into (17) we obtain

$$\lim_{t\to\infty}\operatorname{Prob}\left\{\left(\frac{Z_t^{(1,c)}}{r_t},\frac{\Psi_{t,c}(Z_t^{(1,c)})-a_{r_t}}{d_{r_t}},\frac{Z_t^{(2,c)}}{r_t},\frac{\Psi_{t,c}(Z_{t,c}^{(2,c)})-a_{r_t}}{d_{r_t}}\right\}\in A\right)$$
$$= \int_A p(x_1,y_1,x_2,y_2)\,dx_1\,dy_1\,dx_2\,dy_2.$$

It remains now to generalise this equality to all sets $A\subset\mathbb{R}^d\times\mathbb{R}$ with $\operatorname{Leb}(\partial A)=0$. Since τ can be arbitrarily small, to do so it suffices to show that p integrates to one. We have

$$\int_{\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}} p(x_1,x_2,y_1,y_2)\,dx_1\,dy_1\,dx_2\,dy_2$$
$$= 2^{2d}(\gamma\theta)^{-2d}\int_{-\infty}^{\infty}\int_{y_2}^{\infty}\gamma^2\exp\{-\gamma(y_1+y_2)-2^d(\gamma\theta)^{-d}e^{-\gamma y_2}\}\,dy_1\,dy_2$$
$$= 2^{2d}(\gamma\theta)^{-2d}\int_{-\infty}^{\infty}\gamma\exp\{-2\gamma y_2-2^d(\gamma\theta)^{-d}e^{-\gamma y_2}\}\,dy_2 \tag{19}$$
$$= \int_0^{\infty} ue^{-u}\,du = 1,$$

where in the last line we used the substitution $u=2^d(\gamma\theta)^{-d}e^{-\gamma y_2}$.

(b) This immediately follows from (a) since $d_{r_t}=d_t(1+o(1))$ and $f_t\to 0$, $g_t\to\infty$, $\lambda_t\to 0$. $\square$

REMARK 7. The reason why we need to study a general c rather than $c=0$ is just to show that $|Z_t^{(1,c)}|<r_tg_t$ and $|Z_t^{(2,c)}|<r_tg_t$ with high probability, which is done in part (b) of the proposition above. This will be required later on in Lemma 4.5 with some c identified in Proposition 4.3. The full strength of the convergence result proved in the part (a) of the proposition will only be used for $c=0$.

PROOF OF THEOREM 1.2. The result follows from Proposition 3.2(a) with $c = 0$ by integrating the density p over all possible values of x_2, y_1, and y_2. Similarly to (19), we obtain

$$\begin{aligned} p^{(1)}(x) &= \int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}} p(x, y_1, x_2, y_2)\, dy_1\, dx_2\, dy_2 \\ &= 2^d(\gamma\theta)^{-d} \exp\{-\gamma\theta|x|\} \\ &\quad \times \int_{-\infty}^{\infty}\int_{y_2}^{\infty} \gamma^2 \exp\{-\gamma(y_1 + y_2) - 2^d(\gamma\theta)^{-d} e^{-\gamma y_2}\}\, dy_1\, dy_2 \\ &= 2^{-d} d^{d(1-1/\gamma)} \exp\{-d^{1-1/\gamma}|x|\} \end{aligned}$$

as required. □

4. Negligible paths of the random walk. Throughout this section, we assume that $0 < \gamma < 2$. We introduce three groups of paths of the random walk (X_s) informally described in the Introduction and show that their contribution to the total mass of the solution u of the parabolic Anderson model is negligible.

Denote by J_t the number of jumps the random walk (X_s) makes up to time t and consider the following three groups of paths:

$$E_i(t) = \begin{cases} \Big\{\max_{0\le s\le t} \xi(X_s) = \xi(Z_t^{(1)}),\, J_t > |Z_t^{(1)}|(1+\rho_t)\Big\}, & i = 1, \\ \Big\{\xi_{r_t g_t}^{(k_t)} \le \max_{0\le s\le t} \xi(X_s) \ne \xi(Z_t^{(1)})\Big\}, & i = 2, \\ \Big\{\max_{0\le s\le t} \xi(X_s) < \xi_{r_t g_t}^{(k_t)}\Big\}, & i = 3. \end{cases}$$

Denote by

$$U_i(t) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\, ds\Big\} \mathbb{1}_{E_i(t)}\Big], \qquad 1 \le i \le 3$$

their contributions to the total mass of the solution. The aim of this section is to show that all $U_i(t)$ is negligible with respect to $U(t)$.

We start with Lemma 4.1 where we collect all asymptotic properties of the environment which we use later on. In Lemma 4.2, we prove a simple lower bound for the total mass $U(t)$. Then we prove Proposition 4.3, which is a crucial tool for analysing $U_1(t)$ and $U_2(t)$ as it gives a general upper bound on the total mass corresponding to the paths reaching the maximum of the potential in a certain set and having a lower bound restriction on the number of jumps J_t. Equipped with this result, we show that $U_1(t)$ and $U_2(t)$ are negligible in Lemmas 4.4 and 4.5. Finally, Lemma 4.6 provides a simple proof of the negligibility of $U_3(t)$.

Observe that Proposition 4.3 identifies the constant c, which is then fixed and used throughout the paper afterward.

LEMMA 4.1. *Almost surely,*

(a) $\xi_r^{(\lfloor r^\rho \rfloor)} \sim ((d-\rho)\log r)^{1/\gamma}$ *and* $\xi_r^{(\lfloor r^\sigma \rfloor)} \sim ((d-\sigma)\log r)^{1/\gamma}$ *as* $r \to \infty$;
(b) $\xi_{r_t g_t}^{(k_t)} \sim ((d-\rho)\log t)^{1/\gamma}$ *and* $\xi_{r_t g_t}^{(m_t)} \sim ((d-\sigma)\log t)^{1/\gamma}$ *as* $t \to \infty$;
(c) $\log(\xi_r^{(\lfloor r^\rho \rfloor)} - \xi_r^{(\lfloor r^\sigma \rfloor)}) = \frac{1}{\gamma}\log\log r + O(1)$ *as* $r \to \infty$;
(d) $\log(\xi_{r_t g_t}^{(1)} - \xi_{r_t g_t}^{(m_t)}) = \frac{1}{\gamma}\log\log t + O(1)$ *as* $t \to \infty$;
(e) *the set* G_p *is totally disconnected eventually for all* p.

Further,

(f) *for all* c, $Z_t^{(1)} \in F_{r_t g_t}$ *on the event* $\mathcal{E}_c(t)$ *eventually for all* t;
(g) *for all* c, $\log \xi(Z_t^{(1)}) = \frac{1}{\gamma}\log\log t + O(1)$ *on the event* $\mathcal{E}_c(t)$ *as* $t \to \infty$;
(h) *there exists a constant* $c_1 > 0$ *such that* $|z| > t^{c_1}$ *for all* $z \in F_{r_t g_t}$ *eventually for all* t *almost surely.*

PROOF. (a) It follows from the proof of [17], Lemma 4.7, that for each $\kappa \in (0, d)$ almost surely

$$\xi_r^{(\lfloor r^\kappa \rfloor)} \sim \big((d-\kappa)\log r\big)^{1/\gamma}$$

as $r \to \infty$. It remains to substitute $\kappa = \rho$ and $\kappa = \sigma$.

(b) This follows from (a) since $k_t = \lfloor (r_t g_t)^\rho \rfloor$ and $m_t = \lfloor (r_t g_t)^\sigma \rfloor$.

(c) This follows from (a) since $\rho \neq \sigma$.

(d) This follows from (a) and Lemma 2.1 since $\rho \neq 0$.

(e) This was proved in [12], Lemma 2.2, for Pareto potentials (observe that the proof relies on $\sigma < 1/2$ which is the reason why we have imposed this restriction). It remains to notice that $\xi(z) = (\alpha \log(\zeta(z)))^{1/\gamma}$, where $\{\zeta(z) : z \in \mathbb{Z}^d\}$ is a Pareto-distributed potential with parameter α. As the locations of upper order statistics for ζ and ξ coincide, we obtain that G_p is eventually totally disconnected for Weibull potentials as well.

(f) Denote by w_t the maximiser of ξ in the ball of radius t. Using Lemma 2.1, we obtain

$$\xi\big(Z_t^{(1)}\big) \ge \Psi_t\big(Z_t^{(1)}\big) \ge \Psi_t(w_t) = \xi(w_t) - \frac{|w_t|}{\gamma t}\log\log t$$
$$\ge \xi_t^{(1)} - \frac{1}{\gamma}\log\log t \sim (d\log t)^{1/\gamma}.$$

It remains to observe that $|Z_t^{(1)}| \le r_t g_t$ on the event $\mathcal{E}_c(t)$ and use (a) to get

$$\xi\big(Z_t^{(1)}\big) \ge \big((d-\rho)\log t\big)^{1/\gamma} \sim \xi_{r_t g_t}^{(k_t)}.$$

(g) It follows from (f) that $\log \xi_{r_t g_t}^{(k_t)} \le \log \xi(Z_t^{(1)}) \le \log \xi_{r_t g_t}^{(1)}$ on the event $\mathcal{E}_c(t)$. It remains to observe that $\log \xi_{r_t g_t}^{(k_t)} = \frac{1}{\gamma}\log\log t + O(1)$ according to (a) and $\log \xi_{r_t g_t}^{(1)} = \frac{1}{\gamma}\log\log t + O(1)$ by Lemma 2.1.

(h) Choose c_1 small enough so that $c_1(d+c_1) < d-\rho-c_1$. Then almost surely eventually

$$\xi^{(1)}_{t^{c_1}} \le \big((d+c_1)\log t^{c_1}\big)^{1/\gamma} < \big((d-\rho-c_1)\log t\big)^{1/\gamma} < \xi^{(k_t)}_{r_t g_t},$$

which implies the result. □

LEMMA 4.2. *For each c,*

$$\log U(t) \ge t\Psi_t\big(Z^{(1)}_t\big) - 2dt + O(r_t g_t) \tag{20}$$

on the event $\mathcal{E}_c(t)$ eventually for all t.

PROOF. The idea of the proof is the same as of [17], Lemma 2.1, for Weibull potentials and [12], Proposition 4.2, for Pareto potentials. However, we need to estimate the error term more precisely.

Let $\rho \in (0,1]$ and $z \in \mathbb{Z}^d$, $z \ne 0$. Following the lines of [12], Proposition 4.2, we obtain

$$U(t) \ge \exp\left\{t(1-\rho)\xi(z) - |z|\log\frac{|z|}{e\rho t} - 2dt + O\big(\log|z|\big)\right\}. \tag{21}$$

Take $z = Z^{(1)}_t$ and $\rho = |Z^{(1)}_t|/(t\xi(Z^{(1)}_t))$. Observe that on the event $\mathcal{E}_c(t)$ this ρ belongs to $(0,1]$ eventually as

$$\frac{|Z^{(1)}_t|}{t\xi(Z^{(1)}_t)} \le \frac{r_t g_t}{t\xi^{(k_t)}_{r_t g_t}} = O\left(\frac{g_t}{\log t \cdot \log\log t}\right) = o(1)$$

by Lemma 4.1(f) and according to (12). Substituting this into (21) and using Lemma 4.1(g) we obtain

$$\begin{aligned}\log U(t) &\ge t\xi\big(Z^{(1)}_t\big) - \big|Z^{(1)}_t\big|\log\xi\big(Z^{(1)}_t\big) - 2dt + O(\log t)\\ &= t\Psi_t\big(Z^{(1)}_t\big) - 2dt + O(r_t g_t)\end{aligned}$$

on the event $\mathcal{E}_c(t)$. □

For all sufficiently large t, consider a set $M_t \subset \mathbb{Z}^d$ and a nonnegative function $h_t = O(r_t g_t)$ (which may both depend on ξ). Denote by z_t a point along the trajectory of $(X)_s$, $s \in [0,t]$, where the value of the potential is maximal. Define

$$U_{M,h}(t) = \mathbb{E}_0\left[\exp\left\{\int_0^t \xi(X_s)\,ds\right\}\mathbb{1}\left\{\max_{0\le s\le t}\xi(X_s) \ge \xi^{(k_t)}_{r_t g_t}, z_t \in M_t, J_t \ge h_t\right\}\right].$$

In the sequel, $U_{M,h}(t)$ will correspond to $U_1(t)$ if we choose $M_t = \{Z^{(1)}_t\}$, $h_t = |Z^{(1)}_t|(1+\rho_t)$ and to $U_2(t)$ if we choose $M_t = \mathbb{Z}^d \setminus \{Z^{(1)}_t\}$, $h_t = 0$.

PROPOSITION 4.3. *There is a constant c such that*

$$\log U_{M,h}(t) \le \max\Big\{t\Psi_t(Z_t^{(2)}), \max_{z\in M_t}\Big\{t\Psi_{t,c}(z) - \frac{(h_t-|z|)_+}{2}(\gamma^{-1}-\beta)\log\log t\Big\} + O(r_tg_t)\Big\} - 2dt$$

on the event $\mathcal{E}_c(t)$ eventually for all t.

PROOF. Consider the event $\mathcal{E}_c(t)$ and suppose that t is sufficiently large. Using the notation from Section 2.2, for each $n, p \in \mathbb{N}\cup\{0\}$ and t large enough, we denote

$$\mathcal{P}_{n,p}(t) = \big\{y \in \mathcal{P}_n : y_0 = 0, p(y) = p, q(y) > \xi_{r_tg_t}^{(k_t)}, z(y) \in M_t\big\}.$$

Observe that $q(y) \ge \xi_{r_tg_t}^{(k_t)}$ implies by Lemma 4.1(h) that $p(y) > t^{c_1}$, for some $c_1 > 0$. In particular,

$$\log\log p(y) \ge \log\log t + \log c_1. \tag{22}$$

We have

$$U_{M,h}(t) = \sum_{n\ge h_t}\sum_{t^{c_1}<p\le n}\sum_{y\in\mathcal{P}_{n,p}(t)} U(t,y),$$

where $U(t,y)$ has been defined in (15). Since the number of paths in the set $\mathcal{P}_{n,p}(t)$ is bounded by $(2d)^n$, we obtain

$$\begin{aligned}U_{M,h}(t) &\le \sum_{p>t^{c_1}}\sum_{n\ge p\vee h_t}(2d)^{-n}\max_{y\in\mathcal{P}_{n,p}(t)}\big\{(2d)^{2n}U(t,y)\big\}\\ &\le 4\max_{p>t^{c_1}}\max_{n\ge p\vee h_t}\max_{y\in\mathcal{P}_{n,z}(t)}\big\{(2d)^{2n}U(t,y)\big\}\end{aligned}$$

and so

$$\log U_{M,h}(t) \le \max_{p>t^{c_1}}\max_{n\ge p\vee h_t}\max_{y\in\mathcal{P}_{n,z}(t)}\big\{3n\log(2d) + \log U(t,y)\big\}. \tag{23}$$

Let $p > t^{c_1}$, $n \ge p \vee h_t$, and $y \in \mathcal{P}_{n,p}(t)$. Denote $i(y) = \min\{i : \xi(y_i) = q(y)\}$ and

$$Q(p,y) = q(y) \vee \xi_p^{(\lfloor p^\rho\rfloor)} + \mu_p, \tag{24}$$

where the correction term μ_p has been defined in (11). Define

$$\xi_i^y = \begin{cases}\xi(y_i), & \text{if } i \ne i(y),\\ Q(p,y), & \text{if } i = i(y).\end{cases}$$

Since $\xi_i^y \ge \xi(y_i)$ for all i, we have

$$U(t,y) \le (2d)^{-n} \mathsf{E}\Bigg[\exp\Bigg\{\sum_{i=0}^{n-1} \tau_i \xi_i^y + \Bigg(t - \sum_{i=0}^{n-1} \tau_i\Bigg)\xi_n^y\Bigg\} \times \mathbb{1}\Bigg\{\sum_{i=0}^{n-1} \tau_i < t, \sum_{i=0}^{n} \tau_i > t\Bigg\}\Bigg].$$

This expectation has been bounded from above in (4.16) and (4.17) of [17]. Substituting its bound, we obtain

$$\begin{aligned} U(t,y) &\le \exp\{t\xi_{i(y)}^y - 2dt\} \prod_{i \ne i(y)} \frac{1}{\xi_{i(y)}^y - \xi_i^y} \\ &= \exp\{tQ(p,y) - 2dt\} \prod_{i \ne i(y)} \frac{1}{Q(p,y) - \xi(y_i)} \end{aligned}$$

and hence

$$\log U(t,y) \le tQ(p,y) - 2dt - \sum_{i \ne i(y)} \log\big(Q(p,y) - \xi(y_i)\big). \tag{25}$$

The set G_p consists of $\lfloor p^\sigma \rfloor$ elements and is totally disconnected by Lemma 4.1(e). Hence, by Lemma 2.3 we have

$$n_+(y, G_p) \le \frac{n-p}{2} + p^\sigma. \tag{26}$$

In each point $y_i \in G_p$ we use (24) to estimate

$$\log\big(Q(p,y) - \xi(y_i)\big) \ge \log \mu_p = -\beta \log\log p. \tag{27}$$

On the other hand,

$$\begin{aligned} n_-(y, G_p) &= n + 1 - n_+(y, G_p) \\ &\ge n + 1 - \frac{n-p}{2} - p^\sigma = p - p^\sigma + \frac{n-p}{2} + 1 \end{aligned} \tag{28}$$

and in each point $y_i \notin G_p$ we obtain by Lemma 4.1(c)

$$\log\big(Q(p,y) - \xi(y_i)\big) \ge \log\big(\xi_p^{(\lfloor p^\rho \rfloor)} - \xi_p^{(\lfloor p^\sigma \rfloor)}\big) \ge \gamma^{-1} \log\log p + c_2 \tag{29}$$

with some constant c_2. Using (27) and (29) together with (25), we obtain

$$\begin{aligned} \log U(t,y) \le\ & tQ(p,y) - 2dt + n_+(y, G_p)\beta \log\log p \\ & - \big(n_-(y, G_p) - 1\big)\big(\gamma^{-1} \log\log p + c_2\big). \end{aligned}$$

Substituting (26) and (28) and using $p^\sigma \log\log p \le n$, we obtain

$$
\begin{aligned}
&3n\log(2d) + \log U(t,y)\\
&\quad \le 3n\log(2d) + tQ(p,y) - 2dt\\
&\qquad + \left[\frac{n-p}{2} + p^\sigma\right]\beta\log\log p - \left[p - p^\sigma + \frac{n-p}{2}\right]\left(\gamma^{-1}\log\log p + c_2\right)\\
&\quad \le tQ(p,y) - \frac{p}{\gamma}\log\log p - 2dt - \frac{n-p}{2}\left(\gamma^{-1}-\beta\right)\log\log p + c_3 n
\end{aligned}
\tag{30}
$$

with some constant c_3.

Now we distinguish between the following two cases.

Case 1. Suppose $q(y) \ge \xi_p^{(\lfloor p^\rho\rfloor)}$. Then $Q(p,y) = \xi(z(y)) + \mu_p$ and estimating $p \ge |z(y)|$ we get

$$
\begin{aligned}
3n\log(2d) + \log U(t,y) \le t\xi\big(z(y)\big) + t\mu_p - \frac{|z(y)|}{\gamma}\log\log p - 2dt\\
- \frac{n-|z(y)|}{2}\left(\gamma^{-1}-\beta\right)\log\log p + c_3 n.
\end{aligned}
$$

Observe that $t\mu_p \le t\mu_{t^{c_1}} = t(c_1\log t)^{-\beta} = o(r_t g_t)$ since $\beta > 1 - 1/\gamma$ and according to (12). Using monotonicity in n and $n \ge |z(y)| \vee h_t$ together with (22), we obtain

$$
\begin{aligned}
&3n\log(2d) + \log U(t,y)\\
&\quad \le t\Psi_t\big(z(y)\big) + c\big|z(y)\big| - 2dt\\
&\qquad - \frac{(h_t - |z(y)|)_+}{2}\left(\gamma^{-1}-\beta\right)\log\log t + ch_t + o(r_t g_t)\\
&\quad \le \max_{z\in M_t}\left\{t\Psi_{t,c}(z) - \frac{(h_t-|z|)_+}{2}\left(\gamma^{-1}-\beta\right)\log\log t\right\} - 2dt + O(r_t g_t)
\end{aligned}
\tag{31}
$$

with some constant c.

Case 2. Suppose $q(y) < \xi_p^{(\lfloor p^\rho\rfloor)}$. Then $Q(p,y) = \xi_p^{(\lfloor p^\rho\rfloor)} + \mu_p$. Now (30) implies

$$
\begin{aligned}
3n\log(2d) + \log U(t,y) \le t\xi_p^{(\lfloor p^\rho\rfloor)} + t\mu_p - \frac{p}{\gamma}\log\log p - 2dt\\
- \frac{n-p}{2}\left(\gamma^{-1}-\beta\right)\log\log p + c_4 n
\end{aligned}
$$

with some constant c_4. Using monotonicity in n and $n \ge p$, we get

$$
3n\log(2d) + \log U(t,y) \le t\xi_p^{(\lfloor p^\rho\rfloor)} + t(\log p)^{-\beta} - \frac{p}{\gamma}\log\log p - 2dt + c_4 p.
$$

By Lemma 4.1(a) and using $\beta \geq 0$, we obtain that the second term is dominated by the first one, the fifth by the third one, and so

$$3n \log(2d) + \log U(t, y) \leq t\big((d - \rho/2) \log p\big)^{1/\gamma} - c_5 p \log\log p - 2dt \tag{32}$$

with some constant $c_5 > 0$. Differentiating, we obtain the following equation for the maximiser p_t of the expression on the right-hand side of (32):

$$\frac{t(d-\rho/2)((d-\rho/2)\log p_t)^{1/\gamma-1}}{\gamma p_t} - c_5 \log\log p_t - \frac{c_5}{\log p_t} = 0.$$

Resolving this asymptotics, we obtain

$$p_t = r_t (d - \rho/2)^{1/\gamma}\big(1 + o(1)\big).$$

Finally, substituting this into (32) yields

$$\begin{aligned} 3n \log(2d) + \log U(t, y) &\leq t\big((d - \rho/3) \log r_t\big)^{1/\gamma} - 2dt \\ &\leq \big(1 - \rho/(3d)\big)^{1/\gamma} t a_{r_t} - 2dt \qquad (33) \\ &\leq t \Psi_t\big(Z_t^{(2)}\big) - 2dt \qquad (34) \end{aligned}$$

on the event $\mathcal{E}_c(t)$. It remains to substitute (31) and (33) into (23) to complete the proof. □

REMARK 8. Observe that the scaling function μ_p, being part of $Q(p, y)$, appears both in the main and in the logarithmic term of (25). Being part of the main term, $t\mu_p$ needs to be as small as $O(r_t g_t)$ in order to not imbalance the significant terms. This leads to the restriction $\beta > 1 - 1/\gamma$. However, as a part of the logarithmic term, μ_p needs to be large enough so that the contribution $\gamma^{-1} \log\log p$ of "good" points $y_i \notin G_p$ dominates over the contribution $\beta \log\log p$ of "bad" points $y_i \in G_p$. This imposes the restriction $\beta < 1/\gamma$. The combination of these two conditions only allows to choose such β if $0 < \gamma < 2$.

From now on, we assume that the constant c is fixed and chosen according to Proposition 4.3.

LEMMA 4.4. *Almost surely,*

$$\frac{U_1(t)}{U(t)} \mathbb{1}_{\mathcal{E}_c(t)} \to 0 \qquad \text{as } t \to \infty.$$

PROOF. We use Proposition 4.3 with $M_t = \{Z_t^{(1)}\}$ and $h_t = |Z_t^{(1)}|(1 + \rho_t)$. Clearly $h_t = O(r_t g_t)$ on the event $\mathcal{E}_c(t)$. By Lemma 4.1(f), we have $Z_t^{(1)} \in F_{r_t g_t}$, which implies $U_{M,h}(t) = U_1(t)$ eventually for all t. Since $|Z_t^{(1)}| \le r_t g_t$ and so $t\Psi_{t,c}(Z_t^{(1)}) = t\Psi_t(Z_t^{(1)}) + O(r_t g_t)$, we obtain

$$\log U_1(t) \le \max\Big\{t\Psi_t\big(Z_t^{(2)}\big), t\Psi_t\big(Z_t^{(1)}\big) - \frac{|Z_t^{(1)}|\rho_t}{2}(1/\gamma - \beta)\log\log t + O(r_t g_t)\Big\} - 2dt. \tag{35}$$

In order to show that

$$\log U_1(t) - \log U(t) \to -\infty \tag{36}$$

we consider the terms under the maximum in (35) separately. Using the lower bound for the total mass given by Lemma 4.2 and taking into account that $\Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)}) > d_t\lambda_t$ on the event $\mathcal{E}_c(t)$, we get for the first term

$$\begin{aligned} t\Psi_t\big(Z_t^{(2)}\big) - 2dt - \log U(t) &\le t\Psi_t\big(Z_t^{(2)}\big) - t\Psi_t\big(Z_t^{(1)}\big) + O(r_t g_t) \\ &< -td_t\lambda_t + O(r_t g_t) \to -\infty \end{aligned} \tag{37}$$

according to (12). For the second term, we again use the lower bound from Lemma 4.2 and take into account that $|Z_t^{(1)}| \ge r_t f_t$ on the event $\mathcal{E}_c(t)$. This implies

$$\begin{aligned} t\Psi_t\big(Z_t^{(1)}\big) &- \frac{|Z_t^{(1)}|\rho_t}{2}(1/\gamma - \beta)\log\log t + O(r_t g_t) - 2dt - \log U(t) \\ &\le -\frac{|Z_t^{(1)}|\rho_t}{2}(1/\gamma - \beta)\log\log t + O(r_t g_t) \\ &\le -\frac{r_t f_t \rho_t}{2}(1/\gamma - \beta)\log\log t + O(r_t g_t) \to -\infty \end{aligned} \tag{38}$$

by (12). Combining (37), (38) and (35) we get (36) on the event $\mathcal{E}_c(t)$. □

LEMMA 4.5. *Almost surely,*

$$\frac{U_2(t)}{U(t)}\mathbb{1}_{\mathcal{E}_c(t)} \to 0 \qquad \text{as } t \to \infty.$$

PROOF. We use Proposition 4.3 with $M_t = \mathbb{Z}^d \setminus \{Z_t^{(1)}\}$ and $h_t = 0$. In this case $U_{M,h}(t) = U_2(t)$, and we have

$$\log U_2(t) \le \max\Big\{t\Psi_t\big(Z_t^{(2)}\big), t\max_{z\ne Z_t^{(1)}}\Psi_{t,c}(z) + O(r_t g_t)\Big\} - 2dt. \tag{39}$$

Since $|Z_t^{(1,c)}| \le r_t g_t$ and $|Z_t^{(2,c)}| \le r_t g_t$ on the event $\mathcal{E}_c(t)$, we have for $i \in \{1,2\}$

$$t\Psi_{t,c}\big(Z_t^{(i,c)}\big) = t\Psi_t\big(Z_t^{(i,c)}\big) + c|Z_t^{(i,c)}| = t\Psi_t\big(Z_t^{(i,c)}\big) + O(r_t g_t).$$

Substituting this into (39) and observing that $z \neq Z_t^{(1)}$, we obtain

$$\log U_2(t) \le t\Psi_t\big(Z_t^{(2)}\big) + O(r_t g_t) - 2dt.$$

Using the lower bound for the total mass given by Lemma 4.2 and taking into account that $\Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)}) > d_t\lambda_t$ on the event $\mathcal{E}_c(t)$, we get

$$\begin{aligned}\log U_2(t) - \log U(t) &\le t\Psi_t\big(Z_t^{(2)}\big) - t\Psi_t\big(Z_t^{(1)}\big) + O(r_t g_t)\\ &\le -td_t\lambda_t + O(r_t g_t) \to -\infty\end{aligned}$$

according to (12) on the event $\mathcal{E}_c(t)$. □

LEMMA 4.6. *Almost surely,*

$$\frac{U_3(t)}{U(t)}\mathbb{1}_{\mathcal{E}_c(t)} \to 0 \qquad \text{as } t \to \infty.$$

PROOF. We can estimate the integral in the Feynman–Kac formula for $U_3(t)$ by $t\xi_{r_t g_t}^{(k_t)}$ and get

$$\log U_3(t) \le t\xi_{r_t g_t}^{(k_t)} \sim t\big((d-\rho)\log t\big)^{1/\gamma} \le (1-\delta)ta_{r_t}$$

with some $\delta > 0$ eventually for all t by Lemma 4.1(b). Using the lower bound for $U(t)$ from Lemma 4.2, we have

$$\begin{aligned}\log U_3(t) - \log U(t) &\le (1-\delta)ta_{r_t} - t\Psi_t\big(Z_t^{(1)}\big) + 2dt + O(r_t g_t)\\ &\le -\delta t a_{r_t} + td_t g_t + 2dt + O(r_t g_t) \to -\infty\end{aligned}$$

since $\Psi_t(Z_t^{(1)}) > a_{r_t} - d_t g_t$ on the event $\mathcal{E}_c(t)$. □

5. Localisation. The aim of this section is to prove Theorem 1.1. We assume throughout this section that $0 < \gamma < 2$ and we suppose that c is chosen according to Proposition 4.3.

Let

$$B_t = \big\{z \in \mathbb{Z}^d : |z| \le \big|Z_t^{(1)}\big|(1+\rho_t)\big\}.$$

For any set $A \subset \mathbb{Z}^d$ denote by $A^c = \mathbb{Z}^d \setminus A$ its complement and by $\tau(A)$ the hitting time of A by the random walk (X_s), and we write $\tau(z)$ for $\tau(\{z\})$ for any point $z \in \mathbb{Z}^d$. Let us decompose the solution u into $u = u_1 + u_2$ according to the two groups of paths (I) and (II) mentioned in the Introduction

$$\begin{aligned}u_1(t,z) &= \mathbb{E}_0\bigg[\exp\bigg\{\int_0^t \xi(X_s)\,ds\bigg\}\mathbb{1}\{X_t = z\}\mathbb{1}\big\{\tau\big(Z_t^{(1)}\big) \le t, \tau\big(B_t^c\big) > t\big\}\bigg],\\ u_2(t,z) &= \mathbb{E}_0\bigg[\exp\bigg\{\int_0^t \xi(X_s)\,ds\bigg\}\mathbb{1}\{X_t = z\}\mathbb{1}\big\{\tau\big(Z_t^{(1)}\big) > t \text{ or } \tau\big(B_t^c\big) \le t\big\}\bigg].\end{aligned}$$

In Lemma 5.1 below, we use the results from Section 4 to prove that the total mass of u_2 is negligible. In order to prove that u_1 localises around $Z_t^{(1)}$, we introduce the gap

$$\mathfrak{g}_t = \xi\big(Z_t^{(1)}\big) - \max\big\{\xi(z) : z \in B_t \setminus \big\{Z_t^{(1)}\big\}\big\}$$

between the value of the potential ξ at the point $Z_t^{(1)}$ and in the rest of the ball B_t. In Lemma 5.2 we find a lower bound for $\mathfrak{g}_t$. This bound tends to infinity for $\gamma < 1$ but is going to zero for $1 \le \gamma < 2$. However, the lower bound turns out to be just large enough to provide localisation of the principal eigenfunction of the Anderson Hamiltonian $\Delta + \xi$ around $Z_t^{(1)}$, which is proved in Lemma 5.3. This easily implies the localisation of u_1 around $Z_t^{(1)}$ and allows us to prove Theorem 1.1 in the end of this section.

LEMMA 5.1. *Almost surely*,

$$\Big\{U(t)^{-1} \sum_{z \in \mathbb{Z}^d} u_2(t,z)\Big\} \mathbb{1}_{\mathcal{E}_c(t)} \to 0 \qquad \textit{as } t \to \infty.$$

PROOF. We have

$$\sum_{z \in \mathbb{Z}^d} u_2(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,ds\Big\} \mathbb{1}\big\{\tau\big(Z_t^{(1)}\big) > t \text{ or } \tau\big(B_t^c\big) \le t\big\}\Big]. \tag{40}$$

Observe that if a path belongs to the set in the indicator function above then either it passes through $Z_t^{(1)}$ and reaches the maximum of the potential there but leaves the ball B_t thus belonging to $E_1(t)$, or it reaches the maximum of the potential not in $Z_t^{(1)}$ thus belonging to $E_2(t)$ or $E_3(t)$, depending on whether the maximum of the potential over the path exceeds the value $\xi_{r_t g_t}^{(k_t)}$. Hence, we have on the event $\mathcal{E}_c(t)$

$$\begin{aligned}\sum_{z \in \mathbb{Z}^d} u_2(t,z) &\le \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,ds\Big\} \mathbb{1}_{E_1(t) \cup E_2(t) \cup E_3(t)}\Big] \\ &= U_1(t) + U_2(t) + U_3(t).\end{aligned}$$

The statement of the lemma now follows from Lemmas 4.4, 4.5 and 4.6. □

LEMMA 5.2. *On the event $\mathcal{E}_c(t)$, the gap $\mathfrak{g}_t$ is positive and, for any $\varepsilon > 0$*,

$$\log \mathfrak{g}_t > (1/\gamma - 1 - \varepsilon) \log\log t$$

eventually for all t.

PROOF. Let $z \in B_t \setminus \{Z_t^{(1)}\}$. Then $\Psi_t(z) \le \Psi_t(Z_t^{(2)})$ and we have on the event $\mathcal{E}_c(t)$

$$d_t\lambda_t \le \Psi_t(Z_t^{(1)}) - \Psi_t(Z_t^{(2)}) \le \Psi_t(Z_t^{(1)}) - \Psi_t(z)$$
$$= \xi(Z_t^{(1)}) - \xi(z) + \frac{|z| - |Z_t^{(1)}|}{\gamma t} \log\log t.$$

Since $|Z_t^{(1)}| < r_t g_t$ on the event $\mathcal{E}_c(t)$, the last term satisfies

$$\frac{|z| - |Z_t^{(1)}|}{\gamma t} \log\log t \le \frac{|Z_t^{(1)}|\rho_t}{\gamma t} \log\log t \le \frac{r_t g_t \rho_t}{\gamma t} \log\log t = O(d_t g_t \rho_t).$$

We obtain uniformly for all $z \in B_t \setminus \{Z_t^{(1)}\}$

$$d_t\lambda_t \le \xi(Z_t^{(1)}) - \xi(z) + O(d_t g_t \rho_t)$$

and so

$$\mathfrak{g}_t \ge d_t\lambda_t + O(d_t g_t \rho_t) = d_t\lambda_t + o(d_t\lambda_t)$$

on according to (13). This estimate implies the statement of the lemma since $\log d_t \sim (\frac{1}{\gamma} - 1)\log\log t$ and λ_t is negligible according to (12). $\square$

Let γ_t and v_t be the principal eigenvalue and eigenfunction of $\Delta + \xi$ with zero boundary conditions in the ball B_t. We extend v_t by zero to the whole space $\mathbb{Z}^d$ and we assume that v_t is normalised so that $v_t(Z_t^{(1)}) = 1$. The eigenfunction v_t has the following probabilistic representation

$$v_t(z) = \mathbb{E}_z\left[\exp\left\{\int_0^{\tau(Z_t^{(1)})} (\xi(X_s) - \gamma_t)\, ds\right\}\mathbb{1}\{\tau(Z_t^{(1)}) < \tau(\mathbb{Z}^d \setminus B_t)\}\right].$$

LEMMA 5.3. *Almost surely,*

$$\left\{\|v_t\|_2^2 \sum_{z \in B_t \setminus \{Z_t^{(1)}\}} v_t(z)\right\}\mathbb{1}_{\mathcal{E}_c(t)} \to 0 \qquad \text{as } t \to \infty.$$

PROOF. Consider the event $\mathcal{E}_c(t)$ and suppose that t is sufficiently large. For each $n, p \in \mathbb{N}$ and $z \in B_t \setminus \{Z_t^{(1)}\}$ denote

$$\mathcal{P}_{n,p}(t,z) = \{y \in \mathcal{P}_n : y_0 = z, y_n = Z_t^{(1)}, y_i \in B_t \setminus Z_t^{(1)}\ \forall i < n, p(y) = p\}.$$

Integrating with respect to the waiting times (τ_i) of the random walk, which are independent and exponentially distributed with parameter $2d$ and observing that the probability of the first n steps of the random walk to follow a given geometric path is $(2d)^{-n}$ we get

$$\begin{aligned} v_t(z) &= \sum_{n\geq|z-Z_t^{(1)}|}\sum_{p\leq n}\sum_{y\in\mathcal{P}_{n,p}(t,z)}(2d)^{-n}\mathsf{E}\Bigg[\exp\Bigg\{\sum_{i=0}^{n-1}\big(\xi(y_i)-\gamma_t\big)\tau_i\Bigg\}\Bigg] \\ &= \sum_{n\geq|z-Z_t^{(1)}|}\sum_{1\leq p\leq n}\sum_{y\in\mathcal{P}_{n,p}(t,z)}\prod_{i=0}^{n-1}\int_0^\infty \exp\{-(\gamma_t+2d-\xi(y_i))t\}\,dt. \end{aligned}$$

The Rayleigh–Ritz formula implies

$$\begin{aligned} \gamma_t &= \sup\{\langle(\Delta+\xi)\varphi,\varphi\rangle : \varphi\in\ell^2(B_t), \varphi|_{\partial B_t}=0, \|\varphi\|_2=1\} \\ &\geq \langle(\Delta+\xi)\mathbb{1}_{\{Z_t^{(1)}\}},\mathbb{1}_{\{Z_t^{(1)}\}}\rangle = \xi\big(Z_t^{(1)}\big)-2d \end{aligned}$$

and so for all i

$$\gamma_t+2d-\xi(y_i)\geq\xi\big(Z_t^{(1)}\big)-\xi(y_i)\geq\mathfrak{g}_t. \tag{41}$$

Since $\mathfrak{g}_t>0$ eventually on the event $\mathcal{E}_c(t)$ by Lemma 5.2, we use (41) to compute

$$\begin{aligned} v_t(z) &= \sum_{n=|z-Z_t^{(1)}|}^{\infty}\sum_{p\leq n}\sum_{y\in\mathcal{P}_{n,p}(t,z)}\prod_{i=0}^{n-1}\frac{1}{\gamma_t+2d-\xi(y_i)} \\ &\leq \sum_{p\geq|z-Z_t^{(1)}|}\sum_{n\geq p}\sum_{y\in\mathcal{P}_{n,p}(t,z)}\prod_{i=0}^{n-1}\frac{1}{\xi(Z_t^{(1)})-\xi(y_i)} \\ &\leq \sum_{p\geq|z-Z_t^{(1)}|}\sum_{n\geq p}(2d)^{-n}\max_{y\in\mathcal{P}_{n,p}(t,z)}\Bigg\{(2d)^{2n}\prod_{i=0}^{n-1}\frac{1}{\xi(Z_t^{(1)})-\xi(y_i)}\Bigg\} \\ &\leq \sum_{p\geq|z-Z_t^{(1)}|}\exp\max_{n\geq p}\max_{y\in\mathcal{P}_{n,p}(t,z)}\Bigg\{2n\log(2d)-\sum_{i=0}^{n-1}\log\big(\xi\big(Z_t^{(1)}\big)-\xi(y_i)\big)\Bigg\} \end{aligned} \tag{42}$$

since $\sum_{n\geq p}(2d)^{-n}\leq 1$ for $p\geq 1$. Fix some positive $\varepsilon\in(\frac{1}{\gamma}-1,\frac{1}{\gamma}-\frac{1}{2})$. Notice that this is possible since $\gamma<2$ and so $\frac{1}{\gamma}-\frac{1}{2}>0$. Let $p\geq|z-Z_t^{(1)}|$, $n\geq p$, and $y\in\mathcal{P}_{n,p}(t,z)$. By Lemma 4.1(e), the set $G_{r_t g_t}$ is totally disconnected and so

$$n_+(y,G_{r_t g_t})\leq\left\lceil\frac{n+1}{2}\right\rceil\leq\frac{n}{2}+1. \tag{43}$$

In each point $y_i\in G_{r_t g_t}$, we can estimate by Lemma 5.2

$$\log\big(\xi\big(Z_t^{(1)}\big)-\xi(y_i)\big)\geq\log\mathfrak{g}_t>(1/\gamma-1-\varepsilon)\log\log t. \tag{44}$$

On the other hand,

$$n_-(y, G_{r_t g_t}) = n + 1 - n_+(y, G_{r_t g_t}) \geq \frac{n}{2} \tag{45}$$

and in each point $y_i \notin G_{r_t g_t}$ we get by Lemma 4.1(d)

$$\log\bigl(\xi(Z_t^{(1)}) - \xi(y_i)\bigr) = \log\bigl(\xi_{r_t g_t}^{(k_t)} - \xi_{r_t g_t}^{(m_t)}\bigr) > (1/\gamma - \varepsilon)\log\log t \tag{46}$$

by Lemma 5.2. Using (44) and (46) and taking into account that the last point $Z_t^{(1)}$ of the path belongs to $G_{r_t g_t}$ but does not contribute to the sum, we obtain

$$\begin{aligned} 2n\log(2d) &- \sum_{i=0}^{n-1}\log\bigl(\xi(Z_t^{(1)}) - \xi(y_i)\bigr) \\ &\leq 2n\log(2d) - \bigl(n_+(y, G_{r_t g_t}) - 1\bigr)(1/\gamma - 1 - \varepsilon)\log\log t \\ &\quad - n_-(y, G_{r_t g_t})(1/\gamma - \varepsilon)\log\log t. \end{aligned}$$

Since $\frac{1}{\gamma} - 1 - \varepsilon < 0$ and $\frac{1}{\gamma} - \varepsilon > 0$, we can estimate further using (43) and (45)

$$\begin{aligned} 2n\log(2d) &- \sum_{i=0}^{n-1}\log\bigl(\xi(Z_t^{(1)}) - \xi(y_i)\bigr) \\ &\leq 2n\log(2d) - \frac{n}{2}(1/\gamma - 1 - \varepsilon)\log\log t - \frac{n}{2}(1/\gamma - \varepsilon)\log\log t \\ &= 2n\log(2d) - n(1/\gamma - 1/2 - \varepsilon)\log\log t. \end{aligned}$$

Since $\frac{1}{\gamma} - \frac{1}{2} - \varepsilon > 0$, this function is decreasing in n and can be estimated by its value at $n = p$. This implies

$$\begin{aligned} 2n\log(2d) &- \sum_{i=0}^{n-1}\log\bigl(\xi(Z_t^{(1)}) - \xi(y_i)\bigr) \\ &\leq 2p\log(2d) - p(1/\gamma - 1/2 - \varepsilon)\log\log t \leq -p\delta\log\log t \end{aligned}$$

with some $\delta > 0$. Substituting this into (42), we obtain

$$v_t(z) \leq \sum_{p \geq |z - Z_t^{(1)}|} (\log t)^{-p\delta} \leq 2(\log t)^{-\delta|z - Z_t^{(1)}|}.$$

Since $v_t(z)$ decays geometrically in distance of z from $Z_t^{(1)}$, $(\log t)^{-\delta} \to 0$, and $v_t(Z_t^{(1)}) = 1$, the statement of the lemma is now obvious. □

REMARK 9. Observe that, similarly to the proof of Proposition 4.3, we have a competition of the positive and negative terms in the sum in (42), and we want the negative terms to dominate. The contribution of the positive terms is of order $(1/\gamma - 1)\log\log t$ and the contribution of the negative terms is roughly $(1/\gamma)\log\log t$. This leads to the condition $1 - 1/\gamma < 1/\gamma$, which restricts our proof to the case $0 < \gamma < 2$.

PROOF OF THEOREM 1.1. We have

$$
\begin{aligned}
1-\frac{u(t,Z_t^{(1)})}{U(t)} &= U(t)^{-1}\sum_{z\neq Z_t^{(1)}} u(t,z) \\
&\le U(t)^{-1}\sum_{z\neq Z_t^{(1)}} u_1(t,z) + U(t)^{-1}\sum_{z\in\mathbb{Z}^d} u_2(t,z).
\end{aligned} \tag{47}
$$

The second term converges to zero on the event $\mathcal{E}_c(t)$ by Lemma 5.1. The first term satisfies the conditions of [8], Theorem 4.1, with $B=B_t$, $V=\xi$, and $\Gamma=\{Z_t^{(1)}\}$, which implies that, for all $z\in B_t$,

$$
u_1(t,z)\le u_1\big(t,Z_t^{(1)}\big)\|v_t\|_2^2 v_t(z).
$$

Observing that $U(t)\ge u_1(t,Z_t^{(1)})$ and $u_1(t,z)=0$ for $z\notin B_t$, we obtain

$$
U(t)^{-1}\sum_{z\neq Z_t^{(1)}} u_1(t,z)\le \|v_t\|_2^2\sum_{z\in B_t\setminus\{Z_t^{(1)}\}} v_t(z),
$$

which converges to zero on the event $\mathcal{E}_c(t)$ by Lemma 5.3. As both terms in (47) converge to zero on the event $\mathcal{E}_c(t)$ and $\mathrm{Prob}\{\mathcal{E}_c(t)\}\to 1$ by Proposition 3.2(b), we obtain that

$$
1-\frac{u(t,Z_t^{(1)})}{U(t)}\to 0 \qquad \text{as } t\to\infty
$$

in probability. □

6. Ageing. In this section, we discuss the ageing behaviour of the parabolic Anderson model. Throughout this section, we assume that $\gamma>0$. As we pointed out in the Introduction, although the results proved in this section hold for all $\gamma>0$, they only imply ageing of the parabolic Anderson model for $0<\gamma<2$ as otherwise the solution u may not be localised at $Z_t^{(1)}$.

We begin by showing that whenever the maximiser of Ψ has moved from one point to another, it cannot go back to the original point.

LEMMA 6.1. *For $s>0$, $\{T_t>s\}=\{Z_t^{(1)}=Z_{t+s}^{(1)}\}$ eventually for all t.*

PROOF. If $T_t>s$, then $Z_t^{(1)}=Z_{t+s}^{(1)}$ by the definition of T_t. Suppose $Z_t^{(1)}=Z_{t+s}^{(1)}$ but there is $u\in(t,t+s)$ such that $Z_t^{(1)}\neq Z_u^{(1)}$. Consider an auxiliary function $\varphi:[t,t+s]\to\mathbb{R}$ given by

$$
\varphi(x)=\Psi_x\big(Z_t^{(1)}\big)-\Psi_x\big(Z_u^{(1)}\big)=\xi\big(Z_t^{(1)}\big)-\xi\big(Z_u^{(1)}\big)-\frac{|Z_t^{(1)}|-|Z_u^{(1)}|}{\gamma x}\log\log x.
$$

Observe that

$$\varphi'(x) = \frac{|Z_t^{(1)}| - |Z_u^{(1)}|}{\gamma x^2 \log x}(\log x \log\log x - 1)$$

and so φ' does not change the sign on the interval $[t, t+s]$ if t is large enough. Hence, φ is strictly monotone on $[t, t+s]$. However, this contradicts the observation that $\varphi(t) \geq 0$ (since $Z_t^{(1)}$ is the maximiser of Ψ_t and $Z_u^{(1)} \neq Z_t^{(1)}$), $\varphi(u) \leq 0$ (since $Z_u^{(1)}$ is the maximiser of Ψ_u and $Z_t^{(1)} \neq Z_u^{(1)}$), and $\varphi(t+s) \geq 0$ (since $Z_t^{(1)} = Z_{t+s}^{(1)}$ is the maximiser of Ψ_{t+s} and $Z_u^{(1)} \neq Z_t^{(1)}$). $\square$

Now we are going to compute the probability of $\{Z_t^{(1)} = Z_{t+wt}^{(1)}\}$, $w > 0$, using the point processes $\Pi_t \equiv \Pi_{t,0}$ studied in Section 3. However, we need to restrict them to a finite box growing to infinity to justify integration and passing to the limit. In order to do so, for each $n \in \mathbb{N}$, we define the event

$$\mathcal{A}(n, w, t) = \{Y_{t, Z_t^{(1)}} \geq -n, \Psi_{t+wt}(z) \leq \Psi_{t+wt}(Z_t^{(1)})\ \forall z \in \mathbb{Z}^d \text{ s.t. } Y_{t,z} \geq -n\}$$

and show that $\text{Prob}\{Z_t^{(1)} = Z_{t+wt}^{(1)}\}$ is captured by the probabilities of these events.

LEMMA 6.2. *For any* $w > 0$,

$$\lim_{t\to\infty} \text{Prob}\{Z_t^{(1)} = Z_{t+wt}^{(1)}\} = \lim_{n\to\infty}\lim_{t\to\infty} \text{Prob}\{\mathcal{A}(n, w, t)\},$$

provided the limit on the right-hand side exists.

PROOF. To obtain an upper bound, observe that

$$\text{Prob}\{Z_t^{(1)} = Z_{t+wt}^{(1)}\} \leq \text{Prob}\{\mathcal{A}(n, w, t)\} + \text{Prob}\{Y_{t, Z_t^{(1)}} \leq -n\}. \tag{48}$$

By Proposition 3.2,

$$\lim_{n\to\infty}\lim_{t\to\infty} \text{Prob}\{Y_{t, Z_t^{(1)}} \leq -n\} = \lim_{n\to\infty} \text{Prob}\{Y^{(1)} \leq -n\} = 0. \tag{49}$$

For a lower bound, we have

$$\text{Prob}\{Z_t^{(1)} = Z_{t+wt}^{(1)}\} \geq \text{Prob}\{\mathcal{A}(n, w, t)\} - \text{Prob}\{Y_{t, Z_{t+wt}^{(1)}} \leq -n\}. \tag{50}$$

Observe that for all z we have, as $t \to \infty$,

$$\begin{aligned}
\Psi_{t+wt}(z) &= \xi(z) - \frac{|z|}{\gamma(t+wt)} \log\log(t+wt) \\
&= \Psi_t(z) + \frac{w|z|}{(1+w)\gamma t}(\log\log t + o(1)) \qquad (51) \\
&= \Psi_t(z) + d_{r_t} \frac{w\theta}{1+w} \frac{|z|}{r_t}(1 + o(1))
\end{aligned}$$

and so the condition $Y_{t,Z^{(1)}_{t+wt}} \le -n$ is equivalent to

$$\frac{\Psi_{t+wt}(Z^{(1)}_{t+wt}) - a_{r_t}}{d_{r_t}} - \frac{w\theta}{1+w}\frac{|Z^{(1)}_{t+wt}|}{r_t}\big(1+o(1)\big) \le -n. \tag{52}$$

It is easy to see that $r_{t+wt} \sim (1+w)r_t$. This implies that $d_{r_{t+wt}} \sim d_{r_t}$ and

$$a_{r_{t+wt}} - a_{r_t} \sim d_{r_t}\gamma^{-1} d\log(1+w).$$

Now condition (52) is equivalent to

$$\left[\frac{\Psi_{t+wt}(Z^{(1)}_{t+wt}) - a_{r_{t+wt}}}{d_{r_{t+wt}}} + \gamma^{-1}d\log(1+w) - w\theta\frac{|Z^{(1)}_{t+wt}|}{r_{t+wt}}\right]\big(1+o(1)\big) \le -n$$

and by Proposition 3.2 we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{t\to\infty}\operatorname{Prob}\{Y_{t,Z^{(1)}_{t+wt}} \le -n\} \\
&\quad = \lim_{n\to\infty}\lim_{t\to\infty}\operatorname{Prob}\left\{\left[Y_{t+wt,Z^{(1)}_{t+wt}} + \gamma^{-1}d\log(1+w) - w\theta\frac{|Z^{(1)}_{t+wt}|}{r_{t+wt}}\right]\right. \\
&\qquad\qquad \left.\times\big(1+o(1)\big) \le -n\right\} \\
&\quad = \lim_{n\to\infty}\operatorname{Prob}\big\{Y^{(1)} + \gamma^{-1}d\log(1+w) - w\theta|X^{(1)}| \le -n\big\} = 0.
\end{aligned} \tag{53}$$

Combining the bounds (48) and (50) with the convergence results (49) and (53), we obtain the required statement. □

Now we show that the probabilities of the events $\mathcal{A}(n,w,t)$ converge to a finite explicit integral.

LEMMA 6.3. *For any* $w \ge 0$,

$$\lim_{n\to\infty}\lim_{t\to\infty}\operatorname{Prob}\big\{\mathcal{A}(n,w,t)\big\} = \int_{\mathbb{R}^d\times\mathbb{R}} \exp\big\{-\nu\big(D_w(x,y)\big)\big\}\nu(dx,dy) < \infty,$$

where $D_w(x,y)$ *has been defined in* (8).

PROOF. We have

$$\begin{aligned}
&\operatorname{Prob}\big\{\mathcal{A}(n,w,t)\big\} \\
&\quad = \int_{\mathbb{R}^d\times[-n,\infty)} \operatorname{Prob}\big\{\big(Z^{(1)}_t r_t^{-1}, Y_{t,Z^{(1)}_t}\big) \in dx\times dy, \\
&\qquad\qquad \Psi_{t+wt}(z) \le \Psi_{t+wt}\big(Z^{(1)}_t\big)\ \forall z\in\mathbb{Z}^d \text{ s.t. } Y_{t,z} \ge -n\big\}.
\end{aligned}$$

Observe that according to (51) the condition $\Psi_{t+wt}(z) \le \Psi_{t+wt}(Z_t^{(1)})$ is equivalent to

$$\Psi_t(z) + d_{r_t} \frac{w\theta}{1+w} \frac{|z|}{r_t}\big(1+o(1)\big) \le \Psi_t\big(Z_t^{(1)}\big) + d_{r_t} \frac{w\theta}{1+w} \frac{|Z_t^{(1)}|}{r_t}\big(1+o(1)\big),$$

that is, to

$$Y_{t,z} + \frac{w\theta}{1+w} \frac{|z|}{r_t}\big(1+o(1)\big) \le Y_{t,Z_t^{(1)}} + \frac{w\theta}{1+w} \frac{|Z_t^{(1)}|}{r_t}\big(1+o(1)\big).$$

Consider the point process Π_t on $\hat{H}_{-n}^{-\alpha}$, where $\alpha \in (\theta \frac{w}{1+w}, \theta)$. The requirement

$$\big\{\big(Z_t^{(1)} r_t^{-1}, Y_{t,Z_t^{(1)}}\big) \in dx \times dy, \Psi_{t+wt}(z) \le \Psi_{t+wt}\big(Z_t^{(1)}\big)\ \forall z \in \mathbb{Z}^d \text{ s.t. } Y_{t,z} \ge -n\big\}$$

means that Π_t has one point in $dx \times dy$ and no points in the domain

$$\begin{aligned} D_{n,w,t}(x,y) = {} & \big(\mathbb{R}^d \times [y,\infty)\big) \\ & \cup \bigg\{(\bar{x},\bar{y}) \in \mathbb{R}^d \times [-n,\infty): \\ & \quad y + \frac{w\theta|x|}{1+w}\big(1+o(1)\big) \le \bar{y} + \frac{w\theta|\bar{x}|}{1+w}\big(1+o(1)\big)\bigg\}. \end{aligned}$$

Hence, by Lemma 3.1,

$$\begin{aligned} & \lim_{t\to\infty} \text{Prob}\{\mathcal{A}(n,w,t)\} \\ & \quad = \lim_{t\to\infty} \int_{\mathbb{R}^d \times [-n,\infty)} \text{Prob}\big\{\Pi_t(dx \times dy) = 1, \Pi_t\big(D_{n,w,t}(x,y)\big) = 0\big\} \\ & \quad = \int_{\mathbb{R}^d \times [-n,\infty)} \text{Prob}\big\{\Pi(dx \times dy) = 1, \Pi\big(D_{n,w}(x,y)\big) = 0\big\} \\ & \quad = \int_{\mathbb{R}^d \times [-n,\infty)} \exp\big\{-\nu\big(D_{n,w}(x,y)\big)\big\}\nu(dx,dy), \end{aligned}$$

where

$$D_{n,w}(x,y) = D_w(x,y) \cap \big(\mathbb{R}^d \times [-n,\infty)\big).$$

Taking the limit in this way is justified as $\hat{H}_{-n}^{-\alpha}$ is compact and contains $\mathbb{R}^d \times [-n,\infty)$.

It remains to show that

$$\begin{aligned} & \lim_{n\to\infty} \int_{\mathbb{R}^d \times [-n,\infty)} \exp\big\{-\nu\big(D_{n,w}(x,y)\big)\big\}\nu(dx,dy) \\ & \qquad = \int_{\mathbb{R}^d \times \mathbb{R}} \exp\big\{-\nu\big(D_w(x,y)\big)\big\}\nu(dx,dy) < \infty. \end{aligned} \tag{54}$$

Observe that $\nu(D_{n,w}(x,y)) \ge \nu(\mathbb{R}^d \times (y,\infty))$ for all $x \in \mathbb{R}^d$ and $y \ge -n$. Then

$$\mathbb{1}_{\mathbb{R}^d \times [-n,\infty)}(x,y) \exp\{-\nu(D_{n,w}(x,y))\} \le \exp\{-\nu(\mathbb{R}^d \times (y,\infty))\}.$$

It is easy to see that $\exp\{-\nu(\mathbb{R}^d \times (y,\infty))\}$ is integrable with respect to the measure ν on $\mathbb{R}^d \times \mathbb{R}$ since using (18) and the substitution $u = e^{-\gamma y}$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \mathbb{R}} \exp\{-\nu(\mathbb{R}^d \times (y,\infty))\} \nu(dx, dy) \\
&\quad = \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \gamma \exp\{-\gamma y - \gamma\theta|x| - 2^d(\gamma\theta)^{-d} e^{-\gamma y}\} \, dx \, dy \\
&\quad = 2^d(\gamma\theta)^{-d} \int_{-\infty}^{\infty} \gamma \exp\{-\gamma y - 2^d(\gamma\theta)^{-d} e^{-\gamma y}\} \, dy \\
&\quad = 2^d(\gamma\theta)^{-d} \int_0^{\infty} \exp\{-2^d(\gamma\theta)^{-d} u\} \, du = 1.
\end{aligned} \tag{55}$$

Now (54) follows from the dominated convergence theorem. □

Finally, we combine all results of this section to prove ageing.

PROOF OF THEOREM 1.3. For any $w > 0$, we have by Lemmas 6.1, 6.2 and 6.3,

$$\begin{aligned}
F(w) &:= \lim_{t\to\infty} \mathrm{Prob}\{T_t/t \le w\} = 1 - \lim_{t\to\infty} \mathrm{Prob}\{Z_t^{(1)} = Z_{t+wt}^{(1)}\} \\
&= 1 - \lim_{n\to\infty} \lim_{t\to\infty} \mathrm{Prob}\{\mathcal{A}(n,w,t)\} \\
&= 1 - \int_{\mathbb{R}^d \times \mathbb{R}} \exp\{-\nu(D_w(x,y))\} \nu(dx, dy).
\end{aligned}$$

Observe that $\exp\{-\nu(D_w(x,y))\} \le \exp\{-\nu(\mathbb{R}^d \times (y,\infty))\}$ which is integrable with respect to the measure ν by (55). Since $\nu(D_w(x,y)) \to \nu(D_{w_0}(x,y))$ whenever $w \to w_0 \in (0,\infty)$ the function F is continuous.

If $w \to 0+$ then $\nu(D_w(x,y) \to \nu(\mathbb{R}^d \times (y,\infty))$ and by (55) we obtain

$$\lim_{w\to 0+} F(w) = 1 - \int_{\mathbb{R}^d \times \mathbb{R}} \exp\{-\nu(\mathbb{R}^d \times (y,\infty))\} \nu(dx, dy) = 0.$$

Finally, if $w \to \infty$ then $\nu(D_w(x,y)) \to \nu(D_\infty(x,y))$, where

$$D_\infty(x,y) = \{(\bar{x},\bar{y}) \in \mathbb{R}^d \times \mathbb{R} : y + \theta|x| \le \bar{y} + \theta|\bar{x}|\} \cup (\mathbb{R}^d \times [y,\infty)).$$

Compute

$$\begin{aligned}
\nu(D_\infty(x,y)) &\ge \int_{|\bar{x}|>|x|} \int_{y+\theta|x|-\theta|\bar{x}|}^{\infty} \gamma \exp\{-\gamma\bar{y} - \gamma\theta|\bar{x}|\} \, d\bar{y} \, d\bar{x} \\
&= \exp\{-\gamma y - \gamma\theta|x|\} \int_{|\bar{x}|>|x|} d\bar{x} = \infty.
\end{aligned}$$

Hence, $F(w) \to 1$ as $w \to \infty$. □

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