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ROBUST DIMENSION FREE ISOPERIMETRY IN GAUSSIAN SPACE[1]

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We prove the first robust dimension free isoperimetric result for the standard Gaussian measure γ_n and the corresponding boundary measure γ_n^+ in $\mathbb{R}^n$. The main result in the theory of Gaussian isoperimetry (proven in the 1970s by Sudakov and Tsirelson, and independently by Borell) states that if $\gamma_n(A) = 1/2$ then the surface area of A is bounded by the surface area of a half-space with the same measure, $\gamma_n^+(A) \le (2\pi)^{-1/2}$. Our results imply in particular that if $A \subset \mathbb{R}^n$ satisfies $\gamma_n(A) = 1/2$ and $\gamma_n^+(A) \le (2\pi)^{-1/2} + \delta$ then there exists a half-space $B \subset \mathbb{R}^n$ such that $\gamma_n(A \Delta B) \le C \log^{-1/2}(1/\delta)$ for an absolute constant C. Since the Gaussian isoperimetric result was established, only recently a robust version of the Gaussian isoperimetric result was obtained by Cianchi et al., who showed that $\gamma_n(A \Delta B) \le C(n)\sqrt{\delta}$ for some function $C(n)$ with no effective bounds. Compared to the results of Cianchi et al., our results have optimal (i.e., no) dependence on the dimension, but worse dependence on δ.

1. Introduction. Gaussian isoperimetric theory is an extensive and rich theory. It connects numerous areas of mathematics including probability, geometry [21], concentration and high dimensional phenomena [19], rearrangement inequalities [6] and more. For an introduction to Gaussian isoperimetry and its many applications, see Ledoux's St.-Flour lecture notes [16].

The main result in this area is that half-spaces minimize the surface area among all sets with a given Gaussian measure. This fact, originally proven by Sudakov and Tsirelson [26] and independently by Borell [3], now has several other proofs: Ehrhard [9–11] developed a symmetrization technique, Bakry and Ledoux [1] and Ledoux [17] used semigroup methods and Bobkov [2] gave a proof based on an isoperimetric inequality on the discrete cube.

Some of the strongest results in this area deal with extensions of this basic theorem. For example, Borell [5] proved that half-spaces minimize a more global version of surface area called Gaussian noise sensitivity. This fact has recently found applications in areas such as quantitative social choice and theoretical computer science; see, for example, [14, 22, 24, 25].

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A second direction of extension is characterizing the case of equality or almost equality. It took about a decade until the equality case was addressed. Erhard [11] showed that if a sufficiently nice set achieves equality then it is a half-space. It took 15 more years until the same result was proven for general measurable sets by Carlen and Kerce [6].

Prior to our work, the harder question of almost equality was only recently addressed by Cianchi et al. [7] who showed that if the Gaussian boundary of set A is within δ of the optimal value then there exists a half space B such that the Gaussian measure of the symmetric difference between A and B is at most $C(n)\sqrt{\delta}$. Their result gives no bound on the function $C(n)$. Indeed the techniques of [7] are not appropriate for deriving any bound on $C(n)$.

Our goal in this paper is to establish a robustness result that is *dimension independent*. Not only such result is more elegant, it is much in the spirit of Gaussian isoperimetric theory, where the statement of most results are dimension independent. In particular, the results of Sudakov and Tsirelson [26] and Borell [3] are dimension independent: the bound they give on the size of the boundary in terms of the measure of the set are dimension independent. Similarly, the results of Borell [5] are stated in a dimension-free way.

1.1. *Gaussian isoperimetry.* The Gaussian isoperimetric inequality was first proved by Sudakov and Tsirelson [26], and independently by Borell [3]. It states that in $\mathbb{R}^n$ with the standard Gaussian measure, the isoperimetric sets are half-spaces. To be more precise, let $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ be the standard Gaussian density, and define $\Phi(x) = \int_{-\infty}^{x} \phi(y)\,dy$. Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$, and define the boundary measure γ_n^+ by

$$\gamma_n^+(A) = \sup\left\{\int_A (\nabla - x)\cdot v(x)\,d\gamma_n : v \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^n), \left|v(x)\right| \le 1 \text{ for all } x\right\},$$

where $\mathcal{S}(\mathbb{R}^n, \mathbb{R}^n)$ is the set of smooth functions $\mathbb{R}^n \to \mathbb{R}^n$ such that all derivatives vanish at infinity. (This definition of boundary measure coincides with Minkowski content and the $(n-1)$-dimensional Gaussian-weighted Hausdorff measure for sufficiently nice sets [6].) Then the Gaussian isoperimetric inequality states that for every measurable A, $\phi(\Phi^{-1}(\gamma_n(A))) \le \gamma_n^+(A)$. It is not hard to verify that equality is attained if A is an affine half-space (i.e., a set of the form $\{x \in \mathbb{R}^n : x \cdot a \ge b\}$). In the theory, it is common to define the isoperimetric profile $I = \phi \circ \Phi^{-1}$, so that the Gaussian isoperimetric inequality reads

$$I\bigl(\gamma_n(A)\bigr) \le \gamma_n^+(A). \tag{1}$$

1.2. *Bobkov's inequality.* Bobkov's inequality [2] is a functional generalization of the Gaussian isoperimetric inequality. The equality case was proved by Carlen and Kerce [6]. Here and for the rest of this article, we will write "$\mathbb{E}$" for the integral with respect to γ_n and $\|\cdot\|$ for the Euclidean norm on $\mathbb{R}^n$.

THEOREM 1.1 ([2, 6]). *For any smooth function* $f : \mathbb{R}^n \to [0, 1]$ *of bounded variation*,

$$I(\mathbb{E}f) \le \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2}. \tag{2}$$

Equality is attained only if $f(x) = \Phi(a \cdot x + b)$ *for some* $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.

Using standard approximation techniques [6], one can make sense of Theorem 1.1 for functions f that are not smooth. In particular, it is possible to take f to be the indicator function of a set A; in that case, $I(f)$ is identically zero and so (2) becomes

$$I\bigl(\gamma_n(A)\bigr) \le \mathbb{E}\|\nabla 1_A\| = \gamma_n^+(A)$$

whenever A is nice enough. This is just the Gaussian isoperimetric inequality again. (We will make the above connection rigorous in Section 5.) In this limiting case, the nonsmooth equality cases $1_{\{a \cdot x + b \ge 0\}}$ appear. These are easily seen to be limits of the equality cases $\Phi(a \cdot x + b)$ in Theorem 1.1.

1.3. *Robustness*. Our goal in this article is to study the robustness of Theorem 1.1: suppose that we have a function f which almost achieves equality. Must there be some a and b for which f is close to a function of the form $x \mapsto \Phi(a \cdot x + b)$? For the case of sets—which is perhaps the most interesting case—this question was previously studied by Cianchi et al. [7], who gave a dimension-dependent estimate.

THEOREM 1.2 ([7]). *If* $A \subset \mathbb{R}^n$ *satisfies* $I(\gamma_n(A)) \ge \gamma_n^+(A) - \delta$, *then there is a half-space* B *such that*

$$\gamma_n(A \Delta B) \le C\bigl(n, \gamma_n(A)\bigr)\sqrt{\delta},$$

where $C(n, r)$ *is some function of* n *and* r.

Due to use of compactness arguments, there are no effective bounds on the function $C(n, r)$.

Theorem 1.2 is sharp in its δ-dependence; however, the n-dependence is certainly not sharp. Indeed, one often finds things in Gaussian space to be independent of the dimension. The isoperimetric inequality itself is an example of this phenomenon, as the Gaussian isoperimetric profile I does not depend on the dimension.

Note that the situation is quite different in Euclidean space with the Lebesgue measure—for which the techniques used in [7] were originally developed—where the isoperimetric profile $x \mapsto n\omega_n^{1/n} x^{(n-1)/n}$ does depend on n.

1.4. *Our results*: *Robust and dimension-free*. Dimension-free estimates are satisfying in themselves, but they are also crucial for certain applications. As an example, consider Borell's noise stability inequality [5]: take $X, Y \in \mathbb{R}^n$ jointly Gaussian with $X, Y \sim \mathcal{N}(0, I)$ and $\mathbb{E}X_i Y_j = \rho\delta_{ij}$. Then $\Pr(1_A(X) \neq 1_A(Y))$ is minimized, over all sets A with prescribed Gaussian volume, by affine half-spaces. Ledoux showed [15] that this generalizes the Gaussian isoperimetric inequality, which is recovered in the limit as $\rho \to 1$. As mentioned above, for applications of this result it is crucial that is dimension-free.

A robust dimension-free version of Borell's result would immediately imply a number of important results. For example, it would show that if a balanced low influence Boolean function is almost as stable as the majority function, then the function is close to a weighted majority. Similarly, it would show that if a balanced low influence function has a Condorcet paradox probability that is almost as small as that of a majority then it must be close to a weight majority of a subset of the coordinates. (Both of the statements above follow from the arguments of [24].)

The potential applications above further motivate our main result which is a dimension-independent stability result for Bobkov's inequality in Gaussian space. We note, however, that our dependence on δ is much worse than the one in Theorem 1.2; improving this dependence is therefore a natural open problem.

Our main functional result is the following.

THEOREM 1.3. *There exists a universal constant C such that the following holds. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and define*

$$\delta = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).$$

There exists a function g of the form $g(x) = \Phi(a \cdot x + b)$ such that

$$\mathbb{E}(f - g)^2 \le C \frac{1}{\sqrt{\log(1/\delta)}}.$$

Of course, the most interesting special case of Theorem 1.3 is when f is the indicator function of some set. Such an f is not smooth, of course, but the same arguments that reduced Theorem 1.1 to the Gaussian isoperimetric inequation can be employed here. Thus, we obtain a robustness result for the Gaussian isoperimetric inequality.

THEOREM 1.4. *There exists an absolute constant C such that the following holds. For any measurable set $A \subset \mathbb{R}^n$, let $\delta = \gamma_n^+(A) - I(\gamma_n(A))$. There exists an affine half-space B such that*

$$\gamma_n(A \Delta B) \le C \frac{1}{\sqrt{\log(1/\delta)}}.$$

1.5. *Proof techniques*. Our approach builds on the work of Carlen and Kerce [6] (which extends ideas of Ledoux [15]). Carlen and Kerce [6] write an integral formula [equation (4) below] which bounds $\delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f)$ from below.

The "main term" in the integral is the Frobenius norm of the Hessian of $h_t = \Phi^{-1} \circ (P_t f)$, where P_t is Ornstein–Uhlenbeck semigroup. It is easy to verify that if f is an indicator of a half-space or if $f = \Phi(a \cdot x + b)$ then h_t is linear. Our first step in the proof is to utilize a second-order Poincaré inequality which implies that if the Frobenius norm of the Hessian of h_t is sufficiently small, then h_t is close to a linear function.

The main effort in our approach is devoted to controlling the "secondary terms" in (4). This main effort is established in a sequence of analytic results using the smoothness of the semigroup P_t and involving—among other techniques—concentration of measure and reverse hypercontractivity. Using the approach above, we show that if $\delta = \delta(f)$ is small then there exists some t, not too large, such that h_t is $\varepsilon(\delta)$-close to a linear function.

The next step of the proof requires applying $P_t^{-1} \circ \Phi$ to conclude that f is close to a linear function. There is an obvious obstacle in this approach: P_t^{-1} is not a bounded operator.

Fortunately, using the smoothness of the original function f, or the fact that we may assume that the original set A has small boundary, we may deduce a decay in the Hermite expansion of f (or A). Thus, we show that for the functions under consideration, P_t^{-1} is "effectively" bounded which allows us to conclude that f is close to a Gaussian (or A is close to a half space), proving the result.

2. Semigroup proof of Bobkov's inequality. Our work begins with Ledoux's short and elementary proof [15] of (2). The main ingredient of this proof is the Ornstein–Uhlenbeck semigroup: for $t \geq 0$, define the operator $P_t : L_\infty(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)$ by

$$(P_t f)(x) = \int_{\mathbb{R}^n} f\big(e^{-t}x + \sqrt{1 - e^{-2t}}\,y\big)\, d\gamma_n(y).$$

Clearly, P_0 is the identity operator and $P_t f$ converges pointwise to $\mathbb{E}f$ as $t \to \infty$. Consider, therefore, the quantity

$$\mathbb{E}\sqrt{I^2(P_t f) + \|\nabla P_t f\|^2}. \tag{3}$$

When $t = 0$, this is exactly the right-hand side of (2); as $t \to \infty$, it approaches the left-hand side of (2) by the dominated convergence theorem and the boundedness of f. To prove (2), Ledoux differentiated (3) with respect to t and showed that the derivative is nonpositive. Thus, a potentially difficult inequality turns into a calculus problem.

Actually, Ledoux only explicitly differentiated (3) in the one-dimensional case. The n-dimensional case of (3) was computed by Carlen and Kerce [6] in their work on the equality case. Our robustness result is based on their calculations, which we will summarize as a lemma.

LEMMA 2.1 ([6]). *Let f be a smooth function $f:\mathbb{R}^n \to [0,1]$. Define $h_t = \Phi^{-1} \circ (P_t f)$ and*

$$\delta(f) = \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} - I(\mathbb{E}f).$$

Then

$$\delta(f) \ge \int_0^\infty \mathbb{E}\frac{\phi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}}\,dt, \tag{4}$$

where $H(h_t)$ is the Hessian matrix of h_t and $\|\cdot\|_F$ denotes the Frobenius norm $\|A\|_F^2 = \operatorname{tr}(A^T A)$.

From now on, $\delta(f)$ will be defined as it was in Lemma 2.1. Where f is clear from the context, we will only write δ.

The equality case in Theorem 1.1 follows fairly easily from Lemma 2.1: if $\delta = 0$, then $H(h_t)$ must be zero for all $t > 0$, which implies that h_t is a linear function for all $t > 0$. A straightforward limiting argument shows that one can take t to zero, and the result follows.

Our proof of Theorem 1.3 works by finding a lower bound for the right-hand side of (4). First, we replace the integral over $[0,\infty)$ by an integral over $[C, C+1]$, where C is a large enough constant. For some $t \in [C, C+1]$, we find an affine function h^* such that h_t is close to h^*. In particular, this means that f_t is close to $\Phi \circ h^*$. This part of the argument will be carried out in Section 3. The second part of the argument, carried out in Section 4, shows that f must be close to $P_{-t}(\Phi \circ h^*)$.

3. Approximation for large t. This section is devoted to the proof of Proposition 3.1, which shows that h_t can be approximated by an affine function for some sufficiently large t.

PROPOSITION 3.1. *There is a universal constant $C > 0$ such that for any measurable $f:\mathbb{R}^n \to [0,1]$ there exists $t \in [C, C+1]$ such that*

$$\mathbb{E}\big(h_t(X) - \mathbb{E}h_t - X \cdot \mathbb{E}\nabla h_t\big)^2 \le C\frac{\delta^{1/4}(f)}{m(f)^{5/4}},$$

where $h_t = \Phi^{-1} \circ (P_t f)$ and $m(f) = (\mathbb{E}f)(1-\mathbb{E}f)$.

NOTE 3.2. In this section and the next, we will not be concerned with the value of universal constants; hence, the letters C and c will denote universal constants, whose values may change from line to line. We will use C to denote constants that must be sufficiently large, while c denotes constants that must be sufficiently small.

Notation. From now on, we fix the notation

$$f_t := P_t f, \qquad h_t := \Phi^{-1} \circ f_t, \qquad k_t = e^{-t}/\sqrt{1-e^{-2t}}.$$

3.1. *A second-order Poincaré inequality.* In proving the equality cases of Bobkov's inequality, Carlen and Kerce used the fact that if $\|H(h_t)\|_F^2$ vanishes then h_t must be a linear function. The first step toward Proposition 3.1 is a quantitative version of this observation.

LEMMA 3.3. *For any smooth function $h:\mathbb{R}^n \to \mathbb{R}$,*

$$\mathbb{E}\big(h(X) - \mathbb{E}h - X \cdot \mathbb{E}\nabla h\big)^2 \le \mathbb{E}\big\|H(h)\big\|_F^2.$$

Since $x \mapsto \mathbb{E}h + x \cdot \mathbb{E}\nabla h$ is a linear function, Lemma 3.3 implies that if $\mathbb{E}\|H(h)\|_F^2$ is small then h is close to linear. This puts us on our way toward the proof of Proposition 3.1. Indeed, if we could remove the $\phi(h_t)(1+|\nabla h_t|^2)^{-3/2}$ term from the right-hand side of (4), we would be done already. The removal of this nuisance term is the topic of the next section.

PROOF OF LEMMA 3.3. Recall Poincaré's inequality [18]

$$\mathbb{E}h^2 - (\mathbb{E}h)^2 \le \mathbb{E}\|\nabla h\|^2. \tag{5}$$

If we apply (5) to the partial derivatives of h, we obtain

$$\mathbb{E}\left(\frac{\partial h}{\partial x_i}\right)^2 - \left(\mathbb{E}\frac{\partial h}{\partial x_i}\right)^2 \le \mathbb{E}\left(\frac{\partial h}{\partial x_i}\right)^2 \le \sum_{j=1}^n \mathbb{E}\left(\frac{\partial^2 h}{\partial x_i\,\partial x_j}\right)^2.$$

Summing over i yields $\mathbb{E}\|\nabla h\|^2 - \|\mathbb{E}\nabla h\|^2 \le \mathbb{E}\|H(h)\|_F^2$. Combining this with (5), we have

$$\mathbb{E}(h - \mathbb{E}h - X\cdot\mathbb{E}\nabla h)^2 = \mathbb{E}h^2 - (\mathbb{E}h)^2 - \|\mathbb{E}\nabla h\|^2 \le \mathbb{E}\big\|H(h)\big\|_F^2,$$

where the first equality follows because integration by parts implies that $\mathbb{E}Xh(X) = \mathbb{E}\nabla h$; hence, the orthogonal projection of h onto the span of linear functions is $X \cdot \mathbb{E}\nabla h$. $\square$

3.2. *First derivative bounds*. We have shown how to use the $\mathbb{E}\|H(h_t)\|_F^2$ term on the right-hand side of (4). In this section, we discuss the $(1+\|\nabla h_t\|^2)^{-3/2}$ term. A result by Bakry and Ledoux [1] shows that this term may be bounded pointwise from below.

THEOREM 3.4 ([1]). *For any measurable function $f:\mathbb{R}^n \to [0,1]$ and any $t > 0$,*

$$\|\nabla f_t\| \le k_t I(f_t)$$

pointwise, where $k_t = e^{-t}/\sqrt{1-e^{-2t}}$ *and* $f_t = P_t f$. *Equivalently,*

$$\|\nabla h_t\| \le k_t$$

pointwise, where $h_t = \Phi^{-1} \circ f_t$.

Note that the second inequality is equivalent to the first by the chain rule, since $\frac{d}{dx}\Phi^{-1}(x) = \frac{1}{I(x)}$. Note also that since $I(x) \sim x\sqrt{2\log(1/x)}$ as $x \to 0$, Theorem 3.4 follows, up to a constant factor, from the reverse log-Sobolev inequality [18]

$$\frac{\|\nabla f_t\|^2}{2k_t^2 f_t} \le P_t(f \log f) - f_t \log f_t. \tag{6}$$

However, the sharp constant in Theorem 3.4 will be useful in Section 4.

3.3. *Reverse-hypercontractivity and reverse-Hölder.* Recall our current task: a lower bound on (4) for large t. We have already shown that $\|\nabla h_t\|$ must be small for large t; our goal for this section is to find a lower bound on $\mathbb{E}\phi(h_t)\|H(h_t)\|_F^2$.

PROPOSITION 3.5. *There exists a constant C such that if $t \ge C$ then*

$$\mathbb{E}\bigl(\phi(h_t)\|H(h_t)\|_F^2\bigr) \ge \tfrac{1}{4}\bigl(I(\mathbb{E}f)\bigr)^2\bigl(\mathbb{E}\|H(h_t)\|_F\bigr)^2.$$

For this, we will use two inequalities: Borell's reverse-hypercontractive inequality and the reverse-Hölder inequality. The reverse-Hölder inequality is classical: for any $p < 1$ and any positive functions f and g,

$$\mathbb{E}fg \ge \bigl(\mathbb{E}f^p\bigr)^{1/p}\bigl(\mathbb{E}g^{p/(p-1)}\bigr)^{(p-1)/p}. \tag{7}$$

The reverse-hypercontractive inequality was proved by Borell [4]: for a positive function f and any $p < 1$, $t > 0$,

$$\bigl(\mathbb{E}(P_t f)^p\bigr)^{1/p} \ge \bigl(\mathbb{E}f^q\bigr)^{1/q}, \tag{8}$$

where $q = 1 + e^{-2t}(p-1)$.

PROOF OF PROPOSITION 3.5. Let $g_t = \|H(h_t)\|_F^2$ and apply (7) with $p = \frac{1}{2}$:

$$\mathbb{E}\bigl(\phi(h_t)\|H(h_t)\|_F^2\bigr) = \mathbb{E}\bigl(I(f_t)g_t\bigr) \ge \bigl(\mathbb{E}\bigl(I(f_t)\bigr)^{-1}\bigr)^{-1}(\mathbb{E}\sqrt{g_t})^2. \tag{9}$$

Now, I is a concave function, and so $I(P_t f) \ge P_{t/2} I(P_{t/2} f)$. Applying (8) with $p = -1$ gives

$$\bigl(\mathbb{E}\bigl(I(f_t)\bigr)^{-1}\bigr)^{-1} \ge \bigl(\mathbb{E}\bigl(P_{t/2}I(f_{t/2})\bigr)^{-1}\bigr)^{-1} \ge \bigl(\mathbb{E}\bigl(I(f_{t/2})\bigr)^q\bigr)^{1/q}, \tag{10}$$

where $q = 1 - 2e^{-t}$. If $t \ge 2$, then $q \ge 1/2$. Hence, we can combine (9) with (10) to obtain

$$\mathbb{E}\bigl(I(f_t)g_t\bigr) \ge \bigl(\mathbb{E}\sqrt{I(f_{t/2})}\bigr)^2(\mathbb{E}\sqrt{g_t})^2.$$

It remains to show that

$$\mathbb{E}\sqrt{I(f_{t/2})} \geq \tfrac{1}{2} I(\mathbb{E}f). \tag{11}$$

Applying (2) to $f_{t/2}$, we have

$$I(\mathbb{E}f) = I(\mathbb{E}f_{t/2}) \leq \mathbb{E}I(f_{t/2}) + \mathbb{E}\|\nabla f_{t/2}\|, \tag{12}$$

while Theorem 3.4 gives

$$\mathbb{E}\|\nabla f_{t/2}\| \leq k_{t/2}\mathbb{E}I(f_{t/2}).$$

For t sufficiently large, $k_{t/2} \leq 1$ and so $\mathbb{E}\|\nabla f_{t/2}\| \leq \mathbb{E}I(f_{t/2})$; by (12), $I(\mathbb{E}f) \leq 2\mathbb{E}I(f_{t/2})$ for large enough t. Now, I is bounded above by $(2\pi)^{-1/2} \leq 1$, and so $\mathbb{E}\sqrt{I(f_{t/2})} \geq \mathbb{E}I(f_{t/2}) \geq \frac{1}{2}I(\mathbb{E}f)$, which proves (11) and the proposition. □

3.4. *Second-derivative estimates*. There is one more ingredient in the proof of Proposition 3.1: an upper bound on second derivatives of h_t. To see why such a bound is useful, note that Lemma 3.3 gives a lower bound on $\mathbb{E}\|H(h_t)\|_F^2$, but Proposition 3.5 contains $\mathbb{E}\|H(h_t)\|_F$. To combine these two results, we must therefore bound the first moment of $\|H(h_t)\|_F$ from below in terms of the second moment. This can be done by Hölder's inequality, as long as we can bound higher moments of $\|H(h_t)\|_F$ from above. Such a bound is the goal of this section.

The main bound of this section is the following proposition.

PROPOSITION 3.6. *There is a constant C such that for all $t > C$,*

$$\left(\mathbb{E}\|H(h_t)\|_F^3\right)^{1/3} \leq \sqrt{\log\frac{1}{m(f)}},$$

where $m(f) = \mathbb{E}f(1 - \mathbb{E}f)$.

Proposition 3.6 essentially follows by integrating a pointwise bound on $\|H(h_t(x))\|_F$:

LEMMA 3.7. *There is a constant C such that for all $x \in \mathbb{R}^n$ and $t > 0$,*

$$\|H(h_t(x))\|_F \leq Ck_t^2\sqrt{\log\frac{1}{f_t(x)(1 - f_t(x))}},$$

where $k_t = e^{-t}/\sqrt{1 - e^{-2t}}$.

And we will also need to relate the median of f_t with its mean.

LEMMA 3.8. *If M_t is a median of f_t, then*

$$\mathbb{E}f(1 - \mathbb{E}f) \leq 2M_t^{(1/(1+k_t))^2}.$$

Before we prove either of these lemmas, we will show how they imply Proposition 3.6.

PROOF OF PROPOSITION 3.6. Let $g_t = \sqrt{\log(1/f_t)}$ (where $f_t = P_t f$); note that $\nabla g_t = -\frac{\nabla f_t}{2 f_t \sqrt{\log(1/f_t)}}$, and so the reverse log-Sobolev inequality (6) implies that g_t is Lipschitz with constant $k_t/\sqrt{2}$. Take N to be a median of g_t. By Gaussian concentration for Lipschitz functions,

$$\mathbb{E}|g_t - N|^p \le \int_0^\infty \gamma_n\{|g_t - N|^p \ge x\}\, dx \le \int_0^\infty e^{-x^{2/p}/(2k_t^2)}\, dx.$$

After the change of variables $y^2 = \frac{x^{2/p}}{k_t^2}$, the right-hand side is just $pk_t^p \mathbb{E}|Y|^{p-1}$, where Y is a standard Gaussian variable. Since $\mathbb{E}|Y|^{p-1} \le (Cp)^{p/2}$, it follows that

$$\left(\mathbb{E}|g_t - N|^p\right)^{1/p} \le \left((Cp)^{p/2+1} k_t^p\right)^{1/p} \le Ck_t\sqrt{p}.$$

Then, by the triangle inequality,

$$\left(\mathbb{E}|g_t|^p\right)^{1/p} \le N + Ck_t\sqrt{p}. \tag{13}$$

By Lemma 3.7, $\|H(h_t)\|_F \le Ck_t^2 g_t$ pointwise whenever $f_t \le \frac{1}{2}$. Thus,

$$\left(\mathbb{E}\|H(h_t)\|_F^p 1_{\{f_t \le 1/2\}}\right)^{1/p} \le Ck_t^2\left(\mathbb{E}|g_t|^p\right)^{1/p} \le Ck_t^2(N + k_t\sqrt{p}), \tag{14}$$

where the second inequality follows from (13).

To relate N to $\mathbb{E}f$, simply note that $M = e^{-N^2}$ is a median for f_t; hence, Lemma 3.8 implies that $N \le C(1 + k_t)\sqrt{\log(1/m(f))}$, where $m(f) = \mathbb{E}f(1 - \mathbb{E}f)$. Plugging these bounds into (14),

$$\left(\mathbb{E}\|H(h_t)\|_F^p 1_{\{f_t \le 1/2\}}\right)^{1/p} \le Ck_t^2\left((1 + k_t)\sqrt{\log\frac{1}{m(f)}} + k_t\sqrt{p}\right). \tag{15}$$

By the same argument with f_t and $1 - f_t$ exchanged, we have

$$\left(\mathbb{E}\|H(h_t)\|_F^p 1_{\{f_t \ge 1/2\}}\right)^{1/p} \le Ck_t^2\left((1 + k_t)\sqrt{\log\frac{1}{m(f)}} + k_t\sqrt{p}\right). \tag{16}$$

Combining (15) and (16) with the triangle inequality

$$\begin{aligned}\left(\mathbb{E}\|H(h_t)\|_F^p\right)^{1/p} &\le \left(\mathbb{E}\|H(h_t)\|_F^p 1_{\{f_t \ge 1/2\}}\right)^{1/p} + \left(\mathbb{E}\|H(h_t)\|_F^p 1_{\{f_t \le 1/2\}}\right)^{1/p} \\ &\le Ck_t^2\left((1 + k_t)\sqrt{\log\frac{1}{m(f)}} + k_t\sqrt{p}\right).\end{aligned}$$

Setting $p = 3$ and taking t sufficiently large completes the proof. □

The fact that $\sqrt{\log(1/f_t)}$ is Lipschitz was noticed by Hino [13], and was also used recently by Ledoux [20]. This fact, which was important in the preceding proof, will also be crucial in the proof of Lemma 3.8:

PROOF OF LEMMA 3.8. Let $g_t = \sqrt{\log(1/f_t)}$; take N to be a median of g_t and let $M = e^{-N^2}$, so that M is a median of f_t. For any $\alpha < 1$,

$$\Pr(f_t \geq M^{\alpha^2}) = \Pr\bigl(g_t \leq \alpha\sqrt{\log(1/M)}\bigr) = \Pr(g_t \leq \alpha N).$$

Recall that g_t is $\frac{1}{\sqrt{2}}k_t$-Lipschitz. Thus,

$$\Pr(f_t \geq M^{\alpha^2}) = \Pr(g_t \leq \alpha N) \leq \exp\left(-\frac{(1-\alpha)^2 N^2}{k_t^2}\right) = M^{(1-\alpha)^2/k_t^2}.$$

Setting $\alpha = \frac{1}{1+k_t}$, we have $\frac{(1-\alpha)^2}{k_t^2} = \alpha^2$. Thus, $\Pr(f_t \geq M^{\alpha^2}) \leq M^{\alpha^2}$. Since $f_t \leq 1$, Markov's inequality implies that $\mathbb{E} f_t \leq 2M^{\alpha^2}$. $\square$

For the rest of the section, we will devote ourselves to proving Lemma 3.7, which we will do very explicitly. The proof of Lemma 3.7 begins with the formula

$$\frac{\partial^2 h_t}{\partial x_i\, \partial x_j} = \frac{1}{I(f_t(x))}\frac{\partial^2 f_t}{\partial x_i\, \partial x_j} + \frac{\Phi^{-1}(f_t(x))}{I^2(f_t(x))}\frac{\partial f_t}{\partial x_i}\frac{\partial f_t}{\partial x_j}. \tag{17}$$

We will bound the two terms on the right-hand side in two different lemmas. But first, we quote a result on the moments of a order-2 Gaussian chaos. To obtain Theorem 3.9 from the result stated in [12], simply note that the operator norm is bounded by the Frobenius norm.

THEOREM 3.9 ([12]). *For any matrix A and any $1 \leq p < \infty$, if Y is a standard Gaussian vector in $\mathbb{R}^n$ then*

$$\bigl(\mathbb{E}|Y^T AY - \operatorname{tr}(A)|^p\bigr)^{1/p} \leq Cp\|A\|_F.$$

Theorem 3.9 will be used to bound the first term of (17).

LEMMA 3.10. *For any matrix $A = (a_{ij})$,*

$$\sum_{ij} a_{ij}\frac{\partial^2 f_t}{\partial x_i\, \partial x_j} \leq Ck_t^2\|A\|_F f_t \log\frac{1}{f_t}.$$

PROOF. We write out derivatives of f_t as integrals: if $i \neq j$ then with the change of variables $y = \frac{z-e^{-t}x}{\sqrt{1-e^{-2t}}}$,

$$\begin{aligned}
\frac{\partial^2 f_t}{\partial x_i\, \partial x_j} &= \int_{\mathbb{R}^n} f(z)\frac{\partial^2}{\partial x_i\, \partial x_j}\phi\left(\frac{z-e^{-t}x}{\sqrt{1-e^{-2t}}}\right)\frac{dz}{\sqrt{1-e^{-2t}}} \\
&= k_t^2\int_{\mathbb{R}^n} f(z) y_i y_j \phi\left(\frac{z-e^{-t}x}{\sqrt{1-e^{-2t}}}\right)\frac{dz}{\sqrt{1-e^{-2t}}} \\
&= k_t^2\int_{\mathbb{R}^n} f\bigl(e^{-t}x + \sqrt{1-e^{-2t}}\,y\bigr) y_i y_j \phi(y)\, dy,
\end{aligned}$$

while if $i = j$ then a similar computation gives

$$\frac{\partial^2 f_t}{\partial x_i \, \partial x_j} = k_t^2 \int_{\mathbb{R}^n} f\left(e^{-t} x + \sqrt{1 - e^{-2t}} y\right)\left(y_i^2 - 1\right)\phi(y)\, dy.$$

Applying Hölder's inequality,

$$\begin{aligned}\sum_{ij} a_{ij} \frac{\partial^2 f_t}{\partial x_i \, \partial x_j} &= k_t^2 \int_{\mathbb{R}^n} f\left(e^{-t} x + \sqrt{1 - e^{-2t}} y\right)\left(y^T A y - \operatorname{tr}(A)\right)\phi(y)\, dy \\ &\le k_t^2 \left(P_t f^p\right)^{1/p} \left(\mathbb{E}\left|Y^T A Y - \operatorname{tr}(A)\right|^q\right)^{1/q} \\ &\le k_t^2 f_t^{1/p} \left(\mathbb{E}\left|Y^T A Y - \operatorname{tr}(A)\right|^q\right)^{1/q},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and Y is distributed as a standard Gaussian variable, and the last line follows because $0 \le f \le 1$ and so $f^p \le f$. By Theorem 3.9,

$$\sum_{ij} a_{ij} \frac{\partial^2 f_t}{\partial x_i \, \partial x_j} \le C k_t^2 q f_t^{1/p} \|A\|_F.$$

Choosing $1/p = 1 - \frac{1}{\log(1/f_t)}$, we have $f_t^{1/p} = e f_t$ and $q = \log \frac{1}{f_t}$, proving the claim. $\square$

Putting Lemma 3.10 and Theorem 3.4 together, we arrive at a proof of Lemma 3.7.

PROOF OF LEMMA 3.7. Note that

$$\|H(h_t)\|_F = \sup_{\|A\|_F = 1} \sum_{ij} a_{ij} \frac{\partial^2 h_t}{\partial x_i \, \partial x_j}.$$

For any fixed matrix A, with $\|A\|_F = 1$, (17) implies that

$$\sum_{ij} a_{ij} \frac{\partial^2 h_t}{\partial x_i \, \partial x_j} = \frac{1}{I(f_t)} \sum_{ij} a_{ij} \frac{\partial^2 f_t}{\partial x_i \, \partial x_j} + \frac{\Phi^{-1}(f_t)}{I^2(f_t)} (\nabla f_t)^T A \nabla f_t.$$

Now, the reverse log-Sobolev inequality (6) implies that $(\nabla f_t)^T A \nabla f_t \le \|A\|_F \times \|\nabla f_t\|^2 \le 2k_t^2 \|A\|_F f_t^2 \log \frac{1}{f_t}$. This, together with Lemma 3.10 and the fact that $\|A\|_F = 1$, implies

$$\sum_{ij} a_{ij} \frac{\partial^2 h_t}{\partial x_i \, \partial x_j} \le C k_t^2 \left(\frac{f_t \log(1/f_t)}{I(f_t)} + \frac{\Phi^{-1}(f_t)}{I^2(f_t)} f_t^2 \log \frac{1}{f_t} \right).$$

Since the right-hand side is independent of A, we can take the supremum over $\{A : \|A\|_F = 1\}$, giving

$$\|H(h_t)\|_F \le C k_t^2 \left(\frac{f_t \log(1/f_t)}{I(f_t)} + \frac{\Phi^{-1}(f_t)}{I^2(f_t)} f_t^2 \log \frac{1}{f_t} \right). \tag{18}$$

Next, we claim that for any $0 < a \le \frac{1}{2}$,

$$a\sqrt{\log(1/a)} \le CI(a), \tag{19}$$

$$a^2\sqrt{\log(1/a)} \le C\left|\frac{I^2(a)}{\Phi^{-1}(a)}\right|. \tag{20}$$

Now, (19) follows from the well-known fact (see, e.g., [1]) that $I(a) \asymp a\sqrt{\log(1/a)}$ as $a \to 0$ [where the notation $f(a) \asymp g(a)$ means that $0 < \liminf \frac{f(a)}{g(a)} \le \limsup \frac{f(a)}{g(a)} < \infty$]. To show (20), set $g(a) = -\frac{I^2(a)}{\Phi^{-1}(a)}$; we then compute

$$g'(a) = 2I(a) + \frac{I(a)}{(\Phi^{-1})^2(a)} \asymp I(a) \asymp a\sqrt{\log\frac{1}{a}}$$

as $a \to 0$. Since

$$\frac{d}{da}a^2\sqrt{\log\frac{1}{a}} = a\sqrt{\log\frac{1}{a}} - \frac{a}{2\sqrt{\log(1/a)}} \asymp a\sqrt{\log\frac{1}{a}},$$

it follows that $g(a) \asymp a^2\sqrt{\log(1/a)}$, proving (20).

Suppose that $f_t \le \frac{1}{2}$. Applying (19) and (20) to (18) with $a = f_t$ completes the proof in this case. If $f_t > \frac{1}{2}$, we apply the same argument to $1 - f_t$. $\square$

3.5. *Proof of Proposition* 3.1. With all of the ingredients laid out, the proof of Proposition 3.1 follows easily.

PROOF OF PROPOSITION 3.1. Suppose C is a large enough universal constant so that for all $t \ge C$:

- $\|\nabla h_t\| \le 1$ (by Theorem 3.4).
- $\mathbb{E}(\phi(h_t)\|H(h_t)\|_F^2) \ge \frac{1}{4}I^2(\mathbb{E}f)(\mathbb{E}\|H(h_t)\|_F)^2$ (by Proposition 3.5).
- $(\mathbb{E}\|H(h_t)\|_F^3)^{1/3} \le \sqrt{\log(1/(m(f)))}$ (by Proposition 3.6), where $m(f) = \mathbb{E}f(1-\mathbb{E}f)$.

By the first two bullet points, for any $t \ge C$

$$\begin{aligned}\mathbb{E}\frac{\phi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} &\ge 2^{-3/2}\mathbb{E}\big(\phi(h_t)\|H(h_t)\|_F^2\big)\\ &\ge 2^{-7/2}I^2(\mathbb{E}f)\big(\mathbb{E}\|H(h_t)\|_F\big)^2.\end{aligned} \tag{21}$$

Now, Hölder's inequality implies that

$$\mathbb{E}X^2 = \mathbb{E}X^{1/2}X^{3/2} \le (\mathbb{E}X)^{1/2}\big(\mathbb{E}X^3\big)^{1/2}$$

for any nonnegative random variable X. Applying this with $X = \|H(h_t)\|_F$, the third bullet point above implies that

$$\begin{aligned}\mathbb{E}\big(\|H(h_t)\|_F^2\big) &\le \big(\mathbb{E}\|H(h_t)\|_F\big)^{1/2}\big(\mathbb{E}\|H(h_t)\|_F^3\big)^{1/2}\\ &\le \big(\mathbb{E}\|H(h_t)\|_F\big)^{1/2}\log^{3/4}\frac{1}{m(f)}.\end{aligned}$$

Plugging this into (21),

$$\mathbb{E}\frac{\phi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}} \ge 2^{-7/2} I^2(\mathbb{E}f)\big(\mathbb{E}\big(\|H(h_t)\|_F^2\big)\big)^4 \log^{-3}\frac{1}{m(f)}. \tag{22}$$

Now, (22) holds for any $t \ge C$. In particular, if we choose $t^* \in [C, C+1]$ to minimize $\mathbb{E}\|H(h_{t^*})\|_F^2$, then by Lemma 2.1 and (22),

$$\begin{aligned}\delta(f) &\ge \int_0^\infty \mathbb{E}\frac{\phi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}}\,dt\\ &\ge \int_C^{C+1} \mathbb{E}\frac{\phi(h_t)\|H(h_t)\|_F^2}{(1+\|\nabla h_t\|^2)^{3/2}}\,dt\\ &\ge 2^{-7/2}\int_C^{C+1} I^2(\mathbb{E}f)\big(\mathbb{E}\|H(h_t)\|_F^2\big)^4\log^{-3}\frac{1}{m(f)}\,dt\\ &\ge 2^{-7/2}\frac{I^2(\mathbb{E}f)}{\log^3(1/m(f))}\big(\mathbb{E}\|H(h_{t^*})\|_F^2\big)^4.\end{aligned} \tag{23}$$

Recall that $I^2(x) \asymp x^2\log(1/x)$ as $x \to 0$. Hence, as $x \to 0$,

$$\frac{I^2(x)}{\log^3(1/(x(1-x)))} \asymp \frac{x^4}{\log(1/x)} \ge x^5.$$

By replacing x with $1-x$ [note that $I(x) = I(1-x)$] and repeating the argument, we see that there is some universal constant $c > 0$ such that $I^2(x)\log^{-3}(1/(x(1-x))) \ge c(x(1-x))^5$ for all $x \in [0, 1]$. Applying this to (23) with $x = \mathbb{E}f$, we have

$$\delta(f) \ge cm(f)^5\big(\mathbb{E}\|H(h_{t^*})\|_F^2\big)^4. \tag{24}$$

Finally, by Lemma 3.3,

$$\mathbb{E}\big(h_{t^*}(X) - \mathbb{E}h_t^* - X\cdot\mathbb{E}\nabla h_t^*\big)^2 \le \mathbb{E}\|H(h_{t^*})\|_F^2 \le C\frac{\delta^{1/4}(f)}{(\mathbb{E}f)^{5/4}},$$

where the last inequality follows from (24). $\square$

4. Approximation for small t. Recall that $f_t = P_t f$ and $h_t = \Phi^{-1} \circ f_t$. Proposition 3.1 shows that if f achieves almost-equality in (2) then h_t—for some t not too large—can be well approximated by a linear function. Since Φ is a contraction, this implies that f_t may be well approximated by a function of the form $\Phi(a \cdot x + b)$. The goal of this section is to complete the proof of Theorem 1.3 by showing that f itself can be approximated by a function of the same form. This will be accomplished mainly with spectral techniques, by expanding f in the Hermite basis.

Let $g_t(x) = \Phi(\mathbb{E}h_t + x \cdot \mathbb{E}\nabla h_t)$, so that Proposition 3.1 implies that $\mathbb{E}(f_t - g_t)^2 \le C\delta^{1/4}(f)m^{-5/4}(f)$. By directly computing P_t applied to the indicator of a half-space, one may check the following lemma (which also appeared implicitly in [6]):

LEMMA 4.1. *If $\|a\| \le k_t$, then the function $g(x) = \Phi(a \cdot x + b)$ is in the range of P_t. Moreover, if $\|a\| = k_t$ then $P_t^{-1} g$ is the indicator function of a half-space, while if $\|a\| < k_t$ then $P_t^{-1} g$ takes the form $\Phi(a' \cdot x + b')$.*

Now, Theorem 3.4 implies that $\|\mathbb{E}\nabla h_t\| \le k_t$ and so by Lemma 4.1, g_t is in the range of P_t, and $P_t^{-1} g_t$ is either the indicator of a half-space or Φ composed with a linear function. Let $g = P_t^{-1} g_t$. Then Proposition 3.1 implies that

$$\mathbb{E}\big(P_t(f-g)\big)^2 = \mathbb{E}(f_t - g_t)^2 \le C\frac{\delta(f)^2}{m(f)^C}.$$

In order to prove Theorem 1.3, it suffices to show that $\mathbb{E}(f-g)^2$ is small. In other words, setting $h = f - g$, we want to bound $\mathbb{E}h^2$ in terms of $\mathbb{E}(P_t h)^2$. For a general function h, this is an impossible task. To see why, consider $h_k(x) = \operatorname{sgn}(\sin(kx))$. Then $\mathbb{E}h_k^2 = 1$ for all k, but for any $t > 0$, $P_t h_k \to 0$ as $k \to \infty$. Hence, $\mathbb{E}(P_t h_k)^2 \to 0$, and so $\mathbb{E}h_k^2$ cannot be bounded in terms of $\mathbb{E}(P_t h_k)^2$.

The key to bounding $\mathbb{E}(f-g)^2$ in terms of $\mathbb{E}(P_t(f-g))^2$ is to exploit some extra information that we have on $f - g$. In particular, we have assumed that f almost minimizes Bobkov's functional $\mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2}$. In particular,

$$\mathbb{E}\|\nabla f\| \le \mathbb{E}\sqrt{I^2(f) + \|\nabla f\|^2} \le I(\mathbb{E}f) + \delta(f).$$

If we assume that $\delta(f) \le 1$ (if not, then Theorem 1.3 is meaningless anyway), then $\mathbb{E}\|\nabla f\| \le 2$. We will translate this smoothness condition into a condition on the Hermite spectrum of f, which will allow us to bound $\mathbb{E}(f-g)^2$ in terms of $\mathbb{E}(P_t(f-g))^2$.

We should remark that for nonnegative functions h, reverse hypercontractive inequalities can be used to bound $\mathbb{E}h^2$ in terms of $\mathbb{E}(P_t h)^2$. The restriction $h \ge 0$ prevents the positive and negative parts of h from canceling out under P_t, rendering examples like $h_k(x) = \operatorname{sgn}\sin(kx)$ impossible. For our application, however, we must consider functions that take positive and negative values.

Our main tool in this section is an inequality by Ledoux, which gives a connection between $\mathbb{E}\|\nabla f\|$ and the action of P_t on f.

THEOREM 4.2 ([15]). *There is a universal constant C such that for any smooth function $f:\mathbb{R}^n \to [-1,1]$ and any $t > 0$,*

$$\mathbb{E}f(f - P_t f) \le C\sqrt{t}\mathbb{E}\|\nabla f\|.$$

This inequality (in a sharper form) was originally derived to show the connection between Borell's noise sensitivity inequality [5] and the Gaussian isoperimetric inequality. We will give another application: Theorem 4.2 implies that for smooth functions, the Hermite coefficients decay at a certain rate.

4.1. *Smoothness and the Hermite expansion.* Recall that the Hermite polynomials $\{H_\alpha : \alpha \in \{0, 1, \dots\}^n\}$ form an orthogonal basis of $(\mathbb{R}^n, \gamma_n)$ [27]. Let

$$G_\alpha = \frac{H_\alpha}{\sqrt{\mathbb{E}H_\alpha^2}}$$

be the corresponding orthonormal basis. We will use the well-known fact that P_t acts diagonally on this basis:

$$P_t G_\alpha = e^{-|\alpha|t} G_\alpha. \tag{25}$$

LEMMA 4.3. *Suppose $f:\mathbb{R}^n \to [-1,1]$ is a smooth function and $f = \sum_\alpha b_\alpha G_\alpha$. Then for any $N \in \{1, 2, \dots\}$,*

$$\sum_{|\alpha| \ge N} b_\alpha^2 \le C N^{-1/2} \mathbb{E}\|\nabla f\|.$$

PROOF. By (25) and Theorem 4.2,

$$\sum_\alpha (1 - e^{-|\alpha|t}) b_\alpha^2 = \mathbb{E}f(f - P_t f) \le C\sqrt{t}\mathbb{E}\|\nabla f\|.$$

If $|\alpha| \ge 1/t$ then $e^{-|\alpha|t} \le 1/e$; hence,

$$(1 - 1/e) \sum_{|\alpha| \ge 1/t} b_\alpha^2 \le \sum_\alpha (1 - e^{-|\alpha|t}) b_\alpha^2 \le C\sqrt{t}\mathbb{E}\|\nabla f\|.$$

Now set $t = \frac{1}{N}$. $\square$

Since we know how the semigroup P_t acts on the Hermite basis and we know how the Hermite coefficients of nice functions are distributed, we are in a position to bound $\mathbb{E}f^2$ in terms of $\mathbb{E}(P_t f)^2$. Essentially, Lemma 4.3 tells us that the high coefficients do not contribute much to $\mathbb{E}f^2$, while (25) implies that the low coefficients contributing to $\mathbb{E}f^2$ also contribute to $\mathbb{E}(P_t f)^2$.

LEMMA 4.4. *For any smooth* $h:\mathbb{R}^n \to [-1,1]$ *and any* $t \geq 1$,

$$\mathbb{E}h^2 \leq C\big(1+\mathbb{E}\|\nabla h\|\big)\sqrt{\frac{t}{\log(1/\mathbb{E}(P_t h)^2)}}.$$

PROOF. Expand $h = \sum_\alpha b_\alpha G_\alpha$ and let $\varepsilon = \mathbb{E}(P_t h)^2$. Then (25) implies that

$$\varepsilon = \mathbb{E}(P_t h)^2 = \sum_\alpha e^{-2t|\alpha|} b_\alpha^2.$$

On the other hand, Lemma 4.3 implies that

$$\begin{aligned}
\mathbb{E}h^2 &= \sum_\alpha b_\alpha^2 \\
&\leq e^{2t(N-1)} \sum_{|\alpha|\leq N-1} b_\alpha^2 e^{-2t|\alpha|} + \sum_{|\alpha|\geq N} b_\alpha^2 \qquad (26)\\
&\leq e^{2t(N-1)}\varepsilon + CN^{-1/2}K,
\end{aligned}$$

where $K = \mathbb{E}\|\nabla h\|$.

Now we choose N to optimize (26). Let $\beta = \frac{1}{2t}\log\frac{1}{\varepsilon}$ and set $N = \lceil \beta - \frac{1}{4t}\log\beta\rceil$. Since $\beta > \log\beta$ and $t \geq 1$, $N \geq \beta/2$ (and in particular, N is a positive integer). Moreover, $N-1 \leq \beta - \frac{1}{4t}\log\beta$ and so (since $e^{2t\beta} = 1/\varepsilon$) $e^{2t(N-1)}\varepsilon \leq \beta^{-1/2}$. Plugging these bounds on N back into (26) yields

$$\mathbb{E}h^2 \leq \beta^{-1/2} + CK\beta^{-1/2} \leq C(1+K)\sqrt{\frac{t}{\log(1/\varepsilon)}}. \qquad \square$$

4.2. *Proof of Theorem* 1.3. Finally, we are ready to prove Theorem 1.3. As we discussed at the beginning of the section, we may assume that $\delta = \delta(f) \leq 1$, which implies that $\mathbb{E}\|\nabla f\| \leq 2$. We may also assume that $m(f) \geq \log^{-1/2}(1/\delta)$: if not, then either $\mathbb{E}f \leq \log^{-1/2}(1/\delta)$ or $(1-\mathbb{E}f) \leq \log^{-1/2}(1/\delta)$. In the first case, f may be approximated well by the zero function, which in turn may be approximated by functions of the form $\Phi(a\cdot x+b)$. Specifically, for any $a \in \mathbb{R}^n$, $\Phi(a\cdot x+b) \to 0$ as $b \to -\infty$ and so

$$\lim_{b\to-\infty} \mathbb{E}\big(f(X) - \Phi(a\cdot X+b)\big)^2 = \mathbb{E}f^2 \leq \mathbb{E}f \leq \frac{1}{\sqrt{\log(1/\delta)}}.$$

That is, if $\mathbb{E}f \leq \log^{-1/2}(1/\delta)$ then the conclusion of Theorem 1.3 holds trivially. A similar argument (but with the zero function replaced by the constant function 1) holds when $(1-\mathbb{E}f) \leq \log^{-1/2}(1/\delta)$. Thus, we may assume that $m(f) \geq \log^{-1/2}(1/\delta)$.

As in the discussion at the beginning of the section, take (by Proposition 3.1) $t \in [C, C+1]$ so that

$$\mathbb{E}(h_t - \mathbb{E}h_t - X\cdot\mathbb{E}\nabla h_t)^2 \leq C\frac{\delta^{1/4}(f)}{m^{5/4}(f)}.$$

Let $g_t(x) = \Phi(x \cdot \mathbb{E}\nabla h_t + \mathbb{E}h_t)$ and $g = P_t^{-1} g_t$ (which exists, recall, by Lemma 4.1 and because $\|\mathbb{E}\nabla h_t\| \leq k_t$).

By Proposition 3.1 and because Φ is a contraction,

$$\mathbb{E}(g_t - f_t)^2 \leq \mathbb{E}(h_t - \mathbb{E}h_t - X \cdot \mathbb{E}\nabla h_t)^2 \leq C \frac{\delta(f)^{1/4}}{m(f)^{5/4}} \leq C\delta^{1/8},$$

where the last inequality follows from because we have assumed that $m(f) \geq \log^{-1/2}(1/\delta)$. Set $h = g - f$. Since g is Φ composed with a linear function, $\mathbb{E}\|\nabla g\| \leq \phi(0) \leq 1$, and hence $\mathbb{E}\|\nabla h\| \leq \mathbb{E}\|\nabla g\| + \mathbb{E}\|\nabla f\| \leq 3$. By Lemma 4.4,

$$\mathbb{E}(g - f)^2 \leq \frac{C}{\sqrt{\log(1/\mathbb{E}(g_t - f_t)^2)}} \leq \frac{C}{\sqrt{\log(1/\delta)}}.$$

This completes the proof of Theorem 1.3.

5. Robust results for sets. There are two pieces needed to get from Theorem 1.3 to Theorem 1.4. First, we need to interpret Theorem 1.3 in the case that f is an indicator function (which is necessarily nonsmooth). For this, we simply apply Theorem 1.3 to $P_t f$, which is smooth, and take t to zero. Fortunately for us, most of the work in this step was done in [6].

LEMMA 5.1 ([6]). *For any measurable set A,*

$$\gamma_n^+(A) = \lim_{t \to 0} \mathbb{E}\sqrt{I^2(P_t 1_A) + \|\nabla P_t 1_A\|^2}.$$

The second piece we require is something that will let us pass from a function $\Phi(a \cdot x + b)$ to an affine half-space. For this piece, we just round $\Phi(a \cdot x + b)$ to $\{0, 1\}$.

LEMMA 5.2. *Let A be a measurable set and $g(x) = \Phi(a \cdot x + b)$. There exists an affine half-space B such that*

$$\gamma_n(A \Delta B) \leq \mathbb{E}(1_A - g)^2.$$

PROOF. Let $B = \{x \in \mathbb{R}^n : a \cdot x + b \geq 0\}$. Since 1_B is obtained by rounding g to $\{0, 1\}$, it follows that $|1_B - g| \leq |1_A - g|$ pointwise. Thus,

$$4\gamma_n(A \Delta B) = \mathbb{E}(1_A - 1_B)^2 \leq 2\mathbb{E}(1_A - g)^2 + 2\mathbb{E}(1_B - g)^2 \leq 4\mathbb{E}(1_A - g)^2. \quad \square$$

PROOF OF THEOREM 1.4. Let $f = 1_A$ and write $f_t = P_t f$; recall that $\delta = \gamma_n^+(A) - I(\gamma_n(A))$. Note that $\mathbb{E}f_t = \mathbb{E}f = \gamma_n(A)$ for all $t > 0$. Since $\mathbb{E}\sqrt{I^2(f_t) + \|\nabla f_t\|^2} \to \gamma_n^+(A)$ by Lemma 5.1,

$$\mathbb{E}\sqrt{I^2(f_t) + \|\nabla f_t\|^2} - I(\mathbb{E}f_t) \leq 2\delta$$

for all small enough t. Now set $\varepsilon = \log^{-1/2}(1/\delta)$. Since the semigroup P_t is strongly continuous in $L_2(\gamma_n)$, we can take t small enough so that $\mathbb{E}(f_t - 1_A)^2 \le \varepsilon$.

Apply Theorem 1.3 to f_t: we receive a function $g(x) = \Phi(a \cdot x + b)$ with $\mathbb{E}(f_t - g)^2 \le C\varepsilon$. By the triangle inequality, $\mathbb{E}(1_A - g)^2 \le C\varepsilon$ and so Lemma 5.2 gives us an affine half-space B with $\gamma_n(A \Delta B) \le C\varepsilon$. $\square$

6. Conclusion.

6.1. *Open problems*. To conclude, we present two natural open problems:

- Is there a result which strengthens both our result and [7]? For sets A of measure $1/2$, such result should give the existence of a half space B with $\gamma_n(A \Delta B) \le C\delta^{1/2}$ where C is an absolute constant.
- Could similar results be obtained for other measures? In particular, log-concave measures? This question was suggested to us independently by Franck Barthe, Michel Ledoux and Shahar Mendelson. We note that the one-dimensional analogue of [7] was established by [8].

6.2. *Subsequent work*. Building on techniques that we develop here, we very recently obtained a robust dimension-free version of Borell's theorem [23], thereby establishing the applications that we mentioned earlier in the Introduction. Moreover, in Borell's theorem, we obtained polynomial rather than logarithmic rates (although we could not achieve the optimal exponent of $\frac{1}{2}$). However, our robust version of Borell's theorem does not imply Theorem 1.4 even though Borell's result implies the isoperimetric inequality as $\rho \to 1$, since we were only able to obtain the correct dependence on ρ for sets A satisfying $\gamma_n(A) = \frac{1}{2}$.

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