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LOCAL UNIVERSALITY OF REPULSIVE PARTICLE SYSTEMS AND RANDOM MATRICES

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We study local correlations of certain interacting particle systems on the real line which show repulsion similar to eigenvalues of random Hermitian matrices. Although the new particle system does not seem to have a natural spectral or determinantal representation, the local correlations in the bulk coincide in the limit of infinitely many particles with those known from random Hermitian matrices; in particular they can be expressed as determinants of the so-called sine kernel. These results may provide an explanation for the appearance of sine kernel correlation statistics in a number of situations which do not have an obvious interpretation in terms of random matrices.

1. Introduction and main results. This paper is motivated by the surprising emergence of sine kernel statistics in many real world observations such as parking cars, perching birds on lines and so on. In the field of random matrices, the sine kernel describes the local correlations of eigenvalues in the bulk of the spectrum of Hermitian random matrices. There it has been shown to be universal to a high extent; that is, it appears for many essentially different matrix distributions. In this article we show that the sine kernel describes the local correlations of more general repulsive particle systems on the real line which only share the repulsion strength exponent $\beta = 2$ with the eigenvalues of (unitary invariant) Hermitian random matrices. We expect that this behavior extends to larger classes of invariant ensembles of random matrices, with repulsion exponents β different from two.

To formulate our results, let us recall the so-called invariant β-ensembles from random matrix theory. Given a continuous function $Q:\mathbb{R} \longrightarrow \mathbb{R}$ of sufficient growth at infinity and $\beta > 0$, set

$$P_{N,Q,\beta}(x) := \frac{1}{Z_{N,Q,\beta}} \prod_{i<j} |x_i - x_j|^\beta e^{-N\sum_{j=1}^N Q(x_j)}. \tag{1}$$

(With a slight abuse of notation, we will not distinguish between a measure and its density.) For the "classical values" $\beta = 1, 2, 4$, $P_{N,Q,\beta}$ is the eigenvalue distribution of a probability ensemble on the space of $(N \times N)$ matrices with real

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symmetric ($\beta = 1$), complex Hermitian ($\beta = 2$) or quaternionic self-dual ($\beta = 4$) entries, respectively. For arbitrary β, only for quadratic Q, $P_{N,Q,\beta}$ is known to be an eigenvalue distribution.

The notion of bulk universality is usually formulated via the correlation functions of the ensemble. For a probability measure $P_N(x)\,dx$ on $\mathbb{R}^N$ and $k = 1, 2, \dots, N$, the *kth correlation function* $\rho_N^k : \mathbb{R}^k \longrightarrow \mathbb{R}$ of P_N is defined as

$$\rho_N^k(x_1, \dots, x_k) := \int_{\mathbb{R}^{N-k}} P_N(x)\,dx_{k+1} \cdots dx_N.$$

The correlation functions ρ_N^k are the densities of the marginals of P_N. The measure $\rho_N^k(t)\,dt$ on $\mathbb{R}^k$ is called *kth correlation measure*.

It is known that under very mild conditions on Q, there is an absolutely continuous probability measure $\mu_{Q,\beta}(t)\,dt$ on $\mathbb{R}$, which is the weak limit of $\rho_{N,Q,\beta}^1(t)\,dt$ as $N \to \infty$.

Now, $P_{N,Q,\beta}$ is said to admit bulk universality, if for all a with $\mu_{Q,\beta}(a) > 0$ and all $t_1, \dots, t_k$ the limit

$$\lim_{N\to\infty} \frac{1}{\mu_{Q,\beta}(a)^k} \rho_N^k\left(a + \frac{t_1}{N\mu_{Q,\beta}(a)}, \dots, a + \frac{t_k}{N\mu_{Q,\beta}(a)}\right) \tag{2}$$

exists and coincides with the one for $P_{N,G,\beta}$, G quadratic (the so-called Gaussian β-ensemble). Universality here should be understood as a coincidence of limit (2) with the corresponding Gaussian β-ensemble. This has been established for large classes of Q. The scaling in (2) is chosen such that the asymptotic mean spacing between consecutive eigenvalues is normalized to 1. However, it is known that the limit depends on β.

In the case $\beta = 2$, which appears frequently in "real world statistical studies," the limiting object (2) is determinantal of type

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{\mu_{Q,2}(a)^k} \rho_N^k\left(a + \frac{t_1}{N\mu_{Q,2}(a)}, \dots, a + \frac{t_k}{N\mu_{Q,2}(a)}\right) \\ &\qquad = \det\left[\frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)}\right]_{1\le i,j\le k}, \end{aligned} \tag{3}$$

involving the sine kernel

$$\mathbb{S}(t) := \frac{\sin(\pi t)}{\pi t}, \qquad t \neq 0, \mathbb{S}(0) := 1.$$

Universality for unitary invariant ensembles, that is, $\beta = 2$ invariant ensembles, was proved in many papers, for example (naming only few) [10, 21, 23, 26, 27]. Recently universality (for general β-ensembles) was proved in [7, 8]. For $\beta = 1, 2$, bulk universality was also proved for Wigner matrices by two groups of authors. Based on earlier work of Johansson [15], universality was shown for general classes of Wigner matrices in a series of papers by Erdős, Yau, Schlein, Yin, Ramirez and Peche (see [12] for a survey on their results) and Tao and Vu; see [30] for a survey on their results. We remark that bulk universality was proved in [13] for the Hermitian fixed trace ensemble, a random matrix which is neither a Wigner matrix nor determinantal.

Writing the density (1) in the Gibbsian form

$$P_{N,Q,\beta} = \frac{1}{Z_{N,Q,\beta}} e^{\beta \sum_{i<j} \log|x_i - x_j| - N \sum_{j=1}^{N} Q(x_j)}, \tag{4}$$

we see that $P_{N,Q,\beta}$ can be interpreted as an interacting particle system on $\mathbb{R}$ in an external field, interacting via a 2d Coulomb potential.

It is believed that many complicated, strongly correlated systems share the local bulk scaling limit (defined again by correlation functions) with some random matrix model. This was conjectured by Wigner who used random matrices to model energy levels of nuclei. By the underlying matrix structure, physical requirements (conserved quantities, time reversal, ...) determine the value of β in the cases $\beta = 1, 2, 4$. The limits with $\beta = 2$ also seem to appear in statistics of distances between parking cars [1], waiting times at bus stops in certain cities [18] (see [5] for a determinantal model) and the pair correlation conjecture of Montgomery [24] for the zeros of the Riemann Zeta function on the critical line. See, for example, [17] for more relations between the Riemann Zeta function and random matrix theory. A common cause for the appearance of sine kernel statistics in a number of statistics about real world repulsive systems and in physics and mathematics still remains to be identified.

We consider here a class of more general interacting particle systems, defined by the density

$$\frac{1}{Z_{N,\varphi,Q}} \prod_{i<j} \varphi(x_i - x_j) e^{-N \sum_{j=1}^{N} Q(x_j)}, \tag{5}$$

where Q is a continuous function of sufficient growth at infinity compared to the continuous function $\varphi : \mathbb{R} \longrightarrow [0, \infty)$. Apart from some technical conditions we will assume that

$$\varphi(0) = 0, \qquad \varphi(t) > 0 \qquad \text{for } t \neq 0 \quad \text{and} \quad \lim_{t \to 0} \frac{\varphi(t)}{|t|^{\beta}} = c > 0, \tag{6}$$

or, in other terms, 0 is the only zero of φ and it is of order β.

We expect that (at least under some smoothness and growth conditions) the bulk scaling limit of (5) coincides with that of the β-ensembles, since in view of the regular local distribution of eigenvalues/particles at $1/N$ spacings only the exponents of the interaction kernel should determine the local universality class.

The purpose of this paper is to prove this for $\beta = 2$ and a special class of φ and Q. From now on, we will always deal with the case $\beta = 2$, therefore omitting the subscript β. To state our results, let h be a continuous even function which is bounded below. Let Q be a continuous even function of sufficient growth at infinity. By $P^h_{N,Q}$ we will denote the probability density on $\mathbb{R}^N$ defined by

$$P^h_{N,Q}(x) := \frac{1}{Z^h_{N,Q}} \prod_{i<j} |x_i - x_j|^2 \exp\Biggl\{-N \sum_{j=1}^N Q(x_j) - \sum_{i<j} h(x_i - x_j)\Biggr\}, \tag{7}$$

where $Z^h_{N,Q}$ denotes the normalizing constant. The density $P^h_{N,Q}$ can also be written in the form (5) with $\varphi(t) := t^2 \exp\{-h(t)\}$. The first result describes the global scaling limit of the correlation measures of $P^h_{N,Q}$. To formulate it, introduce for a twice differentiable convex function Q the quantity $\alpha_Q := \inf_{t\in\mathbb{R}} Q''(t)$. Moreover, denote by $\rho^{h,k}_{N,Q}$ the kth correlation function of $P^h_{N,Q}$.

THEOREM 1.1. *Let h be a real analytic and even Schwartz function. Then there exists a constant $\alpha^h \geq 0$ such that for all real analytic, strictly convex and even Q with $\alpha_Q > \alpha^h$, the following holds:*

There exists a compactly supported probability measure μ^h_Q having a nonzero and continuous density on the interior of its support and for $k = 1, 2, \dots,$ the kth correlation measure of $P^h_{N,Q}$ converges weakly to the k-fold product $(\mu^h_Q)^{\otimes k}$, that is, for any bounded and continuous function $g : \mathbb{R}^k \longrightarrow \mathbb{R}$,

$$\lim_{N\to\infty} \int g\rho^{h,k}_{N,Q}\, d^k t = \int g\, d(\mu^h_Q)^{\otimes k}. \tag{8}$$

REMARK. (a) If h is (additionally) positive semi-definite, then α^h in Theorem 1.1 may be explicitly chosen as $\alpha^h = \sup_{t\in\mathbb{R}} -h''(t)$.

(b) In general, the measure μ^h_Q depends on h.

(c) $P^h_{N,Q}$ does not seem to be either determinantal nor have a natural spectral interpretation; therefore we will speak of particles instead of eigenvalues.

(d) We remark that in [9], macroscopic correlations have been studied in a more general setup.

The next result states the universality of the sine kernel in the local scaling limit in the bulk.

THEOREM 1.2. *Let h and Q satisfy the assumptions of Theorem 1.1. Then for $k = 1, 2, \dots,$ we have*

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{\mu^h_Q(a)^k} \rho^{h,k}_{N,Q}\left(a + \frac{t_1}{N\mu^h_Q(a)}, \dots, a + \frac{t_k}{N\mu^h_Q(a)}\right) \\ &\quad = \det\left[\frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)}\right]_{1\leq i,j\leq k} \end{aligned} \tag{9}$$

uniformly in $t_1, \dots, t_k$ from any compact subset of $\mathbb{R}^k$ and uniformly in the point a from any compact proper subset of the support of μ^h_Q.

REMARK. (a) If h is positive semi-definite, then α^h in Theorem 1.2 may be explicitly chosen as $\alpha^h = \sup_{t\in\mathbb{R}} -h''(t)$.

(b) Bulk universality for ensembles of form (7) with arbitrary $\beta > 0$ replacing the repulsion exponent 2 in (7) has been shown by the second author in [34]. The notion of universality is weaker than in the present paper. The proof of bulk universality uses methods similar to the present work, combined with techniques developed by Erdős, Yau and co-workers; see, for example, [12] for a review.

(c) Similar results hold at the edge of the support of μ_Q^h. An article on edge universality of $P_{N,Q}^h$ is in preparation [20].

We shall demonstrate our approach to bulk universality by means of the following example of functions h and Q.

THEOREM 1.3. *Let $\gamma > 0$ and $\alpha > 0$ be arbitrary. Let $h(t-s) := \gamma(t-s)^2$ and $Q(t) = \alpha t^2$. Then (8) and (9) hold for $(P_{N,Q}^h)_N$ uniformly as in Theorem 1.2. Here μ_Q^h will be the semi-circle distribution with support $[-\omega,\omega]$, $\omega := (\sqrt{\alpha+\gamma})^{-1}$.*

A first step in the proof of Theorems 1.1 and 1.2 is to compare the correlation functions of $P_{N,Q}^h$ with correlation functions of *eigenvalues* of some unitary invariant ensemble. To construct such an ensemble, we first determine μ_Q^h as the equilibrium measure of some external field V (depending on h and Q) using a fixed point argument. The difference between $P_{N,Q}^h$ and this unitary invariant ensemble $P_{N,V}$ consists of (up to normalization) a factor $\exp\{\mathcal{U}(x)\}$, where $\mathcal{U}$ is a quadratic interaction energy which may be expressed as a mixture of linear interaction energy terms using Gaussian processes. This finally leads, after a truncation procedure, to a mixture representation of $P_{N,Q}^h$ by invariant ensembles with the *same* bulk universality.

The paper is organized as follows. In Section 2, the asymptotics of $P_{N,Q}^h$ for $h(t-s) := \gamma(t-s)^2$ and $Q(t) = \alpha t^2$ are investigated, and in particular Theorem 1.3 is proved. In Section 3, we associate to $P_{N,Q}^h$ a unitary invariant ensemble which will turn out to have the same asymptotic behavior as $P_{N,Q}^h$. Section 4 contains concentration of measure inequalities. Section 5 deals with bounds on the first correlation function of a unitary invariant ensemble. The proofs in this section use established techniques which we decided to include in detail for the sake of completeness of the exposition. Theorems 1.1 and 1.2 are proved in Section 6. In the Appendix we recall a number of results on equilibrium measures.

A prior version of these results is based on the Ph.D. thesis of the second author [33].

2. A first example.

In this section, we will study the probability measure

$$P_N^{\alpha,\gamma}(x) := \frac{1}{Z_N^{\alpha,\gamma}} \prod_{1\le i<j\le N} |x_i - x_j|^2 \exp\left\{-\alpha N M_2(x) - \gamma \sum_{i<j} (x_i - x_j)^2\right\}, \tag{10}$$

using the potentials $M_p(x) := \sum_{j=1}^N x_j^p$ with $p = 2$ and constants $\alpha, \gamma > 0$, where $Z_N^{\alpha,\gamma}$ denotes the normalization factor. In the following we shall suppress the dependencies on α and γ.

We will reduce bulk universality of $(P_N^{\alpha,\gamma})_N$ to the well-known bulk universality of the GUE.

It is convenient to introduce the distribution GUE_ω, depending on a parameter $\omega > 0$, as

$$P_{N,\omega}^{\mathrm{GUE}}(x) := \frac{1}{Z_{N,\omega}^{\mathrm{GUE}}} \prod_{j<k} |x_k - x_j|^2 \exp\left\{-\frac{2}{\omega^2} N M_2(x)\right\}.$$

Under this scaling the first correlation measure of $P_{N,\omega}^{\mathrm{GUE}}$ will converge to the semicircle law supported on $[-\omega, \omega]$; for a proof see, for example, [25]. First we rewrite the density $P_N := P_N^{\alpha,\gamma}$ using

$$\gamma \sum_{i<j} (x_i - x_j)^2 = \gamma N M_2(x) - \gamma M_1(x)^2 \quad \text{as}$$

$$P_N(x) = \frac{1}{Z_N} \prod_{1 \le i < j \le N} |x_i - x_j|^2 \times \exp\{-(\alpha + \gamma) N M_2(x) + \gamma M_1(x)^2\}. \tag{11}$$

Using the simple identity

$$\exp\{\gamma t^2\} = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\{\varepsilon \sqrt{\gamma} t\} \exp\{-\varepsilon^2/4\}\, d\varepsilon, \tag{12}$$

we may write

$$\begin{aligned} P_N(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{Z_N} \prod_{1 \le i < j \le N} |x_i - x_j|^2 \\ &\qquad \times \exp\{-(\alpha + \gamma) N M_2(x) + \sqrt{\gamma} \varepsilon M_1(x)\} \\ &\qquad \times \exp\{-\varepsilon^2/4\}\, d\varepsilon \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{Z_N^\varepsilon}{Z_N} P_N^\varepsilon(x) e^{-\varepsilon^2/4}\, d\varepsilon \qquad (13) \end{aligned}$$

where

$$P_N^\varepsilon(x) := \frac{1}{Z_N^\varepsilon} \prod_{1 \le i < j \le N} |x_i - x_j|^2 \times \exp\{-(\alpha + \gamma) N M_2(x) + \sqrt{\gamma} \varepsilon M_1(x)\},$$

$$Z_N^\varepsilon := \int_{\mathbb{R}^N} \prod_{1 \le i < j \le N} |x_i - x_j|^2 \times \exp\{-(\alpha + \gamma) N M_2(x) + \sqrt{\gamma} \varepsilon M_1(x)\}\, dx.$$

We have thus expressed P_N as a probabilistic mixture of the probability measures P_N^{ε}.

The next lemma deals with the ratio Z_N^{ε}/Z_N.

LEMMA 2.1. *For each ε, each N and all $\alpha, \gamma > 0$ we have*

$$Z_N^{\varepsilon}/Z_N = \exp\left\{\frac{\gamma\varepsilon^2}{4(\alpha+\gamma)}\right\}\left(\sqrt{1-\frac{\gamma}{\alpha+\gamma}}\right)^{-1}.$$

PROOF. We first expand the fraction

$$Z_N^{\varepsilon}/Z_N = \left(Z_N^{\varepsilon}/Z_{N,\omega}^{\mathrm{GUE}}\right)/\left(Z_N/Z_{N,\omega}^{\mathrm{GUE}}\right) \qquad \text{where } \omega = (\alpha+\gamma)^{-1/2}.$$

The diagonal elements of a GUE_{ω} matrix are independent Gaussians with mean 0 and variance $\frac{1}{2N(\alpha+\gamma)}$. Using this, we get easily for any ε, any N and any $\alpha, \gamma > 0$

$$Z_N^{\varepsilon}/Z_{N,\omega}^{\mathrm{GUE}} = \mathbb{E}_{N,\mathrm{GUE}_{\omega}} \exp\{\varepsilon\sqrt{\gamma} M_1(x)\} = \exp\{\gamma\varepsilon^2 \cdot (4(\alpha+\gamma))^{-1}\},$$

where $\mathbb{E}_{N,\mathrm{GUE}_{\omega}}$ denotes expectation w.r.t. $P_{N,\omega}^{\mathrm{GUE}}$. Similarly, we get for any N and any $\alpha, \gamma > 0$

$$Z_N/Z_{N,\omega}^{\mathrm{GUE}} = \mathbb{E}_{N,\mathrm{GUE}_{\omega}} \exp\{\gamma M_1(x)^2\} = \left(1-\gamma/(\alpha+\gamma)\right)^{-1/2}. \qquad \square$$

DEFINITION 2.2. For $\omega > 0$, the probability measure σ_{ω} on $\mathbb{R}$ given by

$$\sigma_{\omega}(t)\,dt := \frac{2}{\pi\omega^2}\sqrt{\omega^2 - t^2}\,\mathbb{1}_{[-\omega,\omega]}(x)\,dt$$

is called (*Wigner's*) *semicircle law* (with parameter ω).

By equation (13), P_N is a mixture of P_N^{ε}. We show first that the statement of Theorem 1.3 is true for each $\varepsilon \in \mathbb{R}$ if we replace $P_{N,Q}^h$ by P_N^{ε}. Eventually we will use Lebesgue's dominated convergence theorem.

PROPOSITION 2.3. *Let $\rho_N^{k,\varepsilon}$ denote the kth correlation function of P_N^{ε} and set $\omega = \sqrt{\frac{1}{\alpha+\gamma}}$.*

(a) *For any $\varepsilon \in \mathbb{R}$, any k and any continuous, bounded $g : \mathbb{R}^k \longrightarrow \mathbb{R}$ we have*

$$\lim_{N\to\infty} \int_{\mathbb{R}^k} g\,d\rho_N^{k,\varepsilon} = \int_{[-\omega,\omega]^k} g\,d(\sigma_{\omega})^k.$$

(b) *We have for any ε and any k,*

$$\lim_{N\to\infty} \frac{1}{\sigma_{\omega}(a)^k}\rho_N^{k,\varepsilon}\left(a+\frac{t_1}{N\sigma_{\omega}(a)},\dots,a+\frac{t_k}{N\sigma_{\omega}(a)}\right) = \det\left(\mathbb{S}(t_i-t_j)\right)_{1\le i,j\le k}$$

locally uniformly for all $t_1,\dots,t_k$ and uniformly for a varying in a compact subset of $(-\omega,\omega)$.

PROOF. A proof of the first part can be found in [14]. For the second part we use orthogonal polynomials. Note that the polynomials orthogonal to a Gaussian weight with nonzero mean are normalized shifted Hermite polynomials. Let $\pi_j^{(\mathrm{N})}$ denote the jth Hermite polynomial orthonormal w.r.t. the weight $e^{-N(\alpha+\gamma)t^2}$.

It is easy to check that the set of polynomials orthogonal w.r.t. the weight $e^{-N(\alpha+\gamma)t^2+\varepsilon\sqrt{\gamma}t}$ are the polynomials $(\pi_j^{(\mathrm{N})*})_j$, where

$$\pi_j^{(\mathrm{N})*}(t) := e^{(\omega''\varepsilon^2/2N)}\pi_j^{(\mathrm{N})}\big(t-\omega'\varepsilon/2N\big) \tag{14}$$

with $\omega' := \sqrt{\gamma}/(\alpha+\gamma)$ and $\omega'' := \omega'^2/4$. The ensemble P_N^{ε} is determinantal, that is,

$$\rho_N^{k,\varepsilon}(t_1,\dots,t_k) = (N-k)!/(N!)\det\big(K_N^*(t_i,t_j)\big)_{i,j=1}^k, \tag{15}$$

where $K_N^*(t,s) = \sum_{j=0}^{N-1}\pi_j^{(\mathrm{N})*}(t)\pi_j^{(\mathrm{N})*}(s)$. From (14) we get

$$K_N^*(t,s) = e^{(\omega''\varepsilon^2)/N}K_N\big(t-\omega'\varepsilon/2N, s-\omega'\varepsilon/2N\big), \tag{16}$$

where K_N denotes the kernel corresponding to the ensemble $P_{N,\omega}^{\mathrm{GUE}}$. Hence we have

$$\begin{aligned}
&\frac{1}{\sigma_\omega(a)}K_N^*\left(a+\frac{t}{N\sigma_\omega(a)}, a+\frac{s}{N\sigma_\omega(a)}\right)\\
&\quad=\frac{e^{(\omega''\varepsilon^2)/N}}{\sigma_\omega(a)}K_N\left(a+\frac{t-\omega'\varepsilon\sigma_\omega(a)/2}{N\sigma_\omega(a)}, a+\frac{s-\omega'\varepsilon\sigma_\omega(a)/2}{N\sigma_\omega(a)}\right)\\
&\quad=\frac{e^{(\omega''\varepsilon^2)/N}}{\sigma_\omega(a)}K_N\left(a+\frac{t'}{N\sigma_\omega(a)}, a+\frac{s'}{N\sigma_\omega(a)}\right),
\end{aligned} \tag{17}$$

where $t' := t-\omega'\varepsilon\sigma_\omega(a)/2$ and $s' := s-\omega'\varepsilon\sigma_\omega(a)/2$. It is well known that

$$\lim_{N\to\infty}\frac{1}{\sigma_\omega(a)}K_N\left(a+\frac{t'}{N\sigma_\omega(a)}, a+\frac{s'}{N\sigma_\omega(a)}\right) = \frac{\sin(\pi(t'-s'))}{\pi(t'-s')}. \tag{18}$$

For a proof of (18) see, for example, [11], Chapter 8, or Theorem 6.1. Since $\lim_{N\to\infty}\exp\{(\omega''\varepsilon^2)/N\} = 1$, we get from (17) and (18) that

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{\sigma_\omega(a)}K_N^*\left(a+\frac{t}{N\sigma_\omega(a)}, a+\frac{s}{N\sigma_\omega(a)}\right)\\
&\quad=\frac{\sin(\pi(t'-s'))}{\pi(t'-s')} = \frac{\sin(\pi(t-s))}{\pi(t-s)}.
\end{aligned} \tag{19}$$

Now, by (19) and (15), the second assertion of Proposition 2.3 follows. As (18) is true locally uniformly in t', s' and uniformly in $a \in I$, $I \subset [-\omega,\omega]$ compact, we get (19) locally uniformly in t, s and uniformly in $a \in I$. $\square$

PROOF OF THEOREM 1.3. By equation (13) and Lemma 2.1 we know that

$$P_N(x) = \int_{\mathbb{R}} P_N^{\varepsilon}(x) p(\varepsilon)\, d\varepsilon, \tag{20}$$

where p is an N-independent probability measure on $\mathbb{R}$. Using Fubini's theorem, (20) implies $\int_{\mathbb{R}^k} g\, d\rho_N^k = \int_{\mathbb{R}} \int_{\mathbb{R}^k} g\, d\rho_N^{k,\varepsilon} p(\varepsilon)\, d\varepsilon$ and $\rho_N^k(t_1, \dots, t_k) = \int_{\mathbb{R}} \rho_N^{k,\varepsilon}(t_1, \dots, t_k) p(\varepsilon)\, d\varepsilon$, and hence for each compact $K \subset \mathbb{R}^k$ and each compact $I \subset (-\omega, \omega)$

$$\begin{aligned}
&\sup_{t\in K, a\in I} \left| \sigma_\omega(a)^{-k} \rho_N^k\left(a + \frac{t_1}{N\sigma_w(a)}, \dots, a + \frac{t_k}{N\sigma_w(a)}\right) \right.\\
&\qquad\qquad \left. - \det\bigl(\mathbb{S}(t_i - t_j)\bigr)_{1\le i,j\le k} \right| \\
&\quad = \sup_{t\in K, a\in I} \left| \int_{\mathbb{R}} p(\varepsilon) \left(\sigma_\omega(a)^{-k} \rho_N^{k,\varepsilon}\left(a + \frac{t_1}{N\sigma_w(a)}, \dots, a + \frac{t_k}{N\sigma_w(a)}\right) \right.\right.\\
&\qquad\qquad \left.\left. - \det\bigl(\mathbb{S}(t_i - t_j)\bigr)_{1\le i,j\le k} \right) d\varepsilon \right| \\
&\quad \le \int_{\mathbb{R}} p(\varepsilon) \sup_{t\in K, a\in I} \left| \sigma_\omega(a)^{-k} \rho_N^{k,\varepsilon}\left(a + \frac{t_1}{N\sigma_w(a)}, \dots, a + \frac{t_k}{N\sigma_w(a)}\right) \right.\\
&\qquad\qquad \left. - \det\bigl(\mathbb{S}(t_i - t_j)\bigr)_{1\le i,j\le k} \right| d\varepsilon,
\end{aligned} \tag{21}$$

where we stick to the notation of Proposition 2.3. Theorem 1.3 will follow from Proposition 2.3 if $\int_{\mathbb{R}^k} g\, d\rho_N^{k,\varepsilon}$ and $\sup_{t\in K, a\in I} |\rho_N^{k,\varepsilon}(s_1, \dots, s_k)|$, $s_i := a + t_i/(N\sigma_\omega(a))$, are uniformly bounded in ε. The uniform boundedness of $\int_{\mathbb{R}^k} g\, d\rho_N^{k,\varepsilon}$ is immediate as g is bounded.

To show uniform boundedness of $\rho_N^{k,\varepsilon}(s_1, \dots, s_k)$ uniformly in ε, t and a, we proceed as in the paper by Pastur and Shcherbina [27]. Since all correlation functions are nonnegative, we see by Sylvester's criterion from the determinantal relations (15) that the matrix $(K_N^*(t_i, t_j))_{1\le i,j\le k} =: A$ is positive semi-definite and can hence be written as $A = B^2$ for some matrix B. Now using Hadamard's inequality we get

$$\det A = (\det B)^2 \le \prod_{j=1}^k \sum_{i=1}^k |B_{ij}|^2 = \prod_{j=1}^k A_{jj}.$$

In our case this reads

$$\rho_N^{k,\varepsilon}(s_1, \dots, s_k) \le (N-k)!/(N!) \prod_{j=1}^k K_N(s_j, s_j) \le C^k \prod_{j=1}^k \rho_N^{1,\varepsilon}(s_j), \tag{22}$$

where C is a constant such that $C \geq N/(N-k)$. Using (14), we get

$$\begin{aligned}\rho_N^{1,\varepsilon}(s_j) &= \frac{1}{N}\sum_{i=0}^{N-1}\pi_i^{(\mathrm{N})*}(s_j)^2 e^{-N(\alpha+\gamma)s_j^2+\sqrt{\gamma}\varepsilon s_j} \\ &= \frac{1}{N}\sum_{i=0}^{N-1}\pi_i^{(\mathrm{N})}\big(t-\omega'\varepsilon/2N\big)^2 e^{-N(\alpha+\gamma)(s_j-\omega'\varepsilon/2N)^2} \\ &= \rho_N^{1,\mathrm{GUE}_\omega}\big(s_j-\omega'\varepsilon/2N\big),\end{aligned}$$

where $\rho_N^{1,\mathrm{GUE}_\omega}$ is the first correlation function of the GUE_ω. From Proposition 2.3(b) for $k=1, \varepsilon=0$ we get that $\rho_N^{1,\mathrm{GUE}_\omega}(s_j-\omega'\varepsilon/2N)$ converges (locally) uniformly in t_j and a toward the bounded function $\sigma_\omega(a)$, hence there is a constant C' such that for all N and all $t \in K, a \in I$ we have $\rho_N^{1,\mathrm{GUE}_\omega}(s_j-\omega'\varepsilon/2N) \leq C'$. To see the required uniformity in ε , either adapt the arguments in Section 6 following (77) or use that $\rho_N^{1,\mathrm{GUE}_\omega}(s)$ is bounded uniformly in N and s, as can be seen from its determinantal representation and the well known asymptotics for the Hermite polynomials. This estimate together with (22) finishes the proof of Theorem 1.3. □

3. The associated random matrix ensemble. In this section, we start with the investigation of our main model. Let h be a continuous even function and Q a strictly convex symmetric function and assume that

$$P_{N,Q}^h(x) := \frac{1}{Z_{N,Q}^h}\prod_{1\leq i<j\leq N}|x_i-x_j|^2 e^{-N\sum_{j=1}^N Q(x_j)-\sum_{i<j}h(x_i-x_j)}, \tag{23}$$

defines the density of a probability measure on $\mathbb{R}^N$, where

$$Z_{N,Q}^h := \int_{\mathbb{R}^N}\prod_{1\leq i<j\leq N}|x_i-x_j|^2 e^{-N\sum_{j=1}^N Q(x_j)-\sum_{i<j}h(x_i-x_j)}\,dx$$

denotes the normalizing constant. This is, for example, the case if h is bounded below.

We will frequently use the notation

$$h_\mu(s) := \int h(t-s)\,d\mu(t), \qquad h_{\mu\mu} := \iint h(t-s)\,d\mu(t)\,d\mu(s) \tag{24}$$

for a compactly supported probability measure μ on $\mathbb{R}$. For the statement of the next lemma, $\mathcal{M}_c^1$ will denote the set of compactly supported (Borel) probability measures on $\mathbb{R}$.

LEMMA 3.1. *Let $h:\mathbb{R}\longrightarrow\mathbb{R}$ be even, twice differentiable, bounded and such that $h''(t)\geq -\alpha_Q$ for all t. Define $T_h:\mathcal{M}_c^1\longrightarrow\mathcal{M}_c^1$, $T_h(\mu)$ as the equilibrium measure to the external field $t\mapsto Q(t)+h_\mu(t)$.*

Then T_h has a fixed point, that is there exists a probability measure μ_Q^h which is the equilibrium measure to the external field $t \mapsto Q(t) + \int h(t-s)\, d\mu_Q^h(s)$.

PROOF. We will apply Schauder's fixed point theorem, which states that each continuous mapping $T : C \longrightarrow C$ of a compact, convex and nonempty subset C of a Hausdorff topological vector space has a fixed point.

We consider the topological vector space $\mathcal{M}(K)$ of all signed finite Borel measures on some compact interval K of $\mathbb{R}$, equipped with the topology of vague convergence. This topology is metrizable and hence the space is Hausdorff (see [28], Chapter 0). The subset $\mathcal{M}^1(K)$ of all Borel probability measures on K is nonempty, convex and compact. The compactness follows from Helly's Selection theorem. We will further restrict to measures μ which are symmetric around 0, that is, $\mu(A) = \mu(-A)$ for all Borel sets A. It is easy to see that this subset still fulfills the assumptions of Schauder's fixed point theorem.

Now we show that since $h''(t) \geq -\alpha_Q$ and h is bounded, the support of the equilibrium measure to the external field $Q(t) + h_\mu(t)$ is included in a compact set which can be chosen to be independent of μ. Indeed, by Theorem A.6, the support of the equilibrium measure for $Q(t) + h_\mu(t)$ is the smallest compact set K (w.r.t. inclusion) of positive capacity maximizing the functional

$$
\begin{aligned}
K \mapsto F_{Q+h_\mu}(K) &= \log \operatorname{cap}(K) - 2\int Q(t)\, d\omega_K(t) - 2\int h_\mu(t)\, d\omega_K(t) \\
&= F_Q(K) - 2\int h_\mu(t)\, d\omega_K(t),
\end{aligned}
\tag{25}
$$

in particular we have

$$
F_{Q+h_\mu}(\operatorname{supp}\mu_Q) \geq F_Q(\operatorname{supp}\mu_Q) - 2\|h\|_\infty \in \mathbb{R} \qquad \text{since } |h_\mu| \leq \|h\|_\infty.
\tag{26}
$$

As Q is convex and symmetric, $\operatorname{supp}\mu_Q$ is a symmetric interval; see Theorem A.6. Because h is twice differentiable, h' (and by assumption also h) are bounded on any compact set. Hence, if we choose a probability measure μ with compact support, h_μ is two times differentiable and $(h_\mu)'' = (h'')_\mu$. By the condition $h''(t) \geq -\alpha_Q$, $Q(t) + h_\mu(t)$ is convex for each compactly supported μ. Theorem A.6 implies that the support of the equilibrium measure to $Q(t) + h_\mu(t)$ is a symmetric interval, say $[-l_\mu, l_\mu]$. Using Lemma A.1, we can rewrite (25) for an arbitrary symmetric interval $[-l, l]$ as

$$
\begin{aligned}
F_{Q+h_\mu}([-l,l]) = \log(l/2) &- 2\int_{-l}^{l} Q(t)\frac{1}{\pi\sqrt{l^2 - t^2}}\, dt \\
&- 2\int_{-l}^{l} h_\mu(t)\frac{1}{\pi\sqrt{l^2-t^2}}\, dt.
\end{aligned}
\tag{27}
$$

Since Q is strictly convex and symmetric, we have $Q(t) \geq \alpha_Q t^2 + C$ for some $C \in \mathbb{R}$, and (27) implies (using that the variance of $\omega_{[-l,l]}$ is $l^2/2$) the inequality

$$F_{Q+h_\mu}([-l,l]) \leq \log(l/2) - \alpha_Q l^2 - C + 2\|h\|_\infty, \tag{28}$$

which holds for any μ. Comparing (26) and (28), we see that

$$F_{Q+h_\mu}(\operatorname{supp}\mu_Q) > F_{Q+h_\mu}([-l,l])$$

for all $l > L$, where $L > 0$ does not depend on μ. Hence such an $[-l,l]$ cannot be the support $[-l_\mu, l_\mu]$ of the equilibrium measure for $Q + h_\mu$. Hence $l_\mu \leq L$ for all compactly supported μ.

We have thus seen that T_h maps the set $\mathcal{M}_s^1(K)$ of symmetric probability measures supported in K into itself, if K is chosen large enough. It remains to show continuity of this map. Since we deal with a metric space, it is enough to show that by T_h, converging sequences are mapped to converging sequences. Let $(\mu_n)_n \subset \mathcal{M}^1(K)$ be a sequence converging vaguely, or equivalently, weakly to a probability measure μ. Denote $T_h(\mu_n) =: \nu_n$. Define the sequence of external fields $V_n(t) := Q(t) + h_{\mu_n}(t)$ which converges pointwise to $V(t) := Q(t) + h_\mu(t)$. We may assume that this convergence is uniform: by Theorem A.4, the equilibrium measure does not depend on values of the external field outside of its support (from which we know a priori that it lies in a certain compact set). Since h' is bounded on this compact set by some constant, say C, we also have $|h'_{\mu_n}| \leq C$. This implies that the sequence of functions $(h_{\mu_n})_n$ is uniformly Lipschitz and hence equicontinuous. It follows that the sequence $(V_n)_n$ is also equicontinuous. Since their domain is a compact and V_n converges pointwise, the equicontinuity implies uniform convergence by the Arzela–Ascoli theorem.

Since all ν_n are supported on the same compact set, it follows that $(\nu_n)_n$ is tight and hence has a weakly converging subsequence $(\nu_{n_m})_m$. We will prove that this limit measure, say ν', is in fact $\nu = T_h(\mu)$, the measure belonging to the external field V, and does not depend on the particular subsequence. It follows that the sequence $(\nu_n)_n$ converges to ν weakly as weak convergence is metrizable.

From the uniform convergence of V_n toward V, it follows by Theorem A.5(1) that

$$U^{\nu_{n_m}}(s) = \int \log|t-s|^{-1}\, d\nu_{n_m}(t)$$

converges uniformly (on $\mathbb{C}$) toward $U^\nu(s) := \int \log|t-s|^{-1}\, d\nu(t)$. On the other hand, by Theorem A.5(2) we have for almost all $s \in \mathbb{C}$

$$\lim_{m\to\infty} U^{\nu_{n_m}}(s) = U^{\nu'}(s) = \int \log|t-s|^{-1}\, d\nu'(t).$$

Hence $U^\nu(s) = U^{\nu'}(s)$ almost everywhere on $\mathbb{C}$. Theorem A.5(3) yields that $\nu = \nu'$, implying that the sequence $(\nu_n)_n$ converges weakly to ν. As T_h is a continuous mapping, Schauder's fixed point theorem yields the existence of a fixed point. □

REMARK 3.2 (Uniqueness). So far we did not prove that this fixed point of T_h is unique. Uniqueness will follow for the class of ensembles from Theorem 1.1. For those ensembles we will show that the first correlation measure converges weakly to any fixed point, which shows uniqueness.

We proceed by decomposing the additional interaction term. Let h be as in Lemma 3.1. Choose a fixed point μ_Q^h as in Lemma 3.1. We will stick to this measure from now on and write μ instead of μ_Q^h. We set using the notation (24)

$$\begin{aligned}\sum_{i<j} h(x_i - x_j) &= -\frac{N^2}{2} h_{\mu\mu} - \frac{N}{2} h(0) + N \sum_{j=1}^{N} h_\mu(x_j) \\ &\quad + \frac{1}{2}\left(\sum_{i,j=1}^{N} h(x_i - x_j) - \left[h_\mu(x_i) + h_\mu(x_j) - h_{\mu\mu}\right]\right) \\ &= -\frac{N^2}{2} h_{\mu\mu} - \frac{N}{2} h(0) + N \sum_{j=1}^{N} h_\mu(x_j) - \mathcal{U}(x),\end{aligned}$$

where

$$\mathcal{U}(x) := -\frac{1}{2}\left(\sum_{i,j=1}^{N} h(x_i - x_j) - \left[h_\mu(x_i) + h_\mu(x_j) - h_{\mu\mu}\right]\right). \tag{29}$$

Now we can rewrite $P_{N,Q}^h$ as

$$P_{N,Q}^h(x) = \frac{1}{Z_{N,V,\mathcal{U}}} \prod_{1 \le i < j \le N} |x_i - x_j|^2 e^{-N \sum_{j=1}^{N} V(x_j) + \mathcal{U}(x)}, \tag{30}$$

where we defined the external field

$$V(t) := Q(t) + h_\mu(t)$$

and absorbed the constant $\exp\{-(N^2/2)h_{\mu\mu} - (N/2)h(0)\}$ into the new normalizing constant $Z_{N,V,\mathcal{U}}$. We will from now on work with this representation of the density of $P_{N,Q}^h$. The proofs of Theorems 1.1 and 1.2 rely on comparison with the unitary invariant matrix ensemble

$$P_{N,V}(x) = \frac{1}{Z_{N,V}} \prod_{1 \le i < j \le N} |x_i - x_j|^2 e^{-N \sum_{j=1}^{N} V(x_j)}. \tag{31}$$

We will show that in the large N limit, the correlation measures in the global scaling as well as correlation functions in the local scaling, are the same for $P_{N,Q}^h$ and $P_{N,V}$. In this sense the quantity $\mathcal{U}$ will turn out to be negligible.

4. Concentration of measure inequalities. We will frequently use the following well-known concentration of measure inequality ([4], Section 4.4).

THEOREM 4.1. *Let Q be an external field on an interval $I = (a, b)$ (possibly unbounded) with $Q'' \geq c > 0$ on I. Then we have for any Lipschitz function f on I and any $\varepsilon > 0$*

$$P_{N,Q}\left(\left|\sum_{j=1}^{N} f(x_j) - \mathbb{E}_{N,Q}\sum_{j=1}^{N} f(x_j)\right| > \varepsilon\right) \leq 2\exp\left\{-\frac{c\varepsilon^2}{2|f|_{\mathcal{L}}2}\right\}$$

and

$$\mathbb{E}_{N,Q}\exp\left\{\varepsilon\left(\sum_{j=1}^{N} f(x_j) - \mathbb{E}_{N,Q}\sum_{j=1}^{N} f(x_j)\right)\right\} \leq \exp\left\{\frac{\varepsilon^2|f|_{\mathcal{L}}^2}{2c}\right\},$$

where for any Lipschitz function f we denote its Lipschitz constant by $|f|_{\mathcal{L}}$ (on I).

REMARK 4.2. In [4], only the case $(a, b) = \mathbb{R}$ is stated. As the proof for general (a, b) is completely analogous, we do not give it here.

Theorem 4.1 yields a concentration inequality for linear statistics around their expectations. However, we rather need concentration around their "limiting expectations." It is well known (see, e.g., [14], Theorem 2.1) that for bounded and continuous functions

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\sum_{j=1}^{N} f(x_j) = \int f(t)\, d\mu_Q(t), \tag{32}$$

where μ_Q denotes the equilibrium measure to Q. We need to quantify the rates of convergence in (32). The following is a special case of a result in [29]; see also [19].

PROPOSITION 4.3. *Let Q be a convex external field on $\mathbb{R}$ which is real analytic in a neighborhood of* $\operatorname{supp}(\mu_Q)$. *Let f be a function whose third derivative is bounded on a neighborhood of* $\operatorname{supp}(\mu_Q)$. *Then*

$$\left|\mathbb{E}_{N,Q}\sum_{j=1}^{N} f(x_j) - N\int f\, d\mu_Q\right| \leq C\big(\|f\|_\infty + \|f^{(3)}\|_\infty\big),$$

where C does not depend on N or f, and $\|\cdot\|_\infty$ denotes the bound on the neighborhood of $\operatorname{supp}(\mu_Q)$.

From Theorem 4.1 and Proposition 4.3 we immediately get the following concentration inequality.

COROLLARY 4.4. *Let Q be a real analytic external field with $Q'' \geq c > 0$. Then for any Lipschitz function f whose third derivative is bounded on a neigh-*

borhood of $\operatorname{supp}(\mu_Q)$, *we have for any* $\varepsilon > 0$

$$\mathbb{E}_{N,Q} \exp\left\{\varepsilon\left(\sum_{j=1}^{N} f(x_j) - N\int f(t)\,d\mu_Q(t)\right)\right\} \leq \exp\left\{\frac{\varepsilon^2 |f|_{\mathcal{L}}^2}{2c} + \varepsilon C\big(\|f\|_\infty + \|f^{(3)}\|_\infty\big)\right\}.$$

REMARK 4.5. Proposition 4.3 and Corollary 4.4 remain true up to an error of order e^{-cN} if we replace $\mathbb{R}$ by an interval I which covers the domain of the equilibrium measure μ_Q. It is well known (see, e.g., [6, 27]) that changing the external field outside a small neighborhood of the equilibrium measure results in a change of the first correlation function of order e^{-cN} for some $c > 0$. We will prove this in Lemma 6.3 provided that I is large enough.

The next lemma gives, using Fourier techniques, a representation of the bivariate statistic $\mathcal{U}$ in terms of certain linear statistics. A similar idea is used in [22].

LEMMA 4.6. *The following holds*:

$$\mathcal{U}(x) = -\frac{1}{2\sqrt{2\pi}} \int \big|\mathring{u}_N(t,x)\big|^2 \hat{h}(t)\,dt,$$

where

$$\mathring{u}_N(t,x) := \sum_{j=1}^{N} \cos(tx_j) - N\int \cos(ts)\,d\mu(s) + \sqrt{-1}\sum_{j=1}^{N} \sin(tx_j),$$

$$\hat{h}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-its} h(s)\,ds.$$

PROOF. Recall from (29) that

$$\mathcal{U}(x) = -\frac{1}{2}\left(\sum_{i,j=1}^{N} h(x_i - x_j) - \big[h_\mu(x_i) + h_\mu(x_j) - h_{\mu\mu}\big]\right).$$

Note that

$$\frac{1}{2}\sum_{j,k} h(x_j - x_k) = \frac{1}{2\sqrt{2\pi}} \int \sum_{j,k} e^{i(x_j - x_k)t} \hat{h}(t)\,dt = \frac{1}{2\sqrt{2\pi}} \int \big|u_N(t,x)\big|^2 \hat{h}(t)\,dt$$

with $u_N(t,x) := \sum_{j=1}^{N} e^{itx_j}$. Writing $\mathring{u}_N(t,x) := u_N(t,x) - N\int e^{its}\,d\mu(s)$, it is not hard to check that

$$\mathcal{U}(x) = -\frac{1}{2\sqrt{2\pi}} \int \big|\mathring{u}_N(t,x)\big|^2 \hat{h}(t)\,dt. \tag{33}$$

□

Note that we can write

$$\mathbb{E}^h_{N,Q} f(x) = (Z_{N,V}/Z_{N,V,\mathcal{U}})\mathbb{E}_{N,V} f(x)e^{\mathcal{U}(x)}.$$

With the help of representation (33), we shall bound this ratio of normalizing constants.

PROPOSITION 4.7. *If the constant α_Q is large enough, then there exist constants $C_1, C_2 > 0$ such that for all N*

$$0 < C_1 \le Z_{N,V,\mathcal{U}}/Z_{N,V} = \mathbb{E}_{N,V} \exp\{\mathcal{U}(x)\} \le C_2.$$

PROOF. We start with proving the lower bound. By Jensen's inequality we see

$$\mathbb{E}_{N,V} \exp\{\mathcal{U}(x)\} \ge \exp\{\mathbb{E}_{N,V}\mathcal{U}(x)\}.$$

Using Lemma 4.6 we show that the expectation of $\mathcal{U}$ is bounded in N. Fubini's theorem gives

$$\begin{aligned} -\mathbb{E}_{N,V}\mathcal{U}(x) &= \frac{1}{2\sqrt{2\pi}} \int \mathbb{E}_{N,V} |\mathring{u}_N(t,x)|^2 \hat{h}(t)\,dt \\ &= \frac{1}{2\sqrt{2\pi}} \int \Bigg(\mathbb{E}_{N,V} \Bigg|\sum_{j=1}^N \cos(tx_j) - N\int \cos(ts)\,d\mu(s)\Bigg|^2 \\ &\qquad + \mathbb{E}_{N,V}\Bigg|\sum_{j=1}^N \sin(tx_j)\Bigg|^2\Bigg)\hat{h}(t)\,dt. \end{aligned}$$

By Corollary 4.4, the terms in the parentheses are bounded by a polynomial function in t, as $|\cos(t\cdot)|_{\mathcal{L}}, |\sin(t\cdot)|_{\mathcal{L}} \le t$ and $\|\cos(t\cdot)^{(3)}\|_\infty, \|\sin(t\cdot)^{(3)}\|_\infty \le Ct^3$. Hence, $\hat{h}$ being a Schwartz function, we have $\mathbb{E}_{N,V}\mathcal{U}(x) \ge -C'$ for some $C' > 0$. Thus the lower bound follows choosing $C_1 := \exp(-C')$.

For the upper bound we will again use the representation of Lemma 4.6. Recall that since h is even, $\hat{h}$ is real-valued. Define $\hat{h}_+(y) := \max\{0, \hat{h}(y)\}$ and $\hat{h}_-(y) := \max\{0, -\hat{h}(y)\}$ such that $\hat{h} = \hat{h}_+ - \hat{h}_-$. For $\hat{h}_- = 0$, which corresponds to the case of a positive definite h, there is nothing to prove, so assume that $\hat{h}_- \neq 0$.

Introducing $H_- := (\hat{h}_-)^{1/2} \ge 0$, we obtain by Jensen's inequality and Tonelli's theorem

$$\begin{aligned} &\mathbb{E}_{N,V} \exp\bigg\{-(2\sqrt{2\pi})^{-1}\int \hat{h}(t)|\mathring{u}_N(t,x)|^2\,dt\bigg\} \\ &\quad\le \mathbb{E}_{N,V} \exp\bigg\{(2\sqrt{2\pi})^{-1}\int H_-(t)^2|\mathring{u}_N(t,x)|^2\,dt\bigg\} \\ &\quad= \mathbb{E}_{N,V} \exp\bigg\{(2\sqrt{2\pi})^{-1}\|H_-\|_{L^1}\int (H_-(t)/\|H_-\|_{L^1})H_-(t)|\mathring{u}_N(t,x)|^2\,dt\bigg\} \\ &\quad\le \int (H_-(t)/\|H_-\|_{L^1})\mathbb{E}_{N,V} \exp\{(2\sqrt{2\pi})^{-1}\|H_-\|_{L^1}H_-(t)|\mathring{u}_N(t,x)|^2\}\,dt. \end{aligned} \tag{34}$$

Abbreviating $K_h := (2\sqrt{2\pi})^{-1}\|H_-\|_{L^1}$ and using the Cauchy–Schwarz inequality and representation (33), we find

$$\mathbb{E}_{N,V}\exp\{K_h H_-(t)|\mathring{u}_N(t,x)|^2\} \tag{35}$$

$$\le \mathbb{E}_{N,V}^{1/2}\exp\left\{2K_h H_-(t)\left|\sum_{j=1}^N \cos(tx_j) - N\int \cos(ts)\,d\mu(s)\right|^2\right\} \tag{36}$$

$$\times\, \mathbb{E}_{N,V}^{1/2}\exp\left\{2K_h H_-(t)\left|\sum_{j=1}^N \sin(tx_j)\right|^2\right\}. \tag{37}$$

Since by Corollary 4.4 the distributions of $\sum_{j=1}^N \cos(tx_j) - N\int\cos(ts)\,d\mu(s)$ and $\sum_{j=1}^N \sin(tx_j)$ are sub-Gaussian, we obtain, for example, for the first term for any $\varepsilon > 0$,

$$\begin{aligned}&\mathbb{E}_{N,V}\exp\left\{\varepsilon\cdot\sqrt{2K_h H_-(t)}\left(\sum_{j=1}^N \cos(tx_j) - N\int\cos(ts)\,d\mu(s)\right)\right\}\\ &\quad\le \exp\{\varepsilon^2\cdot 2K_h H_-(t)t^2(2\alpha_V)^{-1} + \varepsilon\sqrt{2K_h H_-(t)}C(1+t^3)\},\end{aligned} \tag{38}$$

where $\alpha_V := \min_t V''(t) > 0$, C does not depend on t or N. For α_Q large enough (hence α_V large enough), we have $2K_h H_-(t)t^2(2\alpha_V)^{-1} < 1/4$ for all t. Since $H_-(t) = \hat{h}_-^{1/2}(t)$ is decaying rapidly, $\sqrt{2K_h H_-(t)}C(1+t^3)$ is bounded in t. Summarizing, if α_Q is large enough, we can bound (38) by

$$\exp\{c\varepsilon^2 + \varepsilon C\}$$

with $0 < c < 1/4$ and c, C do not depend on N or t. We conclude that (36) and (37) and hence (35) are bounded in N. Finally, since $\hat{h}$ is a Schwartz function, it follows from (34) that

$$\mathbb{E}_{N,V}\exp\left\{-\int \hat{h}(t)|\mathring{u}_N(t,x)|^2\,dt\right\} \le C$$

for some constant $C > 0$ independent of N. This proves the upper bound and hence the proposition. □

REMARK 4.8. The proof of Proposition 4.7 actually shows that for each $\lambda > 0$ there is a threshold $\alpha^h(\lambda) > 0$ and constants C_1, C_2 (depending on λ and α^h) such that

$$0 < C_1 < \mathbb{E}_{N,V}\exp\{\lambda\mathcal{U}(x)\} \le C_2 \qquad \text{if } \alpha_Q \ge \alpha^h(\lambda).$$

5. Bounding the first correlation function. This section deals with properties of the first correlation function. We give information on its decay and dependence on additional external fields of lower order.

First of all, we need to introduce some notation from [14]:

$$K_{N,Q}(x) := \sum_{1\le i\ne j\le N} k_Q(x_i, x_j), \tag{39}$$

$$k_Q(t,s) := \log|t-s|^{-1} + \frac{1}{2}Q(t) + \frac{1}{2}Q(s),$$

$$F_Q := I_Q(\mu), \qquad \psi_Q(t) := Q(t) - \log(t^2+1) \tag{40}$$

where $I_Q(\mu)$ is defined in (82).

From the simple inequality $|t-s| \le \sqrt{t^2+1}\sqrt{s^2+1}$ we conclude $\log|t-s|^{-1} \ge -\frac{1}{2}\log(t^2+1)(s^2+1)$ and hence

$$k_Q(t,s) \ge (1/2)\psi_Q(t) + (1/2)\psi_Q(s). \tag{41}$$

We also note that since Q is an external field, there is a constant c_Q such that

$$\psi_Q(t) \ge c_Q. \tag{42}$$

We define a generalized unitary invariant ensemble on $\mathbb{R}^N$ (or some compact $[a,b]^N$) via

$$P^M_{N,Q,f}(x) := \frac{1}{Z^M_{N,Q,f}} \prod_{1\le i<j\le N} |x_i - x_j|^2 e^{-M\sum_{j=1}^N Q(x_j) + \sum_{j=1}^N f(x_j)}, \tag{43}$$

where $N, M \in \mathbb{N}$ and f is a continuous function with $|f(t)| \le Q(t)$ for t large enough. Usually we have $M = N$ or $M = N-1$. If $M = N$, we will write $P_{N,Q,f}$ instead of $P^M_{N,Q,f}$. If $f = 0$, we write $P^M_{N,Q}$. The following result is due to Johansson.

PROPOSITION 5.1. *Let*

$$A_{N,\varepsilon} := \left\{x \in \mathbb{R}^N : \frac{1}{N^2} K_{N,Q}(x) \le F_Q + \varepsilon\right\}.$$

Then there is some constant C such that, if $\lim_{N\to\infty} N/M_N \to 1$,

$$P^{M_N}_{N,Q}(\mathbb{R}^N \setminus A_{N,\varepsilon+a}) \le Ce^{-aN^2} \qquad \textit{for all } N \ge N_0(\varepsilon) \textit{ and all } a \ge 0.$$

PROOF. See [14], Lemma 4.2. □

We now deal with the decay of $\rho^1_{N,Q}$. The following lemma can be found in several papers including [14, 27]. We follow [14].

LEMMA 5.2. *Let Q be a continuous function satisfying $Q(t) \geq (1+\delta)\log(1+t^2)$ for some $\delta > 0$ and all t large enough. Then there is a constant $C > 0$ such that for all t,*

$$\rho^1_{N,Q}(t) \leq e^{CN} e^{-N[Q(t) - \log(1+t^2)]}.$$

PROOF. We will from now on drop the subscript Q, defining

$$P_N^M(x) := \frac{1}{Z_N^M} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 e^{-M \sum_{j=1}^N Q(x_j)}$$

and abbreviating $\rho_N^1 := \rho^1_{N,Q}$, we compute

$$\rho_N^1(t) = \frac{Z_{N-1}^N}{Z_N^N} \mathbb{E}_N^{N-1}\left(\prod_{j=1}^{N-1}(x_j - t)^2\right) e^{-NQ(t)},$$
$$\frac{Z_N^N}{Z_{N-1}^N} = \mathbb{E}_{N-1}^N\left(\int e^{2\sum_{j=1}^{N-1}\log|x_j - t| - NQ(t)}\,dt\right). \tag{44}$$

Since adding a constant to Q does not change the ensemble, we will assume that $Q \geq 0$, which corresponds to considering the potential $Q + C_Q$, where C_Q denotes a lower bound of Q. Setting $Z := \int e^{-Q(t)}\,dt$ we get by Jensen's inequality

$$\begin{aligned} &Z \frac{1}{Z} \int \exp\left\{2 \sum_{j=1}^{N-1} \log|x_j - t| - NQ(t)\right\} dt \\ &\quad \geq Z \exp\left\{\frac{1}{Z} \int \left(2 \sum_{j=1}^{N-1} \log|x_j - t| - (N-1)Q(t)\right) e^{-Q(t)}\,dt\right\}. \end{aligned}$$

Since $Q \geq 0$, we get

$$\int \log|t - x_j| e^{-Q(t)}\,dt \geq \int_{x_j - 1}^{x_j + 1} \log|t - x_j|\,dt = -2.$$

Summarizing we see that

$$Z_N^N / Z_{N-1}^N \geq Z \exp\{-CN\} \qquad \text{for some constant } C > 0. \tag{45}$$

Using the inequality $(x_j - t)^2 \leq (1 + x_j^2)(1 + t^2)$ gives

$$\mathbb{E}_N^{N-1}\left(\prod_{j=1}^{N-1}(x_j - t)^2\right) \leq (1 + t^2)^N \mathbb{E}_N^{N-1}\left(\prod_{j=1}^{N-1}(1 + x_j^2)\right). \tag{46}$$

As before, we can assume (otherwise we add a constant) that Q satisfies $Q(t) \geq (1+\delta)\log(1+t^2)$ for all t and some $\delta > 0$. Using notation (39)–(40) and inequality (41), this condition yields

$$K_{N-1,Q}(x) \geq \delta(N-1) \sum_{j=1}^{N-1} \log(1 + x_j)^2.$$

Proposition 5.1 shows that for A large enough we have

$$P_{N-1,Q}^{N}\left(\sum_{j=1}^{N-1}\log(1+x_j)^2\geq AN\right)$$
$$\leq P_{N-1,Q}^{N}\big(K_{N-1,Q}(x)\geq\delta A(N-1)N\big)\leq e^{-cAN^2} \tag{47}$$

for some constant $c>0$. From this we conclude that for A large enough

$$\mathbb{E}_N^{N-1}\left(\prod_{j=1}^{N-1}(1+x_j^2)\right)\leq e^{AN}+\mathbb{E}_N^{N-1}\left(\prod_{j=1}^{N-1}(1+x_j^2)\mathbb{1}_{\prod_{j=1}^{N-1}(1+x_j^2)\geq e^{AN}}\right).$$

Equation (47) gives that

$$P_{N-1,Q}^{N}\left(\sum_{j=1}^{N-1}\log(1+x_j)^2-AN\geq|y|\right)\leq\exp\{-cAN^2-c|y|N\}.$$

From this bound it is easy to see that $\mathbb{E}_N^{N-1}(\prod_{j=1}^{N-1}(1+x_j^2)\mathbb{1}_{\prod_{j=1}^{N-1}(1+x_j^2)\geq\exp\{AN\}})$ is of order $\exp\{-CN^2\}$ for some $C>0$. Hence we have

$$\mathbb{E}_N^{N-1}\left(\prod_{j=1}^{N-1}(1+x_j^2)\right)\leq\exp\{cAN\}\qquad\text{for some } c. \tag{48}$$

In view of (44) we find combining (45), (46) and (48),

$$\rho_{N,Q}^1(t)\leq\exp\{CN\}\exp\{-N[Q(t)-\log(1+t^2)]\}.$$ □

From the previous lemma we easily deduce the following important corollary; cf. [11, 14, 27].

COROLLARY 5.3. *Let Q be as in Lemma 5.2. Then there are $L,C>0$ such that for all t with $t>L$, we have*

$$\rho_N^1(t)\leq\exp\{-CNQ(t)\}.$$

We finish the section with a useful bound on the first correlation function $\rho_{N,Q,f}^1$ of the unitary invariant ensemble $P_{N,Q,f}$; see (43).

LEMMA 5.4. *Let f be bounded. Then we have*

$$\rho_{N,Q,f}^1(t)\leq\rho_{N,Q}^1(t)e^{2\|f\|_\infty}.$$

PROOF. We use the identity

$$\rho_{N,Q,f}^1(t)=\frac{e^{-NQ+f}}{N\lambda_N(e^{-NQ+f},t)}, \tag{49}$$

where $\lambda_N(e^{-NQ+f}, \cdot)$ is the so-called Nth Christoffel function to the weight e^{-NQ+f} (see [32] for references and more information on Christoffel functions)

$$\lambda_N(W,t) := \inf_{P_{N-1}(t)=1} \int |P_{N-1}(s)|^2 W(s)\, ds, \tag{50}$$

where the infimum is taken over all polynomials P_{N-1} of at most degree $N-1$ with the property that $P_{N-1}(t) = 1$ and W denotes a weight function on $\mathbb{R}$. It is obvious from (50) that $\lambda_N(W_1, \cdot) \leq \lambda_N(W_2, \cdot)$ if $W_1 \leq W_2$. Then the lemma follows easily by $e^{-NQ-\|f\|_\infty} \leq e^{-NQ+f} \leq e^{-NQ+\|f\|_\infty}$. □

6. Proofs of Theorems 1.1 and 1.2. We first cite a general result by Levin and Lubinsky ([21], Theorem 1.1) about bulk universality for unitary invariant ensembles. Recall the definition of $\rho^k_{N,Q,f}$ following (43).

THEOREM 6.1. *Let Q be a continuous external field on the set $\Sigma \subset \mathbb{R}$, which is assumed to consist of at most finitely many intervals. Let f be a bounded continuous function on Σ. Let K_N denote the kernel*

$$K_N(t,s) = \sum_{j=0}^{N-1} \psi_j^{(\mathrm{N})}(t)\psi_j^{(\mathrm{N})}(s),$$

where $(\psi_j^{(\mathrm{N})})_j$ are the orthonormal functions to the weight $e^{-NQ(t)+f(t)}$. Let J be a closed interval lying inside the support of μ_Q. Assume that μ_Q is absolutely continuous in a neighborhood of J and that Q' and the density μ_Q are continuous in that neighborhood, while $\mu_Q > 0$ there. Then uniformly for $a \in J$ and t, s in compacts of the real line, we have

$$\lim_{N\to\infty} \frac{K_N(a + (t/(K_N(a,a))), a + (s/(K_N(a,a)))}{K_N(a,a)} = \frac{\sin(\pi(t-s))}{\pi(t-s)}. \tag{51}$$

We use a notion of bulk universality which slightly differs from (51); namely we scale by the limiting density μ_Q instead of using the N-particle density. The following obvious corollary is a translation of Theorem 6.1 into this setup.

COROLLARY 6.2. *Let Q, f and μ_Q be as in Theorem 6.1. Then bulk universality as defined in (2) holds for the unitary invariant ensemble $P_{N,Q,f}$.*

PROOF. The corollary follows from the well-known determinantal relations for unitary invariant ensembles, the local uniformness of the limit (51) in t, s and the fact that by [32], Theorem 1.2, we have uniformly in compact proper subsets of supp μ_Q

$$\lim_{N\to\infty} \frac{1}{N} K_N(a,a) = \lim_{N\to\infty} \rho^1_{N,Q,f}(a) = \mu_Q(a).$$ □

We will prove Theorems 1.1 and 1.2 together by comparing the correlation functions of the ensembles $P^h_{N,Q}$ [see (30)] and $P_{N,V}$; see (31). We start with $\rho^k_{N,V}$, the kth correlation function of $P_{N,V}$. We obtain $\rho^k_{N,V}(a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)})$ as k-marginal, integrating the density

$$P_{N,V}\left(a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)},x_{k+1},\dots,x_N\right)$$

over $x_{k+1},\dots,x_N$. We have k fixed eigenvalues at positions $a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)}$ and $N-k$ random eigenvalues. We first rewrite $\rho^k_{N,V}$ in terms of these $N-k$ random eigenvalues as follows:

$$\begin{aligned}
&\rho^k_{N,V}\left(a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)}\right)\\
&\quad=\int_{\mathbb{R}^{N-k}}\frac{1}{Z_{N,V}}\exp\left\{-N\sum_{j=k+1}^{N}V(x_j)+2\sum_{i<j;i,j>k}\log|x_j-x_i|\right\}\\
&\qquad\times\exp\left\{-N\sum_{j=1}^{k}V\left(a+\frac{t_j}{N\mu(a)}\right)+2\sum_{i<j;i,j\le k}\log\left|\frac{t_i-t_j}{N\mu(a)}\right|\right\}\qquad(52)\\
&\qquad\times\exp\left\{2\sum_{i\le k,j>k}\log\left|a+\frac{t_i}{N\mu(a)}-x_j\right|\right\}dx_{k+1}\cdots dx_N
\end{aligned}$$

$$=F(a,t)\frac{Z^N_{N-k,V}}{Z_{N,V}}\mathbb{E}^N_{N-k,V}\exp\left\{2\sum_{i\le k,j>k}\log\left|a+\frac{t_i}{N\mu(a)}-x_j\right|\right\},\tag{53}$$

where

$$F(a,t):=\exp\left\{-N\sum_{j=1}^{k}V\left(a+\frac{t_j}{N\mu(a)}\right)+2\sum_{i<j;i,j\le k}\log\left|\frac{t_i-t_j}{N\mu(a)}\right|\right\}\tag{54}$$

is the factor (52), which depends only on the fixed particles, and

$$P^N_{N-k,V}(x_{k+1},\dots,x_N):=\frac{1}{Z^N_{N-k,V}}\prod_{k+1\le i<j\le N}|x_i-x_j|^2e^{-N\sum_{j=k+1}^N V(x_j)}.$$

As before, the subscript $N-k$ indicates that $P^N_{N-k,V}$ is a probability measure in $N-k$ variables, whereas the superscript N indicates that the factor in front of the external field term $\sum_{j=k+1}^N V(x_j)$ of $P^N_{N-k,V}$ is N and not $N-k$. We keep the labeling $x_{k+1},\dots,x_N$. Setting

$$\begin{aligned}
&L^N_{N-k,V}(a,t,x)\\
&\quad:=2\sum_{i\le k,j>k}\log\left|a+\frac{t_i}{N\mu(a)}-x_j\right|+\log\left[F(a,t)\frac{Z^N_{N-k,V}}{Z_{N,V}}\right],
\end{aligned}\tag{55}$$

we get from (53) the equality

$$(56)\quad \rho^k_{N,V}\left(a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)}\right)=\mathbb{E}^N_{N-k,V}\exp\{L^N_{N-k,V}(a,t,x)\}.$$

Similar to (53), we see that the kth correlation function $\rho^{h,k}_{N,Q}$ of $P^h_{N,Q}$ at $a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)}$ can be written as

$$(57)\qquad \frac{1}{\mathbb{E}_{N,V}\exp\{\mathcal{U}(x)\}}\mathbb{E}^N_{N-k,V}\exp\{\mathcal{U}(t,x)+L^N_{N-k,V}(a,t,x)\},$$

where we abbreviated $\mathcal{U}(a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)},x_{k+1},\dots,x_N)$ by $\mathcal{U}(t,x)$.

In the following we shall abbreviate $(t_1,\dots,t_k,x_{k+1},\dots,x_N)$ by (t,x), and by $(t,x)_j$ we will denote the jth component of the vector (t,x). Furthermore, for the sake of brevity, we set

$$(58)\qquad R_a:=L^N_{N-k,V}(a,t,x)\quad\text{and}\quad R:=L^N_{N-k,V}\big(0,N\mu(0)t,x\big).$$

Note that R arises in the global scaling, whereas R_a appears in the local scaling. It will later turn out to be convenient that all the x_j's lie in a compact set. To this end we formulate the following truncation lemma. This procedure is well known for invariant ensembles; see, for instance, [14] or [9].

LEMMA 6.3. *For α_Q large enough, the following holds: for each k there are $L,C>0$ such that for all N and for all $t_1,\dots,t_k$*

$$\left|\rho^{h,k}_{N,Q}(t_1,\dots,t_k)-\frac{1}{\mathbb{E}_{N,V,L}\exp\{\mathcal{U}(x)\}}\mathbb{E}^N_{N-k,V,L}\exp\{\mathcal{U}(t,x)+R_L\}\right|\le e^{-CN},$$

where $\mathbb{E}^M_{N,V,L}$ denotes expectation w.r.t. the ensemble $P^M_{N,V,L}$ obtained by normalizing the ensemble $P^M_{N,V}$ restricted to $[-L,L]^N$ and R_L is the analog of R in which all integrations over $\mathbb{R}$ have been replaced by integrations over $[-L,L]$. Furthermore, for any external field Q on $\mathbb{R}$, the following holds: for each k there are $L',C>0$ such that for all N and all $t_1,\dots,t_k$

$$\left|\rho^k_{N,Q}(t_1,\dots,t_k)-\rho^k_{N,Q,L'}(t_1,\dots,t_k)\right|\le e^{-C'N},$$

where $\rho^k_{N,Q,L'}$ is the kth correlation function of the ensemble $P_{N,Q,L'}$ obtained by normalizing the ensemble $P_{N,Q}$ restricted to $[-L',L']^N$.

PROOF. We will use representation (57) and show that the restriction of integrals to $[-L,L]^N\subset\mathbb{R}^N$, respectively, $[-L,L]^{N-k}\subset\mathbb{R}^{N-k}$ results in an asymptotically negligible error. For $\mathbb{E}_{N,V}e^{\mathcal{U}}$ we use Hölder's inequality to estimate

$$\begin{aligned}&\mathbb{E}_{N,V}\big(\exp\{\mathcal{U}(x)\}\mathbb{1}_{([-L,L]^N)^c}(x)\big)\\&\qquad\le\big(\mathbb{E}_{N,V}\exp\{(1+\varepsilon)\mathcal{U}(x)\}\big)^{1/(1+\varepsilon)}\big(P_{N,V}\big(([-L,L]^N)^c\big)\big)^{1/\varepsilon'},\end{aligned}$$

where $1/(1+\varepsilon)+1/\varepsilon'=1$ and $\varepsilon>0$ is fixed. Now $\mathbb{E}_{N,V}e^{(1+\varepsilon)\mathcal{U}(x)}$ is uniformly bounded in N by Proposition 4.7 provided that α_Q is large enough. Furthermore, by Corollary 5.3 we get for the L defined there

$$P_{N,V}\big(([-L,L]^N)^c\big)\le N\int_{|t|>L}\rho^1_{N,V}(t)\,dt \tag{59}$$
$$\le N\int_{|t|>L}e^{-CNV(t)}\,dt\le e^{-C'N}$$

for some $C'>0$. In fact, C' can be chosen arbitrarily large by increasing L. We conclude that

$$\mathbb{E}_{N,V}\big(\exp\{\mathcal{U}(x)\}\mathbb{1}_{([-L,L]^N)^c}(x)\big)\le\exp\{-C''N\}$$

for some $C''>0$, if L is large enough. It follows by (59) as well that the exchange of the normalizing constants $Z_{N,V}$ and $Z^N_{N-k,V}$ by their counterparts $Z_{N,V,L}$, and $Z^N_{N-k,V,L}$ and hence also the exchange of R by R_L is asymptotically negligible.

In order to bound $\mathbb{E}^N_{N-k,V}(\exp\{\mathcal{U}(t,x)+R\}\mathbb{1}_{([-L,L]^N)^c}(x))$, first use Hölder's inequality as above. It remains to estimate $\mathbb{E}^N_{N-k,V}\exp\{(1+\varepsilon)\mathcal{U}(t,x)+(1+\varepsilon)R\}$ for some fixed $\varepsilon>0$. Again by Hölder's inequality we reduce this to bounding $\mathbb{E}^N_{N-k,V}\exp\{(1+\varepsilon')\mathcal{U}(t,x)\}$ and $\mathbb{E}^N_{N-k,V}\exp\{(1+\varepsilon'')R\}$ for some $\varepsilon',\varepsilon''>0$. Recall from (33) that

$$\mathcal{U}(x)=-\frac{1}{2\sqrt{2\pi}}\int\big|\mathring{u}_N(s,x)\big|^2\hat{h}(s)\,ds,$$

where

$$\mathring{u}_N(s,x)=\sum_{j=1}^N\cos(sx_j)-N\int\cos(st)\,d\mu(t)+\sqrt{-1}\sum_{j=1}^N\sin(sx_j).$$

For any a and any $t_1,\dots,t_k$ we get

$$\begin{aligned}\mathcal{U}(t,x)&\le\frac{1}{2\sqrt{2\pi}}\int\Bigg|\sum_{j=k+1}^N\cos(sx_j)-(N-k)\int\cos(su)\,d\mu(u)\\&\qquad+\sum_{j=1}^k\cos(st_j)-k\int\cos(su)\,d\mu(u)\Bigg|^2\hat{h}_-(s)\,ds\\&\quad+\frac{1}{2\sqrt{2\pi}}\int\Bigg|\sum_{j=k+1}^N\sin(sx_j)+\sum_{j=1}^k\sin(st_j)\Bigg|^2\hat{h}_-(s)\,ds\\&\le\frac{1}{\sqrt{2\pi}}\int\Bigg|\sum_{j=k+1}^N\cos(sx_j)-(N-k)\int\cos(su)\,d\mu(u)\Bigg|^2\hat{h}_-(s)\,ds\\&\quad+\frac{1}{\sqrt{2\pi}}\int\Bigg|\sum_{j=k+1}^N\sin(sx_j)\Bigg|^2\hat{h}_-(s)\,ds+\frac{5k^2}{\sqrt{2\pi}}\int\hat{h}_-(s)\,ds,\end{aligned}\tag{60}$$

where we used the inequalities $(a+b)^2 \le 2(a^2+b^2)$ and $|\cos|, |\sin| \le 1$. From this we conclude as in the proof of Proposition 4.7 that $\mathbb{E}^N_{N-k,V} \exp\{(1+\varepsilon')\mathcal{U}(t, x)\} \le C$ provided that α_Q is large enough (which does not depend on k), and C does not depend on $t_1, \dots, t_k$ or N. To see that Theorem 4.1 also applies for $P^N_{N-k,V}$ is obvious, and for Proposition 4.3 we use that $P^N_{N-k,V} = P^{N-k}_{N-k,V,f}$ with $f(t) := kV(t)$, and the notation introduced in (43). Proposition 4.3 is proved in [29] also for the case of $P_{N,Q,f}$ for real-analytic Q and f, hence it can be applied as in the proof of Proposition 4.7. We may now bound $\mathbb{E}^N_{N-k,V} \exp\{(1+\varepsilon'')R\}$ as in the arguments following (46). Recall that

$$R := 2 \sum_{i\le k, j>k} \log|t_i - x_j| + \log\left[F\bigl(0, N\mu(0)t\bigr)\frac{Z^N_{N-k,V}}{Z_{N,V}}\right],$$

where $F(a,t)$ was defined in (54). Using the same Jensen type trick as in the proof of Lemma 5.2, we find that $Z^N_{N-k,V}/Z_{N,V} \le \exp\{CkN\}$ for some C. As in (46) we get

$$\begin{aligned} &\mathbb{E}^N_{N-k,V} \exp\left\{(2+2\varepsilon'') \sum_{i\le k, j>k} \log|t_i - x_j|\right\} \\ &\quad \le \exp\left\{(N-k)\bigl(1+\varepsilon''\bigr)\sum_{i\le k}\log\bigl(1+t_i^2\bigr)\right\} \\ &\qquad \times \mathbb{E}^N_{N-k,V} \exp\left\{\bigl(1+\varepsilon''\bigr)\sum_{j>k}\log\bigl(1+x_j^2\bigr)\right\}. \end{aligned} \tag{61}$$

Analogously to (48) we conclude that $\mathbb{E}^N_{N-k,V} \exp\{(2+2\varepsilon'')\sum_{j>k}\log(1+x_j^2)\} \le \exp\{cN\}$ for some $c > 0$. Using (42), it is straightforward to bound

$$\begin{aligned} &\exp\left\{(N-k)\bigl(2+2\varepsilon''\bigr)\sum_{i\le k}\log\bigl(1+t_i^2\bigr) + \log\bigl[F\bigl(0, N\mu(0)t\bigr)Z^N_{N-k,V}/Z_{N,V}\bigr]\right\} \\ &\quad \le \exp\left\{-c_1 N \sum_{i=1}^k \bigl[V(t_i) - c_2\log\bigl(1+t_i^2\bigr)\bigr] + CkN\right\}, \end{aligned} \tag{62}$$

where c_1, c_2 are absolute positive constants. Since V is strictly convex, this yields

$$\mathbb{E}^N_{N-k,V} \exp\bigl\{\bigl(1+\varepsilon''\bigr)R\bigr\} \le e^{CN}$$

and hence

$$\mathbb{E}^N_{N-k,V} \exp\bigl\{(1+\varepsilon)\mathcal{U}(t,x) + (1+\varepsilon)R\bigr\} \le e^{C'N}$$

for some C, C'. From (59), we get that for L and N large enough

$$\mathbb{E}^N_{N-k,V}\bigl(\exp\{\mathcal{U}(t,x) + R\}\mathbb{1}_{([-L,L]^N)^c}(x)\bigr) \le e^{-C''N}$$

for some $C'' > 0$ and all $t_1, \dots, t_k$.

From (57), (60) and (61) we also obtain similarly as in Lemma 5.2

$$\rho_{N,Q}^{h,k}(t_1, \dots, t_k) \le \exp\left\{CN - c_1 N \sum_{i=1}^{k} [V(t_i) - c_2 \log(1 + t_i^2)]\right\}$$

for some positive C, c_1, c_2. As before, this implies that we can assume all $t_1, \dots, t_k$ to lie in some compact set.

The second assertion of the lemma follows analogously from (59), (62) and (61) with $\varepsilon'' = 0$. □

PROOF OF THEOREMS 1.1 AND 1.2. We first outline the main idea of the proof. Recall from (29) that

$$\mathcal{U}(x) = -(1/2)\left(\sum_{i,j=1}^{N} h(x_i - x_j) - [h_\mu(x_i) + h_\mu(x_j) - h_{\mu\mu}]\right).$$

Assume for a moment that $-h/2$ is positive semi-definite, or in other words, the covariance function of a centered stationary Gaussian process $(G_t)_{t\in[-L,L]}$, that is, $-h(t - s)/2 = \mathbb{E}(G_t G_s)$. We may linearize the bivariate statistic $-(1/2)\sum_{i,j=1}^{N} h(x_i - x_j)$ via

$$\exp\left\{-(1/2) \sum_{i,j=1}^{N} h(x_i - x_j)\right\} = \mathbb{E}\exp\left\{\sum_{j=1}^{N} G_{x_j}\right\},$$

where $\mathbb{E}$ denotes expectation w.r.t. the underlying probability measure. By definition we conclude that

$$\exp\{\mathcal{U}(x)\} = \mathbb{E}\exp\left\{\sum_{j=1}^{N} G_{x_j} - N \int G_.\, d\mu\right\}, \tag{63}$$

provided that $G_.$ is a.s. integrable w.r.t. μ. Since we would like to apply Corollary 4.4 to the linear statistic in (63), we need that $G_.$ is sufficiently smooth with probability one. To see this, we use the well-known Karhunen–Loève expansion of G. By a classical result due to Mercer, the covariance function h admits an expansion, converging uniformly on $[-L, L]$,

$$-h(t - s)/2 = \sum_{i=1}^{\infty} \lambda_i \theta_i(t)\overline{\theta_i(s)}, \tag{64}$$

where $(\theta_i)_i$ denotes an orthonormal system of eigenfunctions of the integral kernel h with real and positive eigenvalues $(\lambda_i)_i$, that is,

$$\int_{-L}^{L} -(1/2)h(t - s)\theta_i(s)\, ds = \lambda_i \theta_i(t) \qquad \forall i.$$

The Karhunen–Loève expansion of G is then given by

$$G_t = \sum_{i=1}^{\infty} \lambda_i^{1/2} \xi_i \theta_i(t), \tag{65}$$

where $(\xi_i)_i$, $\xi_i := (\lambda_i)^{-1/2} \int_{-L}^{L} \theta_i(t) G_t \, dt$, are independent standard normal variables. The convergence in (65) is a.s. uniform on the compact interval $[-L, L]$; see [3], Theorem 3.1.2. The a.s. continuity of G_t used for this theorem follows, for example, from the Kolmogorov–Chentsov theorem ([16], Theorem 3.23). Since h is analytic on some domain containing the compact set, say $A := [-L, L] \times [-\delta, \delta] \subset \mathbb{C}$, $\delta > 0$, its eigenfunctions (with nonzero eigenvalues) are analytic on A. Hence the uniform convergence in (65) implies that G_w, $w \in A$ is analytic with probability one. Furthermore, recall that the derivative process $(G_t')_{t \in [-L,L]}$ of G is a centered (real-valued) Gaussian process with covariance function $h''/2$; see, for example, [2], Theorem 2.2.2.

To summarize, if $-h$ is positive semi-definite, $\mathcal{U}$ admits the linearization (63) in terms of linear statistics with random test functions which fulfill the prerequisites of Corollary 4.4 if we restrict ourselves to a compact $[-L, L]$. In the following we sketch the main strategy in this case. Let $k \in \mathbb{N}$ be fixed. Eventually we will prove

$$\lim_{N \to \infty} \rho_{N,Q}^{h,k}\left(a + \frac{t_1}{N\mu(a)}, \ldots, a + \frac{t_k}{N\mu(a)}\right) - \mathbb{S}^k(t) = 0 \tag{66}$$

locally uniformly, where

$$\mathbb{S}^k(t) := \mu(a)^k \det\left[\frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)}\right]_{1 \le i,j \le k}.$$

By the boundedness of $\mathbb{E}_{N,V} e^{\mathcal{U}}$ (Proposition 4.7) and Lemma 6.3, (66) converges to zero if and only if

$$\mathbb{E}_{N,V,L} e^{\mathcal{U}} \rho_{N,Q}^{h,k}\left(a + \frac{t_1}{N\mu(a)}, \ldots, a + \frac{t_k}{N\mu(a)}\right) - \mathbb{E}_{N,V,L} e^{\mathcal{U}} \mathbb{S}^k(t)$$

tends to 0, where the $L > 0$ was introduced in Lemma 6.3. But this means, using (56), (57) and the abbreviation $R_{a,L}$, which denotes a version of R_a which is truncated to $[-L, L]$ [see (58)] and Lemma 6.3 that

$$\mathbb{E}_{N-k,V,L}^{N} \exp\{\mathcal{U}(t,x) + R_{a,L}\} - \mathbb{E}_{N,V,L} \exp\{\mathcal{U}\} \mathbb{S}^k(t) \to 0 \tag{67}$$

as $N \to \infty$. The linearization procedure then gives

$$\begin{aligned} &\mathbb{E}_{N-k,V,L}^{N} \exp\{\mathcal{U}(t,x) + R_{a,L}\} - \mathbb{E}_{N,V,L} \exp\{\mathcal{U}\} \mathbb{S}^k(t) \\ &\quad = \mathbb{E}\Bigg[\mathbb{E}_{N-k,V,L}^{N} \exp\Bigg\{\sum_{j=1}^{N} G_{(t,x)_j} + R_{a,L}\Bigg\} \\ &\qquad - \mathbb{E}_{N,V,L} \exp\Bigg\{\sum_{j=1}^{N} G_{x_j}\Bigg\} \mathbb{S}^k(t)\Bigg]. \end{aligned} \tag{68}$$

We find similarly as in (57) that

$$
\begin{aligned}
&\left(\mathbb{E}_{N,V,L}\exp\left\{\sum_{j=1}^{N}G_{x_j}\right\}\right)^{-1}\mathbb{E}_{N-k,V,L}^{N}\exp\left\{\sum_{j=1}^{N}G_{(t,x)_j}+R_{a,L}\right\}\\
&\qquad=\rho_{N,V,G.,L}^{k}\left(a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)}\right),
\end{aligned}
\tag{69}
$$

where $P_{N,V,G.,L}$ denotes the determinantal ensemble on $[-L,L]^N$ with external field $\exp\{-NV(t)+G_t\}$.

With representation (69), we can use the bulk universality of $P_{N,V,G.,L}$ to show convergence of

$$
\mathbb{E}_{N-k,V,L}^{N}\exp\left\{\sum_{j=1}^{N}G_{(t,x)_j}+R_{a,L}\right\}-\mathbb{E}_{N,V,L}\exp\left\{\sum_{j=1}^{N}G_{x_j}\right\}\mathbb{S}^k(t)
\tag{70}
$$

to 0 almost surely. To show that convergence to 0 also holds for the expectation, we will bound (70) in terms of $G_{\cdot}$. Here we can use that G is a Gaussian process and quantities like $\|G_{\cdot}\|_\infty$ and $\|G_{\cdot}'\|_\infty$ have sub-Gaussian tails.

We now turn to the detailed proof. As $-h$ is in general not positive semi-definite, we may extend the previous case by means of the following argument. Recall the decomposition of $\hat{h}$ into nonnegative functions $\hat{h}=(\hat{h})_+-(\hat{h})_-$. By setting $h^+:=\widehat{(\hat{h})_+}$, $h^-:=\widehat{(\hat{h})_-}$, we get a decomposition $h=h^+-h^-$ of h into positive semi-definite, real-analytic functions. Define for a complex parameter $z\in\mathbb{C}$

$$
\mathcal{U}_z(x):=\frac{z}{2}\left(\sum_{i,j=1}^{N}h^+(x_i-x_j)-\left[h_\mu^+(x_i)+h_\mu^+(x_j)-h_{\mu\mu}^+\right]\right)
\tag{71}
$$

$$
+\frac{1}{2}\left(\sum_{i,j=1}^{N}h^-(x_i-x_j)-\left[h_\mu^-(x_i)+h_\mu^-(x_j)-h_{\mu\mu}^-\right]\right).
\tag{72}
$$

Note that $\mathcal{U}_{-1}=\mathcal{U}$. Similar to (67), we have to show that for $z=-1$,

$$
\mathbb{E}_{N-k,V,L}^{N}\exp\{\mathcal{U}_z(t,x)+R_{a,L}\}-\mathbb{E}_{N,V,L}\exp\{\mathcal{U}_z\}\mathbb{S}^k(t)\to 0
$$

as $N\to\infty$. As the linearization procedure only works for nonnegative z, we shall use the following result, known as Vitali's convergence theorem, which can be found, for example, in [31].

THEOREM 6.4 (Vitali's convergence theorem). *Let $f_n(z)$ be a sequence of analytic functions on a region $D\subset\mathbb{C}$ with $|f_n(z)|\le M$ for all n and all $z\in D$. Assume that $\lim_{n\to\infty}f_n(z)$ exists for a set of z having a limit point in D. Then $\lim_{n\to\infty}f_n(z)$ exists for all z in the interior of D and the limit is an analytic function in z.*

We will apply Vitali's convergence theorem to the sequence (in N) of the following analytic functions of z:

$$W_{N,z}(a,t) := \mathbb{E}^N_{N-k,V,L} \exp\{\mathcal{U}_z(t,x) + R_{a,L}\} - \mathbb{E}_{N,V,L} \exp\{\mathcal{U}_z\}\mathbb{S}^k(t). \tag{73}$$

Introduce the domain $D := \{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R}, x < C(\alpha_Q)\}$, where $C(\alpha_Q) > 0$ is a sufficiently small constant such that the following quantity is bounded by some constant C:

$$\mathbb{E}_{N,V,L} \exp\{\mathcal{U}_{C(\alpha_Q)}\} \le C$$

(the existence of such constants follows from the proof of Proposition 4.7). First we shall show uniform boundedness of $W_{N,z}(a,t)$ for all N, a, t and $z \in D$. By the definition of $\mathcal{U}_z$ in (71) and the positivity of (72) and (71) for positive z (being variances of Gaussian random variables) it is clear that it suffices to bound $W_{N,z}(a,t)$ for real, positive z, since for negative real parts of z the boundedness of $W_{N,z}(a,t)$ is obvious. Hence we restrict ourselves to $0 \le z < C(\alpha_Q)$ only. Let G^+ and G^- denote two independent, centered and stationary Gaussian processes on a probability space $(\Omega, \mathcal{A}, P)$ indexed by $A := [-L, L] \times [-\varepsilon, \varepsilon] \subset \mathbb{C}$ with covariance functions $(z/2)h^+$ and $h^-/2$, respectively, where h^+ and h^- are analytic on A. Writing $G_t = G_t^+ - \int G_\cdot^+ \, d\mu + G_t^- - \int G_\cdot^- \, d\mu$ and denoting by $\mathbb{E}$ the expectation w.r.t. P, we can rewrite

$$\begin{aligned} &\mathbb{E}^N_{N-k,V,L} \exp\{\mathcal{U}_z(t,x) + R_{a,L}\} - \mathbb{E}_{N,V,L} \exp\{\mathcal{U}_z\}\mathbb{S}^k(t) \\ &\quad = \mathbb{E}\Bigg[\mathbb{E}^N_{N-k,V,L} \exp\Bigg\{\sum_{j=1}^N G_{(t,x)_j} + R_{a,L}\Bigg\} - \mathbb{E}_{N,V,L} \exp\Bigg\{\sum_{j=1}^N G_{x_j}\Bigg\}\mathbb{S}^k(t)\Bigg]. \end{aligned} \tag{74}$$

Similar to (69), we have

$$\begin{aligned} &\Bigg(\mathbb{E}_{N,V,L} \exp\Bigg\{\sum_{j=1}^N G_{x_j}\Bigg\}\Bigg)^{-1} \mathbb{E}^N_{N-k,V,L} \exp\Bigg\{\sum_{j=1}^N G_{(t,x)_j} + R_{a,L}\Bigg\} \\ &\quad = \rho^k_{N,V,G.,L}\bigg(a + \frac{t_1}{N\mu(a)}, \dots, a + \frac{t_k}{N\mu(a)}\bigg), \end{aligned} \tag{75}$$

where $P_{N,V,G.,L}$ denotes the determinantal ensemble on $[-L, L]^N$ with external field $\exp\{-NV(t) + G_t^+ + G_t^-\}$.

Fix compact sets $E \subset \mathbb{R}^k$ and $I \subset \operatorname{supp} \mu^\circ$. We have

$$\begin{aligned} \sup_{t \in E, a \in I} \Bigg| \mathbb{E}\Bigg[&\mathbb{E}^N_{N-k,V,L} \exp\Bigg\{\sum_{j=1}^N G_{(t,x)_j} + R_{a,L}\Bigg\} \\ &- \mathbb{E}_{N,V,L} \exp\Bigg\{\sum_{j=1}^N G_{x_j}\Bigg\}\mathbb{S}^k(t)\Bigg]\Bigg| \end{aligned} \tag{76}$$

$$\leq \mathbb{E} \sup_{t\in E, a\in I}\Bigg|\mathbb{E}^N_{N-k,V,L} \exp\Bigg\{\sum_{j=1}^N G_{(t,x)_j} + R_{a,L}\Bigg\} - \mathbb{E}_{N,V,L} \exp\Bigg\{\sum_{j=1}^N G_{x_j}\Bigg\}\mathbb{S}^k(t)\Bigg|.$$

Since (75) converges by Theorem 6.1 to $\mathbb{S}^k(t)$ locally uniformly and the term $\mathbb{E}_{N,V,L} \exp\{\sum_{j=1}^N G_{x_j}\}$ is bounded in N by Corollary 4.4 and bounded away from 0 by Proposition 4.3 and Lemma 6.3, we see that the term

$$\sup_{t\in E, a\in I}\Bigg|\mathbb{E}^N_{N-k,V,L} \exp\Bigg\{\sum_{j=1}^N G_{(t,x)_j} + R_{a,L}\Bigg\} - \mathbb{E}_{N,V,L} \exp\Bigg\{\sum_{j=1}^N G_{x_j}\Bigg\}\mathbb{S}^k(t)\Bigg| \tag{77}$$

converges to 0 a.s. w.r.t. P. To show convergence of (76) to 0, it remains to show that (77) is uniformly integrable w.r.t. P. We first consider the term $\mathbb{E}_{N,V,L} \exp\{\sum_{j=1}^N G_{x_j}\}$. In view of Corollary 4.4, we need to determine the distribution of the Lipschitz constant of $G^+ + G^-$ and of

$$\|G^+ + G^-\|_\infty + \|(G^+ + G^-)^{(3)}\|_\infty \tag{78}$$

on $[-L, L]$. The derivative processes $(G^+)'$ and $(G^-)'$ are Gaussian with covariance functions $-(z/2)(h^+)''$ and $-(h^-)''/2$, respectively. Furthermore, it is well known that $\sup_{t\in[-L,L]} |G_t^+|$ and $\sup_{t\in[-L,L]} |G_t^-|$ are sub-Gaussian with certain means and variances $-(z/2)(h^+)''(0)$ and $-(h^-)''(0)/2$, respectively. By the same argument, $\|G^+ + G^-\|_\infty$ and $\|(G^+ + G^-)^{(3)}\|_\infty$ are sub-Gaussian with certain means and the variances given in terms of derivatives of (h^+) and (h^-). For a reference, see, for example, [3], Theorem 2.1.1. From the sub-Gaussianity of these quantities and Corollary 4.4, it is easy to see that

$$\mathbb{E}_{N,V,L} \exp\Bigg\{\sum_{j=1}^N G_{x_j}\Bigg\}, \tag{79}$$

has a P-integrable dominating function, provided that α_Q (and hence α_V) is large enough. Note that the estimates above are uniform in z varying in a small interval. It remains to show that

$$\mathbb{E}^N_{N-k,V,L} \exp\Bigg\{\sum_{j=1}^N G_{(t,x)_j} + R_{a,L}\Bigg\} \tag{80}$$

is uniformly integrable and bounded in z for z varying in a small interval. To this end we use that (80) is equal to

$$\mathbb{E}_{N,V,L} \exp\Bigg\{\sum_{j=1}^N G_{x_j}\Bigg\}\rho^k_{N,V,G_\cdot,L}\bigg(a + \frac{t_1}{N\mu(a)}, \dots, a + \frac{t_k}{N\mu(a)}\bigg).$$

As in the proof of Theorem 1.3, we get

$$\rho^k_{N,V,G.,L}\left(a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)}\right)$$
$$\le C^k\prod_{j=1}^k \rho^1_{N,V,G.,L}\left(a+\frac{t_j}{N\mu(a)}\right),$$

where C is such that $C \ge N/(N-k)$. By Lemma 5.4 we have

$$\rho^1_{N,V,G.,L}\left(a+\frac{t_j}{N\mu(a)}\right) \le \rho^1_{N,V,L}\left(a+\frac{t_j}{N\mu(a)}\right)e^{2\|G.\|_\infty},$$

where $\|G.\|_\infty := \sup_{t\in[-L,L]}|G_t|$. Bulk universality for $k=1$ gives that $\rho^1_{N,V,L}(a+\frac{t_j}{N\mu(a)})$ converges (locally) uniformly toward the bounded function $\mu(a)$. We conclude that there is a constant $C>0$ such that for $t_1,\dots,t_k \in E$, $a \in I$ we have

$$\rho^k_{N,V,G.,L}\left(a+\frac{t_1}{N\mu(a)},\dots,a+\frac{t_k}{N\mu(a)}\right) \le Ce^{2k\|G.\|_\infty}.$$

As $\|G.\|_\infty$ is sub-Gaussian, we get in combination with (79) that (77) is uniformly integrable w.r.t. P, provided that α_Q is large enough. It is clear that this bound is uniform in $z \in [0,\varepsilon)$ for some small $\varepsilon > 0$.

To summarize, we have shown that (76) converges to 0 for (small) positive z, or in other terms, locally uniform convergence in a and t of $W_{N,z}(a,t)$ (for small positive z) as $N\to\infty$. We have also shown uniform boundedness of $W_{N,z}(a,t)$ for arbitrary N, a, t and $z \in (-\infty,\varepsilon)\times\mathbb{R}\subset\mathbb{C}$ and as locally uniform convergence implies pointwise convergence, we get by Vitali's convergence theorem that the sequence (in N) of functions $W_{N,z}(a,t)$ converges to 0 for $z=-1$ pointwise in a and t. To get locally uniform convergence in t and a for $z=-1$, recall that by Arzelà–Ascoli's theorem, a sequence of continuous functions on a compact set has a uniformly converging subsequence if and only if the sequence is uniformly bounded and equicontinuous. Thus it remains to show that $(W_{N,z}(a,t))_N$ is equicontinuous in a and t (boundedness has already been shown). As the convergence of $W_{N,z}(a,t)$ is uniform in a, t for small positive z, Arzelà–Ascoli's theorem implies equicontinuity (in a,t) of $(W_{N,z}(a,t))_N$ for small positive z. To see that this implies equicontinuity (in a,t) of $(W_{N,z}(a,t))_N$ also for $z=-1$, observe that a (real-valued) sequence of functions $(f_N)_N$ on some compact $K\subset\mathbb{R}^d$ is equicontinuous in $x\in K$ if and only if for each sequence $(x_m)_m\subset K$, $\lim_{m\to\infty}x_m = x$ and each sequence $(N_m)_m\subset\mathbb{N}$ we have $\lim_{m\to\infty} f_{N_m}(x_m) - f_{N_m}(x) = 0$. Using this characterisation, equicontinuity for $z=-1$ is easily seen by applying Vitali's convergence theorem to deduce $\lim_{m\to\infty} W_{N_m,-1}(a_m,t_m)=0$ from $\lim_{m\to\infty} W_{N_m,z}(a_m,t_m)=0$ for small positive z. This completes the proof of Theorem 1.2.

To prove Theorem 1.1, take $g:\mathbb{R}^k \longrightarrow \mathbb{R}$ bounded and continuous. With the same arguments as above, we arrive in analogy to (74)–(75) at proving

$$\mathbb{E}\Biggl[\mathbb{E}_{N,V,L}\exp\Biggl\{\sum_{j=1}^N G_{x_j}\Biggr\}\int_{\mathbb{R}^k}\rho_{N,V,G.,L}^k(t_1,\dots,t_k)g(t_1,\dots,t_k)\,dt_1\cdots dt_k$$
$$-\mathbb{E}_{N,V,L}\exp\Biggl\{\sum_{j=1}^N G_{x_j}\Biggr\}\int_{\mathbb{R}^k}g(t_1,\dots,t_k)\mu(t_1)\cdots\mu(t_k)\,dt_1\cdots dt_k\Biggr]\to 0.$$

All the boundedness and integrability arguments above for $\mathbb{E}_{N,V,L}\exp\{\sum_{j=1}^N G_{x_j}\}$ can be used again. The convergence of $\int_{\mathbb{R}^k}\rho_{N,V,G.}^k(t)g(t)\,dt$ toward $\int g(t)\mu(t_1)\cdots\mu(t_k)\,dt$ is given by [14], Theorem 2.1. Lemma 6.3 enables us to transfer Johansson's result to the correlation function $\rho_{N,V,G.,L}^k$. This finishes the proof of Theorem 1.1. □

APPENDIX: EQUILIBRIUM MEASURES WITH EXTERNAL FIELDS

In this appendix, we recall some results about equilibrium measures, mainly from the book by Saff and Totik [28], Section I.1. The following can be found in [28], Section I.1.

Let $\mathcal{M}^1(\Sigma)$ denote the set of Borel probability measures on a set Σ. Define for $\Sigma\subset\mathbb{C}$ compact the *logarithmic energy of* $\mu\in\mathcal{M}^1(\Sigma)$ as

$$I(\mu) := \iint \log|z-t|^{-1}\,d\mu(z)\,d\mu(t) \tag{81}$$

and the *energy* V *of* Σ by $V:=\inf_{\mu\in\mathcal{M}^1(\Sigma)} I(\mu)$. It turns out that V is finite or ∞ and in the finite case there is a unique measure ω_Σ which minimizes (81). This measure ω_Σ is called *equilibrium measure of* Σ and the quantity $\operatorname{cap}(\Sigma):=e^{-V}$ is called *capacity of* Σ. For an arbitrary Borel set Σ, we define the capacity of Σ as

$$\operatorname{cap}(\Sigma) := \sup\{\operatorname{cap}(K): K\subset\Sigma \text{ compact}\}.$$

LEMMA A.1. *If* $\Sigma=[-l,l]$, $l\ge 0$, *then* $\operatorname{cap}(\Sigma)=l/2$ *and the equilibrium measure is the arcsine distribution with support* $[-l,l]$,

$$d\omega_\Sigma(t) = \frac{1}{\pi\sqrt{l^2-t^2}}\,dt, \qquad t\in[-l,l].$$

ω_Σ *has mean* 0 *and variance* $l^2/2$.

PROOF. See [28], Section I.1. □

DEFINITION A.2. Let $\Sigma\subset\mathbb{R}$ be closed. Let $Q:\Sigma\longrightarrow[0,\infty]$ satisfy:

(a) Q is lower semicontinuous;

(b) $\Sigma_0 := \{t \in \Sigma : Q(t) < \infty\}$ has positive capacity;
(c) if Σ is unbounded, then $\lim_{|t|\to\infty, t\in\Sigma} Q(t) - \log|t| = \infty$.

If Q satisfies these properties, we call it *external field* on Σ and $W = e^{-Q}$ its corresponding *weight function*.

Furthermore, define for $\mu \in \mathcal{M}^1(\Sigma)$ the *energy functional*

$$I_Q(\mu) := \int Q(t)\, d\mu(t) + \iint \log|s-t|^{-1}\, d\mu(s)\, d\mu(t). \tag{82}$$

REMARK A.3. In [28] the authors define the energy functional to be (in our notation) I_{2Q} instead of I_Q. It is more convenient for our purposes to use this definition. We note that under this change qualitative results from [28] remain the same but quantitative results involving Q have to be changed by a factor 2 or 1/2, respectively.

$I_Q(\mu)$ might be ∞, but the following theorem holds. The support of a measure μ will be denoted as $\operatorname{supp}(\mu)$.

THEOREM A.4. *Let Q be an external field on Σ.*

(a) *There is a unique probability measure $\mu_Q \in \mathcal{M}^1(\Sigma)$ with*

$$I_Q(\mu_Q) = \inf_{\mu \in \mathcal{M}^1(\Sigma)} I_Q(\mu). \tag{83}$$

(b) *μ_Q has a compact support.*
(c) *Let $\widetilde{Q}$ be an external field on Σ such that $\widetilde{Q} = Q$ on a compact set K with $\operatorname{supp}(\mu_Q) \subset K$ and $\widetilde{Q}(t) = \infty$ for $t \notin K$. Then $\mu_{\widetilde{Q}} = \mu_Q$.*

PROOF. Statements (a) and (b) can be found in [28], Theorem I.1.3, (c) follows from [28], Theorem I.3.3 (also see the remark on page 48 in [28]). □

μ_Q is called the *equilibrium measure* for Q. The next theorem summarizes properties of the *logarithmic potential*

$$U^\mu(z) := \int \log|z-t|^{-1}\, d\mu(t).$$

THEOREM A.5. (a) *Let Q and $\widetilde{Q}$ be external fields on Σ such that $|Q - \widetilde{Q}| \le \varepsilon$ on Σ. Then for all $z \in \mathbb{C}$,*

$$\left|U^{\mu_Q}(z) - U^{\mu_{\widetilde{Q}}}(z)\right| \le 2\varepsilon.$$

(b) *Let $K \subset \mathbb{R}$ be compact and $(\mu_n)_n$ be a sequence in $\mathcal{M}^1(K)$ converging weakly to a probability measure μ. Then for a.e. $z \in \mathbb{C}$ (w.r.t. the Lebesgue measure on $\mathbb{C}$),*

$$\liminf_{n\to\infty} U^{\mu_n}(z) = U^\mu(z).$$

(c) *If* μ *and* ν *are two compactly supported probability measures and their logarithmic potentials* U^{μ} *and* U^{ν} *coincide almost everywhere on* $\mathbb{C}$, *then* $\mu = \nu$.

PROOF. Statement (a) is contained in [28], Corollary I.4.2, statement (b) is [28], Theorem I.6.9, and assertion (c) is [28], Corollary II.2.2. □

THEOREM A.6. *Let* Q *be an external field on* Σ.

(a) *For a compact set* K *of positive capacity*, *define the functional*

$$F_Q(K) := \log \operatorname{cap}(K) - 2 \int Q \, d\omega_K.$$

For any compact K *of positive capacity*, *we have* $F_Q(K) \leq F_Q(\operatorname{supp}(\mu_Q))$. *Furthermore*, *if* K *is compact and of positive capacity and such that* $F_Q(K) = F_Q(\operatorname{supp}(\mu_Q))$, *then* $\operatorname{supp}(\mu_Q) \subset K$.

(b) *If* Q *is convex*, *then* $\operatorname{supp}(\mu_Q)$ *is an interval*.

(c) *If* Q *is even*, *then* $\operatorname{supp}(\mu_Q)$ *is even*.

PROOF. For statement (a), see [28], Theorem IV.1.5, for statements (b) and (c), see [28], Theorem IV.1.10. □

THEOREM A.7. (a) *Let* Q *be an external field on* Σ. *If* Q *is finite on* $\operatorname{supp}(\mu_Q)$ *and locally of class* $C^{1+\varepsilon}$ *for some* $\varepsilon > 0$ (*which means that* Q *is continuously differentiable and the derivative* Q' *is Hölder continuous with parameter* ε), *then* μ_Q *has a continuous density on the interior of* $\operatorname{supp}(\mu_Q)$.

(b) *If* Q *has two Lipschitz derivatives and is strictly convex*, *then* $\operatorname{supp}(\mu_Q) =: [a, b]$ *and the density of* μ_Q *can be represented as*

$$\frac{d\mu(t)}{dt} = r(t)\sqrt{(t-a)(b-t)}\mathbb{1}_{[a,b]}(t), \tag{84}$$

where r *can be extended into an analytic function on a domain containing* $[a, b]$ *and* $r(t) > 0$ *for* $t \in [a, b]$. *In particular*, *the density is positive on* (a, b).

PROOF. Statement (a) is [28], Theorem IV.2.5, and for assertion (b), see, for example, the appendix of the paper by McLaughlin and Miller [23]. □

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