

The cut-tree of large recursive trees

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Abstract. Imagine a graph which is progressively destroyed by cutting its edges one after the other in a uniform random order. The so-called cut-tree records key steps of this destruction process. It can be viewed as a random metric space equipped with a natural probability mass. In this work, we show that the cut-tree of a random recursive tree of size n , rescaled by the factor $n^{-1} \ln n$, converges in probability as $n \rightarrow \infty$ in the sense of Gromov–Hausdorff–Prokhorov, to the unit interval endowed with the usual distance and Lebesgue measure. This enables us to explain and extend some recent results of Kuba and Panholzer (Multiple isolation of nodes in recursive trees (2013) Preprint) on multiple isolation of nodes in large random recursive trees.

Résumé. Imaginons la destruction progressive d'un graphe auquel on retire ses arêtes une à une dans un ordre aléatoire uniforme. Le "cut-tree" permet de coder les étapes essentielles du processus de destruction; il peut être vu comme un espace métrique aléatoire muni d'une mesure de probabilité naturelle. Dans cet article, nous montrons que le cut-tree d'un arbre récursif aléatoire de taille n , et renormalisé par un facteur $n^{-1} \ln n$, converge en probabilité quand $n \rightarrow \infty$ au sens de Gromov–Hausdorff–Prokhorov, vers l'intervalle unité muni de la distance usuelle et de la mesure de Lebesgue. Ceci nous permet d'expliquer et d'étendre des résultats récents de Kuba and Panholzer (Multiple isolation of nodes in recursive trees (2013) Preprint) sur l'isolation multiple de sommets dans un grand arbre récursif aléatoire.

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1. Introduction and main statements

Imagine that we destroy some connected graph $G = (V, E)$ by cutting its edges one after the other and uniformly at random. Meir and Moon initiated the study of the number of steps required to isolate a distinguished vertex, if at each time when a cut induces a further disconnection, the connected component which does not contain the distinguished vertex is discarded forever (in other words, one only takes into account the cuts occurring in the connected component of the distinguished vertex). More precisely, Meir and Moon estimated the first and second moments of this quantity in the cases when G is a Cayley tree [18] and a recursive tree [19]. In the last 10 years or so, several weak limit theorems for the number of cuts have been obtained for Cayley trees (Panholzer [20,21]), complete binary trees (Janson [13]), simply generated trees (Janson [14]), recursive trees (Drmotá et al. [5], Iksanov and Möhle [12]), binary search trees (Holmgren [10]) and split trees (Holmgren [11]).

More recently, some authors have considered a more general version of this problem in which one is now interested in the number of cuts needed to isolate $\ell \geq 2$ distinguished vertices, again discarding the connected components which contain no distinguished points as soon as they are created. See Bertoin [2] and Addario-Berry et al. [1] for Cayley trees, Bertoin and Miermont [4] for simply generated trees, and Kuba and Panholzer [16] for recursive trees. More precisely, the approach of [2] and [4] relies on the study of the so-called *cut-tree* (which will be defined below) whereas [16] uses moment calculations. In short, the present work explains and extends some results of Kuba and Panholzer by describing a limit theorem for the cut-tree of large recursive trees.

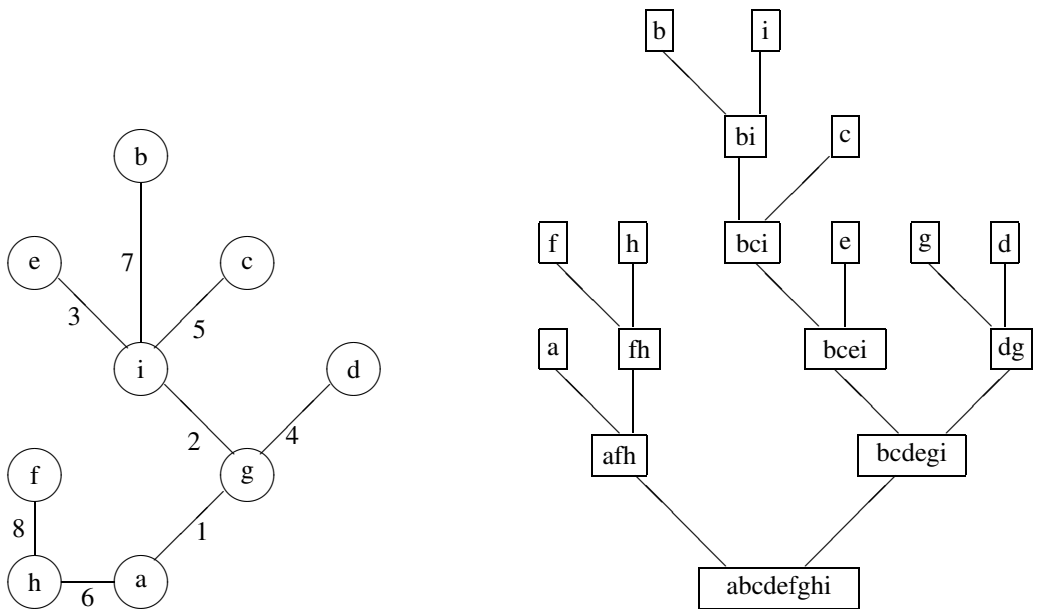


Fig. 1. Left: Tree T with vertices labelled a, \dots, i ; edges are enumerated in order of the cuts. Right: Cut-tree $\text{Cut}(T)$ on the set of blocks recording the destruction of T .

The cut-tree $\text{Cut}(G)$ is a random binary rooted tree¹ which records key informations about the destruction of the graph G ; in particular its nodes correspond to the blocks, i.e., connected components of V , which appear during the destruction process. Specifically, $\text{Cut}(G)$ is rooted at the block V , and its leaves (which correspond to singleton blocks) can be identified with the vertices in V . The basic structure is that each time a block B is split into two sub-blocks B' and B'' (because a pivotal edge of B is cut), then we think of B' and B'' as the two children of B . Figure 1 above should provide a useful illustration of this definition.

Cut-trees can be especially useful when the graph G is itself a tree, a case on which we shall now focus, as then each cut of an edge induces the split of some block. So assume henceforth that $G = T$ is a tree; it should be clear that the number of cuts required to isolate a given vertex v in the destruction of T corresponds precisely to the height of the leaf $\{v\}$ in $\text{Cut}(T)$. More generally, the number of cuts required to isolate k vertices v_1, \dots, v_k coincides with the total length of the cut-tree reduced to its root and the k leaves $\{v_1\}, \dots, \{v_k\}$, where the length is measured as usual by the graph distance on $\text{Cut}(T)$.

We now introduce the family of (random) trees which we are interested in. Recall that a tree T on a totally ordered set of n vertices, say $[n] = \{1, \dots, n\}$, is called increasing when the sequence of vertices along any segment started from 1 increases. There are $(n - 1)!$ increasing trees on $[n]$, and a random recursive tree of size n , T_n , is an increasing tree on $[n]$ picked uniformly at random. A recent result due to Kuba and Panholzer (Theorem 3 in [16]) shows that for every fixed $\ell \geq 1$, if given T_n , we select ℓ vertices of T_n uniformly at random and independently of the destruction process, then the number of cuts needed to isolate these ℓ vertices, normalized by a factor $n^{-1} \ln n$, converges in distribution as $n \rightarrow \infty$ towards a beta variable with parameter ℓ and 1. The motivation of the present work is to point out that this result can be viewed as a consequence of a more general limit theorem for the cut-tree $\text{Cut}(T_n)$.

To give a formal statement, we consider the set of pointed metric spaces equipped with a probability measure, and its equivalence classes induced by measure-preserving isometries. It is well-known that this yields a Polish space \mathbb{M}

¹We stress that [4] uses a slight variation of the present definition.

when equipped the pointed Gromov–Hausdorff–Prokhorov distance d_{GHP}^* . We refer to, e.g., Gromov [8], Greven et al. [7], Haas and Miermont [9] and references therein for background.

We denote by I the element of \mathbb{M} corresponding to the unit interval $[0, 1]$, pointed at 0 and equipped with the usual distance and the Lebesgue measure. It is convenient to agree that, if X is a pointed metric measured space and $a > 0$, then aX denotes the same space endowed with the same measure, but with a distance rescaled by the factor a . Using naturally the graph distance on $\text{Cut}(T)$ and its root V as a distinguished point, and further endowing $\text{Cut}(T)$ with the uniform probability measure on its set of leaves, we view of $\text{Cut}(T)$ as a random variable with values in \mathbb{M} . The main object of interest in the present paper is the sequence of random variables $\text{Cut}(T_n)$ in \mathbb{M} , with T_n a random recursive tree of size n .

It is interesting to recall from [2] that when τ_n is a Cayley tree of size n , then $n^{-1/2} \text{Cut}(\tau_n)$ converges in distribution to a Brownian Continuum Random Tree, in the sense of Gromov–Hausdorff–Prokhorov. This has been extended in [4] to a large family of simply generated trees, except that the convergence is then only established in the sense of Gromov–Prokhorov. We also mention that Haas and Miermont [9] have obtained deep limit theorems for a large class of (rescaled) Markov branching trees. Even though, thanks to the splitting property, $\text{Cut}(T_n)$ is a Markov branching tree, the results of Haas and Miermont do not apply to the present case, cf. the discussion in the last section of [3].

Theorem 1. *As $n \rightarrow \infty$, the sequence $n^{-1} \ln n \text{Cut}(T_n)$ converges in probability to I , in the sense of the pointed Gromov–Hausdorff–Prokhorov distance on \mathbb{M} .*

Remark. *An informal version of Theorem 1 was alluded to in the last paragraph of [3]; more precisely it was written there: “It is easy to deduce from the approach developed in the present work that if we rescale the edge-lengths of $\text{Cut}(T_n)$ by a factor $n^{-1} \ln n$, then the sequence of rescaled random trees converges in probability to a degenerate deterministic real tree which can be identified as the unit interval $[0, 1]$. Details are left to the interested reader.” A couple of years later, it seems to the author that, despite of this rather blunt claim, providing a rigorous proof may nonetheless have some interest as the arguments are not entirely straightforward.*

We now present the consequence of Theorem 1 to the number of cuts needed to isolate a fixed number of distinguished vertices, which has motivated the present work. For a fixed integer $\ell \geq 1$ and for each integer n , let $u_1^{(n)}, \dots, u_\ell^{(n)}$ denote a sequence of i.i.d. uniform variables in $[n] = \{1, \dots, n\}$. We write $Y_{n,\ell}$ for the number of random cuts which are needed to isolate $u_1^{(n)}, \dots, u_\ell^{(n)}$. The following corollary is a multi-dimensional extension of Theorem 3 of Kuba and Panholzer [16].

Corollary 1. *As $n \rightarrow \infty$, the random vector*

$$\left(\frac{\ln n}{n} Y_{n,1}, \dots, \frac{\ln n}{n} Y_{n,\ell} \right)$$

converges in distribution to

$$(u_1, \max(u_1, u_2), \dots, \max(u_1, \dots, u_\ell)),$$

where u_1, \dots, u_ℓ are i.i.d. random variables with the uniform distribution on $[0, 1]$. In particular, $\frac{\ln n}{n} Y_{n,\ell}$ converges in distribution to a beta variable with parameters ℓ and 1.

Much in the same vein, Theorem 2 of Kuba and Panholzer shows that if $Z_{n,\ell}$ denotes the number of random cuts which are needed to isolate the ℓ last vertices of T_n , viz. $n - \ell + 1, \dots, n$, then $\frac{\ln n}{n} Z_{n,\ell}$ converges in distribution to a beta variable with parameters ℓ and 1. We claim the following multi-dimensional extension.

Corollary 2. *As $n \rightarrow \infty$, the random vector*

$$\left(\frac{\ln n}{n} Z_{n,1}, \dots, \frac{\ln n}{n} Z_{n,\ell} \right)$$

converges in distribution to

$$(u_1, \max(u_1, u_2), \dots, \max(u_1, \dots, u_\ell)),$$

where u_1, \dots, u_ℓ are i.i.d. random variables with the uniform distribution on $[0, 1]$.

The rest of this work is organized as follows. In Section 2, as a preparatory step, we shall describe precisely the decomposition $\text{Cut}(T_n)$ into its *trunk* and its *branches*, which may be viewed as the analog of the celebrated backbone decomposition for Galton–Watson trees; see Lyons et al. [17]. Our guiding line is similar to that in [3], and relies crucially on a coupling due to Iksanov and Möhle [12] that connects the destruction of random recursive trees with a remarkable random walk. Then Theorem 1 and Corollaries 1 and 2 will be established in Section 3. For the sake of clarity, we will consider the metric and the measure aspects separately. Roughly speaking, the key point is to prove that, from the point of view of metric spaces, the branches are small compared to the trunk when $n \rightarrow \infty$.

2. Cut-tree, its trunk and its branches

2.1. The trunk

We start by considering the segment of $\text{Cut}(T_n)$ from the root $[n]$ to the leaf $\{1\}$ (recall that T_n is naturally rooted at 1). This segment is given by a nested sequence of blocks

$$B_{n,0} := [n] \supset B_{n,1} \supset \dots \supset B_{n,\zeta(n)} = \{1\},$$

where $\zeta(n)$ is height of $\{1\}$ in $\text{Cut}(T_n)$, or equivalently the number of cuts which are needed to isolate the vertex 1 in the destruction of T_n . At the heart of our argument lies the fact that the statistics of the block-sizes along a main portion of that segment have a simple description in terms of a remarkable random walk.

In this direction, introduce first an integer-valued random variable ξ with distribution

$$\mathbb{P}(\xi = k) = \frac{1}{k(k+1)}, \quad k = 1, 2, \dots,$$

and then a random walk

$$S_j = \xi_1 + \dots + \xi_j, \quad j = 1, 2, \dots,$$

where the ξ_i are i.i.d. copies of ξ . Introduce also the last passage time

$$L(n) = \max\{j \geq 1: S_j < n\}.$$

We shall need the following elementary features.

Lemma 1. (i) *We have*

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} L(n) = 1 \quad \text{in probability.}$$

(ii) *Further, the random point measure*

$$\sum_{1 \leq j \leq L(n)} \delta_{(\ln n/n)\xi_j}(\mathrm{d}x)$$

converges in distribution on the space of locally finite measures on $(0, \infty]$ endowed with the vague topology towards to a Poisson random measure with intensity $x^{-2} \mathrm{d}x$.

(iii) We have also

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} (n - S_{L(n)}) = 0 \quad \text{in probability.}$$

Proof. The first claim derives immediately from Proposition 2 in [12], which provides a finer limit theorem for $L(n) + 1$, the first passage time of the random walk S above level n . The second follows then from the law of rare events, as the number of indices $j \leq k$ such that $\xi_j > an/\ln n$ has the binomial distribution with parameters k and $\lceil an/\ln n \rceil^{-1}$. See Theorem 16.16 in Kallenberg [15].

Then fix $\varepsilon > 0$. Clearly, on the event $(1 - \varepsilon)n \ln^{-1} n \leq L(n) \leq (1 + \varepsilon)n \ln^{-1} n$, the undershoot $n - S_{L(n)}$ can be bounded from above by $\max\{\xi_i : (1 - \varepsilon)n \ln^{-1} n \leq i \leq (1 + \varepsilon)n \ln^{-1} n\}$. Elementary extreme value theory shows that the latter quantity, rescaled by a factor $n^{-1} \ln n$, converges in distribution as $n \rightarrow \infty$ to some random variable with distribution function $x \mapsto \exp(-2\varepsilon/x)$. Since ε can be chosen arbitrarily small, this observation combined with the first claim establishes the third assertion. Alternatively, recall from Theorem 6 of Erickson [6] (see also Lemma 2 in [12]) that $\ln(n - S_{L(n)})/\ln n$ converges in distribution to a uniform variable on $[0, 1]$, which is a stronger statement than the third claim. \square

Iksanov and Möhle [12] introduced a remarkable coupling with the random walk S to explain the asymptotic behavior established in [5] for number $\zeta(n)$ of random cuts needed to isolate the root 1 of a large recursive tree. In short, the coupling follows by iterating two crucial observations made by Meir and Moon [19]. If one removes an edge picked uniformly at random in T_n , then, first, the two subtrees resulting from the split are in turn, conditionally on their sizes, independent random recursive trees, and second, the distribution of the size of the subtree which does not contain the root is the same as ξ conditionally on $\xi < n$. The next lemma is a consequence of the coupling of Iksanov and Möhle; the statement is essentially a reformulation of Lemma 2 in [3].

Lemma 2. *One can construct on the same probability space a random recursive tree T_n with size n and its destruction process, together with a version of the random walk S such that the following hold:*

- (i) *The height $\zeta(n)$ of the leaf $\{1\}$ in $\text{Cut}(T_n)$ is bounded from below by $L(n)$.*
- (ii) *There is the identity*

$$(|B_{n,0}|, |B_{n,1}|, \dots, |B_{n,L(n)}|) = (n, n - S_1, \dots, n - S_{L(n)}).$$

We shall henceforth work in the framework of this coupling, in the sense that we shall implicitly assume that the recursive tree T_n and its destruction process are indeed coupled with the random walk S as in Lemma 2. The segment $[B_{n,0}, B_{n,L(n)-1}]$ of $\text{Cut}(T_n)$ will be called the *trunk* and denoted by $\text{Trunk}(T_n)$. We next turn our attention to the *branches* of $\text{Cut}(T_n)$, i.e., the components corresponding to the complement of the trunk.

2.2. The branches

We introduce the blocks

$$B'_{n,1} = B_{n,0} \setminus B_{n,1}, \dots, B'_{n,L(n)} = B_{n,L(n)-1} \setminus B_{n,L(n)}$$

and also agree that

$$B'_{n,L(n)+1} = B_{n,L(n)}.$$

Note that there are the identities

$$|B'_{n,j}| = \xi_j \quad \text{for } 1 \leq j \leq L(n) \quad \text{and} \quad |B'_{n,L(n)+1}| = n - S_{L(n)}. \tag{1}$$

Further $B_{n,j}$ and $B'_{n,j}$ are the two children of $B_{n,j-1}$ in $\text{Cut}(T_n)$; see Figure 2.

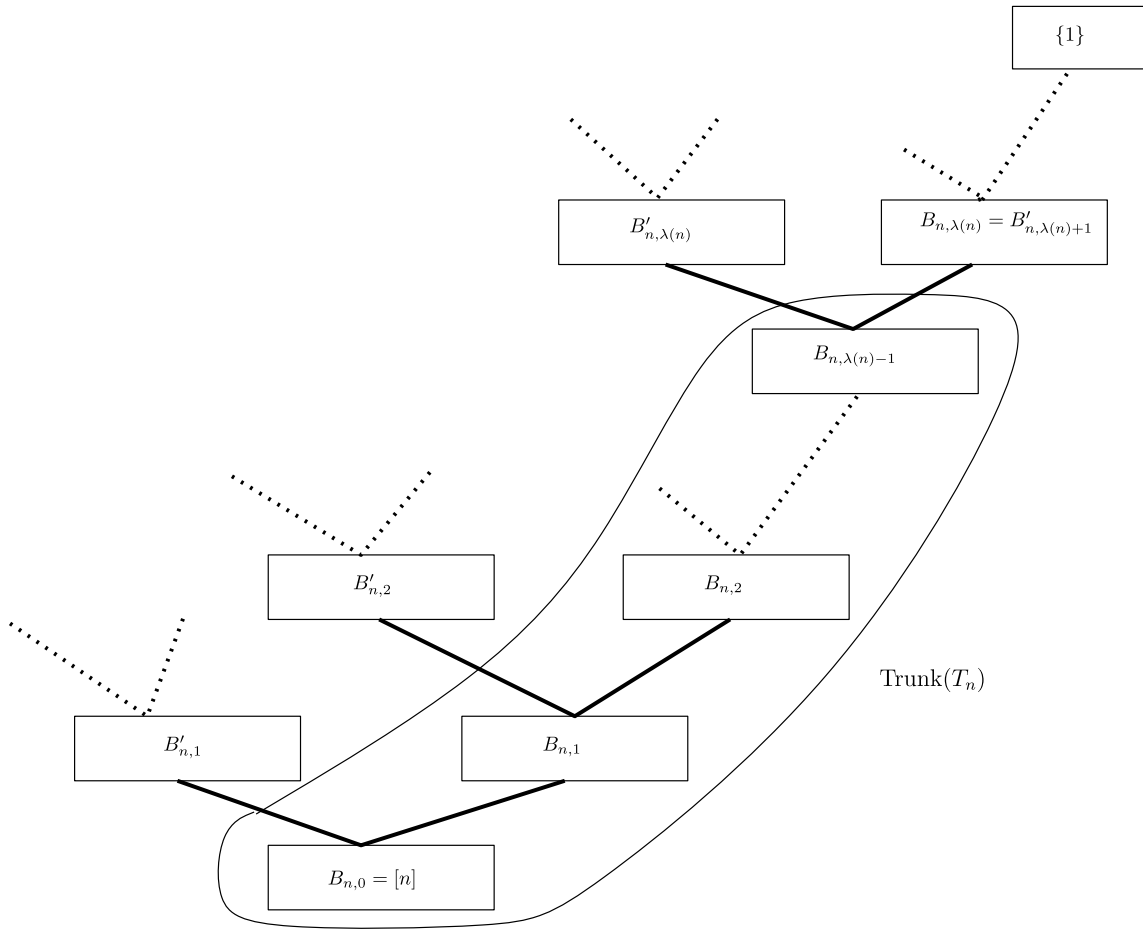


Fig. 2. $\text{Cut}(T_n)$, its trunk and its branches.

Plainly, the blocks $B'_{n,j}$ for $j = 1, \dots, L(n) + 1$ form a partition of $[n]$ into connected components (for the tree T_n), and we write $T'_{n,j}$ for the subtree of T_n restricted to $B'_{n,j}$. It will be convenient to introduce the following terminology. For an arbitrary block B of $[n]$ with size $k \geq 1$, we call canonic relabeling of vertices the bijective map from B to $[k]$ which preserves the order, i.e., the map which assigns to a vertex $v \in B$ its rank in B . Plainly the canonical relabeling transforms canonically an increasing tree on B into an increasing tree on $[k]$.

The following lemma stems from the important splitting property of random recursive trees (called also randomness preservation property in Kuba and Panholzer [16]); see Lemma 1 in [3].

Lemma 3. *Conditionally on the sizes*

$$|B'_{n,1}| = k_1, \dots, |B'_{n,L(n)+1}| = k_{L(n)+1},$$

and upon canonic relabeling of vertices, the subtrees $T'_{n,j}$ for $j = 1, \dots, L(n) + 1$ are independent random recursive trees on $[k_1], \dots, [k_{L(n)+1}]$, respectively.

For $j = 1, \dots, L(n) + 1$, we write $\text{Cut}(T'_{n,j})$ for the cut-tree of $T'_{n,j}$ obtained by restricting the destruction process of T_n to $T'_{n,j}$. Observe that during this restricted destruction process, the edges of $T'_{n,j}$ are indeed cut in a uniform random order, so this notation is consistent with the preceding. We think of $\text{Cut}(T'_{n,j})$ as the j th branch of $\text{Cut}(T_n)$, in the sense that it is the sub-tree that stems from the $\text{Trunk}(T_n)$ at height $j - 1$; see Figure 2 above. We also stress that $\text{Cut}(T'_{n,j})$ is connected to the trunk by an edge between the root $B'_{n,j}$ of $\text{Cut}(T'_{n,j})$ and its parent $B_{n,j-1} \in \text{Trunk}(T_n)$.

Lemma 4. *Conditionally on the sizes*

$$|B'_{n,1}| = k_1, \dots, |B'_{n,L(n)+1}| = k_{L(n)+1},$$

and upon canonic relabeling of vertices, the branches $\text{Cut}(T'_{n,j})$, for $j = 1, \dots, L(n) + 1$, are independent, and each $\text{Cut}(T'_{n,j})$ has the same distributions as the cut-tree of a random recursive tree on $[k_j]$.

Proof. Indeed, given the subtrees $T'_{n,1}, \dots, T'_{n,L(n)+1}$, the destruction processes restricted to $T'_{n,1}, \dots, T'_{n,L(n)+1}$ are independent (imagine that each edge of T_n is cut at an independent exponential time with parameter 1, and then use basic properties of sequences of i.i.d. exponential variables). The statement now follows from Lemma 3. \square

3. Proofs of the main results

The purpose of this section is to prove Theorem 1 and Corollaries 1 and 2. In this direction, we shall establish the convergence of the rescaled version of $\text{Cut}(T_n)$ to I , first in the sense of Gromov–Hausdorff, and then in the sense of Gromov–Prokhorov. In both cases, the key issue is to check that the branches of $\text{Cut}(T_n)$ are asymptotically small compared to the trunk.

3.1. Hausdorff distance

We write d_H (respectively, d_{GH}^*) for the Hausdorff (respectively, pointed Gromov–Hausdorff) distance. We aim at showing that

$$\lim_{n \rightarrow \infty} d_{GH}^* \left(\frac{\ln n}{n} \text{Cut}(T_n), I \right) = 0 \quad \text{in probability,} \tag{2}$$

where $I = [0, 1]$ is equipped with the usual distance and pointed at 0. As $\text{Trunk}(T_n)$ is merely a segment with length $L(n)$, it follows immediately from Lemma 1(i) that

$$\lim_{n \rightarrow \infty} d_{GH}^* \left(\frac{\ln n}{n} \text{Trunk}(T_n), I \right) = 0 \quad \text{in probability.}$$

Therefore, in order to prove (2), it suffices to establish that the whole $\text{Cut}(T_n)$ remains in a relatively small neighborhood of $\text{Trunk}(T_n)$, namely that

$$d_H(\text{Cut}(T_n), \text{Trunk}(T_n)) = o(n/\ln n) \quad \text{in probability.}$$

In turn, the former is a consequence of the fact that the branches of $\text{Cut}(T_n)$ are asymptotically small compared to the trunk, see Proposition 1 below.

In order to make a formal statement, it is convenient to write $\text{Depth}(T)$ for the depth of a rooted tree T , that is the maximal distance from the roof to a leaf of T .

Proposition 1. *We have*

$$\max_{1 \leq j \leq L(n)+1} \text{Depth}(\text{Cut}(T'_{n,j})) = o(n/\ln n) \quad \text{in probability.}$$

The proof of Proposition 1 requires first the following crude estimate.

Lemma 5. *For every fixed $\varepsilon, a > 0$ and every $n \in \mathbb{N}$, set*

$$p(\varepsilon, a, n) = \sup_{k \leq an/\ln n} \mathbb{P}(\text{Depth}(\text{Cut}(T_k)) > \varepsilon n / \ln n),$$

where T_k stands for a random recursive tree on $[k]$. Then

$$\lim_{n \rightarrow \infty} p(\varepsilon, a, n) = 0.$$

Proof. From the decomposition of $\text{Cut}(T_k)$ along its trunk and the fact that the depth of the cut-tree of any tree T is bounded from above by the number of edges of T , we see from (1) that

$$\text{Depth}(\text{Cut}(T_k)) \leq L(k) + \max\{\xi_j: 1 \leq j \leq L(k)\} + (n - S_{L(n)}).$$

Our claim follows now easily from Lemma 1. □

We can now establish Proposition 1.

Proof of Proposition 1. For $b > 0$, set

$$N(b, n) = \text{Card}\{j \leq L(n) + 1: \xi_j > bn/\ln n\}.$$

Then fix $\varepsilon > 0$. Using again the fact that the depth of the cut-tree of any tree T cannot exceed the number of edges of T , we see that the event that $\text{Depth}(\text{Cut}(T'_{n,j})) > \varepsilon n/\ln n$ can only occur when $|B'_{n,j}| > \varepsilon n/\ln n$. It follows from Lemma 4 that

$$\mathbb{P}\left(\max_{1 \leq j \leq L(n)+1} \text{Depth}(\text{Cut}(T'_{n,j})) > \varepsilon n/\ln n\right)$$

can be bounded from above by

$$mp(\varepsilon, a, n) + \mathbb{P}(N(\varepsilon, n) > m) + \mathbb{P}(N(a, n) \geq 1),$$

where $m \in \mathbb{N}$ and $a > 0$ are arbitrary.

Next fix $\eta > 0$. Thanks to Lemma 1, for every fixed $\varepsilon > 0$, we may find m and a sufficiently large so that

$$\lim_{n \rightarrow \infty} \mathbb{P}(N(\varepsilon, n) > m) \leq \eta/2 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{P}(N(a, n) \geq 1) \leq \eta/2.$$

Then using Lemma 5, we conclude that

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\max_{1 \leq j \leq L(n)+1} \text{Depth}(\text{Cut}(T'_{n,j})) > \varepsilon n/\ln n\right) \leq \eta,$$

which establishes our claim. □

3.2. Prokhorov distance

We now endow $\text{Cut}(T_n)$ with the uniform probability measure μ_n on its leaves, $I = [0, 1]$ with the Lebesgue measure λ , and aim at proving that

$$\lim_{n \rightarrow \infty} d_{\text{GP}}^* \left(\left(\frac{\ln n}{n} \text{Cut}(T_n), \mu_n \right) (I, \lambda) \right) = 0 \quad \text{in probability,} \tag{3}$$

where d_{GP}^* stands for the pointed Gromov–Prokhorov distance and $\text{Cut}(T_n)$ and I are pointed respectively at $[n]$ and 0.

In this direction, it is convenient to equip $\text{Trunk}(T_n)$ with the probability measure

$$\nu_n(B_{n,j}) = n^{-1} |B'_{n,j+1}| \quad \text{for } j = 0, 1, \dots, L(n) - 2$$

and

$$\nu_n(B_{n,L(n)-1}) = n^{-1} (|B'_{n,L(n)}| + |B'_{n,L(n)+1}|).$$

In words, v_n is the image of μ_n by the projection $\text{proj}: \text{Cut}(T_n) \rightarrow \text{Trunk}(T_n)$, i.e., the map which associates to each node of $\text{Cut}(T_n)$ its closest ancestor on the trunk. Proposition 1 shows that the maximal distance on the rescaled cut-tree $\frac{\ln n}{n} \text{Cut}(T_n)$ between a leaf v and its projection $\text{proj}(v)$ tends to 0 in probability as $n \rightarrow \infty$, and this entails that

$$\lim_{n \rightarrow \infty} d_{\text{GP}}^* \left(\left(\frac{\ln n}{n} \text{Cut}(T_n), \mu_n \right), \left(\frac{\ln n}{n} \text{Trunk}(T_n), \nu_n \right) \right) = 0 \quad \text{in probability.}$$

The proof of (3) is now reduced to checking the following.

Proposition 2. *We have*

$$\lim_{n \rightarrow \infty} d_{\text{GP}}^* \left(\left(\frac{\ln n}{n} \text{Trunk}(T_n), \nu_n \right), (I, \lambda) \right) = 0 \quad \text{in probability.}$$

Proof. It is convenient to view the rescaled segment $\frac{\ln n}{n} \text{Trunk}(T_n)$ as a (random) subset of $[0, \infty)$, using the obvious embedding

$$B_{n,j} \mapsto \frac{\ln n}{n} j \quad \text{for } j = 0, 1, \dots, L(n) - 1.$$

Then write F_n for the distribution function of ν_n , specifically,

$$F_n(x) = n^{-1} \sum_{0 \leq j \leq \lfloor \frac{n}{\ln n} x \rfloor} |B'_{n,j+1}| \quad \text{when } \left\lfloor \frac{n}{\ln n} x \right\rfloor < L(n) - 1$$

and

$$F_n(x) = 1 \quad \text{when } \left\lfloor \frac{n}{\ln n} x \right\rfloor \geq L(n) - 1.$$

Next, observe from Lemma 1 that the random walk S fulfills the weak law of large numbers

$$\lim_{n \rightarrow \infty} n^{-1} S_{n/\ln n} = 1 \quad \text{in probability.}$$

A standard argument (cf. Theorem 15.17 in Kallenberg [15]) enables us to reinforce the preceding to uniform convergence. Namely, for every $t \geq 0$

$$\lim_{n \rightarrow \infty} \sup_{0 \leq u \leq t} |n^{-1} S_{\lfloor nu/\ln n \rfloor} - u| = 0 \quad \text{in probability.}$$

It follows now readily from (1) that

$$\lim_{n \rightarrow \infty} \sup_{x \geq 0} |F_n(x) - x \wedge 1| = 0 \quad \text{in probability,}$$

that is ν_n , viewed as a random probability measure on $[0, \infty)$, converges in probability to the Lebesgue measure on $[0, 1]$, on the space of probability measures on $[0, \infty)$ endowed with the weak convergence. This yields our claim. \square

3.3. Proof of Corollary 1

The proof of Corollary 1 only requires the convergence of $\frac{\ln n}{n} \text{Cut}(T_n)$ to the unit interval in the sense of Gromov–Prokhorov, that is (3).

Let $u_1^{(n)}, \dots, u_\ell^{(n)}$ denote ℓ independent uniform vertices of T_n , so the singletons $\{u_1^{(n)}\}, \dots, \{u_\ell^{(n)}\}$ form a sequence of ℓ i.i.d. blocks of $\text{Cut}(T_n)$ distributed according to μ_n . Let also u_1, \dots, u_ℓ be a sequence of ℓ i.i.d. uniform variables

on the unit interval I . Denote by $\mathcal{R}_{n,\ell}$ the reduction of $\text{Cut}(T_n)$ to the ℓ leaves $\{u_1^{(n)}\}, \dots, \{u_\ell^{(n)}\}$ and its root $[n]$, i.e., $\mathcal{R}_{n,\ell}$ is the smallest subtree of $\text{Cut}(T_n)$ which connects these nodes. Similarly, write \mathcal{R}_ℓ for the reduction of I to u_1, \dots, u_ℓ and the origin 0. Both reduced trees are viewed as combinatorial trees structures with edge lengths, and (3) implies that $n^{-1} \ln n \mathcal{R}_{n,\ell}$ converges in distribution to \mathcal{R}_ℓ as $n \rightarrow \infty$. In particular, focussing on the lengths of those reduced trees, there is the weak convergence

$$\lim_{n \rightarrow \infty} \left(\frac{\ln n}{n} |\mathcal{R}_{n,1}|, \dots, \frac{\ln n}{n} |\mathcal{R}_{n,\ell}| \right) = (|\mathcal{R}_1|, \dots, |\mathcal{R}_\ell|) \quad \text{in distribution.}$$

It suffices then to observe that for the unit interval,

$$(|\mathcal{R}_1|, \dots, |\mathcal{R}_\ell|) = (u_1, \max\{u_1, u_2\}, \dots, \max\{u_1, \dots, u_\ell\}).$$

3.4. Proof of Corollary 2

If we write $v_1^{(n)}, \dots, v_\ell^{(n)}$ for the parents of $n, \dots, n - \ell + 1$ in T_n , then $v_1^{(n)}, \dots, v_\ell^{(n)}$ are independent and $v_j^{(n)}$ has the uniform distribution on $[n - j]$. The distribution of $v_1^{(n)}, \dots, v_\ell^{(n)}$ is thus close (in the sense of total variation) to that of $u_1^{(n)}, \dots, u_\ell^{(n)}$, a sequence of ℓ i.i.d. uniform vertices in $[n]$, and it follows that the number $Y'_{n,\ell}$ of cuts needed to isolate $v_1^{(n)}, \dots, v_\ell^{(n)}$ has the same asymptotic behavior in law as $Y_{n,\ell}$.

On the other hand, the vertices $n, \dots, n - \ell + 1$ are leaves of T_n with high probability when n is large, and in that case, the number $Z_{n,\ell}$ of cuts required to isolate $n, \dots, n - \ell + 1$ is plainly bounded from above by $Y'_{n,\ell}$. According to Theorems 2 and 3 in [16], both $n^{-1} \ln n Z_{n,\ell}$ and $n^{-1} \ln n Y_{n,\ell}$ converge in distribution to a beta variable with parameters ℓ and 1, and it follows from the preceding observations that as a matter of fact

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} (Y'_{n,\ell} - Z_{n,\ell}) = 0 \quad \text{in probability.}$$

Thus our claim now follows from Corollary 1.

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