Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2014, Vol. 50, No. 4, 1347–1370
DOI: 10.1214/13-AIHP559

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

On smoothing properties of transition semigroups associated to a class of SDEs with jumps

Seiichiro Kusuoka[a] and Carlo Marinelli[b]

[a]*Graduate School of Science, Tohoku University, 6-3 Aramaki Aza-Aoba, Aoba-ku Sendai 980-8578, Japan. E-mail: kusuoka@math.tohoku.ac.jp*
[b]*Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom. E-mail: c.marinelli@ucl.ac.uk*

Received 15 August 2012; revised 16 March 2013; accepted 18 March 2013

Abstract. We prove smoothing properties of nonlocal transition semigroups associated to a class of stochastic differential equations (SDE) in $\mathbb{R}^d$ driven by additive pure-jump Lévy noise. In particular, we assume that the Lévy process driving the SDE is the sum of a subordinated Wiener process Y (i.e. $Y = W \circ T$, where T is an increasing pure-jump Lévy process starting at zero and independent of the Wiener process W) and of an arbitrary Lévy process independent of Y, that the drift coefficient is continuous (but not necessarily Lipschitz continuous) and grows not faster than a polynomial, and that the SDE admits a Feller weak solution. By a combination of probabilistic and analytic methods, we provide sufficient conditions for the Markovian semigroup associated to the SDE to be strong Feller and to map $L_p(\mathbb{R}^d)$ to continuous bounded functions. A key intermediate step is the study of regularizing properties of the transition semigroup associated to Y in terms of negative moments of the subordinator T.

Résumé. Nous établissons des propriétés de lissage de semi-groupes de transition non locaux associés à une classe d'équations différentielles stochastiques dans $\mathbb{R}^d$ dirigées par un bruit additif de Lévy sans partie continue. En particulier, nous supposons que le processus de Lévy est la somme d'un processus de Wiener subordonné Y (i.e. $Y = W \circ T$, où T est un processus croissant de Lévy sans partie continue, avec $T_0 = 0$, indépendant du processus de Wiener W) et d'un processus de Lévy arbitraire indépendant de Y; que le coefficient de dérive est continu (mais pas nécessairement lipschitzien) et à croissance polynomiale; et que la EDS admet une solution faible fellerienne. Par une combinaison de méthodes probabilistes et analytiques, nous fournissons des conditions suffisantes pour le semi-groupe markovien associé à l'EDS soit fortement fellérien et envoye $L_p(\mathbb{R}^d)$ dans les fonctions continues bornées. Une étape intermédiaire essentielle est l'étude de certaines propriétés régularisantes du semi-groupe de transition associé à Y qui dépendent de moments négatifs du subordinateur T.

MSC: 60G30; 60G51; 60H07; 60H10; 60J35

Keywords: Lévy processes; Subordination; Transition semigroups; Non-local operators; Malliavin calculus

1. Introduction

The purpose of this work is to prove smoothing properties for the (Markovian) semigroup generated by the (weak) solution to a stochastic differential equation in $\mathbb{R}^d$ of the type

$$dX_t = b(X_t)\,dt + dZ_t, \quad X_0 = x, \tag{1.1}$$

where Z is a pure-jump Lévy process which can be written as $Z = Y + \xi$, where Y is obtained by subordination of a Wiener process W with non-degenerate covariance matrix, and ξ is a further Lévy process independent of Y, on which no further assumption is imposed. In particular, assume that (1.1) admits a Markovian weak solution denoted by $(X_t^x)_{t\geq 0}$, and that the semigroup $(P_t^X)_{t>0}$, $P_t^X f(x) := \mathbb{E} f(X_t^x)$, for f Borel measurable and bounded, is Feller, i.e. that P_t^X leaves invariant the space of bounded continuous functions. We look for sufficient conditions on the Lévy process Y and on the drift coefficient b such that P_t^X is strong Feller, resp. L_p-strong Feller, for all $t > 0$, i.e. that P_t^X maps bounded Borel measurable functions, resp. $L_p(\mathbb{R}^d)$, to bounded continuous functions.

We proceed in two steps: first we study the regularizing properties of the semigroups $(P_t^Y)_{t>0}$ and $(P_t^Z)_{t>0}$ associated, respectively, to the Lévy process Y and Z (an issue which is interesting in its own right); then we show that the semigroup associated to X inherits, at least in part, the regularizing properties of P^Y. In particular, in the former step we provide conditions in terms of the existence of negative moments of the subordinator T_t such that P_t^Y (hence also P_t^Z, as we shall see) maps $\mathscr{B}_b(\mathbb{R}^d)$, the space of bounded Borel measurable functions on $\mathbb{R}^d$, or $L_p(\mathbb{R}^d)$, to $C_b^k(\mathbb{R}^d)$, $k \in \mathbb{N}$. The latter step is a perturbation argument relying on Duhamel's formula.

The strong Feller property for semigroups generated by solutions to stochastic (both ordinary and partial) differential equations with jumps is usually obtained by suitable versions of the Bismut–Elworthy–Li formula (see e.g. [20,25]). However, this method requires the driving noise to have a non-degenerate diffusive component, therefore it is not applicable to our problem. For some special classes of equations driven by pure jump noise other approaches have been devised: for instance, in [26] the authors prove the strong Feller property for the semigroup generated by the solution to a semilinear SPDE driven by an infinite sum of one-dimensional independent stable processes, assuming that the nonlinearity in the drift term is Lipschitz continuous and bounded. Their proofs rely on finite-dimensional projections and specific properties of stable measures.

Let us also mention that the problem we are dealing with admits a clear analytic interpretation. In fact, an application of Itô's formula yields that the generator L of P^X acts on smooth functions as follows:

$$L\phi(x) = \langle b(x), \nabla\phi(x)\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \bigl(\phi(x+y) - \phi(x) - \langle\nabla\phi(x), y\rangle 1_{\{|y|<1\}}\bigr) m^Z(\mathrm{d}y),$$

where m^Z stands for the Lévy measure of Z. Therefore, being somewhat formal, our problem is equivalent to establishing regularity (specifically, continuity and boundedness) of the solution at time $t > 0$ to the non-local parabolic Kolmogorov equation $\partial_t u = Lu$, $u(0) = u_0$, where the initial datum u_0 is taken either Borel measurable and bounded, or belonging to an L_p space, on $\mathbb{R}^d$. Using analytic methods, related problems have already been investigated, e.g. in [23], where it is assumed, roughly speaking, that Z is a perturbation of an α-stable process. For more recent results, covering also nonlinear equations, one could see e.g. [7] and references therein. It does not seem, however, that our results can be recovered by available regularity estimates for non-local parabolic equations.

One should also recall that there exists a rich literature on existence and regularity of densities for solutions to SDEs with jumps, mostly applying suitable versions of Malliavin calculus (see e.g. [1,12,15,17,18] and references therein). Such existence and regularity result may be used, in turn, to prove that a Feller process is strong Feller (see [28], Corollary 2.2, for a general result in this direction). In general, however, applying Malliavin calculus directly to an SDE driven by a Lévy process usually requires that the Lévy measure of the driving noise admits sufficiently many finite moments (see e.g. [12,15]). This problem of course does not occur in the case of SDEs driven by Brownian motion (cf. [16]). Moreover, the coefficients of the SDE are assumed to be sufficiently smooth (usually of class C_b^2 at least), and it is difficult to weaken this hypothesis very much. The smoothing properties proved in this paper, on the other hand, are applicable also to equations driven by Lévy processes whose Lévy measure possesses only moments of very low order (such as stable processes), and with a drift coefficient that is not Lipschitz continuous.

From the analytic point of view, speaking again somewhat formally, the distribution μ_t of the solution X_t^x, $t \ge 0$, to (1.1) solves, in the sense of distributions, the non-local parabolic equation for probability measures $\partial_t \mu = L^*\mu$, with initial datum equal to a Dirac measure centered at x, where L^* stands for the formal adjoint of L. Note that, in our specific situation, assuming for the sake of simplicity that the generator of Z is symmetric (which is certainly the case if $\xi \equiv 0$), one can write

$$L^*\phi(x) = \operatorname{div}(b\phi) + \int_{\mathbb{R}^d\setminus\{0\}} \bigl(\phi(x+y) - \phi(x) - \langle\nabla\phi(x), y\rangle 1_{\{|y|<1\}}\bigr) m^Z(\mathrm{d}y).$$

Unfortunately, however, we are not aware of any existence and regularity results for non-local Fokker–Planck equations of the type $\partial_t \mu = L^*\mu$. Nonetheless, it is interesting to note that, if one knows a priori (or assumes, as we do) that X has the Feller property, such results would imply regularity properties of the solution to the Kolmogorov equation $\partial_t u = Lu$.

Let us also recall that subordination has already been used to establish the strong Feller property for some classes of Markov processes with jumps (cf. [8,17,18]). We would like to stress, however, that it seems difficult to deduce properties of semigroups generated by operators such as L from the properties of semigroups generated by corresponding local operators of the type $L_\ell\phi = \langle b, \nabla\phi\rangle + \Delta\phi$. Using more probabilistic language, it is not clear at all whether one can establish properties of the solution to an SDE of the type (1.1) (assuming $\xi \equiv 0$ for simplicity) studying the process obtained by subordination with T of the solution to the same SDE with Y replaced by the Wiener process W. These considerations and the need to treat semigroups generated by non-local operators with drift are the main motivations for our approach.

Smoothing properties of equations with multiplicative noise (i.e. with a "diffusion" coefficient depending on X in front of the noise in (1.1)) are also an interesting problem, but unfortunately it seems difficult to adapt our techniques to this case. On the other hand, if both the drift and the diffusion coefficients are sufficiently smooth (i.e. at least of class C_b^2), and the noise Z is α-stable, we show that one can apply Malliavin calculus methods to prove that the solution generates a strong Feller semigroup.

The paper is organized as follows: we collect in Section 2 some basic preliminaries, and, in Section 3, we extensively study regularizing properties of semigroups associated to subordinate Wiener processes. In particular, we derive estimates on the kth order Fréchet derivative of such semigroups in terms of negative moments of the corresponding subordinators. These estimates are an essential ingredient for the proof of the main results in Section 4. Finally, in Section 5 we consider the case of equations with multiplicative stable noise: under smoothness assumptions on the coefficients, we prove the strong Feller property of the transition semigroup, applying some results that were obtained in [17] by a suitable version of Malliavin calculus.

2. Preliminaries

2.1. *Notation and terminology*

The standard scalar product in $\mathbb{R}^d$ will be denoted by $\langle\cdot,\cdot\rangle$. We shall denote the set of bounded Borel measurable functions on $\mathbb{R}^d$ by $\mathcal{B}_b(\mathbb{R}^d)$. Note that $\mathcal{B}_b(\mathbb{R}^d)$, endowed with the norm $\|\phi\|_\infty := \sup_{x\in\mathbb{R}^d} |\phi(x)|_{\mathbb{R}^d}$, is a Banach space. The subset of $\mathcal{B}_b(\mathbb{R}^d)$ consisting of functions with compact support is denoted by $\mathcal{B}_{b,c}(\mathbb{R}^d)$. The space of bounded continuous functions on $\mathbb{R}^d$ will be denoted by $C_b(\mathbb{R}^d)$, and, similarly, $C_b^k(\mathbb{R}^d)$, $k \in \mathbb{N}$, will denote the space of continuously differentiable functions with bounded derivatives up to order k. The space of infinitely differentiable functions with compact support is denoted by $C_c^\infty(\mathbb{R}^d)$. Given a function $f : \mathbb{R}^d \to \mathbb{R}$ and a multiindex $\alpha \in \mathbb{N}_0^d$ (where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), we shall use the standard notation

$$\partial^\alpha f(x_0) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f(x_0), \qquad x_0 \in \mathbb{R}^d,$$

and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. Given $n \in \mathbb{N}$, the nth Fréchet derivative of f at a point $x_0 \in \mathbb{R}^d$ will be denoted by $D^n f(x_0)$. Recall that $D^n f(x_0)$ can be identified with an element of $\mathcal{L}_n(\mathbb{R}^d)$, the space of n-multilinear mappings on $\mathbb{R}^d$.

Lebesgue spaces are denoted by $L_p(\mathbb{R}^d)$, $1 \le p \le \infty$, and the corresponding Sobolev spaces by $W_p^m(\mathbb{R}^d)$, $m \in \mathbb{N}$. In the following we shall sometimes denote function spaces without mentioning the underlying space $\mathbb{R}^d$. An expression of the type $E \hookrightarrow F$ means that the space E is continuously embedded into the space F. If $a \le Nb$ for some positive constant N we shall often write $a \lesssim b$.

We recall standard terminology, plus some slightly non-standard one needed for the purposes of this work. Let us recall that a linear positivity preserving operator $A : \mathcal{B}_b(\mathbb{R}^d) \to \mathcal{B}_b(\mathbb{R}^d)$ is called sub-Markovian if it is contracting, i.e. if $\|A\phi\|_\infty \le \|\phi\|_\infty$ for all $\phi \in \mathcal{B}_b(\mathbb{R}^d)$.

Definition 2.1. *A sub-Markovian operator A on $\mathcal{B}_b(\mathbb{R}^d)$ is called*:

(i) Feller *if* $A(C_b(\mathbb{R}^d)) \subseteq C_b(\mathbb{R}^d)$;
(ii) strong Feller *if* $A(\mathcal{B}_b(\mathbb{R}^d)) \subseteq C_b(\mathbb{R}^d)$;
(iii) c-strong Feller *if* $A(\mathcal{B}_{b,c}(\mathbb{R}^d)) \subseteq C_b(\mathbb{R}^d)$;

(iv) k-smoothing, $k \in \mathbb{N}$, *if* $A(\mathcal{B}_b(\mathbb{R}^d)) \subseteq C_b^k(\mathbb{R}^d)$.

Definition 2.2. *Given* $1 \le p \le \infty$, *a linear bounded operator* A *from* $L_p(\mathbb{R}^d)$ *to* $C_b(\mathbb{R}^d)$ *will be called* L_p*-strong Feller.*

Remark 2.3. *Note that, in general, a sub-Markovian operator on* $\mathcal{B}_b(\mathbb{R}^d)$ *may not even be defined on any space* $L_p(\mathbb{R}^d)$, $1 \le p \le \infty$. *Conversely, a map from* $L_p(\mathbb{R}^d)$ *to* $C_b(\mathbb{R}^d)$ *may not be defined on* $\mathcal{B}_b(\mathbb{R}^d)$. *Therefore, in general, an* L_p*-strong Feller operator may not be Feller, and viceversa. However, by Lemma 2.4 below, if an operator is Feller and* L_p*-strong Feller, then it is strong Feller. A necessary condition for the last definition to make sense is that the operator* A *maps indicator functions of sets of Lebesgue measure zero to zero* (*i.e. to the continuous function equal to zero*). *This condition is clearly not satisfied by all sub-Markovian operators on* $\mathcal{B}_b(\mathbb{R}^d)$, *but it is indeed satisfied if* A *is of the type*

$$Af = \int_{\mathbb{R}^d} k(\cdot, y) f(y)\,\mathrm{d}y \quad \forall f \in \mathcal{B}_b\big(\mathbb{R}^d\big),$$

with appropriate measurability conditions on k.

As it is customary, one says that a Markov process is Feller (or strong Feller, etc.) to mean that its transition semigroup is made of Feller operators.

In the following lemma we provide a simple yet useful criterion to establish that a Feller operator is strong Feller.

Lemma 2.4. *Let* A *be a Feller operator on* $\mathcal{B}_b(\mathbb{R}^d)$. *Then* A *is strong Feller if and only if it is c-strong Feller.*

Proof. We only have to prove that the c-strong Feller property implies the strong Feller property. Let $f \in \mathcal{B}_b$, and $\{\chi_k\}_{k\in\mathbb{N}} \subset C_c^\infty$ a sequence of cutoff functions such that $0 \le \chi_k \le 1$ for all k and $\chi_k \uparrow 1$ as $k \to \infty$. Since A is positivity preserving, we have $A\chi_k \uparrow A1$ as $k \to \infty$, and $A\chi_k \in C_b$ for all k because (obviously) $\chi_k \in \mathcal{B}_{b,c}$. If an increasing sequence of continuous functions converges pointwise to a continuous function, the convergence is locally uniform by Dini's theorem. Therefore, we infer that $A\chi_k \to A1$ locally uniformly as $k \to \infty$. Using again that A is sub-Markovian and $\chi_k \le 1$ for all k, we have

$$\big|Af - A(\chi_k f)\big| = \big|A\big((1-\chi_k)f\big)\big| \le \|f\|_\infty |A1 - A\chi_k|,$$

which implies that $A(\chi_k f) \to Af$ locally uniformly. But $\chi_k f \in \mathcal{B}_{b,c}$, hence $A(\chi_k f) \in C_b$, and we can conclude that Af is continuous as it is the local uniform limit of a sequence of continuous functions. That Af is bounded is obvious by sub-Markovianity of A. □

By inspection of the proof, one realizes that one could also assume that A is Markovian (i.e. sub-Markovian and conservative[1]), rather than Feller. The previous lemma then has the following immediate consequence, which can indeed be quite useful.

Corollary 2.5. *Let* A *be a Markovian operator on* $\mathcal{B}_b(\mathbb{R}^d)$. *Then* A *is strong Feller if and only if it is c-strong Feller.*

2.2. *Subordinators*

By subordinator we shall always understand an increasing Lévy process $T : \mathbb{R}_+ \to \mathbb{R}_+$ such that $T_0 = 0$ and $T_t > 0$ for $t > 0$. Then one has, for $\lambda \ge 0$,

$$\mathbb{E}\mathrm{e}^{-\lambda T_t} = \mathrm{e}^{-t\Phi(\lambda)},$$

[1]The operator A is conservative if $A1 = 1$.

where $\Phi : [0, \infty[\to [0, \infty[$ is such that

$$\Phi(\lambda) = \int_{]0,\infty[} \left(1 - \mathrm{e}^{-\lambda x}\right) m(\mathrm{d}x).$$

Here m is the Lévy measure of T, whose support is contained in $[0, \infty[$, and satisfies

$$\int_{]0,\infty[} (1 \wedge x) m(\mathrm{d}x) < \infty.$$

For a proof of the above facts (and much more) one can consult e.g. [2].

2.3. *Function spaces*

We recall some definitions and results on Hölder and Bessel potential spaces, referring to [29] for a complete treatment as well as for all unexplained notation. Bessel spaces are only used in Section 3.1.

Given a real non-integer number $s > 0$, let us set $s = [s] + \{s\}$, with $[s] \in \mathbb{N}$ and $0 < \{s\} < 1$. The Hölder space $C_b^s(\mathbb{R}^d)$ is defined as the set of functions $f \in C_b^{[s]}(\mathbb{R}^d)$ such that

$$\|f\|_{C_b^s(\mathbb{R}^d)} := \sum_{|\alpha| \le [s]} \left\|\partial^\alpha f\right\|_{L_\infty(\mathbb{R}^d)} + \sum_{|\beta| = [s]} \sup_{x \ne y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x - y|^{\{s\}}} < \infty.$$

The Zygmund space on $\mathbb{R}^d$ of order $s \in \mathbb{R}$, cf. [29], p. 36, will be denoted by $\mathcal{C}^s(\mathbb{R}^d)$. Recall that one has $\mathcal{C}^s(\mathbb{R}^d) = C^s(\mathbb{R}^d)$ for all real non-integer $s > 0$ (see [29], Remark 3, p. 38).

The Bessel potential space $H_p^s(\mathbb{R}^d)$, with $1 < p < \infty$ and $s \in \mathbb{R}$, is the space of Schwartz distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $(I - \Delta)^{s/2} f \in L_p(\mathbb{R}^d)$, with

$$\|f\|_{H_p^s(\mathbb{R}^d)} = \left\|(I - \Delta)^{s/2} f\right\|_{L_p(\mathbb{R}^d)}.$$

For convenience, let us also define the homogeneous norm

$$\|f\|_{\dot{H}_p^s(\mathbb{R}^d)} := \left\|(-\Delta)^{s/2} f\right\|_{L_p(\mathbb{R}^d)}.$$

The one has

$$\|f\|_{H_p^s(\mathbb{R}^d)} \eqsim \|f\|_{L_p(\mathbb{R}^d)} + \|f\|_{\dot{H}_p^s(\mathbb{R}^d)}. \tag{2.1}$$

As is well known, if $m \in \mathbb{N}$ one has $W_p^m(\mathbb{R}^d) = H_p^m(\mathbb{R}^d)$.

The following embedding result for Bessel potential spaces is certainly known, but we have not been able to find it anywhere in this formulation. We include a proof (admittedly very cryptic, but with precise references) for completeness.

Lemma 2.6 (Sobolev embedding theorem). *Let $1 < p < \infty$ and $s > d/p$. Then $H_p^s(\mathbb{R}^d) \hookrightarrow \mathcal{C}^{s-d/p}(\mathbb{R}^d)$. In particular, if $s - d/p \notin \mathbb{N}$, then $H_p^s(\mathbb{R}^d) \hookrightarrow C_b^{s-d/p}(\mathbb{R}^d)$.*

Proof. We have

$$H_p^s(\mathbb{R}^d) = F_{p,2}^s(\mathbb{R}^d) \hookrightarrow F_{p,\infty}^s(\mathbb{R}^d) \hookrightarrow B_{p,\infty}^s(\mathbb{R}^d),$$

where we have used [29], Theorem (i), p. 88, and [29], Proposition 2, p. 47, in this order. By [29], Theorem (i), p. 129, we also have

$$B_{p,\infty}^s(\mathbb{R}^d) \hookrightarrow B_{\infty,\infty}^{s_1}(\mathbb{R}^d),$$

where $s_1 = s - d/p$. Since $s_1 > 0$, [29], Corollary (i), p. 113, implies $\mathcal{C}^{s_1}(\mathbb{R}^d) = B^{s_1}_{\infty,\infty}(\mathbb{R}^d)$, hence also $H^s_p(\mathbb{R}^d) \hookrightarrow \mathcal{C}^{s_1}(\mathbb{R}^d)$. The proof is concluded recalling that, as already mentioned above, one has $\mathcal{C}^{s_1}(\mathbb{R}^d) = C^{s_1}_b(\mathbb{R}^d)$, if s_1 is not integer. □

3. Smoothing properties of subordinated Wiener processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, on which all random variables and processes will be defined. Let W be a standard $\mathbb{R}^d$-valued Wiener process (i.e. with covariance operator equal to the identity) and T be a subordinator with infinite lifetime and independent from W. Let us define the Markovian stochastic process $Y := W \circ T$, i.e. $Y_t := W_{T_t}$ for all $t \geq 0$, and its associated semigroup

$$P^Y_t f(x) := \mathbb{E} f(x + Y_t), \quad f \in \mathcal{B}_b\bigl(\mathbb{R}^d\bigr).$$

The process Y is often referred to as the Wiener process W subordinated to T.

In the sequel we shall denote the density of the random variable W_t, $t \in [0, \infty[$, by $p_t : \mathbb{R}^d \to \mathbb{R}$, with

$$p_t(y) = \frac{1}{(2\pi t)^{d/2}} \mathrm{e}^{-|y|^2/(2t)}.$$

With a slight (but innocuous) abuse of terminology, we shall refer to the function $p : (t, y) \mapsto p_t(y)$ as the transition density of W (and similarly for other translation-invariant processes), or as the heat kernel on $\mathbb{R}^d$.

The following elementary lemma relates the transition density of the Lévy process Y to the one of W and shows that the strong Feller property of W is inherited by Y.

Lemma 3.1. *The process $Y = W \circ T$ admits a transition density $(t, y) \mapsto p^Y_t(y)$ given by*

$$p^Y_t(y) = \int_0^\infty p_s(y)\nu_t(\mathrm{d}s),$$

where $\nu_t := \mathbb{P} \circ T_t^{-1}$, $t > 0$, stands for the law of the random variable T_t. In particular, P^Y_t is strong Feller for all $t > 0$.

Proof. Let $f \in \mathcal{B}_b(\mathbb{R}^d)$. Then, using properties of conditional expectation and recalling that W and T are independent, one has

$$\mathbb{E} f(Y_t) = \int_0^\infty \int_{\mathbb{R}^d} f(y) p_s(y)\, \mathrm{d}y\, \nu_t(\mathrm{d}s).$$

The conclusion then follows by Fubini's theorem, since p is positive and

$$\int_0^\infty \int_{\mathbb{R}^d} \bigl|f(y)\bigr| p_s(y)\, \mathrm{d}y\, \nu_t(\mathrm{d}s) \leq \|f\|_\infty.$$

Another immediate application of Tonelli's theorem (or just recalling that Y_t is finite $\mathbb{P}$-a.s. for all $t \geq 0$) shows that $p^Y_t \in L_1$ and $\|p^Y_t\|_1 = 1$. In particular, $P^Y_t f = f * p^Y_t$, with $f \in L_\infty$ and $p^Y_t \in L_1$, hence $\|P^Y_t f\|_{L_\infty} \leq \|p^Y\|_{L_1} \|f\|_\infty$ by Young's inequality. We only have to show that $P^Y_t f$ is continuous: let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C^\infty_c(\mathbb{R}^d)$ such that $\phi_n \to p^Y_t$ in $L_1(\mathbb{R}^d)$. Then clearly $f * \phi_n \in C_b(\mathbb{R}^d)$ for all $n \in \mathbb{N}$, and

$$\bigl\|f * p^Y_t - f * \phi_n\bigr\|_{L_\infty} \leq \|f\|_\infty \bigl\|p^Y_t - \phi_n\bigr\|_{L_1} \xrightarrow{n\to\infty} 0,$$

which implies that $P^Y_t f \in C(\mathbb{R}^d)$ as uniform limit of continuous functions. □

We are going to use some well-known properties of the heat kernel on finite dimensional Euclidean spaces. In particular, observing that one can write

$$p_t(x) = \frac{1}{(2\pi)^{d/2}} t^{-d/2} \phi\left(\frac{|x|^2}{t}\right), \quad \phi(r) = \mathrm{e}^{-r/2}, \tag{3.1}$$

it is immediately seen (and well known) that $x \mapsto p_t(x) \in \mathcal{S}(\mathbb{R}^d)$ for all $t > 0$, where $\mathcal{S}(\mathbb{R}^d)$ stands for the Schwartz space of smooth functions with rapid decrease at infinity.

Before we proceed, we need to recall some facts about Hermite polynomials (see e.g. [4], p. 7, but note that we use a different normalization). For $n \in \mathbb{N}_0$, the Hermite polynomial of degree n is

$$H_n(y) = (-1)^n \mathrm{e}^{y^2/2} \frac{\mathrm{d}^n}{\mathrm{d}y^n} \mathrm{e}^{-y^2/2}, \quad y \in \mathbb{R}.$$

Let us recall that, if n is even, one has

$$H_n(y) = \sum_{j=0}^{n/2} a_{n,n-2j} y^{n-2j},$$

and, for n odd,

$$H_n(y) = \sum_{j=0}^{(n-1)/2} a_{n,n-2j} y^{n-2j},$$

where $|a_{n,m}|$ is the number of unordered partitions of the set $\{1, 2, \dots, n\}$ into m singletons and $(n-m)/2$ unordered pairs. Given a multiindex $\alpha \in \mathbb{N}_0^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we set

$$H_\alpha(x) := \prod_{k=1}^d H_{\alpha_k}(x_k).$$

Let us now give an expression for the general mixed partial derivatives of p_t.

Lemma 3.2. *Let $x \in \mathbb{R}^d$ and $t > 0$. For any $\alpha \in \mathbb{N}_0^d$, one has*

$$\partial^\alpha p_t(x) = t^{-|\alpha|/2} (-1)^{|\alpha|} H_\alpha\big(t^{-1/2} x\big) p_t(x).$$

Proof. Writing $x = (x_1, \dots, x_d)$, one has

$$p_t(x) = t^{-d/2} p_1(x/\sqrt{t}) = t^{-d/2} \prod_{k=1}^d p_1(x_k/\sqrt{t}),$$

therefore

$$\partial^\alpha p_t(x) = t^{-d/2} \prod_{k=1}^d D_{x_k}^{\alpha_k} p_1(x_k/\sqrt{t}) = t^{-d/2} \prod_{k=1}^d t^{-\alpha_k/2} p_1^{(\alpha_k)}(x_k/\sqrt{t}).$$

Recalling the definition of Hermite polynomials, one has

$$p_1^{(n)}(y) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2} (-1)^n H_n(y) = (-1)^n H_n(y) p_1(y), \quad y \in \mathbb{R}$$

for all $n \in \mathbb{N}$. This yields

$$\partial^\alpha p_t(x) = t^{-d/2} t^{-|\alpha|/2} (-1)^{|\alpha|} \prod_{k=1}^{d} H_{\alpha_k}\big(t^{-1/2} x_k\big) p_1\big(t^{-1/2} x_k\big)$$
$$= t^{-|\alpha|/2} (-1)^{|\alpha|} H_\alpha\big(t^{-1/2} x\big) p_t(x). \qquad \square$$

With some more effort one can obtain an expression for the general Fréchet derivative of p_t of order $n \in \mathbb{N}$. To this purpose, given any $x \in \mathbb{R}^d$, let us first associate to the Hermite polynomial H_n an n-linear operator $\tilde{H}_n(x) \in \mathcal{L}_n(\mathbb{R}^d)$. It is sufficient to associate to any monomial of the form $a_{n,m} x^m$, $0 \le m \le n$, the following operator in $\mathcal{L}_n(\mathbb{R}^d)$:

$$\bigotimes_{k=1}^{n} \mathbb{R}^d \to \mathbb{R},$$
$$(h_1, \dots, h_n) \mapsto \sum_{\beta \in B(n,m)} \langle x, h_{\beta_1}\rangle \langle x, h_{\beta_2}\rangle \cdots \langle x, h_{\beta_m}\rangle \langle h_{\beta_{m+1}}, h_{\beta_{m+2}}\rangle \cdots \langle h_{\beta n-1}, h_{\beta_n}\rangle,$$

where $B(n,m)$ is the set of all unordered partitions of the set $\{1, 2, \dots, n\}$ into m singletons and $(n-m)/2$ unordered pairs, and we identify $\beta \in B(n,m)$ with the corresponding rearrangement of the set $\{1, 2, \dots, n\}$. Let us give an explicit example: given the Hermite polynomial

$$H_4(y) = y^4 - 6y^2 + 3,$$

one has, for any $x \in \mathbb{R}^d$, $\tilde{H}_4(x) = \tilde{H}_{41}(x) - \tilde{H}_{42}(x) + \tilde{H}_{43}(x)$, where

$$\begin{aligned}
\tilde{H}_{41}(x) &: (h_1, \dots, h_4) \mapsto \langle x, h_1\rangle \cdots \langle x, h_4\rangle, \\
\tilde{H}_{42}(x) &: (h_1, \dots, h_4) \mapsto \langle x, h_1\rangle\langle x, h_2\rangle\langle h_3, h_4\rangle + \langle x, h_1\rangle\langle x, h_3\rangle\langle h_2, h_4\rangle \\
&\qquad + \langle x, h_1\rangle\langle x, h_4\rangle\langle h_2, h_3\rangle \\
&\qquad + \langle x, h_2\rangle\langle x, h_3\rangle\langle h_1, h_4\rangle + \langle x, h_2\rangle\langle x, h_4\rangle\langle h_1, h_3\rangle \\
&\qquad + \langle x, h_3\rangle\langle x, h_4\rangle\langle h_1, h_2\rangle, \\
\tilde{H}_{43}(x) &: (h_1, \dots, h_4) \mapsto \langle h_1, h_2\rangle\langle h_3, h_4\rangle + \langle h_1, h_3\rangle\langle h_2, h_4\rangle + \langle h_1, h_4\rangle\langle h_2, h_3\rangle.
\end{aligned}$$

With these preparations, we can state the following lemma.

Lemma 3.3. *Let $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Then one has*

$$D^n p_t(x) = (-1)^n t^{-n/2} p_t(x) \tilde{H}_n\big(t^{-1/2} x\big). \tag{3.2}$$

Proof. Repeating the computations leading to the definition of the Hermite polynomials, replacing the usual derivative on the real line with the Fréchet derivative, one arrives at

$$D^n p_1(x) = (-1)^n p_1(x) \tilde{H}_n(x).$$

Recalling that $p_t(x) = t^{-d/2} p_1(t^{-1/2} x)$, hence

$$D^n p_t(x) = t^{-n/2} t^{-d/2} D^n p_1\big(t^{-1/2} x\big),$$

we are left with

$$D^n p_t(x) = (-1)^n t^{-n/2} t^{-d/2} p_1\big(t^{-1/2} x\big) \tilde{H}_n\big(t^{-1/2} x\big)$$
$$= (-1)^n t^{-n/2} p_t(x) \tilde{H}_n\big(t^{-1/2} x\big). \qquad \square$$

It should be noted that, for lower values of n, an expression for $D^n p_t(x)$ can be easily obtained by (Fréchet) differentiation of (3.1). For instance,

$$
\begin{aligned}
Dp_t(x) &= -\frac{1}{(2\pi)^{d/2}} t^{-d/2-1} \phi\left(\frac{|x|^2}{t}\right)\langle x, \cdot\rangle = -t^{-1} p_t(x)\langle x, \cdot\rangle, \\
D^2 p_t(x) &= \frac{1}{(2\pi)^{d/2}} t^{-d/2-2} \phi\left(\frac{|x|^2}{t}\right)\langle x, \cdot\rangle\langle x, \cdot\rangle - \frac{1}{(2\pi)^{d/2}} t^{-d/2+1} \phi\left(\frac{|x|^2}{t}\right)\langle \cdot, \cdot\rangle \\
&= t^{-2} p_t(x)\langle x, \cdot\rangle\langle x, \cdot\rangle - t^{-1} p_t(x)\langle \cdot, \cdot\rangle.
\end{aligned}
$$

The following estimate is of central importance for most of the results of this paper.

Theorem 3.4. *Let $k \in \mathbb{N}$, $\ell \geq 0$, $p, q \in [1, \infty]$, and $t > 0$. If $f \in L_p(\mathbb{R}^d)$ is such that $y \mapsto |y|^\ell f(y) \in L_q(\mathbb{R}^d)$, then, setting*

$$
\|f\|_{p,q,\ell} = \|f\|_{L_p(\mathbb{R}^d)} + \big\||\cdot|^\ell f\big\|_{L_q(\mathbb{R}^d)},
$$

one has

$$
|x|^\ell \big\|D^k P_t^Y f(x)\big\|_{\mathcal{L}_k(\mathbb{R}^d)} \lesssim \|f\|_{p,q,\ell}\big(\mathbb{E}T_t^{(\ell-k)/2-d/(2p)} + \mathbb{E}T_t^{-k/2-d/(2q)}\big) \quad \forall x \in \mathbb{R}^d.
$$

Proof. Taking the norm in $\mathcal{L}_k(\mathbb{R}^d)$ on both sides of (3.2) yields

$$
\big\|D^k p_t(x)\big\|_{\mathcal{L}_k(\mathbb{R}^d)} \lesssim \big(t^{-k}|x|^k + t^{-k/2}\big) p_t(x) \quad \forall x \in \mathbb{R}^d, \tag{3.3}
$$

thus also

$$
\begin{aligned}
\big\|D^k P_t^W f(x)\big\|_{\mathcal{L}_k(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} \big|f(y)\big| \big\|D^k p_t(x-y)\big\|_{\mathcal{L}_k(\mathbb{R}^d)}\,\mathrm{d}y \\
&\lesssim t^{-k/2} \int_{\mathbb{R}^d} \big|f(y)\big|\big(1 + t^{-k/2}|x-y|^k\big) p_t(x-y)\,\mathrm{d}y.
\end{aligned}
$$

Multiplying both sides by $|x|^\ell$ and using the triangle inequality, one gets

$$
\begin{aligned}
&|x|^\ell \big\|D^k P_t^W f(x)\big\|_{\mathcal{L}_k(\mathbb{R}^d)} \\
&\quad\lesssim t^{-k/2} \int_{\mathbb{R}^d} \big|f(y)\big||x-y|^\ell \big(1 + t^{-k/2}|x-y|^k\big) p_t(x-y)\,\mathrm{d}y \\
&\qquad + t^{-k/2} \int_{\mathbb{R}^d} \big|f(y)\big||y|^\ell \big(1 + t^{-k/2}|x-y|^k\big) p_t(x-y)\,\mathrm{d}y \\
&\quad=: t^{-k/2}(I_1 + I_2).
\end{aligned}
$$

Thanks to Hölder's and Minkowski's inequalities, denoting by $p' \in [1, \infty]$ the conjugate exponent of p, it holds

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} f(y)\big(|x-y|^\ell + t^{-k/2}|x-y|^{k+\ell}\big) p_t(x-y)\,\mathrm{d}y \\
&\leq \|f\|_{L_p(\mathbb{R}^d)}\big(\big\||\cdot|^\ell p_t\big\|_{L_{p'}(\mathbb{R}^d)} + t^{-k/2}\big\||\cdot|^{k+\ell} p_t\big\|_{L_{p'}(\mathbb{R}^d)}\big) \\
&=: \|f\|_{L_p(\mathbb{R}^d)}(I_{11} + I_{12}).
\end{aligned}
$$

Note that one has, by well-known scaling properties of the heat kernel p_t,

$$\begin{aligned}\big\||\cdot|^\ell p_t\big\|_{L_{p'}(\mathbb{R}^d)} &= t^{-d/2}\bigg(\int_{\mathbb{R}^d}|x|^{\ell p'}p_1^{p'}\big(t^{-1/2}x\big)\,\mathrm{d}x\bigg)^{1/p'}\\ &= t^{-d/2}\bigg(\int_{\mathbb{R}^d}\big|t^{1/2}y\big|^{\ell p'}p_1^{p'}(y)t^{d/2}\,\mathrm{d}y\bigg)^{1/p'}\\ &= t^{\ell/2+d/(2p')-d/2}\big\||\cdot|^\ell p_1\big\|_{L_{p'}(\mathbb{R}^d)},\end{aligned}$$

hence

$$I_{11} = t^{\ell/2+d/(2p')-d/2}\big\||\cdot|^\ell p_1\big\|_{L_{p'}(\mathbb{R}^d)}, \qquad I_{12} = t^{\ell/2+d/(2p')-d/2}\big\||\cdot|^{k+\ell} p_1\big\|_{L_{p'}(\mathbb{R}^d)},$$

and

$$I_1 \le \|f\|_{L_p(\mathbb{R}^d)}t^{\ell/2+d/(2p')-d/2}\big(\big\||\cdot|^\ell p_1\big\|_{L_{p'}(\mathbb{R}^d)} + \big\||\cdot|^{k+\ell} p_1\big\|_{L_{p'}(\mathbb{R}^d)}\big).$$

Similarly, one has

$$\begin{aligned}I_2 &\le \big\||\cdot|^\ell f\big\|_{L_q(\mathbb{R}^d)}\big(\|p_t\|_{L_{q'}(\mathbb{R}^d)} + t^{-k/2}\big\||\cdot|^k p_t\big\|_{L_{q'}(\mathbb{R}^d)}\big)\\ &= \big\||\cdot|^\ell f\big\|_{L_q(\mathbb{R}^d)}t^{d/(2q')-d/2}\big(\|p_1\|_{L_{q'}(\mathbb{R}^d)} + \big\||\cdot|^k p_1\big\|_{L_{q'}(\mathbb{R}^d)}\big).\end{aligned}$$

Collecting estimates, we are left with

$$|x|^\ell\big\|D^kP_t^Wf(x)\big\|_{\mathcal{L}_k(\mathbb{R}^d)} \lesssim N_1\|f\|_{L_p(\mathbb{R}^d)}t^{(\ell-k)/2+d/(2p')-d/2} + N_2\big\||\cdot|^\ell f\big\|_{L_q(\mathbb{R}^d)}t^{-k/2+d/(2q')-d/2},$$

which implies

$$\begin{aligned}&|x|^\ell\big\|D^kP_t^Yf(x)\big\|_{\mathcal{L}_k(\mathbb{R}^d)}\\ &\quad\le \int_0^\infty |x|^\ell\big\|D^kP_s^Wf(x)\big\|_{\mathcal{L}_k(\mathbb{R}^d)}\nu_t(\mathrm{d}s)\\ &\quad\lesssim \big(\|f\|_{L_p(\mathbb{R}^d)} + \big\||\cdot|^\ell f\big\|_{L_q(\mathbb{R}^d)}\big)\int_0^\infty\big(s^{(\ell-k)/2+d/(2p')-d/2} + s^{-k/2+d/(2q')-d/2}\big)\nu_t(\mathrm{d}s)\\ &\quad= \big(\|f\|_{L_p(\mathbb{R}^d)} + \big\||\cdot|^\ell f\big\|_{L_q(\mathbb{R}^d)}\big)\big(\mathbb{E}T_t^{(\ell-k)/2-d/(2p)} + \mathbb{E}T_t^{-k/2-d/(2q)}\big).\end{aligned}$$

□

Taking $p = q = \infty$, one immediately has the following regularizing property.

Corollary 3.5. *Let $k \in \mathbb{N}$, $\ell \ge 0$, and $t > 0$. If $f \in \mathcal{B}_{b,c}(\mathbb{R}^d)$, then, setting*

$$M_\ell = \|f\|_\infty + \sup_{y\in\mathbb{R}^d}|y|^\ell\big|f(y)\big|,$$

one has

$$|x|^\ell\big\|D^kP_t^Yf(x)\big\|_{\mathcal{L}_k(\mathbb{R}^d)} \lesssim M_\ell\mathbb{E}\big[T_t^{-k/2} + T_t^{(\ell-k)/2}\big] \quad \forall x\in\mathbb{R}^d.$$

As a further immediate consequence of the previous theorem (taking $p = q = \infty$ and $\ell = 0$) we obtain a sufficient condition for the semigroup associated to a Lévy process, obtained by subordination of a Wiener process, to be k-smoothing.

Corollary 3.6. *Let $k \in \mathbb{N}$ and $t > 0$. If*

$$\mathbb{E}T_t^{-k/2} < \infty, \tag{3.4}$$

then P_t^Y is k-smoothing, i.e. $P_t^Y f \in C_b^k(\mathbb{R}^d)$ for any $f \in \mathcal{B}_b(\mathbb{R}^d)$.

Similarly, taking $\ell = 0$ and $p = q$, one gets sufficient conditions for the semigroup P^Y to be L_p-strong Feller, $1 \le p \le \infty$, or, more generally, to map $L_p(\mathbb{R}^d)$ into $C_b^k(\mathbb{R}^d)$.

Corollary 3.7. *Let $p \in [1, \infty]$, $k \in \mathbb{N}_0$ and $t > 0$. If*

$$\mathbb{E}T_t^{-(1/2)(k+d/p)} < \infty,$$

then $P_t^Y f \in C_b^k(\mathbb{R}^d)$ for all $f \in L_p(\mathbb{R}^d)$. In particular, if $\mathbb{E}T_t^{-d/2p} < \infty$, then P_t^Y is L_p-strong Feller.

This result can be extended to (positive) real values of k, in which case the space of differentiable functions C_b^k has to replaced by Hölder spaces.

Proposition 3.8. *Let $p \in [1, \infty]$, $\beta > 0$ real, and $t > 0$. If*

$$\mathbb{E}T_t^{-(1/2)(\beta+d/p)} < \infty,$$

then $P_t^Y f \in C_b^\beta(\mathbb{R}^d)$ for all $f \in L_p(\mathbb{R}^d)$.

Proof. It is easily seen that it is enough to consider the case $0 < \beta < 1$. Denoting the conjugate exponent of p by p', one has

$$\begin{aligned}
&\frac{|P_t^W f(x_1) - P_t^W f(x_2)|}{|x_1 - x_2|^\beta} \\
&\quad \le \int_{\mathbb{R}^d} \bigl|f(y)\bigr| \frac{|p_t(x_1 - y) - p_t(x_2 - y)|}{|x_1 - x_2|^\beta}\, \mathrm{d}y \\
&\quad = \int_{\mathbb{R}^d} \bigl|f(y)\bigr| \left|\frac{p_t(x_1 - y) - p_t(x_2 - y)}{(x_1 - y) - (x_2 - y)}\right|^\beta \bigl|p_t(x_1 - y) - p_t(x_2 - y)\bigr|^{1-\beta}\, \mathrm{d}y \\
&\quad \le \|f\|_{L_p} \left(\int_{\mathbb{R}^d} \left|\frac{p_t(x_1 - y) - p_t(x_2 - y)}{(x_1 - y) - (x_2 - y)}\right|^{\beta p'} \bigl|p_t(x_1 - y) - p_t(x_2 - y)\bigr|^{(1-\beta)p'}\, \mathrm{d}y\right)^{1/p'}.
\end{aligned}$$

Moreover, recalling the scaling properties of p_t, one has

$$\begin{aligned}
\frac{|p_t(x) - p_t(y)|}{|x - y|} &= t^{-d/2} \frac{|p_1(t^{-1/2}x) - p_1(t^{-1/2}y)|}{|x - y|} \\
&= t^{-d/2} t^{-1/2} \frac{|p_1(t^{-1/2}x) - p_1(t^{-1/2}y)|}{|t^{-1/2}x - t^{-1/2}y|} \\
&\le t^{-d/2} t^{-1/2} \|p_1\|_{\dot{C}^{0,1}},
\end{aligned}$$

where $\|p_1\|_{\dot{C}^{0,1}}$ stands for the Lipschitz constant of p_1. The latter implies

$$\begin{aligned}
&\frac{|P_t^W f(x_1) - P_t^W f(x_2)|}{|x_1 - x_2|^\beta} \\
&\quad \le \|f\|_\infty \|p_1\|_{\dot{C}^{0,1}}^\beta t^{-\beta/2} t^{-d\beta/2} \left(\int_{\mathbb{R}^d} \bigl|p_t(x_1 - y) - p_t(x_2 - y)\bigr|^{(1-\beta)p'}\, \mathrm{d}y\right)^{1/p'}.
\end{aligned}$$

Again by the scaling properties of p_t, as well as elementary inequalities, and changing variable a few times, one gets

$$\left(\int_{\mathbb{R}^d} \left|p_t(x_1-y)-p_t(x_2-y)\right|^{(1-\beta)p'}\,\mathrm{d}y\right)^{1/p'} \lesssim \left(\int_{\mathbb{R}^d} \left|p_t(y)\right|^{(1-\beta)p'}\,\mathrm{d}y\right)^{1/p'}$$
$$\leq t^{-d(1-\beta)/2} t^{(d/2)(1/p')} \left\| |p_1|^{1-\beta} \right\|_{L_{p'}}.$$

We have thus obtained

$$\sup_{x_1\neq x_2} \frac{|P_t^W f(x_1) - P_t^W f(x_2)|}{|x_1-x_2|^\beta} \lesssim \|f\|_{L_p(\mathbb{R}^d)} t^{-\beta/2-d/(2p)}.$$

By subordination and Minkowski's inequality, this implies

$$\|P_t^Y f\|_{\dot{C}_b^\beta} := \sup_{x_1\neq x_2} \frac{|P_t^Y f(x_1) - P_t^Y f(x_2)|}{|x_1-x_2|^\beta} \lesssim \|f\|_{L_p(\mathbb{R}^d)} \mathbb{E}T_t^{-(1/2)(\beta-d/p)}.$$

Since $\mathbb{E}T_t^{-d/2p} \leq \mathbb{E}T_t^{-(1/2)(\beta-d/p)} < \infty$, the previous corollary yields $P_t^Y f \in C_b$, allowing us to conclude that $\|P_t^Y f\|_{C_b^\beta} = \|P_t^Y f\|_\infty + \|P_t^Y f\|_{\dot{C}_b^\beta} < \infty$, which is the desired result. □

We shall see in Section 3.1 (in particular, cf. Corollary 3.16) that the last corollary can be obtained, at least for $1 < p < \infty$, also by results on embeddings of Sobolev spaces.

Remark 3.9. *Theorem 3.4, thus also its corollaries, continue to hold also in the more general case that the covariance matrix of W is $Q \neq I$ with $\det Q \neq 0$. In fact, in this case the density of the Gaussian random variable W_t, $t > 0$, is*

$$p_t^Q(x) = \frac{1}{t^{d/2}} \frac{1}{\sqrt{\det Q}} \frac{1}{(2\pi)^{d/2}} \exp\left(\frac{\langle Q^{-1}x, x\rangle}{2t}\right),$$

which can be written as

$$p_t^Q(x) = \frac{1}{\sqrt{\det Q}} p_t\bigl(Q^{-1/2}x\bigr).$$

One can now establish a corresponding version of Lemma 3.3, e.g. introducing the following equivalent scalar product and norm in $\mathbb{R}^d$:

$$\langle x, y\rangle_Q := \bigl\langle Q^{-1}x, y\bigr\rangle, \qquad \|x\|^2 := \langle x, x\rangle_Q,$$

and computing the Fréchet derivatives under the "new" topology (recall that Fréchet differentiability does not depend on the metric properties of the underlying space). It is easily seen that an expression completely analogous to (3.2) still holds, if one uses the scalar product $\langle\cdot,\cdot\rangle_Q$ in place of the natural one. One gets, for instance,

$$D^2 p_t^Q(x) = t^{-2} p_t^Q(x) \bigl\langle Q^{-1}x, \cdot\bigr\rangle\bigl\langle Q^{-1}x, \cdot\bigr\rangle - t^{-1} p_t^Q(x)\bigl\langle Q^{-1}\cdot, \cdot\bigr\rangle.$$

Moreover, since the norms $|\cdot|$ and $\|\cdot\|$ are equivalent, the estimate (3.3) continues to hold also if p_t is replaced by p_t^Q.

We are now going to provide a simple, yet very useful, sufficient and necessary condition for the finiteness of negative moments of subordinators in terms of their Laplace exponent.

Proposition 3.10. *Let $1 \leq p < \infty$ and $t > 0$. Then $\mathbb{E}T_t^{-p} < \infty$ if and only if $\lambda \mapsto \lambda^{p-1} \mathrm{e}^{-t\Phi(\lambda)} \in L_1$.*

Proof. By definition of gamma function, that is

$$\Gamma(p)=\int_0^\infty z^{p-1}\mathrm{e}^{-z}\,\mathrm{d}z,$$

one gets, by the change of variable $z=as$, $a>0$,

$$a^{-p}=\frac{1}{\Gamma(p)}\int_0^\infty s^{p-1}\mathrm{e}^{-as}\,\mathrm{d}s.$$

This implies, by Tonelli's theorem,

$$\mathbb{E}T_t^{-p}=\frac{1}{\Gamma(p)}\int_0^\infty \lambda^{p-1}\mathbb{E}\mathrm{e}^{-\lambda T_t}\,\mathrm{d}\lambda=\frac{1}{\Gamma(p)}\int_0^\infty \lambda^{p-1}\mathrm{e}^{-t\Phi(\lambda)}\,\mathrm{d}\lambda,$$

thus finishing the proof. □

It is also possible to give a sufficient condition for the finiteness of negative moments of subordinators in terms of their Lévy measure. The following proposition is a special case of a Tauberian theorem due to Bismut [3], Theorem 4.15, p. 208.

Proposition 3.11. *Let* $0<p<1$, $C>0$. *If*

$$m\bigl(]x,+\infty[\bigr)\sim Cx^{-p}\quad \textit{as } x\to 0^+, \tag{3.5}$$

then

$$\Phi(\lambda)\sim -C\Gamma(1-p)\lambda^{-p}\quad \textit{as } \lambda\to+\infty.$$

In particular, (3.5) *implies* $\mathbb{E}T_t^{-p}<\infty$ *for all* $t>0$.

As a noteworthy application of the criteria just proved we recover a (known) result on smoothness of α-stable densities.

Corollary 3.12. *Let Y be an $\mathbb{R}^d$-valued rotation-invariant α-stable process. Then, for any $f\in\mathcal{B}_b(\mathbb{R}^d)$, one has $P_t^Y f\in C_b^\infty(\mathbb{R}^d)$ for all $t>0$.*

Proof. One can write $Y=W\circ T$ in distribution, where T is an $\alpha/2$-stable subordinator (see e.g. [27]). In particular one has $\Phi(\lambda)\propto\lambda^{\alpha/2}$, hence $\mathrm{e}^{-t\Phi(\cdot)}$ is rapidly decreasing for all $t>0$ and $\lambda\mapsto\lambda^{p-1}\mathrm{e}^{-t\Phi(\lambda)}\in L_1$ for all $p\geq 1$. The desired result then follows by Corollary 3.6 and Proposition 3.10. □

Example 3.13 (Variance-gamma processes). *Let $Y=W\circ T$, where T is a Gamma process, independent of W, with parameters a and b. Such a process Y is often called (especially in the literature on mathematical finance) a variance-gamma process. It is known (see e.g.* [2], *p.* 73) *that, for any $t>0$, the random variable T_t admits the density*

$$x\mapsto\frac{b^{at}}{\Gamma(at)}x^{at-1}\mathrm{e}^{-bx},\quad x\geq 0.$$

Elementary calculations based on the definition and properties of the Gamma function yield that $\mathbb{E}T_t^{-k/2}<\infty$ if and only if $t>k/(2a)$. Therefore, according to Corollary 3.6, *P_t^Y is k-smoothing for all $t>k/(2a)$ (and of course it is strong Feller for all $t>0$). As we shall see in the next section, the semigroup associated to the variance-gamma process Y is indeed only "eventually" regularizing.*

3.1. *Smoothing in scales of Sobolev and Bessel spaces*

The results of this subsection are not used in the rest of the paper. We have nonetheless included them here because they are, in our opinion, an interesting complement to the smoothing properties of the semigroup P^Y proved above.

We first consider mapping properties of P^Y from $L_p(\mathbb{R}^d)$ to integer-order Sobolev spaces $W_p^m(\mathbb{R}^d)$. Note that also the endpoint cases $p=1$ and $p=\infty$ are included.

Proposition 3.14. *Let $1 \le p \le \infty$, $f \in L_p(\mathbb{R}^d)$, and $m \in \mathbb{N}_0$. If $\mathbb{E}T_t^{-m/2} < \infty$, then $P_t^Y f \in W_p^m(\mathbb{R}^d)$ with*

$$\left\|\partial^\alpha P_t^Y f\right\|_{L_p(\mathbb{R}^d)} \lesssim \|f\|_{L_p(\mathbb{R}^d)} \mathbb{E}T_t^{-m/2}$$

for all multiindices α such that $|\alpha| = m$.

Proof. For any multiindex α, Lemma 3.2 and the scaling properties of the heat kernel p_t yield

$$\begin{aligned}\partial^\alpha P_t^W f &= t^{-|\alpha|/2}(-1)^{|\alpha|}\int_{\mathbb{R}^d} f(y) H_\alpha\big(t^{-1/2}(x-y)\big) p_t(x-y)\,\mathrm{d}y \\ &= t^{-|\alpha|/2} t^{-d/2}(-1)^{|\alpha|}\int_{\mathbb{R}^d} f(y) H_\alpha\big(t^{-1/2}(x-y)\big) p_1\big(t^{-1/2}(x-y)\big)\,\mathrm{d}y,\end{aligned}$$

hence, by Young's inequality and the change of variable formula, one obtains

$$\begin{aligned}\left\|\partial^\alpha P_t^W f\right\|_{L_p(\mathbb{R}^d)} &\le t^{-|\alpha|/2} t^{-d/2}\|f\|_{L_p(\mathbb{R}^d)}\int_{\mathbb{R}^d}\big|H_\alpha\big(t^{-1/2}x\big)\big| p_1\big(t^{-1/2}x\big)\,\mathrm{d}x \\ &= t^{-|\alpha|/2}\|f\|_{L_p(\mathbb{R}^d)}\int_{\mathbb{R}^d}\big|H_\alpha(x)\big| p_1(x)\,\mathrm{d}x.\end{aligned}$$

We have thus shown that one has

$$\left\|\partial^\alpha P_t^W f\right\|_{L_p} \lesssim \|f\|_{L_p} t^{-|\alpha|/2},$$

which also implies, for any α such that $|\alpha| = m$,

$$\left\|\partial^\alpha P_t^Y f\right\|_{L_p} \lesssim \|f\|_{L_p}\mathbb{E}T_t^{-m/2}.$$

In fact, by Minkowski's inequality, one has

$$\begin{aligned}\left\|\partial^\alpha P_t^Y f\right\|_{L_p} &= \left\|\int_0^\infty \partial^\alpha P_s^W f\,\nu_t(\mathrm{d}s)\right\|_{L_p} \le \int_0^\infty \left\|\partial^\alpha P_s^W f\right\|_{L_p}\nu_t(\mathrm{d}s) \\ &\lesssim \|f\|_{L_p}\int_0^\infty s^{-|\alpha|/2}\nu_t(\mathrm{d}s) = \|f\|_{L_p}\mathbb{E}T_t^{-|\alpha|/2}.\end{aligned}$$

This immediately implies that $P_t^Y f \in W_p^m$ by virtue of the well-known estimate $\|\phi\|_{W_p^m} \lesssim \|\phi\|_{L_p} + \sum_{|\alpha|=m}\|\partial^\alpha\phi\|_{L_p}$. □

An analogous result can be obtained in the scale of Bessel potential spaces $H_p^r(\mathbb{R}^d)$, $1 < p < \infty$, $r \ge 0$ real, thus generalizing the previous result, even though the endpoint cases $p=1$ and $p=\infty$ are not included. For the proof we need to recall some facts about analytic semigroups. Let $-A$ be a linear operator on a Banach space E, generating an analytic semigroup of contractions $S_t = \mathrm{e}^{-tA}$, $t > 0$ (see e.g. [24] for details). Then one has, for any $\alpha \ge 0$ and $t > 0$,

$$\left\|A^\alpha S(t) f\right\|_E \lesssim \frac{1}{t^\alpha}\|f\|_E. \tag{3.6}$$

Letting $p \in]1,\infty[$, $E = L_p$, and $A = -\Delta$, one can show that existence of negative moments of the subordinator T_t implies that P_t^Y maps L_p to a Bessel potential space.

Proposition 3.15. *Let $1 < p < \infty$. Assume that there exist $r > 0$ and $t > 0$ such that $\mathbb{E}T_t^{-r/2} < \infty$. Then $P_t^Y(L_p) \subseteq H_p^r$, with*

$$\|P_t^Y\|_{L_p \to \dot{H}_p^r} \lesssim \mathbb{E}T_t^{-r/2}.$$

Proof. Since $-\frac{1}{2}\Delta$ is the generator of P^W, which is an analytic semigroup of contractions in L_p, estimate (3.6) reads

$$\|(-\Delta)^{r/2} P_t^W f\|_{L_p} \lesssim \frac{1}{t^{r/2}} \|f\|_{L_p},$$

which in turn implies

$$\begin{aligned}\|P_t^Y f\|_{\dot{H}_p^r} &= \|(-\Delta)^{r/2} P_t^Y f\|_{L_p} = \left\|(-\Delta)^{r/2} \int_0^\infty P_s^W f \, \nu_t(\mathrm{d}s)\right\|_{L_p} \\ &\le \int_0^\infty \|(-\Delta)^{r/2} P_s^W f\|_{L_p} \nu_t(\mathrm{d}s) \\ &\lesssim \|f\|_{L_p} \int_0^\infty \frac{1}{s^{r/2}} \nu_t(\mathrm{d}s) = \|f\|_{L_p} \mathbb{E}T_t^{-r/2}.\end{aligned}$$

The proof is concluded, upon recalling (2.1) and that the semigroup associated to any Lévy process is contracting in L_p (the latter fact follows easily by Young's inequality for convolutions). □

As a consequence, we partially recover the result of Proposition 3.8 on mapping properties of P^Y from L_p spaces to Hölder spaces (note that the endpoint cases $p = 1$ and $p = \infty$ are excluded).

Corollary 3.16. *Let $1 < p < \infty$, $f \in L_p(\mathbb{R}^d)$, and assume that $\mathbb{E}T_t^{-r/2} < \infty$ for some $r \in \mathbb{R}$ such that $\sigma := r - d/p > 0$. Then $P_t^Y f \in C_b^\sigma(\mathbb{R}^d)$.*

Proof. If $\sigma \notin \mathbb{N}$, the result follows by Sobolev embedding (Lemma 2.6), and by Corollary 3.7 if $\sigma \in \mathbb{N}$. □

Remark 3.17. *If Y is a rotationally invariant α-stable process on $\mathbb{R}^d$, then the generator of the (analytic) semigroup P^Y coincides with $(-\Delta)^{\alpha/2}$. Therefore, by well-known properties of analytic semigroups, one has $P_t^Y f \in \mathrm{dom}((-\Delta)^{k\alpha})$ for all $k > 0$ and $t > 0$, i.e. $f \in L_p(\mathbb{R}^d)$, $1 < p < \infty$, implies $P_t^Y f \in \dot{H}_p^\sigma(\mathbb{R}^d)$ for all $\sigma > 0$ and $t > 0$, hence also $P_t^Y f \in H_p^\sigma(\mathbb{R}^d)$ because P_t^Y is contracting in $L_p(\mathbb{R}^d)$. By the Sobolev embedding theorem, this implies that $P_t^Y f \in C_b^\infty(\mathbb{R}^d)$ for all $f \in L_p(\mathbb{R}^d)$. The same conclusion can be reached applying the previous corollary, recalling that stable subordinators have finite negative moments of all orders (cf. the proof of Corollary 3.12).*

3.2. *Smoothing properties of P_t^Z*

Thus far we have only considered smoothing properties of Markovian semigroups associated to subordinated Wiener processes. In this subsection we provide a simple argument which allows to extend the results of the previous section to a much larger class of semigroups.

In particular, let $Z = Y + \xi$, where $Y = W \circ T$ is a subordinated Wiener process, and ξ is a further Lévy process, independent of Y. Then Z is a Lévy process, and we denote by P^Z its associated semigroup, defined in the usual way, i.e.

$$P_t^Z f(x) := \mathbb{E} f(x + Z_t) \equiv \mathbb{E} f(x + Y_t + \xi_t), \quad f \in \mathcal{B}_b\big(\mathbb{R}^d\big).$$

This notation will be used throughout this subsection without further notice.

Proposition 3.18. *Let $t > 0$ and assume that P_t^Y is k-smoothing. Then P_t^Z is also k-smoothing. Moreover, if there exists $\beta > 0$ such that*

$$\left\| D^k P_t^Y f(x) \right\|_{\mathcal{L}_k(\mathbb{R}^d)} \lesssim t^{-\beta} \|f\|_\infty \quad \forall x \in \mathbb{R}^d, \forall f \in \mathcal{B}_b\big(\mathbb{R}^d\big),$$

then the same estimate is satisfied with P_t^Y replaced by P_t^Z.

Proof. Let $f \in \mathcal{B}_b(\mathbb{R}^d)$ and α be a multiindex such that $|\alpha| \le k$. Since ξ is independent of Y, we have

$$\begin{aligned} \partial^\alpha P_t^Z f(x) &= \partial^\alpha \mathbb{E} f(x + Y_t + \xi_t) \\ &= \partial^\alpha \mathbb{E}\mathbb{E}\big[f(x + Y_t + \xi_t)|\xi_t\big] \\ &= \partial^\alpha \int_{\mathbb{R}^d} P_t^Y f(x+y)\mu_t(\mathrm{d}y), \end{aligned}$$

where μ_t, the distribution of the random variable ξ_t, is a probability measure on $\mathbb{R}^d$. Since $P_t^Y f \in C_b^k(\mathbb{R}^d)$, the dominated convergence theorem implies that one can differentiate under the integral sign, obtaining

$$\partial^\alpha P_t^Z f(x) = \int_{\mathbb{R}^d} \big[\partial^\alpha P^Y f\big](x+y)\mu_t(\mathrm{d}y). \tag{3.7}$$

Let $(x_n)_{n\in\mathbb{N}}$ a sequence converging to x in $\mathbb{R}^d$. The dominated convergence theorem yields

$$\lim_{x_n \to x} \int_{\mathbb{R}^d} \big[\partial^\alpha P_t^Y f\big](x_n+y)\mu_t(\mathrm{d}y) = \int_{\mathbb{R}^d} \big[\partial^\alpha P_t^Y f\big](x+y)\mu_t(\mathrm{d}y),$$

hence, by (3.7), that $\partial^\alpha P_t^Z f$ is continuous, or, equivalently, that $P_t^Z f \in C^k(\mathbb{R}^d)$. Moreover, by Minkowski's inequality, one has

$$\left\|\partial^\alpha P_t^Z f\right\|_\infty \le \int_{\mathbb{R}^d} \left\|\partial^\alpha P_t^Y f\right\|_\infty \mu_t(\mathrm{d}y) = \left\|\partial^\alpha P_t^Y f\right\|_\infty, \tag{3.8}$$

which implies $\partial^\alpha P_t^Z f \in C_b(\mathbb{R}^d)$, hence also $P_t^Z f \in C_b^k(\mathbb{R}^d)$. The second assertion is an obvious consequence of (3.8). □

Thanks to the previous proposition, we have the following smoothing result, generalizing Corollary 3.5.

Corollary 3.19. *Let $k \in \mathbb{N}$ and $t > 0$. Assume that $Z = Y + \xi$, where $Y = W \circ T$ and*

$$\mathbb{E} T_t^{-k/2} < \infty.$$

Then P_t^Z is k-smoothing, i.e. $P_t^Z f \in C_b^k(\mathbb{R}^d)$ for any $f \in \mathcal{B}_b(\mathbb{R}^d)$. Moreover, one has

$$\left\| D^k P_t^Z f(x) \right\|_{\mathcal{L}_k(\mathbb{R}^d)} \lesssim \|f\|_\infty \mathbb{E} T_t^{-k/2} \quad \forall x \in \mathbb{R}^d, \forall f \in \mathcal{B}_b\big(\mathbb{R}^d\big).$$

The above argument holds even if ξ is not Markovian. Let $\tilde{Z}_t^x = x + Y_t + \eta_t$ where Y is a subordinated Wiener process as above and η is *any* stochastic process independent of Y such that $\eta_0 = 0$ almost surely (in particular it is not necessary to assume that η is a Lévy nor a Markov process). Let us define, for $t \ge 0$, define a bounded linear operator $A_t^{\tilde{Z}}$ on $\mathcal{B}_b(\mathbb{R}^d)$ by

$$A_t^{\tilde{Z}} f(x) := \mathbb{E} f\big(\tilde{Z}_t^x\big) = \mathbb{E} f(x + Y_t + \eta_t), \quad f \in \mathcal{B}_b\big(\mathbb{R}^d\big).$$

We remark that $t \mapsto A_t^{\tilde{Z}}$ is, in general, not a semigroup. Nonetheless, the above results still hold in this case, with exactly the same proofs, and we have the following proposition.

Proposition 3.20. *Let $k \in \mathbb{N}$ and $t > 0$. Assume that*

$$\mathbb{E}T_t^{-k/2} < \infty.$$

Then $A_t^{\tilde{Z}}$ is k-smoothing. Moreover, one has

$$\left\| D^k A_t^{\tilde{Z}} f(x) \right\|_{\mathcal{L}_k(\mathbb{R}^d)} \le C \|f\|_\infty \mathbb{E}T_t^{-k/2} \quad \forall x \in \mathbb{R}^d, \forall f \in B_b\big(\mathbb{R}^d\big),$$

where C is a constant depending only on d and k.

Remark 3.21. *Proposition 3.20 implies that smoothing properties holds for $A_t^{\tilde{Z}}$, even if the driving noise is not Lévy process. For example, let ξ be a $\mathbb{R}^d$-valued fractional Brownian motion. Then, the driving noise is no more Lévy noise, but the estimate in Proposition 3.20 holds.*

3.3. *On smoothing properties of general Lévy processes*

It is well known (and it was used in the proof of Lemma 3.1) that the operation of convolution with an L_1 function is strong Feller. In fact the converse result is true as well, as it was proved by Hawkes [11] (the result was actually already proved, using different terminology, by Brainerd and Edwards [6]). We state their result, and provide the (short) proof for completeness.

Proposition 3.22 (Brainerd and Edwards, Hawkes). *Let μ be a finite measure on $\mathbb{R}^d$, and consider the linear operator A_μ defined as*

$$[A_\mu f](x) = \int_{\mathbb{R}^d} f(x-y)\mu(\mathrm{d}y).$$

Then the following assertions are equivalent:

(a) *μ is absolutely continuous with respect to Lebesgue measure*;
(b) *A_μ is strong Feller*;
(c) *A_μ is c-strong Feller.*

Proof. (a) implies (b): let us denote, with an harmless abuse of notation, the density of μ again by μ. Then $\mu \in L_1$, hence $A_\mu f = f * \mu$, and, as already recalled above, A_μ maps $\mathcal{B}_b$ to C_b. Obviously (b) implies (c). In order to conclude, we only have to show that (c) implies (a). Define the measure $\tilde{\mu}$ by $B \mapsto \tilde{\mu}(B) := \mu(-B)$, and observe that the linear operator

$$[\tilde{A}_\mu f](x) := \int_{\mathbb{R}^d} f(x-y)\tilde{\mu}(\mathrm{d}y)$$

is the formal adjoint of A_μ, in the sense that, for any $f, g \in \mathcal{B}_b$, one has

$$\int_{\mathbb{R}^d} A_\mu f g = \int_{\mathbb{R}^d} f \tilde{A}_\mu g.$$

Assume that $B \subset \mathbb{R}^d$ has zero Lebesgue measure, and write $B = \bigcup_{n\in\mathbb{N}} B_n$, with $B_n := B \cap E_n$, where $\bigcup_{n\in\mathbb{N}} E_n = \mathbb{R}^d$, and E_n is bounded for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary but fixed, and set $f := 1_{B_n}$. Then $f \in \mathcal{B}_{b,c}$ and thus, by hypothesis, $A_\mu f \in C_b$ and

$$\int_{\mathbb{R}^d} A_\mu f g = 0 \quad \forall g \in C_c^\infty,$$

hence $A_\mu f(x) = 0$ for almost all $x \in \mathbb{R}^d$ with respect to the Lebesgue measure. In particular,

$$0 = A_\mu f(0) = \int_{\mathbb{R}^d} 1_{B_n} \, \mathrm{d}\mu = \mu(B_n),$$

thus also $\mu(B) = 0$ because B is the union of countably many sets of μ-measure zero. □

Remark 3.23. (i) *The assumption that μ is a finite measure is essential in the previous proposition. In fact, if one just assumes that μ has a density in $L_{1,\mathrm{loc}}$, it is easily seen that* (a) *does not imply* (b), *and that* (b) *actually implies $\mu \in L_1$* (*e.g. choosing $f \equiv 1$*). *Moreover, without assuming that μ is a finite measure*, (c) *does indeed only imply $\mu \in L_{1,\mathrm{loc}}$.*

(ii) *As a consequence of the above proposition, Hawkes* [11] *shows that the semigroup generated by a* (*finite dimensional*) *Lévy process is strong Feller if and only if its transition densities are absolutely continuous with respect to Lebesgue measure and that, in this case, the densities are lower semicontinuous.*

An immediate, but nonetheless quite useful consequence of the previous proposition is the following.

Corollary 3.24. *Let $k \in \mathbb{N} \cup \{0\}$. If μ admits a density belonging to W_1^k, then A_μ is k-smoothing.*

Unfortunately, however, it is not possible to assert that if A_μ is k-smoothing, then $\mu \in W_1^k$. This follows immediately from Theorem 3.25 below.

The following result by Knopova and Schilling [14], which extends a "classical" criterion by Hartman and Wintner [10], says, among other things, that the transition semigroup of a Lévy process is 1-smoothing at all (positive) times if and only if it is ∞-smoothing at all (positive) times.

Theorem 3.25 ([14]). *Let Y be an $\mathbb{R}^d$-valued Lévy process without Gaussian component, with characteristic function $\mathbb{E}\exp(\mathrm{i}t\langle\xi, Y_t\rangle) =: \mathrm{e}^{-t\psi(\xi)}$ and transition kernel $\pi_t : B \mapsto \mathbb{P}(Y_t \in B)$. The following assertions are equivalent*:

(a) *It holds*

$$\lim_{|\xi|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\log(1+|\xi|)} = \infty; \tag{3.9}$$

(b) *π_t is absolutely continuous for all $t > 0$, with density p_t^Y such that $\partial^\alpha p_t^Y \in L_1 \cap C_0$ for any multiindex $\alpha \geq 0$*;
(c) *π_t is absolutely continuous for all $t > 0$, with density p_t^Y such that $\nabla p_t^Y \in L_1$.*

Moreover, if there exists an increasing function g such that $\psi(\xi) = g(|\xi|^2)$, then the above assertions are also equivalent to the following ones:

(d) *π_t is absolutely continuous for all $t > 0$, with density $p_t^Y \in C_0$*;
(e) *π_t is absolutely continuous for all $t > 0$, with density $p_t^Y \in L_\infty$*;
(f) *$\mathrm{e}^{-t\psi} \in L_1$ for all $t > 0$.*

Example 3.26 (Stable processes). *If Y is an isotropic stable process, then $\psi(\xi) = |\xi|^\alpha$, so that* (3.9) *is clearly satisfied, and we get that the transition density of Y_t belongs to C_b^∞ at all positive times, hence P_t^Y, $t > 0$, is infinitely smoothing. We reached exactly the same conclusion by our method based on estimates of negative moments of stable subordinators, cf. Corollary* 3.12.

Example 3.27 (Variance-gamma processes). *Let us consider the variance-gamma process Y defined in Example* 3.13. *We have $\mathbb{E}\mathrm{e}^{-\lambda T_t} = \mathrm{e}^{-t\Phi(\lambda)}$, with*

$$\Phi(\lambda) = a\log(1+\lambda/b)$$

(*see e.g.* [2], *p.* 73), *thus also*

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\langle\xi, W(T_t)\rangle} = \mathrm{e}^{-t\psi(\xi)}, \quad \psi(\xi) = \Phi\bigl(|\xi|^2\bigr) = a\log\bigl(1+|\xi|^2/b\bigr),$$

that is

$$\mathbb{E}e^{i\langle\xi,(W(T_t))\rangle} = \left(1+|\xi|^2/b\right)^{-at}.$$

In particular, we have

$$\liminf_{|\xi|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\log(1+|\xi|)} = 2a, \tag{3.10}$$

hence the density of the variance-gamma process (when it exists) is not in C_b^∞ for all $t>0$. Let us also recall that Hartman and Wintner [10] proved that if there exists $t_0 \geq 0$ such that

$$\liminf_{|\xi|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\log(1+|\xi|)} > \frac{d}{t_0},$$

then π_t is absolutely continuous for all $t \geq t_0$ with density $p_t^Y \in L_1 \cap C_0$. Therefore (3.10) implies that the variance-gamma process Y admits a density in $L_1 \cap C_0$ for all $t > d/(2a)$. As a matter of fact this condition is sharp, as one can verify by a direct calculation: since

$$p_t^Y(x) \propto \int_0^\infty e^{-|x|^2/(2s)} s^{at-1-d/2} e^{-bs}\, ds,$$

it is easily seen that the integral is finite for all $x \neq 0$, while for $x=0$ it is finite if and only if $t > d/(2a)$. In other words, the density of the variance-gamma process has a singularity at the origin for $t \leq d/(2a)$. Equivalently, since $\xi \mapsto e^{-t\psi(\xi)} \in L_1$ for all $t > d/(2a)$, one could also conclude by the Riemann–Lebesgue lemma that $p_t^Y \in L_1 \cap C_0$ for all $t > d/(2a)$. Our method using subordination instead yields that $p_t^Y \in L_1$ and $\nabla p_t^Y \in L_1$ for all $t > 1/(2a)$. Of course these properties do not imply that $p_t^Y \in C_0$.

Remark 3.28. *The discussion of the variance-gamma process in the above example can be generalized in a rather straightforward way to geometric strictly α-stable processes, for which $\psi(\xi) = \log(1+|\xi|^\alpha)$. See e.g. [5], Chapter 5, for more information about this class of processes.*

As it will become clear in the next section, the above results, while powerful and interesting in their own right, seem to be of little help for establishing smoothing properties of the semigroup generated by the solution to a SDE driven by a Lévy process. The main obstruction is of course that the transition kernels of the solution to a SDE are not translation invariant. Moreover the law of the solution is not infinitely divisible, with the exception of very simple situations. Even restricting our attention to proving smoothing properties of semigroups generated by Lévy processes, our approach through subordination and negative moments of the subordinator is, in general, not comparable to the criteria quoted above. In fact, while we can cover only a particular class of Lévy processes, our results give smoothing estimates that can depend on time. Most importantly, our method gives explicit estimates on the rate of blow-up as $t \to 0$ of the norm of $P_t^Y f$, $f \in \mathscr{B}_b$, in spaces of type C_b^k. As we shall see, this is essential for the developments in the next section.

4. Smoothing for the SDE (1.1)

The main result of this section is the following theorem, where we establish the strong Feller property of the semigroup associated to the solution of an SDE driven by a subordinated Wiener process, assuming that the subordinator satisfies a suitable integrability condition.

Throughout this section we shall tacitly assume that P_t^X is Feller for all $t \geq 0$. As is well known, this is always the case if the solution to (1.1) depends continuously on the initial datum x. This condition is satisfied very often, e.g. when the drift term b is Lipschitz continuous, or, more generally, when b is continuous, dissipative, and with polynomial growth (for the latter case see e.g. [9,13,21], as well [22], Chapter 8, and [19]).

We first consider the case that $\xi = 0$, so that the noise is a Lévy process obtained as subordination of a Wiener process.

Theorem 4.1. *Let $\ell \geq 0$. Assume that $\xi = 0$, there exists $\delta > 0$ such that*

$$\int_0^\delta \mathbb{E}\bigl(T_s^{-1/2} + T_s^{(\ell-1)/2}\bigr)\,\mathrm{d}s < \infty, \tag{4.1}$$

and $x \mapsto b(x)(1+|x|)^{-\ell} \in C_b(\mathbb{R}^d)$. Then P_t^X is strong Feller for all $t > 0$.

Proof. Let $f \in \mathscr{B}_{b,c}$ and $t > 0$. That $x \mapsto P_t^X f(x)$ is bounded is immediate by the stochastic representation $P_t^X f(x) = \mathbb{E} f(X_t^x)$. Therefore we just have to prove that $P_t^X f$ is continuous. Assumption (4.1) implies that $\mathbb{E}T_s^{-1/2}\,\mathrm{d}s < \infty$ for all $s \in]0,\delta[$, therefore, by Corollary 3.6, P_s^Y is 1-smoothing for all $s \in]0,\delta[$. In particular, one has $DP_s^Y f \in C_b$ and, by virtue of Theorem 3.4,

$$\bigl\| DP_s^Y f \bigr\|_\infty \lesssim \|f\|_\infty \mathbb{E}T_s^{-1/2} \tag{4.2}$$

for all $s \in]0,\delta[$. For any $t \in]0,\delta[$, we have, by Duhamel's formula,

$$P_t^X f = P_t^Y f + \int_0^t P_{t-s}^X \bigl\langle b, DP_s^Y f\bigr\rangle\,\mathrm{d}s. \tag{4.3}$$

Since $P_t^Y f \in C_b^1$, it is enough to prove that the integral on the right-hand side is a continuous function. Note that $P_t^Y f \in C_b^1$ and $b \in C$ imply $\langle b, DP_s^Y f\rangle \in C$, hence we have that P_t^X is c-strong Feller, if we can show that the sup-norm of the integral is finite. To this purpose, note that we can write

$$\begin{aligned}
\bigl\langle b, DP_s^Y f\bigr\rangle(x) &= \bigl\langle b(x)\bigl(1+|x|\bigr)^{-\ell}, \bigl(1+|x|\bigr)^{\ell} DP_s^Y f(x)\bigr\rangle \\
&\lesssim \bigl\langle b(x)\bigl(1+|x|\bigr)^{-\ell}, DP_s^Y f(x)\bigr\rangle + \bigl\langle b(x)\bigl(1+|x|\bigr)^{-\ell}, |x|^{\ell} DP_s^Y f(x)\bigr\rangle \\
&=: I_s^1 + I_s^2,
\end{aligned}$$

hence also

$$\left\| \int_0^t P_{t-s}^X \bigl\langle b, DP_s^Y f\bigr\rangle\,\mathrm{d}s \right\|_\infty \lesssim \int_0^t \bigl\| P_{t-s}^X \bigl(I_s^1 + I_s^2\bigr)\bigr\|_\infty\,\mathrm{d}s.$$

Since $b(1+|\cdot|)^{-\ell} \in C_b$, taking (4.2) into account, and recalling that P_t^X is contracting in L_∞ because it is Markovian, we obtain

$$\int_0^t \bigl\| P_{t-s}^X I_s^1 \bigr\|_\infty\,\mathrm{d}s \lesssim \|f\|_\infty \bigl\|b\bigl(1+|\cdot|\bigr)^{-\ell}\bigr\|_\infty \int_0^t \mathbb{E}T_s^{-1/2}\,\mathrm{d}s,$$

which is finite by hypothesis. Analogously, appealing to Theorem 3.4, we have

$$\int_0^t \bigl\| P_{t-s}^X I_s^2 \bigr\|_\infty\,\mathrm{d}s \lesssim \bigl(\|f\|_\infty + \bigl\||\cdot|^{\ell} f\bigr\|_\infty\bigr) \bigl\|b\bigl(1+|\cdot|\bigr)^{-\ell}\bigr\|_\infty \int_0^t \mathbb{E}\bigl(T_s^{-1/2} + T_s^{(\ell-1)/2}\bigr)\,\mathrm{d}s,$$

which is finite by hypothesis, recalling that $f \in \mathscr{B}_{b,c}$. We have thus established that P_t^X maps $\mathscr{B}_{b,c}$ to C_b for all $t \in]0,\delta[$. Lemma 2.4 implies that P_t^X is strong Feller for all $t \in]0,\delta[$, hence for all $t > 0$: in fact, if $t > \delta$, one can write $P_t^X f = P_{t-\delta/2}^X P_{\delta/2}^X f$, from which it follows that $P_t^X f \in C_b$ because $P_{t-\delta/2}^X$ is Feller and $P_{\delta/2}^X$ is strong Feller. □

Here is a result about the L_p-strong Feller property of P_t^X.

Theorem 4.2. *Let $1 \leq p \leq \infty$. Assume that $\xi = 0$, $b \in C_b(\mathbb{R}^d)$ and that there exists $\delta > 0$ such that*

$$\int_0^\delta \mathbb{E}T_s^{-1/2-d/(2p)}\,\mathrm{d}s < \infty. \tag{4.4}$$

Then $P_t^X f \in C_b(\mathbb{R}^d)$ for all $f \in L_p(\mathbb{R}^d)$.

Proof. The proof is similar to the one of the previous theorem, hence we omit some detail. By Corollary 3.7 one has $DP_s^Y f \in C_b$ with

$$\|DP_s^Y f\|_\infty \lesssim \|f\|_{L_p} \mathbb{E}T_s^{-1/2-d/(2p)} \quad \forall s \in]0, \delta[.$$

For any $t \in]0, \delta[$, we have, by Duhamel's formula,

$$P_t^X f = P_t^Y f + \int_0^t P_{t-s}^X \langle b, DP_s^Y f \rangle \, \mathrm{d}s.$$

Since $P_t^Y f \in C_b^1$, it is enough to prove that the integral on the right-hand side belongs to C_b. Since $P_t^Y f \in C_b^1$ and $b \in C$ imply $\langle b, DP_s^Y f \rangle \in C$, it is enough to show that the sup-norm of the integral is finite: one has

$$\|\langle b, DP_s^Y f \rangle\|_\infty \lesssim \|b\|_\infty \|f\|_{L_p} \mathbb{E}T_s^{-1/2-d/(2p)} \quad \forall s \in]0, \delta[,$$

hence also, by Minkowski's inequality,

$$\left\| \int_0^t P_{t-s}^X \langle b, DP_s^Y f \rangle \, \mathrm{d}s \right\|_\infty \lesssim \|b\|_\infty \|f\|_{L_p} \int_0^t \mathbb{E}T_s^{-1/2-d/(2p)} \, \mathrm{d}s,$$

which is finite by assumption (4.4). This proves that P_t^X maps L_p to C_b for all $t \in]0, \delta[$, hence also for all $t > 0$ by the same argument used above. □

Remark 4.3. *Note that, choosing f equal to zero outside a set of Lebesgue measure zero, it is immediately seen that $P_t^X f_1 = P_t^X f_2$ everywhere if $f_1 = f_2$ almost everywhere.*

Example 4.4. *Assume that T is self-similar with self-similarity index β, i.e. $T_t = t^\beta T_1$ in distribution, and $\mathbb{E}T_1^{-1/2} < \infty$. Then (4.1) certainly holds if $\beta < 2$ and $\ell \in \{0, 1\}$. In fact, one has $\mathbb{E}T_t^{-1/2} = t^{-\beta/2} \mathbb{E}T_1^{-1/2}$, which is integrable with respect to t around zero if (and only if) $\beta < 2$. In particular, assuming $\ell \in \{0, 1\}$, (4.1) always holds if T is an $\alpha/2$-stable subordinator, which is self-similar with index $\alpha/2$, $\alpha < 2$, and whose inverse moments (of any order) are finite, as already seen above. This in turn implies the strong Feller property for the semigroup generated by the solution to an SDE with linearly growing drift (e.g. of Ornstein–Uhlenbeck type) driven by a rotationally invariant stable process.*

Finally, we state a result for the case that $\xi \neq 0$ in (1.1).

Theorem 4.5. *Assume that there exists $\delta > 0$ such that*

$$\int_0^\delta \mathbb{E}T_s^{-1/2} \, \mathrm{d}s < \infty, \tag{4.5}$$

and $b \in C_b(\mathbb{R}^d)$. Then, $P_t^X f$ is strong Feller for $t > 0$.

Proof. The proof is completely analogous to that of Theorem 4.1 for $\ell = 0$. The only difference is that one has to appeal to Proposition 3.20 instead of Theorem 3.4. □

Remark 4.6. *Theorem 4.5 implies that if Z can be decomposed into the independent sum of Lévy processes Y and ξ, if Y is a subordinated Wiener process, and if the subordinator satisfies the integrable condition of the negative moment (4.5), P_t^X has the strong Feller property for $t > 0$. Here, note that nothing is assumed on ξ. This means that the part Y of the noise determines the smoothing properties of P_t^X.*

5. Strong Feller property via Malliavin calculus: A special case

Unfortunately it does not seem possible to adapt the method of the previous section to the case of equations with multiplicative noise of the type

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_{t-})\,\mathrm{d}Y_t, \quad X_0 = x \in \mathbb{R}^d, \tag{5.1}$$

essentially because one would need to have quantitative control on the smoothing properties at small time of the semigroup generated by the solution to the corresponding SDE without drift.

In the following we obtain the strong Feller property for the semigroup generated by X, by a completely different method. In particular, adapting some techniques based on Malliavin calculus that were developed in [17], we consider equations driven by rotationally-invariant stable processes. Some smoothness of the coefficients b and σ has also to be imposed (cf. Theorem 5.4 below).

Let us recall that, for SDEs driven by Brownian motions, general existence and regularity results of transition probability densities (implying the strong Feller property) are obtained in [16]. A crucial role in the argument of [16] is played by existence and integrability properties (with respect to the probability measure) of the stochastic flow. In the case of equations driven by stable processes, a major obstruction to the extension of this method comes from the fact that α-stable laws have infinite moments of order α and higher. In [17] a version of Malliavin calculus for SDEs driven by stable noise is developed, via subordination techniques, avoiding the problem of integrability of the stochastic flow.

Before turning to the main result of this section, we provide a sufficient condition on the transition densities of a Markovian semigroup to be strong Feller.

Lemma 5.1. *Let $(P_t)_{t>0}$ be a Markovian semigroup on $\mathcal{B}_b(\mathbb{R}^d)$ such that*

$$P_t f(x) = \int_{\mathbb{R}^d} f(y) q_t(x, y)\,\mathrm{d}y,$$

for some function $q_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Assume that P_t has the Feller property and that for all $M > 0$ and $x_0 \in \mathbb{R}^d$, there exists $p > 1$ and $\gamma > 0$ such that

$$\sup_{x\in\mathbb{R}^d;|x-x_0|<\gamma} \int_{\{y\in\mathbb{R}^d;|y|\le M\}} q_t(x, y)^p\,\mathrm{d}y < \infty. \tag{5.2}$$

Then P_t has the strong Feller property.

Proof. Let $f \in \mathcal{B}_{b,c}(\mathbb{R}^d)$ be given. Take $M > 0$ such that $\operatorname{supp} f \subset \{x \in \mathbb{R}^d; |x| < M/2\}$. Let $x_0 \in \mathbb{R}^d$ given. Choose $p > 1$ and $\gamma > 0$ such that (5.2) holds and let

$$C := \sup_{x\in\mathbb{R}^d;|x-x_0|<\gamma} \int_{\{y\in\mathbb{R}^d;|y|\le M\}} q_t(x, y)^p\,\mathrm{d}y.$$

Let p' be the conjugate exponent of p, i.e. $1/p + 1/p' = 1$. For any $\varepsilon > 0$ there exists $g \in C_c^\infty(\mathbb{R}^d)$ such that $\|f - g\|_{p'} < (4C^{1/p})^{-1}\varepsilon$ and $\operatorname{supp} g \subset \{x \in \mathbb{R}^d; |x| < M\}$. Since P_t has the Feller property, there exists $\delta > 0$ such that $|P_t g(x) - P_t g(x_0)| < \varepsilon/2$ if $|x - x_0| < \delta$. Therefore we have, for $|x - x_0| < \min(\delta, \gamma)$,

$$\begin{aligned}
&\bigl|P_t f(x) - P_t f(x_0)\bigr| \\
&\quad\le \bigl|P_t f(x) - P_t g(x)\bigr| + \bigl|P_t g(x) - P_t g(x_0)\bigr| + \bigl|P_t g(x_0) - P_t f(x_0)\bigr| \\
&\quad< \frac{\varepsilon}{2} + \int_{\mathbb{R}^d} \bigl|f(y) - g(y)\bigr| q_t(x, y)\,\mathrm{d}y + \int_{\mathbb{R}^d} \bigl|f(y) - g(y)\bigr| q_t(x_0, y)\,\mathrm{d}y \\
&\quad\le \frac{\varepsilon}{2} + 2\|f - g\|_{p'} \left(\sup_{x\in\mathbb{R}^d;|x-x_0|<\gamma} \int_{\{y\in\mathbb{R}^d;|y|\le M\}} q_t(x, y)^p\,\mathrm{d}y\right)^{1/p} \\
&\quad< \varepsilon.
\end{aligned}$$

Thus, we have the continuity of $P_t f$ at x_0. Since x_0 is an arbitrary point in $\mathbb{R}^d$, we have $P_t f \in C(\mathbb{R}^d)$. The boundedness of $P_t f$ follows immediately from the Markov property of P_t. Hence, P_t is c-strong Feller, and the proof is completed thanks to Lemma 2.4. □

Remark 5.2. *Lemma 5.1 is a criterion for Markovian semigroups to be strong Feller, and the criterion is similar to that obtained in* [28], *Corollary* 2.2. *The advantage of Lemma 5.1 is that it is applicable to the case that the transition probability density is not bounded. For example, the gamma process with certain parameters has unbounded transition density, but it satisfies the strong Feller property. We also remark that, on the other hand,* [28], *Corollary* 2.2, *would suffice to prove Theorem 5.4.*

Similarly to Lemma 5.1, we have the following sufficient condition for the L_p-strong Feller property.

Corollary 5.3. *Let $t \in (0, \infty)$. Assume that P_t has the Feller property and that for all $x_0 \in \mathbb{R}^d$, there exists $p > 1$ and $\gamma > 0$ such that*

$$\sup_{x\in\mathbb{R}^d;|x-x_0|<\gamma} \int_{\mathbb{R}^d} q_t(x, y)^p \, \mathrm{d}y < \infty.$$

Then, P_t has the $L_{p'}$-strong Feller property, where p' be the conjugate exponent of p.

Proof. The proof is almost the same as the one of Lemma 5.1. The difference is that we take $f \in L_{p'}$ instead of taking $f \in \mathcal{B}_{b,c}(\mathbb{R}^d)$, and that we do not need either to take $M > 0$ nor to apply Lemma 2.4. □

We can now state and prove the main result of this section, which asserts that if the coefficients of (5.1) are sufficiently smooth and if the diffusion coefficient is uniformly elliptic, we have the strong Feller property of the associated Markovian semigroup.

Let X be the unique solution to (5.1), where Y is d-dimensional rotationally-invariant α-stable process, $\sigma \in C([0, \infty[\times \mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$, $b \in C([0, \infty[\times \mathbb{R}^d; \mathbb{R}^d)$, and there exists $K > 0$ such that

$$\bigl|\sigma(t, x) - \sigma(t, y)\bigr|_{\mathbb{R}^d\otimes\mathbb{R}^d} + \bigl|b(t, x) - b(t, y)\bigr|_{\mathbb{R}^d} \le K|x - y|, \quad \forall x, y \in \mathbb{R}^d, t \in [0, \infty[.$$

As usual, we shall denote by P_t^X the Markovian semigroup defined by

$$P_t^X f(x) := \mathbb{E} f\bigl(X_t^x\bigr), \quad f \in \mathcal{B}_b\bigl(\mathbb{R}^d\bigr).$$

Note that, thanks to the Lipschitz continuity hypothesis on b and σ, it is well known that the solution X_t^x to (5.1) depends continuously on the initial datum x, which in turn implies that P_t^X is Feller.

Theorem 5.4. *Assume that there exist positive numbers δ and ε such that $\sigma \in C^{0,2}([0, \delta] \times \mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$, $\nabla\sigma \in C_b^{0,1}([0, \delta] \times \mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d)$, $b \in C^{0,2}([0, \delta] \times \mathbb{R}^d; \mathbb{R}^d)$, $\nabla b \in C_b^{0,1}([0, \delta] \times \mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$, and*

$$\bigl|\sigma(t, x)\xi\bigr|^2 \ge \varepsilon|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, t \in [0, \delta], x \in \mathbb{R}^d.$$

Then P_t^X is strong Feller for all $t > 0$.

Proof. Let $f \in \mathcal{B}_b(\mathbb{R}^d)$. Let us assume, for the time being, that $t \in (0, \delta]$. By [17], Theorem 6.2, for each $x \in \mathbb{R}^d$ the density function $q_t(x, \cdot)$ of the distribution of X_t^x exists and it belongs to $C_b(\mathbb{R}^d)$. Furthermore, checking the dependence of the estimate for $q_t(x, y)$ in the proof [17], Theorem 6.2, one infers that $q_t \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$, hence also that (5.2) is satisfied. We can then apply Lemma 5.1 obtaining that $P_t^X f \in C_b(\mathbb{R}^d)$ for all $t \in (0, \delta]$. Let now assume $t > \delta$. Since $P_t^X f = P_{t-\delta}^X(P_\delta^X f)$ and $P_\delta^X f \in C_b(\mathbb{R}^d)$, the Feller property of P_t^X for all $t > 0$ yields $P_t^X f \in C_b(\mathbb{R}^d)$. The theorem is thus proved. □

Acknowledgements

Part of the work for this paper was carried out when both authors where visiting the Institute for Applied Mathematics of the University of Bonn. We are particularly grateful to our hosts Sergio Albeverio and Andreas Eberle. The first-named author also gratefully acknowledges the support of the Japan Society for the Promotion of Science through the Excellent Young Researcher Overseas Visit Program.

References

[1] V. Bally and E. Clément. Integration by parts formula and applications to equations with jumps. *Probab. Theory Related Fields* **151** (3–4) (2011) 6133–657. MR2851695

[2] J. Bertoin. *Lévy Processes*. Cambridge Univ. Press, Cambridge, 1996. MR1406564

[3] J.-M. Bismut. Calcul des variations stochastique et processus de sauts. *Z. Wahrsch. Verw. Gebiete* **63** (2) (1983) 147–235. MR0701527

[4] V. I. Bogachev. *Gaussian Measures*. *Mathematical Surveys and Monographs* **62**. American Mathematical Society, Providence, RI, 1998. MR1642391

[5] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song and Z. Vondraček. *Potential Analysis of Stable Processes and Its Extensions*. *Lecture Notes in Mathematics* **1980**. Springer, Berlin, 2009. MR2569321

[6] B. Brainerd and R. E. Edwards. Linear operators which commute with translations. I. Representation theorems. *J. Austral. Math. Soc.* **6** (1966) 289–327. MR0206725

[7] L. Caffarelli, C. H. Chan and A. Vasseur. Regularity theory for parabolic nonlinear integral operators. *J. Amer. Math. Soc.* **24** (3) (2011) 849–869. MR2784330

[8] M. Gordina, M. Röckner and F.-Y. Wang. Dimension-independent Harnack inequalities for subordinated semigroups. *Potential Anal.* **34** (3) (2011) 293–307. MR2782975

[9] I. Gyöngy and N. V. Krylov. On stochastic equations with respect to semimartingales. I. *Stochastics* **4** (1) (1980/81) 1–21. MR0587426

[10] P. Hartman and A. Wintner. On the infinitesimal generators of integral convolutions. *Amer. J. Math.* **64** (1942) 273–298. MR0006635

[11] J. Hawkes. Potential theory of Lévy processes. *Proc. London Math. Soc. (3)* **38** (2) (1979) 335–352. MR0531166

[12] Y. Ishikawa and H. Kunita. Malliavin calculus on the Wiener–Poisson space and its application to canonical SDE with jumps. *Stochastic Process. Appl.* **116** (12) (2006) 1743–1769. MR2307057

[13] J. Jacod. Une condition d'existence et d'unicité pour les solutions fortes d'équations différentielles stochastiques. *Stochastics* **4** (1) (1980/81) 23–38. MR0587427

[14] V. Knopova and R. Schilling. A note on the existence of transition probability densities of Lévy processes. *Forum Math.* **25** 125–149. MR3010850

[15] H. Kunita. Smooth density of canonical stochastic differential equation with jumps. *Astérisque* **327** (2009) 69–91. MR2642353

[16] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32** (1) (1985) 1–76. MR0783181

[17] S. Kusuoka. Malliavin calculus for stochastic differential equations driven by subordinated Brownian motions. *Kyoto J. Math.* **50** (3) (2010) 491–520. MR2723861

[18] R. Léandre. Calcul des variations sur un Brownien subordonné. In *Séminaire de Probabilités, XXII* 414–433. *Lecture Notes in Math.* **1321**. Springer, Berlin, 1988. MR0960537

[19] J.-P. Lepeltier and B. Marchal. Problème des martingales et équations différentielles stochastiques associées à un opérateur intégro-différentiel. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **12** (1) (1976) 43–103. MR0413288

[20] C. Marinelli, C. Prévôt and M. Röckner. Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise. *J. Funct. Anal.* **258** (2) (2010) 616–649. MR2557949

[21] C. Marinelli and M. Röckner. Well-posedness and asymptotic behavior for stochastic reaction-diffusion equations with multiplicative Poisson noise. *Electron. J. Probab.* **15** (49) (2010) 1528–1555. MR2727320

[22] M. Métivier. *Semimartingales*. de Gruyter, Berlin, 1982. MR0688144

[23] R. Mikulevičius and H. Pragarauskas. On the Cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces. *Liet. Mat. Rink.* **32** (2) (1992) 299–331. MR1246036

[24] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, New York, 1983. MR0710486

[25] E. Priola and J. Zabczyk. Liouville theorems for non-local operators. *J. Funct. Anal.* **216** (2) (2004) 455–490. MR2095690

[26] E. Priola and J. Zabczyk. Structural properties of semilinear SPDEs driven by cylindrical stable processes. *Probab. Theory Related Fields* **149** (2011) (1–2), 97–137. MR2773026

[27] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Random Processes*. Chapman & Hall, New York, 1994. MR1280932

[28] R. L. Schilling and J. Wang. Strong Feller continuity of Feller processes and semigroups. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **15** (2) (2012) 1250010. MR2957137

[29] H. Triebel. *Theory of Function Spaces*. Birkhäuser, Basel, 1983. MR0781540