Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2014, Vol. 50, No. 4, 1371–1384
DOI: 10.1214/13-AIHP556

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Hausdorff dimension of affine random covering sets in torus

Esa Järvenpää[a], Maarit Järvenpää[a], Henna Koivusalo[a], Bing Li[b,a] and Ville Suomala[a]

[a] *Department of Mathematical Sciences, P.O. Box 3000, 90014 University of Oulu, Finland. E-mail: esa.jarvenpaa@oulu.fi; maarit.jarvenpaa@oulu.fi; henna.koivusalo@oulu.fi; ville.suomala@oulu.fi*
[b] *Department of Mathematics, South China University of Technology, Guangzhou, 510641, P.R. China. E-mail: libing0826@gmail.com*

Received 2 July 2012; revised 20 February 2013; accepted 27 February 2013

Abstract. We calculate the almost sure Hausdorff dimension of the random covering set $\limsup_{n\to\infty}(g_n + \xi_n)$ in d-dimensional torus $\mathbb{T}^d$, where the sets $g_n \subset \mathbb{T}^d$ are parallelepipeds, or more generally, linear images of a set with nonempty interior, and $\xi_n \in \mathbb{T}^d$ are independent and uniformly distributed random points. The dimension formula, derived from the singular values of the linear mappings, holds provided that the sequences of the singular values are decreasing.

Résumé. Nous calculons presque sûrement la dimension de Hausdorff de l'ensemble de recouvrement aléatoire $\limsup_{n\to\infty}(g_n + \xi_n)$ dans le tore $\mathbb{T}^d$ de dimension d, où $g_n \subset \mathbb{T}^d$ sont des parallélépipèdes, ou plus généralement, des images linéaires d'un ensemble d'intérieur non vide et $\xi_n \in \mathbb{T}^d$ sont des points aléatoires indépendants et uniformément distribués. La formule de dimension, exprimée en fonction des valeurs singulières des applications linéaires, est valable à condition que la suite de ces valeurs singulières soit décroissante.

MSC: 60D05; 28A80

Keywords: Random covering set; Hausdorff dimension; Affine Cantor set

1. Introduction

Given a sequence of positive numbers (l_n) and a sequence of independent random variables (ξ_n) uniformly distributed on the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, define *the random covering set* E as follows:

$$E = \{x \in \mathbb{T}^1 \mid x \in [\xi_n, \xi_n + l_n] \text{ for infinitely many } n\} = \limsup_{n\to\infty}[\xi_n, \xi_n + l_n].$$

Denoting the Lebesgue measure by $\mathcal{L}$ and using the Borel–Cantelli lemma and Fubini's theorem, it follows that, almost surely, the following dichotomy holds:

$$\mathcal{L}(E) = \begin{cases} 0, & \text{when } \sum_{n=1}^{\infty} l_n < \infty, \\ 1, & \text{when } \sum_{n=1}^{\infty} l_n = \infty, \end{cases} \tag{1.1}$$

that is, almost all or almost no points of the circle are covered, depending on whether or not the series of the lengths of the covering intervals diverges.

The case of full Lebesgue measure has been extensively studied. It was a long-standing problem to find conditions on (l_n) guaranteeing that the whole circle is covered almost surely, that is,

$$P(E = \mathbb{T}^1) = 1. \tag{1.2}$$

This problem, known in literature as the Dvoretzky covering problem, was first posed by Dvoretzky [5] in 1956. After substantial contribution of many, including Kahane [18], Erdős [7], Billard [3] and Mandelbrot [24], the full answer was given by Shepp [29] in 1972. He proved that (1.2) holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(l_1 + \cdots + l_n) = \infty,$$

where the lengths (l_n) are in decreasing order. After this, a natural problem, raised by Carleson (private communication to Kahane), is to describe the growth of the covering number of a given point $x \in \mathbb{T}^1$, that is, to study the asymptotic behaviour of the sums

$$C_N(x) = \sum_{n=1}^{N} \chi_{[\xi_n, \xi_n + l_n]}(x),$$

where χ_A is the characteristic function of a set A. Obviously, the expectation $\mathbb{E}(C_N(x)) = \sum_{n=1}^{N} l_n$. In the case $l_n = \frac{\gamma}{n}$ with $\gamma > 1$, Fan and Kahane [10] proved that almost surely the order of the covering number $C_N(x)$ is $\log N$ for every $x \in \mathbb{T}^1$, meaning that for sufficiently large N

$$A_\gamma \log N \le \min_{x \in \mathbb{T}^1} C_N(x) \le \max_{x \in \mathbb{T}^1} C_N(x) \le B_\gamma \log N$$

with positive and finite constants A_γ and B_γ. Furthermore, Fan [9] verified that the set

$$F_\beta = \left\{ x \in \mathbb{T}^1 \,\middle|\, \lim_{N \to \infty} \frac{C_N(x)}{\sum_{n=1}^{N} l_n} = \beta \right\}$$

has positive Hausdorff dimension for a certain interval of $\beta > 0$ in the case $l_n = \frac{\gamma}{n}$ with $\gamma > 0$. For general l_n, Barral and Fan [2] answered Carleson's problem by identifying three kinds of phenomena depending whether the index $\bar{\gamma} = \limsup_{N \to \infty} \frac{\sum_{n=1}^{N} l_n}{-\log l_N}$ is zero, positive and finite or infinite. More precisely, when $\bar{\gamma} = 0$, $\dim_{\mathrm{H}} F_\beta = 1$ almost surely for all $\beta \ge 0$, when $\bar{\gamma} = \infty$, $F_1 = \mathbb{T}^1$ almost surely, and when $0 < \bar{\gamma} < \infty$, $\dim_{\mathrm{H}} F_\beta$ depends on β. Here the Hausdorff dimension is denoted by $\dim_{\mathrm{H}}$.

For the case of zero Lebesgue measure, the Hausdorff dimension of E was first calculated by Fan and Wu [12] in the case $l_n = 1/n^\alpha$. When studying the Hausdorff measure and the large intersection properties of E for general l_n, Durand [4] gave another, independent proof of the dimension result. According to [12] and [4], the almost sure Hausdorff dimension of E is given by

$$\dim_{\mathrm{H}} E = \inf\left\{ t \ge 0 \,\middle|\, \sum_{n=1}^{\infty} l_n^t < \infty \right\} = \limsup_{n \to \infty} \frac{\log n}{-\log l_n}, \tag{1.3}$$

where the lengths l_n are in decreasing order. In [4], the author also proved that the packing dimension of E equals 1 almost surely. When considering the hitting probability property of the random set E, Li, Shieh and Xiao [22] provided an alternative way to obtain the Hausdorff and packing dimension results under some additional conditions. The result (1.3) can be also proven as a consequence of the mass transference principle due to Beresnevich and Velani [1] (see Proposition 4.7). The fact that both packing and box counting dimensions are equal to 1 almost surely follows since E is almost surely a dense G_δ-set in $\mathbb{T}^1$ (see [19], Chapter 5, Proposition 11, and [27], Section 2).

In this paper we study random covering sets in d-dimensional torus $\mathbb{T}^d$. Letting (g_n) be a sequence of subsets of $\mathbb{T}^d$ and letting (ξ_n) be a sequence of independent random variables, uniformly distributed on $\mathbb{T}^d$, define the random covering set by

$$E = \limsup_{n \to \infty} (g_n + \xi_n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (g_k + \xi_k).$$

Notice that a counterpart of (1.1) is easily obtained, that is, almost surely

$$\mathcal{L}(E) = \begin{cases} 0, & \text{when } \sum_{n=1}^{\infty} \mathcal{L}(g_n) < \infty, \\ 1, & \text{when } \sum_{n=1}^{\infty} \mathcal{L}(g_n) = \infty, \end{cases}$$

where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{T}^d$.

On the d-dimensional torus the Dvoretzky covering problem has been studied by El Hélou [6] and Kahane [20] among others. In [20] Kahane gave a complete solution for the problem when the sets g_n are similar simplexes (see also Janson [16]). However, in the general case the covering problem has not been completely solved.

For an overview on the research on random covering sets and related topics, we refer to [19], Chapter 11, the survey [21] and the references therein. Here we only mention a few variations on the classical random covering model. For example, Hawkes [13] considered under which conditions all the points in $K \subset \mathbb{T}^1$ are covered with probability one (or zero). Mandelbrot [25], in turn, introduced Poisson covering of the real line (see also Shepp [28]). In general metric spaces, the random coverings by balls have been studied by Hoffman-Jörgensen [15]. Recent contributions to the topic include various types of dynamical models, see Fan, Schmeling and Troubetzkoy [11], Jonasson and Steif [17] and Liao and Seuret [23].

We address the question of determining the analogue of (1.3) in higher dimensional case. In [12] the method is strongly adapted to the 1-dimensional case whereas the argument based on the mass transference principle [1] can be carried through in any dimension provided that the sets g_n are uniformly ball like (see Proposition 4.7). Our main interest is the case where the sets g_n are not uniformly ball like, and therefore, the mass transference principle cannot be applied. It turns out that almost surely the Hausdorff dimension of the covering set E is given in terms of the singular value functions of the linear mappings related to the system, see Theorem 2.1.

To this end, in Section 2 we introduce our setting, state our main result and prove preliminary lemmas including the upper bound for the dimension. In Section 3 we construct a random subset of the covering set E having large dimension with positive probability which, in turn, gives the lower bound of the dimension in Section 4.

2. Preliminaries and statement of main theorem

Denote the closed ball of radius r and centre x in $\mathbb{R}^d$ by $B(x, r)$. Letting $L : \mathbb{R}^d \to \mathbb{R}^d$ be a contractive linear injection, the image $L(B(0, 1))$ of the unit ball $B(0, 1)$ is an ellipse whose semiaxes are non degenerated. The singular values $0 < \alpha_d(L) \le \cdots \le \alpha_1(L) < 1$ of L are the lengths of the semiaxes of $L(B(0, 1))$ in decreasing order. Given $0 < s \le d$, define the *singular value function* by

$$\Phi^s(L) = \alpha_1(L) \cdots \alpha_{m-1}(L) \alpha_m(L)^{s-(m-1)},$$

where m is the integer such that $m - 1 < s \le m$.

We use the notations $\mathbb{T}^d$ for the d-dimensional torus and $\mathcal{L}$ for the Lebesgue measure on $\mathbb{T}^d$. Consider a probability space $(\Omega, \mathcal{A}, P)$ and let (ξ_n) be a sequence of independent random variables which are uniformly distributed on $\mathbb{T}^d$, that is, $(\xi_n)_* P = \mathcal{L}$, where $(\xi_n)_* P$ is the image measure of P under ξ_n. Letting (g_n) be a sequence of subsets of $\mathbb{T}^d$, we use the notation G_n for the random translates $G_n = g_n + \xi_n \subset \mathbb{T}^d$ and define the random covering set generated by (g_n) by

$$E = E^\omega = \limsup_{n \to \infty} G_n.$$

In this paper we consider the case $g_n = \Pi(L_n(R))$, where $R \subset [0, 1]^d$ has non-empty interior, $L_n : \mathbb{R}^d \to \mathbb{R}^d$ is a contractive linear injection for all $n \in \mathbb{N}$ and $\Pi : \mathbb{R}^d \to \mathbb{T}^d$ is the natural covering map. Moreover, we assume that for all $i = 1, \ldots, d$ the sequence of singular values $\alpha_i(L_n)$ decreases to 0 as n tends to infinity. Defining

$$s_0 = \inf\left\{ 0 < s \le d \,\middle|\, \sum_{n=1}^{\infty} \Phi^s(L_n) < \infty \right\}, \tag{2.1}$$

with the interpretation $s_0 = d$ if $\sum_{n=1}^{\infty} \Phi^d(L_n) = \infty$, we are ready to state our main theorem.

Theorem 2.1. *For P-almost all $\omega \in \Omega$ we have*

$$\dim_{\mathrm{H}} E^{\omega} = s_0. \tag{2.2}$$

Theorem 2.1 is an immediate consequence of the following proposition concerning the case where each generating set g_n is a rectangular parallelepiped in $\mathbb{T}^d$ meaning that there exist a parallelepiped $\tilde{g}_n \subset \mathbb{R}^d$ such that $g_n = \Pi(\tilde{g}_n)$. In what follows rectangular parallelepipeds will consistently be called rectangles.

Let $E(g_n) = E^{\omega}(g_n)$ be the covering set generated by a sequence (g_n) of rectangles. For all rectangles g and for all $0 < s \le d$ define

$$\Phi^s(g) = \alpha_1(g) \cdots \alpha_{m-1}(g) \alpha_m(g)^{s-(m-1)},$$

where $0 < \alpha_d(g) \le \cdots \le \alpha_1(g) < 1$ are the lengths the edges of g in decreasing order and m is the integer such that $m - 1 < s \le m$.

Proposition 2.2. *Assume that (g_n) is a sequence of rectangles such that for all $i = 1, \ldots, d$ the sequence of lengths $\alpha_i(g_n)$ decreases to 0 as n tends to infinity. Then almost surely*

$$\dim_{\mathrm{H}} E(g_n) = s_0(g_n), \tag{2.3}$$

where

$$s_0(g_n) = \inf\left\{0 < s \le d \,\middle|\, \sum_{n=1}^{\infty} \Phi^s(g_n) < \infty\right\}$$

with the interpretation $s_0 = d$ if $\sum_{n=1}^{\infty} \Phi^d(g_n) = \infty$.

We proceed by verifying first that Theorem 2.1 follows from Proposition 2.2.

Proof of Theorem 2.1 as a consequence of Proposition 2.2. Letting (L_n), R and E be as in Theorem 2.1, there are sequences (g_n') and (g_n) of rectangles such that $g_n' \subset \Pi(L_n(R)) \subset g_n$, and moreover, $\alpha_i(g_n') = c'\alpha_i(L_n)$ and $\alpha_i(g_n) = c\alpha_i(L_n)$ for all $i = 1, \ldots, d$. Here the constants c' and c are independent of n and i. Since $E(g_n') \subset E \subset E(g_n)$ we have

$$\dim_{\mathrm{H}} E(g_n') \le \dim_{\mathrm{H}} E \le \dim_{\mathrm{H}} E(g_n).$$

Applying Proposition 2.2 to the sequences (g_n') and (g_n) and noting that $s_0(g_n') = s_0(g_n) = s_0$, gives (2.2). □

It remains to prove Proposition 2.2. As the first step we verify the following lemma according to which the Hausdorff dimension of $E(g_n)$ is always bounded above by $s_0(g_n)$. The proof is standard following, for example, the ideas in [8].

Lemma 2.3. *Assume that (g_n) and $s_0(g_n)$ are as in Proposition 2.2. Then for all $\omega \in \Omega$ we have $\dim_{\mathrm{H}} E^{\omega}(g_n) \le s_0(g_n)$.*

Proof. We may assume that $s_0(g_n) < d$. Let $s_0(g_n) < s < d$ and let m be the integer with $m - 1 < s \le m$. For each $n \in \mathbb{N}$ we estimate the number of cubes of side length $\alpha_m(g_n)$ needed to cover G_n. By expanding the last $d - m + 1$ edges of G_n to length $\alpha_m(g_n)$ and by dividing the expanded rectangle to cubes of side length $\alpha_m(g_n)$, we end up with an upper bound

$$\left(\left\lfloor \frac{\alpha_1(g_n)}{\alpha_m(g_n)} \right\rfloor + 1\right) \cdots \left(\left\lfloor \frac{\alpha_{m-1}(g_n)}{\alpha_m(g_n)} \right\rfloor + 1\right) \le 2^{m-1} \alpha_1(g_n) \cdots \alpha_{m-1}(g_n) \alpha_m(g_n)^{-m+1},$$

where the integer part of any $x \ge 0$ is denoted by $\lfloor x \rfloor$.

Recalling that for all $N \in \mathbb{N}$

$$E(g_n) \subset \bigcup_{n=N}^{\infty} G_n,$$

gives the following estimate for the s-dimensional Hausdorff measure

$$\begin{aligned}\mathcal{H}^s(E) &\le \liminf_{N\to\infty} \sum_{n=N}^{\infty} 2^{m-1}\big(\sqrt{d}\alpha_m(g_n)\big)^s \alpha_1(g_n)\cdots\alpha_{m-1}(g_n)\alpha_m(g_n)^{-m+1} \\ &= \liminf_{N\to\infty} \sum_{n=N}^{\infty} 2^{m-1}(\sqrt{d})^s \Phi^s(g_n) = 0.\end{aligned}$$

This implies that $\dim_{\mathrm{H}} E(g_n) \le s_0(g_n)$. □

We continue by proving two auxiliary results.

Lemma 2.4. *Assume that (L_n) is a sequence of contractive linear injections $L_n : \mathbb{R}^d \to \mathbb{R}^d$. Let s_0 be as in (2.1) and let $m-1 < s_0 \le m$. Defining for all $m-1 < s < s_0$*

$$f(s) := \limsup_{n\to\infty} \frac{\log n}{-\log \Phi^s(L_n)},$$

we have $f(s) > 1$.

Proof. We will show that $f(s) \ge 1$ for all $m-1 < s < s_0$ and f is strictly decreasing. This clearly implies the claim.

Let $m-1 < s < s_0$. The fact that $\sum_{n=1}^{\infty} \Phi^s(L_n) = \infty$ implies that for all $\varepsilon > 0$ there exists a subsequence (n_k) such that $\Phi^s(L_{n_k}) > \frac{1}{n_k^{1+\varepsilon}}$ for all k. From this we deduce that $f(s) \ge \frac{1}{1+\varepsilon}$, and letting ε go to 0 yields $f(s) \ge 1$.

Consider $\delta > 0$ such that $m-1 < s+\delta < s_0$. Since $\Phi^s(L_n) \ge \alpha_m(L_n)^s$ we obtain

$$\Phi^{s+\delta}(L_n) = \Phi^s(L_n)\alpha_m(L_n)^{\delta} \le \Phi^s(L_n)^{1+\delta/s},$$

giving

$$f(s+\delta) \le \limsup_{n\to\infty} \frac{\log n}{(1+\delta/s)(-\log \Phi^s(L_n))} = \frac{f(s)}{1+\delta/s} < f(s).$$

Hence f is strictly decreasing. □

Remark 2.5. *Lemma 2.4 holds for all $0 < s < s_0$, but this stronger claim is not necessary for our purposes.*

Proposition 2.6. *Assume that $G \subset \mathbb{T}^d$ and $\mathcal{L}(G) > 0$. Let $\xi_1, \dots, \xi_n$ be independent, uniformly distributed random variables on $\mathbb{T}^d$. Let*

$$M_n = \#\big\{i \in \{1,\dots,n\} \mid \xi_i \in G\big\},$$

where $\#A$ denotes the number of elements in a set A. Then

$$P\bigg(M_n \le \frac{1}{2} n\mathcal{L}(G)\bigg) \le \frac{4(1-\mathcal{L}(G))}{n\mathcal{L}(G)}.$$

Proof. Denote by χ_A the characteristic function of a set A. Calculating the first and second moments of M_n gives

$$\mathbb{E}(M_n) = \mathbb{E}\left(\sum_{i=1}^{n} \chi_{\{\xi_i \in G\}}\right) = n\mathcal{L}(G)$$

and

$$\mathbb{E}\big(M_n^2\big) = \mathbb{E}\left(\left(\sum_{i=1}^{n} \chi_{\{\xi_i \in G\}}\right)^2\right) = \mathbb{E}\left(\sum_{i=1}^{n} \chi_{\{\xi_i \in G\}} + \sum_{j \neq i} \chi_{\{\xi_i \in G\}} \chi_{\{\xi_j \in G\}}\right)$$
$$= n\mathcal{L}(G) + \big(n^2 - n\big)\mathcal{L}(G)^2.$$

From Chebyshev's inequality we deduce

$$P\left(M_n \leq \frac{1}{2}\mathbb{E}(M_n)\right) \leq P\left(\big|M_n - \mathbb{E}(M_n)\big| \geq \frac{1}{2}\mathbb{E}(M_n)\right)$$
$$\leq \frac{4(\mathbb{E}(M_n^2) - \mathbb{E}(M_n)^2)}{\mathbb{E}(M_n)^2} = \frac{4(1 - \mathcal{L}(G))}{n\mathcal{L}(G)}$$

which completes the proof. □

3. Construction of random Cantor sets

Let (g_n) and $s_0(g_n)$ be as in Proposition 2.2. Consider an integer m such that $m - 1 < s_0(g_n) \leq m$. For notational simplicity, we assume that 0 is a vertex of each g_n. Indeed, by choosing suitable deterministic translates, we find an isomorphic probability space $(\Omega', \mathcal{A}', P')$ where this is the case since the random variables (ξ_n) are uniformly distributed and the rectangles (g_n) are deterministic. For each n, let $T_n : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map such that $\Pi(T_n([0,1]^d)) = g_n$. Observe that $\alpha_i(T_n) = \alpha_i(g_n)$ for all $i = 1, \ldots, d$. Let $m - 1 < s < s_0(g_n)$. For the purpose of proving Proposition 2.2 we construct in this section an event $\Omega(\infty) \subset \Omega$, having positive probability, and a random Cantor like set C^ω such that $C^\omega \subset E^\omega$ for all $\omega \in \Omega(\infty)$. In Section 4 we prove that $\dim_{\mathrm{H}} C^\omega \geq s$ almost surely conditioned on $\Omega(\infty)$.

Let $a_0 = \frac{1}{2}$. Consider a sequence (a_l) of real numbers larger than $1/2$ increasing to 1 with $\prod_{l=1}^{\infty} \frac{1}{a_l} < \infty$. By Lemma 2.4, there exists a sequence (n_k) of natural numbers satisfying

$$\lim_{k \to \infty} \frac{\log n_k}{-\log \Phi^s(T_{n_k})} = f(s) > 1. \tag{3.1}$$

Moreover, by considering a suitable subsequence of (n_k), we may assume that for all $k \in \mathbb{N}$

$$\operatorname{diam}(g_{n_k}) \leq \frac{1}{2}(1 - a_{k-1})\alpha_d(g_{n_{k-1}}), \tag{3.2}$$

$$n_k \mathcal{L}(g_{n_{k-1}}) \geq n_k^{(3+f(s))/(2+2f(s))} \quad \text{and} \tag{3.3}$$

$$\log n_k \geq n_{k-1}, \tag{3.4}$$

where $n_0 = 0$ and $g_0 = \mathbb{T}^d$. Notice that since the sequence (n_k) is deterministic it is independent of $\omega \in \Omega$.

We proceed by constructing inductively a random nested sequence of finite collections $\mathcal{C}_k$ of rectangles as follows: Let $\mathcal{C}_0 = \{\mathbb{T}^d\}$ and $N_0 = 1$. Define $N_1 = \lfloor \frac{1}{2} a_0^d n_1 \rfloor$ and $I(1, \mathbb{T}^d) = \{1, \ldots, N_1\}$. For all $i \in I(1, \mathbb{T}^d)$, let g_i' be a linear isometric copy of g_{n_1} contained in g_i. The existence of g_i' follows from the fact that $\alpha_j(g_{n_1}) \leq \alpha_j(g_i)$ for all $i \leq n_1$ and $j = 1, \ldots, d$. For each $i \in I(1, \mathbb{T}^d)$, set $G_i' = g_i' + \xi_i$. Then $G_i' \subset G_i$. Defining $\mathcal{C}_1 = \{G_i' | i \in I(1, \mathbb{T}^d)\}$, we have

$$\bigcup_{G \in \mathcal{C}_1} G \subset \bigcup_{i=1}^{n_1} G_i.$$

Furthermore, the collection $\mathcal{C}_1$ can be chosen for any $\omega \in \Omega =: \Omega(1)$ giving $P(\Omega(1)) = q_1$ with $q_1 = 1$.

Assume that there exist events $\Omega(1), \dots, \Omega(k-1)$ with $P(\bigcap_{j=1}^{k-1} \Omega(j)) = q_1 \cdots q_{k-1}$ such that for all $\omega \in \bigcap_{j=1}^{k-1} \Omega(j)$ there are collections $\mathcal{C}_1, \dots, \mathcal{C}_{k-1}$ having the following properties for all $j = 1, \dots, k-1$

$$\left(\frac{1}{2}\right)^3 a_{j-1}^d (n_j - n_{j-1}) \mathcal{L}(g_{n_{j-1}}) \le N_j \le (n_j - n_{j-1}) \mathcal{L}(g_{n_{j-1}}), \quad \text{where } N_j = \#\mathcal{C}_j, \tag{3.5}$$

$$\bigcup_{G \in \mathcal{C}_j} G \subset \bigcup_{G \in \mathcal{C}_{j-1}} G, \tag{3.6}$$

$$\#\{G' \in \mathcal{C}_j | G' \subset G\} = \left\lfloor \frac{1}{2} a_{j-1}^d m_j \mathcal{L}(g_{n_{j-1}}) \right\rfloor \quad \text{for each } G \in \mathcal{C}_{j-1}$$
$$\text{where } m_j = \left\lfloor \frac{n_j - n_{j-1}}{N_{j-1}} \right\rfloor, \tag{3.7}$$

$$\mathcal{C}_j \text{ is a finite collection of isometric copies of } g_{n_j} \quad \text{and} \tag{3.8}$$

$$\bigcup_{G \in \mathcal{C}_j} G \subset \bigcup_{l=n_{j-1}+1}^{n_j} G_l. \tag{3.9}$$

We define an event $\Omega(k)$ such that $P(\bigcap_{j=1}^{k} \Omega(j)) = q_1 \cdots q_k$ and for all $\omega \in \bigcap_{j=1}^{k} \Omega(j)$ there is a collection $\mathcal{C}_k$ satisfying (3.5)–(3.9). Write $\mathcal{C}_{k-1} = \{\widetilde{G}_1, \dots, \widetilde{G}_{N_{k-1}}\}$ and set $m_k = \lfloor \frac{n_k - n_{k-1}}{N_{k-1}} \rfloor$. For $l = 1, \dots, N_{k-1}$, define random sets

$$\widetilde{I}(k, \widetilde{G}_l) = \{i \in \{n_{k-1} + 1 + (l-1)m_k, \dots, n_{k-1} + l m_k\} | \xi_i \in a_{k-1} \widetilde{G}_l\},$$

where aG is the similar copy of G with similarity ratio a and with the same centre as G. Let

$$\Omega(k) = \left\{\omega \in \Omega \,\middle|\, \#\widetilde{I}(k, G) > \frac{1}{2} a_{k-1}^d m_k \mathcal{L}(g_{n_{k-1}}) \text{ for all } G \in \mathcal{C}_{k-1}\right\}$$

and

$$q_k = P(\Omega(k) | \Omega(1), \dots, \Omega(k-1)).$$

Note that $q_k > 0$. For each $G \in \mathcal{C}_{k-1}$ we denote by $I(k, G)$ the collection of the first $\lfloor \frac{1}{2} a_{k-1}^d m_k \mathcal{L}(g_{n_{k-1}}) \rfloor$ elements in $\widetilde{I}(k, G)$ and set

$$\mathcal{C}_k = \{G_i' | G \in \mathcal{C}_{k-1}, i \in I(k, G)\} \quad \text{and} \quad N_k = \#\mathcal{C}_k,$$

where $G_i' = g_i' + \xi_i$ and g_i' is a linear isometric copy of g_{n_k} contained in g_i. (See Fig. 1.) Observe that N_k is deterministic. As above, g_i' exists since $\alpha_j(g_{n_k}) \le \alpha_j(g_i)$ for all $j = 1, \dots, d$ and $i \le n_k$. Clearly, (3.7) and (3.8) are valid for $\mathcal{C}_k$. Since, by inequality (3.2), we have $g_i' + \xi_i \subset G \in \mathcal{C}_{k-1}$ provided that $\xi_i \in a_{k-1} G$, property (3.6) holds for $\mathcal{C}_k$. Furthermore, the choices of m_k and $I(k, G_l)$ imply (3.9). The choice of m_k gives

$$\begin{aligned}\left(\frac{1}{2}\right)^3 a_{k-1}^d (n_k - n_{k-1}) \mathcal{L}(g_{n_{k-1}}) &\le N_{k-1} \left\lfloor \frac{1}{2} a_{k-1}^d m_k \mathcal{L}(g_{n_{k-1}}) \right\rfloor = N_k \\ &\le (n_k - n_{k-1}) \mathcal{L}(g_{n_{k-1}}),\end{aligned}$$

and therefore, condition (3.5) is satisfied for $\mathcal{C}_k$. Finally,

$$P\left(\bigcap_{l=1}^{k} \Omega(l)\right) = P(\Omega(k) | \Omega(1), \dots, \Omega(k-1)) P\left(\bigcap_{l=1}^{k-1} \Omega(l)\right) = q_1 \cdots q_k.$$

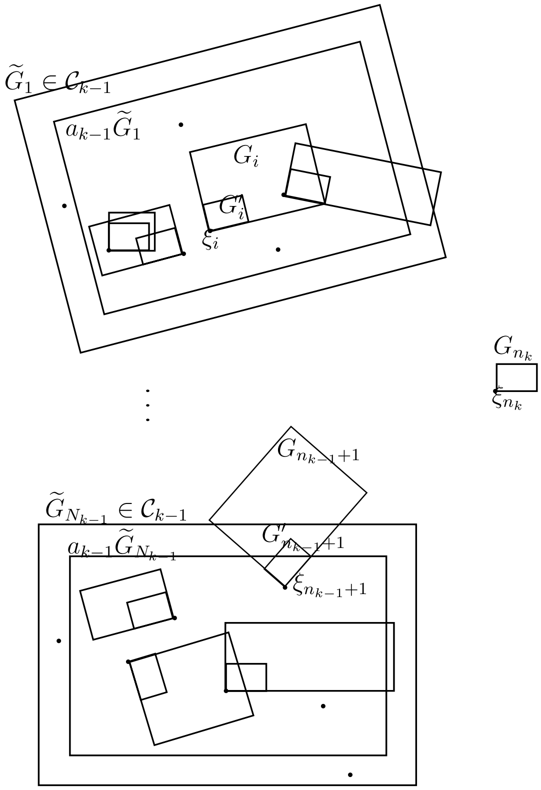

Fig. 1. Construction of $\mathcal{C}_k$.

Letting $\Omega(\infty) = \bigcap_{n=1}^{\infty} \Omega(n)$, we have $P(\Omega(\infty)) = \prod_{n=1}^{\infty} q_n$. Define for all $\omega \in \Omega(\infty)$

$$C^{\omega} = \bigcap_{n=1}^{\infty} \bigcup_{G \in \mathcal{C}_n} G \subset E^{\omega}.$$

Next we verify that the Cantor like set $C^{\omega} \subset E^{\omega}$ exists with positive probability. We use the notation $\mathcal{F}_k$ for the σ-algebra generated by the random variables $\xi_1, \ldots, \xi_{n_k}$.

Proposition 3.1. *With the above notation we have* $P(\Omega(\infty)) > 0$.

Proof. We have

$$\begin{aligned}
q_k &= P\big(\Omega(k) | \Omega(1), \ldots, \Omega(k-1)\big) \\
&= \frac{1}{P(\bigcap_{l=1}^{k-1} \Omega(l))} P\left(\Omega(k) \cap \bigcap_{l=1}^{k-1} \Omega(l)\right) \\
&= \frac{1}{P(\bigcap_{l=1}^{k-1} \Omega(l))} \mathbb{E}\big(\mathbb{E}(\chi_{\Omega(k)} \chi_{\bigcap_{l=1}^{k-1} \Omega(l)} | \mathcal{F}_{k-1})\big) \\
&= \frac{1}{P(\bigcap_{l=1}^{k-1} \Omega(l))} \mathbb{E}\big(\chi_{\bigcap_{l=1}^{k-1} \Omega(l)} \mathbb{E}(\chi_{\Omega(k)} | \mathcal{F}_{k-1})\big)
\end{aligned}$$

$$= \frac{1}{P(\bigcap_{l=1}^{k-1}\Omega(l))}\mathbb{E}\big(\chi_{\bigcap_{l=1}^{k-1}\Omega(l)}\mathbb{E}(\chi_{\bigcap_{G\in\mathcal{C}_{k-1}}\{\#\widetilde{I}(k,G)>(1/2)a_{k-1}^d m_k\mathcal{L}(g_{n_{k-1}})\}}|\mathcal{F}_{k-1})\big)$$

$$\geq \frac{1}{P(\bigcap_{l=1}^{k-1}\Omega(l))}\mathbb{E}\bigg(\chi_{\bigcap_{l=1}^{k-1}\Omega(l)}\bigg(1-\sum_{G\in\mathcal{C}_{k-1}}\mathbb{E}(\chi_{\{\#\widetilde{I}(k,G)\leq(1/2)m_k\mathcal{L}(a_{k-1}g_{n_{k-1}})\}}\Big|\mathcal{F}_{k-1})\bigg)\bigg),$$

and applying Proposition 2.6 hence gives

$$q_k \geq 1 - N_{k-1}^2\frac{8(1-\mathcal{L}(a_{k-1}g_{n_{k-1}}))}{(n_k-n_{k-1})\mathcal{L}(a_{k-1}g_{n_{k-1}})} =: 1-p_k.$$

Inequalities (3.5) and (3.4), in turn, imply that $N_{k-1}\leq(n_{k-1}-n_{k-2})\mathcal{L}(g_{n_{k-2}})\leq n_{k-1}\leq\log n_k$, and therefore, noting that $n_k-n_{k-1}\geq\frac{1}{2}n_k$ by (3.4) and using (3.3), we obtain

$$\sum_{k=1}^{\infty}p_k\leq\sum_{k=1}^{\infty}\frac{8(1-\mathcal{L}(a_{k-1}g_{n_{k-1}}))(\log n_k)^2}{(1/2)a_{k-1}^d n_k\mathcal{L}(g_{n_{k-1}})}\leq\sum_{k=1}^{\infty}\frac{8(\log n_k)^2}{(1/2)a_{k-1}^d n_k^{(3+f(s))/(2+2f(s))}}<\infty,$$

where the convergence follows since by (3.4) the sequence (n_k) is growing exponentially fast. Letting $k_0\in\mathbb{N}$ be such that $p_k<1$ for all $k\geq k_0$, we have $\prod_{k=1}^{\infty}q_k\geq\prod_{k=1}^{k_0}q_k\prod_{k=k_0+1}^{\infty}(1-p_k)>0$. □

Remark 3.2. *The idea of finding a large-dimensional Cantor subset of the random covering set was already exploited in the dimension calculation of Fan and Wu* [12] *in the case of* $\mathbb{T}^1$. *In their proof it is essential that the sets* C^ω *are homogeneous and the construction intervals are well-separated, which follows from well-known results on random spacings of uniform random samples* [14]. *Structure of the set allows them then to directly estimate sizes of intersections of balls with the set* C^ω, *giving the dimension bound from below. In our choice of the subset* C^ω, *however, separation of the generating sets plays no role. Indeed, it is a well-known fact that for self-affine sets no separation condition guarantees the dimension formula. Also a direct estimate for measures of balls is probably hopeless. Instead a potential theoretic method based on a transversality argument is the key, see Lemma* 4.3 *below. In the implementation of this idea we need the assumption* (3.8).

4. Dimension estimate

Using the notation introduced in Section 3, we prove that for $s<s_0(g_n)$ the event $\{\omega\in\Omega(\infty)\,|\,\dim_{\mathrm{H}}C^\omega\geq s\}$ has positive probability. To obtain the dimension bound, we use potential theoretic methods and define a measure supported on C^ω with finite s-energy. In what follows, we consider only the event $\Omega(\infty)$ and denote the expectation over $\Omega(\infty)$ simply by $\mathbb{E}$.

For any $\omega\in\Omega(\infty)$, $k\in\mathbb{N}$ and $G\in\mathcal{C}_{k-1}$, let $M_k=\#I(k,G)=\lfloor\frac{1}{2}a_{k-1}^d m_k\mathcal{L}(g_{n_{k-1}})\rfloor$ be the number of level k construction rectangles contained in G. Notice that M_k is a deterministic number depending only on k. For later notational simplicity, we will relabel the random variables ξ_i using a deterministic tree structure.

For all $l\in\mathbb{N}$, consider the sets $\mathbf{J}_l=\{i_1\ldots i_l|i_j\in\{1,\ldots,M_j\}$ for all $j\in\{1,\ldots,l\}\}$ and define $\mathbf{J}=\bigcup_{l=0}^{\infty}\mathbf{J}_l$, with the convention $\mathbf{J}_0=\{\varnothing\}$. For $\mathbf{i},\mathbf{j}\in\mathbf{J}$, denote by $\mathbf{i}\wedge\mathbf{j}$ the maximal common initial sequence of $\mathbf{i}$ and $\mathbf{j}$ and let $\mathbf{ij}\in\mathbf{J}$ be the word obtained by juxtaposing the words $\mathbf{i}$ and $\mathbf{j}$. Further, we denote by $|\mathbf{i}|$ the length of $\mathbf{i}\in\mathbf{J}$, that is, $|\mathbf{i}|=l$ if $\mathbf{i}\in\mathbf{J}_l$. For each $l\leq k$ and $\mathbf{i}\in\mathbf{J}_l$, define the *cylinder* of length l and of depth k by $C(\mathbf{i},k)=\{\mathbf{j}\in\mathbf{J}_k|\mathbf{i}\wedge\mathbf{j}=\mathbf{i}\}$. For $i\in\{1,\ldots,M_1\}$, define $\phi_i=\xi_i$ and $G(i)=g_i'+\phi_i$ and let T_i' be a linear map such that $\Pi(T_i'([0,1]^d)=g_i'$. Assume that we have defined the random variables $\phi_{\mathbf{i}}$ and the rectangles $G(\mathbf{i})\in\mathcal{C}_{k-1}$ for all $\mathbf{i}\in\mathbf{J}_{k-1}$. Let $I(k,G(\mathbf{i}))=\{j_1,\ldots,j_{M_k}\}$ where $j_i<j_{i+1}$ in the natural order given by the construction. For all $i\in\{1,\ldots,M_k\}$, define $\phi_{\mathbf{i}i}=\xi_{j_i}$, $g_{\mathbf{i}i}'=g_{j_i}'$ and $G(\mathbf{i}i)=g_{\mathbf{i}i}'+\phi_{\mathbf{i}i}$ and let $T_{\mathbf{i}i}'$ be a linear map satisfying $\Pi(T_{\mathbf{i}i}'([0,1]^d))=g_{\mathbf{i}i}'$. Then $\det(T_{n_{|\mathbf{i}|}})=\mathcal{L}(G(\mathbf{i}))$ and $\Phi^s(T_{\mathbf{i}}')=\Phi^s(T_{n_{|\mathbf{i}|}})$ for all $\mathbf{i}\in\mathbf{J}$. For notational purposes set $G(\varnothing)=\mathbb{T}^d$ and $\phi_\varnothing=0$. When necessary we view $T_{\mathbf{i}}'$ as a map on $\mathbb{T}^d$ by identifying $\mathbb{T}^d$ with $[0,1[^d$. Finally, for $\mathbf{i}_1,\ldots,\mathbf{i}_k\in\mathbf{J}$, denote by $\mathcal{F}(\mathbf{i}_1,\ldots,\mathbf{i}_k)$ the σ-algebra generated by the events $\{\omega\in\Omega(\infty)|G(\mathbf{i}_l)=Q_l$ for all $l=1,\ldots,k\}$, where each $Q_l\subset\mathbb{T}^d$ is an isometric copy of $g_{n_{|\mathbf{i}_l|}}$.

Remark 4.1. *Note that* $\{\phi_{\mathbf{i}}|\mathbf{i}\in C(\mathbf{j},k)\}=\{\xi_i|i\in I(k,G(\mathbf{j}))\}$ *for any* $\mathbf{j}\in\mathbf{J}_{k-1}$ *and* $\{\phi_{\mathbf{i}}|\mathbf{i}\in\mathbf{J}_k\}=\{\xi_i|i\in\bigcup_{G\in\mathcal{C}_{k-1}}I(k,G)\}$. *Let* $A\subset\mathbb{T}^d$ *be a Borel set with* $\mathcal{L}(A)>0$. *Since* ξ_j *is uniformly distributed on* $\mathbb{T}^d$ *for given* j, *every ξ_j is uniformly distributed on A when conditioned on the event $\xi_j \in A$. Let $i \in \mathbb{N}$ and let $\mathbf{i}i \in \mathbf{J}_{k+1}$. By definition $\phi_{\mathbf{i}i} = \xi_j$ for some $j \in \{n_k+1, \dots, n_{k+1}\}$ with $\xi_j \in G(\mathbf{i})$, and hence the random variable $\phi_{\mathbf{i}i}$ is uniformly distributed on $a_k G(\mathbf{i})$ when conditioned on $\phi_{\mathbf{i}i} = \xi_j$ and the σ-algebra $\mathcal{F}(\mathbf{i})$. Furthermore,*

$$\begin{aligned}\mathbb{E}\big(\chi_{\{\phi_{\mathbf{i}i}\in A\}}|\mathcal{F}(\mathbf{i})\big) &= \sum_{j=n_k+1}^{n_{k+1}} \mathbb{E}\big(\chi_{\{\phi_{\mathbf{i}i}\in A\}}|\mathcal{F}(\mathbf{i}), \phi_{\mathbf{i}i}=\xi_j\big)\mathbb{E}\big(\chi_{\{\phi_{\mathbf{i}i}=\xi_j\}}|\mathcal{F}(\mathbf{i})\big) \\ &= \frac{\mathcal{L}(A\cap a_k G(\mathbf{i}))}{\mathcal{L}(a_k G(\mathbf{i}))} \sum_{j=n_k+1}^{n_{k+1}} \mathbb{E}\big(\chi_{\{\phi_{\mathbf{i}i}=\xi_j\}}|\mathcal{F}(\mathbf{i})\big) = \frac{\mathcal{L}(A\cap a_k G(\mathbf{i}))}{\mathcal{L}(a_k G(\mathbf{i}))}.\end{aligned}$$

Hence $\phi_{\mathbf{i}i}$ is uniformly distributed inside $a_k G(\mathbf{i})$ when conditioned on $\mathcal{F}(\mathbf{i})$. Moreover, if $\mathbf{j}$ satisfies $\mathbf{j} \wedge \mathbf{i}i \neq \mathbf{i}i$, conditioning on $\mathcal{F}(\mathbf{i},\mathbf{j})$ instead of $\mathcal{F}(\mathbf{i})$ does not change the uniform distribution of $\phi_{\mathbf{i}i}$ on $a_k G(\mathbf{i})$, since ξ_j and ξ_l are independent for $j \neq l$. Recall that even though the corner points $\phi_{\mathbf{i}i}$ and $\phi_{\mathbf{i}h}$ are independent for $i \neq h$, the rectangles $G(\mathbf{i}i)$ and $G(\mathbf{i}h)$ are not, since the orientation of $g'_{\mathbf{i}i}$ is determined by the index j_i.

Lemma 4.2. *The sequence of measures μ_l^ω on $\mathbb{T}^d$ given by*

$$\mu_l^\omega = \frac{\sum_{\mathbf{i}\in\mathbf{J}_l}(T'_{\mathbf{i}}+\phi_{\mathbf{i}})_*\mathcal{L}}{N_l} \tag{4.1}$$

converges in weak-topology to a measure μ^ω supported on C^ω.*

Proof. By the Riesz representation theorem a weak*-limit μ^ω exists, if we prove that for all positive, continuous functions f on $\mathbb{T}^d$ the sequence $\int f\,\mathrm{d}\mu_l^\omega$ converges.

To that end, fix a positive, continuous function f on $\mathbb{T}^d$ and $\varepsilon > 0$. Since $\mathbb{T}^d$ is compact, there exists $\delta > 0$ with $|f(x) - f(y)| < \varepsilon$ for all $|x - y| < \delta$. Let K be so large that $\operatorname{diam}(g_{n_K}) < \delta$, and fix $k \geq K$. Write μ_k^ω as a sum of measures $\mu_{\mathbf{i},k}^\omega$ defined by

$$\mu_k^\omega = \sum_{\mathbf{i}\in\mathbf{J}_K}\sum_{\mathbf{j}\in C(\mathbf{i},k)} \frac{(T'_{\mathbf{j}}+\phi_{\mathbf{j}})_*\mathcal{L}}{N_k} = \sum_{\mathbf{i}\in\mathbf{J}_K}\mu_{\mathbf{i},k}^\omega.$$

For all $\mathbf{i} \in \mathbf{J}_K$, we have $\mu_{\mathbf{i},k}^\omega(G(\mathbf{i})) = \frac{1}{N_K} = \mu_{\mathbf{i},K}^\omega(G(\mathbf{i}))$ and $\operatorname{spt}\mu_{\mathbf{i},k}^\omega \subset G(\mathbf{i})$. Therefore,

$$\left|\int f\,\mathrm{d}\mu_k^\omega - \int f\,\mathrm{d}\mu_K^\omega\right| \leq \sum_{\mathbf{i}\in\mathbf{J}_K}\left|\int_{G(\mathbf{i})} f\,\mathrm{d}\mu_{\mathbf{i},k}^\omega - \int_{G(\mathbf{i})} f\,\mathrm{d}\mu_{\mathbf{i},K}^\omega\right| \leq \varepsilon,$$

since $\operatorname{diam} G(\mathbf{i}) = \operatorname{diam}(g_{n_K}) < \delta$. Thus sequence $\int f\,\mathrm{d}\mu_l^\omega$ converges. The claim $\operatorname{spt}\mu^\omega \subset C^\omega$ holds since C^ω is compact and $\operatorname{spt}\mu_l^\omega \subset \bigcup_{G\in\mathcal{C}_l} G$ for all l. □

Next we show that for all $s < s_0(g_n)$ the s-energy $I^s(\mu^\omega) = \int\int \frac{\mathrm{d}\mu^\omega(x)\,\mathrm{d}\mu^\omega(y)}{|x-y|^s}$ of μ^ω is finite almost surely. In the energy estimate we will make use of the following lemma [8], Lemma 2.2.

Lemma 4.3 (Falconer). *Let s be non-integral with $0 < s < d$ and let $T:\mathbb{R}^d \to \mathbb{R}^d$ be an affine injection. Then there exists a number $0 < D_0 < \infty$, depending only on d and s, such that*

$$\int_{[0,1]^d} \frac{\mathrm{d}\mathcal{L}(x)}{|T(x)|^s} \leq \frac{D_0}{\Phi^s(T)}.$$

Lemma 4.4. *For all $\mathbf{i},\mathbf{j} \in \mathbf{J}$ and $x, y \in \mathbb{T}^d$ we have*

$$\mathbb{E}\big(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{i})}(x)\big) \leq \left(\prod_{l=1}^{\infty}\frac{1}{a_l}\right)^{2d} \frac{\det(T_{n_{|\mathbf{i}|}})\det(T_{n_{|\mathbf{j}|}})}{\det(T_{n_{|\mathbf{i}\wedge\mathbf{j}|}})^2}\mathbb{E}\big(\chi_{G(\mathbf{i}\wedge\mathbf{j})}(y)\chi_{G(\mathbf{i}\wedge\mathbf{j})}(x)\big).$$

Proof. Since $(\prod_{l=1}^{\infty}\frac{1}{a_l}) > 1$, the claim holds when $\mathbf{i}=\mathbf{j}$. Consider $\mathbf{i}\neq\mathbf{j}\in\mathbf{J}$. Without loss of generality, we may assume that $|\mathbf{i}|\geq|\mathbf{j}|$. Letting $\mathbf{k}\in\mathbf{J}$ and $i\in\mathbb{N}$ satisfy $\mathbf{i}=\mathbf{k}i\in\mathbf{J}$, we obtain for any $x,y\in\mathbb{T}^d$ that

$$\begin{aligned}\mathbb{E}\big(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{i})}(x)\big)&=\mathbb{E}\big(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{k})}(x)\chi_{G(\mathbf{i})}(x)\big)\\&=\mathbb{E}\big(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{k})}(x)\mathbb{E}\big(\chi_{G(\mathbf{i})}(x)|\mathcal{F}(\mathbf{j},\mathbf{k})\big)\big)\\&=\mathbb{E}\big(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{k})}(x)\mathbb{E}\big(\chi_{x-g'_{\mathbf{i}}}(\phi_{\mathbf{i}})|\mathcal{F}(\mathbf{j},\mathbf{k})\big)\big).\end{aligned}$$

Even though the orientation of $g'_{\mathbf{i}}$ depends on $\omega\in\Omega(\infty)$, the volume $\mathcal{L}(g'_{\mathbf{i}})$ does not. Therefore, from Remark 4.1 we get

$$\mathbb{E}\big(\chi_{x-g'_{\mathbf{i}}}(\phi_{\mathbf{i}})|\mathcal{F}(\mathbf{j},\mathbf{k})\big)\leq\frac{\mathcal{L}(g_{n_{|\mathbf{i}|}})}{\mathcal{L}(a_{|\mathbf{k}|}g_{n_{|\mathbf{k}|}})},$$

and therefore,

$$\begin{aligned}\mathbb{E}\big(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{i})}(x)\big)&\leq\mathbb{E}\left(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{k})}(x)\frac{\mathcal{L}(g_{n_{|\mathbf{i}|}})}{\mathcal{L}(a_{|\mathbf{k}|}g_{n_{|\mathbf{k}|}})}\right)\\&=\frac{\det(T_{n_{|\mathbf{i}|}})}{a_{|\mathbf{k}|}^d\det(T_{n_{|\mathbf{k}|}})}\mathbb{E}\big(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{k})}(x)\big).\end{aligned}$$

Iterating this with respect to $\mathbf{k}$, if necessary, gives

$$\mathbb{E}\big(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{i})}(x)\big)\leq\left(\prod_{l=1}^{\infty}\frac{1}{a_l}\right)^d\frac{\det(T_{n_{|\mathbf{i}|}})}{\det(T_{n_{|\mathbf{i}\wedge\mathbf{j}|}})}\mathbb{E}\big(\chi_{G(\mathbf{j})}(y)\chi_{G(\mathbf{i}\wedge\mathbf{j})}(x)\big).\tag{4.2}$$

Inequality (4.2) completes the proof provided that $\mathbf{j}=\mathbf{i}\wedge\mathbf{j}$. If this is not the case, we apply the above argument with $\mathbf{j}$ playing the role of $\mathbf{i}$ and $\mathbf{i}\wedge\mathbf{j}$ playing that of $\mathbf{j}$. □

Lemmas 4.3 and 4.4 lead to the following energy estimate.

Proposition 4.5. *Letting $s<s_0(g_n)$, there exists a constant $C<\infty$ such that $\int_{\Omega(\infty)}I^s(\mu_l^\omega)\,\mathrm{d}P(\omega)<C$ for all $l\in\mathbb{N}$. In particular, $I^s(\mu^\omega)<\infty$ for P-almost all $\omega\in\Omega(\infty)$.*

Proof. Let $s<s_0(g_n)$ and let $\mathbf{i},\mathbf{j}\in\mathbf{J}$. Define

$$H(\mathbf{i},\mathbf{j},s)=\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{1}{|x-y|^s}\,\mathrm{d}\big(T'_{\mathbf{i}}+\phi_{\mathbf{i}}\big)_*\mathcal{L}(x)\,\mathrm{d}\big(T'_{\mathbf{j}}+\phi_{\mathbf{j}}\big)_*\mathcal{L}(y).$$

As the functions involved are clearly measurable, use of Fubini's theorem and Lemmas 4.4 and 4.3 yields the following estimate

$$\begin{aligned}\int_{\Omega(\infty)}H(\mathbf{i},\mathbf{j},s)\,\mathrm{d}P&=(\det T_{n_{|\mathbf{i}|}}\det T_{n_{|\mathbf{j}|}})^{-1}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\mathbb{E}(\chi_{G(\mathbf{i})}(x)\chi_{G(\mathbf{j})}(y))}{|x-y|^s}\,\mathrm{d}\mathcal{L}(x)\,\mathrm{d}\mathcal{L}(y)\\&\leq\left(\prod_{l=1}^{\infty}\frac{1}{a_l}\right)^{2d}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\mathbb{E}(\chi_{G(\mathbf{i}\wedge\mathbf{j})}(x)\chi_{G(\mathbf{i}\wedge\mathbf{j})}(y))}{\det(T'_{\mathbf{i}\wedge\mathbf{j}})^2|x-y|^s}\,\mathrm{d}\mathcal{L}(x)\,\mathrm{d}\mathcal{L}(y)\\&=\left(\prod_{l=1}^{\infty}\frac{1}{a_l}\right)^{2d}\int_{\Omega(\infty)}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\mathrm{d}\mathcal{L}(x)\,\mathrm{d}\mathcal{L}(y)\,\mathrm{d}P}{|T'_{\mathbf{i}\wedge\mathbf{j}}(x-y)|^s}\leq\frac{D}{\Phi^s(T_{n_{|\mathbf{i}\wedge\mathbf{j}|}})},\end{aligned}$$

where D depends on D_0 of Lemma 4.3. Combining this with (4.1) gives

$$\begin{aligned}\int_{\Omega(\infty)} I^s\big(\mu_l^\omega\big)\,\mathrm{d}P(\omega) &= \int_{\Omega(\infty)}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{1}{|x-y|^s}\,\mathrm{d}\mu_l^\omega(x)\,\mathrm{d}\mu_l^\omega(y)\,\mathrm{d}P(\omega)\\ &= \frac{\sum_{\mathbf{i}\in\mathbf{J}_l}\sum_{\mathbf{j}\in\mathbf{J}_l}}{N_l^2}\int H(\mathbf{i},\mathbf{j},s)\,\mathrm{d}P \le N_l^{-2}\sum_{\mathbf{i}\in\mathbf{J}_l}\sum_{\mathbf{j}\in\mathbf{J}_l}\frac{D}{\Phi^s(T_{n_{|\mathbf{i}\wedge\mathbf{j}|}})}\\ &\le N_l^{-2}\sum_{K=0}^{l}\sum_{\mathbf{k}\in\mathbf{J}_K}\sum_{\mathbf{i}\in C(\mathbf{k},l)}\sum_{\mathbf{j}\in C(\mathbf{k},l)}\frac{D}{\Phi^s(T_{n_K})} = \sum_{K=0}^{l}\frac{D}{N_K\Phi^s(T_{n_K})}.\end{aligned}$$

From (3.1) we deduce that $\Phi^s(T_{n_k}) > n_k^{-2/(1+f(s))}$ for large k. Recalling (3.5), (3.4) and (3.3), gives for large k that

$$\begin{aligned}N_k\Phi^s(T_{n_k}) &\ge \left(\frac{1}{2}\right)^3 a_{k-1}^d(n_k-n_{k-1})\mathcal{L}(g_{n_{k-1}})\Phi^s(T_{n_k})\\ &\ge \left(\frac{1}{2}\right)^4 a_{k-1}^d n_k\mathcal{L}(g_{n_{k-1}})\Phi^s(T_{n_k}) \ge \left(\frac{1}{2}\right)^4 a_{k-1}^d n_k^{(3+f(s))/(2+2f(s))}n_k^{-2/(1+f(s))}\\ &= \left(\frac{1}{2}\right)^4 a_{k-1}^d n_k^{(f(s)-1)/(2+2f(s))}.\end{aligned}\tag{4.3}$$

By (3.4) the sequence (n_k) is growing exponentially fast. Therefore, recalling that $f(s)-1<0$, inequality (4.3) implies that the series $\sum_{K=0}^{\infty}\frac{D}{N_K\Phi^s(T_{n_K})}$ converges. The final claim follows by approximating the kernel $|x|^{-s}$ by kernels $\min\{|x|^{-s},A\}$, where $A\in\mathbb{N}$. □

Now Proposition 2.2 follows in a straightforward manner.

Proof of Proposition 2.2. By Lemma 2.3 it suffices to prove that $\dim_{\mathrm{H}} E \ge s_0(g_n)$. Consider $m-1<s<s_0(g_n)\le m$ where m is an integer. Lemma 4.2 and Proposition 4.5 combined with [26], Theorem 8.7, imply that $\dim_{\mathrm{H}} C^\omega \ge s$ almost surely conditioned on $\Omega(\infty)$ which, in turn, gives

$$P\big(\dim_{\mathrm{H}} E^\omega \ge s\big) > 0.$$

Since $\{\dim_{\mathrm{H}} E\ge s\}$ is a tail event, from the Kolmogorov zero–one law we deduce that $P(\dim_{\mathrm{H}} E\ge s)=1$. Approaching $s_0(g_n)$ along an increasing sequence of real numbers s gives $\dim_{\mathrm{H}} E^\omega \ge s_0(g_n)$ for P-almost all $\omega\in\Omega$. □

As we mentioned in the Introduction, for ball like covering sets the dimension formula is an easy consequence of the mass transference principle of Beresnevich and Velani. Since the proof is quite simple in this case, we give the details here.

For a ball $B=B(x,r)\subset\mathbb{R}^d$ and $0<s<d$, write $B^s=B(x,r^{\frac{s}{d}})$. We recall a special case of the mass transference principle [1], Theorem 2, suitable for our purposes.

Theorem 4.6 (Beresnevich–Velani). *Let $(B_n)\subset\mathbb{R}^d$ be a sequence of balls whose radii converge to zero. Suppose that for any ball $B\subset\mathbb{R}^d$*

$$\mathcal{H}^d\Big(B\cap\limsup_{n\to\infty} B_n^s\Big)=\mathcal{H}^d(B).$$

Then for any ball B in $\mathbb{R}^d$,

$$\mathcal{H}^s\Big(B\cap\limsup_{n\to\infty} B_n\Big)=\infty.$$

Proposition 4.7. *Consider a sequence (g_n) of subsets of $\mathbb{T}^d$ satisfying $B(x_n, r_n) \subset g_n$ for sequences of points (x_n) and radii (r_n). Letting ρ_n be the diameter of g_n with $\rho_n \downarrow 0$, assume that there exists $C < \infty$ such that $\frac{\rho_n}{r_n} \le C$ for all $n \in \mathbb{N}$. Let (ξ_n) be a sequence of independent random variables, uniformly distributed on $\mathbb{T}^d$. Then for $E = \limsup_{n\to\infty}(g_n + \xi_n)$, almost surely*

$$\dim_{\mathrm{H}} E = \min\{s_0, d\},$$

where $s_0 = \inf\{s \ge 0 | \sum_{n=1}^{\infty} \rho_n^s < \infty\}$.

Proof. Let $s > s_0$. Set $G_n = g_n + \xi_n$. Since $E \subset \bigcup_{n=N}^{\infty} G_n$ for all N, we obtain

$$\mathcal{H}^s(E) \le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \rho_n^s = 0,$$

giving $\dim_{\mathrm{H}} E \le \min\{s_0, d\}$.

Obviously, $E \supset \limsup_{n\to\infty} B_n$ where $B_n = B(x_n + \xi_n, r_n)$. Consider $s < \min\{s_0, d\}$. Letting $K = \mathcal{L}(B(0,1))$, we have

$$\sum_{n=1}^{\infty} \mathcal{L}\big(B_n^s\big) = K \sum_{n=1}^{\infty} r_n^s \ge K C^{-s} \sum_{n=1}^{\infty} \rho_n^s = \infty. \tag{4.4}$$

Since $P(x \in B_n^s) = \mathcal{L}(B_n^s)$ for all $x \in \mathbb{T}^d$ and $n \in \mathbb{N}$, Borel–Cantelli lemma and (4.4) imply $P(x \in \limsup_{n\to\infty} B_n^s) = 1$. Applying Fubini's theorem, gives $\mathcal{L}(\limsup_{n\to\infty} B_n^s) = 1$ almost surely, implying $\mathcal{L}(\limsup_{n\to\infty} B_n^s \cap B) = \mathcal{L}(B)$ for any ball $B \subset \mathbb{T}^d$. From Theorem 4.6 we get $\mathcal{H}^s(\limsup_{n\to\infty} B_n) = \infty$, which leads to $\dim_{\mathrm{H}} E \ge \min\{s_0, d\}$, almost surely. □

Remark 4.8. *In $\mathbb{T}^1$ one may assume without loss of generality that (l_n) is a decreasing sequence by reordering the sequence if necessary whereas in $\mathbb{T}^d$ with $d > 1$ one cannot always reorder $\alpha_i(L_n)$ simultaneously for all $i = 1, \dots, d$. However, we do not know whether this assumption is necessary for the validity of Theorem 2.1.*

Acknowledgements

We acknowledge the support of Academy of Finland, the Centre of Excellence in Analysis and Dynamics Research. HK acknowledges the support of Jenny and Antti Wihuri Foundation. BL acknowledges the support of NSFC (No. 11201155). We thank Ai-Hua Fan and the referee for useful comments.

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