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FILTRATION SHRINKAGE, STRICT LOCAL MARTINGALES AND THE FÖLLMER MEASURE

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When a strict local martingale is projected onto a subfiltration to which it is not adapted, the local martingale property may be lost, and the finite variation part of the projection may have singular paths. This phenomenon has consequences for arbitrage theory in mathematical finance. In this paper it is shown that the loss of the local martingale property is related to a measure extension problem for the associated Föllmer measure. When a solution exists, the finite variation part of the projection can be interpreted as the compensator, under the extended measure, of the explosion time of the original local martingale. In a topological setting, this leads to intuitive conditions under which its paths are singular. The measure extension problem is then solved in a Brownian framework, allowing an explicit treatment of several interesting examples.

1. Introduction. It is a simple fact that the optional projection of a martingale onto a subfiltration is again a martingale. However, for local martingales the situation is different, and this was the starting point for Föllmer and Protter in [10]. They consider, among other things, three-dimensional Brownian motion $B = (B^1, B^2, B^3)$ starting from $(1, 0, 0)$, defined on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, P)$ where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is generated by B. In this setting they study optional projections of the process $N = 1/\|B\|$ onto subfiltrations $\mathbb{F}^1 = (\mathcal{F}^1_t)_{t\geq 0}$ and $\mathbb{F}^{1,2} = (\mathcal{F}^{1,2}_t)_{t\geq 0}$ generated by B^1 and (B^1, B^2), respectively. It is well known that N, the reciprocal of a BES(3) process, is a local martingale in $\mathbb{G}$. The same turns out to be true for its optional projection onto $\mathbb{F}^{1,2}$. However, the optional projection onto $\mathbb{F}^1$ *is not a local martingale*. Indeed, it was shown in [10], Theorem 5.1, that the equality

$$E^P[N_t \mid \mathcal{F}^1_t] = 1 + \int_0^t u_x(s, B^1_s)\, dB^1_s - \int_0^t \frac{1}{s}\, dL^0_s, \qquad t \geq 0, \tag{1}$$

holds P-a.s., where the function u is given by

$$u(t, x) = \sqrt{\frac{2\pi}{t}} \exp\left(\frac{x^2}{2t}\right)\left(1 - \Phi\left(|x|/\sqrt{t}\right)\right)$$

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and L^0 is the local time of B^1 at level zero. Here $\Phi(\cdot)$ is the standard Normal cumulative distribution function. A superficial reason for the appearance of the local time is the nondifferentiability of u at $x = 0$, but this is of course highly specific to this particular example. The main goal of the present paper is to shed further light on when the optional projection of a general positive local martingale N fails to be a local martingale, and, when this is the case, what can be said about the behavior of its finite variation part. The basic structural result holds for arbitrary positive local martingales, subject only to a weak regularity condition on the filtration.

A crucial tool in the analysis is a variant of the *Föllmer measure* Q_0 associated with N, whose construction we briefly review in Section 2. A nonuniqueness property of (this variant of) the Föllmer measure leads us to formulate an *equivalent measure extension problem* (Problem 1): find an extension Q of Q_0 that is equivalent to P on each σ-field of the subfiltration under consideration. When a solution exists, one can interpret the finite variation part of the projection of N as the compensator of a certain stopping time (Theorem 1). This stopping time is the explosion time of N, which may be finite under the Föllmer measure. These developments, valid in full generality, are carried out in Section 3. We then proceed in Section 4 to study filtrations generated by the image under some continuous map of the coordinate process Y (we now restrict ourselves to path space), and take N to be a deterministic function of Y. This additional structure makes it possible to obtain more detailed results about the points of increase of the finite variation part of the projection of N (Theorem 2). As a consequence (Corollary 2) we obtain a simple sufficient condition for its paths to be singular. Next, in Section 5, we address the problem of actually finding a solution to the equivalent measure extension problem. The setting is now restricted further: the coordinate process is assumed to be (multidimensional) Brownian motion under P. In this framework we derive explicit conditions under which the equivalent measure extension problem can be solved (Theorem 3). Several illustrating examples are given in Section 6, including the aforementioned example of Föllmer and Protter.

Strict local martingales are fundamental in financial models for asset pricing bubbles and relative arbitrage; see, for instance, [8, 12, 13, 17, 20]. They also appear in the so-called Benchmark approach [26]. The role of filtration shrinkage in this context, in particular the loss of the local martingale property, is discussed in [10]. The authors explain how less informed investors may perceive arbitrage opportunities where there are none; see also [15]. Applications in credit risk include [2] and [16] (the latter relying on the very nice theory article [28]). More generally, filtration shrinkage appears naturally in models with restricted information, and results such as those obtained in the present paper will be instrumental for developing models of this type. This is discussed further in Section 7, which concludes the paper.

1.1. *Notation.* Let us now fix some notation that will be in force throughout the paper. $(\Omega, \mathcal{G}, \mathbb{G}, P)$ is a filtered probability space, where the filtration

$\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the right-continuous modification of a *standard system*. That is, $\mathcal{G}_t = \bigcap_{u>t} \mathcal{G}_u^o$, where each $\mathcal{G}_t^o$ is Standard Borel (see Parthasarathy [25], Definition V.2.2), and any decreasing sequence of atoms has a nonempty intersection.[1] We always assume that $\mathcal{G} = \mathcal{G}_\infty = \bigvee_{t\geq 0} \mathcal{G}_t$. A key example of a standard system is the filtration generated by all right-continuous paths, allowed to explode to an absorbing cemetery state in finite time, and with left limits prior to the explosion time. This example is considered in detail in [23], and will re-appear in Section 4 of the present paper. Note that *we do not augment* $\mathbb{G}$ *with the* P*-nullsets*—but this does not cause any serious complications, due to the following result which allows one to pass painlessly between a filtration and its completion (see also [1] for a discussion of this and related issues).

LEMMA 1. *Let R be a probability measure on $\mathcal{G}$, and denote by $(\overline{\mathcal{G}}, \overline{\mathbb{G}})$ the augmentation of $(\mathcal{G}, \mathbb{G})$ with respect to R. Then*:

(i) *Every $\overline{\mathbb{G}}$ optional* (*predictable*) *process is R-indistinguishable from a $\mathbb{G}$ optional* (*predictable*) *process.*

(ii) *Every right-continuous* ($\mathbb{G}$, R) *martingale is a* ($\overline{\mathbb{G}}$, R) *martingale.*

PROOF. Part (i) is Lemma 7 in Appendix 1 of [6]. Part (ii) follows from Theorem IV.3 in the same reference. □

Next, let N be a local martingale on $(\Omega, \mathcal{G}, \mathbb{G}, P)$ that is càdlàg, strictly positive and satisfies $N_0 = 1$, P-a.s. Define stopping times

$$\tau_n = n \wedge \inf\{t \geq 0 : N_t \geq n\}, \qquad \tau = \lim_{n\to\infty} \tau_n.$$

Since N is a local martingale under P, and hence does not explode in finite time, we have $P(\tau < \infty) = 0$. However, there may be P-nullsets on which τ is finite—in particular this is the case when N is a strict local martingale, as will become clear when we discuss the Föllmer measure.

The reciprocal of N will play a sufficiently important role that it merits its own notation. We thus define a process M by

$$M_t = \frac{1}{N_t}\mathbf{1}_{\{\tau>t\}}, \tag{2}$$

whenever $N_t > 0$, and $M_t = 0$ otherwise (N_t will never be zero under any measure considered in the sequel).

Finally, note that $\mathcal{G}_{\tau-} = \bigvee_{n\geq 1} \mathcal{G}_{\tau_n}$, see, for instance, [5], Theorem IV.56(d).

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[1] This means that if $(t_n)_{n\geq 0}$ is a nonnegative increasing sequence, $A_n \in \mathcal{G}_{t_n}^o$ is an atom for each $n \geq 1$, and $A_n \supset A_{n+1}$, then $\bigcap_n A_n \neq \varnothing$.

2. The Föllmer measure. Following similar ideas as in Delbaen and Schachermayer [4] and Pal and Protter [24], which originated with the paper by Föllmer [9] (who in turn was inspired by Doob [7]), we can construct a new probability Q_0 on $\mathcal{G}_{\tau-}$ as follows. For each $n \geq 1$, the stopped process $N^{\tau_n} = (N_{t\wedge\tau_n})_{t\geq 0}$ is a strictly positive uniformly integrable martingale, so we may define a probability $Q_n \sim P$ on $\mathcal{G}_{\tau_n}$ by $dQ_n = N_{\tau_n}\, dP$. The optional stopping theorem and uniform integrability yield

$$N_{\tau_n} = N_{\tau_n}^{\tau_{n+1}} = E^P\bigl[N_\infty^{\tau_{n+1}} \mid \mathcal{G}_{\tau_n}\bigr] = E^P[N_{\tau_{n+1}} \mid \mathcal{G}_{\tau_n}].$$

The measures $(Q_n)_{n\geq 1}$ thus form a consistent family. Next, by Remark 6.1 in the Appendix of [9], $(\mathcal{G}_{\tau_n-})_{n\geq 1}$ is a standard system, so Parthasarathy's extension theorem (Theorem V.4.2 in [25]) applies: there exists a probability measure Q_0 on $\mathcal{G}_{\tau-}$ that coincides with Q_n on $\mathcal{G}_{\tau_n-}$, for each n.

From now on, Q_0 *will denote the measure on* $\mathcal{G}_{\tau-}$ *obtained from* P *in this way.*

Here is the key point: Q_0 is only defined on $\mathcal{G}_{\tau-}$, not on all of $\mathcal{G}$. There are typically many ways in which Q_0 can be extended to a measure Q on $\mathcal{G}$, and we will see that the choice of extension is crucial in the context of filtration shrinkage. In particular, the existence of an extension with certain properties is intimately connected with the behavior of the optional projection of N (under P) onto smaller filtrations $\mathbb{F} \subset \mathbb{G}$.

The following lemma shows that no matter which extension Q one chooses, M defined in (2) is always the density process relative to P. In particular it is a (true) P martingale.

LEMMA 2. *Suppose* Q *is an extension of* Q_0 *to all of* $\mathcal{G}$. *Then, for each* $t \geq 0$,

$$M_t = \frac{dP}{dQ}\bigg|_{\mathcal{G}_t} \qquad Q\text{-a.s.}$$

PROOF. The argument is well-known. Fix $t \geq 0$ and pick $A \in \mathcal{G}_t$. Using that $M_t = 0$ for $t \geq \tau$, monotone convergence and the fact that $M_{t\wedge\tau_n} = \frac{dP}{dQ}|_{\mathcal{G}_{t\wedge\tau_n}}$ (which relies on the strict positivity of N), we obtain

$$\begin{aligned}
E^Q[M_t \mathbf{1}_A] &= E^Q[M_t \mathbf{1}_{A\cap\{\tau>t\}}] = \lim_{n\to\infty} E^Q[M_t \mathbf{1}_{A\cap\{\tau_n>t\}}] \\
&= \lim_{n\to\infty} E^Q[M_{t\wedge\tau_n} \mathbf{1}_{A\cap\{\tau_n>t\}}] = \lim_{n\to\infty} P\bigl(A\cap\{\tau_n>t\}\bigr) \\
&= P\bigl(A\cap\{\tau>t\}\bigr).
\end{aligned}$$

Since $P(\tau > t) = 1$, the right-hand side equals $P(A)$, as claimed. $\square$

If N is a strict local martingale under P, then $Q(\tau < \infty) > 0$, and vice versa. To see this, simply write

$$Q(\tau > t) = E^Q[M_t N_t] = E^P[N_t],$$

which is strictly less than one for some $t > 0$ if and only if N is a strict local martingale. Our focus will be on this case, and in particular this means that P and Q cannot be equivalent. In fact, they may even be singular, which is the case if $Q(\tau < \infty) = 1$. On the other hand, Lemma 2 guarantees that we always have *local absolute continuity*: for each t, $Q|_{\mathcal{G}_t} \ll P|_{\mathcal{G}_t}$. "Global" absolute continuity, $Q \ll P$, holds when $(M_t)_{t\geq 0}$ is uniformly integrable under P.

The following simple but useful result shows that although N may explode under Q, it does so continuously—it does not jump to infinity.

LEMMA 3. *On* $\{\tau < \infty\}$, *the equality* $M_{\tau-} = 0$ *holds* Q_0-*a.s.*

PROOF. First, note that $\tau_n < \tau$, Q_0-a.s. Indeed, since N^{τ_n} is a martingale under P and τ_n is bounded by construction,

$$Q_0(\tau_n < \tau) = E^{Q_0}[M_{\tau_n} N_{\tau_n}] = E^P[N_{\tau_n}] = 1.$$

Now, on $\{N_{\tau-} < \infty \text{ and } \tau < \infty\}$ there exists a (large) n such that $\tau_n = \tau$. Hence

$$Q_0(N_{\tau-} < \infty \text{ and } \tau < \infty) \leq \sum_{n\geq 1} Q_0(\tau_n = \tau) = 0.$$

Therefore $Q_0(M_{\tau-} > 0 \text{ and } \tau < \infty) = 0$, as claimed. □

Let us mention that the construction of P from Q is straightforward: assuming that M is a Q martingale, the measures P_n on $\mathcal{G}_n$ given by $dP_n = M_n \, dQ$ form a consistent family, extendable to a measure P on $\mathcal{G}$ using Parthasarathy's theorem. Local absolute continuity is immediate, and "global" absolute continuity holds when M is uniformly integrable. Note that P only depends on the behavior of Q on $\mathcal{G}_{\tau-}$, since $P(\tau = \infty) = 1$.

We finally comment on how the question of uniqueness has been treated previously in the literature. In Föllmer's original paper [9], a measure is constructed on the product space $(0, \infty] \times \Omega$, specifically on the predictable σ-field. This measure assigns zero mass to the stochastic interval $(\tau, \infty]$, which is key to obtaining uniqueness. On the other hand, neither [4] nor [24] consider the product space, but work directly on Ω. However, N is now taken to be the coordinate process, with $+\infty$ as an absorbing state. Hence there is "no more randomness" contained in the probability space after τ, which gives uniqueness of Q. In the recent paper [19], Kardaras et al. consider more general probability spaces, and in particular discuss the question of nonuniqueness. A construction of the Föllmer measure when the local martingale N may reach zero is discussed in [1].

3. Filtration shrinkage and a measure extension problem. Consider now a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_t \subset \mathcal{G}_t$, $t \geq 0$, assumed to be the right-continuous modification of a standard system. Again, completeness is not assumed. The focus of this paper is on the object

$$E^P[N_t \mid \mathcal{F}_t], \qquad t \geq 0,$$

interpreted as the optional projection of N onto $\mathbb{F}$ (see below).

We suppose that Q is an extension of Q_0 as discussed in Section 2. By Theorem 6 in Appendix 1 of [6], optional projections of N and M exist under P and Q, respectively. When we write $E^P[N_t \mid \mathcal{F}_t]$ and $E^Q[M_t \mid \mathcal{F}_t]$ we always refer to these optional projections. Moreover, the projections almost surely have càdlàg paths. This follows from the càdlàg property of the optional projections onto the augmentation of $\mathbb{F}$ (under P, resp., Q), together with Lemma 1 and the uniqueness of the projection. A subtlety arises here: the optional projection of N under P is unique up to a P-evanescent set. However, this set need not be Q-evanescent. We will return to this issue momentarily; see Remark 1 below. First, however, we introduce the following *equivalent measure extension problem*, which turns out to be intimately related to properties of the optional projections.

PROBLEM 1 (Equivalent measure extension problem). *Given the probability Q_0 constructed in Section 2, and the subfiltration $\mathbb{F} \subset \mathbb{G}$, find a probability Q on $(\Omega, \mathcal{G})$ such that*:

(i) $Q = Q_0$ *on* $\mathcal{G}_{\tau-}$;
(ii) *The restrictions of P and Q to $\mathcal{F}_t$ are equivalent for each $t \geq 0$.*

REMARK 1. The issue of Q-nonuniqueness of the optional projection of N under P is resolved if Q solves the equivalent measure extension problem. Indeed, if N' and N'' are two versions of $E^P[N_t \mid \mathcal{F}_t]$, then for every $T \geq 0$, $(N'_t)_{t \leq T}$ and $(N''_t)_{t \leq T}$ coincide on a set A_T with $P(A_T) = 1$. But $A_T \in \mathcal{F}_T$, so $Q(A_T) = 1$ as well. It follows that $N' = N''$ Q-a.s.

REMARK 2. If N is a true martingale, then $Q_0(\tau = \infty) = 1$, and the equivalent measure extension problem has a trivial solution: take $Q = Q_0$. Of course, for us the interesting case is when N is a strict local martingale.

The following result clarifies the link between the equivalent measure extension problem and filtration shrinkage.

LEMMA 4. *Fix $t \geq 0$, and let Q be any extension to $\mathcal{G}$ of Q_0. Then the following are equivalent*:

(i) *The restrictions of P and Q to $\mathcal{F}_t$ are equivalent.*
(ii) $E^Q[M_t \mid \mathcal{F}_t] > 0$, *$Q$-a.s.*
(iii) $Q(\tau > t \mid \mathcal{F}_t) > 0$, *$Q$-a.s.*

If either of the above conditions holds, then

$$Q(\tau > t \mid \mathcal{F}_t) = E^Q[M_t \mid \mathcal{F}_t]E^P[N_t \mid \mathcal{F}_t], \qquad P\text{- and } Q\text{-a.s.} \tag{3}$$

PROOF. The equivalence of (i) and (ii) is immediate, since $E^Q[M_t \mid \mathcal{F}_t]$ is the Radon–Nikodym density of $P|_{\mathcal{F}_t}$ with respect to $Q|_{\mathcal{F}_t}$. We now prove that (ii) and (iii) are equivalent. To this end, let $A = \{E^Q[M_t \mid \mathcal{F}_t] = 0\} \in \mathcal{F}_t$. In the following, inclusions and equalities are understood up to Q-nullsets. We have

$$E^Q[\mathbf{1}_A M_t] = E^Q[\mathbf{1}_A E^Q[M_t \mid \mathcal{F}_t]] = 0,$$

so $M_t = 0$ on A. Hence $\tau \le t$ on A, so

$$E^Q\big[\mathbf{1}_A Q(\tau > t \mid \mathcal{F}_t)\big] = Q\big(A \cap \{\tau > t\}\big) = 0$$

and we deduce that $Q(\tau > t \mid \mathcal{F}_t) = 0$ on A. The reverse inclusion, $\{Q(\tau > t \mid \mathcal{F}_t) = 0\} \subset A$, is proved similarly, and this gives (ii) $\Longleftrightarrow$ (iii). To prove formula (3), we use that $P(\tau > t) = 1$, Bayes' rule and the fact that $\frac{dP}{dQ}|_{\mathcal{G}_t} = M_t$ (Lemma 2) to get

$$\begin{aligned} E^P[N_t \mid \mathcal{F}_t] &= E^P\left[\frac{1}{M_t}\mathbf{1}_{\{\tau > t\}} \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{E^Q[M_t(1/M_t)\mathbf{1}_{\{\tau > t\}} \mid \mathcal{F}_t]}{E^Q[M_t \mid \mathcal{F}_t]} = \frac{Q(\tau > t \mid \mathcal{F}_t)}{E^Q[M_t \mid \mathcal{F}_t]}. \end{aligned}$$

This gives the desired conclusion. □

A solution Q to the equivalent measure extension problem, when it exists, leads to an interpretation of the finite variation part of the P optional projection onto $\mathbb{F}$ of the local martingale N. To see how, let us define

$$Z_t = Q(\tau > t \mid \mathcal{F}_t).$$

This is an $(\mathbb{F}, Q)$ supermartingale, therefore it has a càdlàg modification since $\mathbb{F}$ is right-continuous. We choose this modification when defining Z. If in addition it is strictly positive, it has a unique multiplicative Doob–Meyer decomposition

$$Z_t = e^{-\Lambda_t} K_t, \tag{4}$$

where Λ is nondecreasing and predictable with $\Lambda_0 = 0$, and K is an $(\mathbb{F}, Q)$ local martingale with $K_0 = 1$, see Theorem II.8.21 in [14].

PROPOSITION 1. *Suppose Q is a solution to the equivalent measure extension problem* (*Problem* 1). *Then $E^P[N_t \mid \mathcal{F}_t]$ is an $(\mathbb{F}, P)$ supermartingale, with multiplicative decomposition*

$$E^P[N_t \mid \mathcal{F}_t] = e^{-\Lambda_t} U_t,$$

where Λ is as in (4) *and U is an $(\mathbb{F}, P)$ local martingale. It is a true martingale provided K in* (4) *is a true $(\mathbb{F}, Q)$ martingale.*

PROOF. If Q solves the equivalent measure extension problem, Lemma 4 implies that Z is strictly positive, so that the decomposition (4) exists. It also implies that

$$E^Q[M_t \mid \mathcal{F}_t]e^{\Lambda_t}E^P[N_t \mid \mathcal{F}_t] = K_t$$

is an $(\mathbb{F}, Q)$ local martingale. Since $E^Q[M_t \mid \mathcal{F}_t] = \frac{dP}{dQ}|_{\mathcal{F}_t}$ it follows that $e^{\Lambda_t}E^P[N_t \mid \mathcal{F}_t]$ is an $(\mathbb{F}, P)$ local martingale, and a true martingale if K is. Denoting this process by U yields the claimed decomposition. $\square$

REMARK 3. The fact that $E^P[N_t \mid \mathcal{F}_t]$ is an $(\mathbb{F}, P)$ supermartingale also follows from Theorem 2.3 in [10]. Moreover, it is of Class (DL) whenever U is a martingale, and by Proposition 1 this holds if K is a martingale. A simple sufficient condition for this is that Λ does not increase too rapidly, in the sense that $E^Q[e^{\Lambda_t}] < \infty$ for each $t \geq 0$. Indeed, in this case $E^Q[\sup_{s\leq t} K_s] < \infty$ since $Z \leq 1$, implying the martingale property.

The following corollary is simple but nonetheless informative, since it shows that the equivalent measure extension problem certainly does not always have a solution.

COROLLARY 1. *Suppose N is a strict $(\mathbb{G}, P)$ local martingale. If $E^P[N_t \mid \mathcal{F}_t]$ is again an $(\mathbb{F}, P)$ local martingale, then the equivalent measure extension problem has no solution.*

PROOF. Suppose a solution exists. Then, since $E^P[N_t \mid \mathcal{F}_t]$ is a local martingale, the process Λ in Proposition 1 is identically zero, so that K is bounded and hence a true martingale. Therefore $E^P[N_t \mid \mathcal{F}_t] = U_t$ is a true martingale by Proposition 1. It follows that $E^P[N_t] = E^P[E^P[N_t \mid \mathcal{F}_t]] = 1$ for all $t \geq 0$, contradicting that N is a strict local martingale. $\square$

We can now establish our first main result. It shows that the finite variation part Λ appearing when N is projected onto the smaller filtration can be interpreted as the *predictable compensator of* τ, viewed in the appropriate filtration. The key step is an application of the Jeulin–Yor theorem from the theory of filtration expansions.

THEOREM 1. *Let $\mathbb{F}^\tau$ be the progressive expansion of $\mathbb{F}$ with τ, that is, the smallest filtration that contains $\mathbb{F}$, satisfies the usual hypotheses (with respect to Q) and makes τ a stopping time. If Q solves the equivalent measure extension problem, then:*

(i) *the process*

$$\mathbf{1}_{\{\tau\leq t\}} - \Lambda_{t\wedge\tau}$$

is an $(\mathbb{F}^\tau, Q)$ uniformly integrable martingale, where Λ is as in (4);

(ii) τ *is not* $\mathbb{F}^{\tau}$*-predictable*, *provided* $Q(\tau < \infty) > 0$.

PROOF. The proof uses stochastic integration, which can be developed without assuming the usual hypotheses; see, for instance, Chapter I.4 in [14]. Alternatively, one may apply Lemma 1 to first pass to the Q-completion $\overline{\mathbb{F}}$ of $\mathbb{F}$ without losing the semimartingale property of any of the processes involved, carry out the computations there and then go back to $\mathbb{F}$ at the cost of changing things on a Q-nullset.

The integration by parts formula yields

$$Z_t = 1 + \int_0^t e^{-\Lambda_{s-}}\, dK_s + \left[e^{-\Lambda}, K\right]_t - \int_0^t e^{-\Lambda_{s-}} K_{s-}\, d\Lambda_s.$$

By Yoeurp's lemma ([6], Theorem VII.36), $[e^{-\Lambda}, K]$ is a local martingale, so we have the additive Doob–Meyer decomposition $Z_t = \mu_t - a_t$, where

$$\mu_t = 1 + \int_0^t e^{-\Lambda_{s-}}\, dK_s + \left[e^{-\Lambda}, K\right]_t \quad \text{and} \quad a_t = \int_0^t Z_{s-}\, d\Lambda_s.$$

The Jeulin–Yor theorem (see Theorem 1.1 in [11], or the original paper [18]), which is applicable in view of Lemma 1, shows that the process

$$\mathbf{1}_{\{\tau \le t\}} - \int_0^{t\wedge\tau} \frac{1}{Z_{s-}}\, da_s$$

is an $(\mathbb{F}^{\tau}, Q)$ martingale, and indeed uniformly integrable since it is the martingale part of the Doob–Meyer decomposition of the Class (D) submartingale $\mathbf{1}_{\{\tau \le \cdot\}}$. Substituting for da_s yields (i).

To prove (ii), assume for contradiction that there is a strictly increasing sequence of $\mathbb{F}^{\tau}$ stopping times ρ_n such that $\lim_n \rho_n = \tau$. By the lemma on page 370 in [27], there are $\mathbb{F}$ stopping times σ_n such that $\sigma_n \wedge \tau = \rho_n \wedge \tau$. But since $\rho_n < \tau$, this yields $\sigma_n = \rho_n$. It follows that τ is Q-a.s. equal to an $\mathbb{F}$ stopping time, implying that

$$Q(\tau > t \mid \mathcal{F}_t) = \mathbf{1}_{\{\tau > t\}} \qquad Q\text{-a.s.}$$

By Lemma 4 this contradicts the assumption that Q solves the equivalent measure extension problem, since by hypothesis $Q(\tau < \infty) > 0$. □

The significance of Theorem 1 is that it shows *when* the $(\mathbb{F}, P)$ supermartingale $E^P[N_t \mid \mathcal{F}_t]$ loses mass: it happens exactly when the compensator of τ increases, that is, when there is an increased probability, conditionally on $\mathbb{F}$, that τ has already happened. This corresponds to a kind of smoothing over time of the sets $\{\tau \le t\}$ when we pass to the smaller filtration $\mathbb{F}$. This smoothing is necessary to make the restrictions of P and Q equivalent, since $\{\tau \le t\}$ is P-null but not necessarily Q-null.

4. The finite variation term in a topological setting.

In this section we specialize the previous setup as follows. Let E be a locally compact topological space with a countable base, and define $E_\Delta = E \cup \{\Delta\}$, where $\Delta \notin E$ is an isolated point. We take Ω to be all right-continuous paths $\omega : \mathbb{R}_+ \to E_\Delta$ that are absorbed at Δ [i.e., if $\omega(s) = \Delta$ then $\omega(t) = \Delta$ for all $t \geq s$] and have left limits on $(0, \zeta(\omega))$, where the *absorption time* ζ is defined by

$$\zeta(\omega) := \inf\{t \geq 0 : \omega(t) = \Delta\}.$$

Let $Y_t(\omega) = \omega(t)$ be the coordinate process, and define $\mathcal{G}_t^o = \sigma(Y_s : s \leq t)$. Then $\mathbb{G}^o = (\mathcal{G}_t^o)_{t\geq 0}$ is a standard system; see the Appendix in [9]. We let $\mathbb{G}$ be the right-continuous modification of $\mathbb{G}^o$, and $\mathcal{G} = \bigvee_{t\geq 0} \mathcal{G}_t$.

Next, consider a function $h : E_\Delta \to [0, \infty)$ that is continuous on E and satisfies $h(Y_0) = 1$ P-a.s. (In particular, the measure P is such that, almost surely, Y starts at a point where h equals one.) Define stopping times $\tau_n = n \wedge \inf\{t \geq 0 : h(Y_t) \leq 1/n\}$ and $\tau = \lim_{n\to\infty} \tau_n$. We assume that the P local martingale N is given by

$$N_t = \frac{1}{h(Y_t)} \mathbf{1}_{\{\tau > t\}}.$$

Note how this imposes restrictions on the interplay between P and h: they have to be such that N is indeed a local martingale. Note also that given this setup, the definitions of τ_n and τ are consistent with those given in Section 1. Furthermore, we let M be given by (2), and Q_0 as in Section 2.

To describe the smaller filtration $\mathbb{F}$, let D be a metrizable topological space, and let

$$\pi : E \to D$$

be a continuous map. We define $D_\Delta = D \cup \{\Delta\}$ (assuming without loss of generality that $\Delta \notin D$), and set $\pi(\Delta) = \Delta$. If $d(\cdot, \cdot)$ is a metric on D, we extend it D_Δ by setting $d(x, \Delta) = d(\Delta, x) = \infty$ for $x \in D$, and $d(\Delta, \Delta) = 0$. Next, define a D_Δ-valued process X by

$$X_t = \pi(Y_t), \qquad t \geq 0.$$

It is clear that X is $\mathbb{G}$-adapted. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, given by

$$\mathcal{F}_t = \bigcap_{u>t} \sigma(X_s : s \leq u),$$

is therefore a subfiltration of $\mathbb{G}$, right-continuous, but not augmented. The structure imposed by the above conditions (and the flavor of the main theorem below) is primarily of a topological nature, which motivates the title of this section.

Recall the multiplicative decomposition $E^P[N_t \mid \mathcal{F}_t] = e^{-\Lambda_t} U_t$ of the positive $(\mathbb{F}, P)$ supermartingale $E^P[N_t \mid \mathcal{F}_t]$. The finite variation part Λ is related to τ by Proposition 1, provided the equivalent measure extension problem has a solution. In the particular setting of the present section, we can say the following about the points of increase of Λ:

THEOREM 2. *Assume that Q is a solution to the equivalent measure extension problem, and let Λ be as in (4). Then the random measure $d\Lambda_t$ is supported on the set $\{t : X_{t-} \in \overline{D_0}\}$, where $\overline{D_0}$ is the closure in D of*

$$D_0 = \pi \circ h^{-1}(\{0\}) = \{x \in D : x = \pi(y) \text{ for some } y \in E \text{ with } h(y) = 0\}.$$

The proof requires two lemmas.

LEMMA 5. *We have $\pi(Y_{\tau-}) \in D_0$ on $\{\tau < \infty\}$, Q_0-a.s.*

PROOF. To show that $\pi(Y_{\tau-}) \in D_0$, one must find $y \in E$ with $h(y) = 0$ such that $\pi(y) = \pi(Y_{\tau-})$. But $h(Y_{\tau-}) = M_{\tau-} = 0$ on $\{\tau < \infty\}$ by Lemma 3, so we may take $y = Y_{\tau-}$. ☐

LEMMA 6. *For any $\mathbb{F}$ stopping time ρ, the equality $Z_\rho = Q(\tau > \rho \mid \mathcal{F}_\rho)$ holds on $\{\rho < \infty\}$, Q-a.s.*

PROOF. We need to show that $E^Q[Z_\rho \mathbf{1}_{A \cap \{\rho < \infty\}}] = Q(A \cap \{\tau > \rho\})$ for every $\mathbb{F}$-stopping time ρ and every $A \in \mathcal{F}_\rho$. This clearly holds when ρ is constant. Suppose now that ρ is of the form

$$\rho = \sum_{i=1}^{n} t_i \mathbf{1}_{A_i}, \tag{5}$$

where $t_i \in [0, \infty]$, $A_i \in \mathcal{F}_{t_i}$, and the A_i constitute a partition of Ω. Then

$$\begin{aligned} E^Q[Z_\rho \mathbf{1}_A] &= \sum_{i=1}^{n} E^Q[Z_{t_i} \mathbf{1}_{A_i \cap A}] \\ &= \sum_{i=1}^{n} Q(A \cap A_i \cap \{\tau > t_i\}) \\ &= Q(A \cap \{\tau > \rho\}), \end{aligned}$$

where the second equality used that $A_i \cap A \in \mathcal{F}_{t_i}$ and that the result holds for constant times. Finally, let ρ_n be a decreasing sequence of stopping times of the form (5) with $\lim_n \rho_n = \rho$. Right-continuity together with bounded convergence and the result applied to ρ_n now yields the statement of the lemma. ☐

PROOF OF THEOREM 2. Let $0 < \rho \leq \sigma$ be bounded $\mathbb{F}$ stopping times such that $X_- \notin D_0$ on $[\rho, \sigma)$. We claim that $\Lambda_\sigma - \Lambda_\rho = 0$, Q-a.s. To prove this, first write

$$\Lambda_\sigma - \Lambda_\rho = (\Lambda_\sigma - \Lambda_\rho)\mathbf{1}_{\{\tau \leq \sigma\}} + (\Lambda_{\sigma\wedge\tau} - \Lambda_{\rho\wedge\tau})\mathbf{1}_{\{\sigma < \tau\}}. \tag{6}$$

By continuity of π and the choice of ρ and σ, we have

$$\pi(Y_{\tau-}) = X_{\tau-} \notin D_0 \qquad \text{on } \{\rho < \tau \leq \sigma\}.$$

But according to Lemma 5, $\pi(Y_{\tau-}) \in D_0$ on $\{\tau < \infty\}$, Q-a.s., so we get

$$Q(\rho < \tau \leq \sigma) \leq Q\big(\pi(Y_{\tau-}) \notin D_0, \tau < \infty\big) = 0. \tag{7}$$

Next, consider the filtration $\mathbb{F}^\tau$ described in Theorem 1. By that theorem,

$$\mathbf{1}_{\{\tau \leq t\}} - \Lambda_{t\wedge\tau}$$

is an $(\mathbb{F}^\tau, Q)$ martingale. Since ρ and σ are also $\mathbb{F}^\tau$ stopping times, the martingale property and the optional sampling theorem, together with (7), yield

$$E^Q[\Lambda_{\sigma\wedge\tau} - \Lambda_{\rho\wedge\tau}] = E^Q[\mathbf{1}_{\{\rho<\tau\leq\sigma\}}] = 0.$$

Since Λ is nondecreasing, we deduce that $\Lambda_{\sigma\wedge\tau} - \Lambda_{\rho\wedge\tau} = 0$, Q-a.s. Using this in the decomposition (6), we obtain

$$\Lambda_\sigma - \Lambda_\rho = (\Lambda_\sigma - \Lambda_\rho)\mathbf{1}_{\{\tau\leq\rho\}}.$$

This implies that $\tau \leq \rho$ on the $\mathcal{F}_\sigma$-measurable set $\{\Lambda_\sigma - \Lambda_\rho > 0\}$. In conjunction with Lemma 6, this gives the equalities

$$Z_\sigma \mathbf{1}_{\{\Lambda_\sigma-\Lambda_\rho>0\}} = Q\big(\{\tau > \sigma\} \cap \{\Lambda_\sigma - \Lambda_\rho > 0\} \mid \mathcal{F}_\sigma\big) = 0, \qquad Q\text{-a.s.}$$

But Q solves the equivalent measure extension problem, so Z is strictly positive, Q-a.s. Therefore $Q(\Lambda_\sigma - \Lambda_\rho > 0) = 0$, and we have finally proved our claim that $\Lambda_\sigma - \Lambda_\rho = 0$, Q-a.s.

Now, choose a metric $d(\cdot,\cdot)$ on D compatible with its topology. For any subset $A \subset D$ and any $x \in D$, define the distance from x to A by

$$\operatorname{dist}(x, A) = \inf\big\{d\big(x, x'\big) : x' \in A\big\}.$$

It is easy to check that $\operatorname{dist}(\cdot, A)$ is continuous (even Lipschitz), and in particular measurable. For each rational number $r > 0$ and natural number $n > r$, define stopping times

$$\begin{aligned} \rho_r &= \begin{cases} r, & \text{if } \operatorname{dist}(X_{r-}, D_0) > 0, \\ \infty, & \text{otherwise,} \end{cases} \\ \rho_{r,n} &= n \wedge \rho_r, \\ \sigma_r &= n \wedge \inf\big\{t > \rho_{r,n} : \operatorname{dist}(X_{t-}, D_0) = 0\big\}. \end{aligned}$$

Then the stopping times $\rho_{r,n}$ and $\sigma_{r,n}$ are all bounded, and it is a simple matter to check the inclusion

$$[\rho_{r,n}, \sigma_{r,n}) \subset \big\{\operatorname{dist}(X_-, D_0) > 0\big\}.$$

Moreover, if for some (t, ω) with $t > 0$ it holds that $\operatorname{dist}(X_{t-}(\omega), D_0) > 0$, then by left-continuity of $X_-(\omega)$ and continuity of the distance function, there is a rational

$r > 0$ such that $r \le t$ and for all $s \in [r, t]$ we have $\operatorname{dist}(X_{s-}, D_0) > 0$. Thus for any $n > t$, we have $(t, \omega) \in [\rho_{r,n}, \sigma_{r,n})$, and we deduce

$$\bigcup_{\substack{r\in\mathbb{Q}, r>0\\ n\in\mathbb{N}, n>r}} [\rho_{r,n}, \sigma_{r,n}) = \{\operatorname{dist}(X_-, D_0) > 0\}.$$

By the first part of the proof, $d\Lambda_t$ does not charge any of the countably many intervals in the union on the left-hand side. It follows that $d\Lambda_t$ is supported on the set $\{\operatorname{dist}(X_-, D_0) = 0\}$, which coincides with $\{X_- \in \overline{D_0}\}$. $\square$

REMARK 4. Since h is continuous and $D_0 = \pi \circ h^{-1}(\{0\})$, D_0 is a closed set in D if π is a closed map. An example is when π is a linear map on $\mathbb{R}^q$, and $D = \pi(\mathbb{R}^q)$. This case is discussed in Section 5.

We can now give a simple sufficient condition for Λ to have singular paths, as in the example studied by Föllmer and Protter [10] that was mentioned in the Introduction.

COROLLARY 2. *Assume D is a subset of $\mathbb{R}^k$ for some k, and that the law of X_t under Q admits a density for almost every $t > 0$. Then, if $\overline{D_0}$ is a nullset in $\mathbb{R}^k$, the paths of Λ are singular.*

PROOF. Since X_s has a density for almost every s and $\overline{D_0}$ is a nullset, Fubini's theorem yields $E^Q[\int_0^t \mathbf{1}_{\{X_s\in\overline{D_0}\}}\, ds] = \int_0^t Q(X_s \in \overline{D_0})\, ds = 0$. Hence $\int_0^t \mathbf{1}_{\{X_s\in\overline{D_0}\}}\, ds = 0$, Q-a.s. Thus $\{t : X_t \in \overline{D_0}\}$ is a nullset Q-a.s., and it contains the support of $d\Lambda_t$ by Theorem 2. This proves the claim. $\square$

We finish this section with a result intended to emphasize the distinction between ζ, the absorption time of the coordinate process Y, and the explosion time τ of the process N.

PROPOSITION 2. *The following statements hold:*

(i) *Let Q be any extension of Q_0 to all of $\mathcal{G}$. Then $\tau \le \zeta$ on $\{\tau < \infty\}$, Q-a.s.*

(ii) *If Q is a solution to the equivalent measure extension problem and $\tau < \infty$ on $\{\zeta < \infty\}$, Q-a.s., then $Q(\zeta = \infty) = 1$.*

PROOF. Since the coordinate process stops at ζ, it is clear that $\mathcal{G}_\infty = \mathcal{G}_\zeta$. Hence for any stopping time σ, $\mathcal{G}_\sigma = \mathcal{G}_\sigma \cap \mathcal{G}_\zeta \subset \mathcal{G}_{\sigma\wedge\zeta} \subset \mathcal{G}_\sigma$, and thus $\mathcal{G}_{\sigma\wedge\zeta} = \mathcal{G}_\sigma$. Applying this with $\sigma = t$, for any $t \ge 0$, we get

$$M_{t\wedge\zeta} = E^Q[M_t \mid \mathcal{G}_{t\wedge\zeta}] = E^Q[M_t \mid \mathcal{G}_t] = M_t,$$

showing that M is Q-a.s. constant after ζ (note that this holds for any martingale). Now, on $\{\zeta < \tau\}$ we have $\inf_{0\le t\le\zeta} M_t > 0$, and since M is constant after ζ we have $\inf_{t\ge 0} M_t > 0$. Hence $\tau = \infty$, and we deduce (i).

To prove (ii), first note that $X_\zeta = \pi(Y_\zeta) = \pi(\Delta) = \Delta$, and that for $t < \zeta$, $X_t \in D$ so that $X_t \neq \Delta$. The absorption time can therefore alternatively be written

$$\zeta = \inf\{t \ge 0 : X_t = \Delta\},$$

showing that ζ is in fact an $\mathbb{F}$ stopping time. Our hypothesis says that $\tau < \infty$ on $\{\zeta < \infty\}$. Hence, by part (i) above, $\tau \le \zeta$ on $\{\zeta < \infty\}$. But since Lemma 6 implies that $Z_\zeta = Q(\tau > \zeta \mid \mathcal{F}_\zeta)$ on this set, we deduce that $Z_\zeta = 0$ on $\{\zeta < \infty\}$. Now, Q solves the equivalent measure extension problem, so in order to avoid a contradiction we must have $Q(\zeta = \infty) = 1$. ☐

REMARK 5. If N itself is the coordinate process, then τ and ζ coincide, as is the case, for example, in [4]. In this case part (ii) of the above proposition implies that the equivalent measure extension problem lacks a solution *for any subfiltration* $\mathbb{F}$ of the type discussed in this section. At first glance, this seems to imply that the proposition is incorrect: let, for instance, $\mathbb{F}$ be the trivial filtration—then P itself is a solution to the equivalent measure extension problem. The issue here is that the trivial filtration is not of the type introduced above, since we assumed that $\pi(\Delta) = \Delta \neq \pi(y)$ for $y \in E$. In particular, ζ is not a stopping time for the trivial filtration, and this breaks the proof of part (ii). On the other hand, part (i) remains correct even if we allow $\pi(\Delta)$ to lie in D, and also part (ii) remains correct as long as we additionally assume that ζ is an $\mathbb{F}$ stopping time.

5. Solving the equivalent measure extension problem.

So far we have *assumed* that the equivalent measure extension problem has a solution. In this section we specialize the setup from Section 4, imposing further assumption that enable us to prove the existence of a particular solution, and to describe this solution explicitly. This is done in Section 5.1. Some examples where the main result (Theorem 3 below) applies are then discussed in Section 6. The symbol $|\cdot|$ denotes the usual Euclidean norm, and ∇ is the gradient.

5.1. *Linear shrinkage in a Brownian setting.*

We make the following assumptions, within the framework described in Section 4:

- $E = \mathbb{R}^q$, some $q \in \mathbb{N}$.
- P is Wiener measure, turning the coordinate process Y into q-dimensional Brownian motion (possibly starting from $Y_0 \neq 0$).
- h is such that $\frac{1}{h}$ is harmonic on $\mathbb{R}^q \setminus E_0$, where we define

$$E_0 = h^{-1}(\{0\}).$$

- $\pi : E \to E$ is linear, and we set $D = \pi(\mathbb{R}^q)$ and $p = \dim D = \operatorname{rank} \pi$. We assume $p < q$, since otherwise we have $\mathbb{F} = \mathbb{G}$, in which case the equivalent measure extension problem has a solution precisely when N is already a martingale under P.

The main result is the following.

THEOREM 3. *Consider the setup just described, and assume furthermore that h satisfies the following conditions*:

$$t \mapsto E^P\left[\frac{|\nabla \ln h(Y_t)|}{h(Y_t)}\right] \quad \textit{is locally bounded on } [0,\infty), \tag{8}$$

$$(t,x) \mapsto E^P\left[\frac{|\pi(\nabla \ln h(Y_t))|}{h(Y_t)} \,\middle|\, \pi(Y_t) = x\right] \quad \textit{is locally bounded on } (0,\infty) \times D, \tag{9}$$

where the right-hand side of (9) *should be understood in the sense of regular conditional probabilities. Then the equivalent measure extension problem has a solution Q with the property that*

$$W = \left(Y_t - Y_0 + \int_0^{t\wedge\tau} \nabla \ln h(Y_s)\, ds\right)_{t\geq 0} \quad \textit{is } Q\textit{-Brownian motion}, \tag{10}$$

where the integral is well-defined and finite for each $t \geq 0$, Q-a.s.

REMARK 6. The role of condition (8) is primarily to ensure that the optional projection of Y under Q can be computed in a reasonable way. Moreover, since trivially π is a bounded operator, (8) also implies that the conditional expectation in (9) is finite for each $(t,x) \in (0,\infty) \times D$. The role of condition (9) is to ensure that $\mathbb{F}$ is small enough for the projection operation to induce sufficient smoothing. In particular, if D is zero-dimensional, so that $\mathbb{F}$ is the trivial filtration, then (9) automatically holds.

REMARK 7. Unfortunately the assumptions of Theorem 3 are quite restrictive. While they do allow us to treat the example by Föllmer and Protter mentioned in the Introduction, a major open problem for future research is to find more general conditions under which the equivalent measure extension problem can be solved.

REMARK 8. Theorem 3 is a closely related to Doob's h-transform. Indeed, one can view P as being obtained from Q by conditioning Y never to hit the zero set of h. Note, however, that Y is not Markovian under Q due to the presence of τ.

The rest of this section is devoted to the proof of Theorem 3. The strategy can be summarized as follows: we first exhibit an extension Q of Q_0 for which (10) holds. Then we describe the law of $X = \pi(Y)$ under P and under Q. Finally, this description is used to show that the laws are locally equivalent. Since X generates $\mathbb{F}$ this yields the result. We now turn to the details, which are carried out through a sequence of lemmas.

LEMMA 7. *Assume that* (8) *is satisfied. Then the inequality*

$$\int_0^t E^{Q_0}\big[|\nabla \ln h(Y_s)|\mathbf{1}_{\{s<\tau\}}\big]\,ds < \infty \tag{11}$$

holds for every $t \geq 0$. *Consequently, there is an extension* Q *of* Q_0 *for which* (10) *holds.*

PROOF. We have

$$E^{Q_0}\big[|\nabla \ln h(Y_t)|\mathbf{1}_{\{t<\tau\}}\big] = E^P\bigg[\frac{1}{h(Y_t)}|\nabla \ln h(Y_t)|\bigg].$$

By (8), the right-hand side is locally integrable in t on $[0,\infty)$, which implies (11). We may therefore define an E_Δ-valued process W by

$$W_t = Y_t - Y_0 + \int_0^{t\wedge\tau} \nabla \ln h(Y_s)\,ds, \qquad t \geq 0,$$

using (11) to see that the integral on the right-hand side is well defined and finite. Now, for each n, N^{τ_n} is the density process of the restriction of Q_0 to $\mathcal{G}_{\tau_n}$ with respect to P. (Recall that τ_n is the minimum of n and the first time N_t hits level n.) We observe that, by Itô's formula,

$$N_t = \frac{1}{h(Y_t)} = 1 - \int_0^t N_s \nabla \ln h(Y_s)\,dY_s, \qquad t < \tau,$$

so that an application of Girsanov's theorem yields that $(W_{t\wedge\tau_n} : t \geq 0)$ is a local martingale for each n. Since $\langle W^i, W^j\rangle_{t\wedge\tau_n} = (t \wedge \tau_n)\delta_{ij}$, it is in fact a martingale behaving like stopped Brownian motion. A standard argument based on Doob's up- and downcrossing inequalities then shows that the limit $\lim_{t\uparrow\tau} W_t$ exists in $\mathbb{R}^q$ on $\{\tau < \infty\}$, Q_0-a.s. As a consequence, $Y_{\tau-}$ also exists on $\{\tau < \infty\}$, and is different from Δ. We now simply choose the law Q so that $Y_\tau = Y_{\tau-}$ and $(Y_{\tau+t} - Y_\tau : t \geq 0)$ is Brownian motion. □

Since $Y - Y_0$ is Brownian motion under P, it is clear that the same holds for $X - X_0 = \pi(Y - Y_0)$, but with a possibly different quadratic covariation depending on π. The following lemma describes what happens under Q.

LEMMA 8. *Assume that* (8) *is satisfied, and let* Q *be an extension of* Q_0 *for which* (10) *holds* (*it exists by Lemma* 7). *The process* X *can then be decomposed as*

$$X_t = X_0 + B_t + \int_0^t \theta_s\,ds \qquad \textit{for all } t \geq 0,\ Q\textit{-a.s.},$$

where B is $(\mathbb{F}, Q)$ *Brownian motion* (*with the same quadratic covariation as* X), *and* θ_t *satisfies, for every* $t \geq 0$,

$$\theta_t = E^Q[\pi(\nabla \ln h(Y_t))\mathbf{1}_{\{\tau > t\}} \mid \mathcal{F}_t] \qquad Q\text{-}a.s. \quad and \quad \int_0^t E^Q[|\theta_s|]\, ds < \infty.$$

PROOF. Due to Lemma 7, the optional projection of $\pi(\nabla \ln h(Y_t))\mathbf{1}_{\{\tau > t\}}$ onto $\mathbb{F}$ is well defined under Q. Denoting this optional projection by θ it is clear that the given expression for θ and the integrability statement are correct. From (10), the definition of X_t and the linearity of π we obtain

$$\begin{aligned} X_t &= E^Q[\pi(Y_t) \mid \mathcal{F}_t] \\ &= \pi(Y_0) + E^Q[\pi(W_t) \mid \mathcal{F}_t] - E^Q\left[\int_0^t \pi(\nabla \ln h(Y_s))\mathbf{1}_{\{s < \tau\}}\, ds \,\middle|\, \mathcal{F}_t\right] \\ &= X_0 + B_t - \int_0^t \theta_s\, ds, \end{aligned}$$

where we define $B_t = E^Q[\pi(W_t) \mid \mathcal{F}_t] + L_t$ with

$$\begin{aligned} L_t = E^Q&\left[\int_0^t \pi(\nabla \ln h(Y_s))\mathbf{1}_{\{s < \tau\}}\, ds \,\middle|\, \mathcal{F}_t\right] \\ &- \int_0^t E^Q[\pi(\nabla \ln h(Y_s))\mathbf{1}_{\{s < \tau\}} \mid \mathcal{F}_s]\, ds. \end{aligned}$$

Suppose we know B is a (local) martingale. Since its quadratic covariation coincides with that of X, we deduce from Lévy's theorem that B is $(\mathbb{F}, Q)$ Brownian motion with that quadratic covariation. To see that B is indeed a martingale, first note that each component of $E^Q[\pi(W_t) \mid \mathcal{F}_t]$ is the projection of a linear combination of martingales, hence itself a martingale. Next, we make use of the following well-known result from filtering theory (see [22], Theorem 7.12): if ξ is a measurable process with $\int_0^t E^Q[|\xi_s|]\, ds < \infty$ for all $t \geq 0$, then

$$E^Q\left[\int_0^t \xi_s\, ds \,\middle|\, \mathcal{F}_t\right] - \int_0^t E^Q[\xi_s \mid \mathcal{F}_s]\, ds, \qquad t \geq 0,$$

is an $(\mathbb{F}, Q)$ martingale. Applying this to each component of L shows that it is a martingale. This completes the proof. □

We now have a description of the law of X under P and under Q. It remains to show that these laws are locally equivalent, and this is where condition (9) is crucial. A priori, (9) only asserts boundedness on compact sets *bounded away from* $\{0\} \times D$. The following result shows that this can be strengthened without imposing any additional assumptions. The proof uses the Moore–Penrose inverse to decompose Y_t into an observable component and an independent component.

LEMMA 9. *Assume condition* (9) *is satisfied. Then there is some* $\varepsilon > 0$, *and an open set* $O \subset D$ *containing* X_0, *such that the function in* (9) *is bounded on* $(0, \varepsilon] \times O$.

PROOF. Define $G(y) = h(y)^{-1}|\pi(\nabla \ln h(y))|$, and let π^+ be the Moore–Penrose inverse of the linear map π. Since π^+ is invertible on D (its inverse is π), the function in (9) can be written

$$E^P[G(Y_t) \mid \pi(Y_t) = x] = E^P[G(Y_t) \mid U_t = \pi^+(x)],$$

where we set $U_t = \pi^+\pi(Y_t)$. Now decompose Y_t as

$$Y_t = \pi^+\pi(Y_t) + (\mathrm{Id} - \pi^+\pi)(Y_t) = U_t + V_t$$

(V_t is defined by this relation), and note that

$$\pi^+\pi(\mathrm{Id} - \pi^+\pi) = \pi^+\pi - \pi^+\pi\pi^+\pi = \pi^+\pi - \pi^+\pi = 0$$

by basic properties of the Moore–Penrose inverse. Hence $Y_t = U_t + V_t$ is the decomposition of Y_t as a direct sum in $D \oplus D^\perp$. In particular U_t and V_t are independent under P, so

$$E^P[G(Y_t) \mid U_t = \pi^+(x)] = E^P[G(u + V_t)]_{u=\pi^+(x)}.$$

We now focus on bounding $E^P[G(z + V_t)]$. The random variable V_t concentrates on $D^\perp$ and is nondegenerate Normal there, so it has a density with respect to Lebesgue measure on $D^\perp$ given by

$$f_t(v) = \frac{1}{(2\pi t)^{m/2}|\det \Sigma|^{1/2}} \exp\left(-\frac{1}{2t}(v - V_0)^\top \Sigma^{-1}(v - V_0)\right), \qquad v \in D^\perp.$$

Here $m = q - p = \dim D^\perp$ and, by a slight abuse of notation, Σ^{-1} the inverse on $D^\perp$ of the covariance operator of V_t, with $\det \Sigma$ being its determinant.

Now, let $\varepsilon > 0$ be a number to be determined later. We let $\mathcal{B} = \{u \in D : |u - U_0| < \varepsilon\}$ be the ball in D of radius ε centered at U_0, and $\mathcal{E}$ be the ellipsoid in $D^\perp$ given by

$$\mathcal{E} = \left\{v \in D^\perp : \frac{1}{m}(v - V_0)^\top \Sigma^{-1}(v - V_0) < \varepsilon\right\}.$$

The following can be verified by direct differentiation:

Claim: Fix $\alpha > 0$ and $\beta > 0$, and let $\psi(t) = t^{-\alpha/2}\exp(-t^{-1}\beta/2)$. Then ψ is nondecreasing on the interval $[0, \beta/\alpha]$.

The claim shows that whenever $v \notin \mathcal{E}$, $f_t(v)$ decreases as t decreases. This gives us the following bound for any $t \in (0, \varepsilon]$:

$$\begin{aligned} E^P[G(z + V_t)] &= \int_{\mathcal{E}} G(u + v) f_t(v)\, dv + \int_{D^\perp \setminus \mathcal{E}} G(u + v) f_t(v)\, dv \\ &\le \sup_{v \in \mathcal{E}} G(u + v) + \int_{D^\perp \setminus \mathcal{E}} G(u + v) f_\varepsilon(v)\, dv \\ &\le \sup_{v \in \mathcal{E}} G(u + v) + E^P[G(z + V_\varepsilon)]. \end{aligned}$$

Therefore,

$$\sup_{(t,u)\in(0,\varepsilon]\times\mathcal{B}} E^P[G(u+V_t)] \le \sup_{y\in\mathcal{B}\oplus\mathcal{E}} G(y) + \sup_{u\in\mathcal{B}} E^P[G(u+V_\varepsilon)].$$

By smoothness of h outside E_0 and the fact that $h(Y_0)=1$, it is possible to choose $\varepsilon>0$ small enough that the set $\mathcal{B}\oplus\mathcal{E}$, which is a neighborhood of Y_0, is bounded away from E_0. With such an ε, the first term on the right-hand side above is finite. The second term is also finite due to the local boundedness assumption (9). Setting $O=\pi(\mathcal{B})$, which is again open in D, gives the statement of the lemma. □

The same orthogonal decomposition of Y_t as in the proof of Lemma 9 gives the following unsurprising result.

LEMMA 10. *Consider a nonnegative measurable function* $G:\mathbb{E}\to\mathbb{R}_+$. *The equality*

$$E^P[G(Y_t)\mid\mathcal{F}_t] = E^P[G(Y_t)\mid\pi(Y_t)=x]_{x=X_t}$$

holds P-*a.s. for all* $t\ge 0$.

PROOF. With the notation from the proof of Lemma 9 we get, P-a.s.,

$$\begin{aligned}E^P[G(Y_t)\mid\mathcal{F}_t] &= E^P[G(Y_t)\mid X_s:s\le t]\\ &= E^P[G(U_t+V_t)\mid U_s:s\le t]\\ &= E^P[G(u+V_t)]_{u=U_t}\\ &= E^P[G(\pi^+(x)+V_t)]_{x=X_t}.\end{aligned}$$

By means of an analogous calculation, the right-hand side is also seen to be equal to $E^P[G(Y_t)\mid\pi(Y_s)=x]_{x=X_s}$. □

The following simple refinement of Bayes's rule is useful for dealing with nonequivalent measures.

LEMMA 11. *Suppose* $R_1\ll R_2$ *are two probability measures with Radon–Nikodym derivative* $Z=\frac{dR_1}{dR_2}$, *and let* X *be a random variable in* $L^1(R_1)$. *Let* $\mathcal{H}$ *be a sub-*σ*-field, and suppose* $A\in\mathcal{H}$ *satisfies* $A\subset\{E^{R_2}[Z\mid\mathcal{H}]>0\}$. *Then* $E^{R_1}[X\mid\mathcal{H}]$ *is uniquely defined on* A *up to an* R_2*-nullset, and we have*

$$E^{R_2}[Z\mid\mathcal{H}]E^{R_1}[X\mid\mathcal{H}]\mathbf{1}_A = E^{R_2}[ZX\mathbf{1}_A\mid\mathcal{H}]$$

R_2-*a.s.* (*and hence* R_1-*a.s.*).

PROOF. To prove the first statement, let Y and Y' be two versions of $E^{R_1}[X \mid \mathcal{H}]$. Then $R_1(Y \neq Y') = 0$, and we get

$$0 = R_1(\{Y \neq Y'\} \cap A) = E^{R_2}\big[E^{R_2}[Z \mid \mathcal{H}]\mathbf{1}_{\{Y \neq Y'\} \cap A}\big].$$

Since $E^{R_2}[Z \mid \mathcal{H}] > 0$ on A, we get $R_2(\{Y \neq Y'\} \cap A) = 0$, as desired. The second statement follows from the following calculation, where $B \in \mathcal{H}$ is arbitrary:

$$\begin{aligned} E^{R_2}\big[E^{R_2}[Z \mid \mathcal{H}]E^{R_1}[X \mid \mathcal{H}]\mathbf{1}_{A\cap B}\big] &= E^{R_2}\big[Z E^{R_1}[X \mid \mathcal{H}]\mathbf{1}_{A\cap B}\big] \\ &= E^{R_1}[X\mathbf{1}_{A\cap B}] \\ &= E^{R_2}[ZX\mathbf{1}_{A\cap B}]. \qquad \square \end{aligned}$$

The next lemma is the key to proving that the laws of X under P and Q are equivalent. It relies on the strengthening of condition (9) given in Lemma 9.

LEMMA 12. *Assume that* (8) *and* (9) *are satisfied, and let* θ *and* Q *be as in Lemma* 8. *For each* $t \geq 0$, *we have*

$$\int_0^t |\theta_s|^2\, ds < \infty \qquad Q\text{-a.s.}$$

PROOF. We would like to rewrite θ_t using Lemma 11, so we verify the assumptions of that lemma. To this end, define

$$\sigma_0 = \inf\{t \geq 0 : Q(\tau > t \mid \mathcal{F}_t) = 0\} = \inf\{t \geq 0 : E^Q[M_t \mid \mathcal{F}_t] = 0\},$$

where the equality follows from Lemma 4. Then $\tau \leq \sigma_0$, Q-a.s., so the expression for θ yields

$$\theta_t \mathbf{1}_{\{\sigma_0 \leq t\}} = E^Q\big[\pi(\nabla \ln h(Y_t))\mathbf{1}_{\{\tau > t\}\cap\{\sigma_0 \leq t\}} \mid \mathcal{F}_t\big] = 0.$$

Hence $\theta_t = \theta_t \mathbf{1}_{\{\sigma_0 > t\}}$. Now, set $H = \pi(\nabla \ln h(Y_t))\mathbf{1}_{\{\tau > t\}}$. Then

$$E^P\big[|H|\big] = E^Q\big[|M_t H|\big] = E^Q\big[|\theta_t|\big],$$

which is finite by Lemma 8. Since also $E^P[M_t \mid \mathcal{F}_t] > 0$ on $\{\sigma_0 > t\}$, we may apply Lemma 11 with $R_1 = P$ and $R_2 = Q$ to get, Q-a.s.,

$$\begin{aligned} \theta_t &= E^Q\big[\pi(\nabla \ln h(Y_t))\mathbf{1}_{\{\tau > t\}} \mid \mathcal{F}_t\big]\mathbf{1}_{\{\sigma_0 > t\}} \\ &= E^P\left[\frac{1}{h(Y_t)}\pi(\nabla \ln h(Y_t)) \,\middle|\, \mathcal{F}_t\right] E^Q[M_t \mid \mathcal{F}_t]. \end{aligned}$$

Now, since $E^Q[M_t \mid \mathcal{F}_t]$ is a finite, càdlàg process, it is pathwise bounded on each $[0, t]$ (with the bound depending on t and ω in a possibly nonpredictable way). It thus suffices to prove that $\int_0^{t\wedge\sigma_0} |\xi_s|^2\, ds < \infty$, Q-a.s., where

$\xi_s = E^P[h(Y_s)^{-1}\pi(\nabla \ln h(Y_s)) \mid \mathcal{F}_s]$. By Lemma 11 this conditional expectation is uniquely defined P- and Q-a.s. on $\{s < \sigma_0\}$. Therefore, by Lemma 10, the equality

$$\xi_s = E^P\left[\frac{1}{h(Y_s)}\pi\big(\nabla \ln h(Y_s)\big) \,\middle|\, \pi(Y_s) = x\right]_{x=X_s}$$

holds Q-a.s. on $\{s < \sigma_0\}$.

Now, let $O \subset D$ and $\varepsilon > 0$ be the objects obtained from Lemma 9, and define

$$\rho_\varepsilon = \inf\{0 \le t \le \varepsilon \wedge \sigma_0 : X_t \notin O\}.$$

Since O is open and contains X_0, we have $\rho_\varepsilon > 0$, Q-a.s. (Note that $\sigma_0 > 0$ by right continuity of $E^Q[M_t \mid \mathcal{F}_t]$.) The properties of O and ε imply that ξ_s is bounded on $(0, \rho_\varepsilon)$. Furthermore, the local boundedness condition (9) implies that ξ_s is pathwise bounded on $[\rho_\varepsilon, t \wedge \sigma_0)$ (again with a random bound). It follows that ξ is square integrable on $(0, t \wedge \sigma_0)$, which is what we had to show. The proof of the lemma is now complete. □

PROOF OF THEOREM 3. We need to prove that Q and P are equivalent on each $\mathcal{F}_t$. By Lemmas 8 and 12, we can define a strictly positive ($\mathbb{F}$, Q) local martingale Z via

$$Z_t = \exp\left(\int_0^t \theta_s^\top \, dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 \, ds\right), \qquad t \ge 0.$$

Consequently, since $\mathbb{F}$ is a standard system, we can find the Föllmer measure associated with Z. To be precise, define stopping times

$$\rho_n = \inf\{t \ge 0 : Z_t \ge n\}, \qquad \rho = \lim_{n\to\infty} \rho_n.$$

Then there is a unique probability R_0 on $\mathcal{F}_{\rho-}$ such that $\frac{dQ}{dR_0}|_{\rho_n-} = \frac{1}{Z_{\rho_n}}$ for each n. Girsanov's theorem and Lévy's characterization of Brownian motion then imply that the process

$$X_{t\wedge\rho_n} - X_0 = B_{t\wedge\rho_n} - \int_0^{t\wedge\rho_n} \theta_s \, ds, \qquad t \ge 0,$$

is Brownian motion (with some invertible volatility matrix) stopped at ρ_n. Moreover, since X generates the filtration $\mathbb{F}$, ρ_n only depends on the path of X. Therefore the law of $(X_{t\wedge\rho_n} : t \ge 0)$ under R_0 is the same as its law under P. Consequently, since $\int_0^t \theta_s^2 \, ds < \infty$ for all $t \ge 0$, P-a.s., so that $P(\rho = \infty) = 1$, we also have $R_0(\rho = \infty) = 1$. It follows that $X - X_0$ (not stopped this time) is Brownian motion under R_0, and we deduce that $R_0 = P$ on each $\mathcal{F}_t$. This leads to the domination relations

$$P|_{\mathcal{F}_t} \ll Q|_{\mathcal{F}_t} \ll R_0|_{\mathcal{F}_t} = P|_{\mathcal{F}_t},$$

which proves the theorem. □

6. Examples.

In this section we discuss some examples where the conditions of Theorem 3 can be verified explicitly. We also give one recipe for how new examples can be constructed from old ones.

EXAMPLE 1 (The inverse Bessel process). Let $E = \mathbb{R}^3$, and suppose $Y_0 = (1,0,0)$. Take $h(y) = |y|$. Then $1/h$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$, and N is the reciprocal of a BES(3) process. In particular it is a strict local martingale. To specify the smaller filtration, we let π be a projection onto the first coordinate of $\mathbb{R}^3$. This puts us exactly in the example analyzed by Föllmer and Protter [10], mentioned in the Introduction.

Let us verify conditions (8) and (9) of Theorem 3. First, note that $\nabla h(y) = y|y|^{-1}$, so that

$$E^P\left[\frac{1}{h(Y_t)}|\nabla \ln h(Y_t)|\right] = E^P\left[\frac{1}{h(Y_t)^2}\right] = E^P[N_t^2].$$

The well-known fact that $t \mapsto E^P[N_t^2]$ is bounded (see Chapter 1.10 in [3]) directly implies (8). To prove (9), write

$$\begin{aligned} F(t,x) &= E^P\left[\frac{1}{h(Y_t)}|\pi(\nabla \ln h(Y_t))| \,\middle|\, \pi(Y_t) = x\right] = E^P\left[\frac{|Y_t^1|}{|Y_t|^3} \,\middle|\, Y_t^1 = x\right] \\ &= E^P\left[\frac{|x|}{[x^2 + (Y_t^2)^2 + (Y_t^3)^2]^{3/2}}\right], \end{aligned}$$

where the last equality follows from the independence of the components of Y. By the scaling property of Brownian motion, $F(t,x) = t^{-1}F(1, t^{-1/2}x)$. To prove local boundedness of F on $(0,\infty) \times \mathbb{R}$ it is therefore enough to show that $x \mapsto F(1,x)$ is locally bounded on $\mathbb{R}$. Noting that the random variable $Z = (Y_1^2)^2 + (Y_1^3)^2$ is χ_2^2 distributed, we obtain

$$\begin{aligned} F(1,x) &= E^P\left[\frac{|x|}{(x^2+Z)^{3/2}}\right] \\ &= \frac{|x|}{2}\int_0^\infty (x^2+z)^{-3/2} e^{-z/2}\,dz \\ &\le \frac{|x|}{2}\int_0^\infty (x^2+z)^{-3/2}\,dz = 1. \end{aligned}$$

We thus obtain (9), as required.

To connect this example with the theory developed in the previous sections, note that the set $D_0 = \pi \circ h^{-1}(\{0\})$ is simply equal to $\{0\} \subset \mathbb{R}$. Theorem 2 then tells us that the process Λ only increases on the set $\{t : Y_t^1 = 0\}$. In view of Proposition 1, this explains the appearance of the local time in the expression for $E^P[N_t \mid \mathcal{F}_t]$ found by Föllmer and Protter; see (1) in the Introduction.

EXAMPLE 2 (The inverse Bessel process embedded in $\mathbb{R}^4$). We now consider what happens when the previous example is embedded in $\mathbb{R}^4$. Thus, we set $E = \mathbb{R}^4$, and let Y start from $(1, 0, 0, 0)$. The function h is now given by

$$h(y) = |\bar{y}| \qquad \text{where } \bar{y} = (y_1, y_2, y_3).$$

In other words, $h(y)$ is the distance between y and the y_4-axis. Then $N_t = 1/h(Y_t)$ is a again the reciprocal of a BES(3) process, and again a strict local martingale. It is clear that $1/h$ is harmonic outside the y_4-axis, $E_0 = \{y : \bar{y} = 0\}$. We let π be given by the following matrix representation in the canonical basis on $\mathbb{R}^4$:

$$\pi(y) = Ay \qquad \text{where } A = \begin{pmatrix} 1 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & \alpha_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{for some } \alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}.$$

Note that $D = \pi(E)$ can be identified with $\mathbb{R}^2$. We proceed to verify conditions (8) and (9). First, the gradient of h is given by

$$\nabla h(y) = \left(\frac{\bar{y}}{|\bar{y}|}, 0 \right) \in \mathbb{R}^4.$$

Hence

$$E^P\left[\frac{1}{h(Y_t)} |\nabla \ln h(Y_t)| \right] = E^P[N_t^2]$$

and we get (8) as in the previous example. We continue with (9), and define

$$F(t, x) = \begin{pmatrix} F_1(t, x) \\ F_2(t, x) \end{pmatrix}, \qquad F_i(t, x) = E^P\left[\frac{|\pi(\nabla \ln h(Y_t))_i|}{h(Y_t)} \,\middle|\, \pi(Y_t) = x \right].$$

Using the definition of h, the expression for ∇h, and the definition of π, one gets

$$F_i(t, x) = E^P\left[\frac{|Y_t^i|}{|\overline{Y}_t|^3} \,\middle|\, \pi(Y_t) = x \right], \qquad i = 1, 2.$$

The Brownian scaling property again shows that $F(t, x) = t^{-1} F(1, t^{-1/2} x)$, so just as in the previous example we need only consider $F(1, x)$. Next,

$$F_i(1, x) \le E^P\left[\frac{|Y_1^i|}{[(Y_1^i)^2 + (Y_1^3)^2]^{3/2}} \,\middle|\, \pi(Y_1) = x \right]. \tag{12}$$

To continue, we need to know the distribution of (Y_1^i, Y_1^3) conditionally on $\pi(Y_1) = x$, for $i = 1, 2$. This can, for instance, be done using the formula for the conditional multivariate Normal, applied to the multivariate Normal vector $(Y_1^i, Y_1^3, \pi(Y_1))$. The result of this calculation is that Y_1^i and Y_1^3 are conditionally independent, with Y_1^3 having mean zero and unit variance, and Y_1^i, $i = 1, 2$, satisfying

$$\mu_1 = E\big[Y_1^1 \mid \pi(Y_1) = x\big] = 1 + \frac{(\alpha_2^2+1)(x_1-1) - \alpha_1\alpha_2 x_2}{1+\alpha_1^2+\alpha_2^2},$$

$$\mu_2 = E\big[Y_1^2 \mid \pi(Y_1) = x\big] = 1 + \frac{(\alpha_1^2+1)x_2 - \alpha_1\alpha_2 (x_1-1)}{1+\alpha_1^2+\alpha_2^2},$$

$$\sigma_i^2 = \operatorname{Var}\big[Y_1^i \mid \pi(Y_1) = x\big] = \frac{\alpha_i^2}{1+\alpha_1^2+\alpha_2^2}.$$

Continuing from (12) and using that α_1 and α_2 are nonzero,

$$F_i(1,x) \le \frac{1}{2\pi\sigma_i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|u|}{(u^2+v^2)^{3/2}} \exp\left(-\frac{(u-\mu_i)^2}{2\sigma_i^2} - \frac{v^2}{2}\right) du\, dv.$$

Now split the inner integral (with variable u) into two parts: the first over $(-1,1)$ and the second over $\mathbb{R}\setminus(-1,1)$. Starting with the first part, we get

$$\begin{aligned}
&\frac{1}{2\pi\sigma_i}\int_{-\infty}^{\infty}\int_{-1}^{1} \frac{|u|}{(u^2+v^2)^{3/2}} \exp\left(-\frac{(u-\mu_i)^2}{2\sigma_i^2} - \frac{v^2}{2}\right) du\, dv \\
&\quad \le \frac{1}{2\pi\sigma_i}\int_{-\infty}^{\infty}\int_{-1}^{1} \frac{|u|}{(u^2+v^2)^{3/2}}\, du\, e^{-v^2/2}\, dv \\
&\quad = \frac{1}{\pi\sigma_i}\int_{-\infty}^{\infty} \left(\sqrt{1+v^2} - \sqrt{v^2}\right) e^{-v^2/2}\, dv \\
&\quad \le \sqrt{\frac{2}{\pi}}\frac{1}{\sigma_i},
\end{aligned}$$

where the last line used the inequality $\sqrt{a^2+b^2} \le |a| + |b|$ and then the fact that the Normal density integrates to one. We now consider the integral over the complementary set $\mathbb{R}\setminus(-1,1)$. Since $u^2 \ge 1$ there, we get

$$\begin{aligned}
&\frac{1}{2\pi\sigma_i}\int_{-\infty}^{\infty}\int_{\mathbb{R}\setminus(-1,1)} \frac{|u|}{(u^2+v^2)^{3/2}} \exp\left(-\frac{(u-\mu_i)^2}{2\sigma_i^2} - \frac{v^2}{2}\right) du\, dv \\
&\quad \le \frac{1}{2\pi\sigma_i}\int_{-\infty}^{\infty}\int_{\mathbb{R}\setminus(-1,1)} |u| \exp\left(-\frac{(u-\mu_i)^2}{2\sigma_i^2} - \frac{v^2}{2}\right) du\, dv \\
&\quad \le E^P\big[|Y_1^i| \mid \pi(Y_1) = x\big].
\end{aligned}$$

The right-hand side is the expectation of a *folded Normal* distribution, and its value is a smooth function of μ_i; see [21] or compute directly. Consequently it is a locally bounded function of x, and this finally shows that (9) holds.

Finally, note that $D_0 = \pi(E_0) = \{(\lambda\alpha_1, \lambda\alpha_2) : \lambda \in \mathbb{R}\}$. This is a proper subspace in $D = \mathbb{R}^2$, and in particular it is Lebesgue-null. We would therefore expect that the semimartingale decomposition of the projection of N onto $\mathbb{F}$ in this case also has a singular component.

EXAMPLE 3 (A counterexample). Consider again the situation in Example 2, but this time set $\alpha_1 = \alpha_2 = 0$. Then Y^4 does not play any role at all, and $\mathbb{F}$ is generated by (Y^1, Y^2). In this case the equivalent measure extension problem has no solution—indeed, this corresponds to projecting the inverse Bessel process onto the filtration $\mathbb{F}^{1,2}$ mentioned in the Introduction, and according to Föllmer and Protter's results (Theorem 5.2 in [10]) this projection is again a local martingale. Corollary 1 then shows that no solution to the equivalent measure extension problem can be found. Condition (9) can therefore not be satisfied, and this can indeed be verified directly: with $F_i(t, x)$ as in Example 2, we have

$$\begin{aligned}
|F_i(1,x)| &= E^P\left[\frac{|x_i|}{[x_1^2 + x_2^2 + (Y_1^3)^2]^{3/2}}\right] \\
&\geq \frac{1}{\sqrt{2\pi e}} \int_{-1}^{1} \frac{|x_i|}{(x_1^2 + x_2^2 + u^2)^{3/2}}\, du \\
&= \sqrt{\frac{2}{\pi e}} \frac{|x_i|}{(x_1^2 + x_2^2)\sqrt{1 + x_1^2 + x_2^2}}.
\end{aligned}$$

The right-hand side is unbounded near the origin.

EXAMPLE 4 (Building new examples from old). Suppose we have functions $h_1, \ldots, h_m$ such that for each i, $1/h_i$ is harmonic outside $h_i^{-1}(\{0\})$. We define the set

$$E_0 = \bigcup_{i=1}^{m} h_i^{-1}(\{0\})$$

as the collection of points where some h_i vanishes. We may then define h by

$$\frac{1}{h} = \frac{1}{h_1} + \cdots + \frac{1}{h_m} \qquad \text{on } E \setminus E_0$$

and extend it continuously to all of E by setting $h(y) = 0$, $y \in E_0$. We have the following result.

LEMMA 13. *Consider h and E_0 as above. The function $1/h$ is harmonic outside E_0, and we have*

$$\frac{1}{h}\nabla \ln h = \frac{1}{h_1}\nabla \ln h_1 + \cdots + \frac{1}{h_m}\nabla \ln h_m.$$

PROOF. By linearity of the Laplacian it is clear that $\frac{1}{h}$ is harmonic. The second statement follows from the following elementary calculation:

$$\begin{aligned}\nabla h &= \nabla\left[\left(\frac{1}{h_1}+\cdots+\frac{1}{h_m}\right)^{-1}\right]\\ &= -\left(\frac{1}{h_1}+\cdots+\frac{1}{h_m}\right)^{-2}\left(\nabla\left(\frac{1}{h_1}\right)+\cdots+\nabla\left(\frac{1}{h_m}\right)\right)\\ &= h^2\left(\frac{1}{h_1}\nabla\ln h_1+\cdots+\frac{1}{h_m}\nabla\ln h_m\right). \qquad \square\end{aligned}$$

It follows directly from this lemma that if each h_i satisfies (8) and (9), then the same will be true for h. A simple application of this result is that any process N of the form

$$N_t = \frac{1}{|Y_t - y^{(1)}|} + \cdots + \frac{1}{|Y_t - y^{(m)}|},$$

where $y^{(1)}, \dots, y^{(m)} \in \mathbb{R}^3$ are fixed and different from Y_0, induces a Föllmer measure that can be extended to an equivalent measure on the subfiltration generated by Y^1.

7. Applications in finance.

We end with a brief discussion of some consequences for financial modeling and arbitrage. The discussion will be kept on an informal level, and we defer the development and analysis of concrete models to future research. The notation from Sections 1 and 2 will be used freely. The first observation, which has been made in [10] and [15], is that market participants with limited information may perceive arbitrage opportunities even if there are none. This interpretation arises when N is a price process, and less informed investors only see its optional projection.

An alternative situation is the following. Consider a well-informed fund manager with filtration $\mathbb{G}$ who trades on behalf of less informed investors with filtration $\mathbb{F}$, in exchange for a fee. Such arrangements are common, and arise because the fund manager has superior information, and/or because he has cheaper (lower transactions costs) access to the market. Suppose further that the measures P and Q represent competing beliefs regarding the future evolution of the world, and suppose M is the value process of the fund manager's investment strategy, where M reaching zero corresponds to bankruptcy. If the beliefs Q (under which M may in fact hit zero) are correct, M is a very risky investment. In contrast, under P bankruptcy happens with zero probability. The key point is that less informed investors who estimate M via its optional projection will always obtain a strictly positive estimate, even if their beliefs are correct and given by Q (where Q solves the equivalent measure extension problem). In effect, the fund manager can run risky strategies which, *conditionally on no bankruptcy*, achieve superior returns, while convincing investors that bankruptcy is impossible. He can thus charge excessive fees, which allows him to achieve arbitrage profits (for himself) by exploiting the fact that investors are ill-informed.

Any model where effects of this type occur will necessarily include components relating to the contractual relationship between investors and fund manager, the investment horizon, what happens if M does, in fact, reach zero, and so forth. While such domain specific issues fall outside the scope of the present paper, they are the subject of ongoing research.

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