The Annals of Applied Probability
2014, Vol. 24, No. 3, 1002–1048
DOI: 10.1214/13-AAP940

ASYMPTOTICALLY OPTIMAL DISCRETIZATION OF HEDGING STRATEGIES WITH JUMPS

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In this work, we consider the hedging error due to discrete trading in models with jumps. Extending an approach developed by Fukasawa [In *Stochastic Analysis with Financial Applications* (2011) 331–346 Birkhäuser/Springer Basel AG] for continuous processes, we propose a framework enabling us to (asymptotically) optimize the discretization times. More precisely, a discretization rule is said to be optimal if for a given cost function, no strategy has (asymptotically, for large cost) a lower mean square discretization error for a smaller cost. We focus on discretization rules based on hitting times and give explicit expressions for the optimal rules within this class.

1. Introduction. A basic problem in mathematical finance is how to replicate a random claim with $\mathcal{F}_T$-measurable payoff H_T with a portfolio involving only the underlying asset Y and cash. When Y follows a diffusion process of the form

$$dY_t = \mu(t, Y_t)\,dt + \sigma(t, Y_t)\,dW_t, \tag{1}$$

it is known that under minimal assumptions, a random payoff depending only on the terminal value of the asset $H_T = H(Y_T)$ can be replicated with the so-called delta hedging strategy. This means that the number of units of the underlying asset to hold at time t is equal to $X_t = \frac{\partial P(t,Y_t)}{\partial Y}$, where $P(t, Y_t)$ is the price of the option, which is uniquely defined in such a model. However, to implement such a strategy, the hedging portfolio must be readjusted continuously, which is of course physically impossible and irrelevant because of the presence of microstructure effects and transaction costs. For this reason, the optimal strategy is always replaced with a piecewise constant one, leading to a discretization error. The relevant questions are then: (i) how big is this discretization error, and (ii) when are the good times to readjust the hedge?

Assume first that the hedging portfolio is readjusted at regular intervals of length $h = \frac{T}{n}$. A result by Zhang [27] (see also [3, 18]) then shows that for Lipschitz continuous payoff functions, assuming zero interest rates, the discretization error

$$\mathcal{E}_T^n = \int_0^T X_t\,dY_t - \int_0^T X_{h[t/h]}\,dY_t$$

Received August 2011; revised May 2013.
MSC2010 subject classifications. 60H05, 91G20.
Key words and phrases. Discretization of stochastic integrals, asymptotic optimality, hitting times, option hedging, semimartingales with jumps, Blumenthal–Getoor index.

satisfies

$$\lim_{h\to 0} nE\big[(\mathcal{E}_T^n)^2\big] = \frac{T}{2}E\bigg[\int_0^T \left(\frac{\partial^2 P}{\partial Y^2}\right)^2 \sigma(s, Y_s)^4\,ds\bigg]. \tag{2}$$

Of course, it is intuitively clear that readjusting the portfolio at regular deterministic intervals is not optimal. However, the optimal strategy for fixed n is very difficult to compute.

Fukasawa [15] simplifies this problem by assuming that the hedging portfolio is readjusted at high frequency. The performance of different families of strategies can then be compared based on their asymptotic behavior as the number of readjustment dates n tends to infinity, rather than the performance for fixed n. Consider a sequence of discretization strategies

$$0 = T_0^n < T_1^n < \cdots < T_j^n < \cdots,$$

with $\sup_j |T_{j+1}^n - T_j^n| \to 0$ as $n \to \infty$, and let $N_T^n := \max\{j \geq 0; T_j^n \leq T\}$ be the total number of readjustment dates on the interval $[0, T]$ for given n. To compare two such sequences in terms of their asymptotic behavior for large n, Fukasawa [15] uses the functional

$$\lim_{n\to\infty} E\big[N_T^n\big]E\big[\langle\mathcal{E}^n\rangle_T\big], \tag{3}$$

where $\langle\mathcal{E}^n\rangle$ is the quadratic variation of the semimartingale $(\mathcal{E}_t^n)_{t\geq 0}$. He finds that when the underlying asset is a continuous semimartingale, the functional (3) admits a nonzero lower bound over all such sequences, and exhibits a specific sequence which attains this lower bound and is therefore called *asymptotically efficient*.

In the diffusion model (1), the asymptotically efficient sequence takes the form

$$\begin{aligned} T_{j+1}^n &= \inf\bigg\{t > T_j^n; |X_t - X_{T_j^n}|^2 \geq h_n \frac{\partial^2 P(T_j^n, Y_{T_j^n})}{\partial Y^2}\bigg\}, \\ X_t &= \frac{\partial P(t, Y_t)}{\partial Y}, \end{aligned} \tag{4}$$

where h_n is a deterministic sequence with $h_n \to 0$. In this case,

$$\lim_{n\to\infty} E\big[N_T^n\big]E\big[\langle\mathcal{E}^n\rangle_T\big] = \frac{1}{6}E\bigg[\int_0^T \frac{\partial^2 P}{\partial Y^2}\sigma(s, Y_s)^2\,ds\bigg]^2, \tag{5}$$

whereas for readjustment at equally spaced dates, formula (2) yields

$$\lim_{n\to\infty} E\big[N_T^n\big]E\big[\langle\mathcal{E}^n\rangle_T\big] = \frac{T}{2}E\bigg[\int_0^T \left(\frac{\partial^2 P}{\partial Y^2}\right)^2 \sigma(s, Y_s)^4\,ds\bigg]. \tag{6}$$

Using the Cauchy–Schwarz inequality, we then see that the asymptotically efficient discretization leads to a gain of at least a factor 3, compared to readjustment at regularly spaced points.

Remark that the discretization scheme (4) is very different from the classical approximation schemes for stochastic differential equations such as Euler or Milstein schemes. In order to be implemented it requires the continuous observation of (X_t) and (Y_t), which of course makes sense in the mathematical finance context because the prices are, essentially, continuously observable and the need for discretization is due to the presence of transaction costs.

While the above approach is quite natural and provides very explicit results, it fails to take into account important factors of market reality. First, the asymptotic functional (3) is somewhat ad hoc, and does not reflect any specific model for the transaction costs. Yet, transaction costs are one of the main reasons why continuous (or almost continuous) readjustments are not used. Therefore, they should be the determining factor for any discretization algorithm. On the other hand, the continuity assumption, especially at relatively high frequencies, is not realistic. Indeed, it is well known that jumps in the price occur quite frequently and have a significant impact on the hedging error. It can even be argued that high-frequency financial data are best described by pure jump processes; see [7].

The objective of this paper is therefore two-fold. First, we develop a framework for characterizing the asymptotic efficiency of discretization strategies which takes into account the transaction costs. Second, we remove the continuity assumption in order to understand the effect of the activity of small jumps (often quantified by the Blumenthal–Getoor index) on the optimal discretization strategies.

Models with jumps correspond to incomplete markets, where the hedging issue is an approximation problem,

$$\min_X E\left(c + \int_0^T X_{t-}\, dY_t - H_T\right)^2, \tag{7}$$

where Y is now a semimartingale with jumps. The optimal strategy X^* for this problem is known to exist for any $H_T \in L^2$; see [9, 10, 13, 14, 19, 24]. If the expectation in (7) is computed under a martingale probability measure, then for any admissible strategy X',

$$\begin{aligned} E\left(c + \int_0^T X'_{t-}\, dY_t - H_T\right)^2 &= E\left(\int_0^T (X'_{t-} - X^*_{t-})\, dY_t\right)^2 \\ &\quad + E\left(c + \int_0^T X^*_{t-}\, dY_t - H_T\right)^2. \end{aligned} \tag{8}$$

Indeed, $\int X^*_{t-}\, dY_t$ is essentially the orthogonal projection of H_T on the subspace of L^2 constituted by the stochastic integrals of the form $\int X_{t-}\, dY_t$ where X_{t-} is an admissible hedging strategy. Therefore, the quadratic hedging problem (7) and the discretization problem can be studied separately. Given that the quadratic hedging problem has already been studied by many authors, in this paper we concentrate on the discretization problem.

Our goal is to study and compare discretization rules for stochastic integrals of the form

$$\int_0^T X_{t-}\, dY_t,$$

where X_t and Y_t are semimartingales with jumps, with the aim of identifying asymptotically optimal rules. In particular we wish to understand the impact of the small jumps of X on the discretization error, and therefore we assume that X has no continuous local martingale part; see Remark 3.

A *discretization rule* is a family of stopping times $(T_i^\varepsilon)_{i\geq 0}^{\varepsilon>0}$ parameterized by a nonnegative integer i and a positive real ε, such that for every $\varepsilon > 0$, $0 = T_0^\varepsilon < T_1^\varepsilon < T_2^\varepsilon < \cdots$. For a fixed discretization rule and a fixed ε, we let $\eta^\varepsilon(t) = \sup\{T_i^\varepsilon : T_i^\varepsilon \leq t\}$ and $N_T^\varepsilon = \sup\{i : T_i^\varepsilon \leq T\}$. Motivated by decomposition (8), we measure the performance of a discretization rule with the L^2 error functional

$$\mathcal{E}(\varepsilon) := E\biggl[\biggl(\int_0^T (X_{t-} - X_{\eta(t)-})\, dY_t\biggr)^2\biggr]. \tag{9}$$

Also, to each discretization rule we associate a family of cost functionals of the form

$$\mathcal{C}^\beta(\varepsilon) = E\biggl[\sum_{i\geq 1 : T_i^\varepsilon \leq T} |X_{T_i^\varepsilon} - X_{T_{i-1}^\varepsilon}|^\beta\biggr], \tag{10}$$

with $\beta \in [0, 2]$. The case $\beta = 0$ corresponds to a fixed cost per transaction, and the case $\beta = 1$ corresponds to a fixed cost per unit of asset. Other values of β often appear in the market microstructure literature where one considers that transaction costs are explained by the shape of the order book.

In our framework, a discretization rule is said to be optimal for a given cost functional if no strategy has (asymptotically, for large costs) a lower discretization error and a smaller cost.

Motivated by the representation (4) and the readjustment rules used by market practitioners, we focus on discretization strategies based on the exit times of X out of random intervals

$$T_{i+1}^\varepsilon = \inf\bigl\{t > T_i^\varepsilon : X_t \notin (X_{T_i^\varepsilon} - \varepsilon \underline{a}_{T_i^\varepsilon}, X_{T_i^\varepsilon} + \varepsilon \overline{a}_{T_i^\varepsilon})\bigr\}, \tag{11}$$

where $(\overline{a}_t)_{t\geq 0}$ and $(\underline{a}_t)_{t\geq 0}$ are positive $\mathbb{F}$-adapted càdlàg processes.

In Theorems 1 and 2, we characterize explicitly the asymptotic behavior of the errors and costs associated to these random discretization rules, by showing that, under suitable assumptions,

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\mathcal{E}(\varepsilon) = E\biggl[\int_0^T A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\, dt\biggr],$$

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha-\beta}\mathcal{C}^\beta(\varepsilon) = E\biggl[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\, dt\biggr],$$

where, for $\underline{a}, \overline{a} \in (0, \infty)$,

$$f(\underline{a}, \overline{a}) = E\left[\int_0^{\tau^*} (X_t^*)^2 \, dt\right], \qquad g(\underline{a}, \overline{a}) = E[\tau^*] \quad \text{and}$$

$$u^\beta(\underline{a}, \overline{a}) = E\left[|X_{\tau^*}^*|^\beta\right] < \infty,$$

with $\tau^* = \inf\{t \geq 0 : X_t^* \notin (-\underline{a}, \overline{a})\}$, where X^* is a strictly α-stable process determined from X by a limiting procedure, and the processes A and λ are determined from the semimartingale characteristics of X and Y.

This allows us to determine the asymptotically optimal intervals as solutions to a simple optimization problem (Proposition 2). In particular, we show that in the case where the cost functional is given by the expected number of discretization dates, the error associated to our optimal strategy with the cost equal to N, converges to zero as $N \to \infty$ at a faster rate than the error obtained by readjusting at N equally spaced dates.

As applications of our method, we consider the discretization of the hedging strategy for a European option in an exponential Lévy model (Proposition 4) and the discretization of the Merton portfolio strategy (Proposition 5). In the option hedging problem, we obtain an explicit representation for the optimal discretization dates, which is similar to (4), but includes two "tuning" parameters: an index which determines the effect of transaction costs (fixed, proportional, etc.) and the Blumenthal–Getoor index measuring the activity of small jumps.

This paper is structured as follows. In Section 2, we introduce our framework and in particular the notion of asymptotic optimality based on the limiting behavior of the error and cost functionals. The assumptions on the processes X and Y and on the admissible discretization rules are also stated here. Section 3.1 contains the main results of this paper which characterize the limiting behavior of the error and the cost functionals, and Sections 3.2 to 3.4 provide explicit examples of optimal discretization strategies in various contexts. Sections 4 and 6 contain the proofs of the main results and Section 5 gathers some technical lemmas needed in Section 6.

2. Framework. *Asymptotic comparison of discretization rules.* We are interested in comparing different discretization rules, as defined in the Introduction, for the stochastic integral

$$\int_0^T X_{t-} \, dY_t,$$

where X and Y are semimartingales, in terms of their limiting behavior when the number of discretization points tends to infinity.

The performance of a given discretization rule is assessed by the error functional $\mathcal{E}(\varepsilon) : (0, \infty) \to [0, \infty)$ (which measures the discretization error associated to this rule) and a cost functional $\mathcal{C}^\beta(\varepsilon) : (0, \infty) \to [0, \infty)$ (which measures the corresponding transaction cost), as defined in (9) and (10). We assume that the cost functional is such that

$$\lim_{\varepsilon\downarrow 0} \mathcal{C}^{\beta}(\varepsilon) = +\infty.$$

For $C > 0$ sufficiently large, we define

$$\varepsilon(C) = \inf\{\varepsilon > 0 : \mathcal{C}^{\beta}(\varepsilon) < C\}$$

and $\overline{\mathcal{E}}(C) := \mathcal{E}(\varepsilon(C))$.

DEFINITION 1. We say that the discretization rule A asymptotically dominates the rule B if

$$\limsup_{C\to\infty} \frac{\overline{\mathcal{E}}^{A}(C)}{\overline{\mathcal{E}}^{B}(C)} \le 1.$$

To apply Definition 1, the following simple result will be very useful.

LEMMA 1. *Assume that for a given discretization rule, the cost and error functionals are such that there exist $a > 0$ and $b > 0$ with*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-a}\mathcal{E}(\varepsilon) = \hat{\mathcal{E}} \quad \textit{and} \quad \lim_{\varepsilon\downarrow 0} \varepsilon^{b}\mathcal{C}^{\beta}(\varepsilon) = \hat{\mathcal{C}} \tag{12}$$

for some positive constants $\hat{\mathcal{E}}$ and $\hat{\mathcal{C}}$. Then

$$\overline{\mathcal{E}}(C) \sim C^{-a/b}(\hat{\mathcal{C}})^{a/b}\hat{\mathcal{E}} \qquad \textit{as } C \to \infty.$$

We shall consider discretizations based on the hitting times of the process X. Recall that such a discretization rule is characterized by a pair of positive $\mathbb{F}$-adapted càdlàg processes $(\overline{a}_t)_{t\ge 0}$ and $(\underline{a}_t)_{t\ge 0}$, and the discretization dates are then defined by (11).

REMARK 1. Consider the discretization rules $A = (\underline{a}, \overline{a})$ and $B = (k\underline{a}, k\overline{a})$ with $k > 0$. These two strategies satisfy $\overline{\mathcal{E}}^{A}(C) = \overline{\mathcal{E}}^{B}(C)$ for all $C > 0$. Therefore, the optimal strategies will be determined up to a multiplicative constant.

Assumptions on the processes X and Y. Our first main result describing the behavior of the error functional will be obtained under the assumptions (HY), (HX) and (HX^1_{loc}) stated below.

(HY) We assume that the process Y is an $\mathbb{F}$-local martingale, whose predictable quadratic variation satisfies $\langle Y\rangle_t = \int_0^t A_s\,ds$, where the process (A_t) is càdlàg and locally bounded.

(HX) The process X is a semimartingale defined via the stochastic representation

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \int_{|z|\le 1} z(M-\mu)(ds\times dz) + \int_0^t \int_{|z|>1} zM(ds\times dz), \tag{13}$$

where M is the jump measure of X, and μ is its predictable compensator, absolutely continuous with respect to the Lebesgue measure in time, $\mu(dt\times dz) = dt\times\mu_t(dz)$, where the kernel $\mu_t(dz)$ is such that for some $\alpha\in(1,2)$ there exist positive càdlàg processes (λ_t) and $(\widehat{K}_t)$ and constants $c_+\ge 0$ and $c_-\ge 0$ with $c_+ + c_- > 0$ and, almost surely for all $t\in[0,T]$,

$$x^\alpha\mu_t((x,\infty)) \le \widehat{K}_t \quad\text{and}\quad x^\alpha\mu_t((-\infty,-x)) \le \widehat{K}_t \qquad \text{for all } x>0; \tag{14}$$

$$x^\alpha\mu_t((x,\infty)) \to c_+\lambda_t \quad\text{and}\quad x^\alpha\mu_t((-\infty,-x)) \to c_-\lambda_t \qquad \text{when } x\to 0. \tag{15}$$

(HX^ρ_{loc}) There exists a Lévy measure $\nu(dx)$ such that, almost surely, for all t, the kernel $\mu_t(dz)$ is absolutely continuous with respect to $\lambda_t\nu(dz)$,

$$\mu_t(dz) = K_t(z)\lambda_t\nu(dz) \tag{16}$$

for a random function $K_t(z)>0$. Moreover, there exists an increasing sequence of stopping times (τ_n) with $\tau_n\to T$ such that for every n,

$$\int_0^{\tau_n}\int_{\mathbb{R}} \left|\sqrt{K_t(z)}-1\right|^{2\rho}\nu(dz)\,dt < C_n, \tag{17}$$

$\frac{1}{C_n}\le\lambda_t\le C_n$, $\widehat{K}_t\le C_n$ and $|b_t|\le C_n$ for $0\le t\le\tau_n$ and some constant $C_n>0$.

REMARK 2 (Concerning the assumptions on the process Y). The assumption that Y is a local martingale greatly simplifies the treatment of quadratic hedging problems in various settings because it allows us to reduce the problem of minimizing the global quadratic risk to myopic local risk minimization. In particular, under this assumption, the error functional (9) becomes

$$\mathcal{E}(\varepsilon) = E\left[\int_0^T (X_t - X_{\eta^\varepsilon(t)})^2 A_t\,dt\right].$$

While it may be unrealistic to assume that the stock price process is a local martingale *for computing the hedging strategy*, in the present study we have a different objective. We are looking for the asymptotically optimal rule to discretize a *given* strategy, that is, the rule which minimizes, asymptotically for large number of discretization dates, the principal term of the discretization error. In the case of equally spaced discretization dates, it is known (see [25] for a proof in the context of Itô semimartingales with jumps) that this principal term does not depend on the drift part of the processes X and Y. We conjecture that the same kind of behavior holds in the context of random rebalancing dates, which means that the drift terms do not need to be taken into account when computing asymptotically optimal discretization rules. Our methodology allows us to determine asymptotically optimal discretization for a given process X, which may correspond, for example, to a quadratic hedging strategy computed in the nonmartingale setting.

REMARK 3 (Concerning the assumptions on the process X).

- In this paper, we focus on semimartingales for which the local martingale part is purely discontinuous, with the aim of determining the effect of small jumps on the convergence rate of the discretization error. Therefore, we do not include a continuous local martingale part in the dynamics of X. Indeed, it would asymptotically dominate the purely discontinuous part as shown in Proposition 7 in the Appendix. The dynamics of Y can, in principle, include such a continuous local martingale part, however in the usual financial models, when X has no continuous local martingale part, this is also the case for Y. Note that from the practical viewpoint, many exponential Lévy models popular among academics and practitioners (Variance Gamma, CGMY, Normal inverse Gaussian etc.) do not include a continuous diffusion part.
- Assumption (*HX*) defines the structure of the integrand (hedging strategy) X, by saying that the small jumps of X ressemble those of an α-stable process, modulated by a random intensity process (λ_t). This assumption introduces the fundamental parameters which will appear in our limiting results: the coefficients α, c_+ and c_- and the intensity process λ. These parameters are determined uniquely up to multiplying λ by a positive constant and dividing c_+ and c_- by the same constant. Note also that these parameters can be estimated from market data; see [1, 11, 12, 26].
- The parameter α measures the activity of small jumps of the process X. In the case where X is a Lévy process, the parameter α coincides with the Blumenthal–Getoor index of X; see [4].
- The assumption $1 < \alpha < 2$ implies that X has infinite variation and ensures that the local behavior of the process is determined by the jumps rather than by the drift part; see [22]. Note that in a recent statistical study on liquid assets [1], the jump activity index defined similarly to our parameter α was estimated between 1.4 and 1.7. However, this assumption does exclude some interesting models and other statistical studies find that this parameter can be smaller than one for certain asset classes [2, 8].
- The assumption (HX^{ρ}_{loc}) is a technical integrability condition. In the sequel, we shall always impose (HX^{1}_{loc}) and sometimes also (HX^{ρ}_{loc}) with $\rho > 1$. The representation (16) of the compensator μ of the jump measure of X implies that the jump part of X is locally equivalent to a time-changed Lévy process. Indeed,

time-changing the process with a continuous increasing process $\Lambda_t = \int_0^t \lambda_s\, ds$ has the effect of multiplying the compensator by λ_t, and making a change of probability measure with density given by (30) has the effect of dividing the compensator by $K_t(z)$. The objects $\nu(dz)$ and $K_t(z)$ in this representation are not unique, but they do not appear in our limiting results. In particular, it is easy to show that the Lévy measure ν necessarily satisfies a stable-like condition similar to (15),

$$x^\alpha \nu((x,\infty)) \to c_+ \quad \text{and} \quad x^\alpha \nu((-\infty,-x)) \to c_- \qquad \text{when } x \to 0. \tag{18}$$

Indeed, there exists a constant $c > 0$ such that

$$c(\sqrt{f}-1)^2 \ge (f-1)^2 \mathbf{1}_{|f-1|\le 1/2} + |f-1| \mathbf{1}_{|f-1|>1/2} \qquad \text{for all } f > 0.$$

From this simple inequality, and denoting $I_t = \int_{\mathbb{R}} (\sqrt{K_t(z)} - 1)^2 \nu(dz)$, one can easily deduce, using the Cauchy–Schwarz inequality that for another constant C,

$$\left| \int_x^\infty \nu(dz) - \int_x^\infty K_t(z)\nu(dz) \right| \le C I_t + C \left\{ \int_x^\infty \nu(dz) \right\}^{1/2} I_t^{1/2},$$

and also that

$$\left| \left(\int_x^\infty \nu(dz) \right)^{1/2} - \left(\int_x^\infty K_t(z)\nu(dz) \right)^{1/2} \right| \le C I_t$$

for yet another constant C. By (17), under (HX^1_{loc}), $I_t < \infty$ for almost all t. For any such t, we can multiply the above inequality with $x^{\alpha/2}$ and take the limit $x \to 0$; we then get

$$\lim_{x\to 0} x^\alpha \int_x^\infty \nu(dz) = \lim_{x \to 0} x^\alpha \int_x^\infty K_t(z)\nu(dz),$$

but the latter limit is equal to c_+ by assumption (15). Moreover, it is always possible with no loss of generality to choose ν so that it also satisfies

$$x^\alpha \nu((x,\infty)) + x^\alpha \nu((-\infty,-x)) \le C \tag{19}$$

for some constant $C < \infty$ and all $x > 0$. Indeed, by property (18), it is enough to show this for all $x \ge \varepsilon$ with some $\varepsilon > 0$. But for this, it is enough to take

$$K_t(z) = \frac{\hat{K}_t}{\lambda_t} \qquad \text{for } |z| \ge \varepsilon$$

and use (14). Such a choice clearly does not violate condition (17). In the sequel we shall assume that ν has been chosen in such a way.

EXAMPLE 1. In applications, the process X is often defined as solution to a stochastic differential equation rather than through its semimartingale characteristics. We now give an example of an SDE which satisfies our assumptions. Let X be the solution of an SDE driven by a Poisson random measure

$$\begin{aligned}X_t = X_0 + \int_0^t \bar{b}_s\,ds + \int_0^t \int_{|z|\le 1} \gamma_s(z)\tilde{N}(ds\times dz)\\ + \int_0^t \int_{|z|>1} \gamma_s(z) N(ds\times dz),\end{aligned}\tag{20}$$

where N is a Poisson random measure with intensity measure $dt \times \bar{\nu}(dz)$, $\tilde{N}$ is the corresponding compensated measure, and $\gamma : [0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ is a predictable random function.

PROPOSITION 1. *Assume that $\bar{\nu}$ is a Lévy measure which has a compact support U such that $0\in \operatorname{int} U$ and admits a density also denoted by $\bar{\nu}(x)$, which is continuous outside any neighborhood of zero and is such that*

$$x^{\alpha+1}\bar{\nu}(x) = \alpha c_+ + O(x) \quad \textit{and} \quad x^{\alpha+1}\bar{\nu}(-x) = \alpha c_- + O(x) \qquad \textit{when } x\downarrow 0\tag{21}$$

for some $\alpha\in(1,2)$ and constants $c_+>0$ and $c_->0$.

Suppose furthermore that for all $\omega\in\Omega$ and $t\in[0,T]$, $\gamma_t(z)$ is twice differentiable with respect to z, $\gamma_t'(z)>0$ for all $z\in U$, $\gamma_t(0)=0$, and there exists an increasing sequence of stopping times (τ_n) with $\tau_n\to T$ and a sequence of positive constants (C_n) with $C_n<\infty$ for all n, such that for every n, almost surely,

$$|b_t|\le C_n,\qquad \frac{1}{C_n}\le \gamma_t'(z)\le C_n \quad \textit{and} \quad |\gamma_t''(z)|<C_n \qquad \textit{for all } 0\le t\le\tau_n, z\in U.\tag{22}$$

Then the process X satisfies the assumption (HX) with $\lambda_t=\gamma_t'(0)^\alpha$ and the assumption (HX^ρ_{loc}) for all $\rho\ge 1$.

The proof of this result is given in Appendix D.

Assumptions on the discretization rules. Our first main result (asymptotics of the error functional) requires the following assumptions on the discretization rule $(\underline{a},\overline{a})$:

(HA) The integrability condition

$$E\Big[\sup_{0\le s\le T}\max(\underline{a}_s,\overline{a}_s)^2\int_0^T A_t\,dt\Big]<\infty.$$

(HA_{loc}) There exists an increasing sequence of stopping times (τ_n) with $\tau_n \to T$ such that for every n, $\frac{1}{C_n} \le \underline{a}_t, \overline{a}_t \le C_n$ for $0 \le t \le \tau_n$ and some constant $C_n > 0$.

To obtain our second main result concerning the behavior of the cost functional, we shall need the following additional technical assumptions:

(HA_2) For some $\delta \in (0, 1)$ with $\beta(1+\delta) < \alpha$,

$$E\Big[\sup_{0\le s\le T}\big(\max\{\underline{a}_s^{\beta-1}, \overline{a}_s^{\beta-1}\}^{1+\delta} + \max\{\underline{a}_s^{(1+\delta)\beta-1}, \overline{a}_s^{(1+\delta)\beta-1}\}\big)\int_0^T |b_s|^{1+\delta}\,ds\Big]$$

$$+ E\Big[\sup_{0\le s\le T}\max\{\underline{a}_s, \overline{a}_s\}^{(\beta\vee(2-\alpha))(1+\delta)}\min\{\underline{a}_s, \overline{a}_s\}^{((\beta-2)\wedge(-\alpha))(1+\delta)}$$

$$\times \int_0^T \widehat{K}_s^{1+\delta}\,ds\Big] < \infty.$$

(HA_2') For some $\delta \in (0, 1)$,

$$E\Big[\sup_{0\le s\le T}\min(\underline{a}_s, \overline{a}_s)^{-\alpha(1+\delta)}\int_0^T \widehat{K}_t^{1+\delta}\,dt$$

$$+ \sup_{0\le s\le T}\min(\underline{a}_s, \overline{a}_s)^{-1-\delta}\int_0^T |b_t|^{1+\delta}\,dt\Big] < \infty.$$

REMARK 4. Condition (HA_2') replaces condition (HA_2) in the case $\beta = 0$. For given β and given processes X and Y, we shall call a discretization rule $(\underline{a}, \overline{a})$ satisfying assumptions (HA), (HA_{loc}) and (HA_2) (if $\beta > 0$) or assumptions (HA), (HA_{loc}) and (HA_2') (if $\beta = 0$) *an admissible discretization rule*.

3. Main results.

In this section, we first characterize the asymptotic behavior of the error and cost functionals for small ε. From these results we then derive the asymptotically optimal discretization strategies using Lemma 1.

3.1. *Asymptotic behavior of the error and cost functionals.*

THEOREM 1. *Under assumptions* (HY), (HX), (HX^1_{loc}), (HA) *and* (HA_{loc}),

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}\mathcal{E}(\varepsilon) = E\Big[\int_0^T A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\,dt\Big], \tag{23}$$

where, for $\underline{a}, \overline{a} \in (0, \infty)$,

$$f(\underline{a}, \overline{a}) = E\Big[\int_0^{\tau^*} (X_t^*)^2\,dt\Big], \qquad g(\underline{a}, \overline{a}) = E[\tau^*]$$

with $\tau^* = \inf\{t \ge 0 : X_t^* \notin (-\underline{a}, \overline{a})\}$, *where* X^* *is a strictly* α*-stable process with Lévy density*

$$\nu^*(x) = \frac{c_+ 1_{x>0} + c_- 1_{x<0}}{|x|^{1+\alpha}}, \qquad x \ne 0,$$

and the constants c_- and c_+ are defined in assumption (*HX*) [*equation* (15)].

THEOREM 2. *We use the notation of Theorem* 1.

(i) *Let assumptions* (*HY*), (*HX*), (HX^1_{loc}), (*HA*), (HA_{loc}) *and* (HA'_2) *be satisfied. Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha} \mathcal{C}^0(\varepsilon) = E\left[\int_0^T \frac{\lambda_t}{g(\underline{a}_t, \overline{a}_t)}\, dt\right]. \tag{24}$$

(ii) *Let* $\beta \in (0, \alpha)$, *and assume that* (*HY*), (*HX*), (HX^1_{loc}), (HX^{ρ}_{loc}) (*for some* $\rho > \frac{\alpha}{\alpha-\beta} \vee 2$), (*HA*), ($HA_{\mathrm{loc}}$) *and* ($HA_2$) *hold true. Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha-\beta} \mathcal{C}^{\beta}(\varepsilon) = E\left[\int_0^T \lambda_t \frac{u^{\beta}(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\, dt\right], \tag{25}$$

where

$$u^{\beta}(\underline{a}, \overline{a}) = E\left[\left|X^*_{\tau^*}\right|^{\beta}\right] < \infty.$$

REMARK 5. Theorems 1 and 2 enable us to apply Lemma 1 and conclude that *for any admissible discretization rule based on hitting times*, the error functional for fixed cost behaves, for large costs, as

$$\overline{\mathcal{E}}(C) \sim C^{-2/(\alpha-\beta)} E\left[\int_0^T A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\, dt\right] E\left[\int_0^T \lambda_t \frac{u^{\beta}(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\, dt\right]^{2/(\alpha-\beta)}.$$

When the cost is equal to the expected number of rebalancings ($\beta = 0$), the error converges to zero at the rate $C^{-2/\alpha}$. On the other hand, for equidistant rebalancing dates, under sufficient regularity, the L^2 discretization error of the quadratic hedging strategy in exponential Lévy models is *inversely proportional* to the number of rebalancings; see [6]. This means that while in diffusion models, asymptotically optimal hedging reduces the error without modifying the rate at which the error decreases with the number of rebalancings [cf. equations (5) and (6)], in pure jump models, any discretization based on hitting times, and a fortiori the optimal discretization, also improves the rate of convergence.

3.2. *Application*: *Computing the optimal barriers*. In this section, we suppose that the assumptions of Theorem 2 [part (i) or (ii), depending on β] are satisfied. In view of Lemma 1, we shall use the following definition of an asymptotically optimal discretization rule.

DEFINITION 2. A discretization rule $(\underline{a}, \overline{a})$ is said to be asymptotically optimal if it is admissible, and for any other admissible rule $(\underline{a}', \overline{a}')$,

$$\begin{aligned} &E\left[\int_0^T A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\, dt\right] E\left[\int_0^T \lambda_t \frac{u^{\beta}(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\, dt\right]^{2/(\alpha-\beta)} \\ &\quad \le E\left[\int_0^T A_t \frac{f(\underline{a}'_t, \overline{a}'_t)}{g(\underline{a}'_t, \overline{a}'_t)}\, dt\right] E\left[\int_0^T \lambda_t \frac{u^{\beta}(\underline{a}'_t, \overline{a}'_t)}{g(\underline{a}'_t, \overline{a}'_t)}\, dt\right]^{2/(\alpha-\beta)}. \end{aligned} \tag{26}$$

The following result simplifies the characterization of such rules.

PROPOSITION 2. *Let* $(\underline{a}, \overline{a})$ *be an admissible discretization rule, and assume that there exists* $c > 0$ *such that for any other admissible rule* $(\underline{a}', \overline{a}')$,

$$A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} + c\lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} \le A_t \frac{f(\underline{a}'_t, \overline{a}'_t)}{g(\underline{a}'_t, \overline{a}'_t)} + c\lambda_t \frac{u^\beta(\underline{a}'_t, \overline{a}'_t)}{g(\underline{a}'_t, \overline{a}'_t)} \tag{27}$$

a.s. for all $t \in [0, T]$. *Then the rule* $(\underline{a}, \overline{a})$ *is asymptotically optimal.*

PROOF. By the nature of assumptions (HA), (HA_{loc}) and (HA_2) [resp., (HA_2')], for all $\kappa > 0$, the rule $(\kappa\underline{a}, \kappa\overline{a})$ is admissible. In addition, by the scaling property of strictly stable processes,

$$f(\kappa\underline{a}_t, \kappa\overline{a}_t) = \kappa^{2+\alpha} f(\underline{a}_t, \overline{a}_t), \qquad g(\kappa\underline{a}_t, \kappa\overline{a}_t) = \kappa^\alpha g(\underline{a}_t, \overline{a}_t),$$
$$u^\beta(\kappa\underline{a}_t, \kappa\overline{a}_t) = \kappa^\beta u^\beta(\underline{a}_t, \overline{a}_t).$$

Using these identities in the left-hand side of (27) and the fact that (27) holds for any $(\underline{a}', \overline{a}')$, integrating both sides and taking the expectation, we get

$$\begin{aligned} &E\left[\int_0^T A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} dt\right] + c\kappa^{\alpha-\beta} E\left[\int_0^T \lambda_t \frac{u^\beta(\kappa\underline{a}_t, \kappa\overline{a}_t)}{g(\kappa\underline{a}_t, \kappa\overline{a}_t)} dt\right] \\ &\quad \le E\left[\int_0^T A_t \frac{f(\kappa'\underline{a}'_t, \kappa'\overline{a}'_t)}{g(\kappa'\underline{a}'_t, \kappa'\overline{a}'_t)} dt\right] + cE\left[\int_0^T \lambda_t \frac{u^\beta(\kappa'\underline{a}'_t, \kappa'\overline{a}'_t)}{g(\kappa'\underline{a}'_t, \kappa'\overline{a}'_t)} dt\right]. \end{aligned}$$

Under the assumptions of Theorems 1 and 2, all expectations above are finite. Indeed, the limiting error functional is finite by assumption (HA) since clearly $f(\underline{a}, \overline{a}) \le \max(\underline{a}, \overline{a})^2 g(\underline{a}, \overline{a})$. The finiteness of the limiting cost functional is shown by applying Lemma 6 to the limiting strictly stable process to obtain a bound on the function u^β and then using assumption (HA_2) or (HA_2').

Now, choose κ so that

$$E\left[\int_0^T \lambda_t \frac{u^\beta(\kappa\underline{a}_t, \kappa\overline{a}_t)}{g(\kappa\underline{a}_t, \kappa\overline{a}_t)} dt\right] = 1 \quad \Rightarrow \quad \kappa = E\left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} dt\right]^{1/(\beta-\alpha)}$$

and κ' so that

$$\begin{aligned} &E\left[\int_0^T \lambda_t \frac{u^\beta(\kappa'\underline{a}'_t, \kappa'\overline{a}'_t)}{g(\kappa'\underline{a}'_t, \kappa'\overline{a}'_t)} dt\right] = \kappa^{\alpha-\beta} \\ &\quad \Rightarrow \quad \kappa' = \frac{1}{\kappa} E\left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}'_t, \overline{a}'_t)}{g(\underline{a}'_t, \overline{a}'_t)} dt\right]^{1/(\alpha-\beta)}. \end{aligned}$$

This yields

$$E\left[\int_0^T A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} dt\right] \le (\kappa')^2 E\left[\int_0^T A_t \frac{f(\underline{a}'_t, \overline{a}'_t)}{g(\underline{a}'_t, \overline{a}'_t)} dt\right].$$

Substituting the expression for κ', we finally obtain (26). $\square$

The above result shows that we may look for optimal barriers as $\underline{a}$ and $\overline{a}$ as minimizers of

$$\min\left\{A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} + c\lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\right\}, \tag{28}$$

provided that the resulting $\underline{a}_t$ and $\overline{a}_t$ are admissible. Moreover if $(\underline{a}, \overline{a})$ is the solution of (28), then the scaling property shows that the solution of

$$\min\left\{A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} + c'\lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\right\}$$

is given by $(\kappa\underline{a}, \kappa\overline{a})$ with $\kappa = (c'/c)^{1/(\alpha-\beta+2)}$. If $c' > c$, then $\kappa > 1$, resulting in a smaller cost functional and a bigger error functional. Therefore, in practice c may be chosen by the trader depending on the maximum acceptable cost: the bigger c, the smaller will be the cost of the strategy and, consequently the bigger its error.

The functions f, g and u appearing above must in general be computed numerically. However, when the constants c_+ and c_- in (15) are equal, which corresponds for example to the CGMY model very popular in practice [7], the results are completely explicit, as will be shown in the next paragraph.

3.3. *Locally symmetric Lévy measures.* In this section we discuss a case important in applications, when the asymptotically optimal strategy can be computed explicitly in terms of A and λ.

PROPOSITION 3. *Let the cost functional be of the form* (10) *with* $\beta \in [0, 1]$. *Let the processes* X *and* Y *satisfy the assumptions* (*HY*), (*HX*) *with* $c_+ = c_-$, (HX^1_{loc}) *and* (HX^ρ_{loc}) *with* $\rho > \frac{\alpha}{\alpha-\beta} \vee 2$ (*if* $\beta > 0$). *Assume that the processes* A, b *and* λ *satisfy the following integrability conditions for some* $\delta > 0$:

$$E\left[\left(\sup_{0\le t\le T} \frac{\lambda_t}{A_t}\right)^{2/(2+\alpha-\beta)} \int_0^T A_t\, dt\right] < \infty,$$

$$E\left[\left(\inf_{0\le t\le T} \frac{\lambda_t}{A_t}\right)^{(1+\delta)(\beta-\alpha)/(2+\alpha-\beta)} \int_0^T \widehat{K}_t^{1+\delta}\, dt\right] < \infty,$$

and, if $\beta = 1$,

$$E\left[\left(\sup_{0\le t\le T} \frac{\lambda_t}{A_t}\right)^{\delta} \int_0^T |b_t|^{1+\delta}\, dt\right] < \infty,$$

or, if $\beta < 1$,

$$E\left[\left(\inf_{0\le t\le T} \frac{\lambda_t}{A_t}\right)^{(\beta-1)(1+\delta)} \int_0^T |b_t|^{1+\delta}\, dt\right] < \infty.$$

Then the strategy given by

$$\underline{a}_t = \overline{a}_t = c\left(\frac{\lambda_t}{A_t}\right)^{1/(2+\alpha-\beta)}$$

is asymptotically optimal.

PROOF. The fact that X satisfies (HX) with $c_+ = c_-$ means that the limiting process X^* is a symmetric stable process. Let $(\underline{a}, \overline{a})$ be an admissible discretization rule. With a change of notation $a_t := \frac{\underline{a}_t + \overline{a}_t}{2}$ and $\theta_t = \frac{\overline{a}_t - \underline{a}_t}{\overline{a}_t + \underline{a}_t}$ and using the results from Appendix A [Proposition 6, equations (53) and (54)], we can compute

$$\frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} = \frac{\alpha}{(\alpha+2)(\alpha+1)} a_t^2 \big(1 + \theta_t^2(1+\alpha)\big),$$

$$\begin{aligned}\frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} \\ &= \frac{\sigma\Gamma(1+\alpha)\sin \pi\alpha/2}{\pi} \\ &\quad \times \int_0^\infty z^{-\alpha/2}(z+2a_t)^{-\alpha/2}\big(|z + a_t(1+\theta_t)|^{\beta-1} + |z + a_t(1-\theta_t)|^{\beta-1}\big)\,dz.\end{aligned}$$

For fixed a_t, both ratios are minimal when $\theta = 0$ (for the second functional this follows from the convexity of the function $x \mapsto x^{\beta-1}$ for $x \ge 0$ and $\beta \le 1$). Moreover, from the structure of assumptions (HA), (HA_{loc}) and (HA_2) [resp., (HA_2')], it is clear that the strategy obtained by taking $\theta = 0$, that is, the strategy (a, a) is also admissible. Therefore, the asymptotically optimal strategy, if it exists, will be symmetric in this case. By the same arguments as in the previous section, we can show that the optimal strategy, if it exists, minimizes

$$A_t \frac{f(a_t, a_t)}{g(a_t, a_t)} + c\lambda_t \frac{u^\beta(a_t, a_t)}{g(a_t, a_t)}$$

for each t. Plugging in the explicit expressions computed above, we see that this functional is minimized by

$$a_t = c\left(\frac{\lambda_t}{A_t}\right)^{1/(2+\alpha-\beta)}$$

for a different constant c. By the assumptions of the proposition, this strategy is admissible, which completes the proof. □

3.4. *Exponential Lévy models.* In this section we treat the case when the process Y (the asset price or the integrator) is the stochastic exponential of a Lévy process. More precisely, throughout this section we assume that

$$Y_t = Y_0 + \int_0^t Y_{s-}\, dZ_s,$$

where Z is a martingale Lévy process with no diffusion part and with Lévy measure ν which has a compact support $U \in (-1, \infty)$ with $0 \in \operatorname{int} U$ and admits a density $\bar{\nu}(x)$ which is continuous outside any neighborhood of zero and satisfies (21). From the martingale property and the boundedness of jumps of Z, it follows immediately that assumption (HY) is satisfied with $A_t = Y_t^2 \int_{\mathbb{R}} z^2 \bar{\nu}(dz)$. For the choice of the integrator X we consider two examples corresponding to the discretization of hedging strategies on one hand and to the discretization of optimal investment policies on the other hand.

EXAMPLE 2 (Discretization of hedging strategies). In this example we assume that the integrand X (the hedging strategy) is a deterministic function of Y, which is indeed the case for classical strategies (quadratic hedging, delta hedging) and European contingent claims in exponential Lévy models; see [6, 19].

PROPOSITION 4. *Let $X_t = \phi(t, Y_t)$ with $\phi(t, y) \in C^{1,2}([0, T) \times \mathbb{R})$ such that for all $\bar{Y} > 0$ and $T^* \in [0, T)$,*

$$\min_{(t,y)\in[0,T^*]\times[-\bar{Y},\bar{Y}]} \frac{\partial \phi(t, y)}{\partial y} > 0.$$

Then, assumptions (HY), (HX) and (HX^{ρ}_{loc}) (for all $\rho \geq 1$) are satisfied with

$$b_t = \frac{\partial \phi}{\partial t}(s, Y_s) + \frac{\partial \phi}{\partial y}(s, Y_s) Y_s \int_{|z|>1} z \bar{\nu}(dz) \quad \textit{and} \quad \lambda_t = \left(Y_t \frac{\partial \phi}{\partial y}(t, Y_t)\right)^{\alpha}.$$

Assume additionally that the function ϕ is such that the integrability conditions of Proposition 3 are satisfied for some $\delta > 0$. Then the strategy given by

$$\underline{a}_t = \overline{a}_t = c\left(\frac{\partial \phi(t, Y_t)}{\partial y}\right)^{\alpha/(2+\alpha-\beta)} Y_t^{(\alpha-2)/(\alpha-\beta+2)}$$

is asymptotically optimal.

PROOF. Applying Itô's formula to $\phi(t, Y_t)$, we get

$$\begin{aligned} X_t = \phi(0, Y_0) &+ \int_0^t b_s \, ds + \int_0^t \int_{|z|\leq 1} \gamma_s(z) \tilde{N}(ds \times dz) \\ &+ \int_0^t \int_{|z|>1} \gamma_s(z) N(ds \times dz) \end{aligned}$$

with $\gamma_t(z) = \phi(t, Y_t(1 + z)) - \phi(t, Y_t)$, which means that we can apply Proposition 1. The local boundedness conditions required by this proposition follow from the local boundedness of Y and the continuity of the derivatives of ϕ. The second statement is a direct corollary of Proposition 3. $\square$

REMARK 6. Using the Cauchy–Schwarz inequality and the fact that Y admits all moments (because Z has bounded jumps), one can show that the following more compact condition implies the integrability conditions of Proposition 3: for some $\delta > 0$,

$$E\Big[\Big(\sup_{x\in U, 0\le t\le T} \phi'_y(t, Y_t(1+x)) + \sup_{0\le t\le T} |\phi'_t(t, Y_t)|\Big)^{2+\delta} + \Big(\inf_{0\le t\le T} \phi'_t(t, Y_t)\Big)^{-\alpha(2+\delta)}\Big] < \infty.$$

This condition can be checked for specific strategies and specific parametric Lévy models using the explicit formulas for the hedging strateigies given in [6, 19], but these computations are out of scope of the present paper.

REMARK 7. When $\beta = 0$ and $\alpha \to 2$, we find that the optimal size of the rebalancing interval is proportional to the square root of $\frac{\partial \phi(t, Y_t)}{\partial Y}$ (the gamma), which is consistent with the results of Fukasawa [15], quoted in the Introduction.

EXAMPLE 3 (Discretization of Merton's portfolio strategy). A widely popular portfolio strategy, which was shown by Merton [21] to be optimal in the context of power utility maximization, is the so called constant proportion strategy, which consists of investing a fixed fraction of one's wealth into the risky asset. Since the price of the risky asset evolves with time, the number of units which corresponds to a given proportion varies, and in practice the strategy must be discretized. Given the importance of this strategy in applications, it is of interest to compute the asymptotically optimal discretization rule in this setting.

Assuming zero interest rate, the value V_t of a portfolio which invests a proportion π of the wealth into the risky asset Y and the rest into the risk-free bank account has the dynamics

$$V_T = V_0 + \int_0^T \pi V_{t-} \frac{dY_t}{Y_{t-}} = V_0 + \int_0^T X_{t-}\, dY_t \qquad \text{with } X_t = \pi \frac{V_t}{Y_t}. \tag{29}$$

The following result provides the asymptotically optimal discretization rule for this integral.

PROPOSITION 5. *Assume that $U \subset (-\frac{1}{\pi}, \infty)$ if $\pi > 1$ and $U \subset (-1, -\frac{1}{\pi})$ if $\pi < 0$. Then the strategy given by*

$$\underline{a}_t = \overline{a}_t = c V_t^{\alpha/(2+\alpha-\beta)} Y_t^{-(2+\alpha)/(2+\alpha-\beta)}$$

is asymptotically optimal for the integral (29).

PROOF. Applying the Itô's formula, we find the dynamics of the integrator X,

$$X_t = X_0 + (\pi - 1)\int_0^t \int_U \frac{X_{s-} z}{1+z} \tilde{N}(ds \times dz) + (1-\pi)\int_0^t \int_U \frac{X_{s-} z^2}{1+z} \nu(dz)\, ds.$$

Hence, X can be written in the form of (20) with

$$\gamma_s(z) = \frac{(\pi-1)X_{s-z}}{1+z} \quad \text{and} \quad \bar{b}_s = (1-\pi)X_s \int_{\mathbb{R}} \left\{ \frac{z^2}{1+z} 1_{|z|\le 1} + z 1_{|z|>1} \right\} \nu(dz).$$

Under the assumption of this proposition, the process X does not change sign, and we can assume without loss of generality that $(\pi-1)X_s$ is always positive (otherwise all the computations can be done for the process $-X$). Since X is a stochastic exponential of a Lévy process with bounded jumps, it is locally bounded, which means that by Proposition 1, X satisfies the assumption (*HX*) with

$$\lambda_t = \gamma_t'(0)^\alpha = \left|(\pi-1)X_{t-}\right|^\alpha$$

and the assumption (HX^ρ_{loc}) for all $\rho \ge 1$. Moreover, since the compensator of the jump measure of X is absolutely continous with respect to the Lebesgue measure (in time), we can take $\lambda_t = |(\pi-1)X_t|^\alpha$. Also, one can choose $\widehat{K}_t = CX_t$ for C sufficiently large in condition (14).

To check the integrability conditions in Proposition 3, observe that the processes A_t, λ_t, $\widehat{K}_t$ and b_t appearing in these conditions, are powers of stochastic exponentials of Lévy processes with bounded jumps. They can therefore be represented as ordinary exponentials of (other) Lévy processes with bounded jumps, but an exponential of a Lévy process with bounded jumps admits all moments, and its maximum on $[0,T]$ also admits all moments; see Theorem 25.18 in [23]. Therefore, the integrability conditions in Proposition 3 follow by using the Cauchy–Schwarz inequality, and the proof is completed by an application of this proposition. □

4. Proof of Theorem 1. *Step* 1. *Reduction to the case of bounded coefficients*. In the proofs of Theorems 1 and 2, we will replace the local boundedness and integrability assumptions of these theorems with the following stronger one:

(H'_ρ) There exists a constant $B > 0$ such that $\frac{1}{B} \le \lambda_t, \underline{a}_t, \overline{a}_t \le B$, $|A_t| + |b_t| + |\widehat{K}_t| \le B$ for $0 \le t \le T$. There exists a Lévy measure $\nu(dx)$ such that, almost surely for all t, the kernel $\mu_t(dz)$ is absolutely continuous with respect to $\lambda_t \nu(dz)$: $\mu_t(dz) = K_t(dz)\lambda_t \nu(dz)$ for a random function $K_t(z) > 0$. Moreover the process (Z_t) defined by

$$Z_t = \mathcal{E}\left(\int_0^\cdot ((K_s(z))^{-1} - 1)(M-\mu)(ds \times dz) \right)_t, \tag{30}$$

is a martingale and satisfies

$$E^Q\Big[\sup_{0\le t\le T} |Z_t|^{-\rho}\Big] < \infty \quad \text{and} \quad E\Big[\sup_{0\le t\le T} Z_t\Big] < \infty,$$

where Q is the probability measure defined by

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_T} := Z_T.$$

Indeed, we have the following lemma.

LEMMA 2. *Assume that* (23) *holds under the assumptions* (HY), (HX) *and* (H_1'). *Then Theorem* 1 *holds*.

PROOF. First, observe that for every n,

$$E\left[\left\{\int_0^{\tau_n}\int_{\mathbb{R}}\left(\left(K_s(z)\right)^{-1}-1\right)^2 M(ds\times dz)\right\}^{1/2}\right]$$
$$\leq E\left[\left\{\int_0^{\tau_n}\int_{|K_s(z)^{-1}-1|\leq 1/2}\left(\left(K_s(z)\right)^{-1}-1\right)^2 M(ds\times dz)\right\}^{1/2}\right]$$
$$+E\left[\left\{\int_0^{\tau_n}\int_{|K_s(z)^{-1}-1|>1/2}\left(\left(K_s(z)\right)^{-1}-1\right)^2 M(ds\times dz)\right\}^{1/2}\right].$$

Using the Cauchy–Schwarz inequality for the first term and the fact that the second integral is a countable sum together with Proposition II.1.28 in [20] for the second term, we see that this last expression is finite since by assumption (HX^1_{loc}),

$$E\left[\int_0^{\tau_n}\int_{|K_s(z)^{-1}-1|\leq 1/2}\left(\left(K_s(z)\right)^{-1}-1\right)^2\mu(ds\times dz)\right]^{1/2}$$
$$+E\left[\int_0^{\tau_n}\int_{|K_s(z)^{-1}-1|>1/2}\left|\left(K_s(z)\right)^{-1}-1\right|\mu(ds\times dz)\right]<\infty.$$

This implies that the process

$$L_t=\int_0^t\int_{\mathbb{R}}\left(\left(K_s(z)\right)^{-1}-1\right)(M-\mu)(ds\times dz)$$

is a local martingale and satisfies $E[[L]^{1/2}_{T\wedge\tau_n}]<\infty$ for every n; see Definition II.1.27 in [20]. The process $Z_t:=\mathcal{E}(L)_t$ is then also well defined, and we take $\sigma_n:=\tau_n\wedge\inf\{t:Z_t\geq n\}$. Then

$$\sup_{0\leq t\leq T} Z_{t\wedge\sigma_n}\leq n+|\Delta Z_{\sigma_n}|1_{\sigma_n\leq T}\leq n+[Z]^{1/2}_{\sigma_n\wedge T}=n+\left(\int_0^{\sigma_n\wedge T}Z^2_{t-}d[L]_t\right)^{1/2}$$
$$\leq n+n[L]^{1/2}_{\sigma_n\wedge T},$$

the last term being integrable. Therefore, we can define a new probability measure Q^n via

$$\frac{dQ^n}{dP}\bigg|_{\mathcal{F}_t}=Z_{t\wedge\sigma_n}.$$

By Girsanov's theorem (Theorem III.5.24 in [20]), M is a random measure with predictable compensator $\mu^{Q^n}:=dt\times\lambda_t\nu(dz)$ under Q^n on $\{t\leq\sigma_n\}$ and

$$Z^{-1}_{t\wedge\sigma_n}=\mathcal{E}\left(\int_0^{\cdot}(K_s(z)-1)\left(M-\mu^{Q^n}\right)(ds\times dz)\right)_{t\wedge\sigma_n}.$$

Therefore, by similar arguments to above, we can find an increasing sequence of stopping times (γ_n) with $\gamma_n \to T$ and such that both

$$E\Big[\sup_{0\le t\le T} Z_{t\wedge\gamma_n}\Big] < \infty \quad \text{and} \quad E^{Q^n}\Big[\sup_{0\le t\le T} Z^{-1}_{t\wedge\gamma_n}\Big] < \infty.$$

Now we define $Y^n_t = Y_{t\wedge\gamma_n}$ and X^n via equation (13) replacing the coefficients λ_t, b_t and $K_t(z)$ with $\lambda^n_t := \lambda_{t\wedge\gamma_n}$, $b^n_t := b_{t\wedge\gamma_n}$ and $K^n_t(z) = K_t(z)1_{t\le\gamma_n} + 1_{t>\gamma_n}$. Moreover, we define $\underline{a}^n_t := \underline{a}_{t\wedge\gamma_n}$, $\overline{a}^n_t := \overline{a}_{t\wedge\gamma_n}$. The stopping times $T^{\varepsilon,n}_i$ and $\eta^n(t)$ are defined similarly. Note that $A^n_t := A_t 1_{t\le\gamma_n}$ satisfies $\int_0^t A^n_s\,ds = \langle Y^n\rangle_t$, that X^n coincides with X on the interval $[0,\gamma_n]$ and that the new coefficients satisfy assumption (H_1'). Consequently,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\varepsilon^{-2}E\bigg[\int_0^{\gamma_n}(X_t - X_{\eta(t)})^2 A_t\,dt\bigg] &= \lim_{\varepsilon\downarrow 0}\varepsilon^{-2}E\bigg[\bigg(\int_0^T \big(X^n_t - X^n_{\eta^n(t)}\big)^2\,dY^n_t\bigg)^2\bigg]\\ &= E\bigg[\int_0^T A^n_t\frac{f(\underline{a}^n_t,\overline{a}^n_t)}{g(\underline{a}^n_t,\overline{a}^n_t)}\,dt\bigg]\\ &= E\bigg[\int_0^{\gamma_n} A_t\frac{f(\underline{a}_t,\overline{a}_t)}{g(\underline{a}_t,\overline{a}_t)}\,dt\bigg],\end{aligned}$$

which implies, by assumption (HA), that

$$E\bigg[\int_0^{\gamma_n} A_t\frac{f(\underline{a}_t,\overline{a}_t)}{g(\underline{a}_t,\overline{a}_t)}\,dt\bigg] \le E\bigg[\sup_{0\le s\le T}\max(\underline{a}_s,\overline{a}_s)^2\int_0^T A_t\,dt\bigg] < +\infty,$$

and so by Fatou's lemma,

$$E\bigg[\int_0^{T} A_t\frac{f(\underline{a}_t,\overline{a}_t)}{g(\underline{a}_t,\overline{a}_t)}\,dt\bigg] \le E\bigg[\sup_{0\le s\le T}\max(\underline{a}_s,\overline{a}_s)^2\int_0^T A_t\,dt\bigg] < +\infty.$$

Therefore, by dominated convergence

$$\lim_n E\bigg[\int_{\gamma_n}^{T} A_t\frac{f(\underline{a}_t,\overline{a}_t)}{g(\underline{a}_t,\overline{a}_t)}\,dt\bigg] = 0.$$

On the other hand,

$$\varepsilon^{-2}E\int_{\gamma_n}^T (X_t - X_{\eta(t)})^2 A_t\,dt \le E\bigg[\sup_{0\le s\le T}\max(\underline{a}_s,\overline{a}_s)^2\int_{\gamma_n}^T A_t\,dt\bigg].$$

The right-hand side does not depend on ε and converges to zero as $n\to\infty$ by the dominated convergence theorem. Therefore, the left-hand side can be made arbitrarily small independently of ε, and the result follows. □

Step 2. *Change of probability measure*. We first prove the following important lemma.

LEMMA 3. *Under the assumption H_1', almost surely,*

$$\lim_{\varepsilon\to 0}\sup_{i:T_i^\varepsilon\le T}\big(T_{i+1}^\varepsilon - T_i^\varepsilon\big)=0.$$

PROOF. In this proof, let us fix $\omega\in\Omega$. By way of contradiction, assume that there exists a constant $C>0$ and a sequence $\{\varepsilon_n\}_{n\ge 0}$ converging to zero such that for every n, there exists $i(n)$ with $T_{i(n)+1}^{\varepsilon_n}-T_{i(n)}^{\varepsilon_n}>C$. From the sequences $\{T_{i(n)+1}^{\varepsilon_n}\}_n$ and $\{T_{i(n)}^{\varepsilon_n}\}_n$ we can extract two subsequences $\{T_{i(\phi(n))+1}^{\varepsilon_{\phi(n)}}\}_n$ and $\{T_{i(\phi(n))}^{\varepsilon_{\phi(n)}}\}_n$ converging to some limiting values $T_1<T_2$. For n big enough, there exists a nonempty interval $\mathcal{I}$ which is a subset of both (T_1,T_2) and $(T_{i(\phi(n))+1}^{\varepsilon_{\phi(n)}},T_{i(\phi(n))}^{\varepsilon_{\phi(n)}})$. Now using that $\sup_{t,s\in(T_{i(\phi(n))+1}^{\varepsilon_{\phi(n)}},T_{i(\phi(n))}^{\varepsilon_{\phi(n)}})}|X_t-X_s|\le 2B\varepsilon_{\phi(n)}$, we obtain that $\sup_{s,t\in\mathcal{I}}|X_t-X_s|=0$, which cannot hold since X is an infinite activity process. $\square$

Let $\Delta T_{i+1}=T_{i+1}\wedge T-T_i\wedge T$. The goal of this step is to show that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\varepsilon^{-2}E\left[\int_0^T(X_t-X_{\eta(t)})^2A_t\,dt\right]\\
&\quad=\lim_{\varepsilon\downarrow 0}E^Q\left[\sum_{i=1}^\infty Z_{T_i\wedge T}^{-1}A_{T_i\wedge T}\varepsilon^{-2}\int_{T_i\wedge T}^{T_{i+1}\wedge T}(X_t-X_{T_i})^2\,dt\right].
\end{aligned}\tag{31}$$

We have

$$\begin{aligned}
&\varepsilon^{-2}E\left[\int_0^T(X_t-X_{\eta(t)})^2A_t\,dt\right]\\
&\quad=\varepsilon^{-2}\sum_{i=0}^{+\infty}E\left[\int_{T_i\wedge T}^{T_{i+1}\wedge T}(X_t-X_{T_i})^2(A_t-A_{T_i})\,dt\right]\\
&\qquad+\varepsilon^{-2}\sum_{i=0}^{+\infty}E^Q\left[Z_{T_{i+1}\wedge T}^{-1}A_{T_i}\int_{T_i\wedge T}^{T_{i+1}\wedge T}(X_t-X_{T_i})^2\,dt\right].
\end{aligned}$$

Since for $t\in[T_i,T_{i+1})$, $(X_t-X_{T_i})^2\le B^2\varepsilon^2$, using the boundedness of A, (31) will follow, provided we show that

$$\lim_{\varepsilon\downarrow 0}\sum_{i=0}^{+\infty}E\left[\int_{T_i\wedge T}^{T_{i+1}\wedge T}|A_t-A_{T_i}|\,dt\right]=0\tag{32}$$

and

$$\lim_{\varepsilon\downarrow 0}\sum_{i=0}^{+\infty}E^Q\big[|Z_{T_{i+1}\wedge T}^{-1}-Z_{T_i\wedge T}^{-1}|\Delta T_{i+1}\big]=0.\tag{33}$$

Limit (32) follows from the dominated convergence theorem (A is bounded by assumption (H_1') and $A_{\eta(t)} \to A_t$ almost everywhere on $[0, T]$ since A is càdlàg and by Lemma 3). Using the fact that Z^{-1} has finite quadratic variation together with Lemma 3 and the Cauchy–Schwarz inequality, we get that, in probability,

$$\lim_{\varepsilon \downarrow 0} \sum_{i=0}^{+\infty} |Z^{-1}_{T_{i+1}\wedge T} - Z^{-1}_{T_i \wedge T}| \Delta T_{i+1} = 0.$$

Then (33) follows from the integrability of $\sup_{t\in[0,T]} Z_t^{-1}$, which is part of assumption (H_1').

Step 3. First, observe that by the dominated convergence theorem, since $\sup_i \Delta T_i$ tends to zero, (31) is equal to

$$S_1 := \lim_{\varepsilon \downarrow 0} S_1^{\varepsilon}$$

$$\text{with } S_1^{\varepsilon} := E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \le T} A_{T_i} Z_{T_i}^{-1} \varepsilon^{-2} E^Q_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} (X_t - X_{T_i})^2 \, dt\right]\right].$$

For this expression to be well defined we extend the processes λ, b, $\underline{a}$, $\overline{a}$ by arbitrary constant values beyond T and define the process X for $t \ge T$ accordingly.

Define a family of continuous increasing processes $(\Lambda_s(t))_{t\ge 0}$ indexed by $s \ge 0$ by $\Lambda_s(t) = \int_s^{s+t} \lambda_r \, dr$, the family of filtrations $\mathcal{G}_t^i = \mathcal{F}_{T_i + t}$ and of processes $(\tilde{X}_t^i)_{t\ge 0}$ and $(\hat{X}_t^i)_{t\ge 0}$ by

$$\hat{X}_t^i = X_{T_i + \Lambda_{T_i}^{-1}(t)} - X_{T_i} - \int_{T_i}^{T_i + \Lambda_{T_i}^{-1}(t)} b_s \, ds, \qquad \tilde{X}_t^i = X_{T_i + \Lambda_{T_i}^{-1}(t)} - X_{T_i}.$$

The process $(\hat{X}_t^i)_{t\ge 0}$ is a (G_t^i)-semimartingale with (deterministic) characteristics $(0, \nu, 0)$ under Q, and therefore, it is a (G_t^i)-Lévy process under Q (Theorem II.4.15 in [20]).

Let $\tilde{\tau}_i = \inf\{t \ge 0 : \tilde{X}_t^i \notin [-\underline{a}_{T_i}\varepsilon, \overline{a}_{T_i}\varepsilon]\}$. Using a change of variable formula we obtain that

$$\int_{T_i}^{T_{i+1}} (X_t - X_{T_i})^2 \, dt = \int_0^{\tilde{\tau}_i} \frac{(\tilde{X}_s^i)^2}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))} \, ds.$$

Using the càdlàg property of λ together with the various boundedness assumptions and the integrability of $\sup_{0\le t\le T} Z_t^{-1}$, we easily get that

$$S_1 = \lim_{\varepsilon \downarrow 0} E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \le T} \frac{A_{T_i} Z_{T_i}^{-1}}{\lambda_{T_i}} \varepsilon^{-2} E^Q_{\mathcal{F}_{T_i}}\left[\int_0^{\tilde{\tau}_i} (\tilde{X}_t^i)^2 \, dt\right]\right].$$

Then we obviously have that

$$S_1 = \lim_{\varepsilon \downarrow 0} E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \le T} \frac{A_{T_i} Z_{T_i}^{-1}}{\lambda_{T_i}} \frac{T_{i+1} - T_i}{E^Q_{\mathcal{F}_{T_i}}[T_{i+1} - T_i]} \varepsilon^{-2} E^Q_{\mathcal{F}_{T_i}}\left[\int_0^{\tilde{\tau}_i} (\tilde{X}_t^i)^2 \, dt\right]\right].$$

Now note that

$$T_{i+1} - T_i = \int_0^{\tilde{\tau}_i} \frac{ds}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))}. \tag{34}$$

Then

$$\begin{aligned}
&E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \le T} \frac{A_{T_i} Z_{T_i}^{-1}}{\lambda_{T_i}} \frac{T_{i+1} - T_i}{E^Q_{\mathcal{F}_{T_i}}[T_{i+1} - T_i]} \varepsilon^{-2} E^Q_{\mathcal{F}_{T_i}}\left[\int_0^{\tilde{\tau}_i} (\tilde{X}^i_t)^2\, dt\right]\right] \\
&\quad = E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \le T} A_{T_i} Z_{T_i}^{-1} \frac{T_{i+1} - T_i}{E^Q_{\mathcal{F}_{T_i}}[\tilde{\tau}_i]} \varepsilon^{-2} E^Q_{\mathcal{F}_{T_i}}\left[\int_0^{\tilde{\tau}_i} (\tilde{X}^i_t)^2\, dt\right]\right] + R^\varepsilon
\end{aligned}$$

with

$$\begin{aligned}
|R^\varepsilon| \le C E^Q\Bigg[&\sum_{i=0}^{\infty} 1_{T_i \le T} Z_{T_i}^{-1} (T_{i+1} - T_i) \\
&\times \left|\frac{\lambda_{T_i}^{-1} E_{\mathcal{F}_{T_i}}[\tilde{\tau}_i] - E_{\mathcal{F}_{T_i}}[\int_0^{\tilde{\tau}_i} ds/(\lambda(T_i + \Lambda_{T_i}^{-1}(s)))]}{E_{\mathcal{F}_{T_i}}[\int_0^{\tilde{\tau}_i} ds/(\lambda(T_i + \Lambda_{T_i}^{-1}(s)))]}\right|\Bigg].
\end{aligned}$$

Using (34), we obtain that

$$\begin{aligned}
|R^\varepsilon| &\le C E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \le T} Z_{T_i}^{-1} \left|\lambda_{T_i}^{-1} E_{\mathcal{F}_{T_i}}[\tilde{\tau}_i] - E_{\mathcal{F}_{T_i}}\left[\int_0^{\tilde{\tau}_i} \frac{ds}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))}\right]\right|\right] \\
&\le C E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \le T} Z_{T_i}^{-1} \int_0^{\tilde{\tau}_i} \left|\frac{1}{\lambda_{T_i}} - \frac{1}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))}\right| ds\right] \\
&\le C E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \le T} Z_{T_i}^{-1} \int_{T_i}^{T_{i+1}} \left|\frac{1}{\lambda_{T_i}} - \frac{1}{\lambda(s)}\right| ds\right],
\end{aligned}$$

which is easily shown to converge to zero. Consequently, we conclude that

$$S_1 = \lim_{\varepsilon \downarrow 0} E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \le T} A_{T_i} Z_{T_i}^{-1} \frac{T_{i+1} - T_i}{E^Q_{\mathcal{F}_{T_i}}[\tilde{\tau}_i]} \varepsilon^{-2} E^Q_{\mathcal{F}_{T_i}}\left[\int_0^{\tilde{\tau}_i} (\tilde{X}^i_t)^2\, dt\right]\right]. \tag{35}$$

Step 4. *Comparison of hitting times and associated integrals*. We start with the following lemma:

LEMMA 4. *Let* $\kappa \in \mathbb{R}_+$ *and* $n \in \mathbb{N}$. *Then*

$$\underline{f}_\varepsilon^{\kappa,n}(\underline{a}_{T_i}, \overline{a}_{T_i}) \le E^Q_{\mathcal{F}_{T_i}}\left[\left(\int_0^{\tilde{\tau}_i} |\hat{X}^i_t|^\kappa\, dt\right)^n\right] \le \overline{f}_\varepsilon^{\kappa,n}(\underline{a}_{T_i}, \overline{a}_{T_i})$$

whenever the expression in the middle is well defined, where $\underline{f}_\varepsilon$ *and* $\overline{f}_\varepsilon$ *are deterministic functions defined by*

$$\underline{f}_\varepsilon^{\kappa,n}(a,b) = E^Q\left[\left(\int_0^{\hat{\tau}_1} |\hat{X}_t|^\kappa\,dt\right)^n\right] \quad \textit{and}$$

$$\overline{f}_\varepsilon^{\kappa,n}(a,b) = E^Q\left[\left(\int_0^{\hat{\tau}_2\wedge\hat{\tau}^j} |\hat{X}_t|^\kappa\,dt\right)^n\right],$$

with $\hat{X}_t = \hat{X}_t^0$ *and*

$$\hat{\tau}_1 = \inf\{t : \hat{X}_t \le -a\varepsilon + tB^2 \text{ or } \hat{X}_t \ge b\varepsilon - tB^2\},$$

$$\hat{\tau}_2 = \inf\{t : \hat{X}_t \le -a\varepsilon - tB^2 \text{ or } \hat{X}_t \ge b\varepsilon + tB^2\},$$

$$\hat{\tau}^j = \inf\{t : |\Delta\hat{X}_t| \ge \varepsilon(a+b)\}.$$

The proof follows from the fact that $|\tilde{X}_t^i - \hat{X}_t^i| \le tB^2$ and that $\hat{X}$ is a $\mathcal{G}_t^i$-Lévy process under Q, and that a jump of size greater than $\varepsilon(a+b)$ immediately takes the process $\tilde{X}^i$ out of the interval.

LEMMA 5.

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-(\kappa+\alpha)n}\underline{f}_\varepsilon^{\kappa,n}(a,b) = \lim_{\varepsilon\downarrow 0} \varepsilon^{-(\kappa+\alpha)n}\overline{f}_\varepsilon^{\kappa,n}(a,b) = f^{*,\kappa,n}(a,b) \tag{36}$$

uniformly on $(a,b) \in [a_1,a_2]\times[b_1,b_2]$ *for all* $0 < a_1 \le a_2 < \infty$ *and* $0 < b_1 \le b_2 < \infty$, *with*

$$f^{*,\kappa,n}(a,b) = E\left[\left(\int_0^{\tau^*} |X_t^*|^\kappa\,dt\right)^n\right],$$

where X^* *is a strictly* α*-stable process with Lévy density*

$$\nu^*(x) = \frac{c_+ 1_{x>0} + c_- 1_{x<0}}{|x|^{1+\alpha}}$$

and $\tau^* = \inf\{t \ge 0 : X_t^* \notin (-a,b)\}$.

PROOF. For $\varepsilon > 0$, let us define $X_t^\varepsilon = \varepsilon^{-1}\hat{X}_{\varepsilon^\alpha t}$, $X_t^{\varepsilon,1} = X_t^\varepsilon - tB^2\varepsilon^{\alpha-1}$, $X_t^{\varepsilon,2} = X_t^\varepsilon + tB^2\varepsilon^{\alpha-1}$ and

$$\tau_1^{\varepsilon,1} = \inf\{t, X_t^{\varepsilon,1} \le -a\}, \qquad \tau_1^{\varepsilon,2} = \inf\{t, X_t^{\varepsilon,2} \ge b\},$$

$$\tau_2^{\varepsilon,1} = \inf\{t, X_t^{\varepsilon,2} \le -a\}, \qquad \tau_2^{\varepsilon,2} = \inf\{t, X_t^{\varepsilon,1} \ge b\}$$

$$\tau_3^{\varepsilon,1} = \inf\{t, X_t^{\varepsilon} \le -a\}, \qquad \tau_3^{\varepsilon,2} = \inf\{t, X_t^{\varepsilon} \ge b\}.$$

We write $\tau_i^\varepsilon = \tau_i^{\varepsilon,1} \wedge \tau_i^{\varepsilon,2}$ for $i = 1,2,3$. Similarly, we define $\tau^{j,\varepsilon} := \inf\{t : |\Delta X_t^\varepsilon| \geq (a+b)\}$. Observe that by a change of variable in the integral,

$$\varepsilon^{-(\kappa+\alpha)n} \underline{f}_\varepsilon^{\kappa,n}(a,b) = E^Q\biggl[\biggl(\int_0^{\tau_1^\varepsilon} |X_t^\varepsilon|^\kappa \, dt\biggr)^n\biggr],$$

$$\varepsilon^{-(\kappa+\alpha)n} \overline{f}_\varepsilon^{\kappa,n}(a,b) = E^Q\biggl[\biggl(\int_0^{\tau_2^\varepsilon \wedge \tau^{j,\varepsilon}} |X_t^\varepsilon|^\kappa \, dt\biggr)^n\biggr].$$

From Lemma 11, we have that X_t^ε converges to X_t^* in Skorohod topology. From Skorohod representation theorem, there exists some probability space on which are defined a process Y^* and a family of processes Y^ε such that Y^ε and X^ε have the same law, Y^* and X^* have the same law and Y^ε converges to Y^* almost surely, for the Skorohod topology.

This implies that $Y^{\varepsilon,1}$ and $Y^{\varepsilon,2}$ also converge to Y^* almost surely, where $Y_t^{\varepsilon,1} = Y_t^\varepsilon - tB^2\varepsilon^{\alpha-1}$ and $Y_t^{\varepsilon,2} = Y_t^\varepsilon + tB^2\varepsilon^{\alpha-1}$. Now using that the application which to a function f in the Skorohod space associates its first hitting time of a constant barrier is continuous at almost all f which are sample paths of strictly stable processes (see Proposition VI.2.11 in [20] and its use in [22]), we obtain that σ_i^ε converges almost surely to σ^* for $i = 1,2,3$, where σ_i^ε and σ^* are defined through $Y^{\varepsilon,1}$, $Y^{\varepsilon,2}$, Y^* in the same way as τ_i^ε and τ^* through $X^{\varepsilon,1}$, $X^{\varepsilon,2}$, X^*. Moreover, since $\sigma_3^\varepsilon \leq \sigma^{j,\varepsilon}$ for all ε, we also have that $\sigma_2^\varepsilon \wedge \sigma^{j,\varepsilon} \to \sigma^*$ almost surely.

Now remark that, almost surely, Y_t^ε converges almost everywhere in t to Y_t^*; see Proposition VI.2.3 in [20]. Therefore, since $|Y_t^\varepsilon| 1_{t \leq \sigma_1^\varepsilon} \leq \max(a,b)$ and $|Y_t^\varepsilon| 1_{t \leq \sigma^{j,\varepsilon} \wedge \sigma_2^\varepsilon} \leq \max(a,b) + B^2 t$, using the dominated convergence theorem, we obtain that almost surely

$$\biggl(\int_0^{\sigma_1^\varepsilon} |Y_t^\varepsilon|^\kappa \, dt\biggr)^n \to \biggl(\int_0^{\sigma^*} |Y_t^*|^\kappa \, dt\biggr)^n \quad \text{and}$$

$$\biggl(\int_0^{\sigma_2^\varepsilon \wedge \sigma^{j,\varepsilon}} |Y_t^\varepsilon|^\kappa \, dt\biggr)^n \to \biggl(\int_0^{\sigma^*} |Y_t^*|^\kappa \, dt\biggr)^n.$$

Finally, we deduce that

$$\biggl(\int_0^{\tau_1^\varepsilon} |X_t^\varepsilon|^\kappa \, dt\biggr)^n \to \biggl(\int_0^{\tau^*} |X_t^*|^\kappa \, dt\biggr)^n \quad \text{and}$$

$$\biggl(\int_0^{\tau_2^\varepsilon \wedge \tau^{j,\varepsilon}} |X_t^\varepsilon|^\kappa \, dt\biggr)^n \to \biggl(\int_0^{\tau^*} |X_t^*|^\kappa \, dt\biggr)^n,$$

in law.

Now note that $\tau^{j,\varepsilon}$ is the first jump time of a Lévy process with characteristic triplet given by $(0, \varepsilon^\alpha \nu|_{(-(a+b)\varepsilon,(a+b)\varepsilon)^c}, 0)$. Using that this process is a compound Poisson process, we get

$$P[\tau^{j,\varepsilon} > T] \leq \exp\{-T\varepsilon^\alpha \nu((-\infty, -(a+b)\varepsilon] \cup [(a+b)\varepsilon, \infty))\},$$

which, by property (19), implies that the family $(\tau^{j,\varepsilon})_{\varepsilon>0}$ has uniformly bounded exponential moment. This implies that the families

$$\left(\int_0^{\tau_2^\varepsilon\wedge\tau^{j,\varepsilon}} |X_t^\varepsilon|^\kappa\,dt\right)^n \quad\text{and}\quad \left(\int_0^{\tau_1^\varepsilon} |X_t^\varepsilon|^\kappa\,dt\right)^n = \left(\int_0^{\tau_1^\varepsilon\wedge\tau^{j,\varepsilon}} |X_t^\varepsilon|^\kappa\,dt\right)^n,$$

parameterized by ε, are uniformly integrable, and therefore the proof of the convergence in (36) is complete.

It remains to show that the convergence in (36) is uniform in (a,b) over compact sets excluding zero. To do this, first observe that $f^{*,\kappa,n}(a,b)$ is continuous in (a,b) on compact sets excluding zero (this is shown using essentially the same arguments as above: continuity of the exit times for stable processes plus uniform integrability). Second, since both $\underline{f}_\varepsilon^{\kappa,n}$ and $\overline{f}_\varepsilon^{\kappa,n}$ are increasing in a and b, a multidimensional version of Dini's theorem can be used to conclude that the convergence is indeed uniform. $\square$

Step 5. First, let us show that

$$S_1 = \lim_{\varepsilon\downarrow 0} E^Q\left[\sum_{i=0}^\infty 1_{T_i\le T} A_{T_i} Z_{T_i}^{-1} \frac{T_{i+1}-T_i}{E^Q_{\mathcal{F}_{T_i}}[\tilde\tau_i]} \varepsilon^{-2} E^Q_{\mathcal{F}_{T_i}}\left[\int_0^{\tilde\tau_i} \hat X_t^2\,dt\right]\right].$$

Indeed, the absolute value of the difference between the expressions under the limit here and in (35) is bounded from above by

$$E^Q\left[\sum_{i=0}^\infty 1_{T_i\le T} A_{T_i} Z_{T_i}^{-1} \frac{T_{i+1}-T_i}{E^Q_{\mathcal{F}_{T_i}}[\tilde\tau_i]} \varepsilon^{-2} E^Q_{\mathcal{F}_{T_i}}\left[\int_0^{\tilde\tau_i} |(\tilde X_t-\hat X_t)(\tilde X_t+\hat X_t)|\,dt\right]\right]$$

$$\le C E^Q\left[\sum_{i=0}^\infty 1_{T_i\le T} Z_{T_i}^{-1} \frac{T_{i+1}-T_i}{E^Q_{\mathcal{F}_{T_i}}[\tilde\tau_i]} \varepsilon^{-2} E^Q_{\mathcal{F}_{T_i}}[\tilde\tau_i^3+\tilde\tau_i^2\varepsilon]\right] \tag{37}$$

$$\le C E^Q\left[\sum_{i=0}^\infty 1_{T_i\le T} Z_{T_i}^{-1}(T_{i+1}-T_i) \frac{\varepsilon^{-2}\overline{f}_\varepsilon^{0,3}(\underline{a}_{T_i},\overline{a}_{T_i}) + \varepsilon^{-1}\overline{f}_\varepsilon^{0,2}(\underline{a}_{T_i},\overline{a}_{T_i})}{\underline{f}_\varepsilon^{0,1}(\underline{a}_{T_i},\overline{a}_{T_i})}\right],$$

where C is a constant which does not depend on ε. Using Lemma 5 and the fact that $\alpha>1$, we get

$$\sup_{1/B\le a,b\le B} \frac{\varepsilon^{-2}\overline{f}_\varepsilon^{0,3}(a,b)+\varepsilon^{-1}\overline{f}_\varepsilon^{0,2}(a,b)}{\underline{f}_\varepsilon^{0,1}(a,b)} \to 0 \qquad \text{as } \varepsilon\to 0.$$

This, together with the fact that $E^Q[\sup_{0\le t\le T} Z_t^{-1}]<\infty$, enables us to apply the dominated convergence theorem and conclude that (37) goes to zero.

Finally, we have that

$$S_1 \le \limsup_{\varepsilon\downarrow 0} E^Q\left[\sum_{i=0}^\infty 1_{T_i\le T} A_{T_i} Z_{T_i}^{-1}(T_{i+1}-T_i) \frac{\varepsilon^{-2-\alpha}\overline{f}_\varepsilon^{2,1}(\underline{a}_{T_i},\overline{a}_{T_i})}{\varepsilon^{-\alpha}\underline{f}_\varepsilon^{0,1}(\underline{a}_{T_i},\overline{a}_{T_i})}\right],$$

$$S_1 \geq \limsup_{\varepsilon \downarrow 0} E^Q\left[\sum_{i=0}^{\infty} 1_{T_i \leq T} A_{T_i} Z_{T_i}^{-1}(T_{i+1}-T_i) \frac{\varepsilon^{-2-\alpha} \underline{f}_\varepsilon^{2,1}(\underline{a}_{T_i}, \overline{a}_{T_i})}{\varepsilon^{-\alpha} \overline{f}_\varepsilon^{0,1}(\underline{a}_{T_i}, \overline{a}_{T_i})}\right].$$

Using for $(\kappa, n) = (0, 1)$ and $(\kappa, n) = (2, 1)$ the uniform convergence on $[1/B, B]$ of $\varepsilon^{-(\kappa+\alpha)n} \underline{f}_\varepsilon^{\kappa,n}$ and $\varepsilon^{-(\kappa+\alpha)n} \overline{f}_\varepsilon^{\kappa,n}$ toward $f^{*,\kappa,n}$ which is continuous, together with a Riemann-sum type argument and the dominated convergence theorem, we obtain that

$$S_1 = E^Q\left[\int_0^T A_t Z_t^{-1} \frac{f^{*,2,1}(\underline{a}_t, \overline{a}_t)}{f^{*,0,1}(\underline{a}_t, \overline{a}_t)}\, dt\right] = E\left[\int_0^T A_t \frac{f^{*,2,1}(\underline{a}_t, \overline{a}_t)}{f^{*,0,1}(\underline{a}_t, \overline{a}_t)}\, dt\right].$$

5. Preliminaries for the proof of Theorem 2. In this section, we prove some technical lemmas concerning the uniform integrability of the hitting time counts and the overshoots, which are needed for the proof of Theorem 2.

LEMMA 6. *Under the assumption* (*HX*), *for all* $\beta \in [0, \alpha)$ *and* $\varepsilon > 0$,

$$\begin{aligned} &E_{\mathcal{F}_{T_i}}[|X_{T_{i+1}} - X_{T_i}|^\beta] \\ &\quad \leq c\varepsilon^{\beta-1} \max\{\underline{a}_{T_i}^{\beta-1}, \overline{a}_{T_i}^{\beta-1}\} E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} |b_s|\, ds\right] \\ &\qquad + c\varepsilon^{\beta-\alpha} \max\{\underline{a}_{T_i}, \overline{a}_{T_i}\}^{\beta\vee(2-\alpha)} \\ &\qquad \times \min\{\underline{a}_{T_i}, \overline{a}_{T_i}\}^{(\beta-2)\wedge(-\alpha)} E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \widehat{K}_s\, ds\right], \end{aligned} \tag{38}$$

provided that the right-hand side has finite expectation.

COROLLARY 1. *Under the assumption* (*HX*), *for all* $\varepsilon > 0$,

$$\begin{aligned} \varepsilon^\alpha &\leq c\varepsilon^{\alpha-1} \min\{\underline{a}_{T_i}, \overline{a}_{T_i}\}^{-1} E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} |b_s|\, ds\right] \\ &\quad + c \min\{\underline{a}_{T_i}, \overline{a}_{T_i}\}^{-\alpha} E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \widehat{K}_s\, ds\right], \end{aligned}$$

provided that the right-hand side has finite expectation.

PROOF. Apply Lemma 6 with $\beta' = 0$, $\underline{a}'_{T_i} = \overline{a}'_{T_i} = \min\{\underline{a}_{T_i}, \overline{a}_{T_i}\}$; then multiply both sides of (38) by ε^α and use the fact that the hitting time of the new barrier is smaller than T_{i+1}. $\square$

PROOF OF LEMMA 6. First of all, from (14) we easily deduce by integration by parts that

$$\begin{aligned} &\int_{x<|z|\leq 1} |z|\mu(dt \times dz) < C\widehat{K}_t x^{1-\alpha} \quad \text{and} \\ &\int_{|z|\leq x} z^2 \mu(dt \times dz) < C\widehat{K}_t x^{2-\alpha} \end{aligned} \tag{39}$$

for all $x > 0$, for some constant $C < \infty$.

For this proof, let

$$f(x) := x^2 1_{0 \le x \le 2\overline{a}_{T_i}\varepsilon} (2\overline{a}_{T_i}\varepsilon)^{\beta-2} + |x|^\beta 1_{x > 2\overline{a}_{T_i}\varepsilon} + x^2 1_{-2\underline{a}_{T_i}\varepsilon \le x \le 0} (2\underline{a}_{T_i}\varepsilon)^{\beta-2} + |x|^\beta 1_{x < -2\underline{a}_{T_i}\varepsilon}.$$

By Itô's formula,

$$\begin{aligned}
2^{\beta-2} E_{\mathcal{F}_{T_i}}&[|X_{T_{i+1}} - X_{T_i}|^\beta] \\
&\le E_{\mathcal{F}_{T_i}}[f(X_{T_{i+1}} - X_{T_i})] \\
&= E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} f'(X_s - X_{T_i}) b_s \, ds\right] \\
&\quad + E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} \{f(X_s + z - X_{T_i}) - f(X_s - X_{T_i}) \right. \\
&\qquad\qquad \left. - f'(X_s - X_{T_i}) z 1_{|z| \le 1}\} \mu(ds \times dz)\right] \\
&\quad + E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} \{f(X_{s-} + z - X_{T_i}) \right. \\
&\qquad\qquad \left. - f(X_s - X_{T_i})\} (M - \mu)(ds \times dz)\right].
\end{aligned} \tag{40}$$

The first term in the right-hand side satisfies

$$\begin{aligned}
E_{\mathcal{F}_{T_i}}&\left[\int_{T_i}^{T_{i+1}} f'(X_s - X_{T_i}) b_s \, ds\right] \\
&\le (2\varepsilon)^{\beta-1} \max\{\underline{a}_{T_i}^{\beta-1}, \overline{a}_{T_i}^{\beta-1}\} E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} |b_s| \, ds\right].
\end{aligned}$$

For the second term, we denote $A_s := \{z : X_s + z - X_{T_i} \in (-2\underline{a}_{T_i}\varepsilon, 2\overline{a}_{T_i}\varepsilon)\}$ and decompose it into two terms,

$$\begin{aligned}
E_{\mathcal{F}_{T_i}}&\left[\int_{T_i}^{T_{i+1}} \int_{A_s^c} \{f(X_s + z - X_{T_i}) \right. \\
&\qquad \left. - f(X_s - X_{T_i}) - f'(X_s - X_{T_i}) z 1_{|z| \le 1}\} \mu(ds \times dz)\right] \\
&\le C E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{(-\underline{a}_{T_i}\varepsilon, \overline{a}_{T_i}\varepsilon)^c} \{|z|^\beta + \varepsilon^{\beta-1} \max\{\underline{a}_{T_i}^{\beta-1}, \overline{a}_{T_i}^{\beta-1}\} \right. \\
&\qquad \left. \times |z| 1_{|z| \le 1}\} \mu(ds \times dz)\right]
\end{aligned}$$

$$\leq C\varepsilon^{\beta-\alpha}\{\max\{\underline{a}_{T_i}^{\beta-\alpha}, \overline{a}_{T_i}^{\beta-\alpha}\} + \max\{\underline{a}_{T_i}^{\beta-1}, \overline{a}_{T_i}^{\beta-1}\} \max\{\underline{a}_{T_i}^{1-\alpha}, \overline{a}_{T_i}^{1-\alpha}\}\}$$
$$\times E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \widehat{K}_s\, ds\right]$$

and

$$E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{A_s} \{f(X_s + z - X_{T_i}) - f(X_s - X_{T_i}) \right.$$
$$\left. - f'(X_s - X_{T_i}) z 1_{|z|\leq 1}\}\mu(ds \times dz)\right],$$

which is smaller than

$$E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{A_s} \left\{\int_0^z f''(X_s - X_{T_i} + x)(z-x)\, dx\right\}\mu(ds \times dz)\right]$$
$$- E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{A_s} \{f'(X_s - X_{T_i}) z 1_{|z|>1}\}\mu(ds \times dz)\right]$$
$$\leq C\varepsilon^{\beta-2} \max\{\underline{a}_{T_i}^{\beta-2}, \overline{a}_{T_i}^{\beta-2}\} E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{-3\underline{a}_{T_i}\varepsilon}^{3\overline{a}_{T_i}\varepsilon} z^2 \mu(ds \times dz)\right]$$
$$+ C\varepsilon^{\beta-1} \max\{\underline{a}_{T_i}^{\beta-1}, \overline{a}_{T_i}^{\beta-1}\} E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{-3\underline{a}_{T_i}\varepsilon}^{3\overline{a}_{T_i}\varepsilon} z^2 \mu(ds \times dz)\right]$$
$$\leq C\varepsilon^{\beta-\alpha}(\max\{\underline{a}_{T_i}^{\beta-2}, \overline{a}_{T_i}^{\beta-2}\} + \varepsilon \max\{\underline{a}_{T_i}^{\beta-1}, \overline{a}_{T_i}^{\beta-1}\})$$
$$\times \max\{\underline{a}_{T_i}^{2-\alpha}, \overline{a}_{T_i}^{2-\alpha}\}$$
$$\times E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \widehat{K}_s\, ds\right],$$

where we used (39) in the last inequality. Assembling the terms and doing some simple estimations yields the statement of the lemma, provided we can show that the third term on the right-hand side of (40) is equal to zero. Splitting it, once again, in two parts, we then get

$$E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{A_s^c} |f(X_{s-} + z - X_{T_i}) - f(X_s - X_{T_i})| \mu(ds \times dz)\right]$$
$$\leq C E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \int_{(-\infty, -\underline{a}_{T_i}\varepsilon)\cup(\overline{a}_{T_i}\varepsilon, \infty)} |z|^\beta \mu(ds \times dz)\right]$$
$$\leq C \max\{\underline{a}_{T_i}^{\beta-\alpha}, \overline{a}_{T_i}^{\beta-\alpha}\} E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}} \widehat{K}_s\, ds\right],$$

and for the other term, using the "isometry property" of the stochastic integral with respect to the random measure together with (39), we obtain

$$E_{\mathcal{F}_{T_i}}\left[\left(\int_{T_i}^{T_{i+1}}\int_{A_s}\frac{f(X_{s-}+z-X_{T_i})-f(X_s-X_{T_i})}{\max\{\underline{a}_{T_i}^{\beta-2},\overline{a}_{T_i}^{\beta-2}\}}(M-\mu)(ds\times dz)\right)^2\right]$$
$$\le C\varepsilon^{2\beta-4}E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}}\int_{-3\underline{a}_{T_i}\varepsilon}^{3\overline{a}_{T_i}\varepsilon}z^2\mu(ds\times dz)\right]$$
$$\le C\varepsilon^{2\beta-2-\alpha}\max\{\underline{a}_{T_i}^{2-\alpha},\overline{a}_{T_i}^{2-\alpha}\}E_{\mathcal{F}_{T_i}}\left[\int_{T_i}^{T_{i+1}}\widehat{K}_s\,ds\right].$$

Using the fact that both these terms have finite expectation by the assumption of the lemma, we can now apply standard martingale arguments to show that the third term in (40) is equal to zero. $\square$

LEMMA 7. *Assume* (*HX*) *and* (*HA*$_2$). *Let* $\{\tau_n\}$ *be a sequence of stopping times converging to* T *from below. Then there exists* $\varepsilon^*>0$ *such that*

$$\sup_{0<\varepsilon<\varepsilon^*}E\left[\left(\varepsilon^{\alpha-\beta}\sum_{i=1}^{N_T^\varepsilon}|X_{T_i}-X_{T_{i-1}}|^\beta\right)^{1+\delta}\right]<\infty \tag{41}$$

and

$$\lim_{n\to\infty}\lim_{\varepsilon\downarrow 0}E\left[\left(\varepsilon^{\alpha-\beta}\sum_{i=N_{\tau_n}^\varepsilon+1}^{N_T^\varepsilon}|X_{T_i}-X_{T_{i-1}}|^\beta\right)^{1+\delta}\right]=0. \tag{42}$$

PROOF. In this proof, we shall use the notation

$$\overline{\Lambda}_t=\sup_{0\le s\le T}\left(\max\{\underline{a}_s^{\beta-1},\overline{a}_s^{\beta-1}\}^{1+\delta}+\max\{\underline{a}_s^{(1+\delta)\beta-1},\overline{a}_s^{(1+\delta)\beta-1}\}\right)|b_t|^{1+\delta}$$
$$+\sup_{0\le s\le T}\max\{\underline{a}_s,\overline{a}_s\}^{(\beta\vee(2-\alpha))(1+\delta)}\min\{\underline{a}_s,\overline{a}_s\}^{((\beta-2)\wedge(-\alpha))(1+\delta)}\widehat{K}_t^{1+\delta}.$$

We now use a martingale decomposition of the sum of the increments. So we write

$$\sum_{i=1}^n|X_{T_i}-X_{T_{i-1}}|^\beta=M_n^1+M_n^2+Z_n,$$
$$M_n^1=\sum_{i=1}^n\{|X_{T_i}-X_{T_{i-1}}|^\beta-E_{\mathcal{F}_{T_{i-1}}}[|X_{T_i}-X_{T_{i-1}}|^\beta]\},$$
$$M_n^2=\sum_{i=1}^nE_{\mathcal{F}_{T_{i-1}}}[|X_{T_i}-X_{T_{i-1}}|^\beta]\left\{1-\frac{\int_{T_{i-1}}^{T_i}\Lambda_s^{T_{i-1}}\,ds}{E_{\mathcal{F}_{T_{i-1}}}[\int_{T_{i-1}}^{T_i}\Lambda_s^{T_{i-1}}\,ds]}\right\},$$

$$Z_n = \sum_{i=1}^{n} E_{\mathcal{F}_{T_{i-1}}}\big[|X_{T_i} - X_{T_{i-1}}|^\beta\big] \frac{\int_{T_{i-1}}^{T_i} \Lambda_s^{T_{i-1}}\, ds}{E_{\mathcal{F}_{T_{i-1}}}[\int_{T_{i-1}}^{T_i} \Lambda_s^{T_{i-1}}\, ds]},$$

where we write

$$\begin{aligned}\Lambda_s^{T_i} &:= \varepsilon^{\alpha-1} \max\{\underline{a}_{T_i}^{\beta-1}, \overline{a}_{T_i}^{\beta-1}\}|b_s| \\ &\quad + \max\{\underline{a}_{T_i}, \overline{a}_{T_i}\}^{\beta\vee(2-\alpha)} \min\{\underline{a}_{T_i}, \overline{a}_{T_i}\}^{(\beta-2)\wedge(-\alpha)} \widehat{K}_s.\end{aligned}$$

The processes M^1 and M^2 are martingales with respect to the discrete filtration $\mathcal{F}_n^d := \mathcal{F}_{T_n}$. Note that for every $\mathcal{F}$-stopping time $\tau \le T$, N_τ^ε is an $\mathcal{F}^d$-stopping time. The Burkholder inequality for a discrete-time martingale M then writes

$$\begin{aligned}E\big[|M_{N_T^\varepsilon} - M_{N_\tau^\varepsilon}|^{1+\delta}\big] &\le CE\Bigg[\Bigg(\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} (M_i - M_{i-1})^2\Bigg)^{(1+\delta)/2}\Bigg] \\ &\le CE\Bigg[\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} |M_i - M_{i-1}|^{1+\delta}\Bigg],\end{aligned}$$

and therefore,

$$\begin{aligned}&E\big[\big|\varepsilon^{\alpha-\beta}\big(M^1_{N_T^\varepsilon} - M^1_{N_\tau^\varepsilon}\big)\big|^{1+\delta}\big] \\ &\quad \le C\varepsilon^{(\alpha-\beta)(1+\delta)} E\Bigg[\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} \big||X_{T_i} - X_{T_{i-1}}|^\beta - E_{\mathcal{F}_{T_{i-1}}}\big[|X_{T_i} - X_{T_{i-1}}|^\beta\big]\big|^{1+\delta}\Bigg] \\ &\quad \le C\varepsilon^{(\alpha-\beta)(1+\delta)} E\Bigg[\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} E_{\mathcal{F}_{T_{i-1}}}\big[|X_{T_i} - X_{T_{i-1}}|^{\beta(1+\delta)}\big]\Bigg].\end{aligned}$$

By Lemma 6, this is smaller than

$$\begin{aligned}&CE\Bigg[\varepsilon^{\alpha(1+\delta)-1} \sup_{0\le s\le T} \max\{\underline{a}_s^{\beta'-1}, \overline{a}_s^{\beta'-1}\} \int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} |b_s|\, ds \\ &\qquad + \varepsilon^{\alpha\delta} \sup_{0\le s\le T} \max\{\underline{a}_s, \overline{a}_s\}^{\beta'\vee(2-\alpha)} \min\{\underline{a}_s, \overline{a}_s\}^{(\beta'-2)\wedge(-\alpha)} \int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} \widehat{K}_s\, ds\Bigg] \\ &\qquad \le C\varepsilon^{\alpha\delta}\Bigg(E\Bigg[\int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} \overline{\Lambda}_s\, ds\Bigg] + E\Bigg[\int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} \overline{\Lambda}_s\, ds\Bigg]^{1/(1+\delta)}\Bigg),\end{aligned}$$

with $\beta' = \beta(1+\delta)$, where the last estimate can be obtained, for example, by Hölder's inequality.

Similarly, the process M^2 satisfies

$$\begin{aligned}
&E\big[\big|\varepsilon^{\alpha-\beta}(M^2_{N^\varepsilon_T}-M^2_{N^\varepsilon_\tau})\big|^{1+\delta}\big]\\
&\quad\le CE\Bigg[\sum_{i=N^\varepsilon_\tau+1}^{N^\varepsilon_T}\bigg\{\int_{T_{i-1}}^{T_i}\Lambda_s^{T_{i-1}}\,ds-E_{\mathcal{F}_{T_{i-1}}}\bigg[\int_{T_{i-1}}^{T_i}\Lambda_s^{T_{i-1}}\,ds\bigg]\bigg\}^{1+\delta}\Bigg]\\
&\quad\le CE\Bigg[\sum_{i=N^\varepsilon_\tau+1}^{N^\varepsilon_T}\bigg\{\int_{T_{i-1}}^{T_i}\Lambda_s^{T_{i-1}}\,ds\bigg\}^{1+\delta}\Bigg]\le CE\bigg[\int_{T_{N^\varepsilon_\tau}}^{T_{N^\varepsilon_T}}(\Lambda_s^{\eta_s})^{1+\delta}\,ds\bigg]\\
&\quad\le CE\bigg[\int_{T_{N^\varepsilon_\tau}}^{T_{N^\varepsilon_T}}\overline{\Lambda}_s\,ds\bigg].
\end{aligned}$$

The process Z can be treated along the same lines as well, since by Lemma 6,

$$E\big[\big|\varepsilon^{\alpha-\beta}(Z_{N^\varepsilon_T}-Z_{N^\varepsilon_\tau})\big|^{1+\delta}\big]\le CE\Bigg[\bigg\{\int_{T_{N^\varepsilon_\tau}}^{T_{N^\varepsilon_T}}\Lambda_s^{\eta_s}\,ds\bigg\}^{1+\delta}\Bigg]\le CE\bigg[\int_{T_{N^\varepsilon_\tau}}^{T_{N^\varepsilon_T}}\overline{\Lambda}_s\,ds\bigg].$$

The three expressions above are uniformly bounded by the assumption of the lemma, proving (41). To show (42), observe that

$$E\bigg[\int_{T_{N^\varepsilon_{\tau_n}}}^{T_{N^\varepsilon_T}}\overline{\Lambda}_s\,ds\bigg]\le E\bigg[\int_{\tau_n}^{T}\overline{\Lambda}_s\,ds\bigg]+E\bigg[\sup_{i:T_i\le T}\int_{T_{i-1}}^{T_i}\overline{\Lambda}_s\,ds\bigg].$$

The first term does not depend on ε and converges to zero as $n\to\infty$ by the assumption of the lemma and the dominated convergence. For the second term, we use Lemma 3 and the absolute continuity of the integral. □

In the case $\beta=0$, assumption (HA_2) can be somewhat simplified.

LEMMA 8. *Assume* (HX) *and* (HA_2'). *Let* $\{\tau_n\}$ *be a sequence of stopping times converging to* T *from below. Then there exists* $\varepsilon^*>0$ *such that*

$$\sup_{0<\varepsilon<\varepsilon^*}E\big[\big(\varepsilon^\alpha N^\varepsilon_T\big)^{1+\delta}\big]<\infty$$

and

$$\lim_{n\to\infty}\lim_{\varepsilon\downarrow 0}E\big[\big(\varepsilon^\alpha\big(N^\varepsilon_T-N^\varepsilon_{\tau_n}\big)\big)^{1+\delta}\big]=0.$$

PROOF. We follow the proof of Lemma 7, taking $\beta=0$ and

$$\Lambda_s^{T_i}:=\varepsilon^{\alpha-1}\min\{\underline{a}_{T_i},\overline{a}_{T_i}\}^{-1}|b_s|+\min\{\underline{a}_{T_i},\overline{a}_{T_i}\}^{-\alpha}\widehat{K}_s$$

and using Corollary 1 instead of Lemma 6. □

6. Proof of Theorem 2. *Step* 1. *Reduction to the case of bounded coefficients*. As before, we start with the localization procedure.

LEMMA 9. *Assume that* (24) *holds under the assumptions* (*HY*), (*HX*) *and* (H_1') *and* (25) *holds under the assumptions* (*HY*), (*HX*) *and* (H_ρ') *for some* $\rho > \frac{\alpha}{\alpha-\beta} \vee 2$. *Then Theorem* 2 *holds.*

PROOF. The arguments related to the localization of Z are the same or very similar to those in Lemma 2, and so they are omitted. We set $u^0(a,b)=1$ for any (a,b). With the same notation as in the proof of this lemma, and using (42) in the first equality we then get, for $0 \le \beta < \alpha$,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \varepsilon^{\alpha-\beta} E\left[\sum_{i=1}^{N_T^\varepsilon} |X_{T_i} - X_{T_{i-1}}|^\beta\right] \\
&\quad = \lim_{n\to\infty}\lim_{\varepsilon\downarrow 0} \varepsilon^{\alpha-\beta} E\left[\sum_{i=1}^{N_{\gamma_n}^\varepsilon} |X_{T_i} - X_{T_{i-1}}|^\beta\right] \\
&\quad = \lim_{n\to\infty}\lim_{\varepsilon\downarrow 0} \varepsilon^{\alpha-\beta} E\left[\sum_{i\ge 1: T_i^n \le \gamma_n} |X_{T_i}^n - X_{T_{i-1}}^n|^\beta\right] \\
&\quad = \lim_{n\to\infty} E\left[\int_0^{\gamma_n} \lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\,dt\right] \\
&\quad = E\left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\,dt\right],
\end{aligned}$$

where the assumptions of the lemma are used to pass from the second to the third line.

Step 2. *Change of probability measure*. The goal of this step is to show that

$$\begin{aligned}
S_2 &:= \lim_{\varepsilon\downarrow 0} \varepsilon^{\alpha-\beta} E\left[\sum_{i=1}^{N_T^\varepsilon} |X_{T_i} - X_{T_{i-1}}|^\beta\right] \\
&= \lim_{\varepsilon\downarrow 0} \varepsilon^{\alpha-\beta} E^Q\left[\sum_{i=1}^{\infty} 1_{T_{i-1}\le T} Z_{T_{i-1}}^{-1} |X_{T_i} - X_{T_{i-1}}|^\beta\right].
\end{aligned} \tag{43}$$

For the right-hand side to be well defined we extend the processes λ, b, $\underline{a}$, $\overline{a}$ by arbitrary constant values beyond T and define the process X for $t \ge T$ accordingly. The case $\beta = 0$ being straightforward, we assume that $\beta > 0$.

To prove (43), it is enough to show that

$$\lim_{\varepsilon\downarrow 0} E^Q\left[\varepsilon^{\alpha-\beta} \sum_{i=1}^{\infty} 1_{T_i\le T} (Z_{T_i}^{-1} - Z_{T_{i-1}}^{-1}) |X_{T_i} - X_{T_{i-1}}|^\beta\right] = 0 \tag{44}$$

and

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{\alpha-\beta}E^Q\big[Z^{-1}_{T_{N^\varepsilon_T}}|X_{T_{N^\varepsilon_T+1}}-X_{T_{N^\varepsilon_T}}|^\beta\big]=0. \tag{45}$$

The second term can be shown to converge to zero using Lemma 6. For the first term, for $1<\kappa<\frac{\alpha\rho}{\alpha+\beta\rho}$, Hölder's inequality yields

$$\begin{aligned}
&E^Q\Bigg[\Bigg(\varepsilon^{\alpha-\beta}\sum_{i=1}^\infty 1_{T_i\le T}(Z^{-1}_{T_i}-Z^{-1}_{T_{i-1}})|X_{T_i}-X_{T_{i-1}}|^\beta\Bigg)^\kappa\Bigg]\\
&\quad\le E^Q\Big[\sup_{0\le t\le T}Z_t^{-\rho}\Big]^{\kappa/\rho}\\
&\qquad\times E^Q\Bigg[\Bigg(\varepsilon^{\alpha-\beta}\sum_{i=1}^\infty 1_{T_i\le T}|X_{T_i}-X_{T_{i-1}}|^\beta\Bigg)^{\kappa\rho/(\rho-\kappa)}\Bigg]^{(\rho-\kappa)/\rho},
\end{aligned}$$

which is bounded by a constant for ε sufficiently small by Lemma 7 (applied under Q) (the assumptions are satisfied because we are working under H'_ρ and therefore all coefficients are bounded). Therefore, the expression under the expectation in (44) is uniformly integrable under Q as $\varepsilon\downarrow 0$. On the other hand, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\varepsilon^{\alpha-\beta}\sum_{i=1}^\infty 1_{T_i\le T}|Z^{-1}_{T_i}-Z^{-1}_{T_{i-1}}||X_{T_i}-X_{T_{i-1}}|^\beta\\
&\quad\le\varepsilon^{(\alpha-\beta)/2}\Bigg(\sum_{i=1}^{N^\varepsilon_T}(Z^{-1}_{T_i}-Z^{-1}_{T_{i-1}})^2\Bigg)^{1/2}\\
&\qquad\times\sup_{0\le t\le T}|X_t|^{\beta/2}\Bigg(\varepsilon^{\alpha-\beta}\sum_{i=1}^{N^\varepsilon_T}|X_{T_i}-X_{T_{i-1}}|^\beta\Bigg)^{1/2}.
\end{aligned}$$

Since Z^{-1} has finite quadratic variation, and the last factor is uniformly integrable under Q by Lemma 7, due to the first deterministic factor, the whole expression converges to zero in probability, and (44) follows.

Step 3. Using the same notation as in the proof of Theorem 1 (step 3), we have

$$\begin{aligned}
&\varepsilon^{\alpha-\beta}E^Q\Bigg[\sum_{i=1}^\infty 1_{T_{i-1}\le T}Z^{-1}_{T_{i-1}}(T_i-T_{i-1})\frac{E^Q_{\mathcal{F}_{T_{i-1}}}|\tilde X_{\tilde\tau_i}|^\beta}{E^Q_{\mathcal{F}_{T_{i-1}}}[T_i-T_{i-1}]}\Bigg]\\
&\quad=\varepsilon^{\alpha-\beta}E^Q\Bigg[\sum_{i=1}^\infty 1_{T_{i-1}\le T}Z^{-1}_{T_{i-1}}\lambda_{T_{i-1}}(T_i-T_{i-1})\frac{E^Q_{\mathcal{F}_{T_{i-1}}}|\tilde X_{\tilde\tau_i}|^\beta}{E^Q_{\mathcal{F}_{T_{i-1}}}[\tilde\tau_i]}\Bigg]+R^\varepsilon,
\end{aligned}$$

where one can show, using first Lemma 6 and then exactly the same arguments as in the proof of Theorem 1, that $R^\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Then, from the previous step,

$$S_2 = \lim_{\varepsilon\downarrow 0} \varepsilon^{\alpha-\beta} E^Q\left[\sum_{i=1}^\infty 1_{T_{i-1}\le T} Z_{T_{i-1}}^{-1}(T_i - T_{i-1}) \frac{E^Q_{\mathcal{F}_{T_{i-1}}}|X_{T_i} - X_{T_{i-1}}|^\beta}{E^Q_{\mathcal{F}_{T_{i-1}}}[T_i - T_{i-1}]}\right]$$

$$= \lim_{\varepsilon\downarrow 0} \varepsilon^{\alpha-\beta} E^Q\left[\sum_{i=1}^\infty 1_{T_{i-1}\le T} Z_{T_{i-1}}^{-1}\lambda_{T_{i-1}}(T_i - T_{i-1}) \frac{E^Q_{\mathcal{F}_{T_{i-1}}}|\tilde{X}_{\tilde{\tau}_i}|^\beta}{E^Q_{\mathcal{F}_{T_{i-1}}}[\tilde{\tau}_i]}\right].$$

Our next goal is to replace $\tilde{X}_{\tilde{\tau}_i}$ with $\hat{X}_{\hat{\tau}_i}$ in the above expression, where $\hat{\tau}_i = \inf\{t \ge 0 : \hat{X}_t \notin [-\underline{a}_{T_i}\varepsilon, \overline{a}_{T_i}\varepsilon]\}$. Let $a = \min(\underline{a}_{T_i}, \overline{a}_{T_i})$ and define

$$f(x) = (\varepsilon a)^\beta \frac{(\beta - \varepsilon a)(x/(\varepsilon a))^2 + 2 - \beta}{2 - \varepsilon a} 1_{|x|<\varepsilon a} + |x|^\beta 1_{|x|>\varepsilon a}.$$

f is a twice differentiable function satisfying for small enough ε

$$|f'(x)| \le C\varepsilon^{\beta-1} \quad \text{and} \quad |f''(x)| \le C\varepsilon^{\beta-2}, \tag{46}$$

and hence Itô's formula can be applied. Then,

$$\begin{aligned}&\left|E^Q_{\mathcal{F}_{T_{i-1}}}\left[|\tilde{X}_{\tilde{\tau}_i}|^\beta - |\hat{X}_{\hat{\tau}_i}|^\beta\right]\right| \\ &\qquad \le \left|E^Q_{\mathcal{F}_{T_{i-1}}}\left[f(\tilde{X}_{\tilde{\tau}_i}) - f(\hat{X}_{\tilde{\tau}_i})\right]\right| + \left|E^Q_{\mathcal{F}_{T_{i-1}}}\left[f(\hat{X}_{\tilde{\tau}_i}) - f(\hat{X}_{\hat{\tau}_i})\right]\right|.\end{aligned}$$

By definition of $\tilde{X}$ and $\hat{X}$ and because all coefficients are bounded, the first term satisfies

$$\left|E^Q_{\mathcal{F}_{T_{i-1}}}\left[f(\tilde{X}_{\tilde{\tau}_i}) - f(\hat{X}_{\tilde{\tau}_i})\right]\right| \le C\varepsilon^{\beta-1} E^Q_{\mathcal{F}_{T_{i-1}}}[\tilde{\tau}_i].$$

For the second term, we use Itô's formula,

$$\begin{aligned}&E^Q_{\mathcal{F}_{T_{i-1}}}\left[f(\hat{X}_{\tilde{\tau}_i}) - f(\hat{X}_{\hat{\tau}_i})\right] \\ &\quad = E^Q_{\mathcal{F}_{T_{i-1}}}\left[\int_{\hat{\tau}_i\wedge\tilde{\tau}_i}^{\hat{\tau}_i\vee\tilde{\tau}_i} \int_{\mathbb{R}} \{f(\hat{X}_s + z) - f(\hat{X}_s) - z1_{|z|\le 1} f'(\hat{X}_s)\}\nu(dz)\,ds\right] \\ &\qquad + E^Q_{\mathcal{F}_{T_{i-1}}}\left[\int_{\hat{\tau}_i\wedge\tilde{\tau}_i}^{\hat{\tau}_i\vee\tilde{\tau}_i} \int_{\mathbb{R}} \{f(\hat{X}_{s-} + z) - f(\hat{X}_{s-})\}\left(\widehat{M}(ds\times dz) - \nu(dz)\,ds\right)\right],\end{aligned}$$

where $\widehat{M}$ is the jump measure of $\hat{X}$. It follows by standard arguments that the local martingale term has zero expectation. To deal with the first term we use the bounds in (46) and decompose the integrand as follows:

$$\begin{aligned}&\left|\int_{\mathbb{R}} \{f(\hat{X}_s + z) - f(\hat{X}_s) - z1_{|z|\le 1} f'(\hat{X}_s)\}\nu(dz)\right| \\ &\qquad \le C\varepsilon^{\beta-2}\int_{|z|\le\varepsilon} z^2\nu(dz) + C\varepsilon^{\beta-1}\int_{|z|>\varepsilon} |z|\nu(dz) \le C\varepsilon^{\beta-\alpha},\end{aligned}$$

so that finally

$$\left|E^Q_{\mathcal{F}_{T_{i-1}}}[|\tilde{X}_{\tilde{\tau}_i}|^\beta - |\hat{X}_{\hat{\tau}_i}|^\beta]\right| \le C\varepsilon^{\beta-1}E^Q_{\mathcal{F}_{T_{i-1}}}[\tilde{\tau}_i] + C\varepsilon^{\beta-\alpha}E^Q_{\mathcal{F}_{T_{i-1}}}[|\tilde{\tau}_i - \hat{\tau}_i|].$$

Substituting this estimate into the formula for S_2, we then get

$$S_2 = \lim_{\varepsilon\downarrow 0}\varepsilon^{\alpha-\beta}E^Q\left[\sum_{i=1}^{\infty}1_{T_{i-1}\le T}Z^{-1}_{T_{i-1}}\lambda_{T_{i-1}}(T_i - T_{i-1})\frac{E^Q_{\mathcal{F}_{T_{i-1}}}|\hat{X}_{\hat{\tau}_i}|^\beta}{E^Q_{\mathcal{F}_{T_{i-1}}}[\tilde{\tau}_i]}\right] + \lim_{\varepsilon\downarrow 0}R^\varepsilon$$

with

$$\begin{aligned}|R^\varepsilon| \le{}& C\varepsilon^{\alpha-1}E^Q\left[\sum_{i=1}^{\infty}1_{T_{i-1}\le T}Z^{-1}_{T_{i-1}}\lambda_{T_{i-1}}(T_i - T_{i-1})\right]\\ &+ CE^Q\left[\sum_{i=1}^{\infty}1_{T_{i-1}\le T}Z^{-1}_{T_{i-1}}\lambda_{T_{i-1}}(T_i - T_{i-1})\frac{E^Q_{\mathcal{F}_{T_{i-1}}}|\tilde{\tau}_i - \hat{\tau}_i|}{E^Q_{\mathcal{F}_{T_{i-1}}}[\tilde{\tau}_i]}\right].\end{aligned}$$

The first expectation is bounded (because λ is bounded) and Z^{-1} is integrable, and therefore the first term converges to zero. For the second term, we observe (using the notation of the proof of Theorem 1, step 4) that

$$\underline{f}^{0,1}_\varepsilon(\underline{a}_{T_i}, \overline{a}_{T_i}) \le E^Q_{\mathcal{F}_{T_{i-1}}}[\tilde{\tau}_i] \le \overline{f}^{0,1}_\varepsilon(\underline{a}_{T_i}, \overline{a}_{T_i})$$

and

$$E^Q_{\mathcal{F}_{T_{i-1}}}|\tilde{\tau}_i - \hat{\tau}_i| \le E^Q[\hat{\tau}_2 \wedge \hat{\tau}^j - \hat{\tau}_1] \le \overline{f}^{0,1}_\varepsilon(\underline{a}_{T_i}, \overline{a}_{T_i}) - \underline{f}^{0,1}_\varepsilon(\underline{a}_{T_i}, \overline{a}_{T_i}).$$

In view of Lemma 5 we then conclude that the second term converges to zero as well. Finally, we have shown that

$$S_2 = \lim_{\varepsilon\downarrow 0}\varepsilon^{\alpha-\beta}E^Q\left[\sum_{i=1}^{\infty}1_{T_{i-1}\le T}Z^{-1}_{T_{i-1}}\lambda_{T_{i-1}}(T_i - T_{i-1})\frac{u^\beta_\varepsilon(\underline{a}_{T_{i-1}}, \overline{a}_{T_{i-1}})}{E^Q_{\mathcal{F}_{T_{i-1}}}[\tilde{\tau}_i]}\right],$$

where u^β_ε is a deterministic function defined by

$$u^\beta_\varepsilon(a,b) = E[|\hat{X}_{\hat{\tau}}|^\beta], \qquad \hat{\tau} = \inf\{t \ge 0: \hat{X}_t \notin (-a\varepsilon, b\varepsilon)\}.$$

Similar to the last step of the proof of Theorem 1, we can now write

$$S_2 \le \limsup_{\varepsilon\downarrow 0}E^Q\left[\sum_{i=1}^{\infty}1_{T_{i-1}\le T}Z^{-1}_{T_{i-1}}\lambda_{T_{i-1}}(T_i - T_{i-1})\frac{\varepsilon^{-\beta}u^\beta_\varepsilon(\underline{a}_{T_{i-1}}, \overline{a}_{T_{i-1}})}{\varepsilon^{-\alpha}\underline{f}^{0,1}_\varepsilon(\underline{a}_{T_{i-1}}, \overline{a}_{T_{i-1}})}\right],$$

$$S_2 \ge \limsup_{\varepsilon\downarrow 0}E^Q\left[\sum_{i=1}^{\infty}1_{T_{i-1}\le T}Z^{-1}_{T_{i-1}}\lambda_{T_{i-1}}(T_i - T_{i-1})\frac{\varepsilon^{-\beta}u^\beta_\varepsilon(\underline{a}_{T_{i-1}}, \overline{a}_{T_{i-1}})}{\varepsilon^{-\alpha}\overline{f}^{0,1}_\varepsilon(\underline{a}_{T_{i-1}}, \overline{a}_{T_{i-1}})}\right].$$

Using Lemma 11 we obtain uniform convergence of

$$\frac{\varepsilon^{-\beta} u_\varepsilon^\beta(a,b)}{\varepsilon^{-\alpha} \overline{f}_\varepsilon^{0,1}(a,b)}$$

toward $\frac{u^\beta(a,b)}{f^{*,0,1}(a,b)}$ and conclude that

$$S_2 = E^Q\left[\int_0^T \lambda_t Z_t^{-1} \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{f^{*,0,1}(\underline{a}_t, \overline{a}_t)}\, dt\right] = E\left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{f^{*,0,1}(\underline{a}_t, \overline{a}_t)}\, dt\right]. \qquad \square$$

APPENDIX A: SOME COMPUTATIONS FOR STABLE PROCESSES

PROPOSITION 6. *Let X be a symmetric α-stable process on $\mathbb{R}$ with characteristic function $E[e^{iuX_t}] = e^{-t\sigma|u|^\alpha}$, $0 < \alpha < 2$, and $\tau_{a,b} = \inf\{t \geq 0 : X_t \notin (-a, b)\}$ with $a, b > 0$. Then*

$$f(a,b) := E\left[\int_0^{\tau_{a,b}} X_t^2\, dt\right] = \frac{\alpha(ab)^{1+(\alpha/2)}}{2\sigma\Gamma(3+\alpha)} \left\{\left(\frac{a}{b} + \frac{b}{a}\right)\left(1 + \frac{\alpha}{2}\right) - \alpha\right\}.$$

The proof of this result is based on the following lemma, where we consider the exit time from the interval $[-1, 1]$ by a process starting from x.

LEMMA 10. *Let X be as above and $\tau_1 = \inf\{t \geq 0 : X_t \notin (-1, 1)\}$. Then*

$$f(x) := E^x\left[\int_0^{\tau_1} X_t^2\, dt\right] = \frac{1}{\sigma} \frac{2(1 - x^2)^{\alpha/2}\{x^2 + (\alpha/2)\}}{\Gamma(3+\alpha)} 1_{x \in (-1,1)}.$$

PROOF. Without loss of generality, we let $\sigma = 1$ in this proof. Let $\hat{f}(u) = \int_{\mathbb{R}} e^{iux} f(x)\, dx$. Using the arguments similar to the ones in [16], one can show that the function f satisfies the equation $\mathcal{L}^\alpha f(x) = -x^2$ on $x \in (-1, 1)$ with the boundary condition $f(x) = 0$ on $x \notin (-1, 1)$, where $\mathcal{L}^\alpha$ is the fractional Laplace operator

$$\mathcal{L}^\alpha f(x) = \int_{\mathbb{R}} \big(f(x+y) - f(x) - yf'(x)\big) \frac{dy}{|y|^{1+\alpha}}, \qquad 1 < \alpha < 2,$$

$$\mathcal{L}^\alpha f(x) = \int_{\mathbb{R}} \big(f(x+y) - f(x) - y1_{|y|\leq 1} f'(x)\big) \frac{dy}{|y|^{1+\alpha}}, \qquad \alpha = 1,$$

$$\mathcal{L}^\alpha f(x) = \int_{\mathbb{R}} \big(f(x+y) - f(x)\big) \frac{dy}{|y|^{1+\alpha}}, \qquad 0 < \alpha < 1.$$

Moreover, the function $\hat{f}$ satisfies the system of integral equations

$$\frac{1}{\pi} \int_0^\infty \hat{f}(u) |u|^\alpha \cos(ux)\, du = x^2, \qquad |x| < 1,$$

$$\frac{1}{\pi} \int_0^\infty \hat{f}(u) \cos(ux)\, du = 0, \qquad |x| > 1.$$

Let $\hat{f}_1(u) = u^{-(1+\alpha)/2} J_{(1+\alpha)/2}(u)$ and $\hat{f}_2(u) = u^{-(3+\alpha)/2} J_{(3+\alpha)/2}(u)$, where J is the Bessel function; see [17], Section 8.40. Then, from [17], Integral 6.699.2, we get

$$\int_0^\infty \hat{f}_1(u)\cos(ux)\,du = \int_0^\infty \hat{f}_2(u)\cos(ux)\,du = 0, \qquad |x| > 1, \tag{47}$$

$$\int_0^\infty \hat{f}_1(u)|u|^\alpha \cos(ux)\,du = 2^{(\alpha-1)/2}\Gamma\left(\frac{1+\alpha}{2}\right), \qquad |x| < 1, \tag{48}$$

$$\int_0^\infty \hat{f}_2(u)|u|^\alpha \cos(ux)\,du = 2^{(\alpha-3)/2}\Gamma\left(\frac{1+\alpha}{2}\right)\bigl(1-(1+\alpha)x^2\bigr), \qquad |x| < 1, \tag{49}$$

$$\int_0^\infty \hat{f}_1(u)\cos(ux)\,du = 2^{-(\alpha+1)/2}\frac{\Gamma(1/2)}{\Gamma((\alpha+2)/2)}\bigl(1-x^2\bigr)^{\alpha/2}, \qquad |x| < 1, \tag{50}$$

$$\int_0^\infty \hat{f}_2(u)\cos(ux)\,du = 2^{-(\alpha+3)/2}\frac{\Gamma(1/2)}{\Gamma((\alpha+4)/2)}\bigl(1-x^2\bigr)^{1+(\alpha/2)}, \qquad |x| < 1. \tag{51}$$

From (47)–(49),

$$\hat{f}(u) = \pi\frac{\hat{f}_1(u) - 2\hat{f}_2(u)}{2^{(\alpha-1)/2}\Gamma((1+\alpha)/2)(1+\alpha)}.$$

To conclude, we compute the inverse Fourier transform of $\hat{f}$ from (50)–(51). □

PROOF OF PROPOSITION 6. Once again, we set $\sigma = 1$ without loss of generality. Recall a result of Blumenthal, Getoor and Ray [5]: the law of a symmetric stable process starting from the point x with $|x| < 1$ and observed at time τ_1 has density given by

$$\mu(x, y) = \frac{1}{\pi}\sin\frac{\pi\alpha}{2}\bigl(1-x^2\bigr)^{\alpha/2}\bigl(y^2-1\bigr)^{-\alpha/2}|y-x|^{-1}, \qquad |y| \geq 1.$$

By the scaling property, we then deduce that the density of a symmetric stable process starting from zero, and observed at time $\tau_{a,b}$ is given by

$$\mu_{a,b}(z) = \frac{1}{\pi}\sin\frac{\pi\alpha}{2}(ab)^{\alpha/2}\bigl((z-b)(z+a)\bigr)^{-\alpha/2}\frac{1}{|z|}. \tag{52}$$

Similarly, from the preceding lemma, we easily deduce by the scaling property that

$$f_A(x) := E^x\left[\int_0^{\tau_{A,A}} X_t^2\,dt\right] = \frac{2(A^2-x^2)^{\alpha/2}\{x^2+(\alpha/2)A^2\}}{\Gamma(3+\alpha)}1_{x\in(-A,A)}.$$

This function satisfies the equation $\mathcal{L}^\alpha f_A(x) = -x^2$ on $[-A, A]$ with the boundary condition $f_A(x) = 0$ on $x \notin [-A, A]$. Taking $A \geq \max(a, b)$, we then get by Itô's formula

$$E\big[f_A(X_{\tau_{a,b}})\big] = f_A(0) - E\bigg[\int_0^{\tau_{a,b}} X_t^2\,dt\bigg].$$

By symmetry, it is sufficient to prove the proposition for $a \geq b$. Taking $A = a$ in the above formula, we finally get

$$\begin{aligned} E\bigg[\int_0^{\tau_{a,b}} X_t^2\,dt\bigg] \\ &= \frac{\alpha a^{\alpha+2}}{\Gamma(3+\alpha)} - \int_b^a f_A(x)\mu_{a,b}(x)\,dx \\ &= \frac{\alpha a^{\alpha+2}}{\Gamma(3+\alpha)} - \frac{2\sin\pi\alpha/2}{\pi\Gamma(3+\alpha)}(ab)^{\alpha/2}\int_b^a \Big(z^2 + \frac{\alpha}{2}a^2\Big)\Big(\frac{a-z}{z-b}\Big)^{\alpha/2}\frac{dz}{z}. \end{aligned}$$

Computing the integral (using [17], Integral 3.228.1 and the standard integral representation for the beta function) then yields the result. $\square$

REMARK 8. Let us list here several other useful results which are already known from the literature or can be obtained with a simple computation. By a result of Getoor [16]: under the assumptions of Proposition 6,

$$\begin{aligned} E^x[\tau_1] &= \frac{1}{\sigma}\frac{2^{-\alpha}\Gamma(1/2)}{\Gamma((2+\alpha)/2)\Gamma((1+\alpha)/2)}\big(1-x^2\big)^{\alpha/2} \\ &= \frac{1}{\sigma}\frac{(1-x^2)^{\alpha/2}}{\Gamma(1+\alpha)}. \end{aligned}$$

By the scaling property we then deduce that for general barriers

$$E[\tau_{a,b}] = \Big(\frac{a+b}{2}\Big)^\alpha E^{(a-b)/(a+b)}[\tau_1] = \frac{(ab)^{\alpha/2}}{\sigma\Gamma(1+\alpha)}. \tag{53}$$

Similarly, from (52), we easily get, for $\beta < \alpha$,

$$\begin{aligned} E\big[|X_{\tau_{a,b}}|^\beta\big] &= \frac{\sin\pi\alpha/2}{\pi}(ab)^{\alpha/2} \\ &\quad\times \int_0^\infty z^{-\alpha/2}(z+a+b)^{-\alpha/2}\big(|z+a|^{\beta-1} + |z+b|^{\beta-1}\big)\,dz. \end{aligned} \tag{54}$$

This integral can be expressed in terms of special functions and is equal to

$$\begin{aligned} &a^\beta\Big(\frac{b}{a+b}\Big)^{\alpha/2}\frac{\sin\pi\alpha/2}{\pi}B(1-\alpha/2, \alpha-\beta) \\ &\quad\times F\Big(\alpha/2, 1-\alpha/2, \alpha/2+1-\beta, \frac{b}{a+b}\Big) \end{aligned}$$

$$
\begin{aligned}
&+ b^\beta\left(\frac{a}{a+b}\right)^{\alpha/2}\frac{\sin\pi\alpha/2}{\pi}B(1-\alpha/2,\alpha-\beta)\\
&\times F\left(\alpha/2, 1-\alpha/2, \alpha/2+1-\beta, \frac{b}{a+b}\right),
\end{aligned}
$$

where B is the beta function and F is the hypergeometric function; see [17], Integral 3.259.3.

APPENDIX B: CONVERGENCE OF RESCALED LÉVY PROCESSES

LEMMA 11. *Let X be a Lévy process with characteristic triplet $(0,\nu,\gamma)$ with respect to the truncation function $h(x) = -1 \vee x \wedge 1$ with*

$$x^\alpha \nu((x,\infty)) \to c_+ \quad \textit{and} \quad x^\alpha \nu((-\infty,-x)) \to c_- \qquad \textit{when } x \to 0$$

for some $\alpha \in (1,2)$ and constants $c_+ \geq 0$ and $c_- \geq 0$ with $c_+ + c_- > 0$. For $\varepsilon > 0$, define the process X^ε via $X_t^\varepsilon = \varepsilon^{-1}X_{\varepsilon^\alpha t}$. Then X^ε converges in law to a strictly α-stable Lévy process X^* with Lévy density

$$\nu^*(x) = \frac{c_+ 1_{x>0} + c_- 1_{x<0}}{|x|^{1+\alpha}}. \tag{55}$$

Assume in addition that there exists $C < \infty$, such that for all $x > 0$,

$$\nu((-x,x)^c) < Cx^{-\alpha}$$

and for $a, b \in (0,\infty)$ and $\beta \in (0,\alpha)$, let

$$u_\varepsilon^\beta(a,b) = E\big[\big|X_{\tau^\varepsilon}^\varepsilon\big|^\beta\big], \qquad \tau^\varepsilon = \inf\big\{t \geq 0 : X_t^\varepsilon \notin (-a,b)\big\}.$$

Then

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon^\beta(a,b) = u^\beta(a,b)$$

uniformly on $(a,b) \in [B^{-1},B]^2$ for all $B < \infty$, with

$$u^\beta(a,b) = E\big[\big|X_{\tau^*}^*\big|^\beta\big]$$

and $\tau^* = \inf\{t \geq 0 : X_t^* \notin (-a,b)\}$.

PROOF. Part (i). From the Lévy–Khintchine formula it is easy to see that the characteristic triplet $(A^\varepsilon,\nu^\varepsilon,\gamma^\varepsilon)$ of X^ε is given by

$$
\begin{aligned}
A^\varepsilon &= 0,\\
\nu^\varepsilon(B) &= \varepsilon^\alpha \nu\big(\{x : x/\varepsilon \in B\}\big), \qquad B \in \mathcal{B}(\mathbb{R}),\\
\gamma^\varepsilon &= \varepsilon^{\alpha-1}\left\{\gamma + \int_{\mathbb{R}} \nu(dx)\big(\varepsilon h(x/\varepsilon) - h(x)\big)\right\}.
\end{aligned}
$$

Under the conditions of the lemma, by Theorem VII.2.9 and Remark VII.2.10 in [20], in order to prove the convergence in law, we need to check (a) that

$$\gamma^\varepsilon \to -\frac{c_+ - c_-}{\alpha(\alpha-1)},$$

where the right-hand side is the third component of the characteristic triplet of the strictly stable process with Lévy density (55) with respect to the truncation function h, and (b) that $|x|^2 \wedge 1 \cdot \nu^\varepsilon(dx)$ converges weakly to $|x|^2 \wedge 1 \cdot \nu^*(dx)$. Since $\alpha > 1$ and h is bounded, for η sufficiently small, using integration by parts and the assumption of the lemma, we obtain

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}\gamma^\varepsilon &= \lim_{\varepsilon\downarrow 0}\varepsilon^{\alpha-1}\int_{|x|\le\eta}\nu(dx)\big(\varepsilon h(x/\varepsilon) - h(x)\big)\\
&= \lim_{\varepsilon\downarrow 0}\varepsilon^{\alpha-1}\left\{\int_{-\eta}^{-\varepsilon}(-\varepsilon - x)\nu(dx) + \int_{\varepsilon}^{\eta}(\varepsilon - x)\nu(dx)\right\}\\
&= \lim_{\varepsilon\downarrow 0}\varepsilon^{\alpha-1}\left\{\int_{-\eta}^{-\varepsilon}\nu\big([-\eta, x]\big)\,dx - \int_{\varepsilon}^{\eta}\nu\big([x,\eta]\big)\,dx\right\}\\
&= \lim_{\varepsilon\downarrow 0}\varepsilon^{\alpha-1}\left\{\int_{-\eta}^{-\varepsilon}\nu\big((-\infty, x]\big)\,dx - \int_{\varepsilon}^{\eta}\nu\big([x,\infty)\big)\,dx\right\}\\
&= \lim_{\varepsilon\downarrow 0}\varepsilon^{\alpha-1}\left\{\int_{-\eta}^{-\varepsilon}\frac{c_-}{|x|^\alpha}\,dx - \int_{\varepsilon}^{\eta}\frac{c_+}{|x|^\alpha}\,dx\right\} = -\frac{c_+ - c_-}{\alpha(\alpha-1)}.
\end{aligned}$$

For property (b), it is sufficient to show that for all $x \ge 0$,

$$\int_x^\infty |z|^2 \wedge 1 \cdot \nu^\varepsilon(dz) \to \int_x^\infty |z|^2 \wedge 1 \cdot \nu^*(dz) \quad \text{and}$$

$$\int_{-\infty}^{-x} |z|^2 \wedge 1 \cdot \nu^\varepsilon(dz) \to \int_{-\infty}^{-x} |z|^2 \wedge 1 \cdot \nu^*(dz).$$

This is done using integration by parts and the assumption of the lemma as in the previous step.

Part (ii). First, similar to the proof of Proposition 3 in [22], it is easy to show that $X^\varepsilon_{\tau^\varepsilon}$ converges in law to $X^*_{\tau^*}$ as $\varepsilon \downarrow 0$. To complete the proof of the convergence of $u^\beta_\varepsilon(a,b)$ to $u^\beta(a,b)$ for fixed a and b, it remains to show that for all $\beta \in (0,\alpha)$,

$$E\big[|X^\varepsilon_{\tau^\varepsilon}|^\beta\big]$$

is bounded uniformly in ε. From Lemma 6,

$$E\big[|X^\varepsilon_{\tau^\varepsilon}|^\beta\big] \le C\varepsilon^{-\alpha}E\big[\tau^\varepsilon\big]$$

for some constant C which does not depend on ε. On the other hand, for ε small enough,

$$E\big[\tau^\varepsilon\big] \le E\big[\inf\{t : |\Delta X_t| \ge \varepsilon(a+b)\}\big] = \frac{1}{\nu((-\varepsilon a, \varepsilon b)^c)} \le C'\varepsilon^\alpha$$

for a different constant C' [the equality above holds because $\inf\{t : |\Delta X_t| \geq \varepsilon(a + b)\}$ is an exponential random variable with parameter $\nu((-\varepsilon a, \varepsilon b)^c)$ by the Lévy–Itô decomposition].

It remains to show that the convergence is uniform in a and b. First, let us show that $u^\beta(a, b)$ is continuous in (a, b) for $(a, b) \in [B^{-1}, B]^2$ and therefore also uniformly continuous on this set. Let (a_n) and (b_n) be two sequences with $a_n \to a \in [B^{-1}, B]$ and $b_n \to b \in [B^{-1}, B]$. For any process Y, we write $\tau_{(a,b)}(Y) := \inf\{t \geq 0 : Y_t \notin (-a, b)\}$ and $\mathcal{O}_{(a,b)}(Y) := Y_{\tau_{(a,b)}(Y)}$. Then

$$\mathcal{O}_{(a_n,b_n)}(X^*) = \frac{a_n + b_n}{a + b}\mathcal{O}_{(a,b)}(X^n) \qquad \text{where } X^n = \frac{ba_n - ab_n}{a_n + b_n} + \frac{a + b}{a_n + b_n}X^*.$$

Since clearly X^n converges in law (in Skorokhod topology) to X^*, we can once again proceed similar to the proof of Proposition 3 in [22] to show that $\mathcal{O}_{(a_n,b_n)}(X^*)$ converges in law to $\mathcal{O}_{(a,b)}(X^*)$. Then, as above, we use the uniform integrability of $|\mathcal{O}_{(a_n,b_n)}(X^*)|^\beta$ for $\beta \in (0, \alpha)$ to show that $E[|\mathcal{O}_{(a_n,b_n)}(X^*)|^\beta]$ converges to $E[|\mathcal{O}_{(a,b)}(X^*)|^\beta]$.

Next, letting $\delta > 0$, we use the uniform continuity of u^β to choose ρ such that for all (a, b) and (a', b') belonging to $[B^{-1}, B]$, $|a - a'| + |b - b'| \leq \rho$ implies $|u^\beta(a, b) - u^\beta(a', b')| \leq \delta/2$.

Next, for every $\lambda > 0$,

$$u^\beta_\varepsilon(\lambda a, \lambda b) = \lambda^\beta u^\beta_{\varepsilon\lambda}(a, b),$$

which means that $u^\beta_\varepsilon(\lambda a, \lambda b)$ converges to $u^\beta(\lambda a, \lambda b)$ uniformly on $\lambda \in [\lambda_1, \lambda_2]$ for $0 < \lambda_1 < \lambda_2 < \infty$. For $B^{-1} = a_0 < a_1 < \cdots < a_N = B$ with $a_{i+1} - a_i \leq \rho$ for $i = 0, \dots, N - 1$, this enables us to find ε_0 such that for all $\varepsilon < \varepsilon_0$, every $i = 0, \dots, N$ and all $\lambda \in [B^{-2}, 1]$,

$$\left|u^\beta_\varepsilon(\lambda a_i, \lambda B) - u^\beta(\lambda a_i, \lambda B)\right| \leq \frac{\delta}{2}. \tag{56}$$

Now, let $(a, b) \in [B^{-1}, B]$ be arbitrary, but to fix the ideas, assume without loss of generality that $a \leq b$. Since $u^\beta_\varepsilon(a, b)$ is increasing in a on $a \leq b$,

$$u^\beta_\varepsilon(a, b) \in \left[u^\beta_\varepsilon\left(a_i \frac{b}{B}, b\right), u^\beta_\varepsilon\left(a_{i+1} \frac{b}{B}, b\right)\right],$$

where i is such that $a_i \leq a\frac{B}{b} \leq a_{i+1}$, and by the property (56), also

$$u^\beta_\varepsilon(a, b) \in \left[u^\beta\left(a_i \frac{b}{B}, b\right) - \frac{\delta}{2}, u^\beta\left(a_{i+1} \frac{b}{B}, b\right) + \frac{\delta}{2}\right].$$

We finally use the uniform continuity of u^β to conclude that $u^\beta_\varepsilon(a, b) \in [u^\beta(a, b) - \delta, u^\beta(a, b) + \delta]$. $\square$

APPENDIX C: A TOY MODEL WITH A CONTINUOUS COMPONENT

Through a toy model, we show in the next proposition that if we include a continuous local martingale part in X, it dominates the purely discontinuous part.

PROPOSITION 7. *Assume (HY) and there exists $B > 0$, $\sigma > 0$ and $\alpha' \in (1, 2)$ such that $|A_t| \le B$, $\frac{1}{B} \le \underline{a}_t, \overline{a}_t \le B$ and X is a Lévy process with characteristic triplet $(\sigma^2, \nu, 0)$ with respect to the truncation function $h(x) = -1 \vee x \wedge 1$ where ν is a Lévy measure with Lévy density*

$$\nu(x) = \frac{c_+ 1_{x>0} + c_- 1_{x<0}}{|x|^{1+\alpha'}}.$$

Then Theorems 1 and 2 hold with $\lambda \equiv 1$, $\alpha = 2$, $\beta < \alpha'$ and $X_t^* = \sigma W_t$, where W_t is a Brownian motion.

PROOF. We first show that Theorem 1 holds with $X_t^* = \sigma W_t$. We follow the steps of the proof in Section 4. Step 1 follows from the assumptions of the proposition, and there is now no need to change probability. Also, Lemma 3 easily holds in the setting of Proposition 7. For step 3, note that $\lambda_t = 1$ and therefore

$$\hat{X}_t^i = \tilde{X}_t^i = X_{T_i+t} - X_{T_i}, \qquad \tilde{\tau}_i = T_{i+1} - T_i.$$

Thus we easily get (35) with $Q = P$ and $Z_t = 1$. Then for step 4 we have

$$E_{\mathcal{F}_{T_i}}\left[\left(\int_0^{\tilde{\tau}_i} |\hat{X}_t|^\kappa \, dt\right)^n\right] = \underline{f}_\varepsilon^{\kappa,n}(\underline{a}_{T_i}, \overline{a}_{T_i}),$$

with B^2 taken equal to zero in the definition of $\hat{\tau}_1$ defining $\underline{f}_\varepsilon^{\kappa,n}(a, b)$. Then note from [22], τ_1^ε has uniformly bounded polynomial moments of any order and X_t^ε (with $\alpha = 2$) converges toward σW_t. Following the proof of Lemma 5, this gives that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-(\kappa+2)} f_\varepsilon^{\kappa,n}(a, b) = f^{*,\kappa,n}(a, b).$$

Finally, we obtain that the preceding convergence is uniform in (a, b) as in steps 4 and 5 follows easily.

In the same spirit, in order to show that Theorem 2 holds with $X_t^* = \sigma W_t$ and $\alpha = 2$, it is enough to follow the steps of the proof in Section 6. This can be done as in the preceding paragraph. However, we still need to prove part (ii) in Lemma 11 in the case where a Brownian component is present, meaning we take $X_t^* = \sigma W_t$ for the limiting process and $\alpha = 2$ in the definition of X_t^ε. To this end, remark that in the setting of Proposition 7,

$$\left|X_{\tau^\varepsilon}^\varepsilon\right|^\beta \le c\left(1 + \left|\check{X}_{\check{\tau}^\varepsilon}^\varepsilon\right|^\beta\right),$$

with $\check{X}_t = X_t - \sigma W_t$ and $\check{\tau}^\varepsilon = \inf\{t \geq 0 : \check{X}_t^\varepsilon \notin (-(a+b), a+b)\}$. Thus, using Lemma 11, we get that

$$E\big[|X^\varepsilon_{\tau^\varepsilon}|^\beta\big]$$

is bounded uniformly in ε. Then we can replicate the end of the proof of Lemma 11. $\square$

APPENDIX D: PROOF OF PROPOSITION 1

PROOF. The process X can be written as

$$X_t = X_0 + \int_0^t \bar{b}_s\,ds + \int_0^t \int_{|z|\leq 1} z(M-\mu)(ds\times dz) + \int_0^t \int_{|z|>1} zM(ds\times dz),$$

where M is a random measure whose compensator μ is given by $\mu(\omega, dt\times dz) = dt \times \bar{\nu}(\gamma_t^{-1}(dz))1_{z\in\gamma_t(U)} = \frac{\bar{\nu}(\gamma_t^{-1}(z))}{\gamma_t'(\gamma^{-1}(z))}1_{z\in\gamma_t(U)}\,dt \times dz$. Hence,

$$\mu_t((x,\infty)) = \int_{\gamma_t^{-1}(x)}^\infty \bar{\nu}(y)1_{y\in U}\,dy,$$

$$\mu_t((-\infty,-x)) = \int_{-\infty}^{\gamma_t^{-1}(-x)} \bar{\nu}(y)1_{y\in U}\,dy.$$

By assumption (21),

$$\int_x^\infty \bar{\nu}(y)1_{y\in U}\,dy = \frac{c_+}{x^\alpha} + O\big(x^{1-\alpha}\big) \quad \text{and} \quad \int_{-\infty}^{-x} \bar{\nu}(y)1_{y\in U}\,dy = \frac{c_-}{x^\alpha} + O\big(x^{1-\alpha}\big)$$

as $x \to 0$ and

$$\int_x^\infty \bar{\nu}(y)1_{y\in U}\,dy + \int_{-\infty}^{-x} \bar{\nu}(y)1_{y\in U}\,dy \leq \frac{C}{x^\alpha}$$

for some $C < \infty$ and all $x > 0$. On the other hand, by Taylor's theorem, $\gamma_t^{-1}(x) = \frac{x}{\gamma_t'(x^*)}$ with $x^* \in [0, x]$. Therefore, we easily obtain that for some $C < \infty$,

$$x^\alpha \mu_t((x,\infty)) + x^\alpha \mu_t((-\infty,-x)) \leq C \max_{x\in U} \gamma_t'(x)^\alpha \qquad \text{for all } x; \tag{57}$$

$$\begin{aligned} &\lim_{x\downarrow 0} x^\alpha \mu_t((x,\infty)) = c_+ \gamma_t'(0)^\alpha \quad \text{and} \\ &\lim_{x\downarrow 0} x^\alpha \mu_t((-\infty,-x)) = c_- \gamma_t'(0)^\alpha, \end{aligned} \tag{58}$$

which proves assumption (*HX*).

To show (HX^{ρ}_{loc}), let ν be a strictly positive Lévy density satisfying (21), continuous outside any neighborhood of zero. We need to prove that the random function $K_t(z)$ defined by

$$K_t(z) = \frac{\bar\nu(\gamma_t^{-1}(z))1_{z\in\gamma_t(U)}}{\gamma_t'(\gamma_t^{-1}(z))\gamma_t'(0)^\alpha \nu(z)},$$

satisfies the integrability condition (17). Let (τ_n) be the sequence of stopping times from condition (22), let $t < \tau_n$ and ε be small enough so that $\{|z| \le \varepsilon\} \subset \gamma_t(U)$, $t \le \tau_n$. Clearly,

$$\begin{aligned}\int_{\mathbb{R}} \bigl|\sqrt{K_t(z)} - 1\bigr|^{2\rho}\nu(dz) \le \int_{|z|\le\varepsilon} \bigl|\sqrt{K_t(z)} - 1\bigr|^{2\rho}\nu(dz) \\ + \int_{|z|>\varepsilon, z\in\gamma_t(U)} K_t^{\rho}(z)\nu(dz) + \nu\bigl(\{z : |z| > \varepsilon\}\bigr).\end{aligned} \tag{59}$$

The third term above is clearly bounded. To deal with the second term, observe that by the fact that ν and $\bar\nu$ are continuous outside any neighborhood of zero, condition (22) and the fact that U is compact, on the set $\{z : |z| > \varepsilon, z \in \gamma_t(U)\}$ for $t \le \tau_n$,

$$K_t \le C_n^{1+\alpha}\frac{\max\{\bar\nu(z) : z \in U, |z| \ge \varepsilon/C_n\}}{\min\{\nu(z) : |z| \ge \varepsilon, z \in C_nU\}} < \infty.$$

Therefore, the second term in (59) is also bounded for $t \le \tau_n$. We finally focus on the first term in (59). First, observe that on the set where $|z| \le \varepsilon$,

$$\begin{aligned}\bigl|K_t(z) - 1\bigr| \le{}& \left|\frac{|z|^{1+\alpha}}{|\gamma_t^{-1}(z)|^{1+\alpha}\gamma_t'(0)^{1+\alpha}} - 1\right| \frac{\gamma_t'(0)}{\gamma_t'(\gamma_t^{-1}(z))} \frac{|\gamma_t^{-1}(z)|^{1+\alpha}\bar\nu(\gamma_t^{-1}(z))}{|z|^{1+\alpha}\nu(z)} \\ &+ \left|\frac{\gamma_t'(0)}{\gamma_t'(\gamma_t^{-1}(z))} - 1\right| \frac{|\gamma_t^{-1}(z)|^{1+\alpha}\bar\nu(\gamma_t^{-1}(z))}{|z|^{1+\alpha}\nu(z)} \\ &+ \left|\frac{|\gamma_t^{-1}(z)|^{1+\alpha}\bar\nu(\gamma_t^{-1}(z))}{|z|^{1+\alpha}\nu(z)} - 1\right|.\end{aligned} \tag{60}$$

For the first term in (60), by Taylor's formula and using condition (22),

$$\begin{aligned}\left|\frac{|z|^{1+\alpha}}{|\gamma_t^{-1}(z)|^{1+\alpha}\gamma_t'(0)^{1+\alpha}} - 1\right| = \left|\frac{\gamma_t'(z^*)^{1+\alpha}}{\gamma_t'(0)^{1+\alpha}} - 1\right| &\le (1+\alpha)C_n^{2\alpha+1}\bigl|\gamma_t'(z^*) - \gamma_t'(0)\bigr| \\ &\le (1+\alpha)C_n^{2\alpha+2}|z|,\end{aligned}$$

where $z^* \in [z \wedge 0, z \vee 0]$. In the second term, similarly,

$$\left|\frac{\gamma_t'(0)}{\gamma_t'(\gamma_t^{-1}(z))} - 1\right| \le C_n\bigl|\gamma_t'\bigl(\gamma_t^{-1}(z)\bigr) - \gamma_t'(0)\bigr| \le C_n^2\bigl|\gamma_t^{-1}(z)\bigr| \le C^{3n}|z|.$$

For the third term, it follows from (21) that for some constant $C < \infty$,

$$\left|\frac{|\gamma_t^{-1}(z)|^{1+\alpha}\bar{\nu}(\gamma_t^{-1}(z))}{|z|^{1+\alpha}\nu(z)} - 1\right| \le \left|\frac{1 + C|\gamma_t^{-1}(z)|}{1 - C|z|} - 1\right| \le \frac{C}{1 - C\varepsilon}\left(\left|\gamma_t^{-1}(z)\right| + |z|\right)$$
$$\le \frac{C(1 + C_n)}{1 - C\varepsilon}|z|.$$

In addition, assume that ε is chosen small enough so that $C\varepsilon < 1$. Therefore,

$$\left|K_t(z) - 1\right| \le c_n|z|$$

for some constant $c_n < \infty$ (which may later change from line to line). This easily implies that for $\rho \ge 1$,

$$\int_{|z|\le\varepsilon} \left|\sqrt{K_t(z)} - 1\right|^{2\rho} \nu(dz) \le c_n. \qquad \square$$

Acknowledgments. We are very grateful to the Associate Editor and to the three referees for their careful reading of the manuscript and their very relevant remarks.

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