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STABILITY OF SOLITONS UNDER RAPIDLY OSCILLATING RANDOM PERTURBATIONS OF THE INITIAL CONDITIONS

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We use the inverse scattering transform and a diffusion approximation limit theorem to study the stability of soliton components of the solution of the nonlinear Schrödinger and Korteweg–de Vries equations under random perturbations of the initial conditions: for a wide class of rapidly oscillating random perturbations this problem reduces to the study of a canonical system of stochastic differential equations which depends only on the integrated covariance of the perturbation. We finally study the problem when the perturbation is weak, which allows us to analyze the stability of solitons quantitatively.

1. Introduction. The aim of the present work is to study the stability of the soliton components of solutions of completely integrable systems under rapidly oscillating random perturbations of the initial condition. We will consider and compare two important examples of equations widely employed to model nonlinear and dispersive effects in wave propagation: the (1-dimensional) nonlinear Schrödinger (NLS) equation,

$$\frac{\partial U}{\partial t} + \frac{i}{2}\frac{\partial^2 U}{\partial x^2} + i|U|^2 U = 0 \tag{1}$$

and the Korteweg–de Vries (KdV) equation,

$$\frac{\partial U}{\partial t} + 6U\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} = 0. \tag{2}$$

The NLS equation models in particular short pulse propagation in single-mode optical fibers (then t is a propagation distance and x is a time) [14]. The KdV equation models shallow water wave propagation [16].

Explicit results are derived for the case of a square (box-like) initial condition perturbed with a zero mean, stationary, rapidly oscillating process $\nu(x/\varepsilon^2)$,

$$U_0(x) = \left(q + \frac{\sigma}{\varepsilon}\nu\left(x/\varepsilon^2\right)\right)\mathbf{1}_{[0,R]}(x), \tag{3}$$

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but the results on the fast oscillating regime of Section 3 can be extended to the case of perturbation of a more general initial condition defined by a bounded, compactly supported function $q(x)$. The function q must be real for KdV, but is allowed to take complex values for NLS: the computations presented in Sections 3 and 4 are relative to the case of a real q, but can be easily extended to the case of a complex (but with constant phase) function q. The rapidly oscillating fluctuations of the initial condition can model the high frequency additive noise of the light source generating the pulse in nonlinear fiber optics, for instance.

Our approach to both examples relies on the inverse scattering transform (IST), a powerful tool used to study solutions of completely integrable nonlinear equations; see [2]. In this framework, the problem is transformed into a linear system of differential equations where the initial condition enters as a potential, and soliton components correspond to eigenvalues. Indeed, the solution of a nonlinear dispersive equation modeling the propagation of waves may show two components with a very distinct behavior: the soliton components, composed of solitary waves that propagate over arbitrarily large distances with constant velocity and constant profile, and in addition, the radiation component, whose amplitude decays in time as a power law. The identification of the soliton components therefore characterizes the long-time behavior of the solution of the PDE. A short introduction to the IST is presented in Section 2, together with a discussion of the deterministic case [$\sigma = 0$ in equation (3)].

We will show in Section 3 that for rapidly oscillating processes (small values of ε) the limit system governing the stability of the soliton components reads as a set of stochastic differential equations (SDEs), and it is formally equivalent to the system where the initial condition contains a white-noise perturbation,

$$U_0(x) = (q + \sqrt{2\alpha}\sigma \dot{W}_x)\mathbf{1}_{[0,R]}(x), \tag{4}$$

where α is the integrated covariance of the process ν. This shows that to study the soliton components in the limit of rapid oscillations the only required parameter of the statistics of ν is its integrated covariance. Notice that we cannot directly use a white noise to perturb the initial condition, as the IST requires some integrability conditions on the initial condition (e.g., $U_0 \in L^1$), which are not satisfied by a white noise. The main result is presented in Theorem 3.2.

We also obtain that solitons are stable under perturbations of the initial condition for both examples studied; this is shown in Sections 4 and 5. However, a few interesting differences will be pointed out; in particular, thresholding effects for the creation of solitons are present in the NLS case and absent for KdV.

We also provide an easy way to compute the first order corrections to the parameters characterizing the soliton components of the solution.

Results and some future directions of research are discussed in the last section.

2. The inverse scattering transform and the deterministic problem. In the IST framework, a direct scattering problem (known as the Zakharov–Shabat spectral problem, ZSSP) associated to the NLS equation is introduced

$$\begin{cases} \dfrac{\partial \psi_1}{\partial x} = iU_0(x)\psi_2 - i\zeta\psi_1, \\ \dfrac{\partial \psi_2}{\partial x} = iU_0^*(x)\psi_1 + i\zeta\psi_2, \end{cases} \tag{5}$$

where $x \in \mathbb{R}$, $\psi_i(x)$, $i = 1, 2$, are the components of a complex vector eigenfunction $\Psi(x) \in \mathbb{H}^1(\mathbb{R})$ and $\zeta \in \mathbb{C}$ is the spectral parameter. The space $\mathbb{H}^1(\mathbb{R})$ is defined as $\mathbb{H}^1(\mathbb{R}) = \{\Psi | \psi_i \in L^2(\mathbb{R}), \partial_x \psi_i \in L^2(\mathbb{R}), i = 1, 2\}$. When $U_0 = 0$, it is easy to see that the continuous part of the spectrum is composed by the whole real line. The eigenspace associated to the eigenvalue $\zeta \in \mathbb{R}$ has dimension 2, and the functions

$$\Psi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x}, \qquad \Phi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x}$$

define a basis of this space. In this case, the discrete spectrum is empty because the nontrivial solutions of $\partial_x f = i\zeta f$ are not in $L^2(\mathbb{R})$.

When introducing any localized initial condition U_0, by Weyl's theorem the continuous spectrum (unlike the discrete spectrum) remains unchanged, and for $\zeta \in \mathbb{R}$ the solutions Ψ, $\widetilde{\Psi}$, Φ, $\widetilde{\Phi}$ defined by the boundary conditions

$$\begin{aligned} &\Psi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x}, \qquad \widetilde{\Psi} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x}, \qquad x \to -\infty, \\ &\Phi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x}, \qquad \widetilde{\Phi} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x}, \qquad x \to +\infty \end{aligned} \tag{6}$$

produce two sets $\{\Psi, \widetilde{\Psi}\}$ and $\{\Phi, \widetilde{\Phi}\}$ of linearly independent solutions. These functions are related through the system

$$\begin{pmatrix} \Psi \\ \widetilde{\Psi} \end{pmatrix} = \begin{pmatrix} b(\zeta) & a(\zeta) \\ \widetilde{a}(\zeta) & \widetilde{b}(\zeta) \end{pmatrix} \begin{pmatrix} \Phi \\ \widetilde{\Phi} \end{pmatrix}, \tag{7}$$

where a, b are called Jost coefficients and Ψ, Φ are called Jost functions. The Jost coefficients are complex-valued functions, while the Jost functions take values in $\mathbb{C}^2$. Therefore, products in the above equation have to interpreted as "scalar times vector" products, so that one has

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = b(\zeta) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + a(\zeta) \begin{pmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \end{pmatrix}$$

and similarly for $\widetilde{\Psi}$.

If $U_0 \in L^1(\mathbb{R})$, the function $a(\zeta)$ can be continuously extended to the upper half of the complex plane $\mathbb{C}^+ = \{\zeta \in \mathbb{C} | \Im[\zeta] > 0\}$, where it is analytic and can only have a countable number of simple zeros; see [2], Lemma 2.1. These zeros turn out to be the eigenvalues of the discrete spectrum of the ZSSP (5). If ζ_n is a zero of a, then from (7) we obtain that Ψ and Φ are linearly dependent. Due to (6) this implies that the eigenfunction Ψ_n relative to the eigenvalue ζ_n has an exponential decay both at $-\infty$ and $+\infty$.

The IST is a powerful tool which allows us to solve many nonlinear completely integrable systems, and the introduction of the IST formalism is particularly convenient when dealing with soliton components of the solution, as solitons have a very easy representation in terms of the scattering variables: each zero of the Jost coefficient a in the upper complex half-plane ($\zeta = \xi + i\eta$, $\eta > 0$) corresponds to a soliton component of the solution. As we have just remarked, these zeros of the Jost coefficient a correspond to the discrete spectrum of (5).

When we study eigenfunctions of the spectral problem (5) with a potential U_0 of compact support in $[0, R]$ we can rewrite the system (5), which is defined for $x \in \mathbb{R}$, as a system defined for $x \in [0, R]$, with some boundary conditions in $x = 0$ and $x = R$ obtained from (6). This can be done as follows. By inspecting the spectral problem (5) we see that if $\zeta \in \mathbb{C}^+$ is a discrete eigenvalue, then the corresponding eigenfunction $\Psi \in \mathbb{H}^1(\mathbb{R})$ for $x \leq 0$ is given by $\psi_1(x) = e^{-i\zeta x}$ and $\psi_2(x) = 0$. For $x \geq R$ it must satisfy $\partial_x \psi_1 = -i\zeta\psi_1$ and $\partial_x\psi_2 = i\zeta\psi_2$, that is, to say $\psi_1(x) = \psi_1(R)e^{-i\zeta(x-R)}$ and $\psi_2(x) = \psi_2(R)e^{i\zeta(x-R)}$. Since the eigenfunction Ψ must be integrable and $\Im[\zeta] = \eta > 0$, this implies that

$$\psi_1(R) = 0.$$

A pure soliton solution of the NLS equation has the form

$$U(t,x) = 2i\eta \frac{\exp(-2i\xi x - 4i(\xi^2 - \eta^2)t)}{\cosh(2\eta(x + 4\xi t))},$$

up to a shift and a phase.

Similarly, one can link the existence of soliton components of solutions of the KdV equation to the spectral properties of an associated equation, the first equation of the Lax pair,

$$\frac{\partial^2\varphi}{\partial x^2} + (U_0 + \zeta^2)\varphi = 0, \tag{8}$$

where the real function $\varphi(x)$ belongs to the Sobolev space $W^{2,2}(\mathbb{R})$. We need to assume that

$$U_0 \in P_1 := \left\{ f : \mathbb{R} \to \mathbb{R} \middle| \int_{-\infty}^{\infty} (1 + |x|)|U(x)|\,\mathrm{d}x < \infty \right\}; \tag{9}$$

see [1], Chapter 2. Consider the continuous part of the spectrum of equation (8), which is again the real axis. For $\zeta \in \mathbb{R}$, there are two convenient complete sets of bounded functions solutions of (8), defined by their asymptotic behavior:

$$\begin{aligned} &\phi(x,\zeta) \sim e^{-i\zeta x}, \qquad \widetilde{\phi}(x,\zeta) \sim e^{i\zeta x} \qquad \text{for } x \to -\infty; \\ &\psi(x,\zeta) \sim e^{i\zeta x}, \qquad \widetilde{\psi}(x,\zeta) \sim e^{-i\zeta x} \qquad \text{for } x \to +\infty. \end{aligned}$$

It follows from the above definitions that

$$\phi(x,\zeta) = \widetilde{\phi}(x,-\zeta), \qquad \psi(x,\zeta) = \widetilde{\psi}(x,-\zeta)$$

and

$$\begin{aligned} \phi(x,\zeta) &= a(\zeta)\widetilde{\psi}(x,\zeta) + b(\zeta)\psi(x,\zeta), \\ \widetilde{\phi}(x,\zeta) &= -\widetilde{a}(\zeta)\psi(x,\zeta) + \widetilde{b}(\zeta)\widetilde{\psi}(x,\zeta). \end{aligned}$$

The function a can be continuously extended to the upper half of the complex plane $\mathbb{C}^+$, where it is analytic and can only have a finite number of simple zeros located on the imaginary axis $\zeta = i\eta$; see [1], Lemma 2.2.2. These zeros are the eigenvalues of the discrete spectrum, and they correspond to the soliton components of the solution. A pure soliton solution is given by

$$U(t,x) = 2\eta^2 \operatorname{sech}^2\big(\eta\big(x - x(t)\big)\big),$$

where $x(t) = x_0 + 4\eta^2 t$ is the center of the soliton.

2.1. *NLS—deterministic box-shaped initial conditions.* Let the initial condition of the NLS equation be given by $U_0(x) = q\mathbf{1}_{[0,R]}(x)$. Burzlaff proved in [4] that in this case the number of solitons generated is the integer part of $1/2 + qR/\pi$; see also the relevant discussion and generalization of [9]. They remark how physical intuition suggests that the first soliton created when increasing R corresponds to $\zeta = 0$ (this is a single soliton with zero amplitude and velocity, the *quiescent soliton*); this "soliton" is created for $qR = \pi/2$. For values of qR just over this critical threshold the created soliton has zero velocity and nonzero amplitude 2η which can be computed explicitly solving (5) for pure imaginary values of ζ.

In the first part of this subsection we report some computations relative to this case, as the results and explicit formulas will be used below. We then conclude the subsection providing the sketch of an analytical proof of the claimed fact that generated solitons correspond to purely imaginary values of ζ.

When the potential U_0 is of compact support in $[0, R]$ and for purely imaginary values of $\zeta = i\eta$, from the decaying condition of the Jost function Ψ at $-\infty$, one obtains the initial condition

$$\Psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\eta x}\bigg|_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{10}$$

The system (5) for $x \in [0, R]$ reads

$$\begin{cases} \dfrac{\partial \psi_1}{\partial x} = iq\psi_2 - i\zeta\psi_1, \\ \dfrac{\partial \psi_2}{\partial x} = iq\psi_1 + i\zeta\psi_2 \end{cases} \tag{11}$$

and $\Psi = (\psi_1, \psi_2)$ is a solution of the initial value problem for $\zeta \neq iq$ if

$$\psi_1(x) = -\frac{i\zeta}{\sqrt{q^2+\zeta^2}} \sin\bigl(\sqrt{q^2+\zeta^2}x\bigr) + \cos\bigl(\sqrt{q^2+\zeta^2}x\bigr), \tag{12}$$

$$\psi_2(x) = i\frac{q}{\sqrt{q^2+\zeta^2}} \sin\bigl(\sqrt{q^2+\zeta^2}x\bigr). \tag{13}$$

To be an eigenfunction, Ψ needs to be integrable and to satisfy the final condition $\psi_1(R) = 0$ at R. This condition can be rewritten for $\zeta \neq 0$ as

$$f = \tan\bigl(\sqrt{q^2+\zeta^2}R\bigr) + i\frac{\sqrt{q^2+\zeta^2}}{\zeta} = 0. \tag{14}$$

Since $a(\zeta) = \psi_1(R,\zeta)e^{i\zeta R}$, the function f is linked to the first Jost coefficient a by the relation

$$f(\zeta) = ia(\zeta)\frac{e^{-i\zeta R}}{\zeta}\frac{\sqrt{q^2+\zeta^2}}{\cos(\sqrt{q^2+\zeta^2}R)}$$

from which we see that the zeros of f coincide with those of a, except for $\zeta = iq$. However, for $\zeta = iq$ it is possible to compute explicitly the solution of (11) satisfying the initial conditions, which is given by

$$\Psi = \begin{pmatrix} 1+x \\ ix \end{pmatrix}.$$

Since this function does not satisfy the final conditions, no soliton can be created for this particular value of ζ.

To prove in an analytic way that the first soliton component of the solution corresponds to a purely imaginary value of ζ, we can proceed as follows.

Recall that the zeros of f coincide with those of a, and observe that the function $a(\xi, \eta, R)$ is analytic in the domain $\mathbb{R} \times (0,\infty) \times (0,\infty)$ and continuous in $\mathbb{R} \times [0,\infty) \times (0,\infty)$. We use the argument principle to study how the number of zeros in the upper half of the complex plane evolves with increasing R. For any fixed R we proceed as in [5], taking a loop C in the complex ζ-plane composed of the (lower) real axis and the infinite semi-arc in the upper half plane. Then the number of zeros is given by

$$N = \frac{1}{2\pi}\int_C \frac{1}{a}\frac{\partial a}{\partial \zeta}\,\mathrm{d}\zeta.$$

Since $a = 1 + O(1/\zeta)$ for $|\zeta| \gg 1$, the integral over the upper part of the loop is zero. Changing variables $a(\zeta) = \rho(\zeta)\exp(i\alpha(\zeta))$, after some computations one obtains that unless there is a zero on the real axis, also the integral on the lower part of the loop is zero. Therefore, the number of zeros changes for a given R only if $a(\xi, 0, R) = 0$ admits a solution. But zeros of a and f coincide, and since in equation (14) for real values of $\zeta = \xi \neq 0$ the tangent is real, and the second term is purely imaginary and nonzero, solutions of $f(\xi, 0, R) = 0$ can only be found at $\xi = 0$.

Explicit computations easily show that $\zeta = 0$ corresponds to a soliton solution only for $R = \frac{2n+1}{2q}\pi$, $n \in \mathbb{N}$. Computing explicitly the derivative of $a(\zeta)$ at $\zeta = 0$ we get

$$\partial_\zeta a(\zeta) = e^{i\zeta R}\Big[\Big(2\frac{\zeta R}{\sqrt{q^2+\zeta^2}} - i\frac{q^2}{\sqrt{q^2+\zeta^2}^3}\Big)\sin\big(\sqrt{q^2+\zeta^2}R\big) + iR\frac{q^2}{q^2+\zeta^2}\cos\big(\sqrt{q^2+\zeta^2}R\big)\Big],$$

$$\partial_\zeta a(\zeta)|_{\zeta=0} = -i\frac{1}{q}\sin(qR) + iR\cos(qR),$$

so that for $Rq = \frac{2n+1}{2}\pi$ the derivative is equal to $(-1)^{n+1}i/q$ and is never zero. Therefore, new solitons are generated one at a time, and they are immediately pushed (as R increases) toward the interior of the domain. Karpman [8] showed that if $a(\zeta) = 0$, then $a'(\zeta) \neq 0$; from this fact it follows that zeros in the interior of the domain are always simple. Considering the complex conjugate Ψ^*, which is a solution whenever Ψ is, one obtains that zeros not laying on the imaginary axis always come in pairs $\pm\xi + i\eta$. But since zeros move continuously (as R grows) in the upper complex plane, cannot coalesce and cannot leave the imaginary axis ($\xi = 0$) unless they form a pair, we get that they must remain on the imaginary axis.

2.2. *KdV—deterministic box-shaped initial conditions.* In [15], Murray obtained a $\mathcal{C}^\infty$ solution for the KdV equation with a deterministic "box-shaped" initial condition $U_0 = q\mathbf{1}_{[-R,R]}(x)$. He showed that in this case the Jost coefficient a extends to an analytic function in the upper part of the ζ-plane. Only in the case of positive values of q, a has a finite number of zeros on the imaginary axis $\zeta = i\eta$ for $0 < \eta \leq \sqrt{q}$.

We report some explicit computations on our similar deterministic case, as the results will be used below, and study some properties of the soliton components of the solution.

First, let us construct explicitly the eigenfunctions solution of the deterministic equation, which we call φ_0. Take here $U_0 = q\mathbf{1}_{[0,R]}(x)$ for some $q > 0$. Due to [1], Lemma 2.2.2, we can assume that the eigenvalue is given by $\zeta = i\eta$; we need to solve

$$\varphi_{xx} = \begin{cases} \eta^2\varphi, & x < 0, x > R, \\ (-q+\eta^2)\varphi, & x \in [0, R]. \end{cases} \tag{15}$$

For $\eta = 0$ the only integrable solution is $\varphi \equiv 0$. Eigenfunctions corresponding to $\eta > 0$ must satisfy

$$\varphi = c_1 e^{\eta x}, \qquad x < 0, \tag{16}$$

$$\varphi = c_2 e^{-\eta x}, \qquad x > R. \tag{17}$$

For $x \in [0, R]$ one can rewrite the problem as

$$\begin{cases} \partial_x \varphi = \widetilde{\varphi}, \\ \partial_x \widetilde{\varphi} = (-q + \eta^2)\varphi. \end{cases}$$

Note that for $\eta \geq \sqrt{q}$ the solution of (15) is monotone, so that it cannot be a Jost function corresponding to a soliton (which has to be integrable). We therefore look for solutions corresponding to $0 < \eta < \sqrt{q}$. Set $c = \sqrt{q - \eta^2}$. Due to (16) and (17), we only need to solve (15) for $x \in [0, R]$. From (15) and the initial conditions

$$\varphi_0(0) = c_1, \qquad \partial_x \varphi_0(0) = \eta c_1$$

derived from (16), we get

$$\varphi_0(x) = \alpha e^{icx} + \beta e^{-icx}, \qquad \alpha = \frac{c_1}{2}\left(1 - i\frac{\eta}{c}\right), \qquad \beta = \frac{c_1}{2}\left(1 + i\frac{\eta}{c}\right),$$

which is to say

$$\varphi_0(x) = c_1 \cosh(icx) - ic_1\frac{\eta}{c}\sinh(icx) = c_1\left[\cos(cx) + \frac{\eta}{c}\sin(cx)\right]. \tag{18}$$

We can set the global constant c_1 equal to 1. Matching this solution with the final condition (17)

$$\begin{cases} \varphi(R) = \cos(cR) + \dfrac{\eta}{c}\sin(cR) = c_2 e^{-\eta R}, \\ \partial_x \varphi(R) = -c\sin(cR) + \eta\cos(cR) = -\eta c_2 e^{-\eta R}, \end{cases}$$

we obtain an equation for η,

$$\begin{cases} q\sin\left(R\sqrt{q-\eta^2}\right) = 2c_2\eta\sqrt{q-\eta^2}\,e^{-R\eta}, \\ \cos\left(R\sqrt{q-\eta^2}\right) = c_2(q - 2\eta^2)e^{-R\eta}. \end{cases}$$

For $\eta = \sqrt{q/2}$, the only possible solution is such that $R\sqrt{q-\eta^2} = \pi/2 + k\pi$, which means that

$$\sqrt{q}R = (2k+1)\pi/\sqrt{2}. \tag{19}$$

All other solutions can be found solving

$$f(\eta) := \tan\left(R\sqrt{q-\eta^2}\right) - \frac{2\eta\sqrt{q-\eta^2}}{q - 2\eta^2} = 0 \tag{20}$$

for $\eta \in [0, \sqrt{q}) \setminus \{\sqrt{q/2}\}$. The existence and the number of solutions for the above equation depend on the quantity $R\sqrt{q}$. Consider some fixed value of R. As it is shown below, for small values of q a first soliton is created with $\eta^{(1)} \sim 0$. As q increases, the value of $\eta^{(1)}$ increases too and tends to $\sqrt{q/2}$ as q tends to $\pi^2/(2R^2)$. We have already found the solution for this specific value of q [$k = 0$ in equation (19)]. For q larger than $\pi^2/(2R^2)$, $\eta^{(1)}$ continues to grow. A second solution appears ($\eta^{(2)} = 0$) when $q = \pi^2/R^2$. A third solution appears at $q = (2\pi/R)^2$; the values of $\eta^{(i)}$ (corresponding to the ith soliton created) continuously increase as q grows, but remain ordered: $\eta^{(i)} < \eta^{(j)}$ for $i > j$. Therefore, the number of solitons created is $\lfloor R\sqrt{q}/\pi \rfloor + 1$.

A few examples of $f(\eta)$ are plotted in Figure 1. We have taken $R = 1$ and different values of q. The first critical points (when new solitons are created) correspond here to $q^{(2)} = \pi^2 \sim 9,87$, $q^{(3)} = 4\pi^2 \sim 39,48$, $q^{(4)} = 9\pi^2 \sim 88,83$, $q^{(5)} = 16\pi^2$. The almost-vertical line appearing near $\eta = \sqrt{q/2}$ for $q = 44$ reflects the fact that we are near the critical points of (19): from (19) for $k = 1$ we have $q = 9\pi^2/2 \sim 44.41$.

Let us take a closer look at the case $q \to 0$. We assume $q = q_0\varepsilon$ and look for the first terms of the expansion of η in ε: $\eta = \eta_0 + \eta_1\varepsilon + \eta_2\varepsilon^2 + O(\varepsilon^3)$. For finite values of R, by the above considerations on the threshold effect the first term of the expansion must be zero. Indeed, if we consider the expansion in ε of the function f defined by (20), we obtain

$$\text{(ord. 0)} \qquad f(\eta) = \tan\Bigl(R\sqrt{-\eta_0^2}\Bigr) + \frac{\sqrt{-\eta_0^2}}{\eta_0} + O(\varepsilon)$$

and the order-zero term cannot be made equal to zero. We have therefore $\eta = \eta_1\varepsilon + O(\varepsilon^2)$. Looking at equation (20) at first order

$$\text{(ord. 1)} \qquad f(\eta) = (Rq_0 - 2\eta_1)\frac{\sqrt{\varepsilon}}{\sqrt{q_0}} + O(\varepsilon^{3/2}) = 0,$$

we obtain $\eta_1 = \frac{Rq_0}{2}$. Pushing the expansion further, one can obtain the following order coefficients. At order two we have

$$\text{(ord. 2)} \qquad f(\eta) = -\left(\frac{4}{24}R^3q_0^2 + 2\eta_2\right)\frac{\varepsilon^{3/2}}{\sqrt{q_0}} + O(\varepsilon^{5/2}).$$

One has then

$$\eta = \frac{Rq_0}{2}\varepsilon - \frac{R^3q_0^2}{12}\varepsilon^2 + O(\varepsilon^3) \tag{21}$$

showing that in the limit $\varepsilon \to 0$, η is of the same order of q.

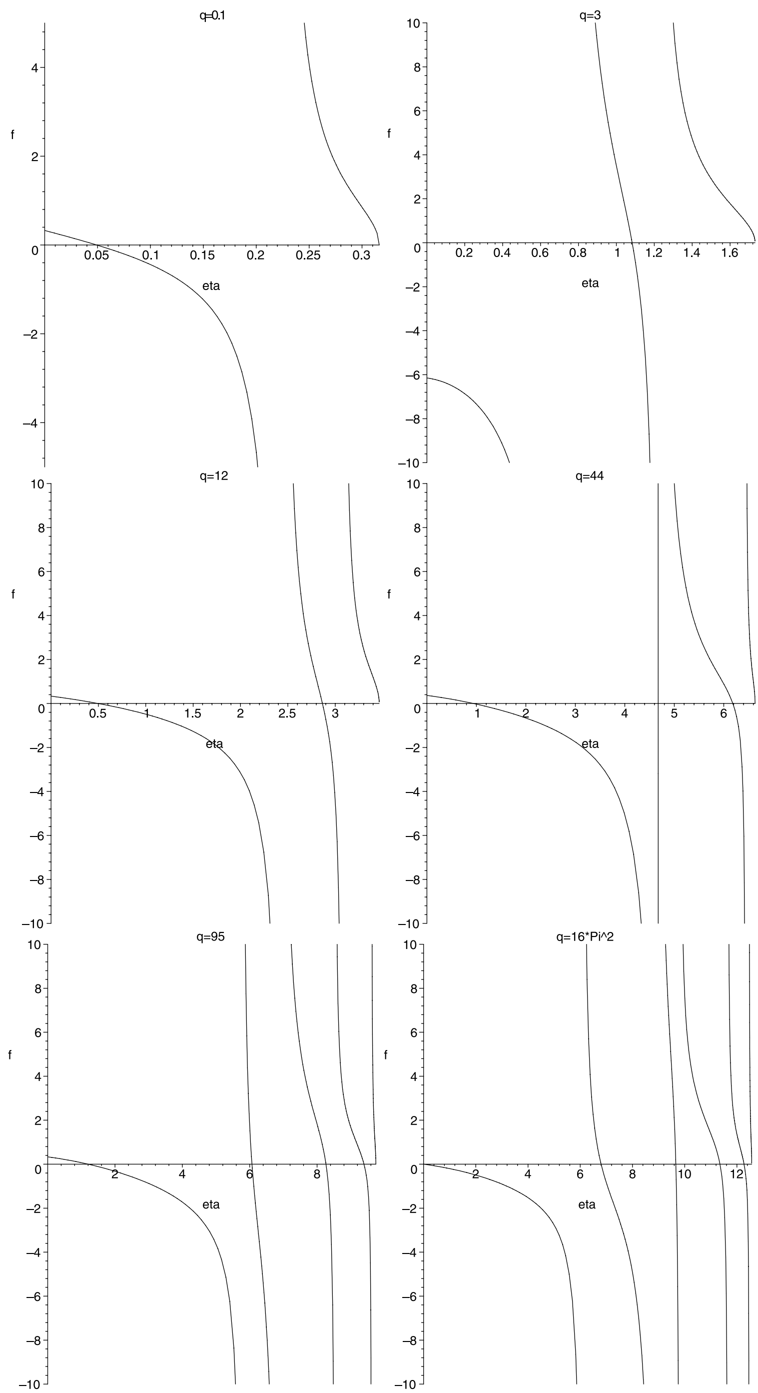

FIG. 1. *Plot of the function $f(\eta)$ for different values of the amplitude q of the initial condition. Each zero of $f(\eta)$ corresponds to a soliton component of the solution identified by the complex number $i\eta$.*

3. Limit of rapidly oscillating processes. This section contains a rigorous justification of the use of the IST when the initial condition contains a rapidly oscillating process. As remarked in the Introduction, to be able to apply the IST, the initial condition U_0 needs to satisfy some integrability conditions, L^1 for NLS and (9) for KdV. For any $\varepsilon > 0$ these hypotheses are satisfied by initial conditions of the form (3) if ν is bounded. Our objective is to show that the IST applied to these random initial conditions gives a problem that reads as a canonical system of SDEs in the limit $\varepsilon \to 0$. Thanks to the convergence result of Theorem 3.2 below, this limit system can be used to study the behavior of rapidly oscillating initial conditions ($0 < \varepsilon \ll 1$), as we shall do in the following sections. We stress that our interest is in the study of rapidly oscillating initial conditions, which are physically more relevant than the limit case of infinitely rapid oscillations and for which the IST can be applied in a rigorous way. We make the following assumptions (standard in the diffusion approximation theory, [7]) on the process ν:

HYPOTHESIS 3.1. *Let $\nu(x)$ be a real, homogeneous, ergodic, centered, bounded, Markov stochastic process, with finite integrated covariance $\int_0^\infty \mathbb{E}[\nu(0)\nu(x)]\,\mathrm{d}x = \alpha < \infty$ and with generator $\mathcal{L}_\nu$ satisfying the Freedholm alternative.*

Set

$$U_0^\varepsilon := \left(q + \frac{\sigma}{\varepsilon}\nu\big(x/\varepsilon^2\big)\right)\mathbf{1}_{[0,R]}(x)$$

and note that for $x \in [0, R]$

$$\int_0^x U_0^\varepsilon(y)\,\mathrm{d}y \xrightarrow{\varepsilon\to 0} \int_0^x U_0(y)\,\mathrm{d}y = qx + \sqrt{2\alpha}\sigma W_x$$

in the space of continuous functions $\mathcal{C}^0([0, R]; \mathbb{R})$, in distribution; see [7]. For every $\varepsilon > 0$, we apply the IST to the NLS and KdV equations with the initial condition U_0^ε, and obtain the associated spectral problem. Then we investigate the passage to the limit of this problem. We point out that this passage to the limit is quite delicate: if it is relatively easy to obtain a pointwise (in ζ) convergence of the spectral data, to obtain fine results for the limit case and for situations near the limit case ($0 < \varepsilon \leq 1$), a much stronger convergence is needed.

We consider the ZSSP associated to the NLS equation: our goal is to identify the points of the upper half of the complex plane, $\zeta \in \mathbb{C}^+$, for which there exists a solution $\Psi \in \mathbb{H}^1$ of the first order system (5) for $x \in [0, R]$, satisfying the boundary conditions

$$\Psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_1(R) = 0$$

derived from the exponentially decaying conditions (6). These particular values of ζ are the discrete eigenvalues of the ZSSP and correspond to the soliton components. The strategy employed is to consider the flow $\Psi(x, \zeta)$, $x \in [0, R]$, $\zeta \in \mathbb{C}^+$, solution of (5) with initial condition

$$\Psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{22}$$

and look for the values of ζ for which the final condition is satisfied.

For a fixed value of ζ, we consider the solution Ψ^ε of the ZSSP obtained from the IST

$$\begin{cases} \dfrac{\partial \psi_1^\varepsilon}{\partial x} = -i\zeta\psi_1^\varepsilon + iU_0^\varepsilon(x)\psi_2^\varepsilon, \\ \dfrac{\partial \psi_2^\varepsilon}{\partial x} = i\big(U_0^\varepsilon(x)\big)^*\psi_1^\varepsilon + i\zeta\psi_2^\varepsilon \end{cases} \tag{23}$$

with initial condition

$$\Psi^\varepsilon(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now, [7], Theorem 6.1, states that the process Ψ^ε converges in distribution in $\mathcal{C}^0([0, R]; \mathbb{C}^2)$ to the process Ψ solution of

$$\begin{cases} d\psi_1 = \big[(-i\zeta - \alpha\sigma^2)\psi_1 + iq\psi_2\big]\,dx + i\sqrt{2\alpha}\sigma\,\psi_2\,dW_x, \\ d\psi_2 = \big[iq\psi_1 + (i\zeta - \alpha\sigma^2)\psi_2\big]\,dx + i\sqrt{2\alpha}\sigma\,\psi_1\,dW_x \end{cases} \tag{24}$$

with initial condition (22). System (24) can be rewritten in Stratonovich form as

$$d\Psi = i\begin{pmatrix} -\zeta & q \\ q & \zeta \end{pmatrix}\Psi\,dx + i\sqrt{2\alpha}\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Psi \circ dW_x. \tag{25}$$

For NLS we can also consider perturbations produced by a complex process: let ν_1, ν_2 be two independent copies of the process ν and set $\widetilde{\nu} := \nu_1 + i\nu_2$. One can define U_0^ε using $\widetilde{\nu}$ instead of ν; proceeding as above, from the IST one obtains again system (23), and from [7], Theorem 6.1, one gets that in this case the limit process is the solution of

$$\begin{aligned} d\Psi = i&\begin{pmatrix} -\zeta & q \\ q & \zeta \end{pmatrix}\Psi\,dx + i\sqrt{2\alpha}\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Psi \circ dW_x^{(1)} \\ &- \sqrt{2\alpha}\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Psi \circ dW_x^{(2)}, \end{aligned} \tag{26}$$

where the $W^{(i)}$ are two independent Wiener processes, and with Ψ having the same initial condition (22).

We apply the same strategy to the KdV equation: the goal is to obtain the values of $\zeta \in \mathbb{C}^+$ for which there exists a solution φ^ε of

$$\varphi_{xx}^\varepsilon + \big(U_0^\varepsilon + \zeta^2\big)\varphi^\varepsilon = 0 \tag{27}$$

with the boundary conditions

$$\varphi^\varepsilon(0) = 1, \qquad \varphi_x^\varepsilon(0) = -i\zeta, \qquad \varphi_x^\varepsilon(R) - i\zeta\varphi^\varepsilon(R) = 0.$$

These conditions correspond to imposing exponential decay of the solution at infinity, so that ζ is an element of the discrete spectrum of the spectral problem (27).

Setting $\Phi^\varepsilon := (\varphi^\varepsilon, \varphi^\varepsilon_x)^T$ this equation can be transformed into

$$d\Phi^\varepsilon = \begin{pmatrix} 0 & 1 \\ -U_0^\varepsilon - \zeta^2 & 0 \end{pmatrix} \Phi^\varepsilon \, dx \tag{28}$$

with boundary conditions

$$\Phi^\varepsilon(0) = \begin{pmatrix} 1 \\ -i\zeta \end{pmatrix}, \qquad \phi_2^\varepsilon(R) - i\zeta\phi_1^\varepsilon(R) = 0.$$

We consider the flow $\Phi^\varepsilon(x,\zeta)$, $x \in [0, R]$, $\zeta \in \mathbb{C}^+$, defined by the above equation with only the initial condition, and look for the values of ζ s.t. the final condition is satisfied. Again by [7], Theorem 6.1, Φ^ε converges in distribution to the solution of

$$d\Phi = \begin{pmatrix} 0 & 1 \\ -q - \zeta^2 & 0 \end{pmatrix} \Phi \, dx + \sqrt{2\alpha}\sigma \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Phi \, dW_x, \tag{29}$$

which, in terms of the function φ, can be rewritten as

$$d\varphi_x = -(q + \zeta^2)\varphi \, dx + \sqrt{2\alpha}\sigma\varphi \, dW_x. \tag{30}$$

The initial condition is

$$\Phi(0) = \begin{pmatrix} 1 \\ -i\zeta \end{pmatrix} \quad \text{or equivalently} \quad \varphi(0) = 1, \qquad \varphi_x(0) = -i\zeta. \tag{31}$$

We remark that in the last two differential equations above the Stratonovich and Itô stochastic integrals coincide.

The convergence obtained above is only for a (finite number of) fixed ζ and σ, but we will need a convergence in $\mathcal{C}^0([0, R]; \mathcal{C}^1(\mathbb{R}^3))$ to be able to differentiate the limit process with respect to the parameters. This is the main result of this section and it is provided by the following theorem. We will focus on the problem of finding the values of ζ for which the limit flows Ψ and Φ match the final conditions in Sections 4 and 5.

THEOREM 3.2. *Assume Hypothesis 3.1. Let $\Psi^\varepsilon := (\psi_1^\varepsilon, \psi_2^\varepsilon)^T$ be the solution of (23) with initial condition (22) and Ψ the solution of (25) with the same initial condition. Let also φ^ε be the solution of (27) with initial condition (31) and φ be the solution of (30) with the same initial condition. Considering these as functions of the space variable x and the parameters ξ, η, σ, we have in the limit of $\varepsilon \to 0$ that $\Psi^\varepsilon(x, \xi, \eta, \sigma) \to \Psi(x, \xi, \eta, \sigma)$ weakly in $\mathcal{C}^0([0, R]; \mathcal{C}^1(\mathbb{R}^3; \mathbb{C}^2))$ and $\varphi^\varepsilon(x, \xi, \eta, \sigma) \to \varphi(x, \xi, \eta, \sigma)$ weakly in $\mathcal{C}^0([0, R]; \mathcal{C}^1(\mathbb{R}^3; \mathbb{C}))$.*

To prove this theorem we need the following standard tightness criteria; see [13], Chapter 2. We will use $\mathcal{D}([0, R]; E)$ to denote the space of CadLag processes defined for $x \in [0, R]$ and with values in the space E. The first lemma is due to Aldous, [3].

LEMMA 3.3. *Let (E,d) be a metric space, and X^ε a process with paths in $\mathcal{D}([0,R];E)$. If for every x in a dense subset of $[0,R]$ the family $(X^\varepsilon(x))_{\varepsilon\in(0,1]}$ is tight in E, and X^ε satisfies the Aldous property*:

A: *For any $\kappa>0$ and $\lambda>0$, there exists $\delta>0$ s.t.*

$$\limsup_{\varepsilon\to 0}\sup_{\tau<R}\sup_{0<\theta<\delta\wedge(R-\tau)}\mathbb{P}\big(\|X^\varepsilon(\tau+\theta)-X^\varepsilon(\tau)\|>\lambda\big)<\kappa$$

where τ is a stopping time;

then the family $(X^\varepsilon)_{\varepsilon\in(0,1]}$ is tight in $\mathcal{D}([0,R];E)$.

To state the next lemma, we need to introduce some notation. If $\mathcal{H}$ is a Hilbert space, and $\mathcal{H}_n$ a subspace of $\mathcal{H}$, we shall use $\pi_{\mathcal{H}_n}$ to denote the projection of $\mathcal{H}$ onto $\mathcal{H}_n$. Also, $d_{\mathcal{H}}$ is used to denote the distance on $\mathcal{H}$ introduced by the inner product.

LEMMA 3.4. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_n$ be an increasing sequence of finite-dimensional subspaces of $\mathcal{H}$ s.t., for any $h\in\mathcal{H}$, $\lim_{n\to\infty}\pi_{\mathcal{H}_n}h=h$. Let $(Z^\varepsilon)_{\varepsilon\in(0,1]}$ be a family of $\mathcal{H}$-valued random variables. Then the family $(Z^\varepsilon)_{\varepsilon\in(0,1]}$ is tight if and only if for any $\kappa>0$ and $\lambda>0$, there exist ρ_κ and a subspace $\mathcal{H}_{\kappa,\lambda}$ s.t.*

$$\sup_{\varepsilon\in(0,1]}\mathbb{P}\big(\|Z^\varepsilon\|\ge\rho_\kappa\big)\le\kappa\quad\textit{and}\quad\sup_{\varepsilon\in(0,1]}\mathbb{P}\big(d_{\mathcal{H}}(Z^\varepsilon,\mathcal{H}_{\kappa,\lambda})>\lambda\big)\le\kappa.\tag{32}$$

The proof of Lemma 3.4 can be found in [12]. For completeness we give it in Appendix, together with other technical results needed for the proof the Theorem 3.2.

PROOF OF THEOREM 3.2. To unify notation, we shall use X^ε to denote both Ψ^ε and Φ^ε. Therefore, X^ε is the solution of what we shall call the approximated system, which is either system (23) or (28), with $\varepsilon>0$.

Since Propositions A.5 and 5.2 ensure that the limit equations for Ψ and Φ have a unique solution which is $\mathcal{C}^0([0,R];\mathcal{C}^1(\mathbb{R}^3))$, it suffices to prove convergence in the space of CadLag processes $\mathcal{D}([0,R];\mathcal{C}^1(\mathbb{R}^3))$. We will do so in three steps.

Step 1 contains a technical result needed for the application in step 2 of Lemma 3.4, namely the proof of the bound (33).

In step 2, using Lemma 3.4, we will show that for every fixed x the sequences $(\Psi^\varepsilon(x))_\varepsilon$ and $(\Phi^\varepsilon(x))_\varepsilon$, denoted $(X^\varepsilon(x))_\varepsilon$ in the following, are tight in the Hilbert space $\mathcal{H}:=W^{3,2}(G)$. Note that, by Sobolev imbedding, $\mathcal{H}\hookrightarrow\mathcal{C}^1(G)$. Here, G is an open, bounded subset of $\mathbb{R}^3$, the space of parameters. For simplicity we take $G=(-N,N)^3$ for some real positive constant N; a justification of the fact that it is not restrictive to assume that the set of parameters G is bounded is given below in the proof of Proposition A.5, where the convergences we are proving here will be used.

In the last step we will use Lemma 3.3, where we take E to be the Hilbert space $\mathcal{H}$. This will provide the desired convergence of the family of processes $(X^\varepsilon)_{\varepsilon\in(0,1]}$ in $\mathcal{D}([0,R];\mathcal{C}^1(\mathbb{R}^3))$.

Since X^ε is the solution of a linear differential equation with coefficients smooth in the parameters $\mu=(\xi,\eta,\sigma)$, from the explicit formula for the solution, we get that $X^\varepsilon(x,\mu)$ is smooth in the parameters. We will soon use its derivatives in the parameters: the vector of X^ε and its first derivatives in μ still satisfy a linear system of ODEs whose coefficients depend linearly on the parameters and on the process $\nu(x/\varepsilon^2)$, and the same result holds adding higher order derivatives.

Step 1 (A preliminary estimate). The key point to show that $(X^\varepsilon(x,\mu))_{\mu\in G}$ is tight in $\mathcal{H}=W^{3,2}(G)$ for every $x\in[0,R]$ is the proof of the bound

$$\limsup_{\varepsilon\to 0}\mathbb{E}\big[\big\|X^\varepsilon(x,\cdot)\big\|_{W^{6,2}(G)}\big]\le C<\infty \tag{33}$$

uniformly in $x\in[0,R]$. This is the content of this step.

Define Y^ε as the vector process of X^ε and all of its derivatives in the parameters $\mu=(\xi,\eta,\sigma)$ up to order 6. As remarked above, this process is the solution of a linear system of ODEs with coefficients (the matrices M_1 and M_2) linear in the parameters

$$\frac{\mathrm{d}}{\mathrm{d}x}Y^\varepsilon=M_1Y^\varepsilon+\frac{1}{\varepsilon}\nu\big(x/\varepsilon^2\big)M_2Y^\varepsilon.$$

Since G is bounded, we only need to check that the second moment of $Y^\varepsilon(x,\mu)$ is uniformly bounded with respect to $\varepsilon\in(0,1]$, $\mu\in G$ and $x\in[0,R]$. Actually, we aim at a stronger result, which we will need later. We are going to show that [recall that $Y_0=Y_0^\varepsilon$ is deterministic since both $\Psi^\varepsilon(0)$, $\Phi^\varepsilon(0)$ and their derivatives in zero are defined by the equation for $x\le 0$, which is deterministic, and the boundary condition at $x\to-\infty$]

$$\mathbb{E}\Big[\sup_{x\in[0,R]}\big|Y^\varepsilon(x)\big|^2\Big]\le C_R\big(1+\big|Y(0)\big|^2\big)<\infty. \tag{34}$$

Following [7], Section 6.3.5, we show this bound with the perturbed test function method. Let $\mathcal{L}^\varepsilon$ be the infinitesimal generator of the process Y^ε and $\mathcal{L}$ the infinitesimal generator of the process Y obtained from the process X solution of the limit system (25) or (29) and its derivatives. Let $m\in N$ be such that $Y^\varepsilon(x)\in\mathbb{C}^m$, and let K be a compact subset of $\mathbb{R}$ containing the image of the bounded process ν. For every $y\in\mathbb{C}^m$ and $z\in K$, $\mathcal{L}^\varepsilon$ has the form

$$\mathcal{L}^\varepsilon g(y,z)=\frac{1}{\varepsilon^2}\mathcal{L}_\nu g(y,z)+\frac{z}{\varepsilon}(M_2y)^T\nabla_y g(y,z)+(M_1y)^T\nabla_y g(y,z),$$

where $\mathcal{L}_\nu$ is the infinitesimal generator of the process ν, as defined in Hypothesis 3.1. Let f be the identity function on $\mathbb{C}^m$ and $f^\varepsilon(y,z)=y+\varepsilon f_1(y,z)$

be the associated perturbed function, which is solution of the Poisson equation $\mathcal{L}_\nu f_1(y,z) = -z(M_2 y)^T \nabla_y f(y)$. In this equation, y plays the role of a frozen parameter, so that f_1 has linear growth in y, uniformly in z, and the same holds for

$$\mathcal{L}^\varepsilon f^\varepsilon(y,z) = z(M_2 y)^T \nabla_y f_1(y,z) + \varepsilon (M_1 y)^T \nabla_y f_1(y,z).$$

Since

$$\begin{aligned} Y^\varepsilon(x) = Y^\varepsilon(0) - \varepsilon\big[f_1\big(Y^\varepsilon(x), \nu^\varepsilon(x)\big) - f_1\big(Y^\varepsilon(0), \nu^\varepsilon(0)\big)\big] \\ + \int_0^x \mathcal{L}^\varepsilon f^\varepsilon\big(Y^\varepsilon(x'), \nu^\varepsilon(x')\big)\,\mathrm{d}x' + M_x^\varepsilon, \end{aligned}$$

where M_x^ε is a vector valued martingale, we get the bound

$$\begin{aligned} \sup_{x\in[0,R]} \big|Y^\varepsilon(x)\big| \le \big|Y^\varepsilon(0)\big| + \varepsilon C\Big[1 + \sup_{x\in[0,R]} \big|Y^\varepsilon(x)\big|\Big] \\ + C\int_0^R 1 + \sup_{x'\in[0,x]} \big|Y^\varepsilon(x')\big|\,\mathrm{d}x + C \sup_{x\in[0,R]} \big|M_x^\varepsilon\big|. \end{aligned}$$

For $\varepsilon \le 1/2C$, applying Gronwall's inequality and renaming constants we get

$$\sup_{x\in[0,R]} \big|Y^\varepsilon(x)\big| \le C_R\Big(1 + \big|Y^\varepsilon(0)\big| + \sup_{x\in[0,R]} \big|M_x^\varepsilon\big|\Big). \tag{35}$$

The quadratic variation of the martingale is given by

$$\langle M^\varepsilon\rangle_x = \int_0^x g^\varepsilon\big(Y^\varepsilon(x'), \nu^\varepsilon(x')\big)\,\mathrm{d}x',$$

where

$$\begin{aligned} g^\varepsilon(y,z) &= \big(\mathcal{L}^\varepsilon f^{\varepsilon 2} - 2 f^\varepsilon \mathcal{L}^\varepsilon f^\varepsilon\big)(y,z) \\ &= \big(\mathcal{L}_\nu f_1^2 - 2 f_1 \mathcal{L}_\nu f_1\big)(y,z) \\ &\quad + 2\varepsilon z\big[(M_2 y)^T f_1(y,z) - (M_2 y)^T\big((\nabla_y f_1)^T f_1\big)(y,z)\big] \\ &\quad + 2\varepsilon^2\big[(M_1 y)^T f_1(y,z) - (M_1 y)^T\big((\nabla_y f_1)^T f_1\big)(y,z)\big] \end{aligned}$$

has quadratic growth in y uniformly in $z \in K$. Therefore, by Doob's inequality,

$$\mathbb{E}\Big[\sup_{x\in[0,R]} \big|M_x^\varepsilon\big|^2\Big] \le C\mathbb{E}\big[\langle M^\varepsilon\rangle_R\big] \le C\int_0^R 1 + \mathbb{E}\big[\big|Y^\varepsilon(x)\big|^2\big]\,\mathrm{d}x.$$

Substituting into the expected value of the square of (35) and using again Gronwall's inequality, we get (34), which gives (33).

Step 2 [Tightness of $X^\varepsilon(x)$]. In this step we show how to obtain the tightness of the family $(X^\varepsilon(x,\mu))_{\mu\in G}$ from (33) using Lemma 3.4. Indeed, from this bound the first part of condition (32) follows by the Markov inequality if we take $\rho_\kappa = C/\kappa$. The bound (33) provides also information on the regularity of $X^\varepsilon(x)$, which can be used to prove the second part of condition (32) as follows. By the Sobolev imbedding $W^{6,2}(G) \hookrightarrow \mathcal{C}^4(G)$ the bound (33) implies that $X^\varepsilon \in \mathcal{C}^4(G)$. An appropriate sequence of finite-dimensional subspaces $(\mathcal{H}_n)_n$ is constructed in the Appendix in Lemma A.1, and Lemma A.2 states that for any function $g \in \mathcal{C}^4(G)$ there exists an arbitrarily good approximation g_n belonging to some $\mathcal{H}_n$. Moreover, the control on the distance between g and the subspace $\mathcal{H}_n$ only depends on the norm $\|g\|_{\mathcal{C}^4}$, so that by (33) the approximation is uniform in ε. This provides the second part of condition (32). Finally, Corollary A.3 provides the last hypothesis of Lemma 3.4, namely that for the sequence of subspaces $\mathcal{H}_n$ constructed in Lemma A.1 and for any $h \in \mathcal{H}$, $\lim_{n\to\infty} \pi_{\mathcal{H}_n} h = h$. Then, Lemma 3.4 gives that for every x, the family $(X^\varepsilon(x,\mu))_{\mu\in G}$ is tight in $\mathcal{H} = W^{3,2}(G)$.

Step 3 (Tightness of X^ε). Thanks to Lemma 3.3, the tightness of the family of processes X^ε in $\mathcal{D}([0,R];\mathcal{H})$ follows if we show that the Aldous property A holds. Since G is bounded, we can prove the Aldous property showing that

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \sup_{\mu\in G} \sup_{\tau\le R} \sup_{0<\theta<\delta} \mathbb{E}\big[\big|Y^\varepsilon(\tau+\theta,\mu) - Y^\varepsilon(\tau,\mu)\big|^2\big] = 0, \tag{36}$$

where Y^ε is the vector process having as components X^ε and its derivatives in μ up to the third order only. We prove the above limit using again the perturbed test function method. With the notation introduced above, we have

$$\begin{aligned}
&\big|Y^\varepsilon(\tau+\theta) - Y^\varepsilon(\tau)\big|^2 \\
&\quad \le C\big|M^\varepsilon_{\tau+\theta} - M^\varepsilon_\tau\big|^2 + C\int_\tau^{\tau+\theta} \big|\mathcal{L}^\varepsilon f^\varepsilon\big(Y^\varepsilon(x), \nu^\varepsilon(x)\big)\big|^2 \,\mathrm{d}x \\
&\qquad + C\varepsilon\Big(1 + \sup_{x\in[\tau,\tau+\theta]} \big|Y^\varepsilon(x)\big|^2\Big) \\
&\quad \le C\big|M^\varepsilon_{\tau+\theta} - M^\varepsilon_\tau\big|^2 + C\int_\tau^{\tau+\theta} \big|\mathcal{L}^\varepsilon f^\varepsilon(Y^\varepsilon,\nu^\varepsilon) - \mathcal{L} f(Y^\varepsilon)\big|^2 \,\mathrm{d}x \\
&\qquad + C\int_\tau^{\tau+\theta} \big|\mathcal{L} f(Y^\varepsilon)\big|^2 \,\mathrm{d}x + C\varepsilon\Big(1 + \sup_{x\in[\tau,\tau+\theta]} \big|Y^\varepsilon(x)\big|^2\Big).
\end{aligned}$$

We have that

$$\mathbb{E}\big[\big|M^\varepsilon_{\tau+\theta} - M^\varepsilon_\tau\big|^2\big] = \mathbb{E}\big[\big(M^\varepsilon_{\tau+\theta}\big)^2 - \big(M^\varepsilon_\tau\big)^2\big] = \mathbb{E}\Big[\int_\tau^{\tau+\theta} \mathrm{d}\langle M^\varepsilon\rangle_x\Big].$$

Since $|\mathcal{L}^\varepsilon f^\varepsilon(y,z) - \mathcal{L} f(y)| \le \varepsilon C(1+|y|)$ and $|\mathcal{L} f(y)| \le C|y|$, for $\theta \le \delta$

$$\mathbb{E}\big[\big|Y^\varepsilon(\tau+\theta) - Y^\varepsilon(\tau)\big|^2\big] \le C_R(\delta+\varepsilon)\Big(1 + \mathbb{E}\Big[\sup_{x\in[0,R]} \big|Y^\varepsilon(x)\big|^2\Big]\Big).$$

The right-hand side is independent of τ, and we can use estimate (34) to bound it uniformly in ε and μ. Therefore, (36) follows, and the proof of Theorem 3.2 is complete. $\square$

REMARK 3.5. Using the notion of pseudo-generators, as introduced in [6], Section 7.4, it is possible to relax the conditions imposed on the driving process $\nu(x)$, assuming that it is just a mixing process instead of Markov.

4. Stability of NLS solitons. This section is devoted to the study of our first example, the NLS equation. We focus on the soliton components of the solution. In the previous section we have obtained the limit equation (25) and we have remarked that every soliton component (soliton, in short) is identified by a complex number $\zeta = \xi + i\eta$ s.t. the flow $\Psi(x,\zeta)$ solution of (25) with initial condition (22) satisfies also a given final condition. The real and imaginary parts of ζ define the velocity and amplitude of the soliton, respectively. In Section 2.1 we presented some classical results on the background deterministic solution. We analyze now how this solution is modified by the introduction of a real, in Section 4.1, or complex, in Section 4.2, small-amplitude white noise perturbation of the initial condition. The main results are contained in Propositions 4.1 and 4.5. We deal with the limit cases of "quiescent" solitons in Corollary 4.4 and Remark 4.7.

For simplicity of exposition, in the present and following sections we will choose the value of the integrated covariance of the process ν to be $\alpha = 1/2$.

4.1. *Small-intensity real white noise.* In this subsection we consider the example of an initial condition composed of a square function perturbed with a small real white noise. First, we use a perturbative approach to study the effects of the perturbation on "true" solitons (Proposition 4.1). Then, in the last part of this subsection, we study the effects of this perturbation on a special structure called "quiescent" soliton (Corollary 4.4).

We have here $U_0(x) = (q + \sigma \dot{W}_x)\mathbf{1}_{[0,R]}(x)$. The initial condition is (10) and the system (5) for $x \in [0, R]$ reads

$$\begin{cases} \mathrm{d}\psi_1 = i(q\psi_2 - \zeta\psi_1)\,\mathrm{d}x + i\sigma\psi_2 \circ \mathrm{d}W_x, \\ \mathrm{d}\psi_2 = i(q\psi_1 + \zeta\psi_2)\,\mathrm{d}x + i\sigma\psi_1 \circ \mathrm{d}W_x. \end{cases} \tag{37}$$

PROPOSITION 4.1. *For $qR > \frac{\pi}{2}$ and in the limit of a small, real, white noise-type stochastic perturbation of the initial condition, the parameter η defining the amplitude of the soliton component of the solution is perturbed at first order by a small, zero-mean, Gaussian random variable, which is given by*

$$\sigma \frac{q\sin(c_0R)W_R + \int_0^R 2(\eta_0 q/c_0)\sin(c_0(R-y))\sin(c_0y)\,\mathrm{d}W_y}{2[q^2/c_0^2 + R\eta_0]\sin(c_0R) - R(\eta_0^2/c_0)\cos(c_0R)}, \tag{38}$$

where $c_0 := \sqrt{q^2 - \eta_0^2}$, and η_0 is the parameter defining the amplitude of the soliton of the unperturbed system.

The velocity of the soliton remains unchanged.

REMARK 4.2. With the same proof, one can show that this result also holds for a purely imaginary (deterministic) initial condition perturbed by a purely imaginary, small white noise. And a simple phase shift [9] allows to extend this result to any complex initial condition $U_0(x) = q\mathbf{1}_{[0,R]}(x)$, $q \in \mathbb{C}$ perturbed with a white noise with the same constant phase of q.

REMARK 4.3. With the same proof, it is possible to show that solitons are stable with respect to small random perturbations under more general hypothesis. In particular, the perturbation need not be rapidly oscillating. For example, substituting the white noise with a general process Q_x in Corollary 4.4 below one obtains the same result: $\partial_\sigma \eta = q \int_0^R Q_x \,\mathrm{d}x$, so that a true soliton is created whenever the integral is positive.

Indeed, another possible and equivalent approach for rapidly oscillating processes would be to work with the original process ν_ε for the IST and carry out the scaling limit only at this stage.

PROOF OF PROPOSITION 4.1. Proposition A.5 ensures that equation (37) with initial condition (10) defines a stochastic flow $\Psi^{(\zeta,\sigma)}(x)$ of $\mathcal{C}^1$-diffeomorphisms, which is $\mathcal{C}^1$ also in the parameters (ξ, η, σ). Looking at the flow at point R we can define a complex-valued function of $\Psi(R)$ as $F(\xi, \eta, \sigma) := \psi_1^{(\zeta,\sigma)}(R)$. We look for the set of values of (ξ, η, σ) corresponding to zeros of the function F: they are the parameters $(\zeta = \xi + i\eta)$ defining the soliton components of the solution of the problem perturbed with a noise of amplitude σ. We claim that a small stochastic perturbation has only the effect of a small variation in the value of $\zeta = \xi + i\eta$ with respect to the value $\zeta_0 = i\eta_0$ of the corresponding soliton in the deterministic case. We will prove this using the implicit function theorem: for any fixed and sufficiently small σ, $F(\xi, \eta, \sigma)$ has a unique zero in some open set containing the point $(0, \eta_0)$.

Since Lemma A.4 in the Appendix guarantees that the Jacobian matrix J of the derivatives of F with respect to ξ and η is invertible, we can apply the implicit function theorem at point $(\xi, \eta, \sigma) = (0, \eta_0, 0)$. Fix $\zeta = i\eta_0$. By Proposition A.5 the flow defined by the system (37) is $\mathcal{C}^1$ in the parameters, so that its derivative in σ coincides with the first term of the Taylor expansion, denoted $\Psi^{(1)}$,

$$\mathrm{d}\Psi^{(1)} = \begin{pmatrix} \eta_0 & iq \\ iq & -\eta_0 \end{pmatrix} \Psi^{(1)} \,\mathrm{d}x + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi^{(0)} \,\mathrm{d}W_x. \tag{39}$$

Here, $\Psi^{(0)}$ denotes the solution of the deterministic problem ($\sigma = 0$). Let M be the matrix appearing in the drift term of the above equation; the solution can be computed explicitly,

$$\Psi^{(1)}(x) = i \int_0^x \exp\bigl(M(x-y)\bigr) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi^{(0)}(y) \,\mathrm{d}W_y,$$

where

$$\begin{aligned}[\exp(M(x-y))\Psi^{(0)}(y)]_1 &= i\frac{q}{c_0}\cos(c_0(x-y))\sin(c_0 y)\\ &\quad + 2i\frac{\eta_0 q}{c_0^2}\sin(c_0(x-y))\sin(c_0 y)\\ &\quad + i\frac{q}{c_0}\sin(c_0(x-y))\cos(c_0 y)\\ &= i\frac{q}{c_0}\sin(c_0 x) + 2i\frac{\eta_0 q}{c_0^2}\sin(c_0(x-y))\sin(c_0 y).\end{aligned}$$

It follows that

$$\partial_\sigma F(0,\eta_0,0) = i\int_0^R [\exp(M(R-y))\Psi_y^{(0)}]_1 \,\mathrm{d}W_y.$$

In the proof of Lemma A.4 the derivatives $\partial_\xi F(0,\eta_0,0)$ and $\partial_\eta F(0,\eta_0,0)$ are computed explicitly: the first one is imaginary pure, while the latter is real. Set $\alpha := \partial_\eta F(0,\eta_0,0) = [\frac{q^2}{c_0^3} + R\frac{\eta_0}{c_0}]\sin(c_0 R) - R\frac{\eta_0^2}{c_0^2}\cos(c_0 R)$. Using the explicit formula for the Jacobian obtained in Lemma A.4 we can write

$$J^{-1} = \begin{pmatrix} 0 & -\dfrac{1}{\alpha} \\ \dfrac{1}{\alpha} & 0 \end{pmatrix}$$

and from the formula

$$v := \begin{pmatrix} \partial_\sigma \xi \\ \partial_\sigma \eta \end{pmatrix}(\sigma = 0) = -J^{-1}\begin{pmatrix} \Re(\partial_\sigma F) \\ \Im(\partial_\sigma F)\end{pmatrix}(0,\eta_0,0)$$

we get that $\partial_\sigma \eta$ is given by (38) and that $\partial_\sigma \xi = 0$. $\square$

When $q = (2n+1)\pi/2R$ for some $n \in \mathbb{N}$, in the deterministic case we have the creation of what is sometimes called a "quiescent" soliton. Also in this case, the stochastic perturbation can modify, at first order, only the amplitude of the soliton.

COROLLARY 4.4. *When the background deterministic solution contains a "quiescent" soliton, the stochastic perturbation destroys it with probability* 1/2 *and transforms it into a true soliton* (*with a positive amplitude*) *with probability* 1/2.

PROOF. In the case of a "quiescent" soliton Lemma A.4 still holds. Indeed we have now $c_0 = q$ and $qR = (2n+1)\pi/2$, so that equation (46) in the proof of the lemma becomes

$$\partial_\xi F(0,0,0) = -i\left[\frac{q^2}{c_0^3} + R\frac{\zeta_0}{c_0}\right]\sin(c_0 R) + iR\frac{\zeta_0^2}{c_0^2}\cos(c_0 R) = \mp i\frac{1}{q} \neq 0$$

and the determinant of the Jacobian is not zero. One has then $\alpha = \pm 1/q$ and

$$\partial_\sigma F(0,0,0) = -\int_0^R \sin(qR)\,\mathrm{d}W_y = \mp W_R,$$

which is real. Therefore,

$$v := \begin{pmatrix} \partial_\sigma \xi \\ \partial_\sigma \eta \end{pmatrix} (\sigma = 0) = -\begin{pmatrix} 0 & \mp q \\ \pm q & 0 \end{pmatrix} \begin{pmatrix} \mp W_R \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ q W_R \end{pmatrix}.$$

A true soliton is created whenever $W_R > 0$. □

4.2. *Small-intensity complex white noise.* The limit case of a small-amplitude complex white noise is similar to the case of the real white noise treated above. Take $U_0(x) = (q + \sigma \dot{W}_x)\mathbf{1}_{[0,R]}(x)$ where $W_x = W_x^{(1)} + i W_x^{(2)}$ is a complex Wiener process. We have the following proposition.

PROPOSITION 4.5. *For $qR > \frac{\pi}{2}$ and in the limit of a small complex white noise-type stochastic perturbation of the initial condition, the parameters ξ, η defining the velocity and amplitude of each soliton are perturbed at first order by small, zero-mean, Gaussian random variables, which are given by*

$$\partial_\sigma \xi = -\frac{q \sin(c_0 R) W_R^{(2)} + \int_0^R 2(\eta_0 q / c_0^2) \sin(c_0(R-y)) \sin(c_0 y)\,\mathrm{d}W_y^{(2)}}{2[q^2/c_0^2 + R\eta_0] \sin(c_0 R) - R(\eta_0^2/c_0)\cos(c_0 R)},$$

$$\partial_\sigma \eta = \frac{q \sin(c_0 R) W_R^{(1)} + \int_0^R 2(\eta_0 q / c_0^2) \sin(c_0(R-y)) \sin(c_0 y)\,\mathrm{d}W_y^{(1)}}{2[q^2/c_0^2 + R\eta_0] \sin(c_0 R) - R(\eta_0^2/c_0)\cos(c_0 R)},$$

where $c_0 := \sqrt{q^2 - \eta_0^2}$, and η_0 is the parameter defining the amplitude of the soliton of the unperturbed system.

PROOF. This proof is similar to that of Proposition 4.1. An analogy of Proposition A.5 holds in this setting, and Lemma A.4 remains unchanged (note that in Lemma A.4 we work on the deterministic equation). From equation (39) onward one has just to remember that W is now complex. □

REMARK 4.6. In this case, since we used a noise with a symmetric law, the first order perturbations of the velocity and amplitude of the soliton have the same law and are independent. We could have taken a nonsymmetric complex noise to perturb the initial condition: $\tilde{\nu} = \nu_1 + i\nu_2$, where ν_1 and ν_2 have different distributions. In this case, the perturbations of the velocity and amplitude of the soliton would still be independent, but not sharing the same law.

REMARK 4.7. "Quiescent" solitons are perturbed in both amplitude and speed, leading to the possible creation of true solitons with nonzero velocity. The same result holds when perturbing the initial data with more general complex processes.

5. Stability of KdV solitons. In this section we study our second example, the KdV equation. We focus on the soliton components of the solution. Solitons of the KdV equation are identified by an imaginary number $\zeta = i\eta$, defining both the velocity and amplitude of the soliton, which are related. Recall that in Section 2.2 we have presented some classical results on the background deterministic solution. Using these results in Section 5.1 we analyze how this solution is modified by the introduction of a small-amplitude white noise perturbation of the initial condition: the main result is contained in Proposition 5.1, while Proposition 5.3 deals with the case of "quiescent" solitons. The last subsection deals with the case of a perturbation of the zero initial condition.

5.1. *Small-amplitude random perturbation with $q > 0$.* Let the initial condition of the KdV equation be given by $U_0 = (q + \sigma \dot{W}_x)\mathbf{1}_{[0,R]}(x)$, where W is a standard Wiener process. Since solitons correspond to zeros of the complex extension of a, which in turn must be located on the imaginary axis, we look for bounded solutions of equation (8) for $\zeta = i\eta$, $\eta \in \mathbb{R}^+$. The first main result is contained in the following proposition.

PROPOSITION 5.1. *In the limit of a small, white noise-type stochastic perturbation of the initial condition, the parameter η defining the velocity and amplitude of the generated soliton is perturbed at first order by a small, zero-mean, Gaussian random variable, which is given by*

$$\frac{\sigma \int_0^R \varphi_0(R-x)\varphi_0(x)\,\mathrm{d}W_x}{\cos(c_0 R)[2 + \eta_0 R - \eta_0^3 R/c_0^2] + \sin(c_0 R)[(3\eta_0 + 2\eta_0^2 R)/c_0 + \eta_0^3/c_0^3]}. \tag{40}$$

Here, φ_0 is the deterministic solution of (15), given by (18), $c_0 := \sqrt{q^2 - \eta_0^2}$, and η_0 is the parameter corresponding to the unperturbed soliton.

Before we start the proof of the above proposition, we shall remark that an analog of Proposition A.5 holds in this setting, and it provides the uniqueness and regularity results for the solution. The result reads in the following way:

PROPOSITION 5.2. *The stochastic differential equation (41) below defines a stochastic flow $\Phi^{(\eta,\sigma)}(x) = (\varphi(x), \widetilde{\varphi}(x))^T$ of $\mathcal{C}^1$-diffeomorphisms, which is $\mathcal{C}^1$ also in the parameters η, σ.*

PROOF OF PROPOSITION 5.1. We need to solve

$$\begin{cases} \mathrm{d}\varphi = \widetilde{\varphi}\,\mathrm{d}x, \\ \mathrm{d}\widetilde{\varphi} = (-q + \eta^2)\varphi\,\mathrm{d}x + \sigma\varphi \circ \mathrm{d}W_x \end{cases} \tag{41}$$

for $x \in [0, R]$ with initial conditions $\varphi(0) = 1$, $\widetilde{\varphi}(0) = \eta$. As we did for the NLS equation, we look at the flow provided by Proposition 5.2 and define a function of the flow at the point $x = R$ as $F(\eta, \sigma) := \widetilde{\varphi}(R) + \eta\varphi(R)$: this remains a function of the two parameters (η, σ). To prove Proposition 5.1 we use the implicit function theorem to show that $F(\eta, \sigma)$ has a unique zero in some open set containing $(\eta_0, 0)$, the point corresponding to the deterministic solution. Since Lemma A.6 in the Appendix guarantees that $\partial_\eta F(\eta_0, 0) \neq 0$, we can apply the implicit function theorem. We have (at η_0)

$$\begin{cases} \mathrm{d}\partial_\sigma\varphi = \partial_\sigma\widetilde{\varphi}\,\mathrm{d}x, \\ \mathrm{d}\partial_\sigma\widetilde{\varphi} = (-q + \eta_0^2)\partial_\sigma\varphi\,\mathrm{d}x - (\varphi + \sigma\partial_\sigma\varphi) \circ \mathrm{d}W_x. \end{cases}$$

By Proposition 5.2 the flow $\Phi_x^{(\eta,\sigma)}$ is $\mathcal{C}^1$ in the parameters, so that its derivative in σ at $\sigma = 0$ coincides with the first term of the Taylor expansion, which is the solution of

$$\begin{cases} \mathrm{d}\partial_\sigma\varphi = \partial_\sigma\widetilde{\varphi}\,\mathrm{d}x, \\ \mathrm{d}\partial_\sigma\widetilde{\varphi} = (-q + \eta_0^2)\partial_\sigma\varphi\,\mathrm{d}x - \varphi_0\,\mathrm{d}W_x. \end{cases}$$

In matrix notation,

$$\mathrm{d}\begin{pmatrix} \partial_\sigma\varphi \\ \partial_\sigma\widetilde{\varphi} \end{pmatrix} = \mathrm{d}\Phi = \begin{bmatrix} 0 & 1 \\ -c^2 & 0 \end{bmatrix} \Phi\,\mathrm{d}x - \begin{pmatrix} 0 \\ \varphi_0\,\mathrm{d}W_x \end{pmatrix}$$

with

$$M = \begin{bmatrix} 0 & 1 \\ -c^2 & 0 \end{bmatrix},$$

$$e^{Mx} = \begin{bmatrix} \cos(cx) & \frac{1}{c}\sin(cx) \\ -c\sin(cx) & \cos(cx) \end{bmatrix}.$$

The solution is

$$\Phi(x) = \Phi(0) - \int_0^x e^{M(x-y)} \begin{pmatrix} 0 \\ \varphi_0(y) \end{pmatrix} \mathrm{d}W_y.$$

We have

$$\partial_\sigma\varphi(x) = -\frac{1}{c}\int_0^x \sin\bigl(c(x-y)\bigr)\Bigl[\cos(cy) + \frac{\eta}{c}\sin(cy)\Bigr]\,\mathrm{d}W_y,$$

$$\partial_\sigma\widetilde{\varphi}(x) = -\int_0^x \cos\bigl(c(x-y)\bigr)\Bigl[\cos(cy) + \frac{\eta}{c}\sin(cy)\Bigr]\,\mathrm{d}W_y.$$

Therefore,

$$\partial_\sigma F(\eta_0, 0) = \partial_\sigma\widetilde{\varphi}(R) + \eta_0\partial_\sigma\varphi(R) = -\int_0^R \varphi_0(R-x)\varphi_0(x)\,\mathrm{d}W_x$$

and we obtain, at first order, the perturbation of the value of the parameter η defining the velocity and amplitude of the soliton: for every (q, R, η_0),

$$
\begin{aligned}
\partial_\sigma \eta(\sigma) &= -\frac{\partial_\sigma F}{\partial_\eta F}(\eta_0, 0) \\
&= \biggl(\int_0^R \Bigl[\cos\bigl(c_0(R-x)\bigr) + \frac{\eta_0}{c_0}\sin\bigl(c_0(R-x)\bigr)\Bigr] \\
&\qquad\qquad \times \Bigl[\cos(c_0 x) + \frac{\eta_0}{c_0}\sin(c_0 x)\Bigr] \mathrm{d}W_x\biggr) \\
&\quad \bigg/ \biggl(\cos(c_0 R)\Bigl[2 + \eta_0 R - \frac{\eta_0^3 R}{c_0^2}\Bigr] + \sin(c_0 R)\Bigl[\frac{3\eta_0 + 2\eta_0^2 R}{c_0} + \frac{\eta_0^3}{c_0^3}\Bigr]\biggr).
\end{aligned}
$$

The proposition is proved. $\square$

We turn now to consider the case when the stochastic perturbation can result in the creation of a new soliton. As for the NLS equation, this happens only for specific "critical" values of q (and R). The result is presented in the next proposition.

PROPOSITION 5.3. *If we are at a critical point, which is to say* $\sqrt{q}R = n\pi$ *for* $n \in \mathbb{N}$, $n > 0$, *a small-amplitude white noise-type stochastic perturbation of the potential* may *create a new small-amplitude soliton. The condition for the creation of a new soliton is that the zero-mean, Gaussian random variable* (43) *is positive.*

PROOF. For a generic η_0, the deterministic background solution is ($c_0 = \sqrt{q - \eta_0^2}$)

$$
\varphi_0 = \cos(c_0 x) + \frac{\eta_0}{c_0}\sin(c_0 x). \tag{42}
$$

We have $U_0 = q + \sigma \dot{W}_x$; we want to apply the implicit function theorem to the function $F(\eta, \sigma)$ defined above, at the point $(0, 0)$. At this point

$$
\begin{aligned}
\partial_\eta \varphi(R) &= \frac{1}{\sqrt{q}}\sin(\sqrt{q}R) = 0, \\
\partial_\eta \widetilde{\varphi}(R) &= \cos(\sqrt{q}R) = \pm 1.
\end{aligned}
$$

Therefore

$$
\partial_\eta F(0,0) = \partial_\eta \widetilde{\varphi} + \varphi + \eta_0 \partial_\eta \varphi = 2\cos(\sqrt{q}R) = \pm 2
$$

and the implicit function theorem is applicable. We also have (again at $\eta_0 = 0$)

$$
\begin{cases}
\mathrm{d}\partial_\sigma \varphi = \partial_\sigma \widetilde{\varphi}\,\mathrm{d}x, \\
\mathrm{d}\partial_\sigma \widetilde{\varphi} = -q\,\partial_\sigma \varphi\,\mathrm{d}x - (\varphi + \sigma\,\partial_\sigma \varphi) \circ \mathrm{d}W_x.
\end{cases}
$$

As seen above, the solution of the above system coincides at $\sigma = 0$ with the solution of

$$\begin{cases} \mathrm{d}\partial_\sigma \varphi = \partial_\sigma \widetilde{\varphi}\, \mathrm{d}x, \\ \mathrm{d}\partial_\sigma \widetilde{\varphi} = -q\,\partial_\sigma \varphi\, \mathrm{d}x - \varphi_0\, \mathrm{d}W_x, \end{cases}$$

which is given by (here, $c = \sqrt{q}$)

$$\partial_\sigma \varphi(x) = -\frac{1}{\sqrt{q}} \int_0^x \sin\bigl(\sqrt{q}(x-y)\bigr) \cos(\sqrt{q}\,y)\, \mathrm{d}W_y,$$

$$\partial_\sigma \widetilde{\varphi}(x) = -\int_0^x \cos\bigl(\sqrt{q}(x-y)\bigr) \cos(\sqrt{q}\,y)\, \mathrm{d}W_y.$$

We have therefore

$$\partial_\sigma F(0,0) = -\int_0^R \cos\bigl(\sqrt{q}(R-x)\bigr) \cos(\sqrt{q}\,x)\, \mathrm{d}W_x$$

and since $\cos(\sqrt{q}R) = \pm 1$

$$\begin{aligned} v := \partial_\sigma \eta(\sigma) &= -\frac{\partial_\sigma F}{\partial_\eta F}(0,0) = \frac{\int_0^R \cos(\sqrt{q}(R-y)) \cos(\sqrt{q}\,y)\, \mathrm{d}W_y}{2\cos(\sqrt{q}R)} \\ &= \frac{1}{2} \int_0^R \cos^2(\sqrt{q}\,x)\, \mathrm{d}W_x. \end{aligned} \tag{43}$$

In this case, a new small-amplitude soliton, corresponding to $\eta = \sigma v$, is created whenever $v > 0$. □

5.2. *Small-amplitude random perturbations with $q = 0$.* We analyze now the case in which the initial condition is the pure stochastic perturbation: contrarily to the NLS equation, even this small initial condition can generate a soliton. In this setting there is no (nontrivial) solution in the deterministic case. This case is obtained at the critical point $\sqrt{q}R = 0$, where the first quiescent soliton is created.

PROPOSITION 5.4. *For $q = 0$, a small-amplitude stochastic perturbation of the potential* may *create a new small-amplitude soliton. For a white noise-type perturbation $\dot{W}_x$, a new soliton is created if $W_R > 0$ (which is an event of probability* 1/2*) and the created soliton corresponds to the value $\eta = \sigma W_R$.*

PROOF. Again, we want to use the implicit function theorem; take the limit of the deterministic solution (42) as $q \to 0$,

$$\varphi_0 = 1, \qquad \widetilde{\varphi}_0 = 0.$$

We have

$$\partial_\eta F(0,0) = \partial_\eta \widetilde{\varphi} + \varphi + \eta_0 \partial_\eta \varphi = 1$$

and ($\eta_0 = 0$)

$$\begin{cases} \mathrm{d}\varphi = \widetilde{\varphi}\,\mathrm{d}x, \\ \mathrm{d}\widetilde{\varphi} = -\sigma\varphi \circ \mathrm{d}W_x, \end{cases}$$

$$\begin{cases} \mathrm{d}\partial_\sigma\varphi = \partial_\sigma\widetilde{\varphi}\,\mathrm{d}x, \\ \mathrm{d}\partial_\sigma\widetilde{\varphi} = -(\varphi + \sigma\partial_\sigma\varphi) \circ \mathrm{d}W_x. \end{cases}$$

Therefore,

$$\partial_\sigma F(0,0) = \partial_\sigma\widetilde{\varphi} + \eta_0\partial_\sigma\varphi = -\int_0^R \varphi\,\mathrm{d}W_x = -W_R.$$

It follows that

$$\partial_\sigma\eta(\sigma) = -\frac{\partial_\sigma F}{\partial_\eta F}(0,0) = W_R$$

and a single soliton corresponding to $\eta = \sigma W_R$ is generated whenever $W_R > 0$, which means with probability $\frac{1}{2}$. $\square$

REMARK 5.5. The same result of Proposition 5.4 holds with more general processes: if, instead of a white noise, we take $U_0(x) = \sigma Q_x$ in (15), where Q_x is a generic stochastic process, we have (at $\eta = 0$)

$$\begin{cases} \mathrm{d}\partial_\sigma\varphi = \partial_\sigma\widetilde{\varphi}\,\mathrm{d}x, \\ \mathrm{d}\partial_\sigma\widetilde{\varphi} = -Q_x(\varphi + \sigma\partial_\sigma\varphi)\,\mathrm{d}x, \end{cases}$$

$$\partial_\sigma F(0,0) = \partial_\sigma\widetilde{\varphi} + \eta_0\partial_\sigma\varphi = -\int_0^R Q_x\,\mathrm{d}x$$

and

$$\partial_\sigma\eta(\sigma) = -\frac{\partial_\sigma F}{\partial_\eta F}(0,0) = \int_0^R Q_x\,\mathrm{d}x.$$

In this case, a single soliton is created whenever the noise introduced has positive mean, and it corresponds to $\eta = \sigma\int_0^R Q_x\,\mathrm{d}x$.

REMARK 5.6. The mass of a soliton of the KdV equation is $M_\eta = \int_{\mathbb{R}} U(x)\,\mathrm{d}x = 4\eta$. Here, we have introduced a perturbation of mass $M_{U_0} = \sigma\int_{\mathbb{R}} Q_x\,\mathrm{d}x = \eta$. We see therefore that the soliton created has a larger mass than the initial perturbation, implying that the radiative part (going in the direction opposite to that of the soliton) has absorbed a total mass of 3η.

The energy conversion efficiency from a small noise source to a soliton is very poor, since the input energy is of order σ^2 ($E_{U_0} = \sigma^2\int_{\mathbb{R}} Q_x^2\,\mathrm{d}x$), while the energy of the created soliton is only $E_\eta = \int_{\mathbb{R}} U^2(x)\,\mathrm{d}x = \frac{16}{3}\eta^3 \sim \sigma^3$, for a smooth source. For a white noise source, the input energy is infinite, while the energy of the created soliton is finite.

6. Conclusion and comments. For the NLS and KdV equations, the study of soliton emergence from a localized, bounded initial condition perturbed by a wide class of rapidly oscillating random processes can be reduced to the study of a canonical system of SDEs, formally corresponding to the white noise perturbation of the initial condition. The integrated covariance is the only parameter of the perturbation process that influences the limit system of SDEs. From the study of this limit system, one obtains quantitative information on the modification of solitons due to the random perturbation.

For the NLS equation we have a threshold effect: if for the deterministic initial condition the integral $\int_{\mathbb{R}} U_0(x)\,\mathrm{d}x$ exceeds $\pi/2$, at least one soliton is created. In this case, a small-amplitude random perturbation of the initial condition results in a small variation in amplitude for the created soliton. However, if the phase of the (possibly complex) perturbation is constant and equal to the phase of the deterministic background initial condition, the speed of the created soliton is not modified. On the contrary, a complex perturbation with varying phase can modify both the amplitudes and speeds of solitons.

Since the KdV equation does not present a threshold phenomenon, a small-amplitude random perturbation always results in a small variation in both speed and amplitude of the created soliton, for any nonnegative initial condition.

As for a "quiescent" soliton, for both NLS and KdV a stochastic perturbation has a positive probability (depending on the type of perturbation used) of creating a new real soliton.

The results of Section 3 on the canonical system of SDEs for rapidly oscillating processes holds for any value σ of the amplitude of the perturbation, and the results on the stability of solitons with respect to small random perturbations ($\sigma \ll 1$) hold also without the assumption of a rapidly oscillating process. However, a general framework to treat the problem of creation of solitons without the assumptions of a rapidly oscillating initial condition or smallness of the random perturbation seems not to be available at the moment. In particular, the result strongly depends on the kind of random process used in the initial condition. However, specific cases can be treated with ad-hoc techniques: some examples will be provided in a subsequent publication.

APPENDIX: TECHNICAL LEMMAS

We collect here a few technical results needed for the proof of Theorem 3.2 and Propositions 4.1 and 5.1. We shall retain the different notation introduced there. We start by showing the proof of Lemma 3.4.

PROOF OF LEMMA 3.4. A subset $\mathcal{A} \subset \mathcal{H}$ of the Hilbert space $\mathcal{H}$ is relatively compact if and only if it is bounded, and for every $\lambda > 0$ there exists a finite-dimensional subspace $\mathcal{H}_n \subset \mathcal{H}$ s.t.

$$\sup_{h \in \mathcal{A}} d_{\mathcal{H}}(h, \mathcal{H}_n) < \lambda.$$

Therefore, the two conditions (32) are necessary to have that for any $\kappa > 0$ there exists a compact subset $\mathcal{A}_\kappa \subset \mathcal{H}$ s.t.

$$\sup_{\varepsilon \in (0,1]} \mathbb{P}\big(X^\varepsilon \in \mathcal{H} \setminus \mathcal{A}_\kappa\big) \le \kappa.$$

The two conditions (32) are also sufficient. Indeed, if they are satisfied for any given $\kappa > 0$ one obtains a compact subset $\mathcal{A}_\kappa$ of $\mathcal{H}$ considering the closure of

$$\mathcal{B}_\kappa = \mathcal{H} \setminus \bigcup_{n\ge 1} \big(\{h \,|\, \|h\| > \rho_\kappa\} \cup \{h \,|\, d_{\mathcal{H}}(h, \mathcal{H}_{\kappa/2^n, 1/n}) > 1/n\}\big).$$

Then one obtains that

$$\mathbb{P}\big(X^\varepsilon(x) \in \mathcal{H} \setminus \mathcal{A}_\kappa\big) \le 2\kappa.$$

The lemma is proved. $\square$

LEMMA A.1. *We construct an explicit example of the subspaces $\mathcal{H}_n \subset \mathcal{H} = W^{3,2}(G)$ to be used in the proof of Theorem 3.2 and Lemma A.2.*

CONSTRUCTION OF $\mathcal{H}_n$. Divide $G = (-N, N)^3$ into cubes with sides of length $1/n$ and add one extra layer of cubes around it:

$$A_{i,j,k} := \big[i/n, (i+1)/n\big) \times \big[j/n, (j+1)/n\big) \times \big[k/n, (k+1)/n\big)$$

for $i, j, k = -(nN+1), \dots, nN$. Define the piecewise (on every cube) polynomials of fourth degree as

$$\widetilde{h}(x) := \sum_{i,j,k=-(Nn+1)}^{Nn} \sum_{m=0}^{4} \frac{1}{m!} \langle a_{i,j,k}^{(m)} | x - y_{i,j,k} \rangle^{(m)} \mathbb{1}_{\{A_{i,j,k}\}}(x), \tag{44}$$

where $y_{i,j,k}$ is the center of the cube $A_{i,j,k}$, $a_{i,j,k}^{(m)}$ are families of m-dimensional tensors and the brackets denote the relative tensor products [so that, e.g., $\langle a_{i,j,k}^{(4)} | x - y_{i,j,k}\rangle^{(4)}$ denotes the product between the four-dimensional tensor $a_{i,j,k}^{(4)}$ and four copies of the vector $(x - y_{i,j,k})$]. With these definitions $\widetilde{h}$ is a function defined on $[-N - \frac{1}{n}, N + \frac{1}{n})^3$, but its restriction to G does not belong to $\mathcal{H}$ in general since it may not even be continuous. Let Γ be a real, nonnegative, smooth function, with compact support contained in $[-1/2, 1/2]^3$ and such that $\int_{[-1/2,1/2]^3} \Gamma(y)\, dy = 1$. Setting $\Gamma^n(y) := n^3 \Gamma(ny)$ we can finally define $\mathcal{H}_n$ as the finite-dimensional space of functions of the form $h(x) := (\widetilde{h} \star \Gamma^n)(x)$. Remark that $\widetilde{h}$ has been defined on a set larger than G, so that the convolution product is well defined for $x \in G$. $\square$

LEMMA A.2. *For every $g \in \mathcal{C}^4(G)$, there exists a $g_n \in \mathcal{H}_n$ s.t.*

$$\|g - g_n\|_{\mathcal{H}} \le C \frac{1}{n} \|g\|_{\mathcal{C}^4(G)}.$$

PROOF. *Step* 1 (Construction of g_n). For $i,j,k=-nN,\dots,nN-1$ (which means that $y_{i,j,k}\in G$) set $a^{(m)}_{i,j,k}:=D^m g(y_{i,j,k})$. For i,j,k such that $y_{i,j,k}\notin G$ set $a^{(m)}_{i,j,k}:=D^m g(y')$, where y' is the nearest cube center; notice that the distance of these two points is at most the diameter of the cubes, which we call $2\delta:=2\sqrt{3}n^{-1}$. With the $a^{(m)}_{i,j,k}$ thus defined we construct the piecewise polynomial function $\widetilde{g}_n$ as in (44): on the cubes the centers of which are not in G, this function is just a copy of the function defined on the nearest cube with center in G. Finally, we define g_n as the convolution product $g_n:=\Gamma^n\star\widetilde{g}_n$.

Step 2 (Estimates). For every multiindex $a\in\mathbb{N}^3$ such that $|a|_1:=a_1+a_2+a_3\le 3$, we need to estimate

$$\int_G\big|\partial^a\big(\Gamma^n\star\widetilde{g}_n\big)(x)-\partial^a g(x)\big|^2\,\mathrm{d}x.$$

To clarify the procedure to obtain an estimate for the above term, we first give explicit computations for the case $a=e_1=(1,0,0)$. Recall that, by definition,

$$\sum_{i,j,k}\int_{A_{i,j,,k}}\Gamma^n(x-y)\,\mathrm{d}y=\int_{[-(N+1/n),N+1/n)^3}\Gamma^n(x-y)\,\mathrm{d}y=1.$$

For $a=(1,0,0)$, using twice the inequality $(a+b)^2\le 2(a^2+b^2)$, we have

$$\begin{aligned}
&\big|\partial_1\big(\Gamma^n\star\widetilde{g}_n\big)(x)-\partial_1 g(x)\big|^2\\
&\quad=\bigg|\sum_{i,j,k}\int_{A_{i,j,k}}-\partial_{y_1}\Gamma^n(x-y)\widetilde{g}_n(y)\,\mathrm{d}y-\partial_1 g(x)\bigg|^2\\
&\quad\le 2\bigg\{\bigg|\sum_{i,j,k}\int_{A_{i,j,k}}\Gamma^n(x-y)\partial_{y_1}\widetilde{g}_n(y)\,\mathrm{d}y-\partial_1 g(x)\bigg|^2\\
&\qquad\qquad+\bigg|\sum_{i,j,k}\int_{\partial_1 A_{i,j,k}}\Gamma^n(x-y)\big[\widetilde{g}_n\big(y_1^+,y_2,y_3\big)-\widetilde{g}_n\big(y_1^-,y_2,y_3\big)\big]\,\mathrm{d}y_2\,\mathrm{d}y_3\bigg|^2\bigg\}\\
&\quad\le 4\bigg\{\bigg|\sum_{i,j,k}\int_{A_{i,j,k}}\Gamma^n(x-y)\big|\partial_{y_1}\widetilde{g}_n(y)-\partial_1\widetilde{g}_n(x)\big|\,\mathrm{d}y\bigg|^2\\
&\qquad\qquad+\big|\partial_1\widetilde{g}_n(x)-\partial_1 g(x)\big|^2\\
&\qquad\qquad+\bigg|\sum_{i,j,k}\int_{\partial_1 A_{i,j,k}}\Gamma^n(x-y)\big[\widetilde{g}_n\big(y_1^+,y_2,y_3\big)-\widetilde{g}_n\big(y_1^-,y_2,y_3\big)\big]\,\mathrm{d}y_2\,\mathrm{d}y_3\bigg|^2\bigg\}\\
&\quad=4\{S_1+S_2+S_3\},
\end{aligned}\tag{45}$$

where $\partial_1 A_{i,j,k}$ denotes the faces of the cubes orthogonal to the direction $e_1:=(1,0,0)$. Notice that the number of nonzero terms in the sums over i,j,k of this proof is limited to 8 because the support of Γ^n can intersect at most 8 cubes. The term S_1 can be bounded by the square of

$$\Big(\sum_{i,j,k} \|\Gamma^n(x-\cdot)\|_{L^1(A_{i,j,k})}\Big)\|\partial_1\widetilde{g}_n(\cdot)-\partial_1\widetilde{g}_n(x)\|_{L^\infty(B_{1/2n}(x))},$$

where the second term is regarded as a function of y (x is fixed), and the L^∞-norm is taken on the ball $B_{1/2n}(x)$, which is the support of $\Gamma^n(x-y)$. The first term above is 1, and to estimate the second term, the worst case is when y does not belong to the same cube as x: let us say that $y \in A^{(1)}$ and $x \in A^{(2)}$, where $y^{(1)}$ and $y^{(2)}$ are the centers of the cubes $A^{(1)}$ and $A^{(2)}$, respectively. We have therefore the bound

$$\begin{aligned}
&\|\partial_1\widetilde{g}_n(\cdot)-\partial_1\widetilde{g}_n(x)\|_{L^\infty(B_{1/(2n)}(x))}\\
&\quad\le \|\partial_1\widetilde{g}_n(\cdot)-\partial_{y_1}\widetilde{g}_n(y^{(1)})\|_{L^\infty}+|\partial_1 g(y^{(1)})-\partial_1 g(y^{(2)})|\\
&\qquad+|\partial_{y_1}\widetilde{g}_n(y^{(2)})-\partial_{y_1}\widetilde{g}_n(x)|\\
&\quad\le 4\delta\|D^2 g\|_{L^\infty(G)}.
\end{aligned}$$

This provides the bound for the term S_1. Similarly, for S_2 we have the bound

$$\begin{aligned}
|\partial_1\widetilde{g}_n(x)-\partial_1 g(x)| &\le |\partial_1\widetilde{g}_n(x)-\partial_1 g(y^{(2)})|+|\partial_1 g(y^{(2)})-\partial_1 g(x)|\\
&\le 2\delta\|D^2 g\|_{L^\infty(G)}.
\end{aligned}$$

We still need to estimate S_3, which contains the boundary terms deriving from the discontinuities of $\widetilde{g}_n$ (and, in the general case, of its derivatives). This term requires more careful estimates. With $C_\Gamma := \sup_x \Gamma(x)$ and using the fact that $\|\Gamma^n(x-\cdot)\|_{L^1(\partial_1 A_{i,j,k})} \le nC_\Gamma$, we have

$$\begin{aligned}
&\sum_{i,j,k}\|\Gamma^n(x-\cdot)\|_{L^1(\partial_1 A_{i,j,k})}\|\widetilde{g}_n(y_1^+,\cdot)-\widetilde{g}_n(y_1^-,\cdot)\|_{L^\infty(\partial_1 A_{i,j,k})}\\
&\quad\le nC_\Gamma\sum_{i,j,k}\{\|\widetilde{g}_n(y_1^+,\cdot)-g(y_1,\cdot)\|_{L^\infty(\partial_1 A_{i,j,k})}\\
&\qquad\qquad+\|g(y_1,\cdot)-\widetilde{g}_n(y_1^-,\cdot)\|_{L^\infty(\partial_1 A_{i,j,k})}\}\\
&\quad\le 8nC_\Gamma\delta^4\|D^4 g\|_{L^\infty(G)}\\
&\quad= C\delta^3\|D^4 g\|_{L^\infty(G)}.
\end{aligned}$$

The sums over i,j,k above are meant as sums over the faces $\partial_1 A_{i,j,k}$ intersecting the support of $\Gamma^n(x-\cdot)$, which are at most 4. Collecting all these results, we have the uniform bound

$$|\partial_1(\Gamma^n\star\widetilde{g}_n)(x)-\partial_1 g(x)|^2 \le C\delta^2(\|D^2 g\|^2_{L^\infty(G)}+\delta^4\|D^4 g\|^2_{L^\infty(G)}).$$

We proceed in the same way with higher order derivatives to get, for a generic derivative a of order $0 \le |a| \le 3$, the estimate

$$\begin{aligned}
&\big|\partial^a\big(\Gamma^n \star \widetilde{g}_n\big)(x) - \partial^a g(x)\big|^2 \\
&\quad \le C\bigg\{\bigg|\sum_{i,j,k}\int_{A_{i,j,k}} \Gamma^n(x-y)\big|\partial^a\widetilde{g}_n(y) - \partial^a\widetilde{g}_n(x)\big|\,\mathrm{d}y\bigg|^2 \\
&\qquad\qquad + \big|\partial^a\widetilde{g}_n(x) - \partial^a g(x)\big|^2 + S_3^a\bigg\} \\
&\quad \le C\big(\|D^{|a|+1}g\|_{L^\infty(G)}^2\delta^2 + S_3^a\big).
\end{aligned}$$

For $a = 0$, $S_3^a = 0$, but in the general case, the estimate of the term S_3^a is a little bit more delicate, since one gets more boundary terms. In particular, when integrating by parts, derivatives along different directions result in terms containing discontinuities of $\widetilde{g}_n$ and its derivatives along the faces (that we denote for brevity ∂A), edges (denoted $\partial^2 A$) or vertices (denoted $\partial^3 A$) of the cubes, while multiple derivatives along the same direction result in derivatives of Γ^n appearing. For example, for $a = (1,1,1)$ we get three kinds of terms,

$$\int_{\partial^3 A_{i,j,k}} \Gamma^n(x-y)\Delta\widetilde{g}_n(y)\,\mathrm{d}y,$$

$$\int_{\partial^2 A_{i,j,k}} \Gamma^n(x-y)\Delta\partial\widetilde{g}_n(y)\,\mathrm{d}y,$$

$$\int_{\partial A_{i,j,k}} \Gamma^n(x-y)\Delta\partial^2\widetilde{g}_n(y)\,\mathrm{d}y,$$

where Δ denotes the jump of the function. For $a = (3,0,0)$ we also have terms like

$$\int_{\partial_1 A_{i,j,k}} \partial_1^2\Gamma^n(x-y)\Delta\widetilde{g}_n(y)\,\mathrm{d}y, \qquad \int_{\partial_1 A_{i,j,k}} \partial_1\Gamma^n(x-y)\Delta\partial_1\widetilde{g}_n(y)\,\mathrm{d}y$$

and in the general case we find also terms like

$$\int_{\partial^2 A_{i,j,k}} \partial\Gamma^n(x-y)\Delta\widetilde{g}_n(y)\,\mathrm{d}y.$$

However, we can bound all this terms in the same way. We have that, for $m \in \mathbb{N}$ and $b, c \in \mathbb{N}^3$,

$$\int_{\partial^m A_{i,j,k}} \partial^b\Gamma^n(x)\,\mathrm{d}x \le C_\Gamma n^{m+|b|} = C\delta^{-(m+|b|)},$$

$$\big|\Delta\partial^c\widetilde{g}_n(y)\big| = \big|\partial^c\widetilde{g}_n(y^+) - \partial^c\widetilde{g}_n(y^-)\big| \le 2\|D^4 g\|_{L^\infty(G)}\delta^{4-|c|}.$$

Therefore

$$S_3^a \le C\|D^4 g\|_{L^\infty(G)}^2\delta^{2(4-|a|)}.$$

Note that for all the terms composing S_3^a we always have $m+|b|+|c|=|a|\le 3$. Summing up, we have obtained the bound

$$\begin{aligned}\|g-g_n\|_{\mathcal{H}}^2&=\sum_a\int_G\bigl|\partial^a\bigl(\Gamma^n\star\widetilde{g}_n\bigr)(x)-\partial^a g(x)\bigr|^2\,\mathrm{d}x\\&\le C\sum_a\bigl(\|D^{|a|+1}g\|_{L^\infty(G)}^2\delta^2+\|D^4 g\|_{L^\infty(G)}^2\delta^{2(4-|a|)}\bigr)\\&\le C\frac{1}{n^2}\|g\|_{\mathcal{C}^4(G)}^2.\end{aligned}$$

The lemma is proved. □

COROLLARY A.3. *For any $h\in\mathcal{H}$, $\lim_{n\to\infty}\pi_{\mathcal{H}_n}h=h$.*

PROOF. Fix any $\varepsilon>0$. By density, there exist a $h_\varepsilon\in\mathcal{C}^4(G)$ s.t. $\|h-h_\varepsilon\|_{\mathcal{H}}\le\varepsilon/2$. Also, by the continuity of the projection, $\|\pi_{\mathcal{H}_n}h-\pi_{\mathcal{H}_n}h_\varepsilon\|_{\mathcal{H}}\le\|h-h_\varepsilon\|_{\mathcal{H}}\le\varepsilon/2$. Since $\|\pi_{\mathcal{H}_n}h_\varepsilon-h_\varepsilon\|_{\mathcal{H}}\le\|h_{\varepsilon,n}-h_\varepsilon\|_{\mathcal{H}}$, by the above lemma we get

$$\begin{aligned}\|\pi_{\mathcal{H}_n}h-h\|_{\mathcal{H}}&\le\bigl\|\pi_{\mathcal{H}_n}(h-h_\varepsilon)\bigr\|_{\mathcal{H}}+\|\pi_{\mathcal{H}_n}h_\varepsilon-h_\varepsilon\|_{\mathcal{H}}+\|h_\varepsilon-h\|_{\mathcal{H}}\\&\le\varepsilon+C\frac{1}{n}\|h_\varepsilon\|_{\mathcal{C}^4(G)}.\end{aligned}$$

Therefore

$$\limsup_{n\to\infty}\|\pi_{\mathcal{H}_n}h-h\|_{\mathcal{H}}\le\varepsilon$$

and since ε is arbitrary, the corollary is proved. □

LEMMA A.4. *Let $F(\xi,\eta,\sigma)$ be the function defined in the proof of Proposition 4.1. Then, whenever $\zeta_0=i\eta_0$ is the value corresponding to a soliton component of the solution of the deterministic NLS equation, the determinant of the Jacobian matrix*

$$J:=\begin{pmatrix}\partial_\xi\Re(F)&\partial_\eta\Re(F)\\\partial_\xi\Im(F)&\partial_\eta\Im(F)\end{pmatrix}$$

at point $(0,\eta_0,0)$ is not zero.

PROOF. For $\sigma=0$ system (37) becomes deterministic, and the solution is given by (12)–(13). Then, setting $c:=\sqrt{q^2-\zeta^2}$,

$$i\partial_\xi\psi_1(\xi,\eta,0)=\partial_\eta\psi_1(\xi,\eta,0)=\left[\frac{q^2}{c^3}-iR\frac{\zeta}{c}\right]\sin(cR)+R\frac{\zeta^2}{c^2}\cos(cR)$$

so that

$$i\partial_\xi F(0,\eta_0,0)=\partial_\eta F(0,\eta_0,0).$$

We are left to verify that

$$
\begin{aligned}
0 \neq \det J(0, \eta_0, 0) &= \big[\partial_\xi \Re(F) \partial_\eta \Im(F) - \partial_\eta \Re(F) \partial_\xi \Im(F)\big](0, \eta_0, 0) \\
&= \big[\Re(\partial_\xi F) \Im(\partial_\eta F) - \Re(\partial_\eta F) \Im(\partial_\xi F)\big](0, \eta_0, 0) \\
&= \big[\big(\Re(\partial_\xi F)\big)^2 + \big(\Im(\partial_\xi F)\big)^2\big](0, \eta_0, 0)
\end{aligned}
$$

or equivalently

$$
\partial_\xi F(0, \eta_0, 0) = -i\left[\frac{q^2}{c_0^3} + R\frac{\eta_0}{c_0}\right]\sin(c_0 R) + i R \frac{\eta_0^2}{c_0^2}\cos(c_0 R) \neq 0, \tag{46}
$$

where $c_0 = \sqrt{q^2 - \eta_0^2}$. Observe that condition (14) implies that $\eta_0 \le q$; c_0 is therefore real. Indeed, for $\eta_0 > q$, c_0 would be purely imaginary, and the function f of equation (14) would become the sum of two purely imaginary terms of the same sign, so that it cannot be zero. In equation (46) the coefficient of the sinus is the sum of two nonzero terms of the same sign, so that to ensure the condition we need to check that

$$
\tan(c_0 R) = \frac{R\eta_0^2 c_0}{q + R\eta_0 c_0^2} \tag{47}
$$

does *not* holds for η_0 solution of (14). The compatibility condition between (14) and (47) is

$$
-\frac{c_0}{\eta_0} = \frac{R\eta_0^2 c_0}{q + R\eta_0 c_0^2}
$$

or equivalently

$$
(q + R\eta_0 c_0^2 + R\eta_0^3) c_0 = 0,
$$

which cannot be satisfied (recall that $\eta_0 \neq q$, so that $c_0 \neq 0$). The lemma is proved. □

PROPOSITION A.5. *The stochastic differential equation* (37) *defines a stochastic flow* $\Psi_x^{(\zeta,\sigma)} = (\psi_1, \psi_2)^T$ *of* $\mathcal{C}^1$*-diffeomorphisms, which is* $\mathcal{C}^1$ *also in the parameters* ξ, η, σ.

PROOF. Write the SDE in Itô and vector form,

$$
\mathrm{d}\Psi = i \begin{pmatrix} -\zeta + i\sigma^2 & q \\ q & \zeta + i\sigma^2 \end{pmatrix} \Psi \,\mathrm{d}x + i\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi \,\mathrm{d}W_x. \tag{48}
$$

The coefficients of the SDE are independent of x and Lipschitz continuous in Ψ for every ζ and σ. Therefore, the existence of a unique solution to the SDE, which defines a stochastic flow of homeomorphisms $\Psi^{(\zeta,\sigma)}(x)$, is a classical fact; see, for example, [10]. Following the notation of [11], we define the local characteristic of the SDE as (a, b, x), where

$$a(\zeta, \zeta', \sigma, \sigma', x) := -\sigma\sigma' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$b(\zeta, \sigma, x) := \begin{pmatrix} -\zeta & q \\ q & \zeta \end{pmatrix}.$$

Fix any $n \in \mathbb{N}$, define the set $G_n := \{(\xi, \eta, \sigma) | |\xi| < n, 0 < \eta < n, 0 < \sigma < n\}$ and consider the SDE only with parameters in G_n. Then, both a and b are uniformly bounded and, together with their first derivatives, are Lipschitz continuous in the parameters. This means that the coefficients satisfy condition $(A.5)_{1,0}$ of [11], Chapter 4.6. It follows from [11], Theorem 4.6.4, that $\Psi^{(\zeta,\sigma)}(x)$ is $\mathcal{C}^1$ in the parameters almost surely on G_n. Let Ω_n be the set of $\omega \in \Omega$ such that $\Psi^{(\zeta,\sigma)}(x) \in \mathcal{C}^1(G_n)$: it is a set of full measure. Since n is arbitrary, $\Psi^{(\zeta,\sigma)}(x)$ is actually $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+)$ for every $\omega \in \bigcap_n \Omega_n$, which is still a set of full measure. This proves the last statement of the proposition.

Since [11], Theorem 4.6.5, states that $\Psi^{(\zeta,\sigma)}(x)$ is actually a stochastic flow of $\mathcal{C}^1$-diffeomorphisms, the proof is complete. □

LEMMA A.6. *Let $F(\eta, \sigma)$ be the function defined in the proof of Proposition 5.1. Then, for all η_0 corresponding to soliton components of solutions of the deterministic problem*, $\partial_\eta F(\eta_0, 0) \neq 0$.

PROOF. For $\eta = \eta_0$ and $\sigma = 0$ we have that $\varphi = \varphi_0$ and (recall that $c_0 = \sqrt{q - \eta_0^2}$)

$$\partial_\eta \varphi(R) = \sin(c_0 R)\left[\frac{1 + \eta_0 R}{c_0} + \frac{\eta_0^2}{c_0^3}\right] - \frac{R\eta_0^2}{c_0^2}\cos(c_0 R),$$

$$\partial_\eta \widetilde{\varphi}(R) = \cos(c_0 R)[1 + R\eta_0] + \frac{\eta_0}{c_0}\sin(c_0 R)[1 + \eta_0 R] = [1 + \eta_0 R]\varphi_0(R).$$

Since

$$\partial_\eta F = \partial_\eta \widetilde{\varphi} + \varphi + \eta \partial_\eta \varphi,$$

$$\partial_\eta F(\eta_0, 0) = \cos(c_0 R)\left[2 + \eta_0 R - \frac{\eta_0^3 R}{c_0^2}\right] + \sin(c_0 R)\left[\frac{3\eta_0 + 2\eta_0^2 R}{c_0} + \frac{\eta_0^3}{c_0^3}\right].$$

The coefficient of the sinus is strictly positive (the coefficient of the cosinus has instead at least one zero for $\eta_0 \in [0, \sqrt{q}]$, since it is positive for $\eta_0 = 0$ and negative for $\eta_0 \to \sqrt{q}$). Therefore, we only need to verify that the equation

$$g(R, q, \eta_0) := \tan(c_0 R) + \frac{2 + \eta_0 R - \eta_0^3 R / c_0^2}{(3\eta_0 + 2\eta_0^2 R)/c_0 + \eta_0^3 / c_0^3} = 0 \tag{49}$$

is *not* satisfied, knowing that η_0 is a value corresponding to a soliton solution of the deterministic equation. As we have seen, we can either have $\eta_0 = \sqrt{q/2}$ if condition (19) is satisfied, or else η_0 is given as the solution of equation (20) in $(0, \sqrt{q}) \setminus \{\sqrt{q/2}\}$.

In the first case we have that $c_0 = \sqrt{q/2}$, so that condition (19) implies that $\cos(c_0 R) = 0$ and $\sin(c_0 R) = \pm 1$. Therefore, $\partial_\eta F \neq 0$.

Consider now the second case. We look for points $\eta \in (0, \sqrt{q}) \setminus \{\sqrt{q/2}\}$ such that $f(\eta) = g(\eta) = 0$. If such a point exists, then

$$\frac{2\eta c}{q - 2\eta^2} = -\frac{2 + \eta R - \eta^3 R/c^2}{(3\eta + 2\eta^2 R)/c + \eta^3/c^3},$$

which also reads

$$\frac{2\eta}{q - 2\eta^2} = -\frac{(2 + \eta R)c^2 - \eta^3 R}{(3\eta + 2\eta^2 R)c^2 + \eta^3}$$

or equivalently

$$\frac{(2\eta^2 + 2\eta^3 R)(q - \eta^2) + 2\eta^4(1 + \eta R) + (2 + \eta R)(q - \eta^2)q - \eta^3 Rq}{(q - 2\eta^2)[(3\eta + 2\eta^2 R)(q - \eta^2) + \eta^3]} = 0. \tag{50}$$

For $\eta < \sqrt{q/2}$ the denominator is positive and the numerator

$$\begin{aligned}&(2\eta^2 + 2\eta^3 R)(q - \eta^2) + 2\eta^4(1 + \eta R) + (2 + \eta R)(q - \eta^2)q - \eta^3 Rq \\ &\quad > \eta R(q - \eta^2) - \eta^3 Rq > 0,\end{aligned}$$

so that fraction (50) is positive and cannot be zero. For $\sqrt{q/2} < \eta < \sqrt{q}$ the denominator is negative and the numerator is larger than $2\eta^5 R - \eta^3 Rq = \eta^3 R(\eta^2 - q) > 0$, so that fraction (50) is negative and cannot be zero. The lemma is proved. □

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