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# Bayes minimax estimators of a location vector for densities in the Berger class

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**Abstract:** We consider Bayesian estimation of the location parameter $\theta$ of a random vector $X$ having a unimodal spherically symmetric density $f(\|x-\theta\|^2)$ when the prior density $\pi(\|\theta\|^2)$ is spherically symmetric and superharmonic. We study minimaxity of the generalized Bayes estimator $\delta_\pi(X) = X + \nabla M(X)/m(X)$ under quadratic loss, where $m$ is the marginal associated to $f(\|x-\theta\|^2)$ and $M$ is the marginal with respect to $F(\|x-\theta\|^2) = 1/2 \int_{\|x-\theta\|^2}^{\infty} f(t)\, dt$ under the condition $\inf_{t \geq 0} F(t)/f(t) = c > 0$ (see Berger [1]). We adopt a common approach to the cases where $F(t)/f(t)$ is nonincreasing or nondecreasing and, although details differ in the two settings, this paper complements the article by Fourdrinier and Strawderman [7] who dealt with only the case where $F(t)/f(t)$ is nondecreasing. When $F(t)/f(t)$ is nonincreasing, we show that the Bayes estimator is minimax provided $a\, \|\nabla\pi(\|\theta\|^2)\|^2/\pi(\|\theta\|^2) + 2\, c^2\, \Delta\pi(\|\theta\|^2) \leq 0$ where $a$ is a constant depending on the sampling density. When $F(t)/f(t)$ is nondecreasing, the first term of that inequality is replaced by $b\, g(\|\theta\|^2)$ where $b$ also depends on $f$ and where $g(\|\theta\|^2)$ is a superharmonic upper bound of $\|\nabla\pi(\|\theta\|^2)\|^2/\pi(\|\theta\|^2)$. Examples illustrate the theory.

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**Contents**



## 1. Introduction

Let $X$ be a random vector in $\mathbb{R}^p$ with spherically symmetric density

$$f(\|x-\theta\|^2) \tag{1.1}$$

around an unknown location parameter $\theta \in \mathbb{R}^p$ which has a spherically symmetric prior

$$\pi(\|\theta\|^2). \tag{1.2}$$

Consider estimation of the parameter $\theta$ under squared error loss

$$\|\delta-\theta\|^2\,, \tag{1.3}$$

for $p \geq 3$. We are mainly interested in the minimaxity of the generalized Bayes estimator $\delta_\pi(X)$ associated with the prior in (1.2).

Maruyama and Takemura [9] proved, through Baranchik type techniques that the Bayes estimator corresponding to the fundamental harmonic prior is minimax for certain $f$ in (1.1). Also Fourdrinier, Kortbi and Strawderman [3] showed such minimaxity results for sampling and prior densities which are both mixtures of normals with a monotone likelihood ratio property when considered as scale families.

By contrast, for spherically symmetric prior densities given in (1.2), Fourdrinier and Strawderman [7] did not utilize the Baranchik representation but used the superharmonicity of the prior $\pi$ more in the spirit of Stein [10]. Setting

$$F(t) = \frac{1}{2}\int_t^\infty f(u)du, \tag{1.4}$$

they treated the case where the sampling densities are such that $F(t)/f(t)$ is nondecreasing in $t$ and bounded from below by some positive constant $c$ (that is, the density in (1.1) belongs to the Berger class [1]). They proved that the risk difference between $\delta_\pi(X)$ and $X$ is bounded above by

$$E_\theta\left[2\,c\,\frac{\Delta M(X)}{m(X)} - 2\,c\frac{\nabla M(X)\cdot\nabla m(X)}{m^2(X)} + \frac{\|\nabla M(X)\|^2}{m^2(X)}\right] \tag{1.5}$$

where $E_\theta$ denotes the expectation with respect to (1.1) and $\cdot$ denotes the inner product in $\mathbb{R}^p$. Then, thanks to the superharmonicity of $\pi(\|\theta\|^2)$, they showed it suffices to prove that

$$\forall x \in \mathbb{R}^p \quad -2\,c\,\frac{\nabla M(x)\cdot\nabla m(x)}{m^2(x)} + \frac{\|\nabla m(x)\|^2}{m^2(x)} \leq 0$$

in order for the risk difference in (1.5) to be non positive. In contrast, in this paper, we demonstrate that

$$\nabla M(x)\cdot\nabla m(x) \geq 0\,, \tag{1.6}$$

and thus, minimaxity of $\delta_\pi(X)$ follows provided that

$$2\,c\frac{\Delta M(x)}{m(x)} + \frac{\|\nabla M(x)\|^2}{m^2(x)} \leq 0\,, \tag{1.7}$$

for all $x \in \mathbb{R}^p$. A benefit of this technique is that it allows us to also deal with the case where $F(t)/f(t)$ is nonincreasing, which constitutes a main contribution of this paper.

Finally, we will see below that, as we need to deal with tractable expressions for the derivatives of the marginals $m$ and $M$, the membership of the prior density $\pi\left(\|\theta\|^2\right)$ in certain Sobolev spaces and the membership of the generating function $f$ in a particular superset of the Schwartz space will be required.

In Section 2, we develop the model and give preliminary calculations involving the derivatives of the marginals related to $f$ and $F$. Section 3 is devoted to the first main result when $F(t)/f(t)$ is nonincreasing while, in Section 4, we deal with the case where $F(t)/f(t)$ is nondecreasing. Section 5 contains examples illustrating the theory developed in Sections 3 and Section 6 illustrates Section 4. Then we give, in Section 7, some concluding remarks and some perspectives. Finally, the last section is an Appendix containing needed technical material.

## 2. Model and generalized Bayes estimators

Let $X$ be a $p\times 1$ random vector having a spherically symmetric distribution as in (1.1) about an unknown vector $\theta \in \mathbb{R}^p$ which has a spherically symmetric prior density as in (1.2). Consider estimation of the parameter $\theta$ under the quadratic loss in (1.3). As in [7], the generalized Bayes estimator is the posterior mean and can be written as

$$\delta_\pi(X) = X + \frac{\nabla M(X)}{m(X)} \tag{2.1}$$

where, for any $x \in \mathbb{R}^p$,

$$m(x) = \int_{\mathbb{R}^p} f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, d\theta \tag{2.2}$$

is the marginal density and

$$M(x) = \int_{\mathbb{R}^p} F(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, d\theta \tag{2.3}$$

with $F(t)$ given in (1.4). Here $\nabla$ denotes the gradient.

As mentioned above, our aim is to study the minimaxity of the Bayes estimator $\delta_\pi(X)$ in (2.1), which follows by proving that $\delta_\pi(X)$ dominates $X$ under the loss (1.3). Of course, this only makes sense if the risk of $X$ is finite, that is, if $E_0(\|X\|^2) < \infty$. It is worth noting that this last condition, in fact, implies the finiteness of the risk of the Bayes estimator $\delta_\pi(X)$ (see Lemma A.1). As stated in Section 1, this domination will be obtained as soon as Inequalities (1.7) and (1.6) will be satisfied.

Note that (1.7) and (1.6) involve the derivatives of the marginals $m$ and $M$ in (2.2) and (2.3). To express these derivatives conveniently, we will rely on formulas of the type

$$\int_{\mathbb{R}^p} D^\alpha \psi\left(\|x-\theta\|^2\right) \pi(\|\theta\|^2)\, d\theta = (-1)^\alpha \int_{\mathbb{R}^p} \psi\left(\|x-\theta\|^2\right) D^\alpha \pi(\|\theta\|^2)\, d\theta \tag{2.4}$$

where $\psi$ will be, either the function $f$, or the function $F$ and where, for a multi-index $\alpha = (\alpha_1, \dots, \alpha_p)$ (a p-uple of nonnegative integers) with lengh $|\alpha| = \alpha_1 + \cdots + \alpha_p$, $D^\alpha$ is the corresponding partial derivative operator.

As proved in Blouza, Fourdrinier and Lepelletier [2], and used in Fourdrinier and Lepelletier [4], general assumptions on $\psi$ and $\pi$ in order for Condition (2.4) to be satisfied are that these functions belong to appropriate Sobolev spaces. Thus it will be needed that the prior $\pi(\|\theta\|^2)$ is in the Sobolev space

$$W^{\alpha,1}_{loc}(\mathbb{R}^p) = \left\{u \in L^1_{loc}(\mathbb{R}^p) \big/ \forall\beta, |\beta| \le \alpha,\, D^\beta u \in L^1_{loc}(\mathbb{R}^p)\right\}$$

and that the function $\psi\left(\|x-\theta\|^2\right)$ is in the Sobolev space

$$W^{\alpha,\infty}(\mathbb{R}^p) = \left\{u \in L^\infty(\mathbb{R}^p) \big/ \forall\beta, |\beta| \le \alpha,\, D^\beta u \in L^\infty(\mathbb{R}^p)\right\}.$$

The aim of the membership in the above Sobolev spaces is mainly to control the functions around the origin. However we also need some regularity in a neighborhood of infinity and, to this end, we also assume that there exits $r > 0$ such that $\pi(\|\theta\|^2)$ belongs to $C^\alpha_b(\mathbb{R}^p \setminus B_r)$ (the space of functions $\alpha$-times continuously differentiable and bounded on $\mathbb{R}^p \setminus B_r$ where $B_r$ is the ball of radius $r$ centered at 0) and there exists $\epsilon > 0$ such that $\psi(\|x-\theta\|^2)$ is in $S^{\alpha,p+\epsilon}(\mathbb{R}^p \setminus B_r)$ (the space of functions $\alpha$-times continuously differentiable on $\mathbb{R}^p \setminus B_r$ such that $\sup_{x\in\mathbb{R}^p\setminus B_r;|\beta|\le\alpha;\gamma\le p+\epsilon} \|x\|^\gamma \left|D^\beta \psi\left(\|x-\theta\|^2\right)\right| < \infty$).

Finally the sufficient conditions we use in order that formula (2.4) is satisfied are

- the function $\theta \longmapsto \pi(\|\theta\|^2)$ belongs to $W^{\alpha,1}_{loc}(\mathbb{R}^p) \cap C^\alpha_b(\mathbb{R}^p \setminus B_r)$ for a certain $r > 0$.
- the function $\theta \longmapsto F\left(\|x-\theta\|^2\right)$ belongs to $W^{\alpha,\infty}(\mathbb{R}^p) \cap S^{\alpha,p+\epsilon}(\mathbb{R}^p \setminus B_r)$ for a certain $\epsilon > 0$.

It is more convenient to express the above condition on $F\left(\|x-\theta\|^2\right)$ in terms of the generating function $f$. This is done in the following lemma whose the proof is similar to the proofs of Lemma 5 and Lemma 6 in [4].

**Lemma 2.1.** *Let $\alpha \ge 1$, $x \in \mathbb{R}^p$ fixed and $r_0 = max(2, 2\|x\|)$. Assume that, for a certain $\epsilon > 0$, we have $f(t) \in S^{\alpha-1,p/2+1+\epsilon}(\mathbb{R}_+ \setminus \{0\})$. Then the function $\theta \longmapsto F\left(\|x-\theta\|^2\right)$ belongs to $W^{\alpha,\infty}(\mathbb{R}^p) \cap S^{\alpha,p+\epsilon}(\mathbb{R}^p \setminus B_{r_0})$.*

The derivatives of the marginals $m(x)$ and $M(x)$ are involved through $\nabla m(x)$, $\nabla M(x)$, $\Delta M(x)$. Clearly this is the highest order of derivation which matters and it can be seen, for $\alpha = 2$, that Lemma 2.1 applies to express these last quantities as expectations with respect to the posterior distribution given $x$. This is stated in the following lemma whose the proof follows directly from (2.4), as in [6], for various values of $\alpha$.

**Lemma 2.2.** *Assume that, there exist $r > 0$ such that $\pi(\|\theta\|^2) \in W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b(\mathbb{R}^p \setminus B_r)$ and $\epsilon > 0$ such that $f(t) \in S^{1,p/2+1+\epsilon}(\mathbb{R}_+ \setminus \{0\})$. We have*

$$\frac{\nabla m(x)}{m(x)} = E^x\left[\frac{\nabla\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right] \tag{2.5}$$

$$\frac{\nabla M(x)}{m(x)} = E^x\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{\nabla\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right] \tag{2.6}$$

$$\frac{\Delta M(x)}{m(x)} = E^x\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{\Delta\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right] \tag{2.7}$$

*where $E^x(\cdot)$ denotes the conditional expectation of $\theta$ given $x$.*

Similarly to what was noticed in [4], Lemma 2.1 and Lemma 2.2 still hold when requiring only that the assumptions on the generating function $f(t)$ are satisfied except on a finite set $T$ of values of $t$.

Note also that particular attention should be paid to the membership of the priors to the Sobolev space in Lemma 2.2. As an example, the fundamental harmonic prior $\pi(\|\theta\|^2) = \|\theta\|^{2-p}$, although it is a smooth function in $R^p \setminus \{0\}$, is not in the Sobolev space $W^{2,1}_{loc}(\mathbb{R}^p)$ so that Equality (2.7) is not valid for that prior. See Subsection 5.2 for more details.

## 3. A minimaxity result for nonincreasing $\mathbf{F(t)/f(t)}$

We can now formulate our first result about the minimaxity of the generalized Bayes estimator $\delta_\pi(X)$ in (2.1). This minimaxity will be obtained through improvement on the usual estimator $X$ using a condition which relates the prior $\pi(\|\theta\|^2)$ and the generating function $f(t)$.

**Theorem 3.1.** *Assume that $X$ has a spherically symmetric unimodal density $f(\|x-\theta\|^2)$ such that $E_0[\|X\|^2] < \infty$ and such that, for some $\epsilon > 0$, the generating function $f$ is in $S^{1,p/2+1+\epsilon}(\mathbb{R}_+ \setminus \{0\})$. Assume also that the function $F(t)/f(t)$ is nonincreasing and that $c = \lim_{t\to\infty} F(t)/f(t) > 0$.*

*Suppose that, for some $r > 0$, the spherical prior density $\pi(\|\theta\|^2)$ in (1.2) belongs to the space $W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b(\mathbb{R}^p \setminus B_r)$.*

*Then, under the quadratic loss (1.3), a sufficient condition for the Bayes estimator $\delta_\pi$ to dominate $X$ (and hence to be minimax) is that*

$$\frac{F^2(0)}{f^2(0)} \frac{\left\|\nabla\pi(\|\theta\|^2)\right\|^2}{\pi(\|\theta\|^2)} + 2\,c^2\,\Delta\pi(\|\theta\|^2) \le 0 \tag{3.1}$$

*for all $\theta \in \mathbb{R}^p$.*

*Proof.* First, note that Inequality (3.1) implies that $\Delta\pi(\|\theta\|^2) \le 0$ and hence that the prior density $\pi(\|\theta\|^2)$ is superharmonic.

The proof consists in proving (1.7) and (1.6). Consider first (1.7). According to Lemma 2.1, this inequality can be expressed as

$$\left\| E^x\!\left[\frac{F(\|x-\theta\|^2)\nabla\pi(\|\theta\|^2)}{f(\|x-\theta\|^2)\pi(\|\theta\|^2)}\right]\right\|^2 + 2\,c\,E^x\!\left[\frac{F(\|x-\theta\|^2)\Delta\pi(\|\theta\|^2)}{f(\|x-\theta\|^2)\pi(\|\theta\|^2)}\right] \le 0\,.$$

Through Jensen's inequality, it suffices to prove that

$$E^x\!\left[\left(\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\right)^2 \left\|\frac{\nabla\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right\|^2\right] + 2\,c\,E^x\!\left[\frac{F(\|x-\theta\|^2)\Delta\pi(\|\theta\|^2)}{f(\|x-\theta\|^2)\pi(\|\theta\|^2)}\right] \le 0\,. \tag{3.2}$$

Now it is clear that, the conditions on $f$ implying

$$\forall t \in \mathbb{R}_+ \quad 0 < c \le \frac{F(t)}{f(t)} \le \frac{F(0)}{f(0)}\,,$$

the superharmonicity of $\pi(\|\theta\|^2)$ entails that Inequality (3.2) holds as soon as

$$\frac{F^2(0)}{f^2(0)}\,E^x\!\left[\left\|\frac{\nabla\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right\|^2\right] + 2\,c^2\,E^x\!\left[\frac{\Delta\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right] \le 0\,,$$

and hence, as soon as

$$\frac{F^2(0)}{f^2(0)}\left\|\frac{\nabla\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right\|^2 + 2\,c^2\,\frac{\Delta\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)} \le 0\,,$$

which is Condition (3.1). Therefore (1.7) is proved. □

We now turn our attention to Inequality (1.6). Note that $F(t)$ is nonincreasing and that, by unimodality of $f(\|x-\theta\|^2)$, the function $f(t)$ is nonincreasing in $t$ as well. As an immediate consequence of (2.5) and (2.6) in Lemma 2.1, the right hand side of (1.6) equals, for any $x \in \mathbb{R}^p$,

$$\begin{aligned}\nabla m(x) \cdot \nabla M(x) &= \int_{\mathbb{R}^p} f(\|x-\theta\|^2)\, \nabla\pi(\|\theta\|^2)\, d\theta \cdot \int_{\mathbb{R}^p} F(\|x-\theta\|^2)\, \nabla\pi(\|\theta\|^2)\, d\theta \\ &= 4 \int_{\mathbb{R}^p} f(\|x-\theta\|^2)\, \pi'(\|\theta\|^2)\, \theta\, d\theta \cdot \int_{\mathbb{R}^p} F(\|x-\theta\|^2)\, \pi'(\|\theta\|^2)\, \theta\, d\theta\end{aligned} \tag{3.3}$$

expressing the gradient of $\pi(\|\theta\|^2)$. Now, by superharmonicity of the prior density $\pi(\|\theta\|^2)$, we have $\pi'(\|\theta\|^2) \le 0$ so that, by Lemma A.2 in the Appendix, each integral in (3.3) equals $x$ multiplied by a non positive function. Hence the right hand side of (3.3) is nonnegative and (1.6) is satisfied. This ends the proof of Theorem 3.1. □

## 4. The case where F(t)/f(t) is nondecreasing

In the case where the ratio $F(t)/f(t)$ is nondecreasing, we need an additionnal assumption on the prior $\pi(\|\theta\|^2)$, that is, the function $\|\nabla\pi(\|\theta\|^2)\|^2/\pi(\|\theta\|^2)$ is bounded above by a superharmonic function. Note that the constant $F(0)/f(0)$ in (3.1) will be replaced by the constant $d$ defined by

$$d^2 = \int_0^\infty \frac{F^2(r^2)}{f^2(r^2)}\, \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1}\, f(r^2)\, dr\,, \tag{4.1}$$

provided this integral is finite. This is stated in the following theorem.

**Theorem 4.1.** *Assume that $X$ has a spherically symmetric unimodal density $f(\|x-\theta\|^2)$ as in (1.1) such that $E_0[\|X\|^2] < \infty$ and $f(t) \in S^{1,p/2+1+\epsilon}(\mathbb{R}_+\backslash\{0\})$, for some $\epsilon > 0$. Assume also that the function $F(t)/f(t)$ is nondecreasing and bounded from below by a constant $c > 0$ and that the constant $d$ in (4.1) is finite.*

*Suppose that, for some $r > 0$, the spherical prior density $\pi(\|\theta\|^2)$ in (1.2) belongs to the space $W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b(\mathbb{R}^p\backslash B_r)$ and is such that $\|\nabla\pi(\|\theta\|^2)\|^2/\pi(\|\theta\|^2) \le g(\|\theta\|^2)$ where $g(\|\theta\|^2)$ is a superharmonic function.*

*Then, under the quadratic loss (1.2), a sufficient condition for the Bayes estimator $\delta_\pi$ to dominate $X$ (and hence to be minimax) is that*

$$d^2\, g(\|\theta\|^2) + 2\, c^2\, \Delta\pi(\|\theta\|^2) \le 0 \tag{4.2}$$

*for all $\theta \in \mathbb{R}^p$, with $d$ in (4.1).*

*Proof.* Note that the conditions which have led, in Theorem 3.1, to Inequality (1.6) remain unchanged; hence we just prove (1.7). Returning to Inequality (3.2), the first term can be written as

$$
\begin{aligned}
C(x) &= E^x\left[\left(\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\right)^2 \left\|\frac{\nabla\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right\|^2\right] \\
&= \frac{1}{m(x)}\int_{R^p}\left(\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\right)^2 \left\|\frac{\nabla\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right\|^2 f(\|x-\theta\|^2)\,\pi(\|\theta\|^2)\,d\theta \\
&= \frac{1}{m(x)}\int_0^\infty \int_{S_{r,x}} \frac{\left\|\nabla\pi(\|\theta\|^2)\right\|^2}{\pi(\|\theta\|^2)}\,dU_{r,x}(\theta)\left(\frac{F(r^2)}{f(r^2)}\right)^2 \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\,r^{p-1}\,f(r^2)\,dr \\
&\le \frac{1}{m(x)}\int_0^\infty \int_{S_{r,x}} g(\|\theta\|^2)\,dU_{r,x}(\theta)\left(\frac{F(r^2)}{f(r^2)}\right)^2 \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\,r^{p-1}\,f(r^2)\,dr \qquad (4.3)
\end{aligned}
$$

by assumption on $\|\nabla\pi(\|\theta\|^2)\|^2/\pi(\|\theta\|^2)$ and where $U_{r,x}$ is the uniform measure distribution on the sphere of radius $r$ centered at $x$. Now, since $g(\|\theta\|^2)$ is superharmonic, the function

$$\int_{S_{r,x}} g(\|\theta\|^2)\,dU_{r,x}$$

is nonincreasing in $r$ and, since $F(r^2)/f(r^2)$ is nondecreasing in $r$, by the covariance inequality, it follows from the last inequality that

$$C(x) \le d^2\,E^x\left[\frac{g(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right] \qquad (4.4)$$

with $d$ given by (4.1). Finally, using (4.4), by superharmonicity of $\pi(\|\theta\|^2)$ and the fact that $F(t)/f(t) \ge c > 0$, Inequality (3.2) holds as soon as

$$d^2\,E^x\left[\frac{g(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right] + 2\,c^2\,E^x\left[\frac{\Delta\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right] \le 0,$$

which is satisfied as soon as (4.2) holds. This ends the proof of the theorem. □

**Remark 4.1.** Bounding $\|\nabla\pi(\|\theta\|^2)\|^2/\pi(\|\theta\|^2)$ by the superharmonic function $g(\|\theta\|^2)$ in the statement of Theorem 4.1 may appear somewhat ad hoc, but it is quite useful in that it dramatically increases the class of priors to which the theorem applies. To see this, note that, for a prior density of the form $(\|\theta\|^2+b)^{-k}$ with $b>0$ and $k>0$, we have

$$\left\|\frac{\nabla\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right\|^2 = 4\,k^2\,\|\theta\|^2\,(\|\theta\|^2+b)^{-k-2}\,,$$

which is never superharmonic. However the upper bound

$$g(\|\theta\|^2) = (\|\theta\|^2+b)^{-k-1}$$

is superharmonic for $p-2\,(k+2)\ge 0$. It is easily shown that Inequality (4.2) is satisfied for $p-(d^2/c^2+2)\,k-2\ge 0$. Thus the replacement of the assumption that $\|\nabla\pi(\|\theta\|^2)\|^2/\pi(\|\theta\|^2)$ is superharmonic by superharmonicity of the upper bound $g(\|\theta\|^2)$ allows inclusion of priors of the form $(\|\theta\|^2+b)^{-k}$ with $b>0$.

Of additional interest is the fact that, if $\|\nabla\pi(\|\theta\|^2)\|^2/\pi(\|\theta\|^2)$ has $g(\|\theta\|^2)$ as an upper bound, then Inequality (4.2) implies

$$d^2\,\frac{\left\|\nabla\pi(\|\theta\|^2)\right\|^2}{\pi(\|\theta\|^2)}+2\,c^2\,\Delta\pi(\|\theta\|^2)\le 0\,,$$

so that a natural candidate for $g(\|\theta\|^2)$ is

$$g(\|\theta\|^2)=\,-2\,\frac{c^2}{d^2}\,\Delta\pi(\|\theta\|^2)\,,$$

provided that the Laplacian $\Delta\pi(\|\theta\|^2)$ is subharmonic. This is formulated in the following corollary.

**Corollary 4.1.** *Assume that $X$ has a spherically symmetric unimodal density $f(\|x-\theta\|^2)$ as in (1.1) such that $E_0[\|X\|^2]<\infty$ and $f(t)\in S^{1,p/2+1+\epsilon}(\mathbb{R}_+\backslash\{0\})$, for some $\epsilon>0$. Assume also that the function $F(t)/f(t)$ is nondecreasing and bounded from below by a constant $c>0$ and that the constant $d$ in (4.1) is finite.*

*Suppose that, for some $r>0$, the spherical prior density $\pi(\|\theta\|^2)$ in (1.2) belongs to the space $W^{2,1}_{loc}(\mathbb{R}^p)\cap C^2_b(\mathbb{R}^p\setminus B_r)$ and that the Laplacian $\Delta\pi(\|\theta\|^2)$ is subharmonic.*

*Then, under the quadratic loss (1.2), a sufficient condition for the Bayes estimator $\delta_\pi$ to dominate $X$ (and hence to be minimax) is that*

$$d^2\,\frac{\|\nabla\pi(\|\theta\|^2)\|^2}{\pi(\|\theta\|^2)}+2\,c^2\,\Delta\pi(\|\theta\|^2)\le 0 \tag{4.5}$$

*for all $\theta\in\mathbb{R}^p$, with $d$ in (4.1).*

*Proof.* The corollary immediately follows, as in the proof of Theorem 4.1, from the fact that the upper bound of $C(x)$ in (4.3) is now

$$-2\,c^2\,E^x\left[\frac{\Delta\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)}\right],$$

thanks to Inequality (4.5) and to the superharmonicity of $-\Delta\pi(\|\theta\|^2)$. □

Corollary 4.1 is applicable to the above example of prior density since it can be shown (see Subsection 6.2) that the bi-Laplacian of $(\|\theta\|^2+b)^{-k}$ is nonnegative for $p-2\,(k+3)\ge 0$. Also, it is easily seen that Inequality (4.5) is satisfied for $p-(d^2/c^2+2)\,k-2\ge 0$ (equivalently $k\le(p-2)/(d^2/c^2+2)$). Note that this is the same condition encountered for Inequality (4.2) to hold with $g(\|\theta\|^2)=(\|\theta\|^2+b)^{-k-1}$. In each case, since $d^2/c^2\ge 1$, the latter inequality predominates so that either bound gives the same result.

## 5. Illustrating Theorem 3.1

We first consider various examples of sampling densities. Then we provide general conditions for prior densities, which are mixtures of normals, to belong to the space $W^{2,1}_{loc}(\mathbb{R}^p)$ and to satisfy Condition (3.1).

### 5.1. Sampling densities

In the following, we give examples of generating functions $f(t)$ such that $f'(t) \leq 0$ and such that $F(t)/f(t)$ is nonincreasing and bounded from below by a constant $c > 0$, with $F(t)$ in (1.4). Note that, for these examples, it is guaranteed that $f \in S^{1,p/2+1+\epsilon}(\mathbb{R}_+ \setminus \{0\})$ for some $\epsilon > 0$ since the densities are elementary functions of the exponential function. For each example, we provide the values of $F(0)/f(0)$ and of $c = \lim_{t\to\infty} F(t)/f(t)$.

**Example 1.** Let

$$f(t) \propto \frac{\exp(-\beta t - \gamma)}{(1 + \exp(-\beta\, t - \gamma))^2}$$

where $\beta > 0$ and $\gamma > 0$. As $f(t)$ can be written as

$$f(t) \propto g\left(\exp\left(-\frac{1}{2}(\beta t + \gamma)\right)\right)$$

with $g(x) = 1/x + x$, it is clear that $f'(t) < 0$ since $g'(x) < 0$ for $0 < x < 1$. Also we have

$$F(t) = \left[\frac{1}{2\,\beta}\,\frac{1}{1 + \exp(-\beta\, u - \gamma)}\right]_t^\infty = \frac{1}{2\,\beta}\,\frac{\exp(-\beta t - \gamma)}{1 + \exp(-\beta\, t - \gamma)}$$

and hence

$$\frac{F(t)}{f(t)} = \frac{1}{2\beta}\,(1 + \exp(-\beta t - \gamma))$$

which is nonincreasing in $t$. It follows then that

$$\frac{F(0)}{f(0)} = \frac{1 + e^{-\gamma}}{2\beta} \tag{5.1}$$

and

$$c = \lim_{t\to\infty} \frac{F(t)}{f(t)} = \frac{1}{2\beta}\,. \tag{5.2}$$

**Example 2.** Let

$$f(t) \propto \frac{1}{\cosh(\beta\, t + \gamma)}$$

where $\beta > 0$ and $\gamma > 0$. Clearly, this function is nonincreasing since the hyperbolic cosine is nondecreasing, and hence, the sampling density $f(\|x - \theta\|^2)$ is unimodal. Also we have

$$\begin{aligned} F(t) &= \frac{1}{2}\int_t^\infty \frac{1}{\cosh(\beta\, u + \gamma)} du \\ &= \left[\frac{1}{\beta}\arctan(e^{\beta\, u + \gamma})\right]_t^\infty \\ &= \frac{1}{\beta}\left[\frac{\pi}{2} - \arctan(e^{\beta\, t + \gamma})\right] \\ &= \frac{1}{\beta}\arctan(e^{-\beta\, t - \gamma})\,, \end{aligned}$$

so that

$$\frac{F(t)}{f(t)} = \frac{1}{\beta}\arctan(e^{-\beta\,t-\gamma})\,\cosh(\beta\,t+\gamma)\,. \tag{5.3}$$

The nonincreasing monotonicity of $F(t)/f(t)$ can be comfortably found through the one of $f'(t)/f(t)$. Indeed,

$$\frac{f'(t)}{f(t)} = -\beta\tanh(\beta\,t+\gamma)$$

is nonincreasing which implies that $F(t)/f(t)$ is nonincreasing as well (see Fourdrinier and Strawderman [7]). As for the bounds of that function, first, it follows from (5.3) that

$$\frac{F(0)}{f(0)} = \frac{1}{\beta}\arctan(e^{-\gamma})\,\cosh(\gamma)\,. \tag{5.4}$$

Secondly, expressing the hyperbolic cosine in (5.3) as

$$\cosh(\beta\,t+\gamma) = 1/2\,\{e^{\beta\,t+\gamma}+e^{-\beta\,t-\gamma}\}$$

it is clear that

$$\begin{aligned} c = \lim_{t\to\infty}\frac{F(t)}{f(t)} &= \frac{1}{2\,\beta}\lim_{t\to\infty} e^{\beta\,t+\gamma}\arctan(e^{-\beta\,t-\gamma}) \\ &= \frac{1}{2\,\beta}\lim_{x\to 0}\frac{1}{x}\arctan x \\ &= \frac{1}{2\,\beta}\,. \end{aligned} \tag{5.5}$$

**Example 3.** Let

$$f(t) \propto \exp\left(\frac{-\beta t}{2}\right) - a\exp\left(\frac{-\beta t}{2c}\right)$$

where $0 \le a \le c < 1$ and $\beta > 0$. We have

$$f'(t) \propto \frac{a}{c}\,\exp\left(\frac{-\beta t}{2c}\right) - \exp\left(\frac{-\beta t}{2}\right) \le 0\,,$$

for the range of $a$ and $c$. We also have

$$\frac{F(t)}{f(t)} = \frac{1}{\beta}\,\frac{\exp\left(\frac{-\beta t}{2}\right) - a\,c\,\exp\left(\frac{-\beta t}{2c}\right)}{\exp\left(\frac{-\beta t}{2}\right) - a\,\exp\left(\frac{-\beta t}{2c}\right)}$$

which is nonincreasing in $t$. Finally, we have

$$\frac{F(0)}{f(0)} = \frac{1}{\beta}\,\frac{1-ac}{1-a} \tag{5.6}$$

and

$$c = \lim_{t \to \infty} \frac{F(t)}{f(t)} = \frac{1}{\beta}\,. \tag{5.7}$$

**Example 4.** Let

$$f(t) \propto (t + A)\, \exp(-\frac{t}{2})$$

with $A > 2$. We have

$$f'(t) \propto (2 - A - t)\, \exp(-\frac{t}{2}) \leq 0$$

for the range of $A$. We also have

$$\frac{F(t)}{f(t)} = 1 + \frac{2}{t + A}$$

which is nonincreasing in $t$. Finally, we have

$$\frac{F(0)}{f(0)} = 1 + \frac{2}{A} \tag{5.8}$$

and

$$c = \lim_{t \to \infty} \frac{F(t)}{f(t)} = 1\,. \tag{5.9}$$

### *5.2. Prior densities*

**Example 1 - Generalized $t$ densities**

A first basic example of spherical density prior is

$$\pi(\|\theta\|^2) = \frac{1}{(\|\theta\|^2 + b)^k} \tag{5.10}$$

with $b > 0$ and $k > 0$. Considering the membership of that density in the Sobolev space $W^{2,1}_{loc}(\mathbb{R}^p)$, it is clear that for $b > 0$, the function $\theta \mapsto \left(\|\theta\|^2 + b\right)^{-k}$ is indefinitely differentiable and is bounded, so that its membership in the space $W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b(\mathbb{R}^p \setminus B_r)$ is automatically satisfied. As for the case $b = 0$, the membership in $C^2_b(\mathbb{R}^p \setminus B_r)$ obviously remains and it can be directly checked from the expression $\|\theta\|^{-2k}$ that the density belongs to the Sobolev space $W^{2,1}_{loc}(\mathbb{R}^p)$ as soon as $k < p/2 - 1$, that is, $p > 2\,(k+1)$ (see below an alternative proof through Corollary 5.1).

Through straightforward calculations, it can be seen that Ineqality (3.1) is expressed as

$$\frac{F^2(0)}{f^2(0)} \times 4\,k^2\, \frac{\|\theta\|^2}{(\|\theta\|^2 + b)^2} + 2\,c^2 \times \frac{-2\,k}{\|\theta\|^2 + b} \left(p - 2\,(k+1)\, \frac{\|\theta\|^2}{\|\theta\|^2 + b}\right) \leq 0$$

which, after simplification, reduces to

$$\left(2\,(k+1) + k\,\frac{F^2(0)/f^2(0)}{c^2}\right)\frac{\|\theta\|^2}{\|\theta\|^2+b} \le p$$

and hence to

$$2\,(k+1) + k\,\frac{F^2(0)/f^2(0)}{c^2} \le p\,,$$

that is, to

$$k \le \frac{p-2}{\frac{F^2(0)/f^2(0)}{c^2}+2}\,. \tag{5.11}$$

**Example 2 - General mixtures of normals**

As the density prior in (5.10) is a mixture of normals densities, we consider prior densities of the form

$$\pi(\|\theta\|^2) \propto \int_0^\infty t^{p/2}\exp\left(-\|\theta\|^2\,t\right)h(t)\,dt \tag{5.12}$$

and we investigate conditions on the function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that the density in (5.12) belongs to $W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b(\mathbb{R}^p \setminus B_r)$, for some $r > 0$, and such that Inequality (3.1) is satisfied. Thus we assume that the mixing function $h$ is such that the function $t \mapsto t^{p/2}\,h(t)$ has, for any $s \ge 0$, a Laplace transform $\mathcal{L}\big(t^{p/2}\,h(t)\big)(s)$ at $s$ so that the prior in (5.12) can be seen as

$$\pi(\|\theta\|^2) \propto \mathcal{L}\Big(t^{p/2}\,h(t)\Big)(\|\theta\|^2)\,. \tag{5.13}$$

Thus

$$\mathcal{L}\Big(t^{p/2}\,h(t)\Big)(s) = \int_0^\infty t^{p/2}\,h(t)\exp(-s\,t)dt$$

is analytic in the open interval $(0,\infty)$ so that the, for any $n \ge 1$, the integral

$$\int_0^\infty t^{p/2+n}\,h(t)\exp(-s\,t)dt$$

converges and we have

$$\lim_{t\to 0} t^{p/2+n}\,h(t) = 0 \tag{5.14}$$

and, for any $u \ge 0$,

$$\lim_{t\to\infty} t^{p/2+n}\,h(t)\exp(-u\,t) = 0\,. \tag{5.15}$$

Also the prior in (5.12) is guaranteed to belong to the space $C^2_b(\mathbb{R}^p \setminus B_r)$. However, as it is not a Laplace transform of a multivariate function, it remains to deal with the membership of this prior in the Sobolev space $W^{2,1}_{loc}(\mathbb{R}^p)$.

Note that formal derivatives of $\pi(\|\theta\|^2)$ can be taken under the integral sign, so that the membership of the prior in (5.12) in the Sobolev space $W^{2,1}_{loc}(\mathbb{R}^p)$ reduces to verifying the local integrability of the formal derivatives. Using straightforward calculations, we have successively, for $i = 1, \ldots, p$,

$$\left|\partial_i(\pi(\|\theta\|^2))\right| \leq 2\,\|\theta\| \int_0^\infty t^{p/2+1} \exp\left(-t\|\theta\|^2\right) h(t)\, dt\,,$$

$$\left|\partial_{ii}(\pi(\|\theta\|^2))\right| \leq 2 \int_0^\infty t^{p/2+1}\, \exp\left(-t\|\theta\|^2\right) h(t)\, dt + 4\|\theta\|^2 \int_0^\infty t^{p/2+2} \exp\left(-t\|\theta\|^2\right) h(t)\, dt$$

and, for $j = 1, \ldots, p$ and $j \neq i$,

$$\left|\partial_{ji}(\pi(\|\theta\|^2))\right| \leq \|\theta\|^2 \int_0^\infty t^{p/2+2} \exp\left(-t\|\theta\|^2\right) h(t)\, dt\,,$$

where $\partial_i(.)$ and $\partial_{ji}(.)$ denote the partial derivative operators of the first and second order respectively.

Clearly, to obtain the desired local integrability of $\pi(\|\theta\|^2)$ and its partial derivatives, it suffices to establish the local integrability of functions of the form

$$\psi_{a,b}(\|\theta\|^2) = \|\theta\|^b \int_0^\infty t^{p/2+a} \exp\left(-t\|\theta\|^2\right) h(t)\, dt \tag{5.16}$$

for $a, b = 0, 1, 2$. This is related to the behavior of $h$ at $\infty$ and can be obtained through an Abelian theorem which relates the behavior of a function at $\infty$ to the behavior of its Laplace transform at 0. As can be seen, for instance, in Corollary 1.a page 182 of Widder [11]), if $\alpha$ is a function of bounded variation and normalized in any interval $[0, b]$ of $\mathbb{R}$ (that is, such that, for any $x$, $\alpha(x) = (\alpha(x_-) + \alpha(x_+))/2$ and such that $\alpha(0) = 0$) and if, for some $\gamma \geq 0$, there exists a constant $A$ such that

$$\alpha(t) \sim \frac{A\, t^\gamma}{\Gamma(\gamma+1)} \qquad t \to \infty\,, \tag{5.17}$$

then

$$\int_0^\infty e^{-st}\, d\alpha(t) \sim \frac{A}{s^\gamma} \qquad s \to 0_+\,,$$

and hence

$$\int_0^\infty e^{-st}\, \alpha(t)\, dt \sim \frac{A}{s^{\gamma+1}} \qquad s \to 0_+\,, \tag{5.18}$$

through an integration by parts. In the case where $A = 0$, Condition (5.17) and Condition (5.18) should be understood as $\alpha(t) = o(t^\gamma)$ and $\mathcal{L}(\alpha)(s) = o\big(s^{-\gamma-1}\big)$ respectively. With this result, we will prove the following lemma.

**Proposition 5.1.** *Let $\psi_{a,b}(\|\theta\|^2)$ be a function as in (5.16). If there exist a constant $A$ and $\gamma \geq 0$ such that*

$$\lim_{t\to\infty} t^{p/2+a-\gamma}\, h(t) \;=\; \frac{A}{\Gamma(\gamma+1)}\,, \tag{5.19}$$

*then $\psi_{a,b}(\|\theta\|^2)$ is locally integrable as soon as $\gamma < (p+b)/2-1$.*

*Proof.* According to (5.16) the function $\psi_{a,b}(\|\theta\|^2)$ can be expressed through the Laplace transform $\mathcal{L}(t^{p/2+a}\, h(u))(\|\theta\|^2)$ and what we have to show is that, for any ball $B_R$ centered at 0 and of radius $R$,

$$\int_{B_R} \psi_{a,b}(\|\theta\|^2)\, d\theta < \infty$$

or, equivalently, for any $c > 0$,

$$\int_0^c \mathcal{L}\Big(t^{p/2+a}\, h(t)\Big)(s)\, s^{(p+b)/2-1}\, ds \;<\; \infty. \tag{5.20}$$

Now, according to Condition (5.19), Condition (5.17) is satisfied with $\alpha(t) = t^{p/2+a}\, h(t)$, and hence, according to (5.18), we have

$$\mathcal{L}\Big(t^{p/2+a}\, h(t)\Big)(s) \sim \frac{A}{s^{\gamma+1}}, \qquad s \to 0_+\,,$$

so that

$$\mathcal{L}\Big(t^{p/2+a}\, h(t)\Big)(s)\, s^{(p+b)/2-1} \sim A\, s^{(p+b)/2-\gamma-2}\,, \qquad s \to 0_+\,, \tag{5.21}$$

Clearly, Condition (5.21) allows Condition (5.20) to be satisfied if and only if $(p+b)/2-\gamma-2 > -1$, that is, if and only if $\gamma < (p+b)/2-1$. □

We are now in a position to give condtions on the prior density in (5.12) to belong to $W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b\,(\mathbb{R}^p \setminus B_r)$.

**Corollary 5.1.** *If there exist a constant $A$ and $\gamma \geq 0$ such that*

$$\lim_{t\to\infty} t^{p/2+2-\gamma}\, h(t) \;=\; \frac{A}{\Gamma(\gamma+1)}\,, \tag{5.22}$$

*then the prior density in (5.12) belongs to $W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b\,(\mathbb{R}^p \setminus B_r)$ as soon as $\gamma < p/2$ (Condition (5.22) being understood as $t^{p/2+2-\gamma}\, h(t) = o(1)$ when $A = 0$).*

In order that the generalized Bayes estimator $\delta_\pi(X)$, associated to the prior in (5.12), dominate $X$ (and hence to minimax), it remains to give conditions on the mixing density $h$ so that Condition (3.1) is satisfied. We will see below that the main assumption on $h$ is that the function $t \longmapsto t\, h'(t)/h(t)$ is non increasing. This condition has been noticed by [8] and [3] as being equivalent to the property of nondecreasing monotone likelihood ratio when $h$ is considered as a scale family. The following proposition gives conditions on the mixing density $h$ for minimaxity to follow via Theorem 3.1.

**Proposition 5.2.** *Assume that the function* $t \longmapsto t\, h'(t)/h(t)$ *is nonincreasing. Then Condition (3.1) is satisfied as soon as, for all* $t \geq 0$,

$$t\, \frac{h'(t)}{h(t)} \leq -\varphi_0 \tag{5.23}$$

*where*

$$\varphi_0 = \frac{1}{2}\, \frac{(p+2)\, F^2(0)/f^2(0) + 8\, c^2}{F^2(0)/f^2(0) + 2\, c^2}\,. \tag{5.24}$$

*Proof.* Through straightforward calculations, we have

$$\left\| \frac{\nabla \pi(\|\theta\|^2)}{\pi(\|\theta\|^2)} \right\|^2 = 4\, \|\theta\|^2 \left( \frac{\pi'(\|\theta\|^2)}{\pi(\|\theta\|^2)} \right)^2$$

and

$$\frac{\Delta \pi(\|\theta\|^2)}{\pi(\|\theta\|^2)} = 2 \left\{ p\, \frac{\pi'(\|\theta\|^2)}{\pi(\|\theta\|^2)} + 2\, \|\theta\|^2\, \frac{\pi''(\|\theta\|^2)}{\pi(\|\theta\|^2)} \right\}$$

which are functions of the ratios

$$\frac{\pi'(\|\theta\|^2)}{\pi(\|\theta\|^2)} \qquad \text{and} \qquad \frac{\pi''(\|\theta\|^2)}{\pi(\|\theta\|^2)}\,.$$

Since

$$\pi'(\|\theta\|^2) = -\int_0^\infty t^{p/2+1}\, h(t)\, \exp(-t\, \|\theta\|^2)\, dt\,,$$

we have

$$\frac{\pi'(\|\theta\|^2)}{\pi(\|\theta\|^2)} = E_{\|\theta\|^2}[-T]$$

where $E_{\|\theta\|^2}$ denotes the expectation with respect to the distribution of $T$ with density

$$t \mapsto \frac{t^{p/2}\, h(t)\, \exp(-\|\theta\|^2\, t)}{\int_0^\infty u^{p/2}\, h(u)\, \exp(-\|\theta\|^2\, u)\, du}\,. \tag{5.25}$$

On the other hand, by integration by parts, we can also express

$$\pi'(\|\theta\|^2) = \frac{-1}{\|\theta\|^2} \int_0^\infty \left[ \frac{p}{2} + 1 + t\, \frac{h'(t)}{h(t)} \right] t^{p/2}\, h(t)\, \exp(-t\, \|\theta\|^2)\, dt$$

according to (5.14) and (5.15), so that

$$\frac{\pi'(\|\theta\|^2)}{\pi(\|\theta\|^2)} = \frac{-1}{\|\theta\|^2} E_{\|\theta\|^2} \left[ \frac{p}{2} + 1 + T\, \frac{h'(T)}{h(T)} \right].$$

Therefore

$$\left\| \frac{\nabla \pi(\|\theta\|^2)}{\pi(\|\theta\|^2)} \right\|^2 = 2\, E_{\|\theta\|^2}[T]\, E_{\|\theta\|^2} \left[ p + 2 + 2\, T\, \frac{h'(T)}{h(T)} \right].$$

Similarly, as

$$\pi''(\|\theta\|^2) = \int_0^\infty t^{p/2+2}\, h(t)\, \exp(-t\,\|\theta\|^2)\, dt\,,$$

an integration by parts leads to

$$\pi''(\|\theta\|^2) = \frac{1}{\|\theta\|^2}\int_0^\infty \left[\left(\frac{p}{2}+2\right)t + t^2\,\frac{h'(t)}{h(t)}\right] t^{p/2}\, h(t)\, \exp(-t\,\|\theta\|^2)\, dt$$

so that

$$\begin{aligned}\frac{\Delta\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)} &= 2\,E_{\|\theta\|^2}\left[-p\,T + (p+4)\,T + 2\,T^2\,\frac{h'(T)}{h(T)}\right] \\ &= 4\,E_{\|\theta\|^2}\left[2\,T + \,T^2\,\frac{h'(T)}{h(T)}\right]. \qquad (5.26)\end{aligned}$$

Thus Inequality (3.1) may be expressed as

$$\frac{F^2(0)}{f^2(0)}\,E_{\|\theta\|^2}[T]\,E_{\|\theta\|^2}\left[p+2+2\,T\,\frac{h'(T)}{h(T)}\right] + 4\,c^2 E_{\|\theta\|^2}\left[2\,T + T^2\,\frac{h'(T)}{h(T)}\right] \le 0\,. \qquad (5.27)$$

Since, by assumption, $t \mapsto t\,h'(t)/h(t)$ is nonincreasing, by the covariance inequality, we have

$$E_{\|\theta\|^2}\left[2\,T + T^2\,\frac{h'(T)}{h(T)}\right] \le E_{\|\theta\|^2}[T]\,E_{\|\theta\|^2}\left[2 + T\,\frac{h'(T)}{h(T)}\right].$$

Hence (5.27) is satisfied as soon as, for all $t$,

$$\frac{F^2(0)}{f^2(0)}\left[p+2+2\,t\,\frac{h'(t)}{h(t)}\right] + 4\,c^2\left[2 + t\,\frac{h'(t)}{h(t)}\right] \le 0,$$

which can be seen to be equivalent to (5.23). □

It is worth noting that, the upper bound of $u\,h'(u)\,/h(u)$ given by Condition (5.23) satisfies $\varphi_0 < -2$ for $p \ge 3$. Note that Fourdrinier and Strawderman [7], to obtain the superharmonicity of priors $\pi(\|\theta\|^2)$ in (5.12), required that $t\,h'(t)\,/t < -2$, which is, of course satisfied here. However, they do not impose a monotone likelihood ratio property on the mixing density $h$, which we require in this paper.

**Example of mixing densities**

*1- A simple example of mixing density* $h$ is given by

$$h(t) \,\propto\, t^{k-p/2-1}\,\exp\left(-b\,t\right)$$

where $b \ge 0$ and $k > 0$. This mixing density gives rise to the prior $\left(\|\theta\|^2 + b\right)^{-k}$ considered in Example 1. We can retrieve the Sobolev membership of that prior applying Corollary 5.1. Indeed, when $b = 0$, with $\gamma = k + 1$, we see that the prior belongs to the Sobolev space $W^{2,1}_{loc}(\mathbb{R}^p)$ as soon as $k < p/2 - 1$, that is, $p > 2\,(k+1)$. The case where $b > 0$ is obvious.

Now it is easy to see that $h$ satisfies the monotonicity condition (5.23) required in Proposition 5.2 since,

$$t\,\frac{h'(t)}{h(t)} = -b\,t + k - p/2 - 1$$

which nonincreasing in $t$. Thus, a sufficient condition to satisfy Inequality (5.23) is

$$k - \frac{p}{2} - 1 \leq -\varphi_0\,.$$

which is, according to (5.24), equivalent to

$$0 \,<\, k \,\leq\, (p-2)\,\frac{c^2}{F^2(0)/f^2(0) + 2\,c^2}\,. \tag{5.28}$$

Note that, as expected, the upper bound on $k$ in (5.28) is smaller than the one in (5.11) ((5.23) is a sufficient condition for (3.1)).

*2- A constructive approach to obtain functions $h$ satisfying (5.23).*

Let $\varphi$ be a nondecreasing and nonnegative function $\varphi(t)$ such that a primitive of the function $t \mapsto \varphi(t)/t$ exists. To any such function $\varphi$, associate a mixing density

$$h(t) = \exp\left(-\int_{t_0}^{t} \frac{\varphi(u)}{u}\,du\right)$$

where $t_0$ is some nonnegative number. It is clear that

$$t\,\frac{h'(t)}{h(t)} \;=\; -\varphi(t)$$

so that it suffices to choose $\varphi$ such that $\varphi(t) \geq \varphi_0$ in order (5.23) to be satisfied.

Now, noticing that, $h(t)$ is bounded from above by a positive multiple of

$$\exp\left(\int_{t_0}^{t} -\frac{\varphi_0}{u}\,du\right) \propto t^{-\varphi_0}$$

so that, for $\gamma < p/2$, up to a positive multiplicative constant, $h(t)\,t^{p/2+2-\gamma}$ is bounded from above by

$$t^{-\varphi_0+p/2+2-\gamma}\,,$$

the needed Sobolev membership can be obtained using Corollary 5.1.

Note that the previous example corresponds to the choice of $\varphi(t) = b\,t - k + p/2 + 1$ and to $\varphi_0 = -k + p/2 + 1$. As another application, the choice of $\varphi(t) = a\,t^2 + b\,t + c$ with $a$, $b$ and $c$ being positive constants such that $b^2 - 4\,a\,c < 0$ leads to the mixing function $h(t) \propto \exp(-a\,t^2/2)\,\exp(-b\,t)\,t^{-c}$.

## 6. Illustrating Theorem 4.1

Here the improvement on the result of Fourdrinier and Strawderman [7] results from weaker assumptions on the sampling density.

### 6.1. Sampling densities

In [7], examples of sampling densities are given which satisfy stronger conditions than the assumptions of Theorem 4.1 (that is, $F(t)/f(t) \geq c > 0$ and the nondecreasing monotonicity of $F(t)/f(t)$). In particular, they require nondecreasing monotonicity of $f'(t)/f(t)$ (which implies the nondecreasing monotonicity of $F(t)/f(t)$) in addition to the inequality

$$\int_0^\infty f(t)\, t^{p/2}\, dt \leq 4\, c \int_0^\infty -f'(t)\, t^{p/2}\, dt < \infty\,. \tag{6.1}$$

Here, we adapt these examples in order that Condition (6.1) is not necessarily satisfied, but nevertheless $d$ as defined in (4.1) is finite.

**Example 1.** Consider the generating function

$$f(t) = \frac{1}{(2\,\pi)^{p/2}} \int_0^\infty v^{-p/2} \exp\biggl(-\frac{t}{2v}\biggr)\, g(v)\, dv \tag{6.2}$$

corresponding to a variance mixture of normals, where $g(v)$ is the mixing density of a random variable $V$. In [7], it is proved that, provided that $E[V^{-p/2}] < \infty$, Condition (6.1) is equivalent to

$$\frac{E[V]\, E[V^{-p/2}]}{E[V^{-p/2+1}]} \leq 2\,. \tag{6.3}$$

Therefore relaxing Condition (6.3) on the mixing function $g$ extends the scope of sampling densities which gives rise to minimaxity of generalized Bayes estimators in (2.1).

In [7], it is shown that, when the distribution of $V$ is an inverse gamma $\mathcal{IG}(\alpha, \beta)$, with $\alpha > 0$ and $\beta > 0$ (i.e. $g(v) = \beta^\alpha/\Gamma(\alpha)\, v^{-\alpha-1} \exp(-\beta/v)$) Condition (6.1) reduces to $p/2+1 \leq \alpha$ (with no role for $\beta$), so that the values of $\alpha$ for which $p/2+1 > \alpha$ provide additional new examples of sampling densities. In particular, if the distribution of $X$ is a multivariate $t$ with $m$ degrees of freedom, corresponding to $\alpha = \beta = m/2$, relaxing Condition (6.3) gives rise to $m < p+2$ allowing the inclusion of more Student distributions. More precisely, it is shown in Proposition A.1, that the constant $d$ in (4.1) is finite and equals

$$d^2 = \frac{\beta^2}{(\alpha-1)(\alpha-2)}\, \frac{p/2+\alpha-2}{p/2+\alpha-1}\,,$$

provided that $\alpha > 2$, so that, in the Student case, we have

$$d^2 = \frac{m^2}{(m-2)(m-4)}\, \frac{p+m-4}{p+m-2}\,,$$

for $m \geq 5$. Note that, when $m$ goes to infinity, $d$ goes to one, which corresponds to the normal case.

**Example 2.** Consider the the generating function

$$f(t) = K \, \exp\left(-\alpha \, t^{\beta}\right)$$

where $\alpha > 0$, $0 < \beta \leq 1$ and $K$ is the normalizing constant. In [7], it is shown that, Condition (6.1) is satisfied for $\beta$ in a neighborhood of the form $(1 - \epsilon, 1]$ with $\epsilon > 0$ and is not satisfied for $\beta = 1/2$.

For the constant $d$, we have

$$\begin{aligned} d^2 &\propto \int_0^{\infty} \frac{F^2(r^2)}{f(r^2)} \, r^{p-1} \, dr \\ &\propto \int_0^{\infty} \frac{\left(\int_{r^2}^{\infty} \exp(-\alpha \, t^{\beta}) \, dt\right)^2}{\exp(-\alpha \, r^{2\beta})} \, r^{p-1} \, dr, \end{aligned}$$

and, through the change of variable $v = \alpha t^{\beta}$,

$$d^2 \propto \int_0^{\infty} \frac{(\int_{\alpha \, r^{2\beta}}^{\infty} \, v^{1/\beta - 1} \, \exp(-v) \, dv)^2}{\exp(-\alpha \, r^{2\beta})} \, r^{p-1} \, dr.$$

Also, by the change of variable $v = s + \alpha \, r^{2\beta}$, we obtain

$$d^2 \propto \int_0^{\infty} \exp(-\alpha \, r^{2\,\beta}) \left( \int_0^{\infty} (s + \alpha \, r^{2\,\beta})^{1/\beta - 1} \, e^{-s} \, ds \right)^2 r^{p-1} \, dr \, ,$$

from the above expression. It is clear that these integrals are finite and hence $d$ is finite.

**Example 3.** Consider the generating function

$$f(t) = K \exp\left(-\alpha \, t + \beta \, \varphi(t)\right)$$

where $\alpha > 0$, $\beta > 0$, $\varphi(0) < \infty$, $\varphi(t) \geq 0$, $\varphi'(t) \leq 0$, $\varphi''(t) \geq 0$,

$$\int_0^{\infty} \exp(-\alpha \, t) \, t^{p/2} \, |\varphi'(t)| \, dt < \infty$$

and $K$ is the normalizing constant.

In [7], it is shown that, Condition (6.1) reduces to

$$\int_0^{\infty} f(t) \, t^{p/2} \, dt \leq 4 \, c \, \alpha \int_0^{\infty} f(t) \, t^{p/2} \, dt + 4 \, c \, \beta \int_0^{\infty} -\varphi'(t) \, f(t) \, t^{p/2} \, dt \, . \quad (6.4)$$

It is clear that, for small values of $\alpha$ and $\beta$, Inequality (6.4) cannot be satisfied.

However, we show below that the constant $d$ is finite. We have

$$\begin{aligned} d^2 &\propto \int_0^{\infty} \frac{\left(\int_{r^2}^{\infty} \, \exp(-\alpha \, u + \beta \, \varphi(u)) \, du\right)^2}{\exp(-\alpha r^2 + \beta \varphi(r^2))} \, r^{p-1} \, dr \\ &\leq \int_0^{\infty} \frac{\int_{r^2}^{\infty} \, \exp(-2 \, \alpha \, u + 2\beta \varphi(u)) \, du}{\exp(-\alpha \, r^2 + \beta \, \varphi(r^2))} \, r^{p-1} \, dr. \end{aligned}$$

Since, $\varphi(\cdot)$ is a nonincreasing positive function, $\varphi(u) \leq \varphi(0)$, and hence

$$\begin{aligned} d_1^2 &= \int_0^\infty \frac{\int_{r^2}^\infty \exp(-2\,\alpha\,u + 2\beta\,\varphi(u))\,du}{\exp(-\alpha r^2 + \beta\,\varphi(r^2))}\, r^{p-1}\,dr \\ &\leq \exp(2\beta\varphi(0)) \int_0^\infty \frac{\int_{r^2}^\infty \exp(-2\,\alpha u)\,du}{\exp(-\alpha\,r^2 + \beta\,\varphi(r^2))}\, r^{p-1}\,dr \\ &= \frac{1}{2\alpha}\exp(2\,\beta\,\varphi(0)) \int_0^\infty \exp(-\beta\,\varphi(r^2))\,\exp(-\alpha\,r^2)\,r^{p-1}\,dr\,. \end{aligned}$$

Also, by monotonicity of $\varphi(\cdot)$, we have $\varphi(u) \geq \lim_{u\to+\infty}\varphi(u) = \mu$, and so

$$d_1^2 \leq \frac{1}{2\,\alpha}\exp(2\,\beta\,\varphi(0) - \beta\,\mu) \int_0^\infty \exp(-\alpha\,r^2)\,r^{p-1}\,dr\,.$$

By evaluating the above integral we obtain

$$d_1^2 \leq \frac{\Gamma(p/2)}{4\,\alpha^{p/2+1}}\exp(2\,\beta\,(\varphi(0) - \mu)),$$

which is finite, and hence $d^2$ is finite.

### *6.2. Prior densities*

**Example 1 - Generalized $t$ densities**

We have seen, in Remark 4.1, that the prior $\pi(\|\theta\|^2) = \left(\|\theta\|^2 + b\right)^{-k}$ considered in (5.10) satisfies the conditions of Theorem 4.1 with $g(\|\theta\|^2) = (\|\theta\|^2 + b)^{-k-1}$, which is superharmonic for $p - 2\,(k+2) \geq 0$. We also noticed that Corollary 4.1 can be applied with $g(\|\theta\|^2) = -2\,c^2/d^2\,\Delta\pi(\|\theta\|^2)$, provided the bi-Laplacian $\Delta^{(2)}\pi(\|\theta\|^2)$ is nonnegative. To see this, note that, through straightforward calculations, we have

$$\Delta\left(\frac{1}{(\|\theta\|^2 + b)^k}\right) = \frac{-2\,k}{(\|\theta\|^2 + b)^{k+1}}\left[p - 2\,(k+1) + \frac{2\,(k+1)\,b}{\|\theta\|^2 + b}\right]$$

which is nonpositive for any $\theta$ when $p - 2\,(k+1) \geq 0$. Hence

$$\begin{aligned} \Delta^{(2)}\!\left(\frac{1}{(\|\theta\|^2 + b)^k}\right) &= -2\,k\left[(p - 2\,(k+1))\,\Delta\!\left(\frac{1}{(\|\theta\|^2 + b)^{k+1}}\right)\right. \\ &\qquad\left. + 2\,(k+1)\,b\,\Delta\!\left(\frac{1}{(\|\theta\|^2 + b)^{k+2}}\right)\right] \end{aligned}$$

and is nonnegative for any $\theta$ when $p - 2\,(k+3) \geq 0$. Recall that condition (4.2) of Theorem 4.1 and condition (4.5) of Corollary 4.1 give rise to the same stronger restriction on the values of $k$, that is, $k \leq (p-2)/(d^2/(c^2+2))$.

**Example 2 - General mixtures of normals**

Condition (4.5) will be satisfied modifying Proposition 5.2 in an obvious way.

**Proposition 6.1.** *Assume that the function $t \longmapsto t\,h'(t)/h(t)$ is nonincreasing. Then Condition (4.5) is satisfied as soon as, for any $t \geq 0$,*

$$t\,\frac{h'(t)}{h(t)} \leq -\varphi_1 \tag{6.5}$$

*where*

$$\varphi_1 = \frac{1}{2}\,\frac{(p+2)\,d^2 + 8\,c^2}{d^2 + 2\,c^2}\,. \tag{6.6}$$

Then the constructive approach in Subsection 5.2 can be pursued choosing a nondecreasing function $\varphi$ such that $\varphi(t) \geq \varphi_1$ for any $t \geq 0$ and such that a primitive of $t \mapsto \varphi(t)/t$ exists. Indeed, it follows from the expression of $\Delta\pi(\|\theta\|^2)/\pi(\|\theta\|^2)$ in (5.26) that

$$\Delta\pi(\|\theta\|^2) = \int_0^\infty g(t)\,t^{p/2}\,\exp\left(-\|\theta\|^2\,t\right)\,dt$$

with

$$g(t) = 4\,t\,(2\,h(t) + t\,h'(t))\,.$$

Therefore the bi-Laplacian can be expressed as

$$\Delta^{(2)}\pi(\|\theta\|^2) = \int_0^\infty 4\,t\,(2\,g(t) + t\,g'(t))\,t^{p/2}\,\exp\left(-\|\theta\|^2\,t\right)\,dt\,,$$

so that a sufficient condition for $\Delta^{(2)}\pi(\|\theta\|^2)$ to be nonnegative is that, for all $t \geq 0$, $2\,g(t) + t\,g'(t) \geq 0$. Now, noticing that $g(t) = 4\,t\,h(t)(2-\varphi(t))$, it can be checked that

$$2\,g(t) + t\,g'(t) = 4\,t\,h(t)\left[(\varphi(t)-2)\,(\varphi(t)-3)\; - t\,\varphi'(t)\right].$$

Hence the required sufficient condition reduces to

$$(\varphi(t)-2)\,(\varphi(t)-3)\; - t\,\varphi'(t) \geq 0\,. \tag{6.7}$$

Inequality (6.7) can be verified directly for the functions $\varphi(t) = b\,t - k + p/2 + 1$ and $\varphi(t) = a\,t^2 + b\,t + c$ considered at the end of Subsection 5.2, provided $-k + p/2 - 2 \geq 0$, $a > 0$, $b > 0$, $c \geq 7/2$ and $b^2 - 4\,a\,c < 0$.

## 7. Concluding remarks

In this paper, we have studied minimaxity of generalized Bayes estimators corresponding to a subclass of superharmonic priors $\pi\big(\|\theta\|^2\big)$ when the sampling density is spherically symmetric and is in Berger's class, i.e. $X \sim f(\|x-\theta\|^2)$ and $F(\|x-\theta\|^2)/f(\|x-\theta\|^2) \geq c > 0$ where $F(\|x-\theta\|^2) = 1/2\int_{\|x-\theta\|^2}^\infty f(t)\,dt$.

This article is closely related to Fourdrinier and Strawderman [7] in that they too studied minimaxity in the same context. However, this paper differs from [7] in several ways. First, while [7] considered only the case $F(t)/f(t)$ nondecreasing, we consider both the nondecreasing and the nonincreasing cases. Second, the method of proof in this paper is quite different from that of [7]. Here, we give conditions for minimaxity that depend jointly on the sampling density and the prior density (and its derivatives) and are expressed through differential inequalities for the prior distributions with coefficients that depend on $f(\cdot)$. In [7], the conditions on the density and the prior were given separately and the key differential inequality was in terms of $f(\cdot)$, not $\pi(\cdot)$. A third difference is that the class of sampling densities, even when $F(t)/f(t)$ is nondecreasing, is considerably enlarged in the present paper. Section 6 indicates the degree to which this paper extends the class covered by [7].

A main disadvantage of the results of this paper relative to [7] is that, when the prior density is of the form $\pi\big(\|\theta\|^2\big) = (\|\theta\|^2 + b)^{-k}$, [7] gives minimaxity for $k$ in the range $[0, (p-2)/2]$ and hence includes the fundamental harmonic prior $\|\theta\|^{2-p}$. In this paper, the range of $k$ is always contained in a proper subinterval of $[0, (p-2)/3]$.

## Appendix A: Miscellaneous results

In the following lemma, we prove that the finiteness of the risk of the generalized Bayes estimator $\delta_\pi(X)$ in (2.1) follows from that of $X$.

**Lemma A.1.** *A sufficient condition for the risk of the Bayes estimator $\delta_\pi(X)$ in (2.1) to be finite is that $E_0[\|X\|^2] < \infty$.*

*Proof.* Recall that the risk of $\delta_\pi(X)$ is finite if

$$E_\theta\left[\left\|\frac{\nabla M(X)}{m(X)}\right\|^2\right] = E_\theta\left[\left\|\frac{\int_{\mathbb{R}^p} (\theta - X)\, f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, d\theta}{\int_{\mathbb{R}^p} f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, d\theta}\right\|^2\right] < \infty\,.$$

Thus it suffices to prove that

$$E_\theta\left[\frac{\int_{\mathbb{R}^p} \|\theta - X\|^2\, f(\|x-\theta\|^2) \pi(\|\theta\|^2)\, d\theta}{\int_{\mathbb{R}^p} f(\|x-\theta\|^2)\, \pi(\|\theta\|^2)\, d\theta}\right] < \infty\,. \tag{A.1}$$

Now write

$$\int_{\mathbb{R}^p} \pi(\|\theta\|^2)\, f(\|x-\theta\|^2)\, d\theta = \int_0^\infty \int_{S_{R,x}} \pi(\|\theta\|^2)\, d\mathcal{U}_{R,x}(\theta)\, \sigma(S)\, R^{p-1} f(R^2)\, dR, \tag{A.2}$$

where $\mathcal{U}_{R,x}$ is the uniform distribution on the sphere $S_{R,x}$ of radius $R$ and centered at $x$ and $\sigma(S)$ is the area of the unit sphere. Through the change of variable $R = \sqrt{v}$, the right hand side of (A.2) can be written as

$$\int_0^\infty \mathcal{S}_\pi(\sqrt{v}, x)\, v^{p/2-1}\, f(v)\, dv,$$

where

$$\mathcal{S}_\pi(\sqrt{v}, x) = \frac{\sigma(S)}{2} \int_{S_{\sqrt{v},x}} \pi(\|\theta\|^2)\, d\mathcal{U}_{\sqrt{v},x}(\theta)$$

is nonincreasing in $v$ by the superharmonicity of $\pi(\|\theta\|^2)$.

Now we can express the quantity in brackets in (A.1) as

$$\begin{aligned} \frac{\int_o^\infty \mathcal{S}_\pi(\sqrt{v}, x)\, f(v) v^{p/2}\, dv}{\int_o^\infty \mathcal{S}_\pi(\sqrt{v}, x)\, f(v) v^{p/2-1}\, dv} &= E_1[v] \\ &\leq E_2[v], \end{aligned} \tag{A.3}$$

where $E_1$ is the expectation with respect to the density $f_1(v)$ proportional to

$$\mathcal{S}_\pi(\sqrt{v}, x)\, f(v) v^{p/2-1},$$

and $E_2$ is the expectation with respect to the density $f_2(v)$ proportional to

$$f(v) v^{p/2-1}.$$

Indeed the ratio $f_2(v)/f_1(v)$ is nondecreasing by the monotonicity of $\mathcal{S}_\pi(\sqrt{v}, x)$. In (A.3), $E_2[v]$ is

$$E_2[v] = \frac{\int_o^\infty f(v)\, v^{p/2}\, dv}{\int_o^\infty f(v)\, v^{p/2-1}\, dv},$$

which is finite as soon as

$$\int_o^\infty f(v) v^{p/2} dv < \infty$$

that is,

$$E_0[\|X\|^2] < \infty\,.$$

□

The next lemma was used in the proofs of Theorems 3.1 and 4.1 to show $\nabla M \cdot \nabla m \geq 0$. A proof can be found in Fourdrinier and Righi [5].

**Lemma A.2.** *Let $x \in \mathbb{R}^p$ fixed and let $\Theta$ a random vector in $\mathbb{R}^p$ with unimodal spherically symmetric density $f(\|\theta - x\|^2)$. Denote by $E_x$ the expectation with respect to that density and let $g$ be a function from $\mathbb{R}_+$ into $\mathbb{R}$.*

*Then there exists a function $\Gamma$ from $\mathbb{R}^p$ into $\mathbb{R}$ such that*

$$E_x\left[g\left(\|\Theta\|^2\right)\Theta\right] = \Gamma(x) \cdot x\,, \tag{A.4}$$

*provided this expectation exists. Moreover, if the function $f$ is nonincreasing and if the function $g$ is nonnegative, then the function $\Gamma$ is nonnegative.*

In the following proposition, we provide an expression for the constant $d$ in (4.1) for Example 1 in Section 6.

**Proposition A.1.** *Let*

$$f(t) = \frac{1}{(2\,\pi)^{p/2}} \int_0^\infty v^{-p/2} \,\exp\Big(-\frac{t}{2v}\Big)\, g(v)\, dv$$

*be a generating function as in (6.2). If the mixing density $g$ is the inverse gamma density, $g(v) = \beta^\alpha/\Gamma(\alpha)\, v^{-\alpha-1} \exp(-\beta/v)$ with $\alpha > 0$ and $\beta > 0$, then the constant $d$ in (4.1) is finite and equals*

$$d^2 = \frac{\beta^2}{(\alpha-1)(\alpha-2)}\, \frac{p/2+\alpha-2}{p/2+\alpha-1}\,, \tag{A.5}$$

*provided that $\alpha > 2$.*

*Proof.* First, note that, through Fubini's theorem, for any $t \geq 0$, we have, according to (1.4) and (6.2),

$$F(t) = \frac{1}{(2\,\pi)^{p/2}} \int_0^\infty v^{-p/2+1} \,\exp\Big(-\frac{t}{2v}\Big)\, g(v)\, dv$$

so that, with $g$ being the above inverse gamma density, we have

$$\begin{aligned} \frac{F(t)}{f(t)} &= \frac{\int_0^\infty v^{-p/2-\alpha} \,\exp\Big(-\frac{t/2+\beta}{v}\Big)\, dv}{\int_0^\infty v^{-p/2-\alpha-1} \,\exp\Big(-\frac{t/2+\beta}{v}\Big)\, dv} \\ &= \frac{t/2+\beta}{p/2+\alpha-1}\,, \end{aligned}$$

since

$$\begin{aligned} f(t) &= \frac{1}{(2\,\pi)^{p/2}}\, \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty v^{-p/2-\alpha-1} \,\exp\Big(-\frac{t/2+\beta}{v}\Big)\, dv \\ &= \frac{1}{(2\,\pi)^{p/2}}\, \frac{\beta^\alpha}{\Gamma(\alpha)}\, \frac{\Gamma(p/2+\alpha)}{(t/2+\beta)^{p/2+\alpha}} \end{aligned}$$

and, according to (1.4) and through Fubini theorem, it is easily seen that

$$\begin{aligned} F(t) &= \frac{1}{(2\,\pi)^{p/2}}\, \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty v^{-p/2-\alpha} \,\exp\Big(-\frac{t/2+\beta}{v}\Big)\, dv \\ &= \frac{1}{(2\,\pi)^{p/2}}\, \frac{\beta^\alpha}{\Gamma(\alpha)}\, \frac{\Gamma(p/2+\alpha-1)}{(t/2+\beta)^{p/2+\alpha-1}}\,. \end{aligned}$$

Hence, according to (4.1) and through the change of variable $t = r^2$, we have

$$\begin{aligned} d^2 &= \frac{\pi^{p/2}}{\Gamma(p/2)}\, \frac{1}{(p/2+\alpha-1)^2} \int_0^\infty (t/2+\beta)^2\, t^{p/2-1}\, f(t)\, dt \\ &= \frac{1}{2^{p/2}}\, \frac{1}{\Gamma(p/2)}\, \frac{1}{(p/2+\alpha-1)^2}\, \frac{\beta^\alpha}{\Gamma(\alpha)}\, \Gamma(p/2+\alpha) \\ &\qquad\qquad \int_0^\infty t^{p/2-1}\, (t/2+\beta)^{-p/2-\alpha+2}\, dt\,, \end{aligned}$$

according to the above expression of $f(t)$. Now, to evaluate the last integral, making the change of variable $(t/2)/(t/2+\beta) = u$ (which leads to $t = 2\,\beta\,u/(1-u)$, $t/2+\beta = \beta/(1-u)$ and $dt = 2\,\beta/(1-u)^2\,du$), we have

$$\begin{aligned}\int_0^\infty & t^{p/2-1}\,(t/2+\beta)^{-p/2-\alpha+2}\,dt \\ &= \int_0^1 2^{p/2-1}\,\beta^{p/2-1}\left(\frac{u}{1-u}\right)^{p/2-1}\left(\frac{1-u}{\beta}\right)^{p/2+\alpha-2}\frac{2\,\beta}{(1-u)^2}\,du \\ &= 2^{p/2}\,\beta^{2-\alpha}\int_0^1 u^{p/2-1}\,(1-u)^{\alpha-3}\,du \\ &= 2^{p/2}\,\beta^{2-\alpha}\,\frac{\Gamma(p/2)\,\Gamma(\alpha-2)}{\Gamma(p/2+\alpha-2)}\,,\end{aligned}$$

as soon as $\alpha > 2$. Therefore $d^2$ equals

$$\begin{aligned}d^2 &= \frac{\beta^2}{(p/2+\alpha-1)^2}\,\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}\,\frac{\Gamma(p/2+\alpha)}{\Gamma(p/2+\alpha-2)} \\ &= \frac{\beta^2}{(\alpha-2)\,(\alpha-1)}\,\frac{p/2+\alpha-2}{p/2+\alpha-1}\,.\end{aligned}$$

□

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## References

[1] Berger, J. O. (1975) Minimax estimation of location vectors for a wide class of densities. *Annals of Statistics*, **6** 1318–1328. MR0386080

[2] Blouza A. Fourdrinier D. and P. Lepelletier P. (2010) A weak functional framework for applications in statistics. *IMS Collections, Institute of Mathematical Statistics, A Festschrift for William E. Strawderman*, **8** 90–103.

[3] Fourdrinier, D., Kortbi, O. and Strawderman, W. E. (2008) Bayes minimax estimators of the mean of a scale mixture of multivariate normal distributions. *Journal of Multivariate Analysis*, **99** 1 74–93. MR2408444

[4] Fourdrinier, D. and Lepelletier, P. (2008) Estimating a general function of a quadratic function. *Annals of the Institute of Statistical Mathematics*, **60** 85–119. MR2400062

[5] Fourdrinier, D. and Righi, A. (2011) Improved generalized Bayes estimators of loss. *Unpublished paper, Université de Rouen*.

[6] Fourdrinier, D. and Strawderman, W. E. (2003) On Bayes and unbiased estimators of loss. *Annals of the Institute of Statistical Mathematics*, **55** 803–816. MR2028618

[7] Fourdrinier, D. and Strawderman, W. E. (2008) Generalized Bayes minimax estimators of location vector for spherically symmetric distributions. *Journal of Multivariate Analysis*, **99** 4 735–750. MR2406080

[8] Maruyama, Y. (2003) Admissible minimax estimators of a mean vector of scale mixtures of multivariate normal distributions. *Journal of Multivariate Analysis*, **84** 274–283. MR1965222

[9] Maruyama, Y. and Takemura, A. (2008) Admissibility and minimaxity of generalized Bayes estimators for spherically symmetric family, *Journal of Multivariate Analysis*, **99** 50–73. MR2408443

[10] Stein, C. (1981) Estimation of the mean of multivariate normal distribution. *Annals of Statistics*, **9** 1135–1151. MR0630098

[11] Widder, D. W. (1946) *The Laplace Transform*. Princeton University Press.