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The beta log-logistic distribution

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Abstract. A new continuous distribution, so-called the beta log-logistic distribution, that extends the log-logistic distribution and some other distributions is proposed and studied. The new model is quite flexible to analyze positive data. Various structural properties of the new distribution are derived, including explicit expressions for the moments, mean deviations and Rényi and Shannon entropies. The score function is derived and the estimation of the model parameters is performed by maximum likelihood. We also determine the expected information matrix. The usefulness of the new model is illustrated by means of two real data sets. We hope that the new distribution proposed here will serve as an alternative model to other models available in the literature for modeling positive real data in many areas.

1 Introduction

Numerous classical distributions have been extensively used over the past decades for modeling data in several areas. In fact, the statistics literature is filled with hundreds of continuous univariate distributions (see, e.g, Johnson et al., 1994, 1995). However, in many applied areas, such as lifetime analysis, finance and insurance, there is a clear need for extended forms of these distributions. Consequently, a significant progress has been made toward the generalization of some well-known lifetime distributions and their successful application to problems in areas such as engineering, environmetrics, economics and biomedical sciences, among others. Recent developments focus on new techniques for building meaningful distributions, including the two-piece approach introduced by Hansen (1994) and the generator approach pioneered by Eugene et al. (2002) and Jones (2004). In particular, Eugene et al. (2002) introduced the beta normal distribution, denoted by $\mathrm{BN}(a, b, \mu, \sigma)$, where $\mu \in \mathbb{R}$, $\sigma > 0$ and a and b are positive shape parameters, which control skewness through the relative tail weights. The BN distribution is symmetric if $a = b$, it has negative skewness when $a < b$ and positive skewness when $a > b$. For $a = b > 1$, it has positive excess kurtosis and, for $a = b < 1$, it has negative excess kurtosis (Eugene et al., 2002).

The generator approach introduced by Eugene et al. (2002) is as follows. For any continuous baseline cumulative distribution function (c.d.f.) $G(x) = G(x; \boldsymbol{\tau})$

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and parameter vector $\boldsymbol{\tau}$, the cumulative function of the beta-G distribution, $F(x)$ say, is defined by

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} \omega^{a-1}(1-\omega)^{b-1}\, d\omega, \tag{1.1}$$

where $a > 0$ and $b > 0$ are additional shape parameters to those in $\boldsymbol{\tau}$ that aim to introduce skewness and to provide greater flexibility of its tails. Here, $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the beta function, $\Gamma(\cdot)$ is the gamma function, $I_y(p,q) = B_y(p,q)/B(p,q)$ is the incomplete beta function ratio and $B_y(p,q) = \int_0^y \omega^{p-1}(1-\omega)^{q-1}\, d\omega$ is the incomplete beta function. The cumulative function (1.1) can also be expressed in terms of the hypergeometric function as $F(x) = G(x)^a {}_2F_1(a, 1-b; a+1; G(x)^a)/[aB(a,b)]$. Thus, for any parent $G(x)$, the properties of $F(x)$ could, in principle, be obtained from the well established properties of the hypergeometric function (see, Gradshteyn and Ryzhik, 2007).

The probability density function (p.d.f.) corresponding to (1.1) takes the form

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1}\big[1 - G(x)\big]^{b-1}, \tag{1.2}$$

whereas the hazard rate function associated to (1.1) is given by

$$r(x) = \frac{g(x)G(x)^{a-1}[1-G(x)]^{b-1}}{B(a,b)[1 - I_{G(x)}(a,b)]}.$$

The p.d.f. (1.2) will be most tractable when both functions $G(x)$ and $g(x)$ have simple analytic expressions. Its major benefit is the ability of fitting skewed data that cannot be properly fitted by existing distributions. Let $Q_G(u)$ be the quantile function of the G distribution, for $u \in (0,1)$. Application of $X = Q_G(V)$ to a beta random variable V with positive parameters a and b generates X with density function (1.2).

By using the probability integral transform (1.1), some beta-G distributions have been proposed in the last few years. In particular, the reader is refereed to Nadarajah and Gupta (2004), Nadarajah and Kotz (2004, 2006), Lee et al. (2007), Akinsete et al. (2008), Silva et al. (2010), Barreto–Souza et al. (2010), Pescim et al. (2010), Paranaíba et al. (2011), Cordeiro et al. (2012, 2013, 2014), Cordeiro and Lemonte (2011a, 2011b, 2011c) and Cordeiro and Brito (2012), among others. In the same way, we can extend the log-logistic distribution because it has closed form cumulative function.

In this article, we use the generator approach suggested by Eugene et al. (2002) to define a new model, so-called the beta log-logistic (BLLog) distribution, which generalizes the log-logistic (LLog) model. In addition, we investigate some structural properties of the new model, discuss maximum likelihood estimation of its parameters and derive the expected information matrix. The proposed model is much more flexible than the LLog distribution and can be used effectively for modeling positive real data sets. We shall see later that the new model may be an appealing alternative to the log-normal, Birnbaum–Saunders, gamma, Weibull, exponentiated Weibull, beta Weibull and Kumaraswamy Weibull models in two applications to real data.

The LLog distribution (also known as the Fisk distribution in economics) has c.d.f. in the form

$$G(x) = \frac{x^\beta}{\alpha^\beta + x^\beta}, \qquad x > 0, \tag{1.3}$$

where $\alpha > 0$ is the scale parameter and is also the median of the distribution, $\beta > 0$ is the shape parameter and the distribution is unimodal when $\beta > 1$. It has shapes similar to the ones of the log-normal distribution but has heavier tails. This distribution is used in survival analysis as a parametric model for events whose hazard rate increases initially and decreases later, for example mortality from cancer following diagnosis or treatment. It has also been used in hydrology to model stream flow and precipitation (Shoukri et al., 1988; Ashkar and Mahdi, 2006) and for modeling flood frequency (Ahmad et al., 1988). Additionally, it is used in economics as a simple model of the distribution of wealth or income (Fisk, 1961). Recently, Dey and Kundu (2010) used the ratio of maximized likelihood for discriminating between the log-normal and log-logistic distributions.

The p.d.f. corresponding to (1.3) is given by

$$g(x) = \frac{\beta (x/\alpha)^{\beta-1}}{\alpha[1 + (x/\alpha)^\beta]^2}, \qquad x > 0. \tag{1.4}$$

The sth moment for $s < \beta$ comes from (1.4) as $\mu'_s = \alpha^s B(1 - s/\beta, 1 + s/\beta)$. Let $\eta = \pi/\beta$ be for convenience, the mean can be expressed as $\mathrm{E}(X) = \alpha\eta/\sin(\eta)$, for $\beta > 1$, and the variance is $\mathrm{var}(X) = \alpha^2(2\eta/\sin(2\eta) - \eta^2/\sin^2(\eta))$, for $\beta > 2$. As β tends to infinity the mean tends to α, the variance and skewness tend to zero and the excess kurtosis tends to $6/5$. If X has a LLog distribution with scale parameter α and shape parameter β, $\mathrm{LLog}(\alpha, \beta)$ say, then $Y = \log(X)$ has a logistic distribution with location parameter $\log(\alpha)$ and scale parameter $1/\beta$. If $X \sim \mathrm{LLog}(\alpha, \beta)$, then $kX \sim \mathrm{LLog}(k\alpha, \beta)$, for $k > 0$.

The article is outlined as follows. In Section 2, we introduce the BLLog distribution and provide plots of the density and hazard rate functions. An explicit expression for the moments is provided in Section 3. Section 4 deals with nonstandard measures for the skewness and kurtosis. The mean deviations are calculated in Section 5. The Rényi and Shannon entropies are determined in Section 6. Estimation by the method of maximum likelihood and an explicit expression for the expected information matrix are presented in Section 7. Two empirical applications to real data are considered in Section 8. Finally, Section 9 offers some concluding remarks.

2 The BLLog model

By inserting (1.3) and (1.4) in (1.2), we obtain the BLLog density function with positive parameters a, b, α and β, say BLLog(a, b, α, β), given by

$$f(x) = \frac{(\beta/\alpha)}{B(a,b)} \frac{(x/\alpha)^{a\beta-1}}{[1+(x/\alpha)^{\beta}]^{a+b}}, \qquad x > 0. \tag{2.1}$$

Evidently, the BLLog density function does not involve any complicated function and can be easily computed from equation (2.1). Also, there is no functional relationship between the parameters and they vary freely in the parameter space. The c.d.f. corresponding to (2.1) is $F(x) = I_{x^{\beta}/(\alpha^{\beta}+x^{\beta})}(a, b)$, the survival function is $S(x) = 1 - I_{x^{\beta}/(\alpha^{\beta}+x^{\beta})}(a, b)$ and the associated hazard rate function takes the form

$$r(x) = \frac{(\beta/\alpha)(x/\alpha)^{a\beta-1}[1+(x/\alpha)^{\beta}]^{-(a+b)}}{B(a,b)S(x)}, \qquad x > 0. \tag{2.2}$$

The BLLog distribution can be applied in survival analysis, hydrology, economics, among others, as the LLog distribution and can be used to model reliability problems. The BLLog distribution allows for greater flexibility of its tails and can be widely applied in many areas. If $X \sim$ BLLog(a, b, α, β), then $kX \sim$ LLog$(a, b, k\alpha, \beta)$, for $k > 0$, that is, the class of BLLog distributions is closed under scale transformations, as in the case of the LLog distribution.

The study of the new density (2.1) is important since it also includes as special sub-models some distributions not previously considered in the literature. The LLog distribution arises as the basic exemplar when $a = b = 1$. The new exponentiated LLog (ELLog) distribution corresponds to $b = 1$. Other special sub-model arises for $a = 1$ as the new Lehmann type II LLog (LeLLog) distribution. For a and b positive integers, the BLLog density function becomes the density function of the ath order statistic from the LLog distribution in a sample of size $a + b - 1$. However, equation (2.1) can also alternatively be extended, when a and b are real non-integers, to define fractional LLog order statistics distributions.

Let $Q_{a,b}(u)$ be the beta quantile function with parameters a and b. The quantile function of the BLLog(a, b, α, β) distribution, say $x = Q(u)$, can be easily obtained as

$$x = Q(u) = \alpha\left[\frac{Q_{a,b}(u)}{1 - Q_{a,b}(u)}\right]^{1/\beta}, \qquad u \in (0, 1). \tag{2.3}$$

This scheme is useful to generate BLLog random variates because of the existence of fast generators for beta random variables in most statistical packages, that is, if V is a beta random variable with parameters a and b, then $X = \alpha[V/(1-V)]^{1/\beta}$ follows the BLLog(a, b, α, β) distribution. From (2.3), we conclude that the median m of X is $m = Q(1/2)$. The mode of X, ξ say, is given by $\xi = \alpha[(a\beta - 1)/(b\beta + 1)]^{1/\beta}$, for $\beta > a^{-1}$.

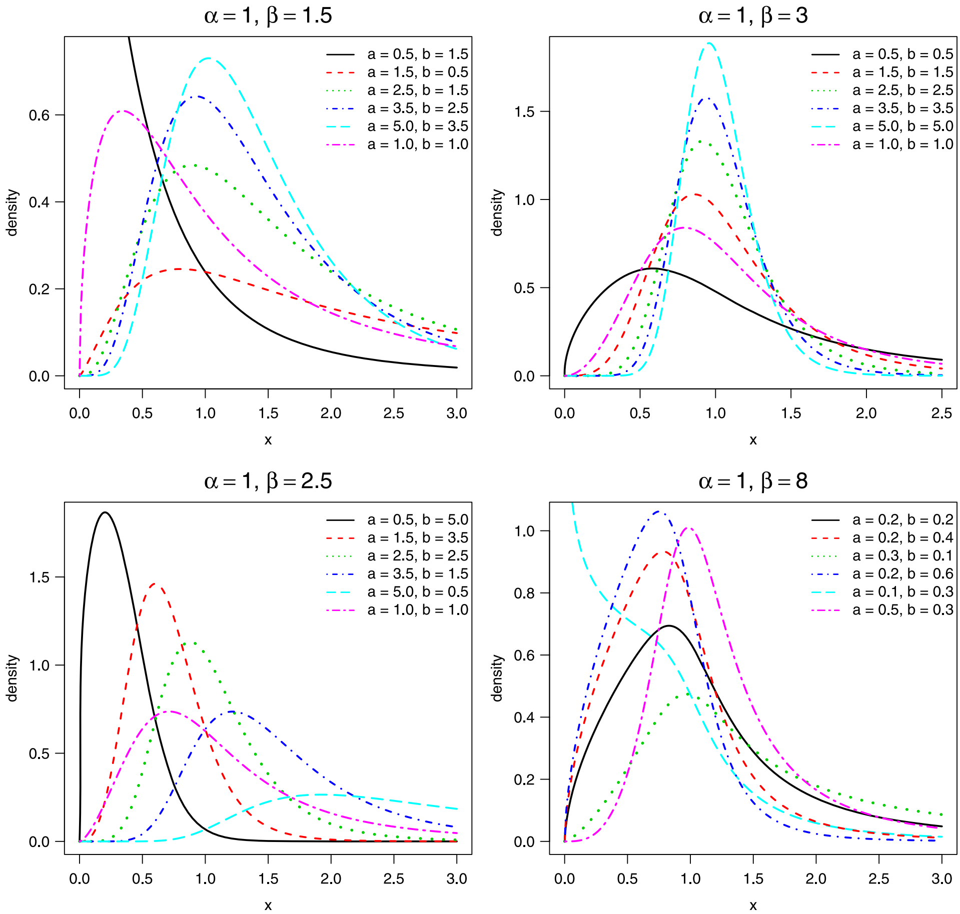

Figure 1 *Plots of the density function* (2.1) *for some parameter values.*

Figures 1 and 2 illustrate some of the possible shapes of the density function (2.1) and hazard rate function (2.2), respectively, for selected parameter values. The density function and hazard rate function can take various forms depending on the parameter values. It is evident that the BLLog distribution is much more flexible than the LLog distribution, that is, the additional shape parameters (a and b) allow for a high degree of flexibility of the BLLog distribution. So, the new model can be very useful in many practical situations for modeling positive real data sets.

3 Moments

We hardly need to emphasize the necessity and importance of moments in any statistical analysis especially in applied work. Some of the most important features

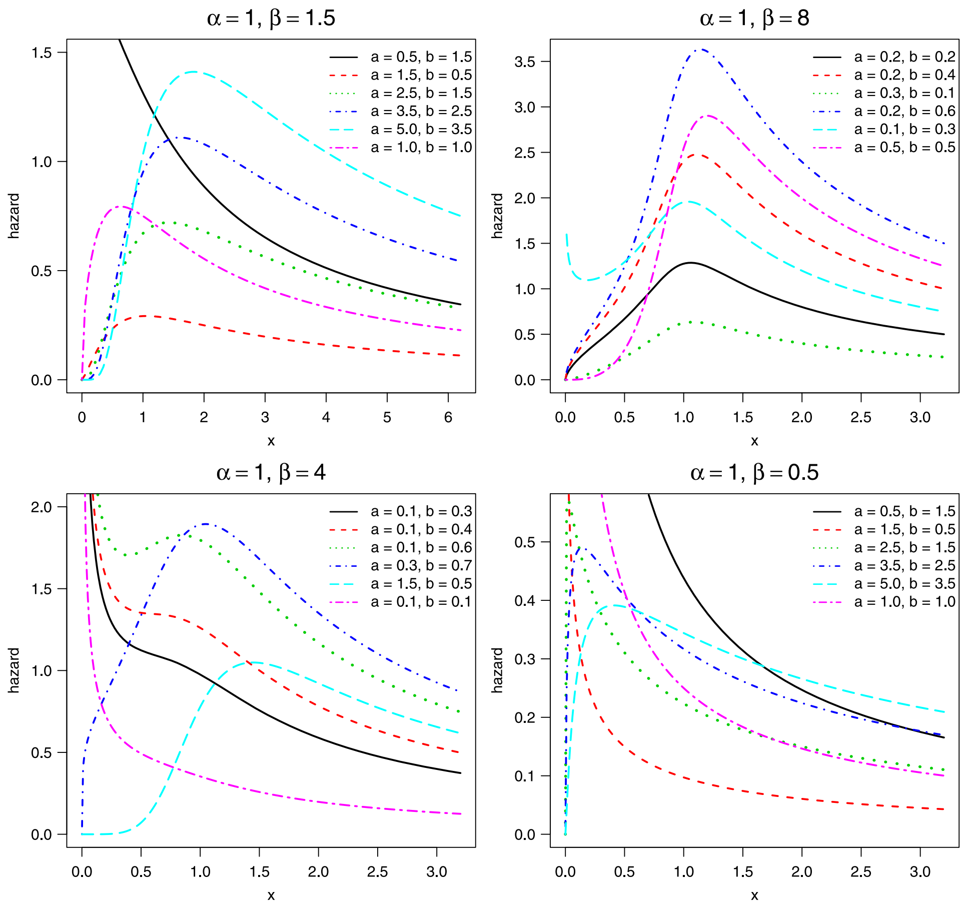

Figure 2 *Plots of the hazard rate function* (2.2) *for some parameter values.*

and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis). Let $X \sim \mathrm{BLLog}(a, b, \alpha, \beta)$. We derive a simple expression for the moments of X, $\mu_s' = \mathrm{E}(X^s)$. The sth moment of X for $s < b\beta$ is given by

$$\mu_s' = \alpha^s \frac{B(b - s/\beta, a + s/\beta)}{B(a, b)}. \tag{3.1}$$

The moments of the ELLog and LeLLog distributions are obtained from (3.1) when $b = 1$ and $a = 1$, respectively. For $a = b = 1$, expression (3.1) reduces to the moments of the LLog distribution. The central moments (μ_s) and cumulants (κ_s) of X can be determined from (3.1) as

$$\mu_s = \sum_{k=0}^{s} \binom{s}{k} (-1)^k \mu_1'^{s} \mu_{s-k}', \qquad \kappa_s = \mu_s' - \sum_{k=1}^{s-1} \binom{s-1}{k-1} \kappa_k \mu_{s-k}',$$

Table 1 *Moments of the* BLLog *distribution for some parameter values*; $\alpha = 1$ *and* $\beta = 15$

	BLLog$(a, b, 1, 15)$ distribution			
μ'_s	$a = 0.5, b = 0.5$	$a = 0.5, b = 1.5$	$a = 1.5, b = 0.5$	$a = 1.5, b = 1.5$
μ'_1	1.02234	0.88603	1.15865	1.00417
μ'_2	1.09464	0.80273	1.38654	1.01680
μ'_3	1.23607	0.74164	1.73050	1.03830
μ'_4	1.49448	0.69742	2.29153	1.06938
μ'_5	2.00000	0.66667	3.33333	1.11111
μ'_6	3.23607	0.64721	5.82492	1.16498
Variance	0.04946	0.01769	0.04406	0.00845
Skewness	1.44159	−0.40205	2.36443	0.38901
Kurtosis	8.07763	0.72897	14.00074	1.16176

respectively, where $\kappa_1 = \mu'_1$. Thus, $\kappa_2 = \mu'_2 - \mu'^2_1$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$, $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu'^2_2 + 12\mu'_2\mu'^2_1 - 6\mu'^4_1$, etc. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ can be calculated from the third and fourth standardized cumulants. Table 1 lists the first six ordinary moments, variance, skewness and kurtosis for selected parameter values of the BLLog(a, b, α, β) distribution by fixing $\alpha = 1$ and $\beta = 15$.

The pth descending factorial moment of X is $\mu'_{(p)} = \mathrm{E}[X^{(p)}] = \mathrm{E}[X(X-1) \times \cdots \times (X-p+1)] = \sum_{m=0}^{p} s(p,m)\mu'_m$, where $s(p,m) = (m!)^{-1}[d^m x^{(p)}/dx^m]_{x=0}$ is the Stirling number of the first kind. They count the number of ways to permute a list of p items into m cycles. The factorial moments of X are given by (for $m < b\beta$)

$$\mu'_{(p)} = \sum_{m=0}^{p} \alpha^m s(p,m) \frac{B(b - m/\beta, a + m/\beta)}{B(a,b)}.$$

Other kinds of moments related to the L-moments (Hosking, 1990) may also be obtained in closed form, but we consider only these moments for reasons of space. An expansion for the moment generating function of X, $M(t)$ say, can be expressed in the form (for $j < b\beta$)

$$M(t) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} B(b - j/\beta, a + j/\beta) \frac{(\alpha t)^j}{j!}.$$

4 Quantile measures

The BLLog quantile function, say $Q(u) = F^{-1}(u)$, can be determined from the beta quantile function as given in (2.3). The effects of the shape parameters a and b on the skewness and kurtosis can be considered based on quantile measures. The shortcomings of the classical kurtosis measure are well-known. The Bowley skewness (Kenney and Keeping, 1962) is one of the earliest skewness measures defined by the average of the quartiles minus the median, divided by half the interquartile range, namely

$$\mathcal{B} = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}.$$

Since only the middle two quartiles are considered and the outer two quartiles are ignored, this adds robustness to the measure. The Moors kurtosis (Moors, 1998) is based on octiles

$$\mathcal{M} = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}.$$

Clearly, $\mathcal{M} > 0$ and there is good concordance with the classical kurtosis measures for some distributions.

These measures are less sensitive to outliers and they exist even for distributions without moments. Because $\mathcal{M}$ is based on the octiles, it is not sensitive to variations of the values in the tails or to variations of the values around the median. The basic justification of $\mathcal{M}$ as an alternative measure of kurtosis is the following: keeping $Q(6/8) - Q(2/8)$ fixed, $\mathcal{M}$ clearly decreases as $Q(3/8) - Q(1/8)$ and $Q(7/8) - Q(5/8)$ decrease. If $Q(3/8) - Q(1/8) \to 0$ and $Q(7/8) - Q(5/8) \to 0$, then $\mathcal{M} \to 0$ and half of the total probability mass is concentrated in the neighborhoods of the octiles $Q(2/8)$ and $Q(6/8)$. In Figures 3 and 4, we plot the measures $\mathcal{B}$ and $\mathcal{M}$ for some parameter values. These plots indicate that both measures $\mathcal{B}$ and $\mathcal{M}$ depend on all shape parameters.

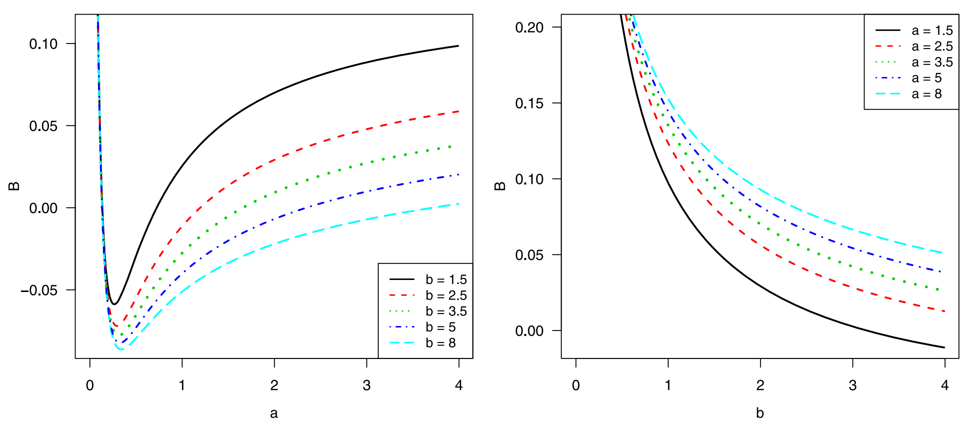

Figure 3 *Plots of the measure $\mathcal{B}$ for some parameter values*; $\alpha = 1.5$ *and* $\beta = 8$.

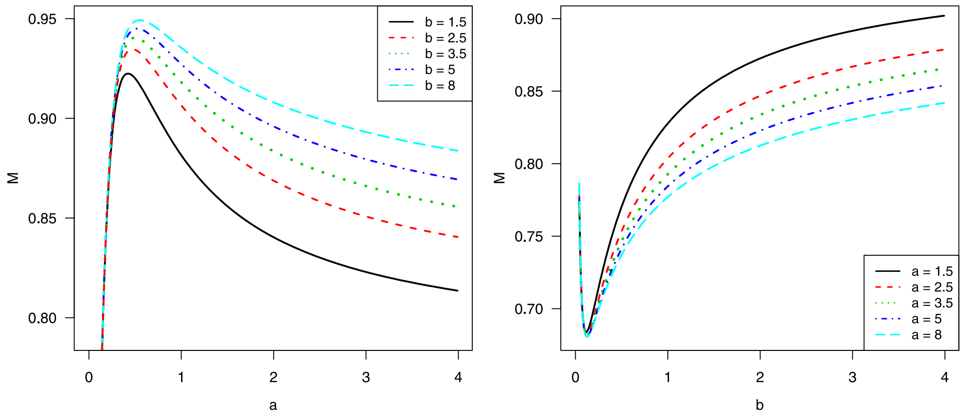

Figure 4 *Plots of the measure* $\mathcal{M}$ *for some parameter values*; $\alpha = 1.5$ *and* $\beta = 8$.

5 Mean deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. If $X \sim \text{BLLog}(a, b, \alpha, \beta)$, we can derive the mean deviations about the mean and about the median from the relations $\delta_1(X) = \text{E}(|X - \mu|)$ and $\delta_2(X) = \text{E}(|X - m|)$, respectively, with

$$\mu = \alpha \frac{B(b - 1/\beta, a + 1/\beta)}{B(a, b)}, \qquad m = \alpha \left[\frac{Q_{a,b}(0.5)}{1 - Q_{a,b}(0.5)} \right]^{1/\beta},$$

where $Q_{a,b}(\cdot)$ is the beta quantile function with parameters a and b. The measures $\delta_1(X)$ and $\delta_2(X)$ can be expressed as $\delta_1(X) = 2\mu F(\mu) - 2J(\mu)$ and $\delta_2(X) = \mu - 2J(m)$, where $J(q) = \int_0^q x f(x)\, dx$. After some algebra, the integral $J(q)$ can be expressed as

$$J(q) = \frac{\alpha\beta}{B(a, b)} \int_0^{q/\alpha} \frac{z^{\beta a}}{(1 + z^{\beta})^{a+b}}\, dz,$$

which can be easily computed numerically in software such as MAPLE (Garvan, 2002), MATLAB (Sigmon and Davis, 2002), MATHEMATICA (Wolfram, 2003), Ox (Doornik, 2006) and R (R Development Core Team, 2012). The Ox (for academic purposes) and R are freely distributed and available at http://www.doornik.com and http://www.r-project.org, respectively. Some numerical values of $J(q)$ are listed in Table 2.

From the mean deviations, we can construct Lorenz and Bonferroni curves, which are important in several fields such as economics, reliability, demography, insurance and medicine. They are defined (for a given probability π) by $L(\pi) = J(q)/\mu$ and $B(\pi) = J(q)/(\pi\mu)$, respectively, where $q = Q(\pi)$ can be determined from (2.3). It is easy to verify that $L(\pi) \geq \pi$, $L(0) = 0$ and $L(1) = 1$. In economics, if $\pi = F(q)$ is the proportion of units whose income is lower than

Table 2 *Numerical values of $J(q)$; $\alpha = 1$, $\beta = 2$ and different values of q*

	BLLog($a, b, 1, 2$) distribution			
q	$a = 0.5, b = 0.5$	$a = 1.5, b = 0.5$	$a = 0.5, b = 1.5$	$a = 1.5, b = 1.5$
1	0.22064	0.12296	0.31831	0.31831
2	0.51230	0.51530	0.50930	0.81487
3	0.73294	0.89291	0.57296	1.03132
4	0.90184	1.20451	0.59917	1.12785
5	1.03708	1.46203	0.61213	1.17718
6	1.14939	1.67937	0.61941	1.20535

or equal to q, $L(\pi)$ gives the proportion of total income volume accumulated by the set of units with an income lower than or equal to q.

6 Entropy measures

Entropy has been used in various situations in science as a measure of variation of the uncertainty. Numerous measures of entropy have been studied and compared in the literature. Here, we derive explicit expressions for two most important entropies of $X \sim \text{BLLog}(a, b, \alpha, \beta)$: Shannon entropy and Rényi entropy.

First, we consider the Shannon entropy which plays a similar role as the kurtosis measure in comparing the shapes of various densities and measuring heaviness of tails. It is defined by $\mathcal{I}_S = \mathrm{E}[-\log(f(X))]$, which implies

$$\mathcal{I}_S = -\log(\beta/\alpha) + \log\big[B(a,b)\big] - (\beta a - 1)\mathrm{E}\left[\log\left(\frac{X}{\alpha}\right)\right] + (a+b)\mathrm{E}\left[\log\left(1 + \left(\frac{X}{\alpha}\right)^{\beta}\right)\right].$$

We can show that

$$\mathrm{E}\left[\log\left(\frac{X}{\alpha}\right)\right] = \frac{\psi(a) - \psi(b)}{\beta}, \qquad \mathrm{E}\left[\log\left(1 + \left(\frac{X}{\alpha}\right)^{\beta}\right)\right] = \psi(a+b) - \psi(b),$$

where $\psi(\cdot)$ is the digamma function. Hence, $\mathcal{I}_S$ reduces to

$$\mathcal{I}_S = \log(\alpha/\beta) + \log\big[B(a,b)\big] - \frac{(\beta a - 1)}{\beta}\big[\psi(a) - \psi(b)\big] + (a+b)\big[\psi(a+b) - \psi(b)\big]. \tag{6.1}$$

The Rényi entropy of X can be expressed as $\mathcal{I}_R = (1-\delta)^{-1}\log(\int_0^\infty f(x)^\delta\, dx)$, where $\delta > 0$ and $\delta \neq 1$. We have that

$$\int_0^\infty f(x)^\delta\, dx = \frac{(\beta/\alpha)^\delta}{B(a,b)^\delta}\int_0^\infty \frac{(x/\alpha)^{\delta(\beta a - 1)}}{[1 + (x/\alpha)^\beta]^{\delta(a+b)}}\, dx.$$

After some algebra, we can show that

$$\int_0^\infty f(x)^\delta\,dx = \frac{(\beta/\alpha)^{\delta-1}}{B(a,b)^\delta} B\big(\delta b + (\delta-1)/\beta, \delta a - (\delta-1)/\beta\big),$$

and hence the Rényi entropy takes the form

$$\mathcal{I}_R = (1-\delta)^{-1}\log\biggl(\frac{(\beta/\alpha)^{\delta-1}}{B(a,b)^\delta} B\big(\delta b + (\delta-1)/\beta, \delta a - (\delta-1)/\beta\big)\biggr). \tag{6.2}$$

As can be seen from equations (6.1) and (6.2), the Shannon and Rényi entropies for the BLLog model have very simple expressions and can be easily computed. The Shannon entropy for the ELLog and LeLLog models are obtained from (6.1) with $b = 1$ and $a = 1$, respectively. Likewise, for the Rényi entropy.

7 Maximum likelihood estimation

In what follows, we shall consider estimation of the model parameters of the BLLog distribution by the method of maximum likelihood. However, some of the other estimators like the percentile estimators, estimators based on order statistics, weighted least squares and estimators based on L-moments can also be explored. We assume that X follows the BLLog distribution and let $\boldsymbol{\theta} = (a, b, \alpha, \beta)^\top$ be the parameter vector of interest. The log-likelihood function $\ell = \ell(\boldsymbol{\theta})$ for a single observation x of X is given by

$$\ell = \log(\beta/\alpha) - \log\big[B(a,b)\big] + (\beta a - 1)\log\biggl(\frac{x}{\alpha}\biggr) - (a+b)\log\biggl[1 + \biggl(\frac{x}{\alpha}\biggr)^\beta\biggr].$$

The components of the unit score vector $\mathbf{U} = \mathbf{U}(\boldsymbol{\theta}) = (\partial\ell/\partial a, \partial\ell/\partial b, \partial\ell/\partial\alpha, \partial\ell/\partial\beta)^\top$ are given by

$$\frac{\partial\ell}{\partial a} = \psi(a+b) - \psi(a) + \beta\log\biggl(\frac{x}{\alpha}\biggr) - \log\biggl[1 + \biggl(\frac{x}{\alpha}\biggr)^\beta\biggr],$$

$$\frac{\partial\ell}{\partial b} = \psi(a+b) - \psi(b) - \log\biggl[1 + \biggl(\frac{x}{\alpha}\biggr)^\beta\biggr],$$

$$\frac{\partial\ell}{\partial\alpha} = -\frac{\beta a}{\alpha} + \frac{\beta(a+b)}{\alpha}\frac{(x/\alpha)^\beta}{[1 + (x/\alpha)^\beta]},$$

$$\frac{\partial\ell}{\partial\beta} = \frac{1}{\beta} + a\log\biggl(\frac{x}{\alpha}\biggr) - (a+b)\frac{(x/\alpha)^\beta\log(x/\alpha)}{[1 + (x/\alpha)^\beta]}.$$

The expected value of the score function vanishes and hence, for example, we have

$$\mathrm{E}\biggl[\frac{(X/\alpha)^\beta}{[1 + (X/\alpha)^\beta]}\biggr] = \frac{a}{a+b}.$$

For interval estimation and hypotheses tests on the model parameters, we require the 4×4 unit expected information matrix, $\mathbf{K}(\boldsymbol{\theta})$ say, given by

$$\mathbf{K}(\boldsymbol{\theta}) = \begin{bmatrix} \kappa_{aa} & \kappa_{ab} & \kappa_{a\alpha} & \kappa_{a\beta} \\ \kappa_{ab} & \kappa_{bb} & \kappa_{b\alpha} & \kappa_{b\beta} \\ \kappa_{a\alpha} & \kappa_{b\alpha} & \kappa_{\alpha\alpha} & \kappa_{\alpha\alpha} \\ \kappa_{a\beta} & \kappa_{b\beta} & \kappa_{\alpha\beta} & \kappa_{\beta\beta} \end{bmatrix},$$

whose elements are

$$\kappa_{aa} = \psi'(a) - \psi'(a+b), \qquad \kappa_{ab} = -\psi'(a+b), \qquad \kappa_{a\alpha} = \frac{b\beta}{\alpha(a+b)},$$

$$\kappa_{a\beta} = \frac{1 - b[\psi(a) - \psi(b)]}{\beta(a+b)}, \qquad \kappa_{bb} = \psi'(b) - \psi'(a+b),$$

$$\kappa_{b\alpha} = -\frac{a\beta}{\alpha(a+b)}, \qquad \kappa_{b\beta} = \frac{1 + a[\psi(a) - \psi(b)]}{\beta(a+b)},$$

$$\kappa_{\alpha\alpha} = \frac{\beta^2(a+b)B(a+1, b+1)}{\alpha^2 B(a,b)},$$

$$\kappa_{\alpha\beta} = -\frac{(a+b)B(a+1, b+1)}{ab\alpha B(a,b)}\{b - a + ab[\psi(a) - \psi(b)]\},$$

$$\kappa_{\beta\beta} = \frac{1}{\beta^2} + \frac{(a+b)B(a+1, b+1)}{ab\beta^2 B(a,b)}\{-2 + 2(a-b)[\psi(b) - \psi(a)] + ab[(\psi(a) - \psi(b))^2 + \psi'(a) + \psi'(b)]\},$$

where $\psi'(\cdot)$ is the trigamma function.

For a random sample $\mathbf{x} = (x_1, \dots, x_n)^\top$ of size n from the BLLog model, the total log-likelihood function for the parameter vector $\boldsymbol{\theta} = (a, b, \alpha, \beta)^\top$ is $\ell_n(\boldsymbol{\theta}) = \sum_{i=1}^n \ell^{(i)}$, where $\ell^{(i)}$ is the log-likelihood for the ith observation ($i = 1, \dots, n$) as given before. The total score function is $\mathbf{U}_n(\boldsymbol{\theta}) = \sum_{i=1}^n \mathbf{U}^{(i)}$, where $\mathbf{U}^{(i)}$ ($i = 1, \dots, n$) has the form given earlier and the total expected information matrix is $\mathbf{K}_n(\boldsymbol{\theta}) = n\mathbf{K}(\boldsymbol{\theta})$. The maximum likelihood estimate (MLE) $\widehat{\boldsymbol{\theta}} = (\widehat{a}, \widehat{b}, \widehat{\alpha}, \widehat{\beta})^\top$ of $\boldsymbol{\theta} = (a, b, \alpha, \beta)^\top$ is obtained by setting $\mathbf{U}_n(\boldsymbol{\theta}) = \mathbf{0}$ and solving these equations numerically using iterative methods, such as Newton–Raphson and Fisher scoring algorithms. Under conditions that are fulfilled for parameters in the interior of the parameter space, the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $\mathcal{N}_4(\mathbf{0}, \mathbf{K}(\boldsymbol{\theta})^{-1})$. The asymptotic multivariate normal $\mathcal{N}_4(\mathbf{0}, \mathbf{K}_n(\widehat{\boldsymbol{\theta}})^{-1})$ distribution can be used to construct approximate confidence intervals and confidence regions for the parameters. The asymptotic $100(1-\eta)\%$ confidence intervals for a, b, α and β are given, respectively, by $\widehat{a} \pm z_{\eta/2} \times [\widehat{\text{var}}(\widehat{a})]^{1/2}$, $\widehat{b} \pm z_{\eta/2} \times [\widehat{\text{var}}(\widehat{b})]^{1/2}$, $\widehat{\alpha} \pm z_{\eta/2} \times [\widehat{\text{var}}(\widehat{\alpha})]^{1/2}$ and $\widehat{\beta} \pm z_{\eta/2} \times [\widehat{\text{var}}(\widehat{\beta})]^{1/2}$, where $\text{var}(\cdot)$ is the diagonal element of $\mathbf{K}_n(\widehat{\boldsymbol{\theta}})^{-1}$ corresponding to each parameter, and $z_{\eta/2}$ is the quantile $(1 - \eta/2)$ of the standard normal distribution.

8 Empirical illustrations

In this section, we present two applications of the proposed BLLog distribution (and their sub-models: ELLog, LeLLog and LLog distributions) in two real data sets to illustrate its potentiality. The first real data set corresponds to an uncensored data set from Nichols and Padgett (2006) on breaking stress of carbon fibres (in Gba): 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65. The second real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003): 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

In the following, we shall compare the proposed BLLog distribution (and their sub-models) with several other four-, three- and two-parameter lifetime distributions, namely:

- Beta Weibull (BW) distribution (Lee et al., 2007). The BW density function is ($x > 0$)

$$f(x) = \frac{\alpha\beta}{B(a,b)} x^{\beta-1} \exp(-b\alpha x^{\beta})\{1 - \exp(-\alpha x^{\beta})\}^{a-1},$$

  where $a > 0$, $b > 0$ and $\beta > 0$ are shape parameters and $\alpha > 0$ is the scale parameter.
- Kumaraswamy Weibull (KW) distribution (Cordeiro et al., 2010). The KW density function is ($x > 0$)

$$f(x) = abc\lambda x^{c-1} \exp(-\lambda x^{c})\{1 - \exp(-\lambda x^{c})\}^{a-1}[1 - \{1 - \exp(-\lambda x^{c})\}^{a}]^{b-1},$$

  where $a > 0$, $b > 0$ and $c > 0$ are shape parameters and $\lambda > 0$ is the scale parameter.

- Exponentiated Weibull (EW) distribution (Mudholkar and Srivastava, 1993). The EW density function is $(x > 0)$

$$f(x) = a\alpha\beta x^{\beta-1}\exp(-\alpha x^{\beta})\{1-\exp(-\alpha x^{\beta})\}^{a-1},$$

  where $a > 0$ and $\beta > 0$ are shape parameters and $\alpha > 0$ is the scale parameter.
- Marshall–Olkin Weibull (MOW) distribution (Ghitany et al., 2005). The MOW survival function is $(x > 0)$

$$\bar{F}(x) = \frac{1-\mathrm{e}^{-\lambda x^{\gamma}}}{(1-\alpha)\mathrm{e}^{-\lambda x^{\gamma}}},$$

  where $\alpha > 0$ and $\gamma > 0$ are shape parameters and λ is the scale parameter.
- Beta half-Cauchy (BHC) distribution (Cordeiro and Lemonte, 2011c). The BHC density function is $(x > 0)$

$$f(x) = \frac{2^{a}[1+(x/\phi)^{2}]^{-1}}{\phi\pi^{a}B(a,b)}\left[\arctan\left(\frac{x}{\phi}\right)\right]^{a-1}\left\{1-\frac{2}{\pi}\left[\arctan\left(\frac{x}{\phi}\right)\right]\right\}^{b-1},$$

  where $a > 0$ and $b > 0$ are shape parameters and $\phi > 0$ is a scale parameter.
- Weibull distribution. The Weibull cumulative function is $(x > 0)$

$$F(x) = 1-\exp(-\alpha x^{\beta}), \qquad x > 0,$$

  where $\beta > 0$ is the shape parameter and $\alpha > 0$ is the scale parameter.
- Gamma distribution. The gamma cumulative function is $(x > 0)$

$$F(x) = \frac{\gamma(\eta,\lambda x)}{\Gamma(\eta)},$$

  where $\eta > 0$ is the shape parameter, $\lambda > 0$ is the scale parameter and $\gamma(\cdot,\cdot)$ denotes the lower incomplete gamma function.
- Log-normal (LN) distribution. The LN cumulative function is $(x > 0)$

$$F(x) = \Phi\left(\frac{\ln(x)-\mu}{\sigma}\right),$$

  where $\Phi(\cdot)$ is the cumulative function of the standard normal distribution, $\sigma > 0$ is the shape parameter and $\mu \in \mathbb{R}$ is the location parameter.
- Birnbaum–Saunders (BS) distribution (Birnbaum and Saunders, 1969). The BS cumulative function is $(x > 0)$

$$F(x) = \Phi\left(\frac{1}{\alpha}\left[\sqrt{\frac{x}{\beta}}-\sqrt{\frac{\beta}{x}}\right]\right),$$

  where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

We estimate the unknown parameters of each model by the maximum likelihood method. All the computations were done using the `Ox` matrix programming language (Doornik, 2006), which is freely distributed for academic purposes and is available at http://www.doornik.com. In order to compare the models above with the proposed BLLog model (and their sub-models), we shall apply formal goodness-of-fit tests to verify which distribution fits better the real data sets. We consider the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics. The statistics W^* and A^* are described in details in Chen and Balakrishnan (1995). In general, the smaller the values of these statistics, the better the fit to the data. Let $H(x;\boldsymbol{\theta})$ be the c.d.f., where the form of H is known but $\boldsymbol{\theta}$ (a k-dimensional parameter vector, say) is unknown. To obtain the statistics W^* and A^*, we can proceed as follows: (i) Compute $v_i = H(x_i;\widehat{\boldsymbol{\theta}})$, where the x_i's are in ascending order, and then $y_i = \Phi^{-1}(v_i)$, where $\Phi(\cdot)$ is the standard normal c.d.f. and $\Phi^{-1}(\cdot)$ its inverse; (ii) Compute $u_i = \Phi\{(y_i - \bar{y})/s_y\}$, where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$; (iii) Calculate $W^2 = \sum_{i=1}^{n}\{u_i - (2i-1)/(2n)\}^2 + 1/(12n)$ and $A^2 = -n - (1/n)\sum_{i=1}^{n}\{(2i-1)\ln(u_i) + (2n+1-2i)\ln(1-u_i)\}$ and then $W^* = W^2(1+0.5/n)$ and $A^* = A^2(1+0.75/n+2.25/n^2)$.

Table 3 lists the MLEs (and the corresponding standard errors in parentheses) of the parameters of all the models for the first data set (breaking stress of carbon fibres). The statistics W^* and A^* are also listed in this table for all the models. As can be seen from the figures of this table, the new BLLog model proposed in this paper presents the smallest values of the statistics W^* and A^* among all the models, that is, the new model fits the breaking stress of carbon fibres data better than all the other models considered. More information is provided by a visual comparison in Figure 5(a) of the histogram of the data with the fitted BLLog density function. Clearly, the BLLog distribution provides a closer fit to the histogram. The Kaplan–Meier (K–M) estimate and the estimated survival function of the fitted BLLog distribution is shown in Figure 5(b). From this plot, note that the BLLog model fits the data adequately and hence can be adequate for these data.

Table 4 lists the MLEs (and the corresponding standard errors in parentheses) of the parameters of all the models and the statistics W^* and A^* for the second data set (remission times). From Table 4, notice that the proposed ELLog model presents the smallest values of the statistics W^* and A^*, and hence should be chosen as the best model among all the distributions to fit the remission times data. The histogram of the data with the fitted ELLog density function is presented in Figure 6(a). The K–M estimate and the estimated survival function of the fitted ELLog distribution is displayed in Figure 6(b), which clearly shows that the ELLog model fits the data adequately and hence can be adequate for modeling these data. In summary, the new BLLog model (and their sub-models) may be an interesting alternative to the other models available in the literature for modeling positive real data.

Table 3 *MLEs (standard errors in parentheses) and the statistics W^* and A^*; first data set*

Distribution	Estimates				W^*	A^*
BLLog(a, b, α, β)	0.0900	0.2254	3.1486	25.417	**0.03867**	**0.27763**
	(0.1700)	(0.4452)	(0.1851)	(46.670)		
BW(α, β, a, b)	0.1013	2.4231	1.3080	0.8907	0.07039	0.41325
	(0.3160)	(0.7389)	(0.6133)	(3.5611)		
KW(a, b, c, λ)	1.9447	12.030	1.6217	0.0561	0.06938	0.40705
	(5.7460)	(146.64)	(4.6401)	(0.1776)		
ELLog(a, α, β)	0.3339	3.3815	7.4714		0.04627	0.30190
	(0.0998)	(0.2270)	(1.4975)			
LeLLog(b, α, β)	7.8795	5.6426	3.0234		0.06717	0.38989
	(11.370)	(3.3334)	(0.3873)			
EW(α, β, a)	0.0928	2.4091	1.3168		0.07036	0.41313
	(0.0904)	(0.5930)	(0.5969)			
MOW$(\alpha, \gamma, \lambda)$	0.6926	3.0094	0.0309		0.07052	0.43016
	(0.8310)	(0.7181)	(0.0472)			
BHC(ϕ, a, b)	15.194	5.5944	46.116		0.13860	0.70838
	(20.687)	(0.8087)	(70.775)			
LLog(α, β)	2.4984	4.1179			0.23903	1.24090
	(0.1051)	(0.3444)				
Weibull(α, β)	0.0490	2.7929			0.06227	0.41581
	(0.0138)	(0.2131)				
Gamma(λ, η)	5.9526	2.2708			0.14802	0.75721
	(0.8193)	(0.3261)				
LN(μ, σ)	0.8774	0.4439			0.27734	1.48332
	(0.0444)	(0.0314)				
BS(α, β)	0.4622	2.3660			0.29785	1.61816
	(0.0327)	(0.1064)				

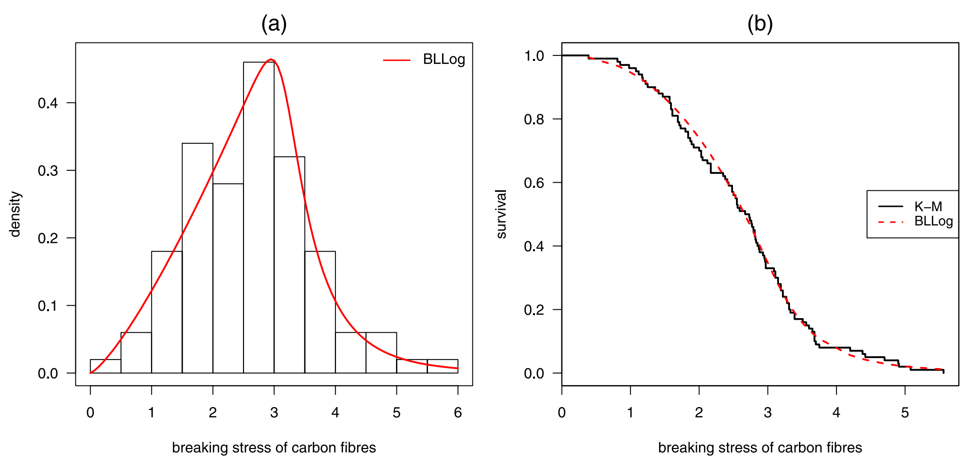

Figure 5 (a) *Estimated p.d.f. of the BLLog model*; (b) *Empirical survival and estimated survival function of the BLLog model; first data set.*

Table 4 *MLEs (standard errors in parentheses) and the statistics W^* and A^*; second data set*

Distribution	Estimates				W^*	A^*
BLLog(a, b, α, β)	0.7201	1.3428	10.211	1.8291	0.01707	0.11110
	(0.5668)	(1.3280)	(4.1701)	(1.0976)		
BW(α, β, a, b)	0.4697	0.6661	2.7348	0.9083	0.04362	0.28825
	(0.3682)	(0.2464)	(1.6122)	(1.5143)		
KW(a, b, c, λ)	4.1178	2.9414	0.4589	0.4949	0.04149	0.27322
	(6.0714)	(8.2687)	(0.5316)	(0.5248)		
ELLog(a, α, β)	0.5858	9.2282	2.1727		**0.01629**	**0.10549**
	(0.1625)	(1.9223)	(0.3280)			
LeLLog(b, α, β)	2.0701	12.033	1.4276		0.01946	0.13077
	(0.9583)	(5.7180)	(0.1744)			
EW(α, β, a)	0.4537	0.6544	2.7960		0.04367	0.28848
	(0.2395)	(0.1354)	(1.2645)			
MOW$(\alpha, \gamma, \lambda)$	0.0639	1.6042	0.0033		0.02842	0.20409
	(0.0627)	(0.1641)	(0.0039)			
BHC(ϕ, a, b)	10.981	1.3725	2.4456		0.01766	0.11679
	(3.9737)	(0.2030)	(0.6640)			
LLog(α, β)	6.0898	1.7252			0.04301	0.31113
	(0.5404)	(0.1275)				
Weibull(α, β)	0.0939	1.0478			0.13137	0.78648
	(0.0191)	(0.0675)				
Gamma(λ, η)	1.1725	0.1252			0.11988	0.71928
	(0.1308)	(0.0173)				
LN(μ, σ)	1.7535	1.0731			0.12230	0.82632
	(0.1013)	(0.1030)				
BS(α, β)	1.3742	4.5711			0.41359	2.56150
	(0.0861)	(0.4460)				

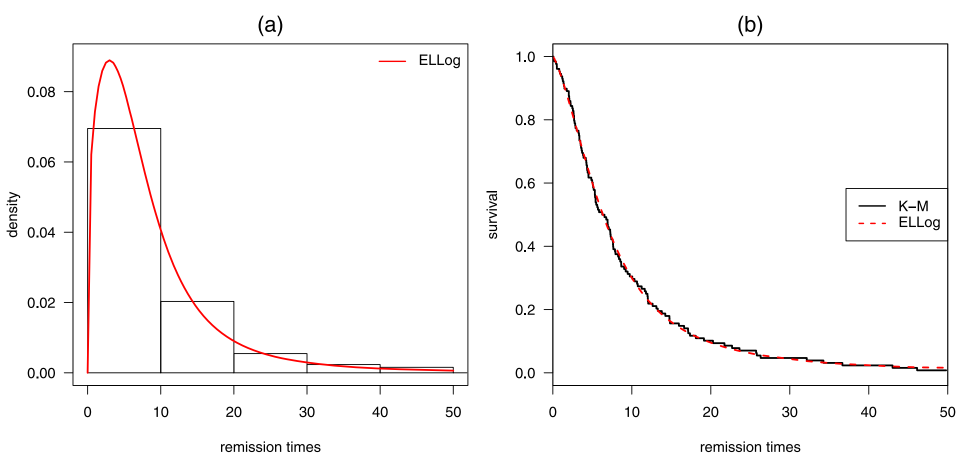

Figure 6 (a) *Estimated p.d.f. of the ELLog model*; (b) *Empirical survival and estimated survival function of the ELLog model*; *second data set*.

9 Concluding remarks

In this paper, we propose a new distribution which generalizes the log-logistic distribution. Further, the new distribution includes as special sub-models other distributions. We refer to the new model as the beta log-logistic (BLLog) distribution and study some of its mathematical and statistical properties. We provide for the new distribution explicit expressions for the moments, mean deviations, Rényi entropy and Shannon entropy. The model parameters are estimated by maximum likelihood and the expected information matrix is derived. The usefulness of the new model is illustrated in two applications to real data using goodness-of-fit tests. The new model provides consistently better fit than other models available in the literature. The formulae related with the new model are manageable and may turn into adequate tools comprising the arsenal of applied statisticians. We hope that the proposed model may attract wider applications for modeling positive real data sets in many areas such as engineering, survival analysis, hydrology, economics, and so on.

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