Brazilian Journal of Probability and Statistics
2013, Vol. 27, No. 3, 322–356
DOI: 10.1214/12-BJPS184

Copulas related to Manneville–Pomeau processes

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Abstract. In this work, we derive the copulas related to Manneville–Pomeau processes. We examine both bidimensional and multidimensional cases and derive some properties for the related copulas. Computational issues, approximations and random variate generation problems are addressed and simple numerical experiments to test the approximations developed are also performed. In particular, we propose an approximation to the derived copulas which we show to converge uniformly to the true one. To illustrate the usefulness of the theory, we derive a fast procedure to estimate the underlying parameter in Manneville–Pomeau processes.

1 Introduction

The statistics of stochastic processes derived from dynamical systems has seen a grown attention in the last decade or so (see Chazottes et al. (2005) and references therein). The relationship between copulas and areas such as ergodic theory and dynamical systems also have seen some development, especially in the last few years (see, for instance, Kolesárová et al. (2008)). In this work, our aim is to contribute with the area by identifying and studying the copulas related to random vectors coming from the so-called Manneville–Pomeau processes, which are obtained as iterations of the Manneville–Pomeau transformation to a specific chosen random variable (see Definitions 2.1 and 2.2). We cover both, bidimensional and n-dimensional cases, which share a lot more in common than one could expect.

The copulas derived here depend on a probability measure which has no closed formula. In order to minimize this deficiency, we propose an approximation to the copula which we show to converge uniformly to the true copula. The copula also depend on several functions which have to be approximated as well, so the approximation depends on several intermediate steps. The results related to the convergence of the proposed approximation presented here are far more general than we need and actually allows one to change these intermediate approximations and still obtain the uniform convergence result for the approximated copula. We also address problems related to random variate generation of the copula and present the results of some simple numerical experiments in order to assess the stability and precision of the intermediate approximations. The usefulness of the theory

Key words and phrases. Copulas, Manneville–Pomeau processes, invariant measures.
Received December 2010; accepted January 2012.

is illustrated by a simple application to the problem of estimating the underlying parameter in Manneville–Pomeau processes.

The paper is organized as follows: in the next section, we briefly review some concepts and results on Manneville–Pomeau transformations and processes and on copulas. Section 3 is devoted to determine the copulas related to any pair (X_t, X_{t+h}) from a Manneville–Pomeau process and to explore some consequences. In Section 4, the multidimensional extensions are shown. In Section 5, an approximation to the copulas derived in Section 3 is proposed. This approximation, which is shown to converge uniformly to the true copula, is then applied to exploit some characteristics of the copulas related to Manneville–Pomeau process through statistical and graphical analysis. Some computational and random variate generation problems are also addressed. In Section 6, we illustrate the usefulness of the theory by deriving a fast procedure to estimate the underlying parameter in Manneville–Pomeau processes. Conclusions are reserved to Section 7.

2 Some background

In this section, we shall briefly review some basic results on Manneville–Pomeau transformations and related processes as well as some concepts on copulas needed later. We start with the definition of the Manneville–Pomeau transformation.

Definition 2.1. The map $T_s : [0, 1] \longrightarrow [0, 1]$, given by

$$T_s(x) = x + x^{1+s} (\text{mod } 1)$$

for $s > 0$, is called the *Manneville–Pomeau transformation* (MP *transformation*, for short).

In what follows, λ shall denote the Lebesgue measure in $I := [0, 1]$ and the k-fold composition will be denoted, as usual, by $T_s^k = T_s \circ \cdots \circ T_s$. Figure 1 shows the plot of the MP transformation for the values of $s \in \{0.5, 1, 10\}$. The plots show

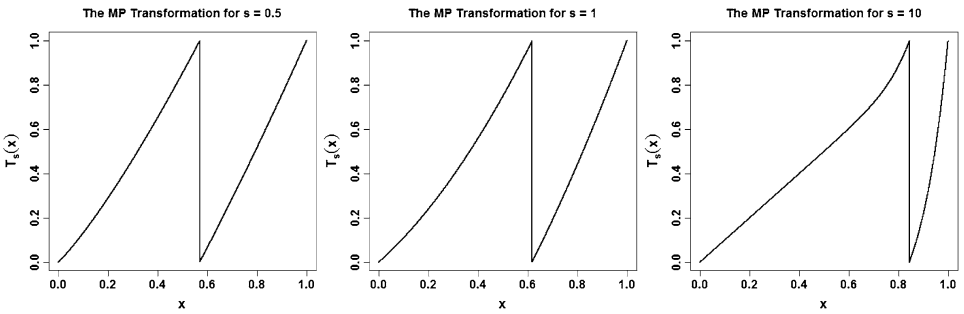

Figure 1 *Plot of the Manneville–Pomeau transformation for different values of* $s \in \{0.5, 1, 10\}$.

the usual behavior of the MP transformations: for any s, they are increasing and differentiable functions by parts in I. Furthermore, for any $s > 0$, the function T_s^k will have exactly 2^k parts.

Pianigiani (1980) shows the existence of a T_s-invariant and absolutely continuous measure with respect to the Lebesgue measure in I which will be denoted henceforth by μ_s. However, the proof uses Perron–Frobenius operator theory and is, for practical purposes, non-constructive so that an explicit form for a T_s-invariant measure is unknown. However, this measure will be a Sinai–Bowen–Ruelle (SBR) measure in the sense that the weak convergence

$$\frac{1}{n}\sum_{k=0}^{n-1}\delta_{T_s^k(x)}(A) \longrightarrow \mu_s(A) \tag{2.1}$$

holds for almost all $x \in I$ and all μ_s-continuity sets[1] A, where $\delta_a(\cdot)$ is the Dirac measure at a.

As a dynamical system, the triple (I, μ_s, T_s) is exact (that is, $\lim_{k\to\infty}(\mu_s \circ T_s^k)(A) = 1$, for all positive μ_s-measurable sets A) which implies ergodicity and strong-mixing. When $s < 1$, μ_s is a probability measure, while if $s \geq 1$, μ_s is no longer finite, but σ-finite (see Fisher and Lopes (2001)). Furthermore, it can be shown that μ_s has a positive, bounded continuous Radon–Nikodym derivative $\mathrm{d}\mu_s = h_s(x)\,\mathrm{d}x$, fact that will be useful later. For further details in the theory of MP transformations and related results, we refer to Pianigiani (1980), Young (1999), Maes et al. (2000) and Fisher and Lopes (2001). For applications, see Zebrowsky (2001), Olbermann et al. (2007) and Lopes and Lopes (1998).

Definition 2.2. Let $s \in (0, 1)$ and let U_0 be a random variable distributed according to (the probability measure) μ_s. Let $\varphi : [0, 1] \longrightarrow \mathbb{R}$ be a function in $\mathcal{L}^1(\mu_s)$. The stochastic process given by

$$X_t = (\varphi \circ T_s^t)(U_0) \qquad \text{for all } t \in \mathbb{N},$$

is called a *Manneville–Pomeau process* (or MP *process*, for short).

The MP process, as defined above, is stationary since μ_s is a T_s-invariant measure and $\mu_s \ll \lambda$. It is also ergodic since μ_s is ergodic for T_s. By its turn, an n-dimensional copula is a distribution function whose marginals are uniformly distributed on I. The copula literature has grown enormously in the last decade, especially in terms of empirical applications and have become standard tools in financial data analysis (see Nelsen (2006) and references therein). The next theorem, known as Sklar's theorem, is the key result for copulas and elucidates the role played by them. See Schweizer and Sklar (2005) for a proof.

[1] Recall that a Borel set A is a μ-continuity set if $\mu(\partial A) = 0$, where ∂A denotes the boundary of A. The measure theoretical results applied here can be found, for instance, in Royden (1988). A good reference in weak convergence of probability measures is Billingsley (1999) and for results related to ergodic theory, see Pollicott and Yuri (1998).

Theorem 2.1 (Sklar's theorem). *Let $X_1, \ldots, X_n$ be random variables with marginals $F_1, \ldots, F_n$, respectively, and joint distribution function H. Then, there exists a copula C such that,*

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \qquad \text{for all } (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

If the F_i's are continuous, then C is unique. Otherwise, C is uniquely determined on $\mathrm{Ran}(F_1) \times \cdots \times \mathrm{Ran}(F_n)$, *where for a function f,* $\mathrm{Ran}(f)$ *denotes the range of f. The converse also holds. Furthermore,*

$$C(u_1, \ldots, u_n) = H\big(F_1^{(-1)}(u_1), \ldots, F_n^{(-1)}(u_n)\big) \qquad \text{for all } (u_1, \ldots, u_n) \in I^n,$$

where for a function F, $F^{(-1)}$ denotes its pseudo-inverse given by $F^{(-1)}(x) := \inf\{u \in \mathrm{Ran}(F) : F(u) \geq x\}$.

The next theorem, whose proof can be found, for instance, throughout Nelsen (2006), shall prove very useful in what follows. Except stated otherwise, the measure implicit to phrases like "almost sure," "almost everywhere" and so on will be the (appropriate) Lebesgue measure.

Theorem 2.2. *Let X and Y be continuous random variables with copula C. If f is an almost everywhere decreasing function then $C_{f(X),Y}(u, v) = u - C_{X,Y}(u, 1 - v)$. Furthermore, if f_1 and f_2 are functions increasing almost everywhere, then $C_{f_1(X),f_2(Y)}(u, v) = C_{X,Y}(u, v)$.*

For an introduction to copulas, we refer the reader to Nelsen (2006). For more details and extensions to the multivariate case with emphasis in modeling and dependence concepts, see Joe (1997). The theory of copulas is also intimately related to the theory of probabilistic metric spaces (see Schweizer and Sklar (2005) for more details in this matter).

3 Copulas and MP processes: Bidimensional case

In this section, we shall investigate the bidimensional copulas associated to pairs of random variables coming from MP processes which we shall call MP *copulas*. As we will see later, the multidimensional case is very similar to the bidimensional case, so we shall give special attention to the latter.

First, let $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0, 1)$ and $\varphi \in \mathcal{L}^1(\mu_s)$ be an increasing almost everywhere function. Throughout this section and in the rest of the paper, we shall treat $s \in (0, 1)$ as a given fixed number. Let

$$F_0(x) := \mathbb{P}(U_0 \leq x) = \mu_s([0, x]).$$

Since $\mu_s \ll \lambda$, μ_s is non-atomic and, therefore, F_0 is (uniformly) continuous. The existence of a positive Radon–Nikodym density for μ_s also shows that F_0

is increasing and its inverse is well defined. Let F_t be the distribution function of $T_s^t(U_0)$, for all $t \in \mathbb{N}$. For $x \in I$, notice that

$$F_t(x) := \mathbb{P}\big(T_s^t(U_0) \leq x\big) = \mu_s((T_s^t)^{-1}([0,x])) = \mu_s([0,x]) = F_0(x), \tag{3.1}$$

since μ_s is a T_s-invariant measure.

In what follows, we shall need the solution for the inequality $T_s^t(X) \leq y$, $y \in (0,1)$, in X, for X a random variable taking values in I. Now, since each of the 2^t parts of T_s^t is one-to-one in its domain, the inverse of T_s^t will also be continuous by parts and each part will also be a one-to-one function in its domain. Let $0 = a_{t,0}, \ldots, a_{t,2^t} = 1$ be the end points of each part of T_s^t. We shall call each interval $[a_{t,k}, a_{t,k+1})$ a *node* of T_s^t, for $k = 0, \ldots, 2^t - 1$ and $t > 0$. The (piecewise) inverse of T_s^t can be conveniently written as

$$\begin{aligned} (T_s^t)^{-1} : I &\longrightarrow I^{2^t}, \\ y &\longmapsto (\mathcal{T}_{t,0}(y), \ldots, \mathcal{T}_{t,2^t-1}(y)), \end{aligned} \tag{3.2}$$

where $\mathcal{T}_{t,k}(y)$ denotes the inverse of T_s^t restricted to its kth node, for all $k \in \{0, \ldots, 2^t - 1\}$. Notice that both $\mathcal{T}_{t,k}$ and $a_{t,k}$ depend on s for each k, but since no confusion will arise, and for the sake of simplicity, we shall omit this dependence from the notation as we shall do in several other occasions. Now, the solution of the inequality $T_s^t(X) \leq y$ in X can be determined and is given by $X \in A_{t,0}(y) \cup \cdots \cup A_{t,2^t-1}(y)$, where

$$A_{t,k}(y) = [a_{t,k}, \mathcal{T}_{t,k}(y)], \tag{3.3}$$

which is a proper closed subinterval of $[a_{t,k}, a_{t,k+1})$, for each $k = 0, \ldots, 2^t - 1$. Notice that $A_{t,k}(y)$ (whose dependence on s was omitted from the notation) is just the inverse image of $[0, y]$ by the transformation T_s^t restricted to the node $[a_{t,k}, a_{t,k+1})$. We can now use this result to prove the following useful lemma.

Lemma 3.1. *Let X be a random variable taking values in I and let T_s be the MP transformation with parameter $s > 0$. Then, for any $t \in \mathbb{N}$ and $x \in I$,*

$$\mathbb{P}\big(T_s^t(X) \leq x\big) = \mathbb{P}\left(X \in \bigcup_{k=0}^{2^t-1} A_{t,k}(x)\right) = \sum_{k=0}^{2^t-1} \mathbb{P}\big(X \in A_{t,k}(x)\big),$$

where $A_{t,k}$'s are given by (3.3).

Proof. The result follows easily from what was just discussed and from the fact that the intervals $A_{t,k}$'s are (pairwise) disjoints. □

As for the copulas related to MP processes, in view of the stationarity of the MP process, the following result follows easily.

Proposition 3.1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0,1)$ and $\varphi \in \mathcal{L}^1(\mu_s)$ be an almost everywhere increasing function. Then, for any $t, h \in \mathbb{N}$,*

$$C_{X_t,X_{t+h}}(u,v) = C_{X_0,X_h}(u,v),$$

everywhere in I^2.

Proof. As a consequence of the stationarity of $\{X_t\}_{t\in\mathbb{N}}$, if we let the joint distribution of the pair (X_p, X_q), for any $p, q \in \mathbb{N}$, $p \neq q$, be denoted by $\widetilde{H}_{p,q}(\cdot,\cdot)$, it follows that for all $x, y \in (0,1)$, $t \in \mathbb{N}$ and $h \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, $\widetilde{H}_{t,t+h}(x,y) = \widetilde{H}_{0,h}(x,y)$. Now, upon applying Sklar's theorem and (3.1), it follows that

$$\begin{aligned} C_{X_t,X_{t+h}}(u,v) &= \widetilde{H}_{t,t+h}(F_t^{-1}(u), F_{t+h}^{-1}(v)) \\ &= \widetilde{H}_{0,h}(F_0^{-1}(u), F_h^{-1}(v)) = C_{X_0,X_h}(u,v), \end{aligned}$$

for all $(u,v) \in I^2$. □

Corollary 3.1. *Let T_s be the MP transformation for some $s \in (0,1)$, μ_s be a T_s-invariant probability measure and let U_0 be distributed as μ_s. Then, for any $t, h \in \mathbb{N}$, $h \neq 0$,*

$$C_{T_s^t(U_0),T_s^{t+h}(U_0)}(u,v) = C_{U_0,T_s^h(U_0)}(u,v)$$

everywhere in I^2.

Proof. Immediate from Theorem 2.2 applied to Proposition 3.1. □

Now we turn our attention to determine the copula associated to any pair of random variables (X_p, X_q), $p, q \in \mathbb{N}$, obtained from an MP process with φ an increasing almost everywhere function. For the sake of simplicity, let us introduce the following functions: let h be a positive integer and for $k = 0, \ldots, 2^h - 1$, let $\mathcal{F}_{h,k} : I \to [F_0(a_{h,k}), F_0(a_{h,k+1})]$ be given by

$$\mathcal{F}_{h,k}(x) := F_0(\mathcal{T}_{h,k}(F_0^{-1}(x))).$$

Notice that for each k, $\mathcal{F}_{h,k}(0) = F_0(a_{h,k})$ and $\mathcal{F}_{h,k}(1) = F_0(a_{h,k+1})$ and $\mathcal{F}_{h,k}$ is a one to one, increasing and uniformly continuous function.

Proposition 3.2. *Let $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0,1)$, $\varphi \in \mathcal{L}^1(\mu_s)$ be an increasing almost everywhere function and let F_0 be the distribution function of U_0. Then, for any $t, h \in \mathbb{N}$, $h \neq 0$ and $(u,v) \in I^2$,*

$$\begin{aligned} C_{X_t,X_{t+h}}(u,v) = &\left(\sum_{k=0}^{n_0-1} \mathcal{F}_{h,k}(v) - F_0(a_{h,k})\right)\delta_{\mathbb{N}^*}(n_0) \\ &+ \min\{u, \mathcal{F}_{h,n_0}(v)\} - F_0(a_{h,n_0}), \end{aligned} \tag{3.4}$$

where $\{a_{h,k}\}_{k=0}^{2^h}$ are the end points of the nodes of T_s^h, $n_0 := n_0(u;h) = \{k : u \in$

$[F_0(a_{h,k}), F_0(a_{h,k+1}))\} \in \{0, \dots, 2^h - 1\}$ *and* $\delta_{\mathbb{N}^*}(x)$ *equals* 1, *if* $x \in \mathbb{N}^*$ *and* 0, *otherwise*.

Proof. By Propositions 3.1 and 2.2, it suffices to derive the copula of the pair $(U_0, T_s^h(U_0))$. So let again $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0,1)$ and $\varphi \in \mathcal{L}^1(\mu_s)$ be an increasing almost everywhere function and let $H_{0,h}(\cdot,\cdot)$ denote the distribution function of the pair $(U_0, T_s^h(U_0))$. Notice that

$$\begin{aligned} H_{0,h}(x,y) &= \mathbb{P}\big(U_0 \le x, T_s^h(U_0) \le y\big) = \mathbb{P}\left(U_0 \le x, U_0 \in \bigcup_{k=0}^{2^h-1} A_{h,k}(y)\right) \\ &= \mathbb{P}\left(U_0 \in [0,x] \cap \bigcup_{k=0}^{2^h-1} A_{h,k}(y)\right) = \mathbb{P}\left(U_0 \in \bigcup_{k=0}^{2^h-1} \big[[0,x] \cap A_{h,k}(y)\big]\right) \\ &= \sum_{k=0}^{2^h-1} \mathbb{P}\big(U_0 \in [0,x] \cap A_{h,k}(y)\big) \end{aligned}$$

for any $x, y \in (0,1)$. Now let $n_1 := n_1(x;h) = \{k : x \in [a_{h,k}, a_{h,k+1})\} \in \{0, \dots, 2^h - 1\}$ and assume for the moment that $n_1 \ge 1$. Since $A_{h,k}(y) = [a_{h,k}, \mathcal{T}_{h,k}(y)]$, it follows that

$$\begin{aligned} H_{0,h}(x,y) &= \sum_{k=0}^{n_1-1} \mathbb{P}\big(U_0 \in A_{h,k}(y)\big) + \mathbb{P}\big(U_0 \in A_{h,n_1}(y) \cap [a_{h,n_1}, x]\big) \\ &= \sum_{k=0}^{n_1-1} \mu_s(A_{h,k}(y)) + \mu_s\big([a_{h,n_1}, \mathcal{T}_{h,n_1}(y)] \cap [a_{h,n_1}, x]\big) \\ &= \sum_{k=0}^{n_1-1} \mu_s([a_{h,k}, \mathcal{T}_{h,k}(y)]) + \mu_s([a_{h,n_1}, \min\{x, \mathcal{T}_{h,n_1}(y)\}]), \end{aligned}$$

which can be written, since $F_0(x) = \mu_s([0,x])$ is increasing, as

$$\begin{aligned} H_{0,h}(x,y) = &\sum_{k=0}^{n_1-1} [F_0(\mathcal{T}_{h,k}(y)) - F_0(a_{h,k})] \\ &+ \min\{F_0(x), F_0(\mathcal{T}_{h,n_1}(y))\} - F_0(a_{h,n_1}). \end{aligned}$$

If $n_1 = 0$, the summation is absent of the formula and we have

$$H_{0,h}(x,y) = \min\{F_0(x), F_0(\mathcal{T}_{h,0}(y))\} - F_0(a_{h,0}),$$

so that, in any case, we have

$$\begin{aligned} H_{0,h}(x,y) = &\left(\sum_{k=0}^{n_1-1} [F_0(\mathcal{T}_{h,k}(y)) - F_0(a_{h,k})]\right)\delta_{\mathbb{N}^*}(n_1) \\ &+ \min\{F_0(x), F_0(\mathcal{T}_{h,n_1}(y))\} - F_0(a_{h,n_1}). \end{aligned}$$

Now upon applying Sklar's theorem, it follows that

$$\begin{aligned} &C_{U_0,T_s^h(U_0)}(u,v) \\ &\quad = H_{0,h}(F_0^{-1}(u), F_h^{-1}(v)) = H_{0,h}(F_0^{-1}(u), F_0^{-1}(v)) \\ &\quad = \left(\sum_{k=0}^{n_0-1} \mathcal{F}_{h,k}(v) - F_0(a_{h,k})\right)\delta_{\mathbb{N}^*}(n_0) + \min\{u, \mathcal{F}_{h,n_0}(v)\} - F_0(a_{h,n_0}), \end{aligned}$$

where $n_0 := n_0(u;h) = n_1(F_0^{-1}(u);h) = \{k : u \in [F_0(a_{h,k}), F_0(a_{h,k+1}))\}$. The result now follows from Proposition 3.1. □

Remark 3.1. Notice that the copula (3.4) can be expressed in terms of μ_s as

$$\begin{aligned} C_{X_t,X_{t+h}}(u,v) = &\left(\sum_{k=0}^{n_0-1} \mu_s([a_{h,k}, \mathcal{T}_{h,k}(F_0^{-1}(v))])\right)\delta_{\mathbb{N}^*}(n_0) \\ &+ \mu_s([a_{h,n_0}, \min\{F_0^{-1}(u), \mathcal{T}_{h,n_0}(F_0^{-1}(v))\}]), \end{aligned} \tag{3.5}$$

which will prove useful in Section 5. Also, expression (3.5) is helpful if one desires to verify directly that the marginals of (3.4) are indeed uniform.

In the next proposition, we address the case where φ is an almost everywhere decreasing function. In view of Theorem 2.2, one could, at first glance, think that a result like $C_{X_0,X_h} = C_{X_t,X_{t+h}}$ would not hold anymore, but in fact it still does, as it is shown in the next proposition.

Proposition 3.3. *Let $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0,1)$, $\varphi \in \mathcal{L}^1(\mu_s)$ be an almost everywhere decreasing function and let F_0 be the distribution function of U_0. Then, $C_{X_0,X_h}(u,v) = C_{X_t,X_{t+h}}(u,v)$ everywhere in I^2 and, for any $t, h \in \mathbb{N}$ and $h \neq 0$,*

$$\begin{aligned} C_{X_t,X_{t+h}}(u,v) = u + v - 1 + &\left(\sum_{k=0}^{n_0-1} [\mathcal{F}_{h,k}(1-v) - F_0(a_{h,k})]\right)\delta_{\mathbb{N}^*}(n_0) \\ &+ \min\{1-u, \mathcal{F}_{h,n_0}(1-v)\} - F_0(a_{h,n_0}), \end{aligned} \tag{3.6}$$

for all $(u,v) \in I^2$, where $\{a_{h,k}\}_{k=0}^{2^h}$ are the end points of the nodes of T_s^h and $n_0 := n_0(u;h) = \{k : u \in (1 - F_0(a_{h,k+1}), 1 - F_0(a_{h,k})]\}$.

Proof. Since the inverse of an almost everywhere decreasing function is still decreasing almost everywhere and $X_t = \varphi(T_s^t(U_0))$, upon applying Theorem 2.2 twice, it follows that

$$\begin{aligned} C_{T_s^t(U_0),T_s^{t+h}(U_0)}(u,v) &= C_{\varphi^{-1}(X_t),\varphi^{-1}(X_{t+h})}(u,v) = u - C_{X_t,\varphi^{-1}(X_{t+h})}(u,1-v) \\ &= u - \big(1 - v - C_{X_t,X_{t+h}}(1-u,1-v)\big), \end{aligned}$$

or, equivalently (changing u by $1-u$ and v by $1-v$),

$$C_{X_t,X_{t+h}}(u,v) = u + v - 1 + C_{T_s^t(U_0),T_s^{t+h}(U_0)}(1-u,1-v). \tag{3.7}$$

Now (3.6) follows upon applying Proposition 3.2 with the identity map and substituting equation (3.4) into (3.7). As for the equality $C_{X_0,X_h}(u,v) = C_{X_t,X_{t+h}}(u,v)$, Corollary 3.1 and Theorem 2.2 applied to (3.7) yield

$$\begin{aligned} C_{X_t,X_{t+h}}(u,v) &= u + v - 1 + C_{U_0,T_s^h(U_0)}(1-u,1-v) \\ &= u + v - 1 + C_{\varphi^{-1}(\varphi(U_0)),\varphi^{-1}(\varphi(T_s^h(U_0)))}(1-u,1-v) \\ &= C_{\varphi(U_0),\varphi(T_s^h(U_0))}(u,v) = C_{X_0,X_h}(u,v), \end{aligned}$$

everywhere in I^2, as desired. □

Remark 3.2. In view of the "stationarity" results of Theorems 3.1 and 3.3, a copula associated to a pair (X_t, X_{t+h}) from an MP process will be referred to as *lag h* MP *copula*.

The copulas in (3.4) and (3.6) are both singular, as it can be readily verified, since $\partial^2 C_{X_t,X_{t+h}}(u,v)/(\partial u\,\partial v) = 0$ everywhere in I^2 on both cases. So the question that naturally arises is, for each h, what is the support of $C_{X_t,X_{t+h}}$? The question is addressed in the next proposition, which will be useful in Sections 5 and 6. For simplicity, for a given MP process and $h > 0$, let $\ell_{h,k}^+, \ell_{h,k}^- : [F_0(a_{h,k}), F_0(a_{h,k+1})) \to I$ be functions defined by

$$\ell_{h,k}^+(x) = \frac{x - F_0(a_{h,k})}{F_0(a_{h,k+1}) - F_0(a_{h,k})} \quad \text{and} \quad \ell_{h,k}^-(x) = \frac{F_0(a_{h,k+1}) - x}{F_0(a_{h,k+1}) - F_0(a_{h,k})}$$

for all $k = 0, \ldots, 2^h - 1$. Notice that, for each k, $\ell_{h,k}^+$ is the linear function connecting the points $(F_0(a_{h,k}), 0)$ and $(F_0(a_{h,k+1}), 1)$, while $\ell_{h,k}^-$ connects the points $(F_0(a_{h,k}), 1)$ and $(F_0(a_{h,k+1}), 0)$.

Proposition 3.4. *Let $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0,1)$, for $\varphi_1 \in \mathcal{L}^1(\mu_s)$ an almost everywhere increasing function and let $\{Y_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0,1)$, for $\varphi_2 \in \mathcal{L}^1(\mu_s)$ an almost everywhere decreasing function. Also let F_0 be the distribution function of U_0. Then, for any $t, h \in \mathbb{N}$, $h > 0$,*

$$\operatorname{supp}\{C_{X_t,X_{t+h}}\} = \bigcup_{k=0}^{2^h-1} \{(u, \ell_{h,k}^+(u)) : u \in [F_0(a_{h,k}), F_0(a_{h,k+1}))\} \tag{3.8}$$

and

$$\operatorname{supp}\{C_{Y_t,Y_{t+h}}\} = \bigcup_{k=0}^{2^h-1} \{(u, \ell_{h,k}^-(u)) : u \in [F_0(a_{h,k}), F_0(a_{h,k+1}))\}. \tag{3.9}$$

Proof. Let $R = [u_1, u_2] \times [v_1, v_2]$ be a rectangle in I^2 and let its $C_{X_t, X_{t+h}}$-volume be denoted by $V_{C_{\mathbf{X}}}(R)$. Let $k \in \{0, \ldots, 2^h - 1\}$ be fixed and suppose that $u_i \in [F_0(a_{h,k}), F_0(a_{h,k+1})]$. This implies that $n_0 = k$ for all four terms in $V_{C_{\mathbf{X}}}(R)$, hence the summands and constants on the copula cancel out so that we have

$$\begin{aligned} V_{C_{\mathbf{X}}}(R) &= \min\{u_1, \mathcal{F}_{h,k}(v_1)\} + \min\{u_2, \mathcal{F}_{h,k}(v_2)\} - \min\{u_1, \mathcal{F}_{h,k}(v_2)\} \\ &\quad - \min\{u_2, \mathcal{F}_{h,k}(v_1)\} \\ &= V_M\big([u_1, u_2] \times [\mathcal{F}_{h,k}(v_1), \mathcal{F}_{h,k}(v_2)]\big), \end{aligned}$$

where $M(u, v) = \min\{u, v\}$ is the Frechèt upper bound copula whose support is the main diagonal in I^2. Observing that by the definition of $\mathcal{F}_{h,k}$, $[u_1, u_2] \times [\mathcal{F}_{h,k}(v_1), \mathcal{F}_{h,k}(v_2)] \subset [F_0(a_{h,k}), F_0(a_{h,k+1})]^2$, $V_{C_{\mathbf{X}}}(R) > 0$ if, and only if, $R \cap \{(u, \ell^+_{h,k}(u)) : u \in [F_0(a_{h,k}), F_0(a_{h,k+1}))\} \neq \varnothing$.

Analogously, denoting the $C_{Y_t, Y_{t+h}}$-volume of R by $V_{C_{\mathbf{Y}}}(R)$, if $u_i \in [1 - F_0(a_{h,k}), 1 - F_0(a_{h,k+1})]$, we have

$$\begin{aligned} V_{C_{\mathbf{Y}}}(R) &= \min\{1 - u_1, \mathcal{F}_{h,k}(1 - v_1)\} + \min\{1 - u_2, \mathcal{F}_{h,k}(1 - v_2)\} \\ &\quad - \min\{1 - u_1, \mathcal{F}_{h,k}(1 - v_2)\} - \min\{1 - u_2, \mathcal{F}_{h,k}(1 - v_1)\} \\ &= V_M\big([1 - u_1, 1 - u_2] \times [\mathcal{F}_{h,k}(1 - v_2), \mathcal{F}_{h,k}(1 - v_1)]\big). \end{aligned} \tag{3.10}$$

Since $[1 - u_1, 1 - u_2] \times [\mathcal{F}_{h,k}(1 - v_1), \mathcal{F}_{h,k}(1 - v_2)] \subset [F_0(a_{h,k}), F_0(a_{h,k+1})]^2$, we observe that $V_{C_{\mathbf{Y}}}(R)$ will be positive if, and only if, $R \cap \{(u, \ell^-_{h,k}(u)) : u \in [F_0(a_{h,k}), F_0(a_{h,k+1}))\} \neq \varnothing$ (notice the terms $1 - v_i$ in expression (3.10), for $i = 1, 2$). Now (3.8) and (3.9) follow by observing that $I = \bigcup_{k=0}^{2^h-1} [F_0(a_{h,k}), F_0(a_{h,k+1})] = \bigcup_{k=0}^{2^h-1} [1 - F_0(a_{h,k+1}), 1 - F_0(a_{h,k})]$. □

Remark 3.3. We end up this section by noticing that as an application of Propositions 3.1 and 3.3, together with the so-called copula version of Hoeffding's lemma (see Nelsen (2006)), we can show in a rather different way that an MP process is weakly stationary. Let F_{X_t} be the distribution function of X_t and notice that $F_{X_t}(x) = F_{X_0}(x)$, for all $t \in \mathbb{N}$, by the stationarity of $\{X_t\}_{t \in \mathbb{N}}$ and since $C_{X_t, X_{t+h}}(u, v) = C_{X_0, X_h}(u, v)$, the result follows immediately.

4 Multidimensional case

In this section, we are interested in extending the results from the previous section to the multidimensional case, that is, in this section we are interested in deriving the copulas associated to n-dimensional vectors $(X_{t_1}, \ldots, X_{t_n})$, $t_1, \ldots, t_n \in \mathbb{N}$, coming from an MP process with φ an increasing almost everywhere function. In view of Theorem 2.2, it suffices to derive the copula associated to the vector $(T_s^{t_1}(U_0), \ldots, T_s^{t_n}(U_0))$. It turns out that there are more similarities between the

bidimensional and multidimensional cases than one could expect. In fact, an expression very similar in form to (3.4) holds for the multidimensional case as well.

Let $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0,1)$ and $\varphi \in \mathcal{L}^1(\mu_s)$ be an almost everywhere increasing function. For the sake of simplicity, we shall use the following notation: let $a, b \in \mathbb{N}$, $a < b$, we shall write $x_{a:b} := (x_a, \ldots, x_b)$ and for a function f, $f(x_{a:b}) := (f(x_a), \ldots, f(x_b))$. Again we shall denote the distribution function of U_0 by F_0.

Theorem 4.1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0,1)$, with $\varphi \in \mathcal{L}^1(\mu_s)$ an almost everywhere increasing function. Let $t_1, \ldots, t_n \in \mathbb{N}$ and set $h_i := t_i - t_1$. Then, for all $(u_1, \ldots, u_n) \in I^n$,*

$$\begin{aligned} C_{X_{t_1},\ldots,X_{t_n}}(u_1,\ldots,u_n) = &\left(\sum_{k=0}^{n_0-1} F_0(b_{h_n,k}(F_0^{-1}(u_{2:n}))) - F_0(a_{h_n,k})\right)\delta_{\mathbb{N}^*}(n_0) \\ &+ \min\{u_1, F_0(b_{h_n,n_0}(F_0^{-1}(u_{2:n})))\} - F_0(a_{h_n,n_0}), \end{aligned} \tag{4.1}$$

where $n_0 := n_0(u_1, n) = \{k : u_1 \in [F_0(a_{h_n,k}), F_0(a_{h_n,k+1}))\}$, $\{a_{h_n,k}\}_{k=0}^{2^h}$ are the end points of the nodes of $T_s^{h_n}$, for $i = 2, \ldots, n$, $j = 0, \ldots, 2^{h_i} - 1$, $\mathcal{T}_{h_i,j}$ is given by (3.2) and for a vector $(x_2, \ldots, x_n) \in I^{n-1}$, $b_{h_n,k}(x_{2:n}) = \min_{i=2,\ldots,n}\{c_i(x_i; h_n, k)\}$, with

$$c_i(x_i; h_n, k) = \begin{cases} a_{h_n,k}, & \text{if } B_i(x_i; h_n, k) = \varnothing; \\ B_i(x_i; h_n, k), & \text{otherwise.} \end{cases}$$

and

$$B_i(x_i; h_n, k) = \min_{j=0,\ldots,2^{h_i}-1} \{\mathcal{T}_{h_i,j}(x_i) : \mathcal{T}_{h_i,j}(x_i) > a_{h_n,k} \text{ and } a_{h_i,j} < a_{h_n,k+1}\}.$$

Proof. Let $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0,1)$ with $\varphi \in \mathcal{L}^1(\mu_s)$ an almost everywhere increasing function and suppose, without loss of generality, that $0 < t_1 < \cdots < t_n$. In view of Theorem 2.2, it suffices to work with the vector $(T_s^{t_1}(U_0), \ldots, T_s^{t_n}(U_0))$. Let $H_{t_1,\ldots,t_n}$ be the distribution function of $(T_s^{t_1}(U_0), \ldots, T_s^{t_n}(U_0))$. Let $h_i = t_i - t_1$, for each $i = 1, \ldots, n$, and notice that $h_i > 0$ since $t_1 < t_i$, for all $i = 2, \ldots, n$. Let $(x_1, \ldots, x_n) \in (0,1)^n$ and for the sake of simplicity, let $Y_{t_1} := T_s^{t_1}(U_0)$, so that we have

$$\begin{aligned} &H_{t_1,\ldots,t_n}(x_1,\ldots,x_n) \\ &\quad= \mathbb{P}\big(T_s^{t_1}(U_0) \le x_1, \ldots, T_s^{t_n}(U_0) \le x_n\big) \\ &\quad= \mathbb{P}\big(Y_{t_1} \le x_1, T_s^{h_2}(Y_{t_1}) \le x_2, \ldots, T_s^{h_n}(Y_{t_1}) \le x_n\big) \\ &\quad= \mathbb{P}\left(Y_{t_1} \in [0, x_1], Y_{t_1} \in \bigcup_{k=0}^{2^{h_2}-1} A_{h_2,k}(x_2), \ldots, Y_{t_1} \in \bigcup_{k=0}^{2^{h_n}-1} A_{h_n,k}(x_n)\right) \end{aligned}$$

$$= \mathbb{P}\left(Y_{t_1} \in [0, x_1] \bigcap_{i=2}^{n} \left[\bigcup_{k=0}^{2^{h_i}-1} A_{h_i,k}(x_i)\right]\right)$$

$$= \mathbb{P}\left(U_0 \in \bigcap_{i=2}^{n} \bigcup_{k=0}^{2^{h_i}-1} \left[[0, x_1] \cap A_{h_i,k}(x_i)\right]\right), \tag{4.2}$$

where $A_{h_i,k}$'s are given by (3.3) and the last equality is a consequence of the T_s-invariance of μ_s. For $k = 0, \ldots, 2^{h_n-1}$, let

$$\widetilde{A}_{h_n,k}(x_{2:n}) = A_{h_n,k}(x_n) \bigcap_{i=2}^{n-1} \left[\bigcup_{j=0}^{2^{h_i}-1} A_{h_i,j}(x_i)\right].$$

In order to simplify the notation, for $i = 2, \ldots, n$ and $k = 0, \ldots, 2^{h_n} - 1$, let $B_i(x_i; h_n, k)$ be as in the enunciate. Notice that, for each k and i, $B_i(x_i; h_n, k)$ is either empty or the smallest $\mathcal{T}_{h_i,j}(x_i)$ which is greater than $a_{h_n,k}$ and such that the correspondent $A_{h_i,j}(x_i)$ has non-empty intersection with $A_{h_n,k}(x_n)$. Let

$$c_i(x_i; h_n, k) = \begin{cases} a_{h_n,k}, & \text{if } B_i(x_i; h_n, k) = \varnothing; \\ B_i(x_i; h_n, k), & \text{otherwise.} \end{cases}$$

Then, for each $k = 1, \ldots, 2^{h_n} - 1$, setting $b_{h_n,k}(x_{2:n}) = \min_{i=2,\ldots,n}\{c_i(x_i; h_n, k)\}$, it follows that

$$\widetilde{A}_{h_n,k}(x_{2:n}) = [a_{h_n,k}, b_{h_n,k}(x_{2:n})],$$

which is a closed subset of $[a_{h_n,k}, a_{h_n,k+1}]$. Also notice that, from the definition of $b_{h_n,k}(x_{2:n})$, we could have $\widetilde{A}_{h_n,k}(x_{2:n}) = \{a_{h_n,k}\}$, in which case we set $\widetilde{A}_{h_n,k}(x_{2:n}) = \varnothing$ (although from a measure-theoretical point of view, this correction makes no difference). Again we are omitting the dependence in s from the notation on both, $b_{h_n,k}$ and $\widetilde{A}_{h_n,k}$. Each $b_{h_n,k}(x_{2:n})$ determines the smallest $\mathcal{T}_{h_i,j}(x_i)$ that lies on the kth node of $T_s^{h_n}$ (which has the smallest nodes among all $T_s^{h_i}$'s), so that $\widetilde{A}_{h_n,k}$'s are just the intersection of all $A_{h_i,k}(x_i)$'s with end point in the kth node of $T_s^{h_n}$. Also notice that the $\widetilde{A}_{h_n,k}$'s are pairwise disjoints. One can rewrite (4.2) as

$$H_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = \mathbb{P}\left(U_0 \in \bigcup_{k=0}^{2^{h_n}-1} \left[\widetilde{A}_{h_n,k}(x_{2:n}) \cap [0, x_1]\right]\right). \tag{4.3}$$

Now, let $n_1 := n_1(x_1; n) = \{k : x_1 \in [a_{h_n,k}, a_{h_n,k+1})\} \in \{0, \ldots, 2^{h_n} - 1\}$, and assume for the moment that $n_1 \geq 1$. Then (4.3) becomes

$$H_{t_1,\ldots,t_n}(x_1, \ldots, x_n)$$

$$= \sum_{k=0}^{n_1-1} \mathbb{P}(U_0 \in \widetilde{A}_{h_n,k}(x_{2:n})) + \mathbb{P}(U_0 \in \widetilde{A}_{h_n,n_1}(x_{2:n}) \cap [a_{h_n,n_1}, x_1])$$

$$
\begin{aligned}
&= \sum_{k=0}^{n_1-1} \mu_s([a_{h_n,k}, b_{h_n,k}(x_{2:n})]) + \mu_s([a_{h_n,n_1}, \min\{x_1, b_{h_n,n_1}(x_{2:n})\}]) \\
&= \sum_{k=0}^{n_1-1} [F_0(b_{h_n,k}(x_{2:n})) - F_0(a_{h_n,k})] \\
&\quad + \min\{F_0(x_1), F_0(b_{h_n,n_1}(x_{2:n}))\} - F_0(a_{h_n,n_1}).
\end{aligned}
$$

If $n_1 = 0$, then

$$
H_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = \min\{F_0(x_1), F_0(b_{h_n,0}(x_{2:n}))\} - F_0(a_{h_n,0}).
$$

In any case, we can write

$$
\begin{aligned}
H_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = &\left(\sum_{k=0}^{n_1-1} F_0(b_{h_n,k}(x_{2:n})) - F_0(a_{h_n,k}) \right) \delta_{\mathbb{N}^*}(n_1) \\
&+ \min\{F_0(x_1), F_0(b_{h_n,n_1}(x_{2:n}))\} - F_0(a_{h_n,n_1}).
\end{aligned}
$$

Recall that the distribution function of $T_s^t(U_0)$ is also F_0 by the T_s-invariance of μ_s. Now applying Sklar's theorem, it follows that,

$$
\begin{aligned}
C_{X_{t_1},\ldots,X_{t_n}}(u_1, \ldots, u_n) &= H_{t_1,\ldots,t_n}(F_0^{-1}(u_1), \ldots, F_0^{-1}(u_n)) \\
&= \left(\sum_{k=0}^{n_0-1} F_0(b_{h_n,k}(F_0^{-1}(u_{2:n}))) - F_0(a_{h_n,k}) \right) \delta_{\mathbb{N}^*}(n_1) \\
&\quad + \min\{u_1, F_0(b_{h_n,n_0}(F_0^{-1}(u_{2:n})))\} - F_0(a_{h_n,n_0}),
\end{aligned}
$$

where $n_0 := n_1(F_0^{-1}(u_1), n) = \{k : u_1 \in [F_0(a_{h_n,k}), F_0(a_{h_n,k+1}))\}$, which is the desired formula. □

Remark 4.1. Notice that the proof of Theorem 4.1 from equation (4.3) on is exactly the same as the one in Proposition 3.2 with the obvious notational adaptations.

Now we turn our attention to the case where φ is an almost everywhere decreasing function. In view of Theorem 2.2, one cannot expect a simple expression for the copula. What happens is that the copula in this case will be the sum of the lower dimensions copulas related to the iterations $T_s^k(U_0)$, as the next proposition shows.

Proposition 4.1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be an MP process with parameter $s \in (0, 1)$, and $\varphi \in \mathcal{L}^1(\mu_s)$ be an almost everywhere decreasing function. Let $t, h_1, \ldots, h_n \in \mathbb{N}$,*

$0 < h_1 < \cdots < h_n$ and set $Y_0 := U_0$ and $Y_k := T_s^{h_k}(U_0)$. Denote the copula associated to the random vector $(X_t, X_{t+h_1}, \ldots, X_{t+h_n})$ by $C_{\mathbf{t}}$. Then the following relation holds

$$\begin{aligned} &C_{\mathbf{t}}(u_0, \ldots, u_n) \\ &\quad = 1 - n + \sum_{i=0}^{n} u_i + \sum_{i=0}^{n} \sum_{j=i+1}^{n} C_{Y_i, Y_j}(1 - u_i, 1 - u_j) + \cdots \\ &\qquad + (-1)^{n-1} \\ &\qquad\quad \times \sum_{k_1=0}^{n} \sum_{k_2=k_1+1}^{n} \cdots \sum_{k_{n-1}=k_{n-2}+1}^{n} C_{Y_{k_1}, \ldots, Y_{k_{n-1}}}(1 - u_{k_1}, \ldots, 1 - u_{k_{n-1}}) \\ &\qquad + (-1)^n C_{U_0, Y_1, \ldots, Y_n}(1 - u_0, \ldots, 1 - u_n), \end{aligned} \tag{4.4}$$

everywhere in I^{n+1}.

Proof. Let $t, h_1, \ldots, h_n \in \mathbb{N}$, $0 < h_1 < \cdots < h_n$, $t \neq 0$. Set $Y_0 := U_0$, $Y_k := T_s^{h_k}(U_0)$ and $y_k := \varphi(x_k)$. We have

$$\begin{aligned} &H_{X_0, X_{h_1}, \ldots, X_{h_n}}(x_0, x_1, \ldots, x_n) \\ &\quad = \mathbb{P}(U_0 \geq y_0, Y_1 \geq y_1, \ldots, Y_n \geq y_n) \\ &\quad = \mathbb{P}(U_0 \geq y_0 | Y_1 \geq y_1, Y_2 \geq y_2, \ldots, Y_n \geq y_n) \mathbb{P}(Y_1 \geq y_1, \ldots, Y_n \geq y_n) \\ &\quad = \mathbb{P}(Y_1 \geq y_1, \ldots, Y_n \geq y_n) - \mathbb{P}(U_0 \leq y_0, Y_1 \geq y_1, \ldots, Y_n \geq y_n). \end{aligned} \tag{4.5}$$

Upon applying a long chain of a conditioning argument on both terms in (4.5), we arrive at

$$\begin{aligned} &H_{X_0, X_{h_1}, \ldots, X_{h_n}}(x_0, x_1, \ldots, x_n) \\ &\quad = 1 - \sum_{i=0}^{n} F_0(y_i) + \sum_{i=0}^{n} \sum_{j=i+1}^{n} H_{Y_i, Y_j}(y_i, y_j) + \cdots \\ &\qquad + (-1)^{n-1} \sum_{k_1=0}^{n} \sum_{k_2=k_1+1}^{n} \cdots \sum_{k_{n-1}=k_{n-2}+1}^{n} H_{Y_{k_1}, \ldots, Y_{k_{n-1}}}(y_{k_1}, \ldots, y_{k_{n-1}}) \\ &\qquad + (-1)^n H_{U_0, Y_1, \ldots, Y_n}(y_0, \ldots, y_n). \end{aligned} \tag{4.6}$$

A simple calculation (using the T_s-invariance of μ_s) shows that, for all $t \in \mathbb{N}^*$ and $x \in (0, 1)$,

$$F_{X_t}(x) = 1 - F_0(\varphi(x)) \quad \text{and} \quad F_{X_t}^{-1}(x) = \varphi^{-1}\big(F_0^{-1}(1 - x)\big),$$

so that, the result follows upon applying Sklar's theorem to (4.6) (recall that $y_k = \varphi(x_k)$). □

Remark 4.2. Notice that the copula in Proposition 4.1 can be explicitly calculated since (4.4) is written as sums of the copulas of vectors containing U_0 and $T^t(U_0)$ for different t's, so that the desired formulas can be deduced in terms of the copulas in Theorem 4.1.

5 Numerical approximations to the MP copulas

The MP copulas derived in the last sections do not have readily computable formulas, especially because μ_s does not have explicit expression and because even apparently simple tasks like determining the discontinuity points of T_s^h or to compute explicit formulas for the branches of T_s^h can be highly complex ones. However, one can still study the copulas derived in the last sections by using appropriate approximations to the functions appearing in the copula expression. Besides the invariant measure μ_s, computation of the bidimensional copulas so far discussed also involves computation of the quantile function F_0^{-1}, the inverse of T_s^h and the end points $\{a_{h,k}\}_{k=0}^{2^h}$ of the nodes of T_s^h.

In this section our goal is to derive simple approximations to these functions in order to obtain an approximation to the copula itself, which we shall prove to converge uniformly in its arguments to the true copula. The approximations presented here are simple ones, usually a linear interpolation based on a grid of values, but the technique and results we shall use and prove here are stronger and cover a wide range of approximations, for instance, all results hold if we use some type of spline interpolation instead of a linear one. This is so because the functions to be approximated are generally very smooth. We also evaluate the stability and performance of the approximations by simple numerical experiments.

5.1 Approximation to μ_s

We start with an approximation to μ_s. In this direction, there are at least two ways to compute approximations to μ_s. One way is by using the ideas and results outlined in Dellnitz and Junge (1999), which are based on a discretization of the Perron–Frobenius operator by means of a Garlekin projection type approximation in order to compute the eigenvectors of the discretized operator corresponding to the eingenvalue 1. Although it can be used to approximate any SBR measure, the method is especially suited to approximate and study (almost) cyclical behavior of dynamical systems. However, its complexity makes the efficient implementation troublesome. A much simpler idea, which we shall adopt here, is to approximate the measure by truncating equation (2.1) for a reasonably large value of n. That is, we consider the approximating measure

$$\mu_n(A; s, x_0) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T_s^k(x_0)}(A) \tag{5.1}$$

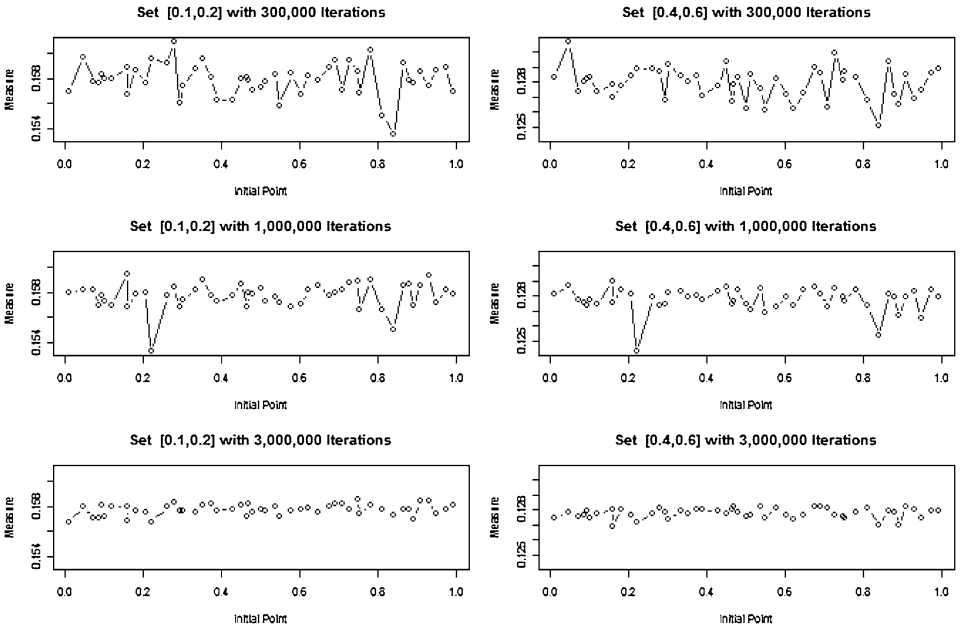

Figure 2 *Performance of the approximation* (5.1) *for truncation points* $n \in \{300{,}000; 1{,}000{,}000; 3{,}000{,}000\}$ (*top*, *middle and bottom*, *resp.*) *and* 50 *different initial points for* $s = 0.5$. *The measured sets are* (*left*) $[0.1, 0.2]$ *and* (*right*) $[0.4, 0.6]$. *All plots within the same set are in the same scale.*

which converges in a weak sense to μ_s as n tends to infinity, for almost all initial points $x_0 \in I$ and all μ_s-continuity sets A. The iterations of T_s are known to be unstable with respect to the initial point in the sense that, given a small $\varepsilon > 0$ and a point $x \in (0, 1)$, the trajectories $T_s^k(x)$ and $T_s^k(x + \varepsilon)$ become far apart exponentially fast. The approximation (5.1), however, is quite stable with respect to the initial point x_0 for large n. For instance, in Figure 2 we show the measure of the sets $[0.1, 0.2]$ and $[0.4, 0.6]$ obtained by using $\mu_n(\cdot; s, x_0)$ with $s = 0.5$, for 50 different initial points x_0 and 3 different truncation points $n \in \{300{,}000; 1{,}000{,}000; 3{,}000{,}000\}$. All plots are in the same scale (within set) in order to make comparison possible. In Table 1, we show basic statistics related to Figure 2. Notice that, in average, the 1,000,000 and 3,000,000 iteration cases are very similar and all cases are fairly stable with respect to the initial points (observe the scale).

Next question is how good is the approximation (5.1)? One way to test this is by testing whether the approximation is invariant under T_s. For given initial points, say $x_1, \ldots, x_k$ and some interval $[a, b]$, we calculate $\mu_n([a, b]; s, x_i)$ and $\mu_n(T_s^{-1}([a, b]); s, x_j)$. If the difference between the two quantities is small for different pairs (x_i, x_j), one can conclude that the approximation is reasonably good. In Table 2, we present the difference $|\mu_n([a, b]; s, x_i) - \mu_n(T_s^{-1}([a, b]); s, x_j)|$ for 7 different initial points and 3 different sets $[a, b]$. The truncation point was taken

Table 1 *Summary statistics for the data presented in Figure 2*

Set	n	300,000	1,000,000	3,000,000
[0.2, 0.3]	[min, max]	[0.12511, 0.13067]	[0.12431, 0.12901]	[0.12688, 0.12825]
	range	0.00556	0.00470	0.00137
	mean	0.12790	0.12775	0.12777
[0.4, 0.6]	[min, max]	[0.15349, 0.16092]	[0.15326, 0.15944]	[0.15676, 0.15857]
	range	0.00743	0.00618	0.00181
	mean	0.15792	0.15771	0.15771

to be 3,000,000 and $s = 0.5$. From Table 2, we conclude that the approximation (5.1) performs very well in all cases and that it can be taken to be T_s-invariant. As expected, when $x_i = x_j$ the differences are the smallest ($<10^{-8}$ in all cases).

In the remaining of this section, we shall assume that $s \in (0, 1)$ has been fixed and $x_0 \in (0, 1)$ has been chosen so that the approximation (5.1) converges to μ_s.

Table 2 *Difference* $|\mu_n([a,b]; s, x_i) - \mu_n(T_s^{-1}([a,b]); s, x_j)|$ *for different values of* x_0 *and sets* $[a, b]$. *The truncation point was taken to be* $n = 3{,}000{,}000$ *and* $s = 0.5$. *The initial points are* $(x_1, \ldots, x_7) = (\pi, \pi/(\sqrt{2}+1), \pi\sqrt{2}, \pi+\sqrt{2}, \sqrt{7}, \pi+\sqrt{7}, \sqrt{11}+\sqrt{7}) (\mathrm{mod}\, 1)$

	initial	x_1	x_2	x_3	x_4	x_5	x_6	x_7
[0.05, 0.2]	x_1	0.00000	0.00019	0.00040	0.00008	0.00004	0.00062	0.00022
	x_2	0.00019	0.00000	0.00020	0.00027	0.00024	0.00043	0.00042
	x_3	0.00040	0.00000	0.00000	0.00047	0.00044	0.00022	0.00062
	x_4	0.00008	0.00030	0.00047	0.00000	0.00003	0.00070	0.00015
	x_5	0.00004	0.00020	0.00044	0.00003	0.00000	0.00066	0.00018
	x_6	0.00062	0.00043	0.00022	0.00070	0.00066	0.00000	0.00084
	x_7	0.00022	0.0004	0.00062	0.00015	0.00018	0.00084	0.00000
[0.3, 0.8]	x_1	0.00000	0.00019	0.00011	0.00009	0.00052	0.00036	0.00155
	x_2	0.00019	0.00000	0.00008	0.00028	0.00033	0.00016	0.00136
	x_3	0.00011	0.00008	0.00000	0.00020	0.00041	0.00024	0.00144
	x_4	0.00009	0.00028	0.00020	0.00000	0.00061	0.00045	0.00164
	x_5	0.00052	0.00033	0.00041	0.00061	0.00000	0.00016	0.00103
	x_6	0.00036	0.00016	0.00024	0.00045	0.00016	0.00000	0.00119
	x_7	0.00155	0.00136	0.00144	0.00164	0.00103	0.00119	0.00000
[0.7, 0.95]	x_1	0.00000	0.00011	0.00005	0.00012	0.00003	0.00012	0.00089
	x_2	0.00011	0.00000	0.00016	0.00022	0.00013	0.00022	0.00078
	x_3	0.00005	0.00016	0.00000	0.00006	0.00003	0.00006	0.00094
	x_4	0.00012	0.00022	0.00006	0.00000	0.00009	0.00000	0.00100
	x_5	0.00003	0.00013	0.00003	0.00009	0.00000	0.00009	0.00091
	x_6	0.00012	0.00022	0.00006	0.00000	0.00009	0.00000	0.00101
	x_7	0.00089	0.00078	0.00094	0.00100	0.00091	0.00101	0.00000

Since no confusion will arise, we shall drop s and x_0 from the notation and write the approximation (5.1), based on a size n iteration vector, just by $\mu_n(\cdot)$.

5.2 Approximating F_0^{-1} and the nodes of T_s^h

In order to approximate F_0^{-1}, one can use an empirical version based on the same iteration vector from which μ_n is derived. First, we need to define an approximation to F_0 from which an approximation to F_0^{-1} will be derived. Let $\widehat{F}_n$ be the empirical distribution based on a size n iteration vector $(x_0, T_s(x_0), \ldots, T_s^{n-1}(x_0))$ and let $x_1, \ldots, x_n$ be the jump points[2] of $\widehat{F}_n$. Consider the set $L_n := \{0 = x_0, x_1, \ldots, x_n, x_{n+1} = 1\}$. Given $x \in I \setminus L_n$, there exists a $k \in \{0, \ldots, n\}$ such that $x \in (x_k, x_{k+1})$. We define the approximate value of $F_0(x)$, denoted by $F_n(x)$, as the linear interpolation of x between the points $(x_k, \widehat{F}_n(x_k))$ and $(x_{k+1}, \widehat{F}_n(x_{k+1}))$, that is, we set

$$F_n(x) := \left(\frac{\widehat{F}_n(x_{k+1}) - \widehat{F}_n(x_k)}{x_{k+1} - x_k}\right)x + \frac{\widehat{F}_n(x_k)x_{k+1} - \widehat{F}_n(x_{k+1})x_k}{x_{k+1} - x_k}. \tag{5.2}$$

If $x \in L_n$, we simply define $F_n(x) := \widehat{F}_n(x)$. Notice that, for each n, $F_n : I \to I$ is a one-to-one, increasing and uniformly continuous function, so that its inverse, F_n^{-1}, is well defined and is also one-to-one and uniformly continuous. In the next proposition, we show that $F_n(x) \to F_0(x)$ and $F_n^{-1}(x) \to F_0^{-1}(x)$, both limits being uniform in x.

Proposition 5.1. *Let $\widehat{F}_n$ denote the empirical distribution based on an iteration vector $(x_0, T_s(x_0), \ldots, T_s^{n-1}(x_0))$ and let $x_1, \ldots, x_n$ be the jump points of $\widehat{F}_n$. Let F_n be the approximation (5.2) based on $\{x_1, \ldots, x_n\}$ and F_n^{-1} be its inverse. Then,*

$$F_n(x) \longrightarrow F_0(x) \quad \textit{and} \quad F_n^{-1}(x) \longrightarrow F_0^{-1}(x),$$

uniformly in x.

Proof. By the Glivenko–Cantelli theorem, $\widehat{F}_n(x) \to F_0(x)$ uniformly in $x \in [0, 1]$, so that, given $\varepsilon > 0$, one can find $n_0 := n_0(\varepsilon) > 0$ such that if $n > n_0$, then $|\widehat{F}_n(x) - F_0(x)| < \varepsilon$ uniformly in x. Now, for $x \in (0, 1)$ (if x equals 0 or 1, the result is trivial), there exists a $k \in \{1, \ldots, n\}$ such that $x \in [x_k, x_{k+1})$. Hence, if $n > n_0$

$$\begin{aligned}
|F_n(x) - F_0(x)| &\le |F_n(x) - \widehat{F}_n(x)| + |\widehat{F}_n(x) - F_0(x)| \\
&< |\widehat{F}_n(x_{k+1}) - \widehat{F}_n(x_k)| + \varepsilon \\
&\le \sup_{i=1,\ldots,n-1} \{|\widehat{F}_n(x_{i+1}) - \widehat{F}_n(x_i)|\} + \varepsilon \\
&\le \frac{1}{n} + \varepsilon,
\end{aligned}$$

[2] By the choice of x_0, there will be exactly n jump points.

uniformly in x. To show the convergence of the inverse, let $y \in [0,1]$ and $\varepsilon > 0$ be given and notice that F_n^{-1} being uniformly continuous, one can find a $\delta := \delta(\varepsilon) > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |F_n^{-1}(x) - F_n^{-1}(y)| < \varepsilon.$$

Now, since F_n converges uniformly to F_0, there exists $n_1 := n_1(\varepsilon) > 0$ such that,

$$n > n_1 \quad \Longrightarrow \quad |F_n(x) - F_0(x)| < \delta$$

for all $x \in I$. Also, since F_0 is one to one, there exists $v_0 \in [0,1]$ such that $y = F_0(v_0)$. Therefore, if $n > n_1$

$$|F_n^{-1}(y) - F_0^{-1}(y)| = |F_n^{-1}(F_0(v_0)) - v_0| = |F_n^{-1}(F_0(v_0)) - F_n^{-1}(F_n(v_0))| < \varepsilon$$

and since n_1 is independent of y, the desired convergence follows. □

As for the end points $\{a_{h,k}\}_{k=0}^{2^h}$ of the nodes of T_s^h, let $\{x_1, \ldots, x_m\} \in (0,1)$, $x_i \neq x_j$ and consider the set $\{T_s^h(x_1), \ldots, T_s^h(x_m)\}$, for $m > 0$ sufficiently large.[3] Note that $a_{h,0} = 0$ and $a_{h,2^h} = 1$, for any h. Let $D = \{i : T_s^h(x_i) > T_s^h(x_{i+1})\} \subset \{1, \ldots, m\}$. The set D contains the indexes $i \in \{1, \ldots, m\}$ for which the interval $[x_i, x_{i+1}]$ contains a discontinuity of T_s^h. Let $\{d_j\}_{j=1}^{2^h-1}$ denote the ordered elements of D, so that the interval $[x_{d_j}, x_{d_{j+1}}]$ contains the jth discontinuity of T_s^h. Now consider the function $T_{i,h;s}^* : [x_{d_i}, x_{d_i+1}] \to [0,2]$ given by $T_{i,h}^*(x;s) := T_s^{h-1}(x) + (T_s^{h-1}(x))^{1+s}$ and notice that we can write

$$T_s^h(x) = T_{i,h}^*(x;s) - \delta_{[1,2]}(T_{i,h}^*(x;s)).$$

Since there is a discontinuity of T_s^h in the interval $[x_{d_i}, x_{d_i+1}]$, we have $T_{i,h}^*(x_{d_i}; s) \leq 1$ and $T_{i,h}^*(x_{d_i+1}; s) \geq 1$ and since $T_{i,h}^*$ is continuous and increasing, there exists a point $x \in [x_{d_i}, x_{d_i+1}]$ such that $T_s^*(x;s) = 1$, which is precisely $a_{h,i}$. With this in mind, let $a_{h,i}^m$ denote the approximation to $a_{h,i}$ obtained from $\{x_1, \ldots, x_m\}$ by using a linear interpolation between the points $(x_{d_i}, T_{i,h}^*(x_{d_i}; s))$ and $(x_{d_i+1}, T_{i,h}^*(x_{d_i+1}; s))$. That is, $a_{h,i}^m$ is given by

$$a_{h,i}^m = x_{d_i} + \frac{x_{d_i+1} - x_{d_i}}{T_{i,h}^*(x_{d_i+1}; s) - T_{i,h}^*(x_{d_i}; s)}\left(1 - T_{i,h}^*(x_{d_i}; s)\right) \tag{5.3}$$

for all $d_i \in D$. Clearly $a_{h,i}^m \underset{m\to\infty}{\longrightarrow} a_{h,i}$, since $|x_{d_i+1} - x_{d_i}| \underset{m\to\infty}{\longrightarrow} 0$ and by the continuity of $T_{i,h}^*$, for each $i \in \{1, \ldots, 2^h - 1\}$.

[3] By "sufficiently large" we mean that m should be at least large enough to guarantee that the set $\{T_s^h(x_1), \ldots, T_s^h(x_m)\}$ reflects the $2^h - 1$ discontinuities of T_s^h, or, in other words, $m \geq 2^h$. The limits in m taken for an approximation are understood to be in terms of partitions, that is, we start with a sufficiently large set of points, say $I_m = \{x_1, \ldots, x_m\}$ and consider refinements of the form $I_{m+1} = I_m \cup \{x_{m+1}\}, \ldots, I_{m+k} = I_{m+k-1} \cup \{x_{m+k}\}$. Suppose that $R_m := R(I_m)$ is an approximation based on I_m. For a sequence of refinements $\{I_k\}_{k=m+1}^{\infty}$, we consider the sequence $\{R(I_k)\}_{k=m+1}^{\infty}$. Whenever the last limit exists, we set $\lim_{m\to\infty} R_m = \lim_{k\to\infty} R(I_k)$.

5.3 Approximating $\mathcal{T}_{h,k}$

Concerning the approximation of $\mathcal{T}_{h,k}$, we shall use an argument based on an empirical inverse and linear interpolation, but we shall also need a doubling argument in order to improve accuracy of the approximation near the discontinuities and guarantee the uniform convergence of the approximation to its target. So let $\{0 = x_1, \ldots, x_m = 1\} \in I$, $x_i < x_j$ and consider the set $\{T_s^h(x_1), \ldots, T_s^h(x_m)\}$, for $m > 0$ sufficiently large. Given $y \in [0, 1]$, recall that the inverse image of y by T_s^h is a size 2^h vector denoted by $(\mathcal{T}_{h,0}, \ldots, \mathcal{T}_{h,2^h-1})$. Let again $D = \{i : T_s^h(x_i) > T_s^h(x_{i+1})\} \subset \{1, \ldots, m\}$ and $\{d_i\}_{i=1}^{2^h-1}$ be the ordered points in D. Suppose that we know exactly or have good estimates for the nodes $\{a_{h,k}\}_{k=0}^{2^h}$ of T_s^h (for instance, we could use $\{a_{h,k}^m\}_{k=0}^{2^h}$, as described before, based on the same set $\{x_1, \ldots, x_m\}$ considered here). For $i = 0, \ldots, 2^h - 1$, let $p_i = d_{i+1} - d_i + 2$ and let

$$R_{h,i} = \{x_{h,i}^{(1)}, \ldots, x_{h,i}^{(p_i)}\} := \{a_{h,i}^m, x_{d_i+1}, \ldots, x_{d_{i+1}}, a_{h,i+1}^m\}$$

and

$$I_{h,i} = \{y_{h,i}^{(1)}, \ldots, y_{h,i}^{(p_i)}\} := \{0, T_s^h(x_{d_i+1}), \ldots, T_s^h(x_{d_{i+1}}), 1\}.$$

Given $y \in [0, 1]$, for each $i = 0, \ldots, 2^h - 1$, there exists a $y_{h,i}^{(k)} \in I_{h,i}$ such that $y \in [y_{h,i}^{(k)}, y_{h,i}^{(k+1)})$. We define the approximation $\mathcal{T}_{h,i}^m(y)$ of $\mathcal{T}_{h,i}(y)$, as being the linear interpolation of y between the points $(x_{h,i}^{(k)}, y_{h,i}^{(k)})$ and $(x_{h,i}^{(k+1)}, y_{h,i}^{(k+1)})$. That is, for each $i = 0, \ldots, 2^h - 1$,

$$\mathcal{T}_{h,i}^m(y) = x_{h,i}^{(k)} + \frac{x_{h,i}^{(k+1)} - x_{h,i}^{(k)}}{y_{h,i}^{(k+1)} - y_{h,i}^{(k)}}(y - y_{h,i}^{(k)}). \tag{5.4}$$

Notice that if y equals 0 or 1, we have $\mathcal{T}_{h,i}^m(y) = \mathcal{T}_{h,i}(y)$. Also, as the partition $\{x_1, \ldots, x_m\}$ increases, $|x_{k+1} - x_k| \underset{m\to\infty}{\longrightarrow} 0$ and the uniform continuity of T_s^h clearly implies $\mathcal{T}_{h,i}^m(y) \underset{m\to\infty}{\longrightarrow} \mathcal{T}_{h,i}(y)$, for each $y \in [0, 1]$, for $i = 0, \ldots, 2^h - 1$. More is true: the convergence is actually uniform in y, as we show in the next proposition.

Proposition 5.2. *Let $\mathcal{T}_{h,k}^m$ be the approximation of $\mathcal{T}_{h,k}$ given by (5.4) based on a partition R_m. Then,*

$$\mathcal{T}_{h,k}^m(y) \longrightarrow \mathcal{T}_{h,k}(y)$$

for each $k = 0, \ldots, 2^h - 1$, as m goes to infinity (i.e., as the partition gets thinner). Moreover, the convergence is uniform in $y \in [0, 1]$.

Proof. Given $\varepsilon > 0$, the uniform continuity of $\mathcal{T}_{h,k}$ implies the existence of a $\delta := \delta(\varepsilon) > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |\mathcal{T}_{h,k}(x) - \mathcal{T}_{h,k}(y)| < \varepsilon$$

for all $x \in [0,1]$. Let $R_0 = \{0 = x_1, \ldots, x_{m_0} = 1\} \in I$ for a sufficiently large $m_0 \in \mathbb{N}^*$ such that

$$\sup_{i=1,\ldots,m_0-1} \{|x_{i+1} - x_i|\} < \delta.$$

For $m > m_0$, let $R_m = \{x_1^*, \ldots, x_m^*\} \supset R_0$ be a size m refinement of R_0. Given $y \in (0,1)$, for each $i = 0, \ldots, 2^h - 1$, let $\mathcal{T}_{h,i}^m$ be the approximation (5.4) based on R_m. By construction and since $y \in (0,1)$, it follows that

$$\mathcal{T}_{h,i}\big(x_{h,i}^{(k)}\big) \le \mathcal{T}_{h,i}^m(y) < \mathcal{T}_{h,i}\big(x_{h,i}^{(k+1)}\big) \quad \text{and} \quad \mathcal{T}_{h,i}\big(x_{h,i}^{(k)}\big) \le \mathcal{T}_{h,i}(y) < \mathcal{T}_{h,i}\big(x_{h,i}^{(k+1)}\big),$$

so that

$$\begin{aligned} |\mathcal{T}_{h,i}^m(y) - \mathcal{T}_{h,i}(y)| &\le \big|\mathcal{T}_{h,i}\big(x_{h,i}^{(k+1)}\big) - \mathcal{T}_{h,i}\big(x_{h,i}^{(k)}\big)\big| \\ &\le \sup_{j=1,\ldots,m_0-1} \{|\mathcal{T}_{h,i}(x_{j+1}) - \mathcal{T}_{h,i}(x_j)|\} < \varepsilon \end{aligned}$$

for all $y \in (0,1)$. If $y \in \{0,1\}$, by construction $\mathcal{T}_{h,i}(y) = \mathcal{T}_{h,i}^m(y)$, so that the result follows uniformly for all $y \in [0,1]$, as desired. □

5.4 Approximating the lag h MP copula

With these approximations in hand, we can now define the approximation for the copula $C_{X_t, X_{t+h}}$ when φ is almost everywhere increasing given in Proposition 3.2 but in the form (3.5). For $(u,v) \in I^2$, $n > 0$ and $m \ge 2^h$, we set

$$\begin{aligned} C_{m,n}(u,v;h) = &\left(\sum_{k=0}^{n_0^*-1} \mu_n([a_{h,k}^m, \mathcal{T}_{h,k}^m(F_n^{-1}(v))])\right)\delta_{\mathbb{N}^*}(n_0^*) \\ &+ \mu_n([a_{h,n_0}^m, \min\{F_n^{-1}(u), \mathcal{T}_{h,n_0}^m(F_n^{-1}(v))\}]), \end{aligned} \tag{5.5}$$

where $n_0^* := n_0(m,n) = \{k : u \in [F_n(a_{h,k}^m), F_n(a_{h,k+1}^m))\}$ and $\lim_{m,n\to\infty} n_0^* = n_0$ since F_n converges uniformly to F_0 and $a_{h,k}^m$ converges to $a_{h,k}$. In the next theorem we establish the convergence of the approximation (5.5) to the true copula.

Theorem 5.1. *Let $C_{m,n}(u,v;h)$ be given by (5.5). Then, for all $(u,v) \in I^2$, $t \ge 0$ and $h > 0$*

$$\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n}(u,v;h) = \lim_{m\to\infty}\lim_{n\to\infty} C_{m,n}(u,v;h) = \lim_{m,n\to\infty} C_{m,n}(u,v;h)$$

and the common limit is $C_{X_t,X_{t+h}}(u,v)$ (given by (3.4)). Furthermore, the above limits are uniform in $(u,v) \in I^2$.

The proof of Theorem 5.1, is a consequence of the following stronger lemma.

Lemma 5.1. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures defined in I such that $\mu_n \xrightarrow{w} \mu$. Let $f_n : I \to I$ be a sequence of continuous functions converging uniformly to a function $f : I \to I$. Let $\{a_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers such that $a_m \in [0, 1]$ for all m and $a_m \to a$. Also let $g_m : [a_m, 1] \to I$ be a sequence of continuous functions converging uniformly to a function $g : I \to I$, $S_{m,n}(v) := [a_m, g_m(f_n(v))]$ and $S(v) := [a, g(f(v))]$. Then,*

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty}\mu_n(S_{m,n}(v)) &= \lim_{n\to\infty}\lim_{m\to\infty}\mu_n(S_{m,n}(v))\\ &= \lim_{m,n\to\infty}\mu_n(S_{m,n}(v)) = \mu(S(v))\end{aligned}$$

uniformly in $v \in I$.

Proof. For all $m, n > 0$ and $v \in [0, 1]$, let $S_{m,n}(v)$ and $S(v)$ be as in the enunciate and let

$$S_n(v) := [a, g(f_n(v))] \quad \text{and} \quad S_m(v) := [a_m, g_m(f(v))].$$

Notice that all sets just defined are μ-continuity sets for all m, n and v. Since the convergence of f_n to f is uniform, we have

$$\lim_{m,n\to\infty} g_m(f_n(v)) = \lim_{n\to\infty}\lim_{m\to\infty} g_m(f_n(v)) = \lim_{m\to\infty}\lim_{n\to\infty} g_m(f_n(v)) = g(f(v))$$

for all v, so that, both, the iterated and the double limits exist and $S_{m,n}(v) \to S(v)$, for all $v \in [0, 1]$. Also notice that we have $\delta_{S_{m,n}}(x) \le \delta_I(x)$ uniformly in m, n and x, and since μ_n converges weakly to μ and I is a μ-continuity set, it follows that

$$\int \delta_I(x)\,\mathrm{d}\mu_n \longrightarrow \int \delta_I(x)\,\mathrm{d}\mu.$$

Now, in one hand, since $S_{m,n}(v) \to S_m(v)$ for all v and $\delta_{S_{m,n}} \le \delta_I$, by the Lebesgue convergence theorem, it follows that

$$\mu_n(S_{m,n}(v)) = \int \delta_{S_{m,n}}(x)\,\mathrm{d}\mu_n \underset{n\to\infty}{\longrightarrow} \int \delta_{S_m}(x)\,\mathrm{d}\mu,$$

and, since $\delta_{S_m} \le \delta_I$ and $\int \delta_I\,\mathrm{d}\mu < \infty$, by the Lebesgue dominated theorem, we conclude that

$$\int \delta_{S_m}(x)\,\mathrm{d}\mu \underset{m\to\infty}{\longrightarrow} \int \delta_S(x)\,\mathrm{d}\mu = \mu(S(v)),$$

which shows that $\lim_{m\to\infty}\lim_{n\to\infty}\mu_n(S_{m,n}(v)) = \mu(S(v))$ and the convergence holds uniformly in $v \in (0, 1)$. On the other hand, since $\delta_{S_{m,n}} \le \delta_I$ and $\int \delta_I\,\mathrm{d}\mu_n < \infty$, by the Lebesgue dominated theorem, it follows that

$$\mu_n(S_{m,n}(v)) = \int \delta_{S_{m,n}}(x)\,\mathrm{d}\mu_n \underset{m\to\infty}{\longrightarrow} \int \delta_{S_n}(x)\,\mathrm{d}\mu_n,$$

and, since $\delta_{S_n} \leq \delta_I$ and $\int \delta_I \, \mathrm{d}\mu_n \to \int \delta_I \, \mathrm{d}\mu$, by the Lebesgue convergence theorem we conclude that,

$$\int \delta_{S_n}(x) \, \mathrm{d}\mu_n \underset{n\to\infty}{\longrightarrow} \int \delta_S(x) \, \mathrm{d}\mu = \mu(S(v)),$$

that is, $\lim_{n\to\infty} \lim_{m\to\infty} \mu_n(S_{m,n}(v)) = \mu(S(v))$, which also holds uniformly in v. Since the iterated limits are established, in order to finish the proof we need to show that the double limit exists and is equal to the iterated ones. Let $\varepsilon > 0$ be given. Since $\mu \ll \lambda$, the Radon–Nikodym theorem implies the existence of a non-negative continuous function h, which will be bounded since we are restricted to the interval I, such that, for any $A \in \mathcal{B}(I)$,

$$\mu(A) = \int_A h(x) \, \mathrm{d}\lambda \leq M\lambda(A),$$

where $M = \sup_{x\in I}\{h(x)\} < \infty$. Now, since $a_m \to a$, one can find $m_1 := m_1(\varepsilon) > 0$ such that, if $m > m_1$,

$$a_m \in K_1(\varepsilon) := \left[a - \frac{\varepsilon}{10M}, a + \frac{\varepsilon}{10M}\right]$$

and

$$\mu(K_1(\varepsilon)) \leq M\lambda\left(\left[a - \frac{\varepsilon}{10M}, a + \frac{\varepsilon}{10M}\right]\right) = \frac{\varepsilon}{5}.$$

The uniform convergence of g_m to g implies the existence of $m_2 := m_2(\varepsilon) > 0$ such that, if $m > m_2$, $|g_m(x) - g(x)| < \varepsilon/(20M)$, for all $x \in I$, or equivalently, taking $x = f_n(v)$, if $m > m_2$

$$g_m(f_n(v)) \in \left[g(f_n(v)) - \frac{\varepsilon}{20M}, g(f_n(v)) + \frac{\varepsilon}{20M}\right].$$

Now, the uniform continuity of g implies the existence of a $\delta := \delta(\varepsilon) > 0$ such that

$$|x - f_n(v)| < \delta \quad \Longrightarrow \quad |g(x) - g(f_n(v))| < \frac{\varepsilon}{20M}.$$

But since f_n converges to f uniformly, there exists a $n_1 = n_1(\delta) > 0$ such that

$$n > n_1 \quad \Longrightarrow \quad |f_n(v) - f(v)| < \delta$$

for all v so that, taking $x = f(v)$, for $n > n_1$, we have

$$g(f_n(v)) \in \left[g(f(v)) - \frac{\varepsilon}{20M}, g(f(v)) + \frac{\varepsilon}{20M}\right]$$

for all $v \in I$. Hence, if we take $m > m_2$ and $n > n_1$,

$$g(f_n(v)) - \frac{\varepsilon}{20M} \in \left[g(f(v)) - \frac{\varepsilon}{10M}, g(f(v))\right]$$

and

$$g(f_n(v)) + \frac{\varepsilon}{20M} \in \left[g(f(v)), g(f(v)) + \frac{\varepsilon}{10M} \right]$$

so that, setting

$$K_2(\varepsilon) := \left[g(f(v)) - \frac{\varepsilon}{10M}, g(f(v)) + \frac{\varepsilon}{10M} \right]$$

for $m > m_2$ and $n > n_1$, it follows that

$$g_m(f_n(v)) \in \left[g(f_n(v)) - \frac{\varepsilon}{20M}, g(f_n(v)) + \frac{\varepsilon}{20M} \right] \subseteq K_2(\varepsilon)$$

for all $v \in I$. Also observe that

$$\mu(K_2(\varepsilon)) \leq M\lambda\left(\left[g(f(v)) - \frac{\varepsilon}{10M}, g(f(v)) + \frac{\varepsilon}{10M} \right] \right) \leq \frac{\varepsilon}{5}.$$

The convergence of μ_n to μ implies the existence of $n_2 := n_2(\varepsilon) > 0$ such that if $n > n_2$ ($K_i(\varepsilon)$ is a μ-continuity set)

$$|\mu_n(K_i(\varepsilon)) - \mu(K_i(\varepsilon))| < \frac{\varepsilon}{5}$$

for $i = 1, 2$. Also, if we set $F_n(x) = \mu_n([0, x])$ and $F_0(x) = \mu([0, x])$, then F_0 is continuous (since $\mu \ll \lambda$), $F_n \to F_0$, and, by Pólya's theorem, there exists a $n_3 := n_3(\varepsilon) > 0$ such that, if $n > n_3$

$$\sup_{x \in I} \{|F_n(x) - F_0(x)|\} < \frac{\varepsilon}{10}.$$

Now, notice that, if $n > n_3$

$$\begin{aligned} |\mu_n(S(v)) - \mu(S(v))| &\leq |F_n(g(f(v))) - F_0(g(f(v)))| + |F_n(a) - F_0(a)| \\ &\leq 2 \sup_{x \in I} \{|F_n(x) - F_0(x)|\} < \frac{\varepsilon}{5} \end{aligned}$$

for all $v \in I$. Observe further that, by construction, if $m > \max\{m_1, m_2\}$ and $n > n_1$,

$$S_{m,n}(v) \setminus S(v) \subset K_1(\varepsilon) \cup K_2(\varepsilon)$$

for all v so that, setting $n_0 = n_0(\varepsilon) := \max\{m_1, m_2, n_1, n_2, n_3\}$, if $m, n > n_0$, we have

$$\begin{aligned} &|\mu_n(S_{m,n}(v)) - \mu(S(v))| \\ &\quad \leq |\mu_n(S_{m,n}(v)) - \mu_n(S(v))| + |\mu_n(S(v)) - \mu(S(v))| \\ &\quad < |\mu_n(K_1(\varepsilon)) + \mu_n(K_2(\varepsilon))| + \frac{\varepsilon}{5} \\ &\quad \leq |\mu_n(K_1(\varepsilon)) - \mu(K_1(\varepsilon))| + \mu(K_1(\varepsilon)) + \mu(K_2(\varepsilon)) \\ &\qquad + |\mu(K_2(\varepsilon)) - \mu_n(K_2(\varepsilon))| + \frac{\varepsilon}{5} \\ &\quad < \varepsilon \end{aligned}$$

for all v, which implies the existence of the double limit, equality with the iterated ones and the desired uniform convergence. □

Proof of Theorem 5.1. First notice that taking $f_n = F_n^{-1}$, $g_m = \mathcal{T}_{h,k}^m$, $a_m = a_{h,k}^m$, it follows from Lemma 5.1 that

$$\mu_n([a_{h,k}^m, \mathcal{T}_{h,k}^m(F_n^{-1}(v))]) \underset{m,n\to\infty}{\longrightarrow} \mu([a_{h,k}, \mathcal{T}_{h,k}(F_0^{-1}(v))])$$

for each $k = 0, \ldots, n_0 - 1$. It remains to show that

$$\lim_{m,n\to\infty} \mu_n([a_{h,n_0}^m, \min\{F_n^{-1}(u), \mathcal{T}_{h,n_0}^m(F_n^{-1}(v))\}])$$
$$= \mu([a_{h,n_0}, \min\{F_0^{-1}(u), \mathcal{T}_{h,n_0}(F_0^{-1}(v))\}]),$$

and that the iterated limits exist and are equal to the double limit. First, since we can write $\min\{u, v\} = \frac{u+v}{2} - \frac{|u-v|}{2}$, it is routine to show that if $f_n \to f$ uniformly, with f_n and f uniformly continuous and $g_m \to g$ uniformly, with g_m and g uniformly continuous, we have $\min\{f_n(u), g_m(f_n(v))\}$ converging uniformly to $\min\{f(u), g(f(v))\}$ in n, m, u and v. So, the problem simplifies to show that if $a_m \to a$, $g_{m,n}(u, v)$ is a sequence of functions such that $g_{m,n}(u, v) \to g(u, v)$ uniformly in u, v, n, m and $a_m \le g_{m,n}(u, v)$ for all u, v, n, m and $\mu_n \xrightarrow{w} \mu$, then

$$\lim_{m,n\to\infty} \mu_n([a_m, g_{m,n}(u, v)]) = \mu([a, g(u, v)]),$$

uniformly in u and v and the double limit above is equal to the iterated limits. A similar argument to the one used in Lemma 5.1 to establish the existence and equality of the iterated limits can be used to show the existence and equality of the iterated limits in this case. As for the double limit, let M be as in the proof of Lemma 5.1. By the uniform convergence of $g_{m,n}(u, v)$ to $g(u, v)$ and since $g_{m,n}$ and g are uniformly continuous for all m, n, it follows that there exists $m_1 := m_1(\varepsilon) > 0$, depending on ε only, such that, if $m, n > m_1$,

$$g_{m,n}(u, v) \in K(\varepsilon) := \left[g(u, v) - \frac{\varepsilon}{10M}, g(u, v) + \frac{\varepsilon}{10M}\right]$$

for all u and v and $\mu(K(\varepsilon)) \le \varepsilon/5$. The rest of the proof is carried out by mimicking the proof of Lemma 5.1 with the obvious adaptations. Identification of $g_{m,n}(u, v)$, $g(u, v)$, a_m and a with $\min\{F_n^{-1}(u), \mathcal{T}_{h,n_0}^m(F_n^{-1}(v))\}$, $\min\{F_0^{-1}(u), \mathcal{T}_{h,n_0}(F_0^{-1}(v))\}$, a_{h,n_0}^m and a_{h,n_0}, respectively, completes the proof. □

Remark 5.1. Notice that neither the convergence proved in Lemma 5.1 nor the one in Theorem 5.1 is uniform in m and n.

As for the case when φ is almost everywhere decreasing, we observe that, in view of (3.7), the function

$$C_{m,n}^*(u, v; h) = u + v - 1 + C_{m,n}(1 - u, 1 - v; h)$$

is an approximation to the copula in (3.6). Clearly $C^*_{m,n}$ converges to the true copula as m and n tends to infinity (view either as an iterated or a double limit) and the convergence is uniform in (u, v).

5.5 Implementation and random variate generation

The implementation of the approximations so far discussed is routine. All the approximations we mentioned can share the same iteration vector, which further improves the efficiency and precision of the task and greatly reduces the computational burden. In the top panel of Figure 3, we show the three dimensional plot of the lag 1 and 2 MP copula for values of $s \in \{0.1, 0.4\}$. The respective level plots are shown in the bottom panel of Figure 3. Notice the non-exchangeability of the copulas in all cases.

Obtaining random samples from an MP copulas is a trivial task in view of Proposition 3.4. There we show that the support of an MP copula is the union of graphs of certain linear functions. The following algorithm can be used to generate a pair of variates from a bidimensional MP copula for φ an almost everywhere increasing function.

1. Generate an uniform (0, 1) variate u.
2. Let κ_0 denote the index for which $u \in [F_0(a_{h,\kappa_0}), F_0(a_{h,\kappa_0+1})]$ and set $v = \ell^+_{h,\kappa_0}(u)$.
3. The desired pair is (u, v).

In practice the T_s-invariant probability measure is unknown and F_0 has to be approximated. Furthermore, most of times the nodes related to T^h_s, for $h > 0$, $s \in (0, 1)$ cannot be analytically obtained. However, we can apply the approximations developed in this section together with the algorithm above to obtain approximated samples from MP copulas. In Figure 4, we show 500 approximated sample points from a lag 1 and 2 MP copula for $s \in \{0.1, 0.4\}$ and φ an almost everywhere. Obvious modifications in the algorithm, allow handling the case where φ is an almost everywhere decreasing function.

Remark 5.2. For small values of the lag, the resemblance of the sample to a piecewise continuous function is very clear, but this is not always the case as it can be seen in Figure 5, where we show 500 approximated sample points of the lag 4, 5 and 7 MP copulas for $s = 0.2$. This is a general principle, for a fixed sample size the higher the lag, the harder to distinguish the support of the copula based on the sample, since the number of branches of T^h_s grow as fast as 2^h. For instance, for $h = 7$ in Figure 5 is difficult to say that the sample came from a singular copula at all.

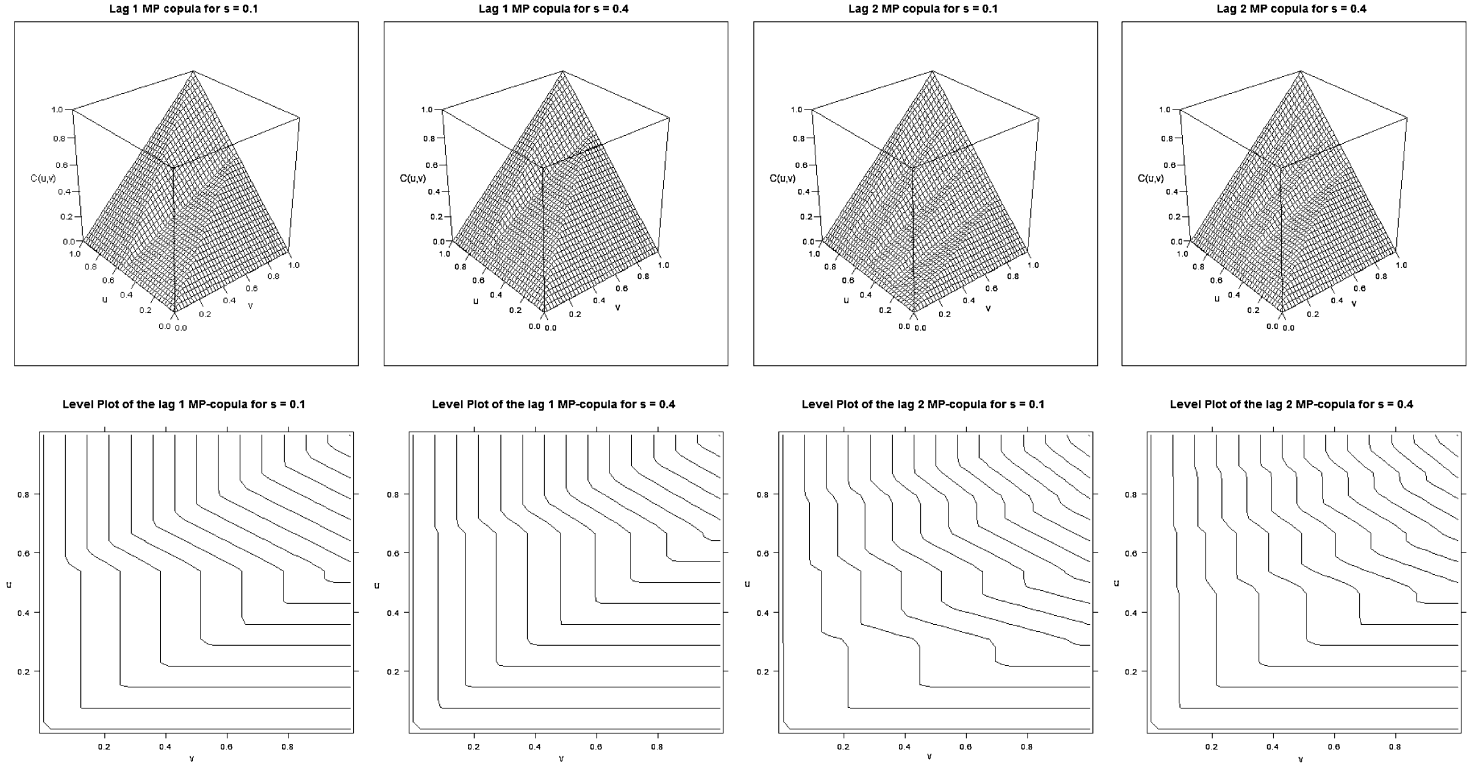

Figure 3 *From left to right, three dimensional plots of the lag* 1 *MP copula for* $s \in \{0.1, 0.4\}$ *and lag* 2 *MP copula for the same parameters* (*top panel*) *and respective level sets* (*bottom panel*) *obtained from approximation* (5.5).

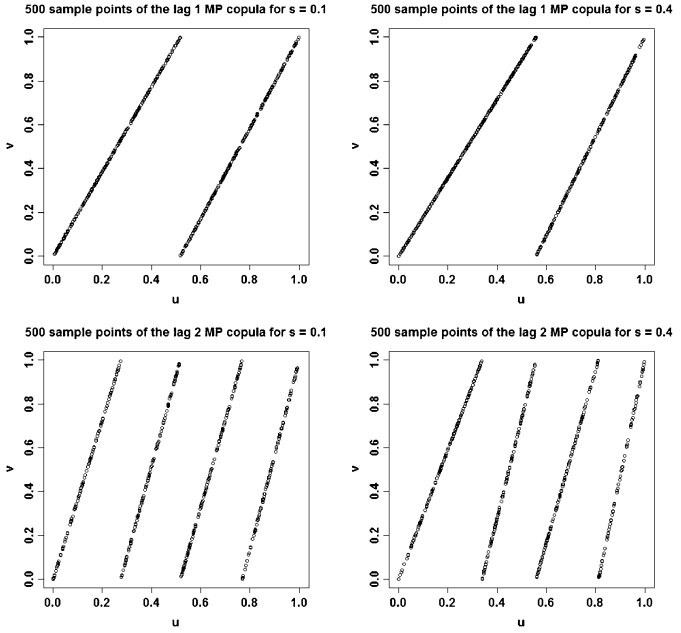

Figure 4 *From left to right*: 500 *approximated sample points from a lag* 1 *MP copula for* $s \in \{0.1, 0.4\}$ (*top panel*) *and lag* 2 *MP copula for the same parameters* (*bottom panel*).

6 Application

In this section, we apply the theory developed in Section 3 to the problem of estimating the parameter s in MP processes. This problem have been studied before in Olbermann et al. (2007), where the authors adapt and apply several estimation methods from the classical theory of long-range dependence to the problem of estimating the parameter s. In this section, we propose an estimator for the parameter s based on the ideas developed in Section 3, which is both, precise and fast.

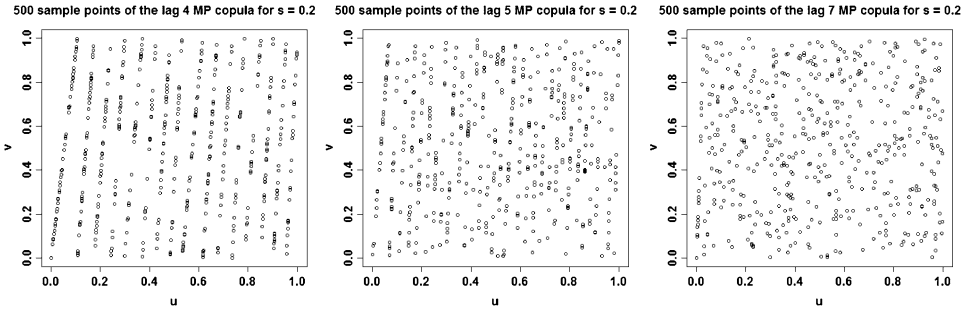

Figure 5 *Left to right*: 500 *approximated sample points from the lag* 4, 5 *and* 7 *MP copulas for* $s = 0.2$.

The mathematical framework is as follows. Let $s \in (0, 1)$ and consider the associated MP process $\{X_n\}_{n\in\mathbb{N}}$ for φ the identity map. Suppose we observe a realization $x_1, \ldots, x_N$ from X_n and our goal is to estimate the unknown parameter s. Let $a := a(s) \in (\frac{1}{2}, \frac{\sqrt{5}-1}{2})$ denote the discontinuity point of the MP transformation and notice that s and a are related by

$$a + a^{1+s} = 1 \quad \Longleftrightarrow \quad s = \frac{\log(1-a)}{\log(a)} - 1.$$

Hence, the problem of estimating s is equivalent to the problem of estimating a.

To define the proposed estimator, we start by observing that Proposition 3.4 for $h = 1$ implies that the lag 1 MP copula's support is given by the graph of the piecewise linear function

$$\ell(x) := \begin{cases} \dfrac{x}{F_0(a)}, & \text{if } x \in [0, F_0(a)), \\ \dfrac{x - F_0(a)}{1 - F_0(a)}, & \text{if } x \in [F_0(a), 1], \end{cases}$$

so that, any (independent or correlated) sample from a lag 1 MP copula consists of points scattered through the lines defined by ℓ (see Figure 4). The discontinuity point of the function ℓ is precisely $F_0(a)$. Let $y_i = F_0(x_i)$, for $i = 1, \ldots, N$, and consider the series $\{u_i := (y_i, y_{i+1})\}_{i=1}^{N-1}$. By Sklar's theorem, $\{u_i\}_{i=1}^{N-1}$ is a (correlated) sample from the lag 1 MP copula, so all points should lie in the graph of the function ℓ.

These considerations suggest the following procedure to obtain s based on a path $x_1, \ldots, x_N$ of X_n within a given accuracy $\varepsilon > 0$. We choose $s_0 \in (0, 1)$ as an initial guess for s and calculate $\hat{y}_i = F_n(x_i; s_0)$, $i = 1, \ldots, N$, where F_n is the approximation of F_0 given in (5.2). Next, we define $\{\hat{u}_i := (\hat{y}_i, \hat{y}_{i+1})\}_{i=1}^{N-1}$, from which we estimate the slope of the two branches of the approximated sample from the lag 1 MP copula obtained this way. The discontinuity point (and hence s) can then be easily calculated. In this manner, we obtain an estimative $\tilde{s}$ which can be compared to s_0. If s_0 is close to the true value s, then the difference between $\tilde{s}$ and s_0 should be small. If not, we choose another starting value and repeat the operation until obtain the desired accuracy. This leads to an optimization procedure to obtain s within a predefined accuracy.

To illustrate the procedure, Figure 6(a) shows a sample path of an MP process for $s = 0.2$, with $N = 200$ while Figure 6(b) shows the sample path $y_i = F_n(x_i; 0.2)$, $i = 1, \ldots, N$. From $\{y_i\}_{i=1}^{N}$, we construct the sequence $\{u_i\}_{i=1}^{N-1}$, where $u_i = (F_n(y_i; s), F_n(y_{i+1}; s))$, $i = 1, \ldots, N-1$, for the correctly specified $s = 0.2$ and for $s = 0.3$. Figure 6(c) presents the graph of $\{u_i\}_{i=1}^{N-1}$ obtained from the correct specification of s, while Figure 6(d) shows the graph of the misspecified one. In Figures 6(c) and 6(d), the solid lines represent the respective theoretical support of the copula given in Proposition 3.4. Some distortion in the points can

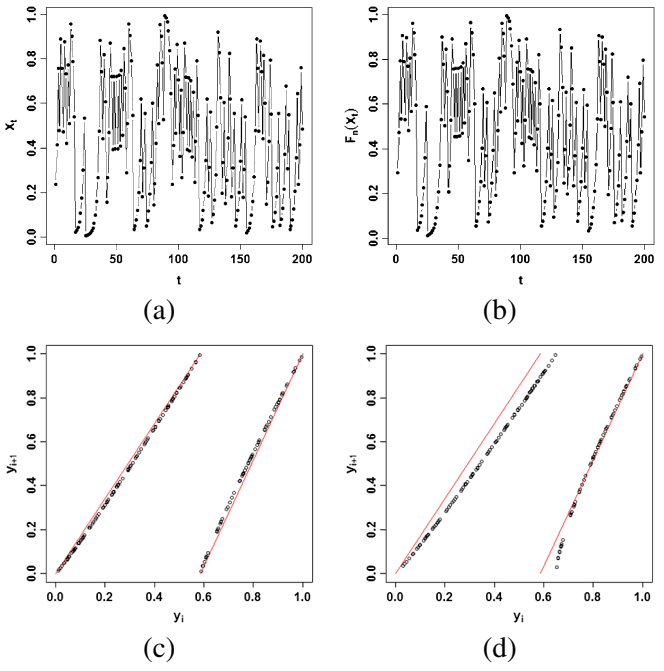

Figure 6 (a) *Sample path* $x_1, \ldots, x_{200}$ *of an MP process with* $s = 0.2$ *starting at* $\sqrt{5} \pmod 1$. (b) *Sample path* $y_i = F_n(x_i)$. *Plot of* $u_i = (y_i, y_{i+1})$ *for the* (c) *correct and* (d) *misspecified* s. *The solid lines correspond to the theoretical support of the respective lag* 1 *MP copula.*

be seen given to the use of the approximation F_n instead of the theoretical F_0, especially in lower quantiles. From Figure 6(d), it is clear that the line obtained from the sequence $\{u_i\}_{i=1}^{N-1}$ and the theoretical one for the chosen value of s_0, namely, 0.3, do not match, while for the correct specified one in Figure 6(c), they do.

The procedure just outlined is, however, computationally expensive given the fact that to calculate the approximation F_n with reasonable stability and accuracy, for each s, it requires the construction of an iteration vector of large size (see Figure 2 and Table 1). Such an optimization procedure can easily take hundreds of evaluations, depending on the desired accuracy, and hence, can be a very time consuming task.

To overcome this difficulty, observe that in Figures 6(a) and 6(b), little differences can be seen between them. In fact, since F_0 is a smooth distribution, an alternative is to apply the previous argument to the points $\hat{v}_i := (x_i, x_{i+1})$, $i = 1, \ldots, N$. There will certainly be some distortion in the lines due to the absence of F_0, but we expect to be able to estimate the discontinuity point a based on v_i by similar idea as before.

As an illustration, Figure 7 shows the plots of $v_i = (x_i, x_{i+1})$, for $i = 1, \ldots, 199$, based on MP processes with $s \in \{0.2, 0.4, 0.6, 0.8\}$ all starting at $\sqrt{5} \pmod 1$. The solid lines correspond to the lines joining the points $(0, 0)$ and $(a, 1)$ and joining

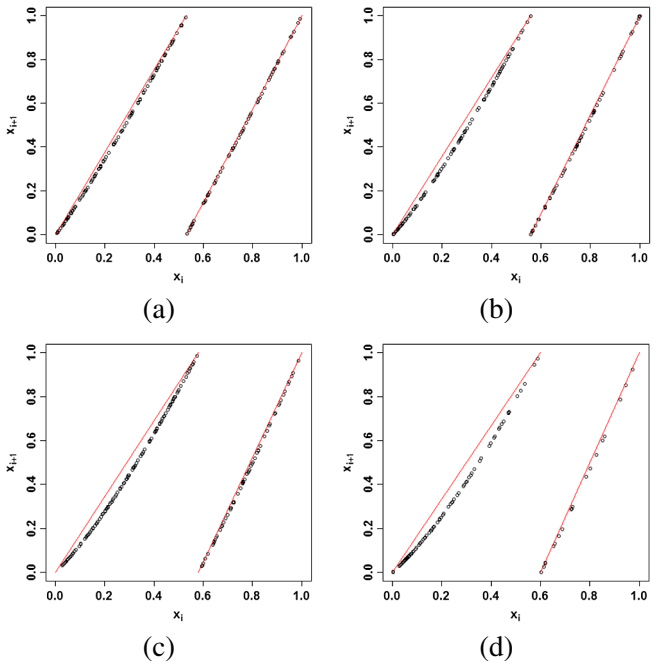

Figure 7 *Plot of* $v_i = (x_i, x_{i+1})$, $i = 1, \ldots, 199$ *from a sample path of an MP process with* (a) $s = 0.2$, (b) $s = 0.4$, (c) $s = 0.6$ *and* (d) $s = 0.8$. *The solid lines correspond to the lines joining the points* $(0, 0)$ *and* $(a, 1)$ *and joining* $(a, 0)$ *and* $(1, 1)$, *where a denotes the correct discontinuity point of the respective MP transformation.*

$(a, 0)$ and $(1, 1)$, where a denotes the correct discontinuity point of the respective MP transformation. From the graphs in Figure 7, we see the identification of the line based on v_i with the correct line, especially in the second branch of the graph. That is so because $a \in (\frac{1}{2}, \frac{\sqrt{5}-1}{2})$, so that the second branch, being smaller, is less affected by the distortion due to the absence of F_0.

In order to assess the performance of the estimation procedure, we perform the following experiment. We randomly select 100 initial points[4] in $(0, 1)$ and for each initial point we generate a path (of size $N = 200$) of an MP process for $s \in \{0.1, 0.15, \ldots, 0.95\}$. For each path, say $x_1, \ldots, x_{200}$, we perform the proposed estimation procedure. In order to estimate a, we applied two methods: the first one is a simple least squares method applied to the points lying in the second branch of (x_i, x_{i+1}). The second method is the following: let (x_{m_0}, x_{m_0+1}) and (x_{m_1}, x_{m_1+1}) denote the points among the ones lying on the second branch of $\{(x_i, x_{i+1})\}_{i=1}^{N-1}$ for which x_{m_0} is minimum and x_{m_1} is maximum. We define the estimator of a, say

[4]Tables with the initial values applied in our experiments and the complete simulation results are available upon request.

$\hat{a}$, as

$$\hat{a} = -\frac{B}{A}, \qquad \text{where } A := \frac{x_{m_1+1} - x_{m_0+1}}{x_{m_1} - x_{m_0}} \text{ and } B := x_{m_0+1} - A x_{m_0}. \tag{6.1}$$

For reference, in the subsequent we shall call this the *min–max procedure*. Geometrically, $\hat{a}$ is the inverse image of 0 by the linear function joining (x_{m_0}, x_{m_0+1}) and (x_{m_1}, x_{m_1+1}).

Table 3 summarizes the experiment results by presenting the mean, range, standard deviation (st.d.) and mean square error (mse) of the results. Figures 8(a) and 8(b) present graphically the results for both methods for $s \in \{0.1, 0.5, 0.9\}$ while in Figures 8(c) and 8(d), the histogram of the results for $s = 0.5$ are presented. From Table 3 and Figure 8, we see that the min–max procedure (MM) outperforms the least squares estimates (LS) obtained. Some bias can be seen for both estimates, especially when s increases.

The min–max procedure can be carried out even for time series of sample size as small as 20, as long as the second branch of $\{(x_i, x_{i+1})\}_{i=1}^{N-1}$ contains at least 2 points, which does not always happen (for instance, for $N = 110$, a sample path of an MP process with $s = 0.8$ starting at $\sqrt{74} (\mathrm{mod}\, 1)$ has only one point in the second branch). In such a situation, a straightforward adaptation of the min–max procedure can be applied to the first branch and still yields reasonable estimates. The closer to 0 and 1 the points x_{m_0} and x_{m_1} in (6.1) are, respectively, the better the estimation performance.

7 Conclusions

In this work, we derive the copulas related to Manneville–Pomeau processes for almost everywhere monotonic functions φ. In the bidimensional case, we find that the copulas of any random pair (X_t, X_{t+h}) depend only on the lag h and are singular. The support of the copulas is derived as well.

As for the multidimensional case, when φ is increasing almost everywhere, the functional form of the copulas are very similar to the ones derived in the bidimensional case. We conclude that the copulas of vectors $(X_{t_1}, \ldots, X_{t_n})$ and $(U_0, T_s^{t_2-t_1}(U_0), \ldots, T_s^{t_n-t_1}(U_0))$ are the same. When φ is decreasing almost everywhere, we find that the copulas of an n-dimensional random vector from an MP process can be deduced from the ones derived for the increasing case.

The copulas derived here depend on the T_s-invariant measure μ_s which has no explicit formula. For the bidimensional case, we propose an approximation to the copula which is shown to converge uniformly to the true copula. From this approximation, we are able to present plots of the copulas for different parameters and lags and to present a simple algorithm to generate approximated samples from the copulas. Some simple numerical calculation are presented to test the steps of the approximation. To illustrate the usefulness of the theory, we derive a fast estimation procedure of the underlying parameter s in Manneville–Pomeau processes.

Table 3 *Summary statistics of the experiment results. Presented are the mean estimate ($\hat{s}$), the range, the standard deviation (st.d.) and the mean square error values (mse) of the estimates. The min–max procedure is denoted by MM while LS denotes the least squares*

Proc.	s	$\hat{s}$	range	st.d.	mse	s	$\hat{s}$	range	st.d.	mse
MM	0.10	0.1008	[0.1000, 0.1024]	0.0006	0*	0.55	0.5581	[0.5500, 0.5888]	0.0069	0.0001
LS		0.1087	[0.1056, 0.1128]	0.0017	0.0001		0.6036	[0.5501, 0.6287]	0.0142	0.0031
MM	0.15	0.1516	[0.1500, 0.1597]	0.0015	0*	0.60	0.6091	[0.6000, 0.6451]	0.0090	0.0002
LS		0.1632	[0.1573, 0.1710]	0.0026	0.0002		0.6545	[0.6012, 0.7023]	0.0186	0.0033
MM	0.20	0.2023	[0.2001, 0.2101]	0.0021	0*	0.65	0.6600	[0.6501, 0.6927]	0.0087	0.0002
LS		0.2179	[0.2098, 0.2315]	0.0038	0.0003		0.7089	[0.6579, 0.7584]	0.0197	0.0039
MM	0.25	0.2534	[0.2501, 0.2632]	0.0027	0*	0.70	0.7125	[0.7001, 0.7726]	0.0119	0.0003
LS		0.2730	[0.2636, 0.2875]	0.0049	0.0006		0.7646	[0.7038, 0.8170]	0.0226	0.0047
MM	0.30	0.3036	[0.3000, 0.3128]	0.0028	0*	0.75	0.7621	[0.7502, 0.8019]	0.0110	0.0003
LS		0.3272	[0.3128, 0.3410]	0.0052	0.0008		0.8177	[0.7505, 0.8612]	0.0246	0.0052
MM	0.35	0.3544	[0.3500, 0.3669]	0.0039	0*	0.80	0.8165	[0.8001, 0.8659]	0.0131	0.0004
LS		0.3835	[0.3550, 0.4024]	0.0078	0.0012		0.8781	[0.8005, 0.9449]	0.0277	0.0069
MM	0.40	0.4050	[0.4000, 0.4214]	0.0049	0*	0.85	0.8677	[0.8500, 0.9428]	0.0151	0.0005
LS		0.4367	[0.4082, 0.4570]	0.0078	0.0014		0.9307	[0.8507, 1.0111]	0.0297	0.0074
MM	0.45	0.4556	[0.4501, 0.4703]	0.0046	0.0001	0.90	0.9172	[0.9002, 0.9704]	0.0141	0.0005
LS		0.4909	[0.4702, 0.5174]	0.0102	0.0018		0.9774	[0.9011, 1.0477]	0.0306	0.0069
MM	0.50	0.5065	[0.5001, 0.5189]	0.0051	0.0001	0.95	0.9706	[0.9500, 1.0517]	0.0189	0.0008
LS		0.5475	[0.5059, 0.5746]	0.0111	0.0024		1.0371	[0.9500, 1.1641]	0.0384	0.0090

Note: 0* means that the mse is smaller than 5×10^{-5}.

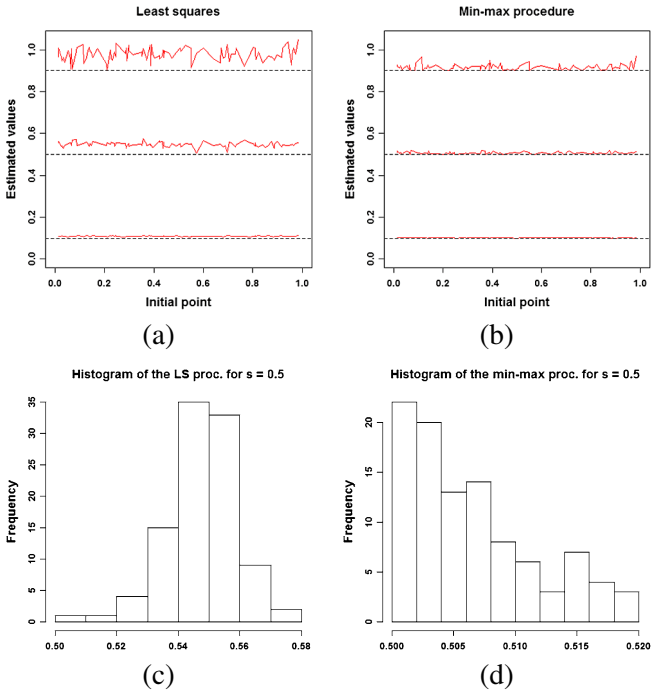

Figure 8 *Plot of the estimated values for* $s \in \{0.1, 0.5, 0.9\}$ *for* 100 *random initial points by using* (a) *the least squares procedure and* (b) *the min–max procedure. The dashed lines correspond to the correct value of* s*. Also shown the histogram of the estimated values for* $s = 0.5$ *by using* (c) *the least squares procedure and* (d) *the min–max procedure.*

Acknowledgments

Sílvia R. C. Lopes research was partially supported by CNPq-Brazil, by CAPES-Brazil, by INCT *em Matemática* and also by Pronex *Probabilidade e Processos Estocásticos* - E-26/170.008/2008 -APQ1. Guilherme Pumi was partially supported by CAPES/Fulbright Grant BEX 2910/06-3 and by CNPq-Brazil.

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