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RANDOM WALKS IN DYNAMIC RANDOM ENVIRONMENTS: A TRANSFERENCE PRINCIPLE[1]

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We study a general class of random walks driven by a uniquely ergodic Markovian environment. Under a coupling condition on the environment we obtain strong ergodicity properties for the environment as seen from the position of the walker, that is, the environment process. We can transfer the rate of mixing in time of the environment to the rate of mixing of the environment process with a loss of at most polynomial order. Therefore the method is applicable to environments with sufficiently fast polynomial mixing. We obtain unique ergodicity of the environment process. Moreover, the unique invariant measure of the environment process depends continuously on the jump rates of the walker.

As a consequence we obtain the law of large numbers and a central limit theorem with nondegenerate variance for the position of the walk.

1. Introduction. In recent days random walks in dynamic random environments have been studied by several authors. Motivation comes among others from nonequilibrium statistical mechanics—derivation of Fourier law—[Dolgopyat and Liverani (2008)] and large deviation theory [Rassoul-Agha, Seppäläinen and Yilmaz (2013)]. In principle, random walks in dynamic random environments contain, as a particular case, a random walk in a static random environment. However, mostly, in turning to dynamic environments, authors concentrate more on environments with sufficient mixing properties. In that case the fact that the environment is dynamic helps to obtain self-averaging properties that ensure standard limiting behavior of the walk, that is, the law of large numbers and the central limit theorem.

In the study of the limiting behavior of the walker, the environment process, that is, the environment as seen from the position of the walker plays a crucial role. See also Joseph and Rassoul-Agha (2011), Rassoul-Agha (2003) for the use of the environment process in related context. In a translation invariant setting, the environment process is a Markov process and its ergodic properties fully determine corresponding ergodic properties of the walk, since the position of the walker equals an additive function of the environment process plus a controllable martingale.

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The main theme of this paper is the following natural question: if the environment is uniquely ergodic, with a sufficient speed of mixing, then the environment process shares similar properties. In several works [Boldrighini et al. (1992), Bandyopadhyay and Zeitouni (2006), Boldrigini, Minlos and Pellegrinotti (2007), Avena, den Hollander and Redig (2011)] this transfer of "good properties of the environment" to "similar properties of the environment process" is made via a perturbative argument, and therefore holds only in a regime where the environment and the walker are weakly coupled. Some nonperturbative results also exist, but those require strong mixing properties of the environment in space and time [Dolgopyat, Keller and Liverani (2008), Dolgopyat and Liverani (2009), Bricmont and Kupiainen (2009)].

In this paper we consider the context of general Markovian uniquely ergodic environments, which are such that the semigroup contracts at a minimal speed in a norm of variation type. Examples of such environments include interacting particle systems in "the $M < \varepsilon$ regime" [Liggett (1985)] and weakly interacting diffusion processes on a compact manifold. Our conditions on the environment are formulated in the language of coupling. More precisely, we impose that for the environment there exists a coupling such that the distance between every pair of initial configurations in this coupling decays fast enough so that multiplied with t^d it is still integrable in time. As a result we then obtain that for the environment process there exists a coupling such that the distance between every pair of initial configurations in this coupling decays at a speed which is at least integrable in time. In fact we show more, namely in going from the environment to the environment process, we essentially loose a factor t^d in the rate of decay to equilibrium. For example, if for the environment there is a coupling where the distance decays exponentially, then the same holds for the environment process (with possibly another rate).

Once we have controllable coupling properties of the environment process, we can draw strong conclusions for the position of the walker, for example, a law of large numbers with an asymptotic speed that depends continuously on the rates, and a central limit theorem. We also prove recurrence in $d = 1$ under condition of zero speed.

Our paper is organized as follows. The model and necessary notation are introduced in Section 2. Section 3 is dedicated to lift properties of the environment to the environment process. The focus is on Theorem 3.1 and its refinements. Based on these results consequences for the walker are summarized in Section 3.5. In Section 4 we give examples for environments to which the results are applicable and present one example which has polynomial mixing in space and time. Section 5 is devoted to proofs.

2. The model.

2.1. *Environment.* A random walk in dynamic random environment is a process $(X_t)_{t\geq 0}$ on the lattice $\mathbb{Z}^d$ which is driven by a second process $(\eta_t)_{t\geq 0}$ on $E^{\mathbb{Z}^d}$,

the (dynamic) environment. This is interpreted as a random walk moving through the environment, with time-dependent transition rates being determined by the local environment around the random walk.

To become more precise, the *environment* $(\eta_t)_{t\geq 0}$ we assume to be a Feller process on the state space $\Omega := E^{\mathbb{Z}^d}$, where (E, ρ) is a compact metric space (examples in mind are $E = \{0, 1\}$ or $E = [0, 1]$). We assume (without loss of generality) that the distance ρ on E is bounded from above by 1. The generator of the Markov process $(\eta_t)_{t\geq 0}$ is denoted by L^E and its semigroup by S_t^E, both considered on the space of continuous functions $\mathcal{C}(\Omega; \mathbb{R})$. We assume that the environment is translation invariant, that is,

$$\mathbb{P}_\eta^E(\theta_x \eta_t \in \cdot) = \mathbb{P}_{\theta_x \eta}^E(\eta_t \in \cdot)$$

with θ_x denoting the shift operator $\theta_x \eta(y) = \eta(y - x)$ and $\mathbb{P}_\eta^E$ the path space measure of the process $(\eta_t)_{t\geq 0}$ starting from η.

2.2. *Lipschitz functions.* Denote, for $x \in \mathbb{Z}^d$,

$$(\Omega \times \Omega)_x := \big\{(\eta, \xi) \in \Omega^2 : \eta(x) \neq \xi(x) \text{ and } \eta(y) = \xi(y)\ \forall y \in \mathbb{Z}^d \setminus \{x\}\big\}, \qquad x \in \mathbb{Z}^d.$$

DEFINITION 2.1. For any $f : \Omega \to \mathbb{R}$, we denote by $\delta_f(x)$ the Lipschitz-constant of f w.r.t. the variable $\eta(x)$,

$$\delta_f(x) := \sup_{(\eta,\xi)\in(\Omega\times\Omega)_x} \frac{f(\eta) - f(\xi)}{\rho(\eta(x), \xi(x))}.$$

We write

$$|||f||| := \sum_{x\in\mathbb{Z}^d} \delta_f(x). \tag{1}$$

Note that $|||f||| < \infty$ implies that f is bounded and continuous, and the value of f is uniformly weakly dependent on sites far away. A weaker semi-norm we also use is the oscillation (semi)-norm

$$\|f\|_{\text{osc}} := \sup_{\eta,\xi\in\Omega} \big(f(\eta) - f(\xi)\big).$$

From telescoping over single site changes one sees $\|f\|_{\text{osc}} \leq |||f|||$.

2.3. *The random walker and assumption on rates.* The random walk X_t is a process on $\mathbb{Z}^d$, whose transition rates depend on the state of the environment as seen from the walker. More precisely, the rate to jump from site x to site $x + z$ given that the environment is in state η is $\alpha(\theta_{-x}\eta, z)$. We make two assumptions

on these jump rates α. First, we guarantee that the position of the walker X_t has a first moment by assuming

$$\|\alpha\|_1 := \sum_{z\in\mathbb{Z}^d} \|z\| \sup_{\eta\in\Omega} \big|\alpha(\eta, z)\big| < \infty. \tag{2}$$

More generally, as sometimes higher moments are necessary, we write

$$\|\alpha\|_p^p := \sum_{z\in\mathbb{Z}^d} \|z\|^p \sup_{\eta\in\Omega} \big|\alpha(\eta, z)\big|, \qquad p \geq 1.$$

Second, we limit the sensitivity of the rates to small changes in the environment by assuming that

$$|||\alpha||| := \sum_{z\in\mathbb{Z}^d} \big|\!\big|\!\big|\alpha(\cdot, z)\big|\!\big|\!\big| < \infty. \tag{3}$$

Finally, sometimes we will have to assume the stronger estimate

$$|||\alpha|||_1 := \sum_{z\in\mathbb{Z}^d} |z| \big|\!\big|\!\big|\alpha(\cdot, z)\big|\!\big|\!\big| < \infty. \tag{4}$$

2.4. *Environment process*. While the random walker X_t itself is not a Markov process due to the dependence on the environment, the pair (η_t, X_t) is a Markov process with generator

$$Lf(\eta, x) = L^E f(\cdot, x)(\eta) + \sum_{z\in\mathbb{Z}^d} \alpha(\theta_{-x}\eta, z)\big[f(\eta, x + z) - f(\eta, x)\big],$$

corresponding semigroup S_t (considered on the space of functions continuous in $\eta \in \Omega$ and Lipschitz continuous in $x \in \mathbb{Z}^d$) and path space measure $\mathbb{P}_{\eta,x}$.

The environment as seen from the walker is of crucial importance to understand the asymptotic behavior of the walker itself. This process, $(\theta_{-X_t}\eta_t)_{t\geq 0}$, is called the *environment process* (not to be confused with the environment η_t). It is a Markov process with generator

$$L^{\mathrm{EP}} f(\eta) = L^E f(\eta) + \sum_{z\in\mathbb{Z}^d} \alpha(\eta, z)\big[f(\theta_{-z}\eta) - f(\eta)\big],$$

corresponding semigroup S_t^{EP} [on $\mathcal{C}(\Omega)$] and path space measure $\mathbb{P}_\eta^{\mathrm{EP}}$. Notice that this process is meaningful only in the translation invariant context.

2.5. *Coupling of the environment*. In the remainder of the paper we will need a coupling $\widehat{\mathbb{P}}_{\eta,\xi}^E$, $\eta, \xi \in \Omega$, of the environment. For η, ξ the coupled pair $(\eta_t^1, \eta_t^2)_{t\geq 0}$ consists of two copies of the environment, started in η and ξ. By definition, a coupling has the marginals $\widehat{\mathbb{P}}_{\eta,\xi}^E(\eta^1 \in \cdot) = \mathbb{P}_\eta(\eta_t \in \cdot)$ and $\widehat{\mathbb{P}}_{\eta,\xi}^E(\eta^2 \in \cdot) = \mathbb{P}_\xi(\eta_t \in \cdot)$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the canonical filtration in the path space of coupled processes. We say such a coupling satisfies the marginal Markov property if, for any $f : \Omega \to \mathbb{R}$,

$$\widehat{\mathbb{E}}_{\eta,\xi}^E\big[f\big(\eta_t^i\big)|\mathcal{F}_s\big] = S_{t-s}^E f\big(\eta_s^i\big), \qquad i = 1, 2; t \geq s \geq 0. \tag{5}$$

We say it satisfies the strong marginal Markov property if, for any $f:\Omega\to\mathbb{R}$ and any stopping time τ,

$$\widehat{\mathbb{E}}^E_{\eta,\xi}\big[\mathbb{1}_{t\geq\tau} f(\eta^i_t)|\mathcal{F}_\tau\big] = \mathbb{1}_{t\geq\tau} S^E_{t-\tau} f(\eta^i_\tau), \qquad i=1,2. \tag{6}$$

Note that the (strong) Markov property for the coupling implies the (strong) marginal Markov property.

3. Ergodicity of the environment process.

3.1. *Assumptions on the environment*. In order to conclude results for the random walk, we need to have sufficient control on the environment. To this end we assume there exists a translation invariant coupling $\widehat{\mathbb{P}}^E_{\eta,\xi}$ of the environment, which satisfies the strong marginal Markov property (6). In this coupling we look at $\widehat{\mathbb{E}}^E_{\eta,\xi}\rho(\eta^1_t,\eta^2)$, measuring the distance of the states at the origin. If this decays sufficiently fast we will be able to obtain ergodicity properties of the environment.

ASSUMPTION 1a. The coupling $\widehat{\mathbb{P}}^E$ satisfies

$$\int_0^\infty t^d \sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta^1_t(0),\eta^2_t(0)\big)\,dt<\infty.$$

This assumption is already sufficient to obtain the law of large numbers for the position of the walker and unique ergodicity of the environment process, but it does not give quite enough control on local fluctuations. The following stronger assumption remedies that.

ASSUMPTION 1b. The coupling $\widehat{\mathbb{P}}^E$ satisfies

$$\int_0^\infty t^d \sum_{x\in\mathbb{Z}^d} \sup_{(\eta,\xi)\in(\Omega\times\Omega)_0} \widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta^1_t(x),\eta^2_t(x)\big)\,dt<\infty.$$

REMARK. Typically, a coupling which satisfies Assumption 1b also satisfies Assumption 1a. It is, however, not automatic. But given a translation invariant coupling $\widehat{\mathbb{P}}^E$ which satisfies Assumption 1b it is possible to construct from $\widehat{\mathbb{P}}^E$ a new coupling $\widetilde{\mathbb{P}}^E$ via a telescoping argument so that $\widetilde{\mathbb{P}}$ satisfies both Assumptions 1b and 1a.

In Section 4 we will discuss some examples which satisfy those assumptions. Beside natural examples where $\widehat{\mathbb{E}}^E_{\eta,\xi}\rho(\eta^1_t(0),\eta^2_t(0))$ decays exponentially fast, we give an example where other decay rates like polynomial decay are obtained.

3.2. *Statement of the main theorem.* The main result of this section is the following theorem, which tells us how the coupling property of the environment lifts to the environment process.

THEOREM 3.1. *Let* $f:\Omega\to\mathbb{R}$ *with* $|||f||| < \infty$.

(a) *Under Assumption* 1a, *there exists a constant* $C_a > 0$ *so that*

$$\sup_{\eta,\xi\in\Omega}\int_0^\infty |S_t^{\mathrm{EP}} f(\eta) - S_t^{\mathrm{EP}} f(\xi)|\,dt \le C_a |||f|||.$$

(b) *Under Assumption* 1b, *there exists a constant* $C_b > 0$ *so that*

$$\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\int_0^\infty |S_t^{\mathrm{EP}} f(\eta) - S_t^{\mathrm{EP}} f(\xi)|\,dt \le C_b |||f|||.$$

This theorem is the key to understanding the limiting behavior of the random walk, that is, the law of large numbers, as well as for the central limit theorem. Section 5 is devoted the proof of Theorem 3.1. In Section 3.4 we generalize this result to give more information about decay in time. Here we continue with results we can obtain using Theorem 3.1. Most results about the environment process just use part (a) of the theorem; part (b) shows how more sophisticated properties lift from the environment to the environment process as well. Those can be necessary to obtain more precise results on the walker, like how likely atypical excursions from the expected trajectory are.

It is possible to lift other properties from the environment to the environment process as well. For example, if Assumption 1b is modified to state

$$\int_0^\infty t^d \sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_0} \widehat{\mathbb{E}}^E_{\eta,\xi}\frac{\rho(\eta_t^1(x),\eta_t^2(x))}{\rho(\eta(0),\xi(0))}\,dt < \infty,$$

then that implies for the environment process

$$\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\int_0^\infty \frac{|S_t^{\mathrm{EP}} f(\eta) - S_t^{\mathrm{EP}} f(\xi)|}{\rho(\eta(x),\xi(x))}\,dt \le C_{b'}|||f|||.$$

This kind of condition can be relevant in the context of diffusive environments to show that small changes in the environment are causing only small changes in the environment process.

3.3. *Existence of a unique ergodic measure and continuity in the rates.* First, the environment process, that is, the environment as seen from the walker, is ergodic.

LEMMA 3.2. *Under Assumption* 1a *the environment process has a unique ergodic probability measure* μ^{EP}.

PROOF. As E is compact, so is Ω, and therefore the space of stationary measures is nonempty. So we must just prove uniqueness.

Assume μ, ν are both stationary measures. Choose an arbitrary $f : \Omega \to \mathbb{R}$ with $|||f||| < \infty$. By Theorem 3.1(a), for any $T > 0$,

$$T|\mu(f) - \nu(f)| \le \iiint_0^T |S_t^{\mathrm{EP}} f(\eta) - S_t^{\mathrm{EP}} f(\xi)|\, dt\, \mu(d\eta)\nu(d\xi)$$
$$\le \sup_{\eta,\xi\in\Omega} \int_0^\infty |S_t^{\mathrm{EP}} f(\eta) - S_t^{\mathrm{EP}} f(\xi)|\, dt < \infty.$$

As T is arbitrary, $\mu(f) = \nu(f)$. As functions f with $|||f||| < \infty$ are dense in $\mathcal{C}(\Omega)$, there is at most one stationary probability measure. □

It is of interest not only to know that the environment process has a unique ergodic measure μ^{EP}, but also to know how this measure depends on the rates α.

THEOREM 3.3. *Under Assumption 1a, the unique ergodic measure μ_α^{EP} depends continuously on the rates α. For two transition rate functions α, α', we have the following estimate*:

$$|\mu_\alpha^{\mathrm{EP}}(f) - \mu_{\alpha'}^{\mathrm{EP}}(f)| \le \frac{C(\alpha)}{p(\alpha)} \|\alpha - \alpha'\|_0 |||f|||,$$

that is,

$$(\alpha, f) \mapsto \mu_\alpha^{\mathrm{EP}}(f)$$

is continuous in $\|\cdot\|_0 \times |||\cdot|||$. The functions $C(\alpha)$, $p(\alpha)$ satisfy $C(\alpha) > 0$, $p(\alpha) \in]0, 1[$. In the case that the rates α do not depend on the environment, that is, $\alpha(\eta, z) = \alpha(z)$, they are given by $p(\alpha) = 1$,

$$C(\alpha) = \int_0^\infty \sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}_{\eta,\xi}^E \rho\big(\eta_t^1(0), \eta_t^2(0)\big)\, dt.$$

As the proof is a variation of the proof of Theorem 3.1, it is delayed to the end of Section 5.

3.4. *Speed of convergence to equilibrium in the environment process*. We already know that under Assumption 1a the environment process has a unique ergodic distribution. However, we do not know at what speed this process converges to its unique stationary measure. Given the speed of convergence for the environment it is natural to believe that the environment process inherits that speed with some form of slowdown due to the additional self-interaction which is induced from the random walk. For example, if the original speed of convergence were exponential, then the environment process would also converge exponentially fast. This is indeed the case.

THEOREM 3.4. *Let $\phi : [0, \infty[\to \mathbb{R}$ be a monotone increasing and continuous function satisfying $\phi(0) = 1$ and $\phi(s + t) \le \phi(s)\phi(t)$.*

(a) *Suppose the coupling $\widehat{\mathbb{P}}^E$ satisfies*

$$\int_0^\infty \phi(t) t^d \sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}^E_{\eta,\xi} \rho\big(\eta_t^1(0), \eta_t^2(0)\big)\, dt < \infty.$$

Then there exists a constant $K_0 > 0$ and a decreasing function $C_a :]K_0, \infty[\to [0, \infty[$ so that for any $K > K_0$ and any $f : \Omega \to \mathbb{R}$ with $|||f||| < \infty$,

$$\sup_{\eta,\xi\in\Omega} \int_0^\infty \phi\left(\frac{t}{K}\right) \big|S_t^{\mathrm{EP}} f(\eta) - S_t^{\mathrm{EP}} f(\xi)\big|\, dt \le C_a(K) |||f|||.$$

(b) *Suppose the coupling $\widehat{\mathbb{P}}^E$ satisfies*

$$\int_0^\infty \phi(t) t^d \sum_{x\in\mathbb{Z}^d} \sup_{(\eta,\xi)\in(\Omega\times\Omega)_0} \widehat{\mathbb{E}}^E_{\eta,\xi} \rho\big(\eta_t^1(x), \eta_t^2(x)\big)\, dt < \infty.$$

Then there exists a constant $K_0 > 0$ and a decreasing function $C_b :]K_0, \infty[\to [0, \infty[$ so that for any $K > K_0$ and any $f : \Omega \to \mathbb{R}$ with $|||f||| < \infty$,

$$\sum_{x\in\mathbb{Z}^d} \sup_{(\eta,\xi)\in(\Omega\times\Omega)_x} \int_0^\infty \phi\left(\frac{t}{K}\right) \big|S_t^{\mathrm{EP}} f(\eta) - S_t^{\mathrm{EP}} f(\xi)\big|\, dt \le C_b(K) |||f|||.$$

Canonical choices for ϕ are $\phi(t) = \exp(\beta t^\alpha)$, $0 < \alpha \le 1$ or $\phi(t) = (1 + t)^\beta$, $\beta > 0$. This leads to the following transfer of convergence speed to equilibrium from the environment to the environment process:

- exponential decay: $e^{-\lambda t} \longrightarrow e^{-\lambda t/(K_0+\varepsilon)}$,
- stretched exponential decay: $e^{-\lambda t^\alpha} \longrightarrow e^{-\lambda t^\alpha/(K_0+\varepsilon)^\alpha}$,
- polynomial decay: $t^{-\lambda} \longrightarrow t^{-(\lambda-d-\varepsilon)}$,

with $\varepsilon > 0$ arbitrary, and in the case of polynomial decay, $\lambda > d + 1$.

3.5. *Consequences for the walker.* The strong convergence of the environment process to its stationary measure obtained in Theorem 3.1 implies various facts for the random walker. The most basic fact is that the random walker has a limiting speed.

PROPOSITION 3.5. *For any $\eta \in \Omega$, $x \in \mathbb{Z}^d$,*

$$v := \lim_{t\to\infty} \frac{X_t}{t} = \int \sum_{z\in\mathbb{Z}^d} z\alpha(\eta, z)\mu^{\mathrm{EP}}(d\eta)$$

in L^1 and almost surely w.r.t. $\mathbb{P}_{\eta,x}$. The L^1-convergence is also uniform w.r.t. η for a given x.

The convergence under $\mathbb{P}_{\mu^{\mathrm{EP}},0}$ is a direct consequence of ergodicity. For the extension to $\mathbb{P}_{\eta,x}$ some ingredients of the proofs in Section 5 are required. Therefore the proof is situated at the end of Section 5.

In the following theorem we prove the functional central limit theorem for the position of the walker. The convergence to Brownian motion via martingales is a rather straightforward consequence of the ergodicity given by Theorem 3.1. The issue of nondegeneracy of the variance is less standard and hence we give a proof.

THEOREM 3.6. *Assume Assumption* 1a, $\|\alpha\|_2 < \infty$, $|||\alpha|||_1 < \infty$. *Then the scaling limit of the random walk is a Brownian motion with drift* v, *that is*,

$$\frac{X_{tT} - vtT}{\sqrt{T}} \underset{T\to\infty}{\longrightarrow} W_D(t),$$

where W_D *is a Brownian motion with covariance matrix* D.

Let $e \in \mathbb{R}^d$ *be a unit vector. Assume that either*:

(a) *there exists a* $z \in \mathbb{Z}^d$, $\langle e, z\rangle \neq 0$, *so that for all* $t > 0$ *and* $\eta \in \Omega$ *the probability* $\mathbb{P}_\eta(\alpha(\eta_t, z) > 0)$ *is positive*;

(b) $\mu^{\mathrm{EP}}(\alpha(\cdot, z)) > 0$ *for* $z \in \mathbb{Z}^d$ *with* $\langle e, z\rangle$ *arbitrary large.*

Then $\lim_{T\to\infty} \frac{1}{T} \operatorname{Var}(\langle X_T, e\rangle) > 0$. *In particular, if* (a) *or* (b) *is satisfied for all* e, *then the covariance matrix* D *is nondegenerate.*

PROOF. Notice that $\sum_{z\in\mathbb{Z}^d} z\alpha(\cdot, z) - v$ is in the domain of $(L^{\mathrm{EP}})^{-1}$ because of Theorem 3.1. Decompose

$$\begin{aligned} X_t - vt = &\left(X_t - \int_0^t \sum_{z\in\mathbb{Z}^d} z\alpha(\theta_{-X_s}\eta_s, z)\,ds\right) \\ &+ \left(\int_0^t \sum_{z\in\mathbb{Z}^d} z\big[\alpha(\theta_{-X_s}\eta_s, z) - \mu^{\mathrm{EP}}(\alpha(\cdot, z))\big]\,ds\right). \end{aligned}$$

The first term on the right-hand side is a martingale, and the second one is one as well, up to a uniformly bounded error. Both converge to Brownian motion with finite variance by standard arguments when $\|\alpha\|_2 < \infty$. However, as the two terms are not independent, an argument is needed to prove that they do not annihilate. To prove that we show that $\frac{1}{T}\operatorname{Var}(\langle X_T, e\rangle)$ is bounded away from 0 under the assumed conditions. Assume $T > 0$ integer, and let $(\mathcal{F}_t)_{t\geq 0}$ be the canonical filtration. Introduce the discrete-time martingale

$$M_n^T := \mathbb{E}\big[\langle X_T, e\rangle|\mathcal{F}_n\big] - \mathbb{E}\big[\langle X_T, e\rangle|\mathcal{F}_0\big] = X_n + \Psi_{T-n}(\theta_{-X_n}\eta_n) - \Psi_t(\eta_0),$$

$$\Psi_S(\eta) := \mathbb{E}_{0,\eta} \int_0^S \sum_{z\in\mathbb{Z}^d} \langle z, e\rangle\alpha(\theta_{-X_t}\eta_t, z)\,dt = \int_0^S S_t^{\mathrm{EP}}\phi(\eta)\,dt;$$

$$\phi(\eta) := \sum_{z\in\mathbb{Z}^d} \langle z, e\rangle\alpha(\eta, z).$$

With this, by stationarity of the environment process started from μ^{EP},

$$\begin{aligned}\operatorname{Var}_{\mu^{\mathrm{EP}}}\big(\langle X_T, e\rangle\big) &\geq \mathbb{E}_{\mu^{\mathrm{EP}}}\big(\langle X_T, e\rangle - \mathbb{E}\big[\langle X_T, e\rangle|\mathcal{F}_0\big]\big)^2 \\ &= \sum_{n=1}^{T} \mathbb{E}_{\mu^{\mathrm{EP}}}(M_n - M_{n-1})^2 \\ &= \sum_{n=1}^{T} \mathbb{E}_{\mu^{\mathrm{EP}}}\big(\langle X_n, e\rangle - \langle X_{n-1}, e\rangle + \Psi_{T-n}(\theta_{-X_n}\eta_n) \\ &\qquad - \Psi_{T-(n-1)}(\theta_{-X_{n-1}}\eta_{n-1})\big)^2 \\ &= \sum_{n=1}^{T} \mathbb{E}_{\mu^{\mathrm{EP}}}\big(\langle X_1, e\rangle + \Psi_{T-n}(\theta_{-X_1}\eta_1) - \Psi_{T-(n-1)}(\eta_0)\big)^2.\end{aligned}$$

What has to be shown is that the above term is not 0. By Theorem 3.1 and $\|\phi\|_\infty \leq |||\alpha|||_1 < \infty$,

$$\sup_{\eta,\xi\in\Omega}\sup_{T\geq 0}\big|\Psi_T(\xi) - \Psi_{T+1}(\eta)\big| =: C < \infty.$$

Therefore, using $|a+b| \geq ||a| - |b||$,

$$\operatorname{Var}_{\mu^{\mathrm{EP}}}\big(\langle X_T, e\rangle\big) \geq T\,\mathbb{E}_{\mu^{\mathrm{EP}}}\mathbb{1}_{|\langle X_1,e\rangle|>C}\big(\big|\langle X_1, e\rangle\big| - C\big)^2.$$

What remains to show is that $\mathbb{P}_{\mu^{\mathrm{EP}}}(|\langle X_1, e\rangle| > C) > 0$. If (b) is satisfied, this is immediate. If (a) is satisfied, then there is a positive probability that X_t performs sufficiently many jumps of size z (and no other jumps) up to time 1. □

REMARK. The convergence to Brownian motion with a nondegenerate variance also provides information about the recurrence behavior of the walker. If $v = 0$, supposing $d = 1$ (in higher dimensions, project onto a line), the limiting Brownian motion is centered. Hence there exists an infinite sequence $t_1 < t_2 < \cdots$ of times with $X_{t_{2n}} < 0$ and $X_{t_{2n+1}} > 0$, $n \in \mathbb{N}$. Supposing the walker has only jumps of size 1, it will traverse the origin between t_n, t_{n+1} for any $n \in \mathbb{N}$; that is, it is recurrent. (If the walker also has larger jumps, then one needs an argument to actually hit the origin with some positive probability in $[t_n, t_{n+1}]$.) Particularly, the recurrence implies that there exists no regime where the random walk is transient but with 0 speed.

4. Examples: Layered environments. There are many examples of environments which satisfy both Assumptions 1a and 1b. Naturally, exponential convergence to the ergodic measure is sufficient, independent of the dimension d. Therefore interacting particle systems in the so-called $M < \varepsilon$-regime or weakly interacting diffusions on a compact manifold belong to the environments to which this method is applicable.

To exploit the fact that only sufficient polynomial decay of correlations is required, we will construct a class of environments which we call *layered environments*. One can think of layered environments as a weighted superposition of a sequence of (independent) environments.

Those kind of environments are fairly natural objects to study. One area where they can appear is an idealization of molecular motors. In molecular motors the walker moves (e.g.) in a potential, where the potential randomly switches between various global states [e.g., related to chemical transitions in the example of kinesine; see Ambaye and Kehr (1999), Donato and Piatnitski (2005), Jarzynski and Mazonka (1999), Jülicher, Ajdari and Prost (1997), Magnasco (1994) for more motivation]. Here each layer is representing the interaction with the environment for one global state. In many realistic situations there are many such states. If the global state changes very quickly compared to the movement of the random walk, what is observed is a weighted superposition with weights given by the relative frequencies of the appearance of the individual global states.

Layered environments could also appear from a multi-scale analysis of a complicated environment, where the layers with a high index represent the long-range interactions. Besides, layered environments are a useful tool because they form a class of environments which are uniformly mixing with arbitrary mixing speed. There are plenty of examples where one has polynomial or stretched exponential mixing, for example, in the context of diffusion processes. However, those examples are not uniformly mixing, in the context of diffusion processes because of an unbounded state space.

Here we focus on layers which still have exponential decay of correlations, but each layer does converge to its stationary measure at a layer specific rate α_n, with n being the index of the layer. When α_n tends to 0 as $n \to \infty$ this introduces some form of arbitrary slow decay of correlations. We counterbalance this by weighting the superposition in such a way that the individual influence of a layer goes to 0 as well. Note that such a counterbalancing is only possible because of the Lipschitz nature of the assumptions. A uniform decay estimate does not hold because of the arbitrary slow decay in deep layers.

More formally, for each $n \in \mathbb{N}$ let $(\eta^n)_{t\geq 0}$ be a Markov process on $\Omega_0 := \{0,1\}^{\mathbb{Z}^d}$, the environment on layer n. This process should have a coupling $\widehat{\mathbb{P}}^n_{\eta,\xi}$ with

$$\sup_{\eta,\xi\in\Omega_0} \widehat{\mathbb{E}}^n_{\eta,\xi}\big|\eta_t^{n,1}(0) - \eta_t^{n,2}(0)\big| \leq 2e^{-\alpha_n t}, \qquad \alpha_n > 0. \tag{7}$$

The *layered environment* $(\eta_t)_{t\geq 0}$ then consists of the stack of independent layers $(\eta_t^n)_{t\geq 0}$. The single site state space is $E = \{0,1\}^{\mathbb{N}}$ and space of all configurations $\Omega = E^{\mathbb{Z}^d}$.

The superposition of the environments is weighted by the distance ρ on E, which we choose in the following way. Fix a sequence $\gamma_1 > \gamma_2 > \cdots > 0$ with

$\sum_{n\in\mathbb{N}} \gamma_n = 1$. For $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in E$ the distance is

$$\rho\big((a_n),(b_n)\big) := \sum_{n\in\mathbb{N}} \gamma_n |a_n - b_n|. \tag{8}$$

The coupling $\widehat{\mathbb{P}}^E$ of the layered environments is simply the independent coupling of the individual layer couplings $\widehat{\mathbb{P}}^n$. The layer decay (7) and the choice of distance (8) then provide the following decay of coupling distance for the layered environment:

$$\sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}^E_{\eta,\xi} \rho\big(\eta_t^1(0), \eta_t^2(0)\big) \le 2 \sum_{n\in\mathbb{N}} \gamma_n e^{-\alpha_n t}. \tag{9}$$

The sum on the right-hand side of (9) can have arbitrary slow decay depending on α_n, γ_n. For example, if one fixes $\alpha_n = n^{-1}$, then $\gamma_n = n^{-\gamma-1}$ leads to decay of order $t^{-\gamma}$.

We did not specify the exact nature of the individual layers, as those did not matter for the construction. A natural example is when individual layers consist of Ising model Glauber dynamics at inverse temperature $\beta_n < \beta_c$, and $\beta_n \to \beta_c$ as $n \to \infty$.

5. Proofs. In this section we always assume that Assumption 1a holds.

We start with an outline of the idea of the proofs. We have a coupling of the environments (η_t^1, η_t^2), which we extend to include two random walkers (X_t^1, X_t^2), driven by their corresponding environment. We maximize the probability of both walkers performing the same jumps. Then Assumption 1a is sufficient to obtain a positive probability of both walkers staying together forever. If the walkers stay together, one just has to account for the difference in environments, but not the walkers as well. When the walkers split, the translation invariance allows for everything to shift so that both walkers are back at the origin, and one can try again. After a geometric number of trials it is then guaranteed that the walkers stay together.

PROPOSITION 5.1 (Coupling construction). *Given the coupling $\widehat{\mathbb{P}}^E_{\eta,\xi}$ of the environments, we extend it to a coupling $\widehat{\mathbb{P}}_{\eta,x;\xi,y}$. This coupling has the following properties*:

(a) (*Marginals*) *The coupling supports two environments and corresponding random walkers*:
   (1) $\widehat{\mathbb{P}}_{\eta,x;\xi,y}((\eta_t^1, X_t^1) \in \cdot) = \mathbb{P}_{\eta,x}((\eta_t, X_t) \in \cdot)$;
   (2) $\widehat{\mathbb{P}}_{\eta,x;\xi,y}((\eta_t^2, X_t^2) \in \cdot) = \mathbb{P}_{\xi,y}((\eta_t, X_t) \in \cdot)$;

(b) (*Extension of* $\widehat{\mathbb{P}}^E_{\eta,\xi}$) *The environments behave as under* $\widehat{\mathbb{P}}^E$,

$$\widehat{\mathbb{P}}_{\eta,x;\xi,y}\big((\eta_t^1,\eta_t^2)\in\cdot\big)=\widehat{\mathbb{P}}^E_{\eta,\xi}\big((\eta_t^1,\eta_t^2)\in\cdot\big);$$

(c) (*Coupling of the walkers*) X_t^1 *and* X_t^2 *perform identical jumps as much as possible, and the rate of performing a different jump is* $\sum_{z\in\mathbb{Z}^d}|\alpha(\theta_{-X_t^1}\eta_t^1,z)-\alpha(\theta_{-X_t^2}\eta_t^2,z)|$;

(d) (*Minimal and maximal walkers*) *In addition to the environments* η_t^1 *and* η_t^2 *and random walkers* X_t^1 *and* X_t^2, *the coupling supports minimal and maximal walkers* Y_t^+, Y_t^- *as well. These two walkers have the following properties*:
   (1) $Y_t^-\le X_t^1-x$, $X_t^2-y\le Y_t^+$ $\widehat{\mathbb{P}}_{\eta,x;\xi,y}$-*a.s.* (*in dimension* $d>1$, *this is to be interpreted coordinate-wise*);
   (2) Y_t^+, Y_t^- *are independent of* η_t^1, η_t^2;
   (3) $\widehat{\mathbb{E}}_{\eta,x;\xi,y}Y_t^+=t\gamma^+$ *for some* $\gamma^+\in\mathbb{R}^d$;
   (4) $\widehat{\mathbb{E}}_{\eta,x;\xi,y}Y_t^-=t\gamma^-$ *for some* $\gamma^-\in\mathbb{R}^d$.

PROOF. The construction of this coupling $\widehat{\mathbb{P}}_{\eta,x;\xi,y}$ is done in the following way: we extend the original coupling $\widehat{\mathbb{P}}^E_{\eta,\xi}$ to contain an independent sequence of Poisson processes N^z, $z\in\mathbb{Z}^d$, with rates $\lambda_z:=\sup_\eta\alpha(\eta,z)$, as well as a sufficient supply of independent uniform $[0,1]$ variables. The walkers X^1, X^2 then start from x (resp., y) and exclusively (but not necessarily) jump when one of the Poisson clocks N^z rings. When the clock N^z rings the walkers jumps from X_t^i to X_t^i+z only if a uniform $[0,1]$ variable U satisfies $U<\alpha(\theta_{-X_t^i}\eta_t^i,z)/\lambda_z$, $i=1,2$. Note that both walkers share the same U, but U's for different rings of the Poisson clocks are independent.

The upper and lower walkers Y_t^+, Y_t^- are constructed from the same Poisson clocks N^z. They always jump on these clocks; however, they jump by $\max(z,0)$ or $\min(z,0)$, respectively.

The properties of the coupling arise directly from the construction plus the fact that $\|\alpha\|_1<\infty$. □

To ease notation we will call $\widehat{\mathbb{P}}_{\eta,0;\xi,0}$ simply $\widehat{\mathbb{P}}_{\eta,\xi}$ and the law of Y_t^+, Y_t^- $\widehat{\mathbb{P}}$ whenever there is no fear of confusion.

Now we show how suitable estimates on the coupling speed of the environment translate to properties of the extended coupling.

LEMMA 5.2.

$$\widehat{\mathbb{E}}_{\eta,x;\xi,y}\rho\big(\eta_t^1(X_t^1),\eta_t^2(X_t^1)\big)<\big(\|\gamma^+-\gamma^-\|_\infty t+1\big)^d\sup_{\eta,\xi\in\Omega}\widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta_t^1(0),\eta_t^2(0)\big).$$

PROOF. Denote with $R_t \subset \mathbb{Z}^d$ the set of sites $z \in \mathbb{Z}^d$ with $Y_t^- \le z \le Y_t^+$ (coordinate-wise). Then

$$\begin{aligned}
&\sup_{\eta,\xi,x,y} \widehat{\mathbb{E}}_{\eta,x;\xi,y}\rho\big(\eta_t^1(X_t^1), \eta_t^2(X_t^1)\big) \\
&\quad\le \sup_{\eta,\xi,x,y} \widehat{\mathbb{E}}_{\eta,x;\xi,y} \sum_{z\in R_t} \rho\big(\eta_t^1(x+z), \eta_t^2(x+z)\big) \\
&\quad\le \widehat{\mathbb{E}}\Big[\sum_{z\in R_t} 1\Big] \sup_{\eta,\xi,z} \widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta_t^1(z), \eta_t^2(z)\big) \\
&\quad\le \big(\|\gamma^+ - \gamma^-\|_\infty t + 1\big)^d \sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta_t^1(0), \eta_t^2(0)\big).
\end{aligned}$$

□

LEMMA 5.3. *Denote by* $\tau := \inf\{t \ge 0 : X_t^1 \ne X_t^2\}$ *the first time the two walkers are not at the same position. Under Assumption* 1a,

$$\inf_{\eta,\xi\in\Omega} \widehat{\mathbb{P}}_{\eta;\xi}(\tau = \infty) > 0,$$

that is, the walkers X^1 *and* X^2 *never decouple with strictly positive probability.*

PROOF. Both walkers start in the origin, therefore $\tau > 0$. The probability that a Poisson clock with time dependent rate λ_t is has not yet rung by time T is $\exp(-\int_0^T \lambda_t\,dt)$. As the rate of decoupling is given by Proposition 5.1(c), we obtain

$$\begin{aligned}
\widehat{\mathbb{P}}_{\eta,\xi}(\tau > T) &= \widehat{\mathbb{E}}_{\eta,\xi} \exp\Big(-\int_0^T \sum_{z\in\mathbb{Z}^d} \big|\alpha\big(\theta_{-X_t^1}\eta_t^1, z\big) - \alpha\big(\theta_{-X_t^1}\eta_t^2, z\big)\big|\,dt\Big) \\
&\ge \exp\Big(-\widehat{\mathbb{E}}_{\eta,\xi}\int_0^T \sum_{z\in\mathbb{Z}^d} \big|\alpha\big(\theta_{-X_t^1}\eta_t^1, z\big) - \alpha\big(\theta_{-X_t^1}\eta_t^2, z\big)\big|\,dt\Big).
\end{aligned} \tag{10}$$

By telescoping over single site changes,

$$\begin{aligned}
&\widehat{\mathbb{E}}_{\eta,\xi} \sum_{z\in\mathbb{Z}^d} \big|\alpha\big(\theta_{-X_t^1}\eta_t^1, z\big) - \alpha\big(\theta_{-X_t^1}\eta_t^2, z\big)\big| \\
&\quad\le \widehat{\mathbb{E}}_{\eta,\xi} \sum_{z\in\mathbb{Z}^d}\sum_{x\in\mathbb{Z}^d} \rho\big(\eta_t^1(X_t^1 + x), \eta_t^2(X_t^1 + x)\big)\delta_{\alpha(\cdot,z)}(x) \\
&\quad\le \sup_{x\in\mathbb{Z}^d} \widehat{\mathbb{E}}_{\eta,\xi}\rho\big(\eta_t^1(X_t^1 + x), \eta_t^2(X_t^1 + x)\big)\,|||\alpha||| \\
&\quad\le |||\alpha|||\big(\|\gamma^+ - \gamma^-\|_\infty t + 1\big)^d \sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta_t^1(0), \eta_t^2(0)\big),
\end{aligned}$$

where the last line follows from Lemma 5.2. With this estimate and Assumption 1a we obtain

$$\begin{aligned}
&\widehat{\mathbb{P}}_{\eta,\xi}(\tau=\infty)\\
&\quad\geq \exp\Bigg(-|||\alpha|||\int_0^\infty \big(\|\gamma^+-\gamma^-\|_\infty t+1\big)^d \sup_{\eta,\xi\in\Omega}\widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta_t^1(0),\eta_t^2(0)\big)\,dt\Bigg)\\
&\quad> 0 \qquad \text{uniformly in } \eta,\xi. \qquad \square
\end{aligned}$$

PROOF OF THEOREM 3.1, PART (a). The idea of the proof is to use the coupling of Proposition 5.1: we wait until the walkers X_t^1 and X_t^2, which are initially at the same position, decouple, and then restart everything and try again. By Lemma 5.3 there is a positive probability of never decoupling, so this scheme is successful. Using the time of decoupling τ (as in Lemma 5.3) and the strong marginal Markov property (6),

$$\begin{aligned}
&\int_0^T \big|\widehat{\mathbb{E}}_{\eta,0;\xi,0}\mathbb{1}_{t\geq\tau}\big(f\big(\theta_{-X_t^1}\eta_t^1\big)-f\big(\theta_{-X_t^2}\eta_t^2\big)\big)\big|\,dt\\
&\quad=\int_0^T \big|\widehat{\mathbb{E}}_{\eta 0;\xi,0}\mathbb{1}_{t\geq\tau}\mathbb{E}\big[f\big(\theta_{-X_t^1}\eta_t^1\big)-f\big(\theta_{-X_t^2}\eta_t^2\big)|\mathfrak{F}_\tau\big]\big|\,dt\\
&\quad\leq\int_0^T \widehat{\mathbb{E}}_{\eta,0;\xi,0}\mathbb{1}_{t\geq\tau}\big|S^{\mathrm{EP}}_{t-\tau}f\big(\theta_{-X_\tau^1}\eta_\tau^1\big)-S^{\mathrm{EP}}_{t-\tau}f\big(\theta_{-X_\tau^2}\eta_\tau^2\big)\big|\,dt\\
&\quad=\widehat{\mathbb{E}}_{\eta,0;\xi,0}\int_0^{(T-\tau)\vee 0}\big|S^{\mathrm{EP}}_{t}f\big(\theta_{-X_\tau^1}\eta_\tau^1\big)-S^{\mathrm{EP}}_{t}f\big(\theta_{-X_\tau^2}\eta_\tau^2\big)\big|\,dt \qquad (11)\\
&\quad\leq\widehat{\mathbb{P}}_{\eta,0;\xi,0}(\tau<\infty)\sup_{\eta,\xi\in\Omega}\int_0^T\big|S_t^{\mathrm{EP}}f(\eta)-S_t^{\mathrm{EP}}f(\xi)\big|\,dt. \qquad (12)
\end{aligned}$$

And therefore

$$\begin{aligned}
&\int_0^T\big|S_t^{\mathrm{EP}}f(\eta)-S_t^{\mathrm{EP}}f(\xi)\big|\,dt\\
&\quad=\int_0^T\big|\widehat{\mathbb{E}}_{\eta,\xi}f\big(\theta_{-X_t^1}\eta_t^1\big)-f\big(\theta_{-X_t^2}\eta_t^2\big)\big|\,dt\\
&\quad\leq\int_0^T\widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{t<\tau}\big|f\big(\theta_{-X_t^1}\eta_t^1\big)-f\big(\theta_{-X_t^1}\eta_t^2\big)\big|\,dt\\
&\qquad+\widehat{\mathbb{P}}_{\eta,\xi}(\tau<\infty)\sup_{\eta,\xi\in\Omega}\int_0^T\big|S_t^{\mathrm{EP}}f(\eta)-S_t^{\mathrm{EP}}f(\xi)\big|\,dt \qquad (13)\\
&\quad\leq\int_0^\infty\widehat{\mathbb{E}}_{\eta,\xi}\big|f\big(\theta_{-X_t^1}\eta_t^1\big)-f\big(\theta_{-X_t^1}\eta_t^2\big)\big|\,dt\\
&\qquad+\widehat{\mathbb{P}}_{\eta,\xi}(\tau<\infty)\sup_{\eta,\xi\in\Omega}\int_0^T\big|S_t^{\mathrm{EP}}f(\eta)-S_t^{\mathrm{EP}}f(\xi)\big|\,dt, \qquad (14)
\end{aligned}$$

which gives us the upper bound

$$\begin{aligned}&\sup_{\eta,\xi\in\Omega}\int_0^\infty |S_t^{\mathrm{EP}} f(\eta) - S_t^{\mathrm{EP}} f(\xi)|\,dt \\ &\quad\le \Big(\inf_{\eta,\xi\in\Omega} \widehat{\mathbb{P}}_{\eta,\xi}(\tau=\infty)\Big)^{-1} \\ &\qquad\times \sup_{\eta,\xi\in\Omega}\int_0^\infty \widehat{\mathbb{E}}_{\eta,\xi}|f(\theta_{-X_t^1}\eta_t^1) - f(\theta_{-X_t^1}\eta_t^2)|\,dt.\end{aligned} \tag{15}$$

To show that the last integral is finite, we telescope over single site changes, and get

$$\begin{aligned}&\int_0^\infty \widehat{\mathbb{E}}_{\eta,\xi}|f(\theta_{-X_t^1}\eta_t^1) - f(\theta_{-X_t^1}\eta_t^2)|\,dt \\ &\quad\le \int_0^\infty \widehat{\mathbb{E}}_{\eta,\xi}\sum_{x\in\mathbb{Z}^d} \rho(\eta_t^1(x+X_t^1),\eta_t^2(x+X_t^1))\delta_f(x)\,dt \\ &\quad\le |||f||| \sup_{\eta,\xi,x}\int_0^\infty \widehat{\mathbb{E}}_{\eta,\xi}\rho(\eta_t^1(x+X_t^1),\eta_t^2(x+X_t^1))\,dt,\end{aligned}$$

which is finite by Lemma 5.2 and Assumption 1a. Choosing

$$C_a = \Big(\inf_{\eta,\xi\in\Omega} \widehat{\mathbb{P}}_{\eta,\xi}(\tau=\infty)\Big)^{-1} \sup_{\eta,\xi,x}\int_0^\infty \widehat{\mathbb{E}}_{\eta,\xi}\rho(\eta_t^1(x+X_t^1),\eta_t^2(x+X_t^1))\,dt \tag{16}$$

completes the proof. □

To prove part (b) of the theorem, we need the following analogue to Lemma 5.3 using Assumption 1b.

LEMMA 5.4. *Under Assumption* 1b, *for every site-weight function* $w:\mathbb{Z}^d\to[0,\infty[$ *with* $\|w\|_1 := \sum_x w(x) < \infty$, *we have*

$$\begin{aligned}&\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\int_0^\infty \sum_{y\in\mathbb{Z}^d} w(y)\widehat{\mathbb{E}}_{\eta,\xi}\rho(\eta_t^1(y+X_t^1),\eta_t^2(y+X_t^1))\,dt \\ &\quad\le \text{const}\cdot\|w\|_1.\end{aligned}$$

PROOF. Denote with $R_t\subset\mathbb{Z}^d$ the set of sites whose jth coordinate lies between $Y_t^{j,-}$ and $Y_t^{j,+}$. Then

$$\begin{aligned}&\sum_{y\in\mathbb{Z}^d} w(y)\widehat{\mathbb{E}}_{\eta,\xi}\rho(\eta_t^1(y+X_t^1),\eta_t^2(y+X_t^1)) \\ &\quad= \sum_{y\in\mathbb{Z}^d}\widehat{\mathbb{E}}_{\eta,\xi} w(y-X_t^1)\rho(\eta_t^1(y),\eta_t^2(y))\end{aligned}$$

$$\leq \sum_{y\in\mathbb{Z}^d} \widehat{\mathbb{E}}_{\eta,\xi} \sum_{z\in R_t} w(y-z)\rho\big(\eta_t^1(y),\eta_t^2(y)\big)$$

$$= \sum_{y\in\mathbb{Z}^d} \widehat{\mathbb{E}}\Big[\sum_{z\in R_t} w(y-z)\Big]\widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta_t^1(y),\eta_t^2(y)\big)$$

by independence of R_t and (η_t^1,η_t^2). Therewith,

$$\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\int_0^\infty \sum_{y\in\mathbb{Z}^d} w(y)\widehat{\mathbb{E}}_{\eta,\xi}\rho\big(\eta_t^1(y+X_t^1),\eta_t^2(y+X_t^1)\big)\,dt$$

$$\leq \int_0^\infty \sum_{y\in\mathbb{Z}^d}\widehat{\mathbb{E}}\Big[\sum_{z\in R_t} w(y-z)\Big]\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta_t^1(y),\eta_t^2(y)\big)\,dt.$$

Note that by translation invariance the right-hand side is equal to

$$\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_0}\widehat{\mathbb{E}}^E_{\eta,\xi}\rho\big(\eta_t^1(x),\eta_t^2(x)\big).$$

By construction of R_t and Proposition 5.1(d),

$$\sum_{y\in\mathbb{Z}^d}\widehat{\mathbb{E}}\Big[\sum_{z\in R_t} w(y-z)\Big] = \widehat{\mathbb{E}}\Big[\sum_{z\in R_t}1\Big]\|w\|_1 = \prod_{j=1}^d\big(\gamma^{j,+}t-\gamma^{j,-}t+1\big)\|w\|_1$$

$$\leq c\big(t^d+1\big)\|w\|_1$$

for some suitable $c>0$. Therefore Assumption 1b completes the proof. □

PROOF OF THEOREM 3.1, PART (b). Let $\tau := \inf\{t\geq 0 : X_t^1 \neq X_t^2\}$. Then we split the integration at τ,

$$\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\int_0^\infty |S_t^{\mathrm{EP}}f(\eta)-S_t^{\mathrm{EP}}f(\xi)|\,dt$$

$$\leq \sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\int_0^\infty |\widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{\tau>t}\big(f\big(\theta_{-X_t^1}\eta_t^1\big)-f\big(\theta_{-X_t^1}\eta_t^2\big)\big)|\,dt$$

$$+\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\int_0^\infty |\widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{\tau\leq t}\big(f\big(\theta_{-X_t^1}\eta_t^1\big)-f\big(\theta_{-X_t^2}\eta_t^2\big)\big)|\,dt.$$

We estimate the first term by moving the expectation out of the absolute value and forgetting the restriction to $\tau>t$,

$$\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\int_0^\infty \sum_{y\in\mathbb{Z}^d}\delta_f(y)\widehat{\mathbb{E}}_{\eta,\xi}\rho\big(\eta_t^1(y+X_t^1),\eta_t^2(y+X_t^1)\big)\,dt.$$

By Lemma 5.4 with $w = \delta_f$, this is bounded by some constant times $|||f|||$. For the second term we start by using the strong marginal Markov property (6),

$$\begin{aligned}
&\int_0^\infty |\widehat{\mathbb{E}}_{\eta,\xi} \mathbb{1}_{\tau \le t} \big(f\big(\theta_{-X_t^1}\eta_t^1\big) - f\big(\theta_{-X_t^2}\eta_t^2\big)\big)|\, dt \\
&\quad = \int_0^\infty |\widehat{\mathbb{E}}_{\eta,\xi} \mathbb{1}_{\tau \le t} \big(S_{t-\tau}^{\mathrm{EP}} f\big(\theta_{-X_\tau^1}\eta_\tau^1\big) - S_{t-\tau}^{\mathrm{EP}} f\big(\theta_{-X_\tau^2}\eta_\tau^2\big)\big)|\, dt \\
&\quad \le \widehat{\mathbb{E}}_{\eta,\xi} \mathbb{1}_{\tau<\infty} \int_\tau^\infty |\big(S_{t-\tau}^{\mathrm{EP}} f\big(\theta_{-X_\tau^1}\eta_\tau^1\big) - S_{t-\tau}^{\mathrm{EP}} f\big(\theta_{-X_\tau^2}\eta_\tau^2\big)\big)|\, dt \qquad (17) \\
&\quad \le \widehat{\mathbb{P}}_{\eta,\xi}(\tau<\infty) \sup_{\eta,\xi\in\Omega} \int_0^\infty |S_t^{\mathrm{EP}} f(\eta) - S_t^{\mathrm{EP}} f(\xi)|\, dt. \qquad (18)
\end{aligned}$$

By part (a) of Theorem 3.1 the integral part is uniformly bounded by $C_a |||f|||$. So what remains to complete the proof is to show that

$$\sum_{x\in\mathbb{Z}^d} \sup_{(\eta,\xi)\in(\Omega\times\Omega)_x} \widehat{\mathbb{P}}_{\eta,\xi}(\tau<\infty) < \infty. \qquad (19)$$

To do so we first use the same idea as in the proof of Lemma 5.3 to obtain

$$\begin{aligned}
&\widehat{\mathbb{P}}_{\eta,\xi}(\tau<\infty) \\
&\quad = 1 - \exp\Big(-\int_0^\infty \widehat{\mathbb{E}}_{\eta,\xi} \sum_{z\in\mathbb{Z}^d} |\alpha\big(\theta_{-X_t^1}\eta_t^1, z\big) - \alpha\big(\theta_{-X_t^1}\eta_t^2, z\big)|\, dt\Big) \\
&\quad \le \int_0^\infty \widehat{\mathbb{E}}_{\eta,\xi} \sum_{z\in\mathbb{Z}^d} |\alpha\big(\theta_{-X_t^1}\eta_t^1, z\big) - \alpha\big(\theta_{-X_t^1}\eta_t^2, z\big)|\, dt \\
&\quad \le \int_0^\infty \sum_{y\in\mathbb{Z}^d} w_\alpha(y) \widehat{\mathbb{E}}_{\eta,\xi} \rho\big(\eta_t^1(y+X_t^1), \eta_t^2(y+X_t^1)\big)\, dt
\end{aligned}$$

with

$$w_\alpha(x) := \sup_{(\eta,\xi)\in(\Omega\times\Omega)_x} \sum_{z\in\mathbb{Z}^d} |\alpha(\eta,z) - \alpha(\xi,z)|$$

and $\sum_{x\in\mathbb{Z}^d} w_\alpha(x) < \infty$. So we get

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d} \sup_{(\eta,\xi)\in(\Omega\times\Omega)_x} \widehat{\mathbb{P}}_{\eta,\xi}(\tau<\infty) \\
&\quad \le \sum_{x\in\mathbb{Z}^d} \sup_{(\eta,\xi)\in(\Omega\times\Omega)_x} \int_0^\infty \sum_{y\in\mathbb{Z}^d} w_\alpha(y) \widehat{\mathbb{E}}_{\eta,\xi} \rho\big(\eta_t^1(y+X_t^1), \eta_t^2(y+X_t^1)\big)\, dt,
\end{aligned}$$

and Lemma 5.4 completes the proof, where C_b is the combination of the various factors in front of $|||f|||$. □

PROOF OF THEOREM 3.3. Let α, α' be two different transition rates. The goal is to show that

$$\left|\mu_{\alpha}^{\mathrm{EP}}(f) - \mu_{\alpha'}^{\mathrm{EP}}(f)\right| \le C\,|||f|||$$

for all $f : \Omega \to \mathbb{R}$ with $|||f||| < \infty$.

The idea is now to use a coupling $\widehat{\mathbb{P}}$ similar to the one in Proposition 5.1. The coupling contains as objects two copies of the environment, η^1 and η^2, and three random walks, X^1, X^{12} and X^2. The random walk X^1 moves on the environment η^1 with rates α, and correspondingly the random walk X^2 moves on η^2 with rates α'. The mixed walker X^{12} moves on the environment η^2 as well, but according to the rates α. The walkers X^1, X^2 will perform the same jumps as X^{12} with maximal probability. This can be achieved with the same construction as in Proposition 5.1, but with Poisson clocks N^z which have rates $\lambda_z = \sup_{\eta\in\Omega} \alpha(\eta, z) \vee \alpha'(\eta, z)$.

We only consider the case where all three walkers start at the origin. We denote by $S_t^{\mathrm{EP},1}$, $S_t^{\mathrm{EP},2}$ the semigroups of the environment process which correspond to the rates α and α'. Let $\tau := \inf\{t \ge 0 : X_t^1 \ne X_t^{12} \text{ or } X_t^{12} \ne X_t^2\}$.

$$\begin{aligned}
&S_t^{\mathrm{EP},1} f(\eta) - S_t^{\mathrm{EP},2} f(\xi) \\
&\quad = \widehat{\mathbb{E}}_{\eta,\xi}\big(f\big(\theta_{-X_t^1}\eta_t^1\big) - f\big(\theta_{-X_t^2}\eta_t^2\big)\big) \\
&\quad = \widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{\tau>t}\big(f\big(\theta_{-X_t^1}\eta_t^1\big) - f\big(\theta_{-X_t^1}\eta_t^2\big)\big) \\
&\qquad + \widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{\tau\le t}\big(f\big(\theta_{-X_t^1}\eta_t^1\big) - f\big(\theta_{-X_t^2}\eta_t^2\big)\big) \\
&\quad = \widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{\tau>t}\big(f\big(\theta_{-X_t^1}\eta_t^1\big) - f\big(\theta_{-X_t^1}\eta_t^2\big)\big) \\
&\qquad + \widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{\tau\le t}\big(S_{t-\tau}^{\mathrm{EP},1} f\big(\theta_{-X_\tau^1}\eta_\tau^1\big) - S_{t-\tau}^{\mathrm{EP},2} f\big(\theta_{-X_\tau^2}\eta_\tau^2\big)\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Psi(T) &:= \sup_{0\le T'\le T}\sup_{\eta,\xi\in\Omega}\int_0^{T'} S_t^{\mathrm{EP},1} f(\eta) - S_t^{\mathrm{EP},2} f(\xi)\,dt \\
&\le \sup_{0\le T'\le T}\sup_{\eta,\xi\in\Omega}\int_0^{T'} \widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{\tau>t}\big(f\big(\theta_{-X_t^1}\eta_t^1\big) - f\big(\theta_{-X_t^1}\eta_t^2\big)\big) \\
&\qquad + \widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{\tau\le t}\sup_{\eta,\xi\in\Omega}\big(S_{t-\tau}^{\mathrm{EP},1} f(\eta) - S_{t-\tau}^{\mathrm{EP},2} f(\xi)\big)\,dt \qquad (20) \\
&\le \sup_{0\le T'\le T}\sup_{\eta,\xi\in\Omega}\Big(\widehat{\mathbb{E}}_{\eta,\xi}\int_0^{\tau} f\big(\theta_{-X_t^1}\eta_t^1\big) - f\big(\theta_{-X_t^1}\eta_t^2\big)\,dt + \mathbb{1}_{\tau\le T'}\Psi\big(T'-\tau\big)\Big) \\
&\le \sup_{\eta,\xi\in\Omega}\widehat{\mathbb{E}}_{\eta,\xi}\Big(\int_0^{\infty} f\big(\theta_{-X_t^1}\eta_t^1\big) - f\big(\theta_{-X_t^1}\eta_t^2\big)\,dt + \mathbb{1}_{\tau\le T}\Psi(T-\tau)\Big).
\end{aligned}$$

We will now exploit this recursive bound on Ψ.

LEMMA 5.5. *Let* $\tau_1 := \inf\{t \geq 0 : X_t^1 \neq X_t^{12}\}$ *and* $\tau_2 := \inf\{t \geq 0 : X_t^{12} \neq X_t^2\}$. *Set*

$$\beta := \sum_{z \in \mathbb{Z}^d} \sup_{\eta \in \mathbb{Z}^d} |\alpha(\eta, z) - \alpha'(\eta, z)|,$$

$$p(\alpha) := \inf_{\eta, \xi \in \Omega} \widehat{\mathbb{P}}_{\eta,\xi}(\tau_1 = \infty),$$

$$C(\alpha) := \int_0^\infty \left(\|\gamma^+(\alpha) - \gamma^-(\alpha)\|_\infty t + 1\right)^d \sup_{\eta,\xi \in \Omega} \widehat{\mathbb{E}}^E_{\eta,\xi} \rho\left(\eta_t^1(0), \eta_t^2(0)\right) dt,$$

where $\gamma^+(\alpha), \gamma^-(\alpha)$ *are as in Proposition* 5.1 *for the rates* α.

Let $Y \in \{0, 1\}$ *be Bernoulli with parameter* $p(\alpha)$ *and* Y' *exponentially distributed with parameter* β. *Let* $Y_1, Y_2, \ldots$ *be i.i.d. copies of* $Y \cdot Y'$ *and* $N(T) := \inf\{N \geq 0 : \sum_{n=1}^N Y_n > T\}$. *Then*

$$\Psi(T) \leq C(\alpha) |||f||| \mathbb{E}N(T).$$

PROOF. By construction of the coupling, τ_2 stochastically dominates Y'. As we have $\tau = \tau_1 \wedge \tau_2$ it follows that $\tau \succeq Y_1$. Using this fact together with the monotonicity of Ψ in (20),

$$\begin{aligned}\Psi(T) &\leq \sup_{\eta,\xi \in \Omega} \left(\widehat{\mathbb{E}}_{\eta,\xi} \int_0^\infty f\left(\theta_{-X_t^1} \eta_t^1\right) - f\left(\theta_{-X_t^1} \eta_t^2\right) dt + \mathbb{1}_{\tau \leq T} \Psi(T - \tau) \right) \\ &\leq \sup_{\eta,\xi \in \Omega} \widehat{\mathbb{E}}_{\eta,\xi} \int_0^\infty f\left(\theta_{-X_t^1} \eta_t^1\right) - f\left(\theta_{-X_t^1} \eta_t^2\right) dt + \mathbb{E}\mathbb{1}_{Y_1 \leq T} \Psi(T - Y_1).\end{aligned}$$

As $p(\alpha) > 0$ by Lemma 5.3 we can iterate this estimate until it terminates after $N(T)$ steps. Therefore we obtain

$$\Psi(T) \leq \mathbb{E}N(T) \sup_{\eta,\xi \in \Omega} \widehat{\mathbb{E}}_{\eta,\xi} \int_0^\infty f\left(\theta_{-X_t^1} \eta_t^1\right) - f\left(\theta_{-X_t^1} \eta_t^2\right) dt.$$

The integral is estimated by telescoping over single site changes and Lemma 5.2 in the usual way, yielding

$$\Psi(T) \leq C(\alpha) |||f||| \mathbb{E}N(T). \qquad \square$$

To finally come back to the original question of continuity,

$$\begin{aligned}\left|\mu_\alpha^{\mathrm{EP}}(f) - \mu_{\alpha'}^{\mathrm{EP}}(f)\right| &= \frac{1}{T} \left| \int\int\int_0^T S_t^{\mathrm{EP},1} f(\eta) - S_t^{\mathrm{EP},2} f(\xi)\, dt\, \mu_\alpha^{\mathrm{EP}}(d\eta) \mu_{\alpha'}^{\mathrm{EP}}(d\xi) \right| \\ &\leq \frac{1}{T} \Psi(T) \leq \frac{1}{T} \mathbb{E}N(T) C(\alpha) |||f||| \underset{T \to \infty}{\longrightarrow} \frac{1}{\mathbb{E}YY'} C(\alpha) |||f||| \\ &= \frac{C(\alpha)}{p(\alpha)} \sum_{z \in \mathbb{Z}^d} \sup_{\eta \in \mathbb{Z}^d} |\alpha(\eta, z) - \alpha'(\eta, z)| |||f|||.\end{aligned}$$

By sending α' to α, the right-hand side tends to 0 so that the ergodic measure of the environment process is indeed continuous in the rates α. It is also interesting to note that both $p(\alpha)$ and $C(\alpha)$ are rather explicit given the original coupling of the environment. Notably when $\alpha(\eta, z) = \alpha(z)$, that is, the rates do not depend on the environment, $p(\alpha) = 1$ and $C(\alpha) = \int_0^\infty \sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}^E_{\eta,\xi} \rho(\eta^1_t(0), \eta^2_t(0))\, dt$. $\square$

PROOF OF THEOREM 3.4. The proof of this theorem is mostly identical to the proof of Theorem 3.1. Hence instead of copying the proof, we just state where details differ.

A first fact is that the conditions for (a) and (b) imply Assumptions 1a and 1b. In the adaptation of the proof for part (a), in most lines it suffices to add a $\phi(\frac{t}{K})$ to the integrals. However, in line (11), we use

$$\phi\left(\frac{t}{K}\right) \le \phi\left(\frac{t-\tau}{K}\right)\phi\left(\frac{\tau}{K}\right) \tag{21}$$

to obtain the estimate

$$\widehat{\mathbb{E}}_{\eta,\xi}\phi\left(\frac{\tau}{K}\right)\int_0^{(T-\tau)\vee 0} \phi\left(\frac{t}{K}\right)\left|S^{\mathrm{EP}}_t f\left(\theta_{-X^1_\tau}\eta^1_\tau\right) - S^{\mathrm{EP}}_t f\left(\theta_{-X^2_\tau}\eta^2_\tau\right)\right| dt$$

instead. Thereby in lines (12), (13) and (14) we have to change $\widehat{\mathbb{P}}_{\eta,\xi}(\tau < \infty)$ to $\widehat{\mathbb{E}}_{\eta,\xi}\phi(\frac{\tau}{K})\mathbb{1}_{\tau<\infty}$. This change then leads to the replacement of

$$\inf_{\eta,\xi\in\Omega} \widehat{\mathbb{P}}_{\eta,\xi}(\tau = \infty)$$

by the term

$$1 - \sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}_{\eta,\xi}\phi\left(\frac{\tau}{K}\right)\mathbb{1}_{\tau<\infty}$$

in the lines (15) and (16) [where naturally C_a becomes $C_a(K)$]. So all we have to prove that for sufficiently big K,

$$\sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}_{\eta,\xi}\phi\left(\frac{\tau}{K}\right)\mathbb{1}_{\tau<\infty} < 1.$$

In a first step, we show that

$$\sup_{\eta,\xi\in\Omega} \widehat{\mathbb{E}}_{\eta,\xi}\phi(\tau)\mathbb{1}_{\tau<\infty} < \infty.$$

As we already saw in the proof of Lemma 5.3, we can view the event of decoupling as the first jump of a Poisson process with time-dependent and random rates

[equation (10)]. Hence we have

$$\begin{aligned}
&\widehat{\mathbb{E}}_{\eta,\xi}\phi(\tau)\mathbb{1}_{\tau<\infty}\\
&\quad=\int_0^\infty \phi(t)\,d\widehat{\mathbb{P}}_{\eta,\xi}(\tau>t)\\
&\quad=\int_0^\infty \phi(t)\widehat{\mathbb{E}}_{\eta,\xi}\sum_{z\in\mathbb{Z}^d}\bigl|\alpha\bigl(\theta_{-X_t^1}\eta_t^1,z\bigr)-\alpha\bigl(\theta_{-X_t^1}\eta_t^2,z\bigr)\bigr|\\
&\qquad\qquad\times\exp\Bigl(-\int_0^t\sum_{z\in\mathbb{Z}^d}\bigl|\alpha\bigl(\theta_{-X_s^1}\eta_s^1,z\bigr)-\alpha\bigl(\theta_{-X_s^1}\eta_s^2,z\bigr)\bigr|\,ds\Bigr)\,dt\\
&\quad\le\int_0^\infty \phi(t)\widehat{\mathbb{E}}_{\eta,\xi}\sum_{z\in\mathbb{Z}^d}\bigl|\alpha\bigl(\theta_{-X_t^1}\eta_t^1,z\bigr)-\alpha\bigl(\theta_{-X_t^1}\eta_t^2,z\bigr)\bigr|\,dt.
\end{aligned}$$

By telescoping over single site discrepancies and using Lemma 5.2, this is less than

$$\int_0^\infty \phi(t)\bigl(\|\gamma^+-\gamma^-\|_\infty+1\bigr)^d t^d\sup_{\eta,\xi\in\Omega}\widehat{\mathbb{E}}^E_{\eta,\xi}\rho\bigl(\eta_t^1(0),\eta_t^2(0)\bigr)\,dt<\infty$$

by assumption. Since $\phi(t/K)$ decreases to 1 as $K\to\infty$, monotone convergence implies

$$\lim_{K\to\infty}\widehat{\mathbb{E}}_{\eta,\xi}\phi\Bigl(\frac{\tau}{K}\Bigr)\mathbb{1}_{\tau<\infty}=\widehat{\mathbb{E}}_{\eta,\xi}\mathbb{1}_{\tau<\infty}<1$$

by Lemma 5.3. Consequently, there exists a $K_0\ge 0$ such that for all $K>K_0$,

$$\widehat{\mathbb{E}}_{\eta,\xi}\phi\Bigl(\frac{\tau}{K}\Bigr)\mathbb{1}_{\tau<\infty}<1.$$

This completes the adaptation of part (a).

The adaptation of the proof of part (b) follows the same scheme, where we add the term $\phi(\frac{t}{K})$ to all integrals. Note that this gives a version of Lemma 5.4 as well. Then, in line (17) we use (21) again and then have to replace $\widehat{\mathbb{P}}_{\eta,\xi}(\tau<\infty)$ by $\widehat{\mathbb{E}}_{\eta,\xi}\phi(\frac{\tau}{K})\mathbb{1}_{\tau<\infty}$ in lines (18) and (19). To estimate (19), we use

$$\begin{aligned}
&\widehat{\mathbb{E}}_{\eta,\xi}\phi\Bigl(\frac{\tau}{K}\Bigr)\mathbb{1}_{\tau<\infty}\\
&\quad\le\int_0^\infty\phi\Bigl(\frac{t}{K}\Bigr)\widehat{\mathbb{E}}_{\eta,\xi}\sum_{z\in\mathbb{Z}^d}\bigl|\alpha\bigl(\theta_{-X_t^1}\eta_t^1,z\bigr)-\alpha\bigl(\theta_{-X_t^1}\eta_t^2,z\bigr)\bigr|\,dt\\
&\quad\le\int_0^\infty\phi\Bigl(\frac{t}{K}\Bigr)\sum_{y\in\mathbb{Z}^d}w_\alpha\widehat{\mathbb{E}}_{\eta,\xi}\rho\bigl(\eta_t^1(y+X_t^1),\eta_t^2(y+X_t^1)\bigr)\,dt
\end{aligned}$$

with w_α as in the original proof. Therefore

$$\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\widehat{\mathbb{E}}_{\eta,\xi}\phi\left(\frac{\tau}{K}\right)\mathbb{1}_{\tau<\infty}$$
$$\leq\sum_{x\in\mathbb{Z}^d}\sup_{(\eta,\xi)\in(\Omega\times\Omega)_x}\int_0^\infty\phi\left(\frac{t}{K}\right)\sum_{y\in\mathbb{Z}^d}w_\alpha(y)\widehat{\mathbb{E}}_{\eta,\xi}\rho\big(\eta_t^1(y+X_t^1),\eta_t^2(y+X_t^1)\big)\,dt,$$

which is finite by Lemma 5.4. □

PROOF OF PROPOSITION 3.5. The LLN for $\mathbb{P}_{\mu^{\mathrm{EP}},0}$ follows directly by ergodicity. To prove the LLN for $\mathbb{P}_{\eta,x}$ we use a slight modification of the coupling $\widehat{\mathbb{P}}$ in Proposition 5.1.

Suppose w.l.o.g. that $x=0$ (otherwise look at $\mathbb{P}_{\theta_{-x}\eta,0}$). In the construction of the modified coupling $\widetilde{\mathbb{P}}$ we look at $X_T^1-X_T^2$. Jump events of $X_T^1-X_T^2$ we call decoupling events, which are events when one walker jumps but the other does not. Up to the first decoupling the coupling $\widetilde{\mathbb{P}}_{\eta,0;\mu^{\mathrm{EP}},0}$ is identical to $\widehat{\mathbb{P}}_{\eta,0;\mu^{\mathrm{EP}},0}$. By Lemma 5.3, there is at least probability $p>0$ uniformly in η to never decouple. At the instant τ of a decoupling event we restart the coupling $\widehat{\mathbb{P}}^E$ of the environment. This is done in the configuration $\theta_{-X_\tau^1}\eta_\tau^1,\theta_{-X_\tau^2}\eta_\tau^2$. That is, instead of coupling $\eta_t^1(x)$ with $\eta_t^2(x)$, we match $\eta_t^1(x+X_\tau^1)$ with $\eta_t^2(x+X_\tau^2)$. This allows us to apply Lemma 5.3 a second time, since for the purpose of decoupling events, both walkers start at the origin at time τ. Iterating, we then have at most a geometric number N of decoupling events at $\tau_1,\dots,\tau_N$. Hence

$$|X_T^1-X_T^2|\leq\sum_{n=1}^N\big(|X_{\tau_n}^1-X_{\tau_n-}^1|+|X_{\tau_n}^2-X_{\tau_n-}^2|\big),$$

which when divided by T converges to 0 in L^1 and almost surely w.r.t. $\widetilde{\mathbb{P}}_{\eta,0;\mu^{\mathrm{EP}},0}$.

Therefore X_T/T converges in $L^1(\mathbb{P}_{\eta,x})$, for given x uniformly in $\eta\in\Omega$, and $\mathbb{P}_{\eta,x}$ almost surely to the same limit, $\lim_{T\to\infty}\frac{1}{T}\mathbb{E}_{\mu^{\mathrm{EP}},0}(X_T)$. □

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