CENTRAL LIMIT THEOREMS FOR *U*-STATISTICS OF POISSON POINT PROCESSES

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A *U*-statistic of a Poisson point process is defined as the sum $\sum f(x_1, \ldots, x_k)$ over all (possibly infinitely many) k-tuples of distinct points of the point process. Using the Malliavin calculus, the Wiener–Itô chaos expansion of such a functional is computed and used to derive a formula for the variance. Central limit theorems for *U*-statistics of Poisson point processes are shown, with explicit bounds for the Wasserstein distance to a Gaussian random variable. As applications, the intersection process of Poisson hyperplanes and the length of a random geometric graph are investigated.

1. Introduction. In recent years, Malliavin calculus, Wiener–Itô chaos expansions and Fock space representations of functionals of Poisson point processes have been a rapidly developing topic. First results already appeared in the classical works of Itô [13, 14] and Wiener [37]. Yet only in the last years prominent contributions produced a deep theory which most probably will have a strong impact on modern theory and applications of Poisson point processes; see, for example, Houdre and Perez-Abreu [12], Last and Penrose [19], Nualart and Vives [25] and Wu [38]. Here in particular we want to point out the groundbreaking paper by Peccati et al. [27] on central limit theorems using Stein's method and Malliavin calculus. These methods were combined the first time by Nourdin and Peccati [24] for functionals depending on Gaussian processes instead of Poisson point processes. Further developments include the book of Peccati and Taqqu [28] about product formulas for multiple Wiener–Itô integrals in the Gaussian and Poisson case and a central limit theorem due to Peccati and Zheng [29] generalizing the main result of [27] to random vectors.

Poisson point processes occur in many branches of probability theory, for example, in the theory of Levy processes, and in the theory of random graphs, in spatial statistics, in communication theory and in stochastic geometry. Hence there is a wide range of potential applications of these new results. In this work, we use the Wiener–Itô chaos expansion and a related result from [27] to prove central limit theorems for a broad class of functionals, namely for *U*-statistics of Poisson point processes.

Let η be a Poisson point process over a state space X. We call a random variable F a U-statistic of η if

$$F(\eta) = \sum_{(x_1, \ldots, x_k) \in \eta_{\neq}^k} f(x_1, \ldots, x_k).$$

By $\eta_{\neq}^k$ we denote the set of all k-tuples of distinct points of the process. One should compare this definition to classical U-statistics defined on a set of n random variables $\{Z_1, \ldots, Z_n\} = \zeta$ where $U(\zeta) = \sum_{\zeta_{\neq}^k} f(x_1, \ldots, x_k)$. For details on classical U-statistics we refer to [11, 16, 20]. From now on, we mean by U-statistic a U-statistic of a Poisson point process.

The first step in this paper is the explicit evaluation of expressions involving Malliavin operators acting on U-statistics of Poisson point processes. The main result of this paper is Theorem 4.7 which gives an explicit bound on the Wasserstein distance between a normalized U-statistic and a standard Gaussian random variable N,

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq 2k^{7/2} \sum_{1 \leq i \leq j \leq k} \frac{\sqrt{M_{ij}(f)}}{\operatorname{Var} F},$$

where $M_{ij}(f)$ are sums of certain fourth moment integrals. If the intensity measure of η is of the form $\mu = \lambda\theta$ with an intensity parameter $\lambda \geq 1$ and a measure θ, one is interested in the behavior of F for increasing λ. In the particular situation that $f : X^k \to \mathbb{R}$ is independent of λ, we conclude in Theorem 5.2 that

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq C_f \lambda^{-1/2}.$$

In general this is the optimal rate in λ because for a set $A \subset X$ with $\theta(A) = 1$ the U-statistic $F = \sum_{x \in \eta} \mathbb{1}(x \in A)$ is Poisson distributed with parameter λ, and it is widely known that a Poisson distributed random variable has this rate of convergence.

As an application of our result we investigate the intrinsic volumes of the intersection process of Poisson hyperplanes in a compact convex window. A central limit theorem for some of these functionals was proved in two long and intricate papers by Heinrich [9] and Heinrich, Schmidt and Schmidt [10]. Here we obtain a general result which in addition gives rates of convergence to Gaussian variables. A second example concerns functionals of Sylvester type by which we mean the question about the probability that k points in a convex set are in convex position. Our last example is about the number of edges of a random geometric graph in a bounded window. Again we obtain a central limit theorem with a rate of convergence. As general references to stochastic geometry and random graphs we refer to [30, 32] and [35].

To prove our central limit theorems, we first use a result of Last and Penrose [19], to expand a U-statistic in a Wiener–Itô chaos expansion as a finite sum

of multiple Wiener–Itô integrals. This enables us to give a formula for the variance of a U-statistic and to compute two operators from Malliavin calculus that are defined by their chaos expansions. Using a theorem for the normal approximation of Poisson functionals due to Peccati et al. [27], we show convergence in the Wasserstein distance. In order to apply their result, we need to compute expected values of products of multiple Wiener–Itô integrals which is well known to be a notorious difficult task. We expect that the same techniques can be used to show central limit theorems for more general functionals of Poisson point processes.

This paper is organized in the following way. In Section 2, we introduce Wiener–Itô chaos expansions for functionals of a Poisson point process and some operators from Malliavin calculus. Then we compute the Wiener–Itô chaos expansion of a U-statistic and its variance in Section 3. Using Malliavin calculus we prove the general version of our central limit theorem for U-statistics in Section 4. Finally, we investigate two special classes of U-statistics and present examples in the Sections 5 and 6.

2. Wiener–Itô chaos expansions for Poisson point processes.

2.1. *Poisson point process.* In this paper, we let η be a Poisson point process on the measure space $(X, \mathcal{B}(X), \mu)$ where X is a Borel space and μ is a σ-finite nonatomic Borel measure. A Borel space is a measurable space which is isomorphic to a Borel subset of $[0, 1]$; see page 7 in [15].

More precisely, let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by $N(X)$ the set of all integer-valued σ-finite measures ν on X, equipped with the smallest σ-algebra $\mathcal{N}(X)$ such that the mappings $\nu \to \nu(A)$ are measurable for all sets $A \in \mathcal{B}(X)$. A random measure $\eta : \Omega \to N(X)$ is called a Poisson point process with intensity measure μ if for $A \in \mathcal{B}(X)$ the random variable $\eta(A)$ is Poisson distributed with parameter $\mu(A)$, and the random variables $\eta(A_1), \ldots, \eta(A_m)$ are independent for pairwise disjoint sets $A_1, \ldots, A_m \in \mathcal{B}(X)$. Since the intensity measure μ is nonatomic, the Poisson point process is simple, that is, $\eta(\{x\}) \leq 1$ for all $x \in X$ almost surely. Thus, we can view η as a random set of points in X.

As usual, $L^p(X^k)$ denotes the space of all measurable functions $f : X^k \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ with

$$\int_{X^k} |f(x_1, \ldots, x_k)|^p \, d\mu(x_1, \ldots, x_k) < \infty,$$

where $d\mu(x_1, \ldots, x_k)$ stands for $d\mu(x_1) \cdots d\mu(x_k)$. Let $L_s^p(X^k)$ be the subset of μ^k-almost everywhere symmetric functions in $L^p(X^k)$. We call a function symmetric if it is invariant under all permutations of its arguments. We denote by $\| \cdot \|$ the norm in $L^2(X^k)$, and by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(X^k)$. Equipped with this inner product, $L^2(X^k)$ and $L_s^2(X^k)$ form Hilbert spaces. Instead of the original probability measure P, we always use the image measure $\mathbb{P} = P \circ \eta$. In the following, $L^p(\mathbb{P})$ stands for the set of all measurable functions $F : N(X) \to \overline{\mathbb{R}}$ with $\mathbb{E}|F|^p < \infty$.

An important property of Poisson point processes is the Slivnyak–Mecke formula (see Corollary 3.2.3 in [32]) which says that

$$(1) \qquad \mathbb{E} \sum_{(x_1,\ldots,x_k)\in\eta_{\neq}^k} f(x_1,\ldots,x_k) = \int_{X^k} f(x_1,\ldots,x_k)\,d\mu(x_1,\ldots,x_k)$$

for $f \in L^1(X^k)$. (Recall the definition of $\eta_{\neq}^k$ in the Introduction.) The sum on the left-hand side is a priori defined as an $L^1(\mathbb{P})$ limit summing only over points in an increasing window. Yet it follows from the Slivnyak–Mecke formula that $f \in L^1(X^k)$ implies that the sum on the left-hand side is absolutely convergent almost surely.

2.2. *Multiple Wiener–Itô integrals.* Now we present the definition of multiple Wiener–Itô integrals of order $k \in \mathbb{N}$ following [36]. One starts with simple functions and extends the definition to arbitrary functions in $L_s^2(X^k)$. A function $f \in L^2(X^k)$ is called simple if:

(1) f is symmetric;

(2) f is constant on a finite number of Cartesian products $B_1 \times \cdots \times B_k \in \mathcal{B}(X)^k$ and vanishes elsewhere;

(3) f vanishes on diagonals, which means $f(x_1,\ldots,x_k) = 0$ if $x_i = x_j$ for some $i \neq j$.

Let $\mathcal{S}(X^k)$ be the space of all simple functions. For $f_0 \in \mathcal{S}(X^k)$ and $k \in \mathbb{N}$, the multiple Wiener–Itô integral $I_k(f_0)$ of f_0 with respect to the compensated Poisson point process $\eta - \mu$ is defined by

$$I_k(f_0) = \int_{X^k} f_0\,d(\eta-\mu)^k = \sum f_0^{B_1\times\cdots\times B_k}(\eta-\mu)(B_1)\cdots(\eta-\mu)(B_k),$$

where we sum over all Cartesian products and $f_0^{B_1\times\cdots\times B_k}$ is the value of f_0 on such a set. For $k = 0$ we put $I_0(f) = f$. By a straightforward computation, one shows

$$(2) \qquad\qquad\qquad \mathbb{E}I_k(f_0)^2 = k!\|f_0\|^2.$$

Thus there is an isometry between $\mathcal{S}(X^k)$ and a subset of $L^2(\mathbb{P})$. Furthermore, $\mathcal{S}(X^k)$ is dense in $L_s^2(X^k)$, whence for every $f \in L_s^2(X^k)$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions with $f_n \to f$ in $L_s^2(X^k)$. Because of the isometry (2), it is possible to define $I_k(f)$ as the limit of $(I_k(f_n))_{n\in\mathbb{N}}$ in $L^2(\mathbb{P})$. Hence for an arbitrary symmetric function $f \in L_s^2(X^k)$ we put $f^0(x_1,\ldots,x_k) = f(x_1,\ldots,x_k)$ if $x_i \neq x_j$ for all $i \neq j$ and $f^0(x_1,\ldots,x_k) = 0$ otherwise and obtain

$$I_k(f) = \int_{X^k} f^0\,d(\eta-\mu)^k.$$

We remark that the denseness of $\mathcal{S}(X^k)$ in $L_s^2(X^k)$ depends on the topological structure of X and the fact that μ is nonatomic. For a definition without these requirements we refer to [19].

It follows directly from the definition that multiple Wiener–Itô integrals have the properties summarized in the following:

LEMMA 2.1. *Let* $f \in L_s^2(X^n)$ *and* $g \in L_s^2(X^m)$ *with* $n, m \geq 1$. *Then*:

(a) $\mathbb{E}I_n(f) = 0$;
(b) $\mathbb{E}I_n(f)I_m(g) = \mathbb{1}(n = m)n!\langle f, g\rangle$.

2.3. *Wiener–Itô chaos expansions.* For a measurable function $F : N(X) \to \overline{\mathbb{R}}$ and $y \in X$ we define the difference operator as

$$D_y F(\eta) = F(\eta + \delta_y) - F(\eta),$$

where δ_y is the Dirac measure at the point y. The difference operator $D_y F$ measures the effect of adding the point $y \in X$ to the Poisson point process, whence it is also denoted as add one cost operator in [19]. The iterated difference operator is defined by

$$D_{y_1,\ldots,y_i} F = D_{y_1} D_{y_2,\ldots,y_i} F.$$

Let the functions $f_i : X^i \to \overline{\mathbb{R}}$ be given by $f_0 = \mathbb{E}F$ and

$$f_i(y_1, \ldots, y_i) = \frac{1}{i!}\mathbb{E}D_{y_1,\ldots,y_i} F, \qquad i \geq 1,$$

if these expectations exist. Because of the symmetry of the iterated difference operator, f_i is symmetric if defined. The following relationships between F, the functions $f_i, i \in \mathbb{N}$, and the variance of F have been shown by Last and Penrose [19].

THEOREM 2.2 (Last and Penrose [19]). *Let* $F \in L^2(\mathbb{P})$. *Then* $f_i \in L_s^2(X^i)$, $i \in \mathbb{N}$ *and*

$$F = \sum_{i=0}^{\infty} I_i(f_i),$$

where the sum converges in $L^2(\mathbb{P})$. *The* $f_i \in L_s^2(X^i), i \in \mathbb{N}$ *are the* μ^i-*almost everywhere unique* $g_i \in L_s^2(X^i), i \in \mathbb{N}$, *satisfying* $F = \sum_{i=0}^{\infty} I_i(g_i)$ *in* $L^2(\mathbb{P})$. *Furthermore,*

$$\text{Var } F = \sum_{i=1}^{\infty} i!\|f_i\|^2.$$

In the following, we call the functions f_i, $i \in \mathbb{N}$, kernels of the Wiener–Itô chaos expansion of F. The class of sequences $(g_i)_{i \in \mathbb{N}}$ with $g_i \in L_s^2(X^i)$ and

$$\sum_{i=0}^{\infty} i! \|g_i\|^2 < \infty$$

composes a Hilbert space isomorphic to the symmetric Fock space associated with $L^2(X)$. In this context, Theorem 2.2 states that there exists an isometry between $L^2(\mathbb{P})$ and a symmetric Fock space.

2.4. *Malliavin calculus.* Our proofs for central limit theorems are based on a result for the normal approximation of Poisson functionals from [27], which uses operators from Malliavin calculus. In the following, we give a short introduction to these operators. For more details we refer to [19, 25, 27].

Let $F \in L^2(\mathbb{P})$ and f_i, $i \in \mathbb{N}$, be the kernels of the Wiener–Itô chaos expansion of F. First of all, we give an alternative definition of the difference operator D_y using the Wiener–Itô chaos expansion of F.

DEFINITION 2.3.   Let

$$(3) \qquad \sum_{i=1}^{\infty} i i! \|f_i\|^2 < \infty.$$

Then the random function $y \mapsto D_y F$, $y \in X$, is given by

$$D_y F = \sum_{i=1}^{\infty} i I_{i-1}(f_i(y, \cdot)).$$

It can be proved (see [25], Theorem 6.2 or [19], Theorem 3.3) that for $F \in L^2(\mathbb{P})$ satisfying (3) this definition coincides with the one introduced in Section 2.3.

DEFINITION 2.4.   If

$$\sum_{i=1}^{\infty} i^2 i! \|f_i\|^2 < \infty,$$

then the Ornstein–Uhlenbeck generator LF is the random variable given by

$$LF = -\sum_{i=1}^{\infty} i I_i(f_i).$$

The Ornstein–Uhlenbeck generator has an inverse operator. Its domain is the space of all centred $F \in L^2(\mathbb{P})$, that is, $F \in L^2(\mathbb{P})$ with $\mathbb{E} F = 0$, and

$$L^{-1} F = -\sum_{i=1}^{\infty} \frac{1}{i} I_i(f_i).$$

If F is in the domain of L, then the Ornstein–Uhlenbeck generator can be written as

$$(4) \qquad LF = \int_X F(\eta - \delta_x) - F(\eta)\, d\eta(x) - \int_X \big(F(\eta) - F(\eta + \delta_z)\big)\, d\mu(z).$$

This follows from the representation of the difference operator and the Skorohod-integral (see [19], formula (3.19)), which is not used in this work.

3. Malliavin calculus and Wiener–Itô chaos expansions for U-statistics. In this section, we define U-statistics of Poisson point processes and investigate their Wiener–Itô chaos expansions. In particular, we apply the Malliavin operators to U-statistics and present explicit formulae for the kernels of the Wiener–Itô chaos expansion and the variance.

3.1. *U-statistics of Poisson point processes.* Recall the definition $\eta_{\neq}^k = \{(x_1, \ldots, x_k) \in \eta^k, x_i \neq x_j \text{ for } i \neq j\}$ from the Introduction.

DEFINITION 3.1. A random variable

$$(5) \qquad\qquad F = \sum_{(x_1, \ldots, x_k) \in \eta_{\neq}^k} f(x_1, \ldots, x_k)$$

with $f \in L_s^1(X^k)$ is called U-statistic of order k.

By the Slivnyak–Mecke formula (1), it holds that

$$\mathbb{E} \sum_{(x_1, \ldots, x_k) \in \eta_{\neq}^k} f(x_1, \ldots, x_k) = \int_X \cdots \int_X f(x_1, \ldots, x_k)\, d\mu(x_1, \ldots, x_k)$$

so that $f \in L_s^1(X^k)$ guarantees $F \in L^1(\mathbb{P})$. Due to the fact that we sum over all permutations of k points in (5), we can assume without loss of generality in Definition 3.1 that f is symmetric.

Since we want to use Wiener–Itô chaos expansions, we always require that F is in $L^2(\mathbb{P})$. For the central limit theorems we additionally assume that F is absolutely convergent.

DEFINITION 3.2. A U-statistic F is absolutely convergent if

$$\overline{F} = \sum_{(x_1, \ldots, x_k) \in \eta_{\neq}^k} \big| f(x_1, \ldots, x_k) \big|$$

is in $L^2(\mathbb{P})$.

Note that F absolutely convergent implies that $F \in L^2(\mathbb{P})$. Obviously every $F \in L^2(\mathbb{P})$ with $f \geq 0$ is absolutely convergent.

3.2. *Malliavin calculus.* We start by calculating the difference operator of a U-statistic F.

LEMMA 3.3. *Let $F \in L^2(\mathbb{P})$ be a U-statistic of order k. Then the difference operator applied to F gives*

$$D_{y_1} F = k \sum_{(x_1,\ldots,x_{k-1}) \in \eta_{\neq}^{k-1}} f(y_1, x_1, \ldots, x_{k-1}).$$

PROOF. By the definition of the difference operator D_y and the symmetry of f, we obtain for a U-statistic

$$D_{y_1} F = \sum_{(x_1,\ldots,x_k) \in (\eta \cup \{y_1\})_{\neq}^k} f(x_1, \ldots, x_k) - \sum_{(x_1,\ldots,x_k) \in \eta_{\neq}^k} f(x_1, \ldots, x_k)$$

$$= \sum_{(x_1,\ldots,x_{k-1}) \in \eta_{\neq}^{k-1}} \left(f(y_1, x_1, \ldots, x_{k-1}) + \cdots + f(x_1, \ldots, x_{k-1}, y_1) \right)$$

$$= k \sum_{(x_1,\ldots,x_{k-1}) \in \eta_{\neq}^{k-1}} f(y_1, x_1, \ldots, x_{k-1}). \qquad \square$$

An analogous straightforward computation using (4) verifies the following lemma.

LEMMA 3.4. *Let $F \in L^2(\mathbb{P})$ be a U-statistic of order k. Then the Ornstein–Uhlenbeck operator applied to F gives*

$$LF = -kF + k \int_X \sum_{(x_1,\ldots,x_{k-1}) \in \eta_{\neq}^{k-1}} f(x_1, \ldots, x_{k-1}, z) \, d\mu(z).$$

Without proof we also state the inverse Ornstein–Uhlenbeck operator of a U-statistic.

$$L^{-1}(F - \mathbb{E}F)$$

$$= \left(\sum_{m=1}^{k} \frac{1}{m} \right) \int_{X^k} f(y_1, \ldots, y_k) \, d\mu(y_1, \ldots, y_k)$$

$$- \sum_{m=1}^{k} \frac{1}{m} \sum_{(x_1,\ldots,x_m) \in \eta_{\neq}^m} \int_{X^{k-m}} f(x_1, \ldots, x_m, y_1, \ldots, y_{k-m})$$

$$d\mu(y_1, \ldots, y_{k-m}).$$

3.3. *Wiener–Itô chaos expansions.* Let us now compute the kernels and the Wiener–Itô chaos expansion of a *U*-statistic $F = \sum_{\eta_{\neq}^k} f$ with $F \in L^2(\mathbb{P})$.

LEMMA 3.5. *Let $F \in L^2(\mathbb{P})$ be a U-statistic of order k. Then the kernels of the Wiener–Itô chaos expansion of F have the form*

$$f_i(y_1, \ldots, y_i) = \begin{cases} \binom{k}{i} \int_{X^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i}) \, d\mu(x_1, \ldots, x_{k-i}), \\ \qquad i \leq k, \\ 0, \qquad i > k, \end{cases}$$

and F has the variance

(6)
$$\begin{aligned} \operatorname{Var} F = \sum_{i=1}^{k} i! \binom{k}{i}^2 \\ \times \int_{X^i} \left(\int_{X^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i}) \, d\mu(x_1, \ldots, x_{k-i}) \right)^2 \\ d\mu(y_1, \ldots, y_i). \end{aligned}$$

For the special case $k = 2$ the formulas for the kernels are already implicit in the paper by Molchanov and Zuyev [22] where ideas closely related to Malliavin calculus have been used.

PROOF OF LEMMA 3.5. In Lemma 3.3, the difference operator of a *U*-statistic was computed. Proceeding by induction, we get

$$D_{y_1, \ldots, y_i} F = \frac{k!}{(k-i)!} \sum_{(x_1, \ldots, x_{k-i}) \in \eta_{\neq}^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i})$$

for $i \leq k$. Hence $D_{y_1, \ldots, y_k} F$ only depends on $y_1, \ldots, y_k$ and is independent of the Poisson point process. This yields

$$D_{y_1, \ldots, y_{k+1}} F = 0 \quad \text{and} \quad D_{y_1, \ldots, y_i} F = 0$$

for all $i > k$. We just proved

$$\begin{aligned} &D_{y_1, \ldots, y_i} F \\ &= \begin{cases} \dfrac{k!}{(k-i)!} \sum_{(x_1, \ldots, x_{k-i}) \in \eta_{\neq}^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i}), & i \leq k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

By the Slivnyak–Mecke formula (1), we obtain

$$
\begin{aligned}
f_i(y_1, \ldots, y_i) &= \frac{1}{i!} \mathbb{E} D_{y_1, \ldots, y_i} F \\
&= \frac{1}{i!} \mathbb{E} \frac{k!}{(k-i)!} \sum_{(x_1, \ldots, x_{k-i}) \in \eta_{\neq}^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i}) \\
&= \frac{k!}{i!(k-i)!} \int_{X^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i}) \, d\mu(x_1, \ldots, x_{k-i})
\end{aligned}
$$

for $i \leq k$. The formula for the variance follows from Proposition 2.2. $\quad\square$

Note that $F \in L^2(\mathbb{P})$ implies $f_i \in L_s^2(X^i)$, and thus that for all $1 \leq i \leq k$

$$
\int_{X^i} \left(\int_{X^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i}) \, d\mu(x_1, \ldots, x_{k-i}) \right)^2 d\mu(y_1, \ldots, y_i) < \infty.
$$

In particular, it holds $f \in L_s^2(X^k)$.

By Lemma 3.5, U-statistics only have a finite number of nonvanishing kernels. The following theorem characterizes a U-statistic by this property. We call a Wiener–Itô chaos expansion finite if only a finite number of kernels do not vanish.

THEOREM 3.6. *Assume $F \in L^2(\mathbb{P})$.*

(1) *If F is a U-statistic, then F has a finite Wiener–Itô chaos expansion with kernels $f_i \in L_s^1(X^i) \cap L_s^2(X^i)$, $i = 1, \ldots, k$.*

(2) *If F has a finite Wiener–Itô chaos expansion with kernels $f_i \in L_s^1(X^i) \cap L_s^2(X^i)$, $i = 1, \ldots, k$, then F is a (finite) sum of U-statistics and a constant.*

PROOF. The fact that a U-statistic $F \in L^2(\mathbb{P})$ has a finite Wiener–Itô chaos expansion with $f_i \in L_s^1(X^i)$ follows from Lemma 3.5 and from $f \in L_s^1(X^k)$.

For the second part of the proof, let $F \in L^2(\mathbb{P})$ have a finite Wiener–Itô chaos expansion, that is,

$$
F = \sum_{i=0}^m I_i(f_i)
$$

with kernels $f_i \in L_s^1(X^i) \cap L_s^2(X^i)$ and $m \in \mathbb{N}$. Now Proposition 4.1 in [36] implies that

$$
I_i(f_i) = \sum_{j=0}^i (-1)^{i-j} \binom{i}{j} \sum_{(x_1, \ldots, x_j) \in \eta_{\neq}^j} f_i^{(j)}(x_1, \ldots, x_j),
$$

where the inner sum is a constant for $j = 0$ and $f_i^{(j)}$ is given by

$$
f_i^{(j)}(x_1, \ldots, x_j) = \int_{X^{i-j}} f_i(x_1, \ldots, x_j, y_1, \ldots, y_{i-j}) \, d\mu(y_1, \ldots, y_{i-j}).
$$

The assumption $f_i \in L_s^1(X^i)$ guarantees $f_i^{(j)} \in L_s^1(X^j)$ for $j = 1, \ldots, i$ and $f_i^{(0)} \in \mathbb{R}$. Hence, every Wiener–Itô integral is a (finite) sum of U-statistics and a constant, and the same holds for F. $\quad\square$

3.4. *Examples.* The following examples show that the assumptions on F and f_i in Theorem 3.6 are necessary. In all examples, we consider a Poisson point process in $\mathbb{R}$ with the Lebesgue measure as intensity measure.

EXAMPLE. There exist random variables in $L^2(\mathbb{P})$ with finite Wiener–Itô chaos expansions which are not sums of U-statistics. This is possible if the kernels f_i are in $L_s^2(X^i) \setminus L_s^1(X^i)$. Define $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = \frac{1}{x} \mathbb{1}(|x| > 1),$$

which is in $L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$. Now we define the random variable $G = I_1(g)$. G is in $L^2(\mathbb{P})$ and has a finite Wiener–Itô chaos expansion. But the formal representation

$$I_1(g) = \sum_{x \in \eta} g(x) - \int_{\mathbb{R}} g(x)\, dx$$

we used in the proof of Theorem 3.6 fails because the integral does not exist.

EXAMPLE. There also exist U-statistics $F \in L^1(\mathbb{P})$ with $f \in L_s^1(X^k) \cap L_s^2(X^k)$ which are not in $L^2(\mathbb{P})$. We construct $f \in L_s^1(\mathbb{R}^2) \cap L_s^2(\mathbb{R}^2)$ with $\|f_1\| = \infty$ by putting

$$f(x_1, x_2) = \mathbb{1}(0 \le x_1 \sqrt{x_2} \le 1) \mathbb{1}(0 \le x_2 \sqrt{x_1} \le 1)$$

and

$$F = \sum_{(x_1, x_2) \in \eta_{\neq}^2} f(x_1, x_2).$$

In this case the first kernel,

$$f_1(y) = \mathbb{E}\left[2 \sum_{x \in \eta} f(y, x) \right] = 2 \int_{\mathbb{R}} f(y, x)\, dx = 2 \mathbb{1}(y \ge 0) \min\left\{ \frac{1}{y^2}, \frac{1}{\sqrt{y}} \right\}$$

is not in $L_s^2(\mathbb{R})$ so that F has no Wiener–Itô chaos expansion and cannot be in $L^2(\mathbb{P})$.

EXAMPLE. By Theorem 3.6(2), a functional $F \in L^2(\mathbb{P})$ with a finite Wiener–Itô chaos expansion and kernels $f_i \in L_s^1(X^i) \cap L_s^2(X^i)$, $i = 1, \ldots, k$, is a (finite) sum of U-statistics. Our next example shows that neither the single U-statistics

are in $L^2(\mathbb{P})$ nor are the summands necessarily in $L_s^2(X^i)$. Set $F = I_2(f)$ with f as above. Then

$$I_2(f) = \int_{\mathbb{R}^2} f(x, y)\, dx\, dy - 2 \sum_{x \in \eta} \int_{\mathbb{R}} f(x, y)\, dy + \sum_{(x_1, x_2) \in \eta_{\neq}^2} f(x_1, x_2),$$

and F is a sum of U-statistics. Since $\mathbb{E}[(\sum_{x \in \eta} \int_{\mathbb{R}} f(x, y)\, dy)^2] = \infty$, we know that the U-statistic

$$\sum_{x \in \eta} \int_{\mathbb{R}} f(x, y)\, dy$$

is not in $L^2(\mathbb{P})$, nor are the summands $\int_{\mathbb{R}} f(x, y)\, dy$ in $L^2(\mathbb{R})$. This is in contrast to the remark after the proof of Lemma 3.5 that for a U-statistic $F \in L^2(\mathbb{P})$, we always have $f \in L^2(X^k)$.

EXAMPLE. To motivate the definition of an absolutely convergent U-statistic, we give an example of a U-statistic that is in $L^2(\mathbb{P})$ but not absolutely convergent. Similarly to the previous examples, we set

$$f(x_1, x_2) = \mathbb{1}\big(0 \le |x_1|\sqrt{|x_2|} \le 1\big)\mathbb{1}\big(0 \le |x_2|\sqrt{|x_1|} \le 1\big)\big(2\mathbb{1}(x_1 x_2 \ge 0) - 1\big)$$

and

$$F = \sum_{(x_1, x_2) \in \eta_{\neq}^2} f(x_1, x_2) \quad \text{and} \quad \overline{F} = \sum_{(x_1, x_2) \in \eta_{\neq}^2} \big|f(x_1, x_2)\big|.$$

Now it is easy to verify that $f_1(x) = 0$ and $f_2(x_1, x_2) = f(x_1, x_2)$ so that $F \in L^2(\mathbb{P})$. But the first kernel of $\overline{F}$ is not in $L^2(\mathbb{R})$ so that $\overline{F} \notin L^2(\mathbb{P})$.

4. Central limit theorems for U-statistics. In this section, we derive a central limit theorem for U-statistics of Poisson point processes. In particular, we are interested in the Wasserstein distance of a normalized U-statistic and a standard Gaussian random variable. Recall that the Wasserstein distance $d_W(Y, Z)$ of two random variables Y and Z is given by

$$d_W(Y, Z) = \sup_{h \in \mathrm{Lip}(1)} \big|\mathbb{E}h(Y) - \mathbb{E}h(Z)\big|,$$

where $\mathrm{Lip}(1)$ is the set of all functions $h : \mathbb{R} \to \mathbb{R}$ with a Lipschitz-constant less than or equal to one. It is important to note that convergence in the Wasserstein distance implies convergence in distribution. In particular, it is known (see [4], e.g.) that for a Gaussian random variable N we have

$$\big|\mathbb{P}(Y \le t) - \mathbb{P}(N \le t)\big| \le 2\sqrt{d_W(Y, N)}$$

for all $t \in \mathbb{R}$. Hence, we can prove central limit theorems by showing convergence to a Gaussian random variable in the Wasserstein distance.

Our main estimate for the distance between $F = \sum_{\eta_{\neq}^k} f$ and a standard Gaussian random variable N is Theorem 4.7 which states that

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq 2k^{7/2} \sum_{1 \leq i \leq j \leq k} \frac{\sqrt{M_{ij}(f)}}{\operatorname{Var} F},$$

where the $M_{ij}(f)$ are sums of certain fourth moment integrals. The precise definition is given in formula (14). In most applications, it is elementary to bound these fourth moments of f. This is carried out in Sections 5 and 6.

4.1. *An abstract CLT.* Our most general result is the following upper bound for the Wasserstein distance of a Poisson functional with a finite Wiener–Itô chaos expansion and a standard Gaussian random variable. To neatly formulate our results and proofs, we use the abbreviations

$$
\begin{aligned}
(7) \quad R_{ij} &= \mathbb{E}\left(\int_X I_{i-1}(f_i(z, \cdot)) I_{j-1}(f_j(z, \cdot)) \, d\mu(z)\right)^2 \\
&\quad - \left[\mathbb{E}\int_X I_{i-1}(f_i(z, \cdot)) I_{j-1}(f_j(z, \cdot)) \, d\mu(z)\right]^2,
\end{aligned}
$$

$$
(8) \quad \tilde{R}_i = \mathbb{E}\int_X I_{i-1}(f_i(z, \cdot))^4 \, d\mu(z)
$$

for $i, j = 1, \ldots, k$. Note that $R_{11} = 0$ and that for $i \neq j$ the second expectation in R_{ij} vanishes.

THEOREM 4.1. *Suppose $F \in L^2(\mathbb{P})$ has a finite Wiener–Itô chaos expansion of order k, and N is a standard Gaussian random variable. Then*

$$
(9) \quad d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq k \sum_{1 \leq i, j \leq k} \frac{\sqrt{R_{ij}}}{\operatorname{Var} F} + k^{7/2} \sum_{i=1}^k \frac{\sqrt{\tilde{R}_i}}{\operatorname{Var} F}
$$

with R_{ij} and $\tilde{R}_i$ defined in (7) *and* (8).

PROOF. Our proof is based on the following result of Peccati et al. (Theorem 3.1 in [27]), which is derived by a combination of Malliavin calculus and Stein's method.

THEOREM 4.2 (Peccati et al. [27]). *Let $G \in L^2(\mathbb{P})$ with $\mathbb{E}G = 0$ be in the domain of D and let N be a standard Gaussian random variable. Then*

$$d_W(G, N) \leq \mathbb{E}|1 - \langle DG, -DL^{-1}G\rangle| + \int_X \mathbb{E}[|D_z G|^2 |D_z L^{-1}G|] \, d\mu(z)$$

$$\leq \sqrt{\mathbb{E}(1 - \langle DG, -DL^{-1}G\rangle)^2} + \int_X \mathbb{E}[|D_z G|^2 |D_z L^{-1}G|] \, d\mu(z).$$

From now on, we denote by

$$G = \frac{F - \mathbb{E}F}{\sqrt{\text{Var } F}}$$

the normalization of F and by $g_i \in L_s^2(X^i), i = 1, \ldots, k$, the kernels of G. Thus, it follows

$$g_i(x_1, \ldots, x_i) = \frac{1}{\sqrt{\text{Var } F}} f_i(x_1, \ldots, x_i)$$

for $i = 1, \ldots, k$ and $\text{Var } G = \sum_{i=1}^{k} i! \|g_i\|^2 = 1$.

Since F has a finite Wiener–Itô chaos expansion, F is in the domain of D, and we can apply the above theorem. By the definitions of the Malliavin operators and the triangle inequality, we obtain

$$\mathbb{E}|1 - \langle DG, -DL^{-1}G \rangle|$$

$$= \mathbb{E}\left| \sum_{i=1}^{k} i! \|g_i\|^2 - \int_X \sum_{i=1}^{k} i I_{i-1}(g_i(z, \cdot)) \sum_{i=1}^{k} I_{i-1}(g_i(z, \cdot)) \, d\mu(z) \right|$$

$$\leq \sum_{i=2}^{k} \mathbb{E}\left| i! \|g_i\|^2 - i \int_X I_{i-1}(g_i(z, \cdot)) I_{i-1}(g_i(z, \cdot)) \, d\mu(z) \right|$$

$$+ \sum_{i,j=1, i \neq j}^{k} i \mathbb{E}\left| \int_X I_{i-1}(g_i(z, \cdot)) I_{j-1}(g_j(z, \cdot)) \, d\mu(z) \right|.$$

The first sum on the right-hand side of the inequality starts with $i = 2$ since the summand for $i = 1$ vanishes. As a consequence of Fubini's theorem and Lemma 2.1, it holds that

$$\mathbb{E}i \int_X I_{i-1}(g_i(z, \cdot)) I_{i-1}(g_i(z, \cdot)) \, d\mu(z) = i! \|g_i\|^2.$$

Combining this with the Cauchy–Schwarz inequality leads to

$$\mathbb{E}\left| i! \|g_i\|^2 - i \int_X I_{i-1}(g_i(z, \cdot)) I_{i-1}(g_i(z, \cdot)) \, d\mu(z) \right|$$

$$\leq \sqrt{\mathbb{E}\left(i! \|g_i\|^2 - i \int_X I_{i-1}(g_i(z, \cdot)) I_{i-1}(g_i(z, \cdot)) \, d\mu(z) \right)^2}$$

$$= \sqrt{i^2 \mathbb{E}\left(\int_X I_{i-1}(g_i(z, \cdot)) I_{i-1}(g_i(z, \cdot)) \, d\mu(z) \right)^2 - (i!)^2 \|g_i\|^4}$$

$$= i \frac{\sqrt{R_{ii}}}{\text{Var } F}$$

and

$$\mathbb{E}\left|\int_X I_{i-1}(g_i(z,\cdot))I_{j-1}(g_j(z,\cdot))\,d\mu(z)\right|$$

$$\leq \sqrt{\mathbb{E}\left(\int_X I_{i-1}(g_i(z,\cdot))I_{j-1}(g_j(z,\cdot))\,d\mu(z)\right)^2} = \frac{\sqrt{R_{ij}}}{\operatorname{Var} F}$$

for $i \neq j$. Now it holds that

$$\mathbb{E}|1 - \langle DG, -DL^{-1}G\rangle| \leq \sum_{i=2}^k i\frac{\sqrt{R_{ii}}}{\operatorname{Var} F} + \sum_{i,j=1,i\neq j}^k i\frac{\sqrt{R_{ij}}}{\operatorname{Var} F}$$

(10)

$$\leq k \sum_{1\leq i,j\leq k} \frac{\sqrt{R_{ij}}}{\operatorname{Var} F}.$$

Furthermore, again by the Cauchy–Schwarz inequality we have

$$\int_X \mathbb{E}[(D_z G)^2 |D_z L^{-1}G|]\,d\mu(z)$$

$$\leq \left(\int_X \mathbb{E}[(D_z G)^4]\,d\mu(z)\right)^{1/2}\left(\int_X \mathbb{E}[(D_z L^{-1}G)^2]\,d\mu(z)\right)^{1/2}.$$

By the definitions of the Malliavin operators and Hölder's inequality, we can rewrite the expressions on the right-hand side as

$$\int_X \mathbb{E}[(D_z L^{-1}G)^2]\,d\mu(z) = \int_X \sum_{i=1}^k \mathbb{E}[I_{i-1}(g_i(z,\cdot))^2]\,d\mu(z)$$

$$= \sum_{i=1}^k (i-1)!\|g_i\|^2 \leq 1$$

and

$$\int_X \mathbb{E}[(D_z G)^4]\,d\mu(z) \leq \int_X k^3 \sum_{i=1}^k i^4 \mathbb{E}[I_{i-1}(g_i(z,\cdot))^4]\,d\mu(z) = k^3 \sum_{i=1}^k i^4 \frac{\tilde{R}_i}{(\operatorname{Var} F)^2}.$$

Hence

$$\int_X \mathbb{E}[(D_z G)^2|D_z L^{-1}G|]\,d\mu(z) \leq \sqrt{k^3 \sum_{i=1}^k i^4 \frac{\tilde{R}_i}{(\operatorname{Var} F)^2}}$$

(11)

$$\leq k\sqrt{k}\sum_{i=1}^k i^2 \frac{\sqrt{\tilde{R}_i}}{\operatorname{Var} F} \leq k^{7/2}\sum_{i=1}^k \frac{\sqrt{\tilde{R}_i}}{\operatorname{Var} F}.$$

Combining Theorem 4.2 with formulas (10) and (11) gives the right-hand side of (9) in Theorem 4.1. $\square$

4.2. *Estimates for the error terms.* To estimate the right-hand side of (9) for a U-statistic $F = \sum_{\eta_{\neq}^k} f$ in terms of the function f, we are interested in the behavior of R_{ij} and $\tilde{R}_i$ for $i, j = 1, \ldots, k$. Thus we need to compute expected values of the type

$$\mathbb{E} \prod_{l=1}^{m} I_{n_l}(f_l), \qquad m \in \mathbb{N}, n_1, \ldots, n_m \in \mathbb{N}$$

with $f_l \in L_s^1(X^{n_l}) \cap L_s^2(X^{n_l})$ for $l = 1, \ldots, m$. Such products of multiple Wiener–Itô integrals are discussed in [28] and [36]. Before stating a result for the expected value of such a product, we introduce some notation. The function $\bigotimes_{l=1}^{m} f_l : X^{\sum n_l} \to \mathbb{R}$ is given by

$$\left(\bigotimes_{l=1}^{m} f_l \right)(z_1^{(1)}, \ldots, z_{n_1}^{(1)}, \ldots, z_1^{(m)}, \ldots, z_{n_m}^{(m)}) = \prod_{l=1}^{m} f_l(z_1^{(l)}, \ldots, z_{n_l}^{(l)}).$$

DEFINITION 4.3. Let $\Pi_{n_1, \ldots, n_m}$ be the set of all partitions of the set of variables $z_1^{(1)}, \ldots, z_{n_m}^{(m)}$ such that two variables $z_i^{(l)}$ and $z_j^{(l)}$ with $i \neq j$ but the same upper index (l) are always in different blocks, and such that every block includes at least two variables.

In this definition, we think of variables as combinatorial objects and partition a set of them. This is slightly different from the approach in [28], where the variables are numbered, and the partitions are defined for a set of numbers. Observe that by definition each block of $\pi \in \Pi_{n_1, \ldots, n_m}$ has at least two and at most m variables each of them with different upper index (l). Subsequently also the following subset of $\Pi_{n_1, \ldots, n_m}$ will play a central role.

DEFINITION 4.4. Let $\overline{\Pi}_{n_1, \ldots, n_m}$ be the set of all partitions $\pi \in \Pi_{n_1, \ldots, n_m}$ such that for any decomposition of $\{1, \ldots, m\}$ into two disjoint nonempty sets M_1, M_2 there are $l_1 \in M_1, l_2 \in M_2$ and two variables $z_i^{(l_1)}, z_j^{(l_2)}$ which are in the same block of π.

By $|\pi|$ we denote the number of blocks of the partition π. For every partition $\pi \in \Pi_{n_1, \ldots, n_m}$ we define the function $(\bigotimes_{l=1}^{m} f_l)_\pi : X^{|\pi|} \to \mathbb{R}$ by replacing all variables of $\bigotimes_{l=1}^{m} f_l$ that belong to the same block of π by a new common variable. The order of the new variables does not matter since we always integrate over all variables.

Let us recall that $\mathcal{S}(X^k)$ stands for the set of simple functions. These are all $f \in L_s^2(X^k)$ that are zero on all diagonals, are constant on a finite number of Cartesian products, and vanish everywhere else. For the product of multiple Wiener–Itô integrals of such functions the following proposition holds; see Corollary 7.2 in [28].

PROPOSITION 4.5. *Let* $f_l \in \mathcal{S}(X^{n_l})$ *for* $l = 1, \ldots, m$. *Then*

$$(12) \quad \mathbb{E} \prod_{l=1}^m I_{n_l}(f_l) = \sum_{\pi \in \Pi_{n_1,\ldots,n_m}} \int_{X^{|\pi|}} \left(\bigotimes_{l=1}^m f_l \right)_\pi (y_1, \ldots, y_{|\pi|}) \, d\mu(y_1, \ldots, y_{|\pi|}).$$

As a consequence of Proposition 3.1 in [36], equation (12) is also true for $f_l \in L_s^2(X^k)$, $l = 1, \ldots, m$, satisfying

$$(13) \quad \left(\bigotimes_{l=1}^m f_l \right)_\pi \in L^2(X^{|\pi|})$$

for all partitions π of the set of variables such that all variables of a function are in different blocks. For some classes of functions f_l it is obvious that (13) holds, for example, if the f_l are bounded and have a support of finite measure. But in general it is difficult to verify condition (13).

In order to avoid this problem, we approximate a general U-statistic by a sequence of U-statistics, whose kernels are simple functions, and apply Proposition 4.5. Afterward, we extend our results to the original U-statistic. From now on, we assume that F is an absolutely convergent U-statistic.

Because of $f \in L_s^1(X^k)$, there exists a sequence $(f^{(n)})_{n \in \mathbb{N}}$ of functions in $\mathcal{S}(X^k)$ such that $|f^{(n)}| \leq |f|$ μ^k-almost everywhere and $(f^{(n)})_{n \in \mathbb{N}}$ converges to f μ^k-almost everywhere on X^k. We define U-statistics $F^{(n)}$, $n \in \mathbb{N}$, by

$$F^{(n)} = \sum_{(x_1,\ldots,x_k) \in \eta_{\neq}^k} f^{(n)}(x_1, \ldots, x_k).$$

Since $(f^{(n)})_{n \in \mathbb{N}}$ converges μ^k-almost everywhere on X^k to f,

$$\lim_{n \to \infty} f^{(n)}(x_1, \ldots, x_k) = f(x_1, \ldots, x_k) \qquad \text{for all } (x_1, \ldots, x_k) \in \eta_{\neq}^k$$

holds with probability 1. Furthermore, the absolute convergence of F implies

$$|F^{(n)}| \leq \sum_{(x_1,\ldots,x_k) \in \eta_{\neq}^k} |f^{(n)}(x_1, \ldots, x_k)| \leq \sum_{(x_1,\ldots,x_k) \in \eta_{\neq}^k} |f(x_1, \ldots, x_k)| \in L^2(\mathbb{P}).$$

Hence, $(F^{(n)})_{n \in \mathbb{N}}$ converges almost surely to F, and the dominated convergence theorem implies even convergence in $L^1(\mathbb{P})$ and $L^2(\mathbb{P})$. Moreover, $F^{(n)} \in L^2(\mathbb{P})$, and every $F^{(n)}$ has a Wiener–Itô chaos expansion with kernels $f_i^{(n)}$ that are simple functions since integration over a variable of a simple function leads to a simple function.

The fact that the kernels of $F^{(n)}$ are simple functions brings us in the position to use Proposition 4.5 to evaluate R_{ij} and $\tilde{R}_i$ for $i, j = 1, \ldots, k$. We start by esti-

mating R_{ii}. By (12), we have

$$
\begin{aligned}
R_{ii} &= \int_{X^2} \mathbb{E} I_{i-1}\big(f_i^{(n)}(s, \cdot)\big)^2 I_{i-1}\big(f_i^{(n)}(t, \cdot)\big)^2 \, d\mu(s, t) - \big[(i-1)! \|f_i^{(n)}\|^2\big]^2 \\
&= \sum_{\pi \in \Pi_{i-1, i-1, i-1, i-1}} \int_{X^{|\pi|+2}} \big(f_i^{(n)}(s, \cdot) \otimes f_i^{(n)}(s, \cdot) \\
&\qquad\qquad\qquad \otimes f_i^{(n)}(t, \cdot) \otimes f_i^{(n)}(t, \cdot)\big)_\pi (y_1, \ldots, y_{|\pi|}) \\
&\qquad\qquad\qquad d\mu(y_1, \ldots, y_{|\pi|}, s, t) \\
&\qquad - \big[(i-1)! \|f_i^{(n)}\|^2\big]^2.
\end{aligned}
$$

The sum over those partitions of $\Pi_{i-1, i-1, i-1, i-1}$ such that every block contains only variables of the first pair or of the second pair of functions leads exactly to $[(i-1)! \|f_i^{(n)}\|^2]^2$. These partitions cancel out with the minus term and we denote the remaining partitions by $\tilde{\Pi}_{i-1, i-1, i-1, i-1}$. Hence,

$$
\begin{aligned}
R_{ii} &= \sum_{\tilde{\pi} \in \tilde{\Pi}_{i-1, i-1, i-1, i-1}} \int_{X^{|\tilde{\pi}|+2}} \big(f_i^{(n)}(s, \cdot) \otimes f_i^{(n)}(s, \cdot) \\
&\qquad\qquad\qquad \otimes f_i^{(n)}(t, \cdot) \otimes f_i^{(n)}(t, \cdot)\big)_{\tilde{\pi}} (y_1, \ldots, y_{|\tilde{\pi}|}) \\
&\qquad\qquad\qquad d\mu(y_1, \ldots, y_{|\tilde{\pi}|}, s, t) \\
&\leq \sum_{\tilde{\pi} \in \tilde{\Pi}_{i-1, i-1, i-1, i-1}} \int_{X^{|\tilde{\pi}|+2}} \big| \big(f_i^{(n)}(s, \cdot) \otimes f_i^{(n)}(s, \cdot) \\
&\qquad\qquad\qquad \otimes f_i^{(n)}(t, \cdot) \otimes f_i^{(n)}(t, \cdot)\big)_{\tilde{\pi}} (y_1, \ldots, y_{|\tilde{\pi}|}) \big| \\
&\qquad\qquad\qquad d\mu(y_1, \ldots, y_{|\tilde{\pi}|}, s, t).
\end{aligned}
$$

In order to simplify our notation, we include s and t into the partitions by adding two blocks generating s and t to the old partition $\tilde{\pi}$ and obtain a new partition $\pi \in \Pi_{i, i, i, i}$. By definition of $\tilde{\pi}$, π has at least one block including variables $z_{i_1}^{(l_1)}$ and $z_{i_2}^{(l_2)}$, $l_1 \in \{1, 2\}$, $l_2 \in \{3, 4\}$. By construction of π, there are also blocks including variables of the first two functions and of the last two functions. Altogether, this implies $\pi \in \overline{\Pi}_{i, i, i, i}$. Since each $\tilde{\pi} \in \tilde{\Pi}_{i-1, i-1, i-1, i-1}$ leads to a different $\pi \in \overline{\Pi}_{i, i, i, i}$, we obtain the upper bound

$$
\begin{aligned}
R_{ii} \leq \sum_{\pi \in \overline{\Pi}_{i, i, i, i}} \int_{X^{|\pi|}} \big| \big(f_i^{(n)} \otimes f_i^{(n)} \otimes f_i^{(n)} \otimes f_i^{(n)}\big)_\pi (y_1, \ldots, y_{|\pi|}) \big| \\
d\mu(y_1, \ldots, y_{|\pi|}).
\end{aligned}
$$

In the very same way, we obtain an upper bound for $R_{ij}, i \neq j$. By (12), it follows

$$
\begin{aligned}
R_{ij} &= \int_{X^2} \mathbb{E}\big[I_{i-1}\big(f_i^{(n)}(s, \cdot)\big) I_{j-1}\big(f_j^{(n)}(s, \cdot)\big) I_{i-1}\big(f_i^{(n)}(t, \cdot)\big) \\
&\qquad\qquad\qquad\qquad \times I_{j-1}\big(f_j^{(n)}(t, \cdot)\big)\big] d\mu(s, t) \\
&= \sum_{\tilde{\pi} \in \Pi_{i-1, j-1, i-1, j-1}} \int_{X^{|\tilde{\pi}|+2}} \big(f_i^{(n)}(s, \cdot) \otimes f_j^{(n)}(s, \cdot) \\
&\qquad\qquad\qquad \otimes f_i^{(n)}(t, \cdot) \otimes f_j^{(n)}(t, \cdot)\big)_{\tilde{\pi}}(y_1, \dots, y_{|\tilde{\pi}|}) \\
&\qquad\qquad\qquad d\mu(y_1, \dots, y_{|\tilde{\pi}|}, s, t) \\
&\leq \sum_{\pi \in \overline{\Pi}_{i,j,i,j}} \int_{X^{|\pi|}} \big|\big(f_i^{(n)} \otimes f_j^{(n)} \otimes f_i^{(n)} \otimes f_j^{(n)}\big)_\pi(y_1, \dots, y_{|\pi|})\big| d\mu(y_1, \dots, y_{|\pi|}).
\end{aligned}
$$

Since $i \neq j$, there exist no $\tilde{\pi} \in \Pi_{i-1, j-1, i-1, j-1}$ with blocks including either variables of the first two or last two functions. Hence, we obtain partitions $\pi \in \overline{\Pi}_{i,j,i,j}$ by the same construction as for R_{ii} and obtain an upper bound by summing over $\overline{\Pi}_{i,j,i,j}$.

The last step is to estimate $\tilde{R}_i$. Here, we have in a similar way

$$
\begin{aligned}
\tilde{R}_i &= \int_X \mathbb{E} I_{i-1}\big(f_i^{(n)}(s, \cdot)\big)^4 d\mu(s) \\
&= \sum_{\tilde{\pi} \in \Pi_{i-1, i-1, i-1, i-1}} \int_{X^{|\tilde{\pi}|+1}} \big(f_i^{(n)}(s, \cdot) \otimes f_i^{(n)}(s, \cdot) \\
&\qquad\qquad\qquad \otimes f_i^{(n)}(s, \cdot) \otimes f_i^{(n)}(s, \cdot)\big)_{\tilde{\pi}}(y_1, \dots, y_{|\tilde{\pi}|}) \\
&\qquad\qquad\qquad d\mu(y_1, \dots, y_{|\tilde{\pi}|}, s) \\
&\leq \sum_{\pi \in \overline{\Pi}_{i,i,i,i}} \int_{X^{|\pi|}} \big|\big(f_i^{(n)} \otimes f_i^{(n)} \otimes f_i^{(n)} \otimes f_i^{(n)}\big)_\pi(y_1, \dots, y_{|\pi|})\big| d\mu(y_1, \dots, y_{|\pi|}).
\end{aligned}
$$

In this case, it is immediate that we obtain a partition $\pi \in \overline{\Pi}_{i,i,i,i}$ by adding s to a partition $\tilde{\pi} \in \Pi_{i-1, i-1, i-1, i-1}$. Thus R_{ij} and $\tilde{R}_i$ are bounded by the same expressions.

Now it remains to estimate the kernels $f_i^{(n)}$. From Lemma 3.5, it follows that

$$
\begin{aligned}
&\big|f_i^{(n)}(y_1, \dots, y_i)\big| \\
&\qquad \leq \binom{k}{i} \int_{X^{k-i}} \big|f^{(n)}(y_1, \dots, y_i, x_1, \dots, x_{k-i})\big| d\mu(x_1, \dots, x_{k-i}).
\end{aligned}
$$

We obtain the following expression as an upper bound for R_{ij} and $\tilde{R}_i$. With $M_{ij}(\cdot)$ defined by

$$
\begin{aligned}
M_{ij}(g) = &\binom{k}{i}^2 \binom{k}{j}^2 \\
&\times \sum_{\pi \in \overline{\Pi}_{i,j,i,j}} \int_{X^{|\pi|+4k-2i-2j}} |(g(\cdot, x_1^{(1)}, \ldots, x_{k-i}^{(1)}) \\
&\qquad\qquad\qquad\qquad \otimes g(\cdot, x_1^{(2)}, \ldots, x_{k-j}^{(2)}) \\
&\qquad\qquad\qquad\qquad \otimes g(\cdot, x_1^{(3)}, \ldots, x_{k-i}^{(3)}) \\
&\qquad\qquad\qquad\qquad \otimes g(\cdot, x_1^{(4)}, \ldots, x_{k-j}^{(4)}))_\pi (y_1, \ldots, y_{|\pi|})| \\
&\qquad\qquad d\mu(x_1^{(1)}, \ldots, x_{k-j}^{(4)}, y_1, \ldots, y_{|\pi|}),
\end{aligned}
$$

(14)

where π acts on the first i, respectively, j variables of $g : X^k \to \mathbb{R}$, we have

$$
R_{ij} \le M_{ij}(f^{(n)}) \quad \text{and} \quad \tilde{R}_i \le M_{ii}(f^{(n)}) \qquad \text{for } 1 \le i, j \le k.
$$

Since in the definition of M_{ij} every block of a partition $\pi \in \overline{\Pi}_{i,j,i,j}$ has at least two elements, the integration in (14) runs over at most $4k - i - j$ variables. For $i = j = 1$ the only partition in $\overline{\Pi}_{i,j,i,j}$ is the partition with one block and the integration runs over $4k - 3$ variables. This observation will be important in Section 5.

Combining our bounds for R_{ij} and $\tilde{R}_i$ with Theorem 4.1 yields:

LEMMA 4.6. *Suppose $F^{(n)} = \sum_{\eta_{\neq}^k} f^{(n)}$ is a U-statistic of order k with $f^{(n)} \in \mathcal{S}(X^k)$ and N is a standard Gaussian random variable. Then*

$$
d_W\left(\frac{F^{(n)} - \mathbb{E}F^{(n)}}{\sqrt{\operatorname{Var} F^{(n)}}}, N\right) \le 2k^{7/2} \sum_{1 \le i \le j \le k} \frac{\sqrt{M_{ij}(f^{(n)})}}{\operatorname{Var} F^{(n)}}.
$$

Together with the fact that $M_{ij}(f^{(n)}) \le M_{ij}(f)$ since $|f^{(n)}| \le |f|$ and the triangle inequality for the Wasserstein distance, we obtain

$$
\begin{aligned}
&d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \\
&\quad \le d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, \frac{F^{(n)} - \mathbb{E}F^{(n)}}{\sqrt{\operatorname{Var} F^{(n)}}}\right) + d_W\left(\frac{F^{(n)} - \mathbb{E}F^{(n)}}{\sqrt{\operatorname{Var} F^{(n)}}}, N\right) \\
&\quad \le d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, \frac{F^{(n)} - \mathbb{E}F^{(n)}}{\sqrt{\operatorname{Var} F^{(n)}}}\right) + 2k^{7/2} \sum_{1 \le i \le j \le k} \frac{\sqrt{M_{ij}(f)}}{\operatorname{Var} F^{(n)}}.
\end{aligned}
$$

By the definition of the Wasserstein distance and some straightforward computations, it follows that

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, \frac{F^{(n)} - \mathbb{E}F^{(n)}}{\sqrt{\operatorname{Var} F^{(n)}}}\right) \leq \mathbb{E}\left|\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}} - \frac{F^{(n)} - \mathbb{E}F^{(n)}}{\sqrt{\operatorname{Var} F^{(n)}}}\right|$$

$$\leq \frac{\mathbb{E}|F^{(n)} - F + \mathbb{E}F - \mathbb{E}F^{(n)}|}{\sqrt{\operatorname{Var} F^{(n)}}}$$

$$+ \left|\frac{\mathbb{E}|F - \mathbb{E}F|}{\sqrt{\operatorname{Var} F}} - \frac{\mathbb{E}|F - \mathbb{E}F|}{\sqrt{\operatorname{Var} F^{(n)}}}\right|.$$

Because of the convergence of $(F^{(n)})_{n \in \mathbb{N}}$ to F in $L^1(\mathbb{P})$ and $L^2(\mathbb{P})$, the right-hand side vanishes for $n \to \infty$, and we get our main result.

THEOREM 4.7. *Suppose F is an absolutely convergent U-statistic of order k, and N is a standard Gaussian random variable. Then*

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq 2k^{7/2} \sum_{1 \leq i \leq j \leq k} \frac{\sqrt{M_{ij}(f)}}{\operatorname{Var} F}$$

with $M_{ij}(f)$ defined in (14).

5. Geometric U-statistics.

5.1. *Central limit theorems for geometric U-statistics.* In this section, we assume that our intensity measure has the form $\mu(\cdot) = \lambda\theta(\cdot)$ with a σ-finite nonatomic measure $\theta(\cdot)$ and $\lambda \geq 1$. We are interested in the behavior of the U-statistic F for $\lambda \to \infty$.

DEFINITION 5.1. A U-statistic $F = \sum_{\eta_{\neq}^k} f$ is a *geometric U-statistic* if it satisfies

$$f(x_1, \ldots, x_k) = g(\lambda)\tilde{f}(x_1, \ldots, x_k)$$

with $g : \mathbb{R} \to \mathbb{R}$, and with $\tilde{f} : X^k \to \mathbb{R}$ not depending on λ.

In the case that $g = 1$ and $f = \tilde{f}$, the value of F for a given realization of the Poisson point process is only determined by the geometry of the points and does not depend on the intensity rate λ of the underlying process. The term "geometric" is used to emphasize this behavior. We slightly generalize this property by allowing our geometric U-statistics to have an intensity related scaling factor since we always consider standardized random variables, where the scaling factor is cancelled out.

By $\tilde{M}_{ij}$ we denote the value of $M_{ij}(\tilde{f})$, which is defined in (14), for $\lambda = 1$. With this notation, the following central limit theorem holds:

THEOREM 5.2. *Suppose F is an absolutely convergent geometric U-statistic of order k with $\|f_1\| > 0$ and N is a standard Gaussian random variable. Then*

$$\lim_{\lambda \to \infty} \frac{\operatorname{Var} F}{\lambda^{2k-1} g(\lambda)^2} = \underbrace{k^2 \int_X \left(\int_{X^{k-1}} \tilde{f}(y, x_1, \ldots, x_{k-1}) \, d\theta(x_1, \ldots, x_{k-1}) \right)^2 d\theta(y)}_{=:\tilde{V}}$$

with $\tilde{V} > 0$, and

$$(15) \qquad d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N \right) \leq \lambda^{-1/2}\left(2k^{7/2} \sum_{1 \leq i \leq j \leq k} \frac{\sqrt{\tilde{M}_{ij}}}{\tilde{V}} \right)$$

for $\lambda \geq 1$.

The main feature of this theorem is that the term in brackets is independent of λ, which means that for $\lambda \to \infty$ the distance to the Gaussian distribution tends to zero at a rate $\lambda^{-1/2}$.

PROOF OF THEOREM 5.2. Because we are interested in the standardized variable $(F - \mathbb{E}F)/\sqrt{\operatorname{Var} F}$ which is independent of $g(\lambda)$, w.l.o.g. we put $g(\lambda) = 1$ and $\tilde{f} = f$. From formula (6) we infer

$$\operatorname{Var} F = \sum_{i=1}^{k} \lambda^{2k-i} i! \binom{k}{i}^2$$

$$\times \int_{X^i} \left(\int_{X^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i}) \, d\theta(x_1, \ldots, x_{k-i}) \right)^2$$

$$d\theta(y_1, \ldots, y_i),$$

which means that the variance is a polynomial of degree $2k - 1$ with the leading term $\tilde{V}\lambda^{2k-1} = \|f_1\|^2 > 0$ and $\operatorname{Var} F \geq \tilde{V}\lambda^{2k-1}$.

As previously mentioned, the integration in $M_{ij}(f)$ runs over at most $4k - i - j \leq 4k - 3$ variables for $(i, j) \neq (1, 1)$ and $4k - 3$ variables for $(i, j) = (1, 1)$, and we see that

$$M_{ij}(f) \leq \tilde{M}_{ij} \lambda^{4k-3}$$

for $\lambda \geq 1$. Hence, Theorem 4.7 leads directly to (15). □

The assumption $\|f_1\| > 0$ cannot be easily dispensed as can be seen from the following example:

EXAMPLE. Let η be a Poisson process on $[-1, 1]$ with intensity measure the Lebesgue measure times intensity $\lambda > 0$. We define the U-statistic $F =$

$\sum_{(x_1,x_2)\in\eta_{\neq}^2} f(x_1,x_2)$ with

$$f(x_1,x_2) = \begin{cases} 1, & x_1 x_2 \geq 0, \\ -1, & x_1 x_2 < 0. \end{cases}$$

Obviously, we obtain $f_1(y) = 0$. It is possible to rewrite F as $F = L(L-1) + R(R-1) - 2LR$ where L and R are the number of points in $[-1,0]$ and $[0,1]$, respectively. This brings us in the position to compute the moments. Elementary calculations show that the variance equals $8\lambda^2$, and the third moment of F is $64\lambda^3$. Thus the third moment of $(F - \mathbb{E}F)/\sqrt{\operatorname{Var} F}$ tends to a constant and hence is too large for convergence of F to a Gaussian distribution. By a technical computation of all moments, using the product formula for multiple Wiener–Itô integrals, for example, and the method of moments, it can be shown that $\sqrt{2}(F - \mathbb{E}F)/\sqrt{\operatorname{Var} F}$ follows a centered chi-square distribution with one degree of freedom as $\lambda \to \infty$.

In the special case $\mu(X) = \lambda\theta(X) < \infty$, it is possible to approximate the Poisson point process η by a binomial point process, that consists of a fixed number of independently distributed points with the probability measure $\theta(\cdot)/\theta(X)$. If we sum over k-tuples of distinct points of the binomial point process instead of a Poisson point process, we obtain a classical U-statistic. This well-known class of random variables satisfies a similar central limit theorem as above with a rate of convergence; see [7, 11, 16, 20]. Although both results are similar, it seems to be difficult to prove one result by the other, especially with keeping rates of convergence.

For classical U-statistics the so-called Hoeffding decomposition which is closely related to the Wiener–Itô chaos expansion plays a crucial role. In the recent paper by Lachiéze-Rey and Peccati [18], this decomposition is applied to U-statistics of Poisson point processes which yields a representation similar to the Wiener–Itô chaos expansion. Combining this with the result of Dynkin and Mandelbaum [6], the authors derive our Theorem 5.2 for the case $\mu(X) < \infty$ (without rates of convergence). They also prove noncentral limit theorems for the case that some of the first kernels of the chaos expansion of a U-statistic vanish, which allows one to deal with situations as in the previous example.

In Sections 5.2 and 5.3, we apply Theorem 5.2 to problems from stochastic geometry. In the recent paper [5] the underlying result from [27] is used to derive a central limit theorem for the number random simplices on a torus. This problem exactly fits in the framework of geometric U-statistics, and some of the results can also be obtained by using Theorem 5.2.

5.2. *Central limit theorems for Poisson hyperplanes.* We use Theorem 5.2 to establish central limit theorems for Poisson hyperplane processes. Let η be a Poisson process on the space $\mathcal{H}$ of all hyperplanes in $\mathbb{R}^d$ with an intensity measure of the form $\mu(\cdot) = \lambda\theta(\cdot)$ with $\lambda \in \mathbb{R}^+$ and a σ-finite nonatomic measure θ. The Poisson hyperplane process is only observed in a compact convex window $W \subset \mathbb{R}^d$

with interior points. Thus, we can view η as a Poisson process on the set $[W]$ defined by

$$[W] = \{h \in \mathcal{H} : h \cap W \neq \varnothing\}.$$

Given the hyperplane process η, we investigate the $(d-k)$-flats in W which occur as the intersection of k hyperplanes of η. In particular, we are interested in the sum of their ith intrinsic volumes given by

$$\Phi_i^k(W) = \frac{1}{k!} \sum_{(h_1,\ldots,h_k) \in \eta_{\neq}^k} V_i(h_1 \cap \cdots \cap h_k \cap W)$$

for $i = 0, \ldots, d - k$ and $k = 1, \ldots, d$. For the definition of the ith intrinsic volume $V_i(\cdot)$ we refer to [31]. We remark that $V_0(K)$ is the Euler characteristic of the set K, and that $V_n(K)$ of an n-dimensional convex set K is the Lebesgue measure $\Lambda_n(K)$. Thus Φ_0^k is the number of $(d-k)$-flats in W, and Φ_{d-k}^k is their $(d-k)$-volume. To ensure that the expectations of these random variables are neither 0 nor infinite, we assume that:

- $0 < \theta([W]) < \infty$;
- $2 \leq k \leq d$ independent random hyperplanes on $[W]$ with probability measure $\theta(\cdot)/\theta([W])$ intersect in a $(d-k)$-flat almost surely and their intersection flat hits the interior of W with positive probability.

For example, these conditions are satisfied if the hyperplane process is stationary and the directional distribution is not concentrated on a great subsphere.

The fact that the summands in the definition of Φ_i^k are bounded and have a bounded support makes sure that the fourth moments in $M_{ij}(\cdot)$ are finite, and we can apply Theorem 5.2:

THEOREM 5.3. *Let N be a standard Gaussian random variable. Then constants $c_\Phi(W, k, i)$ exist such that*

$$d_W\left(\frac{\Phi_i^k(W) - \mathbb{E}\Phi_i^k(W)}{\sqrt{\operatorname{Var}\Phi_i^k(W)}}, N\right) \leq c_\Phi(W, k, i)\lambda^{-1/2}$$

for $\lambda \geq 1$, $i = 0, \ldots, d - k$ and $k = 1, \ldots, d$.

Furthermore, the asymptotic variances are given by

$$\lim_{\lambda \to \infty} \frac{\operatorname{Var}\Phi_i^k(W)}{\lambda^{2k-1}}$$

$$= \frac{1}{(k-1)!^2}$$

$$\times \int_{[W]} \left(\int_{[W]^{k-1}} V_i(h \cap h_1 \cap \cdots \cap h_{k-1} \cap W) \, d\theta(h_1, \ldots, h_{k-1}) \right)^2 d\theta(h).$$

Similar results have first been derived by Paroux [26], and by Heinrich [9] and Heinrich, Schmidt and Schmidt [10] using Hoeffding's decomposition of classical U-statistics. Schulte and Thäle [33] used the Wiener–Itô chaos expansion to compute the moments and cumulants and to formulate central limit theorems for the surface area of Poisson hyperplanes in an increasing window. In their recent paper [34] this approach is further refined to obtain point process convergence for the intrinsic volumes of the intersection process of Poisson k-flats in the unit ball.

5.3. *Convex hulls of random points.* In the following, we assume that the Poisson point process η has an intensity-measure of the form $\mu(\cdot) = \lambda\Lambda_d(\cdot \cap K)$, $\lambda \geq 1$, where Λ_d is Lebesgue measure, and $K \subset \mathbb{R}^d$ a compact convex set with $\Lambda_d(K) = 1$. If we integrate with respect to Λ_d, we omit the measure in our notation.

We consider the following functional related to Sylvester's problem:

$$H = \sum_{(x_1,\ldots,x_k) \in \eta_{\neq}^k} h(x_1, \ldots, x_k)$$

with

$$h(x_1, \ldots, x_k) = \mathbb{1}(x_1, \ldots, x_k \text{ are vertices of } \operatorname{conv}(x_1, \ldots, x_k)),$$

which counts the number of k-tuples of the process such that every point is a vertex of the convex hull, that is, the number of k-tuples in convex position. The expected value of H is then given by

$$\mathbb{E}H = \lambda^k \mathbb{P}(X_1, \ldots, X_k \text{ are vertices of } \operatorname{conv}(X_1, \ldots, X_k)) = \lambda^k p^{(k)}(K),$$

where $X_1, \ldots, X_k$ are independent random points chosen according to the uniform distribution on K.

The question to determine the probability $p^{(k)}(K)$ that k random points in a convex set K are in convex position has a long history; see, for example, the more recent developments by Bárány [1, 2] and Buchta [3]. In our setting, the function H is an estimator for the probability $p^{(k)}(K)$, and we are interested in distributional properties of this estimator.

The asymptotic behavior of $\operatorname{Var} H$ is determined by

$$\tilde{H} = \lim_{\lambda \to \infty} \frac{\operatorname{Var} H}{\lambda^{2k-1}} = k^2 \int_K \left(\int_{K^{k-1}} h(y, x_1, \ldots, x_{k-1}) \, dx_1 \cdots dx_{k-1} \right)^2 dy.$$

By the Cauchy–Schwarz inequality, because $\Lambda_d(K) = 1$ and $h^2 = h$, we obtain

$$\tilde{H} \leq k^2 \int_K \int_{K^{k-1}} h(y, x_1, \ldots, x_{k-1})^2 \, dx_1 \cdots dx_{k-1} \, dy$$
$$= k^2 p^{(k)}(K)$$

and

$$k^2 p^{(k)}(K)^2 = k^2 \left(\int_{K^k} h(x_1, \ldots, x_k) \, dx_1 \cdots dx_k \right)^2$$

$$\leq k^2 \int_K \left(\int_{K^{k-1}} h(x_1, x_2, \ldots, x_k) \, dx_2 \cdots dx_k \right)^2 dx_1 = \tilde{H}.$$

Together with Theorem 5.2, we immediately get the following result showing that the estimator H is asymptotically Gaussian:

THEOREM 5.4. *Let N be a standard Gaussian random variable. Then there exists a constant C such that*

$$d_W\left(\frac{H - \mathbb{E}H}{\sqrt{\operatorname{Var} H}}, N \right) \leq C\lambda^{-1/2}.$$

Furthermore $\operatorname{Var} H = \lambda^{2k-1} \tilde{H} (1 + O(\lambda^{-1}))$ *as* $\lambda \to \infty$ *with*

$$k^2 p^{(k)}(K)^2 \leq \tilde{H} \leq k^2 p^{(k)}(K).$$

6. Local U-statistics.

6.1. *Central limit theorems for local U-statistics.* For a geometric U-statistic the function f is [up to the scaling factor $g(\lambda)$] independent of λ. Now we allow that f is influenced by λ in a more intricate way, but we assume that a k-tupel of points is only in the support of f if the points are close together.

From now on, let X be a metric space, and denote by $B(y, r)$ the ball with center y and radius r. Again, we assume that the intensity measure μ has the form $\mu(\cdot) = \lambda\theta(\cdot)$ with $\lambda \geq 1$ and a σ-finite nonatomic measure $\theta(\cdot)$ on X. We denote the diameter of $A \subset X$ by $\operatorname{diam}(A)$.

DEFINITION 6.1. A U-statistic $F = \sum_{\eta_{\neq}^k} f$ is a *local U-statistic* if it satisfies

(16) $f(x_1, \ldots, x_k) = 0$ if $\operatorname{diam}(\{x_1, \ldots, x_k\}) > \delta.$

Note that in general δ may depend on λ. We denote the L^2-norm on X^i with respect to the measure $\theta(\cdot)$ by $\| \cdot \|_\theta$. Now we can rephrase Theorem 4.7 for local U-statistics as follows:

THEOREM 6.2. *Suppose F is an absolutely convergent local U-statistic of order k with $\|f_1\| > 0$, and N is a standard Gaussian random variable. Putting $\tilde{V} = \|f_1\|^2/\lambda^{2k-1}$ and $b(\delta) = \max_{y \in X} \mu(B(y, 4\delta)) < \infty$, we have*

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N \right) \leq c_k \lambda^{-3k/2+1} \max\{1, b(\delta)^{(3k-3)/2}\} \frac{\|f^2\|_\theta}{\tilde{V}}$$

with a constant $c_k \in \mathbb{R}$ only depending on k.

PROOF. Formula (6) yields $\text{Var } F \geq \|f_1\|^2 = \lambda^{2k-1}\tilde{V}$. The estimate for M_{ij} runs as follows. Since $\pi \in \overline{\Pi}_{i,j,i,j}$, and condition (16) forces all arguments of f to be close, we can rewrite $M_{ij}(f)$ as

$$M_{ij}(f) = \binom{k}{i}^2 \binom{k}{j}^2$$
$$\times \sum_{\pi \in \overline{\Pi}_{i,j,i,j}} \int_{X^{|\pi|+4k-2i-2j}} |(f(\cdot, x_1^{(1)}, \ldots, x_{k-i}^{(1)})$$
$$\otimes f(\cdot, x_1^{(2)}, \ldots, x_{k-j}^{(2)})$$
$$\otimes f(\cdot, x_1^{(3)}, \ldots, x_{k-i}^{(3)})$$
$$\otimes f(\cdot, x_1^{(4)}, \ldots, x_{k-j}^{(4)}))_\pi (y_1, \ldots, y_{|\pi|})|$$
$$\times \mathbb{1}\big(\text{diam}(\{x_1^{(1)}, \ldots, x_{k-j}^{(4)}, y_1, \ldots, y_{|\pi|}\}) \leq 4\delta\big)$$
$$d\mu(x_1^{(1)}, \ldots, x_{k-j}^{(4)}, y_1, \ldots, y_{|\pi|}).$$

By Hölder's inequality, we obtain

$$M_{ij}(f) \leq c_{ij} \sum_{\pi \in \overline{\Pi}_{i,j,i,j}} \int_{X^{|\pi|+4k-2i-2j}} f(z_1, \ldots, z_k)^4$$
$$\times \mathbb{1}\big(\text{diam}(\{z_1, \ldots, z_{|\pi|+4k-2i-2j}\}) \leq 4\delta\big) d\mu(z_1, \ldots, z_{|\pi|+4k-2i-2j})$$
$$\leq c_{ij} \|f^2\|^2 \sum_{\pi \in \overline{\Pi}_{i,j,i,j}} b(\delta)^{|\pi|+3k-2i-2j}$$
$$= c_{ij} \lambda^k \|f^2\|_\theta^2 \sum_{\pi \in \overline{\Pi}_{i,j,i,j}} b(\delta)^{|\pi|+3k-2i-2j}$$

with a constant $c_{ij} \in \mathbb{R}$ depending on i, j, k. One should keep in mind that $\max(i, j) \leq |\pi| \leq i + j$ for all $\pi \in \overline{\Pi}_{i,j,i,j}$ and that the only partition $\pi \in \overline{\Pi}_{1,1,1,1}$ satisfies $|\pi| = 1$. This leads to $|\pi| - 2i - 2j \leq -3$ and

$$2k^{7/2} \sum_{1 \leq i \leq j \leq k} \frac{\sqrt{M_{ij}(f)}}{\text{Var } F} \leq c_k' \sum_{1 \leq i \leq j \leq k} \frac{\sqrt{\lambda^k \|f^2\|_\theta^2 \sum_{\pi \in \overline{\Pi}_{i,j,i,j}} b(\delta)^{|\pi|+3k-2i-2j}}}{\lambda^{2k-1}\tilde{V}}$$
$$\leq c_k' \lambda^{-3k/2+1} \frac{\|f^2\|_\theta}{\tilde{V}} \sum_{1 \leq i \leq j \leq k} \sqrt{\sum_{\pi \in \overline{\Pi}_{i,j,i,j}} b(\delta)^{|\pi|+3k-2i-2j}}$$
$$\leq c_k \lambda^{-3k/2+1} \frac{\|f^2\|_\theta}{\tilde{V}} \max\{1, b(\delta)^{(3k-3)/2}\}$$

with constants $c'_k, c_k \in \mathbb{R}$ only depending on k. Combining this estimate with Theorem 4.1 gives the claimed result. $\square$

The proof rests essentially upon the fact that F is a local U-statistic since this allows us to rewrite $M_{ij}(f)$ such that every function depends on all variables and to split these functions using Hölder's inequality.

6.2. *A central limit theorem for the total edge length of a random geometric graph.* We apply the results of the previous subsection to a problem from random graph theory. We construct a random graph in the following way. Let η be a Poisson process in $X = \mathbb{R}^d$ with an intensity measure of the form

$$\mu(\cdot) = \lambda \Lambda_d(\cdot \cap W)$$

with $\lambda \geq 1$, the d-dimensional Lebesgue-measure $\Lambda_d(\cdot)$ and a compact window $W \subset \mathbb{R}^d$ of volume $\Lambda_d(W) = 1$ containing the origin in its interior. We regard η as a set of points in W. As in (16) we connect two points $x, y \in \eta$ by an edge if

$$\|x - y\| \leq \delta = \delta(\lambda).$$

The resulting graph $G(P_\lambda, \delta)$ is a random geometric graph, sometimes called a Gilbert graph or an interval graph (for $d = 1$) and a disk graph (for $d = 2$). For graph-theoretical properties of $G(P_\lambda, \delta)$ we refer to [30] and to the more recent developments [8, 17, 21, 23]. For our central limit theorem we take $\lambda \to \infty$ and assume that δ is small enough to ensure that

$$\bigcap_{x \in B(0,\delta)} (W + x) \supset \frac{1}{2} W.$$

We are interested in the total edge length $L(\eta)$ of $G(P_\lambda, \delta)$ in the window W, which is given by

$$L(\eta) = \frac{1}{2} \sum_{(x,y) \in \eta_{\neq}^2} g(x - y) \mathbb{1}(\|x - y\| \leq \delta).$$

Here $g : B(0, \delta) \to \mathbb{R}$ is some kind of measure of the length of the edge (x, y). We assume $g \in L^2(B(0, \delta))$ which implies that L is absolutely convergent. The following lemma is immediate from Lemma 3.5.

LEMMA 6.3. *$L(\eta)$ has a Wiener–Itô chaos expansion with kernels*

$$f_1(y) = \lambda \int_{B(0,\delta)} g(x) \mathbb{1}(y + x \in W) \, dx, \qquad y \in W$$

and

$$f_2(x, y) = \tfrac{1}{2} g(x - y) \mathbb{1}(\|x - y\| \leq \delta), \qquad x, y \in W.$$

For the length of this random graph, we obtain the following central limit theorem:

THEOREM 6.4. *Assume* $g \in L^2(B(0,\delta))$ *with* $\int_{B(0,\delta)} g(x)\,dx \neq 0$, *and let* N *be a standard Gaussian random variable. Then there is a constant* c_d *only depending on the dimension d such that*

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq c_d \lambda^{-2} \max\{1, b(\delta)^{3/2}\} \frac{(\int_{B(0,\delta)} g(x)^4\,dx)^{1/2}}{(\int_{B(0,\delta)} g(x)\,dx)^2}.$$

PROOF. We compute the bound from Theorem 6.2. Lemma 6.3 yields

$$\tilde{V} = \frac{\|f_1\|^2}{\lambda^3} = \int_W \left(\int_{B(0,\delta)} g(x)\mathbb{1}(y + x \in W)\,dx\right)^2 dy$$

$$\geq \int_{(1/2)W} \left(\int_{B(0,\delta)} g(x)\,dx\right)^2 dy = 2^{-d}\left(\int_{B(0,\delta)} g(x)\,dx\right)^2$$

and

$$\|f^2\|_\theta^2 = \frac{1}{16} \int_W \int_{B(0,\delta)} g(x)^4 \mathbb{1}(y + x \in W)\,dx\,dy \leq \frac{1}{16} \int_{B(0,\delta)} g(x)^4\,dx. \quad \square$$

As an example we consider the particular case $g = 1$, where $L(\eta)$ reduces to the number of edges of the graph. Then the expectation is of order $\lambda^2 \delta^d$. Lemma 6.3 and Theorem 6.4 tell us that the variance is of order $\max\{\lambda^3 \delta^{2d}, \lambda^2 \delta^d\}$ and that

$$d_W\left(\frac{L - \mathbb{E}L}{\sqrt{\operatorname{Var} L}}, N\right) \leq \tilde{c}_d \lambda^{-2} \delta^{-3d/2} \max\{1, \lambda^{3/2} \delta^{3d/2}\}$$

with a constant $\tilde{c}_d \in \mathbb{R}$ only depending on d. The right-hand side tends to zero if $\lambda^{4/3} \Lambda_d(B(0,\delta)) \to \infty$ as $\lambda \to \infty$. In the maybe most natural case when $\lambda \Lambda_d(B(0,\delta))$ stays constant we have an order $\lambda^{-1/2}$ of convergence to the Gaussian distribution. A central limit theorem without rate of convergence is a special case of Theorem 3.9 in [30].

Similar results to Lemma 6.3 and Theorem 6.4 can be obtained if the intensity measure is of the form $d\mu(x) = \lambda f(x)\,d\Lambda_d(x)$ with $\lambda \in \mathbb{R}^+$ and a density function $f(x)$.

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