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THE OBSTACLE PROBLEM FOR QUASILINEAR STOCHASTIC PDES: ANALYTICAL APPROACH

BY LAURENT DENIS[1], ANIS MATOUSSI[2] AND JING ZHANG

Université d'Evry Val d'Essonne, Université du Maine and CMAP, and Université d'Evry Val d'Essonne

We prove an existence and uniqueness result for quasilinear Stochastic PDEs with obstacle (OSPDE in short). Our method is based on analytical technics coming from the parabolic potential theory. The solution is expressed as a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying the minimal Skohorod condition.

1. Introduction. The starting point of this work is the following parabolic stochastic partial differential equation (in short, SPDE):

$$\begin{aligned} du_t(x) &= \partial_i\big(a_{i,j}(x)\partial_j u_t(x) + g_i\big(t,x,u_t(x),\nabla u_t(x)\big)\big)\,dt \\ &\quad + f\big(t,x,u_t(x),\nabla u_t(x)\big)\,dt \\ &\quad + \sum_{j=1}^{+\infty} h_j\big(t,x,u_t(x),\nabla u_t(x)\big)\,dB_t^j, \end{aligned} \tag{1}$$

where a is a symmetric bounded measurable matrix which defines a second order operator on an open domain $\mathcal{O} \subset \mathbb{R}^d$, with Dirichlet boundary condition. The initial condition is given as $u_0 = \xi$, a $L^2(\mathcal{O})$-valued random variable, and f, $g = (g_1, \dots, g_d)$ and $h = (h_1, \dots, h_i, \dots)$ are nonlinear random functions. Given an obstacle $S : \Omega \times [0, T] \times \mathcal{O} \to \mathbb{R}$, we study the obstacle problem for SPDE (1), that is, we want to find a solution of (1) which satisfies "$u \geq S$" where the obstacle S is regular in some sense and controlled by the solution of a SPDE.

Nualart and Pardoux [16] have studied the obstacle problem for a nonlinear heat equation on the spatial interval $[0, 1]$ with Dirichlet boundary conditions, driven by an additive space–time white noise. They proved the existence and uniqueness

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of the solution and their method relied heavily on the results for a deterministic variational inequality. Donati-Martin and Pardoux [9] generalized the model of Nualart and Pardoux. The nonlinearity appears both in the drift and in the diffusion coefficients. They proved the existence of the solution by penalization method but they did not obtain the uniqueness result. And then in 2009, Xu and Zhang solved the problem of the uniqueness; see [21]. However, in all their models, there is not the term of divergence and they do not consider the case where the coefficients depend on ∇u.

The work of El Karoui et al. [10] treats the obstacle problem for deterministic semilinear PDE's within the framework of backward stochastic differential equations (BSDE in short). Namely, the equation (1) is considered with f depending on u and ∇u, while the function g is null (as well h) and the obstacle v is continuous. They considered the viscosity solution of the obstacle problem for the equation (1), they represented this solution stochastically as a process and the main new object of this BSDE framework is a continuous increasing process that controls the set $\{u = v\}$. Bally et al. [3] (see also [14]) point out that the continuity of this process allows one to extend the classical notion of a strong variational solution (see Theorem 2.2 of [4], page 238) and express the solution to the obstacle as a pair (u, ν) where ν is supported by the set $\{u = v\}$.

Matoussi and Stoica [13] have proved an existence and uniqueness result for the obstacle problem of backward quasilinear stochastic PDE on the whole space $\mathbb{R}^d$ and driven by a finite dimensional Brownian motion. The method is based on the probabilistic interpretation of the solution by using the backward doubly stochastic differential equation (BDSDE in short). They have also proved that the solution is a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying the minimal Skohorod condition. In particular, they gave for the regular measure ν a probabilistic interpretation in terms of the continuous increasing process K where (Y, Z, K) is the solution of a reflected generalized BDSDE.

Michel Pierre [17, 18] has studied the parabolic PDE with obstacle using the parabolic potential as a tool. He proved that the solution uniquely exists and is quasi-continuous. With the help of Pierre's result, under suitable assumptions on f, g and h, our aim is to prove existence and uniqueness for the following SPDE with given obstacle S that we write formally as

$$\begin{cases} du_t(x) = \partial_i\big(a_{i,j}(x)\partial_j u_t(x) + g_i\big(t, x, u_t(x), \nabla u_t(x)\big)\big)\,dt \\ \qquad\qquad + f\big(t, x, u_t(x), \nabla u_t(x)\big)\,dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{+\infty} h_j\big(t, x, u_t(x), \nabla u_t(x)\big)\,dB_t^j, \\ u_t(x) \geq S_t(x), \qquad \forall (t, x) \in \mathbb{R}^+ \times \mathcal{O}, \\ u_0(x) = \xi(x), \qquad \forall x \in \mathcal{O}, \\ u_t(x) = 0, \qquad \forall (t, x) \in \mathbb{R}^+ \times \partial\mathcal{O}. \end{cases} \tag{2}$$

To give a rigorous definition to the notion of a solution to this equation, we will use the technics of parabolic potential theory developed by M. Pierre in the stochastic framework. We first prove a quasi-continuity result for the solution of the SPDE (1) with null Dirichlet condition on given domain $\mathcal{O}$ and driven by an infinite dimensional Brownian motion. This result is not obvious and is based on a mixing pathwise argument and Mignot and Puel [15] existence result of the obstacle problem for some deterministic PDEs. Moreover, we prove in our context that the reflected measure ν is a regular random measure and we give the analytical representation of such a measure in terms of parabolic potential in the sense given by M. Pierre in [17]. The main theorem we obtain is the following:

THEOREM 1. *Assume that f, g and h satisfy some Lipschitz continuity and integrability hypotheses, $\xi \in L^2(\Omega \times \mathcal{O})$, S is quasi-continuous and $S_t \leq S'_t$, where S' is the solution of the linear SPDE with null boundary condition*

$$\begin{cases} dS'_t = LS'_t\,dt + f'_t\,dt + \sum_{i=1}^{d} \partial_i g'_{i,t}\,dt + \sum_{j=1}^{+\infty} h'_{j,t}\,dB^j_t, \\ S'(0) = S'_0, \end{cases}$$

where $S'_0 \in L^2(\Omega \times \mathcal{O})$, and f', g' and h' are square integrable adapted processes.

Then there exists a unique solution (u, ν) of the obstacle problem for the SPDE (2) associated to (ξ, f, g, h, S), that is, u is a predictable continuous process which takes values in a proper Sobolev space, $u \geq S$ and ν is a random regular measure such that:

(1) *The following relation holds almost surely, for all $t \in [0, T]$ and $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$,*

$$\begin{aligned} &(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s)\,ds + \int_0^t \mathcal{E}(u_s, \varphi_s)\,ds \\ &\quad + \sum_{i=1}^{d} \int_0^t \big(g_s^i(u_s, \nabla u_s), \partial_i \varphi_s\big)\,ds \\ &\quad = \int_0^t \big(f_s(u_s, \nabla u_s), \varphi_s\big)\,ds + \sum_{j=1}^{+\infty} \int_0^t \big(h_s^j(u_s, \nabla u_s), \varphi_s\big)\,dB_s^j \\ &\qquad + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds). \end{aligned}$$

(2) *u admits a quasi-continuous version, $\tilde{u}$, and we have the mininal Skohorod condition*

$$\int_0^T \int_{\mathcal{O}} \big(\tilde{u}(s,x) - S(s,x)\big) \nu(dx, ds) = 0 \qquad a.s.$$

This paper is divided as follows: in the second section we set the assumptions, then we introduce in the third section the notion of a regular measure associated to parabolic potentials. The fourth section is devoted to prove the quasi-continuity of the solution of SPDE without obstacle. The fifth section is the main part of the paper in which we prove existence and uniqueness of the solution. To do that, we begin with the linear case, and then by Picard iteration we get the result in the nonlinear case; we also establish an Itô formula. Finally, in the sixth section we prove a comparison theorem for the solution of SPDE with obstacle.

2. Preliminaries. We consider a sequence $((B^i(t))_{t\geq 0})_{i\in\mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions.

Let $\mathcal{O} \subset \mathbb{R}^d$ be an open domain and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$. It is a Hilbert space equipped with the usual scalar product and norm as follows:

$$(u, v) = \int_{\mathcal{O}} u(x)v(x)\,dx, \qquad \|u\| = \left(\int_{\mathcal{O}} u^2(x)\,dx\right)^{1/2}.$$

Let A be a symmetric second order differential operator, with domain $\mathcal{D}(A)$, given by

$$A := -L = -\sum_{i,j=1}^{d} \partial_i\big(a^{i,j}(x)\partial_j\big).$$

We assume that $a(x) = (a^{i,j}(x))_{i,j}$ is a measurable symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{d} a^{i,j}(x)\xi^i\xi^j \leq \Lambda|\xi|^2 \qquad \forall x \in \mathcal{O}, \xi \in \mathbb{R}^d,$$

where λ and Λ are positive constants.

Let $(F, \mathcal{E})$ be the associated Dirichlet form given by $F := \mathcal{D}(A^{1/2}) = H_0^1(\mathcal{O})$ and

$$\mathcal{E}(u, v) := \big(A^{1/2}u, A^{1/2}v\big) \quad \text{and} \quad \mathcal{E}(u) = \big\|A^{1/2}u\big\|^2 \qquad \forall u, v \in F,$$

where $H_0^1(\mathcal{O})$ is the first order Sobolev space of functions vanishing at the boundary. As usual, we shall denote $H^{-1}(\mathcal{O})$ its dual space.

We consider the quasilinear stochastic partial differential equation (1) with initial condition $u(0,\cdot) = \xi(\cdot)$ and Dirichlet boundary condition $u(t, x) = 0$, $\forall (t, x) \in \mathbb{R}^+ \times \partial\mathcal{O}$.

We assume that we have predictable random functions

$$\begin{aligned} f &: \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \\ g = (g_1, \dots, g_d) &: \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \\ h = (h_1, \dots, h_i, \dots) &: \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{\mathbb{N}^*}. \end{aligned}$$

In the sequel, $|\cdot|$ will always denote the underlying Euclidean or l^2-norm. For example,

$$\big|h(t,\omega,x,y,z)\big|^2 = \sum_{i=1}^{+\infty}\big|h_i(t,\omega,x,y,z)\big|^2.$$

ASSUMPTION (H). There exist nonnegative constants C,α,β such that for almost all ω, the following inequalities hold for all $(t,x,y,z)\in\mathbb{R}^+\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d$:

(1) $|f(t,\omega,x,y,z)-f(t,\omega,x,y',z')|\le C(|y-y'|+|z-z'|)$,
(2) $(\sum_{i=1}^d|g_i(t,\omega,x,y,z)-g_i(t,\omega,x,y',z')|^2)^{1/2}\le C|y-y'|+\alpha|z-z'|$,
(3) $(|h(t,\omega,x,y,z)-h(t,\omega,x,y',z')|^2)^{1/2}\le C|y-y'|+\beta|z-z'|$,
(4) the contraction property: $2\alpha+\beta^2<2\lambda$.

REMARK 1. This last contraction property ensures existence and uniqueness for the solution of the SPDE without obstacle (see [8]).

With the uniform ellipticity condition we have the following equivalent conditions:

$$\|f(u,\nabla u)-f(v,\nabla v)\|\le C\|u-v\|+C\lambda^{-1/2}\mathcal{E}^{1/2}(u-v),$$
$$\|g(u,\nabla u)-g(v,\nabla v)\|_{L^2(\mathcal{O};\mathbb{R}^d)}\le C\|u-v\|+\alpha\lambda^{-1/2}\mathcal{E}^{1/2}(u-v),$$
$$\|h(u,\nabla u)-h(v,\nabla v)\|_{L^2(\mathcal{O};\mathbb{R}^{\mathbb{N}^*})}\le C\|u-v\|+\beta\lambda^{-1/2}\mathcal{E}^{1/2}(u-v).$$

Moreover, for simplicity, we fix a terminal time $T>0$, and we assume the following:

ASSUMPTION (I).

$$\xi\in L^2(\Omega\times\mathcal{O})\text{ is an }\mathcal{F}_0\text{-measurable random variable},$$
$$f(\cdot,\cdot,\cdot,0,0):=f^0\in L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}),$$
$$g(\cdot,\cdot,\cdot,0,0):=g^0=(g_1^0,\dots,g_d^0)\in L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^d),$$
$$h(\cdot,\cdot,\cdot,0,0):=h^0=(h_1^0,\dots,h_i^0,\dots)\in L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^{\mathbb{N}^*}).$$

Now we introduce the notion of a weak solution.

We denote by $\mathcal{H}_T$ the space of $H_0^1(\mathcal{O})$-valued predictable $L^2(\mathcal{O})$-continuous processes $(u_t)_{t\ge0}$ which satisfy

$$E\sup_{t\in[0,T]}\|u_t\|^2+E\int_0^T\mathcal{E}(u_t)\,dt<+\infty.$$

It is the natural space for solutions.

The space of test functions is denoted by $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ is the space of all real-valued infinitely differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}_c^2(\mathcal{O})$ is the set of C^2-functions with compact support in $\mathcal{O}$.

Heuristically, a pair (u, ν) is a solution of the obstacle problem for (1) with Dirichlet boundary condition if we have the following:

(1) $u \in \mathcal{H}_T$ and $u(t,x) \geq S(t,x)$, $dP \otimes dt \otimes dx$-a.e. and $u_0(x) = \xi$, $dP \otimes dx$-a.e.;

(2) ν is a random measure defined on $[0, T) \times \mathcal{O}$;

(3) the following relation holds almost surely, for all $t \in [0, T]$ and $\forall \varphi \in \mathcal{D}$,

$$\begin{aligned}
&(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s)\,ds + \int_0^t \mathcal{E}(u_s, \varphi_s)\,ds \\
&\quad + \sum_{i=1}^d \int_0^t \big(g_s^i(u_s, \nabla u_s), \partial_i \varphi_s\big)\,ds \\
&\quad = \int_0^t \big(f_s(u_s, \nabla u_s), \varphi_s\big)\,ds + \sum_{j=1}^{+\infty} \int_0^t \big(h_s^j(u_s, \nabla u_s), \varphi_s\big)\,dB_s^j \\
&\qquad + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds);
\end{aligned}$$

(4)

$$\int_0^T \int_{\mathcal{O}} \big(u(s,x) - S(s,x)\big) \nu(dx, ds) = 0 \qquad \text{a.s.}$$

But, the random measure, which in some sense obliges the solution to stay above the barrier, is a local time so, in general, it is not absolutely continuous w.r.t. the Lebesgue measure. As a consequence, for example, the condition

$$\int_0^T \int_{\mathcal{O}} \big(u(s,x) - S(s,x)\big) \nu(dx\,ds) = 0$$

makes no sense. Hence, we need to consider a precise version of u and S defined ν-almost surely.

In order to tackle this difficulty, we introduce in the next section the notions of parabolic capacity on $[0, T] \times \mathcal{O}$ and a quasi-continuous version of functions introduced by Michel Pierre in several works (see, e.g., [17, 18]). Let us remark that these tools were also used by Klimsiak [11] to get a probabilistic interpretation to semilinear PDEs with obstacle.

Finally and to end this section, we give an important example of stochastic noise which is covered by our framework:

EXAMPLE 1. Let W be a noise white in time and colored in space, defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ whose covariance function is given by

$$\forall s, t \in \mathbb{R}^+, \forall x, y \in \mathcal{O} \qquad E\big[\dot{W}(x,s)\dot{W}(y,t)\big] = \delta(t-s)k(x,y),$$

where $k : \mathcal{O} \times \mathcal{O} \mapsto \mathbb{R}^+$ is a symmetric and measurable function.

Consider the following SPDE driven by W:

$$\begin{aligned} du_t(x) = \Bigg(&\sum_{i,j=1}^{d} \partial_i a_{i,j}(x)\partial_j u_t(x) + f\big(t, x, u_t(x), \nabla u_t(x)\big) \\ &\qquad + \sum_{i=1}^{d} \partial_i g_i\big(t, x, u_t(x), \nabla u_t(x)\big)\Bigg) dt \\ &+ \tilde{h}\big(t, x, u_t(x), \nabla u_t(x)\big) W(dt, x), \end{aligned} \tag{3}$$

where f and g are as above and $\tilde{h}$ is a random real-valued function.

We assume that the covariance function k defines a trace class operator denoted by K in $L^2(\mathcal{O})$. It is well known (see [19]) that there exists an orthogonal basis $(e_i)_{i\in\mathbb{N}^*}$ of $L^2(\mathcal{O})$ consisting of eigenfunctions of K with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}^*}$ such that

$$\sum_{i=1}^{+\infty} \lambda_i < +\infty$$

and

$$k(x,y) = \sum_{i=1}^{+\infty} \lambda_i e_i(x) e_i(y).$$

It is also well known that there exists a sequence $((B^i(t))_{t\geq 0})_{i\in\mathbb{N}^*}$ of independent standard Brownian motions such that

$$W(dt, \cdot) = \sum_{i=1}^{+\infty} \lambda_i^{1/2} e_i B^i(dt),$$

so that equation (3) is equivalent to (1) with $h = (h_i)_{i\in\mathbb{N}^*}$ where

$$\forall i \in \mathbb{N}^* \qquad h_i(s,x,y,z) = \sqrt{\lambda_i}\,\tilde{h}(s,x,y,z) e_i(x).$$

Assume as in [20] that for all $i \in \mathbb{N}^*$, $\|e_i\|_\infty < +\infty$ and

$$\sum_{i=1}^{+\infty} \lambda_i \|e_i\|_\infty^2 < +\infty.$$

Since

$$
\begin{aligned}
&\left(\left|h(t,\omega,x,y,z)-h(t,\omega,x,y',z')\right|^2\right)^{1/2}\\
&\qquad\le\left(\sum_{i=1}^{+\infty}\lambda_i\|e_i\|_\infty^2\right)\left|\tilde h(t,x,y,z)-\tilde h(t,x,y',z')\right|^2,
\end{aligned}
$$

h satisfies the Lipschitz hypothesis (H)-(3) if $\tilde h$ satisfies a similar Lipschitz hypothesis.

3. Parabolic potential analysis.

3.1. *Parabolic capacity and potentials.* In this section we will recall some important definitions and results concerning the obstacle problem for parabolic PDE in [17] and [18].

$\mathcal{K}$ denotes $L^\infty([0,T];L^2(\mathcal{O}))\cap L^2([0,T];H_0^1(\mathcal{O}))$ equipped with the norm

$$
\begin{aligned}
\|v\|_{\mathcal{K}}^2&=\|v\|_{L^\infty([0,T];L^2(\mathcal{O}))}^2+\|v\|_{L^2([0,T];H_0^1(\mathcal{O}))}^2\\
&=\sup_{t\in[0,T[}\|v_t\|^2+\int_0^T\left(\|v_t\|^2+\mathcal{E}(v_t)\right)dt.
\end{aligned}
$$

$\mathcal{C}$ denotes the space of continuous functions on compact support in $[0,T[\times\mathcal{O}$ and, finally,

$$
\mathcal{W}=\left\{\varphi\in L^2\big([0,T];H_0^1(\mathcal{O})\big);\ \frac{\partial\varphi}{\partial t}\in L^2\big([0,T];H^{-1}(\mathcal{O})\big)\right\},
$$

endowed with the norm $\|\varphi\|_{\mathcal{W}}^2=\|\varphi\|_{L^2([0,T];H_0^1(\mathcal{O}))}^2+\|\frac{\partial\varphi}{\partial t}\|_{L^2([0,T];H^{-1}(\mathcal{O}))}^2$.

It is known (see [12]) that $\mathcal{W}$ is continuously embedded in $C([0,T];L^2(\mathcal{O}))$, the set of $L^2(\mathcal{O})$-valued continuous functions on $[0,T]$. So without ambiguity, we will also consider $\mathcal{W}_T=\{\varphi\in\mathcal{W};\varphi(T)=0\}$, $\mathcal{W}^+=\{\varphi\in\mathcal{W};\varphi\ge0\}$, $\mathcal{W}_T^+=\mathcal{W}_T\cap\mathcal{W}^+$.

We now introduce the notion of parabolic potentials and regular measures which permit to define the parabolic capacity.

DEFINITION 1. An element $v\in\mathcal{K}$ is said to be a *parabolic potential* if it satisfies

$$
\forall\varphi\in\mathcal{W}_T^+\qquad\int_0^T-\left(\frac{\partial\varphi_t}{\partial t},v_t\right)dt+\int_0^T\mathcal{E}(\varphi_t,v_t)\,dt\ge0.
$$

We denote by $\mathcal{P}$ the set of all parabolic potentials.

The next representation property is crucial:

PROPOSITION 1 (Proposition 1.1 in [18]). *Let $v \in \mathcal{P}$, then there exists a unique positive Radon measure on* $[0, T[\times\mathcal{O}$, *denoted by* ν^v, *such that*

$$\forall \varphi \in \mathcal{W}_T \cap \mathcal{C} \qquad \int_0^T \left(-\frac{\partial \varphi_t}{\partial t}, v_t\right) dt + \int_0^T \mathcal{E}(\varphi_t, v_t)\, dt = \int_0^T \int_{\mathcal{O}} \varphi(t,x)\, d\nu^v.$$

Moreover, v admits a right-continuous (*resp., left-continuous*) *version* $\hat{v}$ (*resp.,* $\bar{v}$) $: [0,T] \mapsto L^2(\mathcal{O})$.

Such a Radon measure ν^v is called a regular measure *and we write*

$$\nu^v = \frac{\partial v}{\partial t} + Av.$$

REMARK 2. As a consequence, we can also define for all $v \in \mathcal{P}$,

$$v_T = \lim_{t \uparrow T} \bar{v}_t \in L^2(\mathcal{O}).$$

DEFINITION 2. Let $K \subset [0,T[\times\mathcal{O}$ be compact; $v \in \mathcal{P}$ is said to be ν-superior than 1 on K, if there exists a sequence $v_n \in \mathcal{P}$ with $v_n \geq 1$ a.e. on a neighborhood of K converging to v in $L^2([0,T]; H_0^1(\mathcal{O}))$.

We denote

$$\mathscr{S}_K = \{v \in \mathcal{P};\, v \text{ is } \nu\text{-superior to 1 on } K\}.$$

PROPOSITION 2 (Proposition 2.1 in [18]). *Let $K \subset [0,T[\times\mathcal{O}$ be compact, then $\mathscr{S}_K$ admits a smallest $v_K \in \mathcal{P}$ and the measure ν_K^v whose support is in K satisfies*

$$\int_0^T \int_{\mathcal{O}} d\nu_K^v = \inf_{v \in \mathcal{P}} \left\{ \int_0^T \int_{\mathcal{O}} d\nu^v;\, v \in \mathscr{S}_K \right\}.$$

DEFINITION 3 (Parabolic capacity).

- Let $K \subset [0,T[\times\mathcal{O}$ be compact, and we define $\operatorname{cap}(K) = \int_0^T \int_{\mathcal{O}} d\nu_K^v$.
- Let $O \subset [0,T[\times\mathcal{O}$ be open, and we define $\operatorname{cap}(O) = \sup\{\operatorname{cap}(K);\, K \subset O \text{ compact}\}$.
- For any Borelian $E \subset [0,T[\times\mathcal{O}$, we define $\operatorname{cap}(E) = \inf\{\operatorname{cap}(O);\, O \supset E \text{ open}\}$.

DEFINITION 4. A property is said to hold quasi-everywhere (in short, q.e.) if it holds outside a set of null capacity.

DEFINITION 5 (Quasi-continuity). A function $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is called quasi-continuous, if there exists a decreasing sequence of open subsets O_n of $[0,T[\times\mathcal{O}$ with the following:

(1) for all n, the restriction of u_n to the complement of O_n is continuous;

(2) $\lim_{n\to+\infty} \mathrm{cap}(O_n) = 0$.

We say that u admits a quasi-continuous version, if there exists $\tilde{u}$ quasi-continuous such that $\tilde{u} = u$ a.e.

The next proposition, whose proof may be found in [17] or [18], shall play an important role in the sequel:

PROPOSITION 3. *Let $K \subset \mathcal{O}$ be a compact set, then $\forall t \in [0, T[$,*

$$\mathrm{cap}\big(\{t\} \times K\big) = \lambda_d(K),$$

where λ_d is the Lebesgue measure on $\mathcal{O}$.

As a consequence, if $u : [0, T[\times\mathcal{O} \to \mathbb{R}$ is a map defined quasi-everywhere, then it defines uniquely a map from $[0, T[$ into $L^2(\mathcal{O})$. In other words, for any $t \in [0, T[$, u_t is defined without any ambiguity as an element in $L^2(\mathcal{O})$. Moreover, if $u \in \mathcal{P}$, it admits version $\bar{u}$ which is left continuous on $[0, T]$ with values in $L^2(\mathcal{O})$ so that $u_T = \bar{u}_{T^-}$ is also defined without ambiguity.

REMARK 3. The previous proposition applies if, for example, u is quasi-continuous.

PROPOSITION 4 (Theorem III.1 in [18]). *If $\varphi \in \mathcal{W}$, then it admits a unique quasi-continuous version that we denote by $\tilde{\varphi}$. Moreover, for all $v \in \mathcal{P}$, the following relation holds*:

$$\int_{[0,T[\times\mathcal{O}} \tilde{\varphi}\, d\nu^v = \int_0^T (-\partial_t\varphi, v) + \mathcal{E}(\varphi, v)\, dt + (\varphi_T, v_T).$$

3.2. *Applications to PDEs with obstacle.* For any function $\psi : [0, T[\times\mathcal{O} \to \mathbb{R}$ and $u_0 \in L^2(\mathcal{O})$, following M. Pierre [17, 18], F. Mignot and J.P. Puel [15], we define

$$\kappa(\psi, u_0) = \operatorname{ess\,inf}\big\{u \in \mathcal{P};\, u \ge \psi \text{ a.e.}, u(0) \ge u_0\big\}. \tag{4}$$

This lower bound exists and is an element in $\mathcal{P}$. Moreover, when ψ is quasi-continuous, this potential is the solution of the following reflected problem:

$$\kappa \in \mathcal{P}, \kappa \ge \psi, \qquad \frac{\partial\kappa}{\partial t} + A\kappa = 0 \qquad \text{on } \{u > \psi\}, \qquad \kappa(0) = u_0.$$

Mignot and Puel have proved in [15] that $\kappa(\psi, u_0)$ is the limit [increasingly and weakly in $L^2([0, T]; H_0^1(\mathcal{O}))$] when ε tends to 0 of the solution of the following penalized equation:

$$u_\varepsilon \in \mathcal{W}, \qquad u_\varepsilon(0) = u_0, \qquad \frac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon - \frac{(u_\varepsilon - \psi)^-}{\varepsilon} = 0.$$

Let us point out that they obtain this result in the more general case where ψ is only measurable from $[0,T[$ into $L^2(\mathcal{O})$.

For given $f \in L^2([0,T]; H^{-1}(\mathcal{O}))$, we denote by $\kappa_{u_0}^f$ the solution of the following problem:

$$\kappa \in \mathcal{W}, \qquad \kappa(0) = u_0, \qquad \frac{\partial \kappa}{\partial t} + A\kappa = f.$$

The next theorem ensures existence and uniqueness of the solution of parabolic PDE with obstacle; it is proved in [17], Theorem 1.1. The proof is based on a regularization argument of the obstacle, using the results of [5].

THEOREM 2. *Let $\psi : [0,T[\times\mathcal{O} \to \mathbb{R}$ be quasi-continuous, suppose that there exists $\zeta \in \mathcal{P}$ with $|\psi| \le \zeta$ a.e., $f \in L^2([0,T]; H^{-1}(\mathcal{O}))$, and the initial value $u_0 \in L^2(\mathcal{O})$ with $u_0 \ge \psi(0)$, then there exists a unique $u \in \kappa_{u_0}^f + \mathcal{P}$ quasi-continuous such that*

$$u(0) = u_0, \tilde{u} \ge \psi,\ q.e.; \qquad \int_0^T \int_{\mathcal{O}} (\tilde{u} - \tilde{\psi})\, d\nu^{u - \kappa_{u_0}^f} = 0.$$

We end this section by a convergence lemma which plays an important role in our approach (Lemma 3.8 in [18]):

LEMMA 1. *If $(v^n)_n \in \mathcal{P}$ is a bounded sequence in $\mathcal{K}$ and converges weakly to v in $L^2([0,T]; H_0^1(\mathcal{O}))$, and if u is a quasi-continuous function and $|u|$ is bounded by a element in $\mathcal{P}$, then*

$$\lim_{n\to+\infty} \int_0^T \int_{\mathcal{O}} u\, d\nu^{v^n} = \int_0^T \int_{\mathcal{O}} u\, d\nu^{v}.$$

REMARK 4. For the more general case one can see [18], Lemma 3.8.

4. Quasi-continuity of the solution of SPDE without obstacle. As a consequence of well-known results (see, e.g., [8], Theorem 8), we know that under Assumptions (H) and (I), SPDE (1) with zero Dirichlet boundary condition admits a unique solution in $\mathcal{H}_T$ (for the definition of solution see, e.g., Definition 1 in [8]); we denote it by $\mathcal{U}(\xi, f, g, h)$. The main theorem of this section is the following:

THEOREM 3. *Under Assumptions (H) and (I), $u = \mathcal{U}(\xi, f, g, h)$ the solution of SPDE (1) admits a quasi-continuous version denoted by $\tilde{u}$, that is, $u = \tilde{u}\ dP \otimes dt \otimes dx$-a.e. and for almost all $w \in \Omega$, $(t,x) \to \tilde{u}_t(w,x)$ is quasi-continuous.*

Before giving the proof of this theorem, we need the following lemmas. The first one is proved in [18], Lemma 3.3:

LEMMA 2. *There exists $C > 0$ such that, for all open set $\vartheta \subset [0, T[\times\mathcal{O}$ and $v \in \mathcal{P}$ with $v \geq 1$ a.e. on ϑ,*

$$\operatorname{cap}\vartheta \leq C\|v\|_{\mathcal{K}}^2.$$

Let $\kappa = \kappa(u, u^+(0))$ be defined by relation (4). One has to note that κ is a random function. From now on, we always take for κ the following measurable version

$$\kappa = \sup_n v^n,$$

where $(v^n)_n$ is the nondecreasing sequence of random functions given by

$$\begin{cases} \dfrac{\partial v_t^n}{\partial t} = Lv_t^n + n\big(v_t^n - u_t\big)^-, \\ v_0^n = u^+(0). \end{cases} \tag{5}$$

Using the results recalled in Section 3, we know that for almost all $w \in \Omega$, $v^n(w)$ converges weakly to $v(w) = \kappa(u(w), u^+(0)(w))$ in $L^2([0, T]; H_0^1(\mathcal{O}))$ and that $v \geq u$.

LEMMA 3. *We have the following estimate*:

$$E\|\kappa\|_{\mathcal{K}}^2 \leq C\left(E\|u_0^+\|^2 + E\|u_0\|^2 + E\int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2\,dt\right),$$

where C is a constant depending only on the structure constants of the equation.

PROOF. All along this proof, we shall denote by C or C_ε some constant which may change from line to line.

The following estimate for the solution of the SPDE we consider is well known:

$$\begin{aligned} &E\sup_{t\in[0,T]}\|u_t\|^2 + E\int_0^T \mathcal{E}(u_t)\,dt \\ &\qquad \leq CE\left(\|u_0\|^2 + \int_0^T \big(\|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2\big)\,dt\right), \end{aligned} \tag{6}$$

where C is a constant depending only on the structure constants of the equation.

Consider the approximation $(v^n)_n$ defined by (5), P-almost surely, it converges weakly to $v = \kappa(u, u^+(0))$ in $L^2([0, T]; H_0^1(\mathcal{O}))$.

We remark that $v^n - u$ satisfies the following equation:

$$\begin{aligned} &d\big(v_t^n - u_t\big) + A\big(v_t^n - u_t\big)\,dt \\ &\qquad = -f_t(u_t, \nabla u_t)\,dt - \sum_{i=1}^d \partial_i g_t^i(u_t, \nabla u_t)\,dt \\ &\qquad\qquad - \sum_{j=1}^{+\infty} h_t^j(u_t, \nabla u_t)\,dB_t^j + n\big(v_t^n - u_t\big)^-\,dt. \end{aligned}$$

Applying Itô's formula to $(v^n - u)^2$ (see Lemma 7 in [7]), we have almost surely, for all $t \in [0, T]$,

$$\begin{aligned}
&\|v_t^n - u_t\|^2 + 2\int_0^t \mathcal{E}(v_s^n - u_s)\,ds \\
&\quad = \|u_0^+ - u_0\|^2 - 2\int_0^t (v_s^n - u_s, f_s(u_s, \nabla u_s))\,ds \\
&\qquad + 2\sum_{i=1}^d \int_0^t (\partial_i(v_s^n - u_s), g_s^i(u_s, \nabla u_s))\,ds + \int_0^t \||h_s(u_s, \nabla u_s)|\|^2\,ds \\
&\qquad - 2\sum_{j=1}^{+\infty} \int_0^t (v_s^n - u_s, h_s^j(u_s, \nabla u_s))\,dB_s^j + 2\int_0^t (n(v_s^n - u_s)^-, v_s^n - u_s)\,ds.
\end{aligned} \tag{7}$$

The last term in the right member of (7) is obviously nonpositive, so

$$\begin{aligned}
&\|v_t^n - u_t\|^2 + 2\int_0^t \mathcal{E}(v_s^n - u_s)\,ds \\
&\quad \le \|u_0^+ - u_0\|^2 - 2\int_0^t (v_s^n - u_s, f_s(u_s, \nabla u_s))\,ds \\
&\qquad + \int_0^t \||h_s(u_s, \nabla u_s)|\|^2\,ds + 2\sum_{i=1}^d \int_0^t (\partial_i(v_s^n - u_s), g_s^i(u_s, \nabla u_s))\,ds \\
&\qquad - 2\sum_{j=1}^{+\infty} \int_0^t (v_s^n - u_s, h_s^j(u_s, \nabla u_s))\,dB_s^j \qquad \text{a.s.}
\end{aligned} \tag{8}$$

Then taking expectation and using Cauchy–Schwarz's inequality, we get

$$\begin{aligned}
&E\|v_t^n - u_t\|^2 + \left(2 - \frac{\varepsilon}{\lambda}\right) E\int_0^t \mathcal{E}(v_s^n - u_s)\,ds \\
&\quad \le E\|u_0^+ - u_0\|^2 + E\int_0^t \|v_s^n - u_s\|^2\,ds \\
&\qquad + E\int_0^t \|f_s(u_s, \nabla u_s)\|^2\,ds + C_\varepsilon E\int_0^t \||g_s(u_s, \nabla u_s)|\|^2\,ds \\
&\qquad + E\int_0^t \||h_s(u_s, \nabla u_s)|\|^2\,ds.
\end{aligned}$$

Therefore, by using the Lipschitz conditions on the coefficients, we have

$$\begin{aligned}
&E\|v_t^n - u_t\|^2 + \left(2 - \frac{\varepsilon}{\lambda}\right) E\int_0^t \mathcal{E}(v_s^n - u_s)\,ds \\
&\quad \le E\|u_0^+ - u_0\|^2 + E\int_0^t \|v_s^n - u_s\|^2\,ds
\end{aligned}$$

$$+ CE\int_0^t (\|f_s^0\|^2 + \||g_s^0|\|^2 + \||h_s^0|\|^2)\,ds + CE\int_0^t \|u_s\|^2\,ds$$
$$+ \left(\frac{C}{\lambda} + \frac{\alpha}{\lambda} + \frac{\beta^2}{\lambda}\right) E\int_0^t \mathcal{E}(u_s)\,ds.$$

Combining with (6), this yields

$$\begin{aligned} E\|v_t^n - u_t\|^2 &+ \left(2 - \frac{\varepsilon}{\lambda}\right) E\int_0^t \mathcal{E}(v_s^n - u_s)\,ds \\ &\le E\|u_0^+ - u_0\|^2 + E\int_0^t \|v_s^n - u_s\|^2\,ds \\ &\quad + CE\left(\|u_0\|^2 + \int_0^T (\|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2)\,dt\right). \end{aligned}$$

We take now ε small enough such that $(2 - \frac{\varepsilon}{\lambda}) > 0$, then, with Gronwall's lemma, we obtain for each $t \in [0, T]$

$$\begin{aligned} &E\|v_t^n - u_t\|^2 \\ &\quad \le Ce^{c'T}\left(E\|u_0^+ - u_0\|^2 + E\|u_0\|^2 + E\int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2\,dt\right). \end{aligned}$$

As we a priori know that P-almost surely, $(v^n)_n$ tends to κ strongly in $L^2([0, T] \times \mathcal{O})$, the previous estimate yields, thanks to the dominated convergence theorem, that $(v^n)_n$ converges to κ strongly in $L^2(\Omega \times [0, T] \times \mathcal{O})$ and

$$\begin{aligned} &\sup_{t\in[0,T]} E\|\kappa_t - u_t\|^2 \\ &\quad \le Ce^{c'T}\left(E\|u_0^+ - u_0\|^2 + E\|u_0\|^2 + E\int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2\,dt\right). \end{aligned}$$

Moreover, as $(v^n)_n$ tends to κ weakly in $L^2([0, T]; H_0^1(\mathcal{O}))$ P-almost-surely, we have for all $t \in [0, T]$,

$$\begin{aligned} &E\int_0^T \mathcal{E}(\kappa_s - u_s)\,ds \\ &\quad \le \liminf_n E\int_0^T \mathcal{E}(v_s^n - u_s)\,ds \\ &\quad \le TCe^{c'T}\left(E\|u_0^+ - u_0\|^2 + E\|u_0\|^2 + E\int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2\,dt\right). \end{aligned}$$

Let us now study the stochastic term in (8). Let us define the martingales

$$M_t^n = \sum_{j=1}^{+\infty} \int_0^t (v_s^n - u_s, h_s^j)\,dB_s^j \quad \text{and} \quad M_t = \sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s^j)\,dB_s^j.$$

Then

$$
\begin{aligned}
&E\big[\big|M_T^n - M_T\big|^2\big]\\
&\quad = E\int_0^T \sum_{j=1}^{+\infty} \big(\kappa_s - v_s^n, h_s\big)^2\,ds\\
&\quad \le E\int_0^T \|\kappa_s - v_s^n\|^2 \big\||h_s|\big\|^2\,ds.
\end{aligned}
$$

Using the strong convergence of $(v^n)_n$ to κ, we conclude that $(M^n)_n$ tends to M in the L^2 sense. Passing to the limit in (8), we get, almost surely, for all $t \in [0, T]$,

$$
\begin{aligned}
&\|\kappa_t - u_t\|^2 + 2\int_0^t \mathcal{E}(\kappa_s - u_s)\,ds\\
&\quad \le \big\|u_0^+ - u_0\big\|^2 - 2\int_0^t \big(\kappa_s - u_s, f_s(u_s, \nabla u_s)\big)\,ds\\
&\qquad + 2\sum_{i=1}^{d}\int_0^t \big(\partial_i(\kappa_s - u_s), g_s^i(u_s, \nabla u_s)\big)\,ds\\
&\qquad - 2\sum_{j=1}^{+\infty}\int_0^t \big(\kappa_s - u_s, h_s^j(u_s, \nabla u_s)\big)\,dB_s^j\\
&\qquad + \int_0^t \big\||h_s(u_s, \nabla u_s)|\big\|^2\,ds.
\end{aligned}
$$

As a consequence of Burkholder–Davies–Gundy's inequalities, we get

$$
\begin{aligned}
&E\sup_{t\in[0,T]}\left|\sum_{j=1}^{+\infty}\int_0^t \big(\kappa_s - u_s, h_s^j(u_s, \nabla u_s)\big)\,dB_s^j\right|\\
&\quad \le CE\left[\int_0^T \sum_{j=1}^{+\infty}\big(\kappa_s - u_s, h_s^j(u_s, \nabla u_s)\big)^2\,ds\right]^{1/2}\\
&\quad \le CE\left[\int_0^T \sum_{j=1}^{+\infty}\sup_{t\in[0,T]}\|\kappa_t - u_t\|^2\big\|h_s^j(u_s, \nabla u_s)\big\|^2\,ds\right]^{1/2}\\
&\quad \le CE\left[\sup_{t\in[0,T]}\|\kappa_t - u_t\|\left(\int_0^T \big\||h_t(u_t, \nabla u_t)|\big\|^2\,dt\right)^{1/2}\right]\\
&\quad \le \varepsilon E\sup_{t\in[0,T]}\|\kappa_t - u_t\|^2 + C_\varepsilon E\int_0^T \big\||h_t(u_t, \nabla u_t)|\big\|^2\,dt.
\end{aligned}
$$

By Lipschitz conditions on h and (6) this yields

$$\begin{aligned} E \sup_{t\in[0,T]} &\left|\sum_{j=1}^{+\infty}\int_0^t (\kappa_s - u_s, h_s(u_s, \nabla u_s))\, dB_s\right| \\ &\leq \varepsilon E \sup_{t\in[0,T]} \|\kappa_t - u_t\|^2 \\ &\quad + C\biggl(E\|u_0\|^2 + E\int_0^T (\|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2)\, dt\biggr). \end{aligned}$$

Hence,

$$\begin{aligned} (1-\varepsilon)&E \sup_{t\in[0,T]} \|\kappa_t - u_t\|^2 + \left(2 - \frac{\varepsilon}{\lambda}\right) E\int_0^T \mathcal{E}(\kappa_t - u_t)\, dt \\ &\leq C\biggl(E\|u_0^+ - u_0\|^2 + E\|u_0\|^2 + E\int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2\, dt\biggr). \end{aligned}$$

We can take ε small enough such that $1-\varepsilon > 0$ and $2 - \frac{\varepsilon}{\lambda} > 0$, hence,

$$\begin{aligned} E &\sup_{t\in[0,T]} \|\kappa_t - u_t\|^2 + E\int_0^T \mathcal{E}(\kappa_t - u_t)\, dt \\ &\leq C\biggl(E\|u_0^+ - u_0\|^2 + E\|u_0\|^2 + E\int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2\, dt\biggr). \end{aligned}$$

Then, combining with (6), we get the desired estimate:

$$\begin{aligned} E &\sup_{t\in[0,T]} \|\kappa_t\|^2 + E\int_0^T \mathcal{E}(\kappa_t)\, dt \\ &\leq C\biggl(E\|u_0^+\|^2 + E\|u_0\|^2 + E\int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2\, dt\biggr). \qquad \square \end{aligned}$$

PROOF OF THEOREM 3. For simplicity, we put

$$f_t(x) = f(t, x, u_t(x), \nabla u_t(x)), \qquad g_t(x) = g(t, x, u_t(x), \nabla u_t(x)) \quad \text{and}$$
$$h_t(x) = h(t, x, u_t(x), \nabla u_t(x)).$$

We introduce (P_t) the semi-group associated to operator A and put for each $n \in \mathbb{N}^*$, $i \in \{1, \dots, d\}$ and each $j \in \mathbb{N}^*$,

$$u_0^n = P_{1/n}u_0, \qquad f^n = P_{1/n}f, \qquad g_i^n = P_{1/n}g_i \quad \text{and} \quad h_j^n = P_{1/n}h_j.$$

Then $(u_0^n)_n$ converges to u_0 in $L^2(\Omega; L^2(\mathcal{O}))$, and $(f^n)_n$, $(g^n)_n$ and $(h^n)_n$ are sequences of elements in $L^2(\Omega\times[0,T]; \mathcal{D}(A))$ which converge, respectively, to f,

g and h in $L^2(\Omega \times [0,T]; L^2(\mathcal{O}))$. For all $n \in \mathbb{N}^*$ we define

$$u_t^n = P_t u_0^n + \int_0^t P_{t-s} f_s^n \, ds + \sum_{i=1}^d \int_0^t P_{t-s} \partial_i g_{i,s}^n \, ds + \sum_{j=1}^{+\infty} \int_0^t P_{t-s} h_{j,s}^n \, dB_s^j$$
$$= P_{t+(1/n)} u_0 + \int_0^t P_{t+(1/n)-s} f_s \, ds + \sum_{i=1}^d \int_0^t P_{t+(1/n)-s} \partial_i g_{i,s} \, ds$$
$$+ \sum_{j=1}^{+\infty} \int_0^t P_{t+(1/n)-s} h_{j,s} \, dB_s^j.$$

We denote by $G(t,x,s,y)$ the kernel associated to P_t, then

$$\begin{aligned} u^n(t,x) = & \int_{\mathcal{O}} G\Big(t + \frac{1}{n}, x, 0, y\Big) u_0(y)\, dy \\ & + \int_0^t \int_{\mathcal{O}} G\Big(t + \frac{1}{n}, x, s, y\Big) f(s,y) \, dy \, ds \\ & + \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} G\Big(t + \frac{1}{n}, x, s, y\Big) \partial_i g_s^i(y) \, dy \, ds \\ & + \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} G\Big(t + \frac{1}{n}, x, s, y\Big) h_s^j(y) \, dy \, dB_s^i. \end{aligned}$$

But, as A is strictly elliptic, G is uniformly continuous in space–time variables on any compact away from the diagonal in time (see Theorem 6 in [1]) and satisfies Gaussian estimates (see Aronson [2]); this ensures that for all $n \in \mathbb{N}^*$, u^n is P-almost surely continuous in (t,x).

We consider a sequence of random open sets

$$\vartheta_n = \{|u^{n+1} - u^n| > \varepsilon_n\}, \qquad \Theta_p = \bigcup_{n=p}^{+\infty} \vartheta_n.$$

Let $\kappa_n = \kappa(\frac{1}{\varepsilon_n}(u^{n+1} - u^n), \frac{1}{\varepsilon_n}(u^{n+1} - u^n)^+(0)) + \kappa(-\frac{1}{\varepsilon_n}(u^{n+1} - u^n), \frac{1}{\varepsilon_n}(u^{n+1} - u^n)^-(0))$, and from the definition of κ and the relation (see [18])

$$\kappa(|v|) \le \kappa(v, v^+(0)) + \kappa(-v, v^-(0)),$$

we know that κ_n satisfy the conditions of Lemma 2, that is, $\kappa_n \in \mathcal{P}$ et $\kappa_n \ge 1$ a.e. on ϑ_n, thus, we get the following relation:

$$\mathrm{cap}(\Theta_p) \le \sum_{n=p}^{+\infty} \mathrm{cap}(\vartheta_n) \le \sum_{n=p}^{+\infty} \|\kappa_n\|_{\mathcal{K}}^2.$$

Thus, remarking that $u^{n+1}-u^n=\mathcal{U}(u_0^{n+1}-u_0^n, f^{n+1}-f^n, g^{n+1}-g^n, h^{n+1}-h^n)$, we apply Lemma 3 to $\kappa(\frac{1}{\varepsilon_n}(u^{n+1}-u^n), \frac{1}{\varepsilon_n}(u^{n+1}-u^n)^+(0))$ and $\kappa(-\frac{1}{\varepsilon_n}(u^{n+1}-u^n), \frac{1}{\varepsilon_n}(u^{n+1}-u^n)^-(0))$ and obtain

$$\begin{aligned}
&E\bigl[\mathrm{cap}(\Theta_p)\bigr]\\
&\quad\le \sum_{n=p}^{+\infty} E\|\kappa_n\|_{\mathcal{K}}^2\\
&\quad\le 2C\sum_{n=p}^{+\infty}\frac{1}{\varepsilon_n^2}\biggl(E\|u_0^{n+1}-u_0^n\|^2+E\int_0^T\|f_t^{n+1}-f_t^n\|^2+\||g_t^{n+1}-g_t^n|\|^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\||h_t^{n+1}-h_t^n|\|^2\,dt\biggr).
\end{aligned}$$

Then, by extracting a subsequence, we can consider that

$$\begin{aligned}
&E\|u_0^{n+1}-u_0^n\|^2+E\int_0^T\|f_t^{n+1}-f_t^n\|^2+\||g_t^{n+1}-g_t^n|\|^2+\||h_t^{n+1}-h_t^n|\|^2\,dt\\
&\quad\le\frac{1}{2^n}.
\end{aligned}$$

Then we take $\varepsilon_n=\frac{1}{n^2}$ to get

$$E\bigl[\mathrm{cap}(\Theta_p)\bigr]\le\sum_{n=p}^{+\infty}\frac{2Cn^4}{2^n}.$$

Therefore,

$$\lim_{p\to+\infty}E\bigl[\mathrm{cap}(\Theta_p)\bigr]=0.$$

For almost all $\omega\in\Omega$, $u^n(\omega)$ is continuous in (t,x) on $(\Theta_p(w))^c$ and $(u^n(\omega))_n$ converges uniformly to u on $(\Theta_p(w))^c$ for all p, hence, $u(\omega)$ is continuous in (t,x) on $(\Theta_p(w))^c$. Then from the definition of quasi-continuous, we know that $u(\omega)$ admits a quasi-continuous version since $\mathrm{cap}(\Theta_p)$ tends to 0 almost surely as p tends to $+\infty$. $\square$

5. Existence and uniqueness of the solution of the obstacle problem.

5.1. *Weak solution*.

ASSUMPTION (O). The obstacle S is assumed to be an adapted process, quasi-continuous, such that $S_0\le\xi$ P-almost surely and controlled by the solution of a SPDE, that is, $\forall t\in[0,T]$,

$$S_t\le S_t',\tag{9}$$

where S' is the solution of the linear SPDE with Dirichlet boundary condition,

$$
\begin{cases} dS'_t = LS'_t\,dt + f'_t\,dt + \sum_{i=1}^{d} \partial_i g'_{i,t}\,dt + \sum_{j=1}^{+\infty} h'_{j,t}\,dB^j_t, \\ S'(0) = S'_0, \end{cases} \tag{10}
$$

where $S'_0 \in L^2(\Omega \times \mathcal{O})$ is $\mathcal{F}_0$-measurable, and f', g' and h' are adapted processes, respectively, in $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$, $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$ and $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}^*})$.

REMARK 5. Here again, we know that S' uniquely exists and satisfies the following estimate:

$$
\begin{aligned} &E \sup_{t\in[0,T]} \|S'_t\|^2 + E\int_0^T \mathcal{E}(S'_t)\,dt \\ &\qquad \le CE\Big[\|S'_0\|^2 + \int_0^T (\|f'_t\|^2 + \||g'_t|\|^2 + \||h'_t|\|^2)\,dt\Big]. \end{aligned} \tag{11}
$$

Moreover, from Theorem 3, S' admits a quasi-continuous version.

Let us also remark that even if this assumption seems restrictive since S' is driven by the same operator and Brownian motions as u, it encompasses a large class of examples.

We now are able to define rigorously the notion of the solution to the problem with obstacle we consider.

DEFINITION 6. A pair (u, ν) is said to be a solution of the obstacle problem for (1) with Dirichlet boundary condition if:

(1) $u \in \mathcal{H}_T$ and $u(t,x) \ge S(t,x)$, $dP \otimes dt \otimes dx$-a.e. and $u_0(x) = \xi$, $dP \otimes dx$-a.e.;

(2) ν is a random regular measure defined on $[0,T) \times \mathcal{O}$;

(3) the following relation holds almost surely, for all $t \in [0,T]$ and $\forall \varphi \in \mathcal{D}$,

$$
\begin{aligned} &(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s)\,ds + \int_0^t \mathcal{E}(u_s, \varphi_s)\,ds \\ &\quad + \sum_{i=1}^{d} \int_0^t \big(g^i_s(u_s, \nabla u_s), \partial_i \varphi_s\big)\,ds \\ &\quad = \int_0^t \big(f_s(u_s, \nabla u_s), \varphi_s\big)\,ds + \sum_{j=1}^{+\infty} \int_0^t \big(h^j_s(u_s, \nabla u_s), \varphi_s\big)\,dB^j_s \\ &\qquad + \int_0^t \int_{\mathcal{O}} \varphi_s(x)\nu(dx, ds); \end{aligned} \tag{12}
$$

(4) u admits a quasi-continuous version, $\tilde{u}$, and we have

$$\int_0^T \int_{\mathcal{O}} \big(\tilde{u}(s,x) - S(s,x)\big)\nu(dx,ds) = 0 \qquad \text{a.s.}$$

The main result of this paper is the following:

THEOREM 4. *Under Assumptions (H), (I) and (O), there exists a unique weak solution of the obstacle problem for the SPDE (1) associated to (ξ, f, g, h, S).*

We denote by $\mathcal{R}(\xi, f, g, h, S)$ the solution of SPDE (1) with obstacle when it exists and is unique.

As the proof of this theorem is quite long, we split it in several steps: first we prove existence and uniqueness in the linear case, then establish an Itô formula and finally prove the theorem thanks to a fixed point argument.

5.2. *Proof of Theorem 4 in the linear case.* All along this subsection, we assume that f, g and h do not depend on u and ∇u, so we consider that f, g and h are adapted processes, respectively, in $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R})$, $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^d)$ and $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^{\mathbb{N}^*})$.

For $n\in\mathbb{N}^*$, let u^n be the solution of the following SPDE:

$$du_t^n = Lu_t^n\,dt + f_t\,dt + \sum_{i=1}^d \partial_i g_{i,t}\,dt + \sum_{j=1}^{+\infty} h_{j,t}\,dB_t^j + n\big(u_t^n - S_t\big)^-\,dt \tag{13}$$

with initial condition $u_0^n = \xi$ and null Dirichlet boundary condition. We know from Theorem 8 in [8] that this equation admits a unique solution in $\mathcal{H}_T$ and that the solution admits $L^2(\mathcal{O})$-continuous trajectories.

LEMMA 4. *For all $n\in\mathbb{N}^*$, u^n satisfies the following estimate:*

$$E\sup_{t\in[0,T]}\|u_t^n\|^2 + E\int_0^T \mathcal{E}\big(u_t^n\big)\,dt + E\int_0^T n\big\|\big(u_t^n - S_t\big)^-\big\|^2\,dt \le C,$$

where C is a constant depending only on the structure constants of the SPDE.

PROOF. From (13) and (10), we know that $u^n - S'$ satisfies the following equation:

$$d\big(u_t^n - S_t'\big) = L\big(u_t^n - S_t'\big)\,dt + \tilde{f}_t\,dt + \sum_{i=1}^d \partial_i \tilde{g}_t^i\,dt + \sum_{j=1}^{+\infty} \tilde{h}_t^j\,dB_t^j + n\big(u_t^n - S_t\big)^-\,dt,$$

where $\tilde{f} = f - f'$, $\tilde{g} = g - g'$ and $\tilde{h} = h - h'$. Applying Itô's formula to $(u^n - S')^2$, we have

$$
\begin{aligned}
&\|u_t^n - S_t'\|^2 + 2\int_0^t \mathcal{E}(u_s^n - S_s')\,ds \\
&\quad = 2\int_0^t ((u_s^n - S_s'), \tilde{f}_s)\,ds + 2\sum_{j=1}^{+\infty}\int_0^t ((u_s^n - S_s'), \tilde{h}_s^j)\,dB_s^j \\
&\qquad - 2\sum_{i=1}^{d}\int_0^t (\partial_i(u_s^n - \tilde{S}_s), \tilde{g}_s^i)\,ds + 2\int_0^t\int_{\mathcal{O}} (u_s^n - S_s')n(u_s^n - S_s)^-\,ds \\
&\qquad + \int_0^t \||\tilde{h}_s|\|^2\,ds, \qquad \text{a.s.}
\end{aligned}
$$

We remark first

$$
\begin{aligned}
&\int_0^t\int_{\mathcal{O}} (u_s^n - S_s')n(u_s^n - S_s)^-\,ds \\
&\quad = \int_0^t\int_{\mathcal{O}} (u_s^n - S_s + S_s - S_s')n(u_s^n - S_s)^-\,ds \\
&\quad = -\int_0^t\int_{\mathcal{O}} n((u_s^n - S_s)^-)^2\,ds + \int_0^t\int_{\mathcal{O}} (S_s - S_s')n(u_s^n - S_s)^-\,dx\,ds;
\end{aligned}
$$

the last term in the right member is nonpositive because $S_t \le S_t'$, thus,

$$
\begin{aligned}
&\|u_t^n - S_t'\|^2 + 2\int_0^t \mathcal{E}(u_s^n - S_s')\,ds + 2\int_0^t n\|(u_s^n - S)^-\|^2\,ds \\
&\quad \le 2\int_0^t (u_s^n - S_s', \tilde{f}_s)\,ds - 2\sum_{i=1}^{d}\int_0^t (\partial_i(u_s^n - S_s'), \tilde{g}_s^i)\,ds \\
&\qquad + 2\sum_{j=1}^{+\infty}\int_0^t (u_s^n - S_s', \tilde{h}_s^j)\,dB_s^j + \int_0^t \||\tilde{h}_s|\|^2\,ds \qquad \text{a.s.}
\end{aligned}
$$

Then using Cauchy–Schwarz's inequality, we have $\forall t \in [0, T]$,

$$
2\left|\int_0^t (u_s^n - S_s', \tilde{f}_s)\,ds\right| \le \varepsilon\int_0^T \|u_s^n - S_s'\|^2\,ds + \frac{1}{\varepsilon}\int_0^T \|\tilde{f}_s\|^2\,ds
$$

and

$$
\begin{aligned}
&2\left|\sum_{i=1}^{d}\int_0^t (\partial_i(u_s^n - S_s'), \tilde{g}_s^i)\,ds\right| \\
&\qquad \le \varepsilon\int_0^T \|\nabla(u_s^n - S_s')\|^2\,ds + \frac{1}{\varepsilon}\int_0^T \||\tilde{g}|\|^2\,ds.
\end{aligned}
$$

Moreover, thanks to Burkholder–Davies–Gundy's inequality, we get

$$\begin{aligned}
E \sup_{t\in[0,T]}&\left|\sum_{j=1}^{+\infty}\int_0^t \big(u_s^n - S_s', \tilde{h}_s^j\big)\,dB_s^j\right| \\
&\le c_1 E\left[\int_0^T \sum_{j=1}^{+\infty}\big(u_s^n - S_s', \tilde{h}_s^j\big)^2\,ds\right]^{1/2} \\
&\le c_1 E\left[\int_0^T \sum_{j=1}^{+\infty}\sup_{s\in[0,T]}\|u_s^n - S_s'\|^2\|\tilde{h}_s^j\|^2\,ds\right]^{1/2} \\
&\le c_1 E\left[\sup_{s\in[0,T]}\|u_s^n - S_s'\|\left(\int_0^T \||\tilde{h}_s|\|^2\,ds\right)^{1/2}\right] \\
&\le \varepsilon E \sup_{s\in[0,T]}\|u_s^n - S_s'\|^2 + \frac{c_1}{4\varepsilon}E\int_0^T \||\tilde{h}_s|\|^2\,ds.
\end{aligned}$$

Then using the strict ellipticity assumption and the inequalities above, we get

$$\begin{aligned}
&\big(1-2\varepsilon(T+1)\big)E\sup_{t\in[0,T]}\|u_t^n - S_t'\|^2 + (2\lambda-\varepsilon)E\int_0^T \mathcal{E}\big(u_s^n - S_s'\big)\,ds \\
&\quad + 2E\int_0^T n\big\|\big(u_s^n - S_s\big)^-\big\|^2\,ds \\
&\qquad \le C\left(E\|\xi\|^2 + \frac{2}{\varepsilon}E\int_0^T \|\tilde{f}_s\|^2 + \frac{2}{\varepsilon}\||\tilde{g}_s|\|^2 + \left(\frac{c_1}{2\varepsilon}+1\right)\||\tilde{h}_s|\|^2\,ds\right).
\end{aligned}$$

We take ε small enough such that $(1-2\varepsilon(T+1)) > 0$; this yields $(2\lambda-\varepsilon) > 0$,

$$E\sup_{t\in[0,T]}\|u_t^n - S_t'\|^2 + E\int_0^T \mathcal{E}\big(u_t^n - S_t'\big)\,dt + E\int_0^T n\big\|\big(u_t^n - S_t\big)^-\big\|^2\,dt \le C.$$

Then with (11), we obtain the desired estimate. $\square$

We now introduce z, the solution of the corresponding SPDE without obstacle:

$$dz_t + Az_t\,dt = f_t\,dt + \sum_{i=1}^{d}\partial_i g_{i,t}\,dt + \sum_{j=1}^{+\infty}h_{j,t}\,dB_t^j,$$

starting from $z_0 = \xi$, with null Dirichlet condition on the boundary. As a consequence of Theorem 3, we can take for z a quasi-continuous version.

For each $n \in \mathbb{N}^*$, we put $v^n = u^n - z$. Clearly, v^n satisfies

$$dv_t^n + Av_t^n\,dt = n\big(v_t^n - (S_t - z_t)\big)^-\,dt = n\big(u_t^n - S_t\big)^-\,dt.$$

Since $S - z$ is quasi-continuous almost-surely, by the results established by Mignot and Puel in [15], we know that P-almost surely, the sequence $(v^n)_n$ is increasing and converges in $L^2([0,T]\times\mathcal{O})$ P-almost surely to v and that the sequence of random measures $\nu^{v^n}=n(u_t^n-S_t)^-\,dt\,dx$ converges vaguely to a measure associated to v: $\nu=\nu^v$. As a consequence of the previous lemma, $(u^n)_n$ and $(v^n)_n$ are bounded sequences in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$, which is a Hilbert space [equipped the norm $(E\int_0^T\|u_t\|^2_{H_0^1(\mathcal{O})}\,dt)^{1/2}$]. By a double extraction argument, we can construct subsequences $(u^{n_k})_k$ and $(v^{n_k})_k$ such that the first one converges weakly in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$ to an element that we denote u and the second one to an element which necessarily is equal to v since $(v^n)_n$ is increasing. Moreover, we can construct sequences $(\hat{u}^n)_n$ and $(\hat{v}^n)_n$ of convex combinations of elements of the form

$$\hat{u}^n=\sum_{k=1}^{N_n}\alpha_k^n u^{n_k}\quad\text{and}\quad\hat{v}^n=\sum_{k=1}^{N_n}\alpha_k^n v^{n_k}$$

converging strongly to u an v, respectively, in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$.

From the fact that u^n is the weak solution of (13), we get

$$\begin{aligned}&\bigl(u_t^n,\varphi_t\bigr)-(\xi,\varphi_0)-\int_0^t\bigl(u_s^n,\partial_s\varphi_s\bigr)\,ds+\int_0^t\mathcal{E}\bigl(u_s^n,\varphi_s\bigr)\,ds\\&\qquad+\sum_{i=1}^d\int_0^t\bigl(g_s^i,\partial_i\varphi_s\bigr)\,ds\\&\qquad=\int_0^t(f_s,\varphi_s)\,ds+\sum_{j=1}^{+\infty}\int_0^t\bigl(h_s^j,\varphi_s\bigr)\,dB_s^j\\&\qquad\quad+\int_0^t\int_{\mathcal{O}}\varphi_s(x)n\bigl(u_s^n-S_s\bigr)^-\,dx\,ds\qquad\text{a.s.}\end{aligned}\tag{14}$$

Hence,

$$\begin{aligned}&\bigl(\hat{u}_t^n,\varphi_t\bigr)-(\xi,\varphi_0)-\int_0^t\bigl(\hat{u}_s^n,\partial_s\varphi_s\bigr)\,ds+\int_0^t\mathcal{E}\bigl(\hat{u}_s^n,\varphi_s\bigr)\,ds+\sum_{i=1}^d\int_0^t\bigl(g_s^i,\partial_i\varphi_s\bigr)\,ds\\&\quad=\int_0^t(f_s,\varphi_s)\,ds+\sum_{j=1}^{+\infty}\int_0^t\bigl(h_s^j,\varphi_s\bigr)\,dB_s^j\\&\qquad+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\Biggl(\sum_{k=1}^{N_n}n_k\bigl(u_s^{n_k}-S_s\bigr)^-\Biggr)\,dx\,ds\qquad\text{a.s.}\end{aligned}\tag{15}$$

We have

$$\int_0^t\int_{\mathcal{O}}\varphi_s(x)\Biggl(\sum_{k=1}^{N_n}n_k\bigl(u_s^{n_k}-S_s\bigr)^-\Biggr)\,dx\,ds=\int_0^T-\biggl(\frac{\partial\varphi_t}{\partial t},\hat{v}_t^n\biggr)\,dt+\int_0^T\mathcal{E}\bigl(\varphi_t,\hat{v}_t^n\bigr)\,dt$$

so that we have almost surely, at least for a subsequence,

$$\begin{aligned}\lim_{n\to+\infty}\int_0^t\int_{\mathcal{O}}\varphi_s(x)\left(\sum_{k=1}^{N_n}n_k\big(u_s^{n_k}-S_s\big)^-\right)dx\,ds\\ =\int_0^T-\left(\frac{\partial\varphi_t}{\partial t},v_t\right)dt+\int_0^T\mathcal{E}(\varphi_t,v_t)\,dt\\ =\int_0^T\int_{\mathcal{O}}\varphi_t(x)\nu(dx,dt).\end{aligned}$$

As $(\hat{u}^n)_n$ converges to u in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$, by making n tend to $+\infty$ in (15), we obtain

$$\begin{aligned}(u_t,\varphi_t)-(\xi,\varphi_0)-\int_0^t(u_s,\partial_s\varphi_s)\,ds+\int_0^t\mathcal{E}(u_s,\varphi_s)\,ds+\sum_{i=1}^d\int_0^t\big(g_s^i,\partial_i\varphi_s\big)\,ds\\ =\int_0^t(f_s,\varphi_s)\,ds+\sum_{j=1}^{+\infty}\int_0^t\big(h_s^j,\varphi_s\big)\,dB_s^j+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\nu(dx,ds)\qquad\text{a.s.}\end{aligned}$$

In the next subsection we will show that u satisfies an Itô formula. As a consequence by applying it to u_t^2, using standard arguments, we get that $u\in\mathcal{H}_T$ so for almost all $\omega\in\Omega$, $u(\omega)\in\mathcal{K}$. And from Theorem 9 in [8], we know that for almost all $\omega\in\Omega$, $z(\omega)\in\mathcal{K}$. Therefore, for almost all $\omega\in\Omega$, $v(\omega)=u(w)-z(w)\in\mathcal{K}$. Hence, $\nu=\partial_t v+Av$ is a regular measure by definition. Moreover, by [17, 18] we know that v admits a quasi-continuous version $\tilde{v}$ which satisfies the minimality condition

$$\int\int(\tilde{v}-S+\tilde{z})\nu(dx\,dt)=0. \tag{16}$$

z is quasi-continuous version, hence, $\tilde{u}=z+\tilde{v}$ is a quasi-continuous version of u and we can write (16) as

$$\int\int(\tilde{u}-S)\nu(dx\,dt)=0.$$

The fact that $u\ge S$ comes from the fact that $v\ge u-z$, so at this stage we have proved that (u,ν) is a solution to the obstacle problem we consider.

Uniqueness comes from the fact that both z and v are unique, which ends the proof of Theorem 4.

5.3. *Itô's formula.* The following Itô formula for the solution of the obstacle problem is fundamental to get all the results in the nonlinear case. Let us also remark that any solution of the nonlinear equation (1) may be viewed as the solution of a linear one so that it also satisfies the Itô formula.

THEOREM 5. *Under assumptions of the previous Section 5.2, let u be the solution of SPDE (1) with obstacle and $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^{1,2}$. We denote by Φ' and Φ'' the derivatives of Φ with respect to the space variables and by $\frac{\partial \Phi}{\partial t}$ the partial derivative with respect to time. We assume that these derivatives are bounded and $\Phi'(t,0) = 0$ for all $t \geq 0$. Then P-a.s. for all $t \in [0,T]$,*

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi\big(t, u_t(x)\big)\,dx + \int_0^t \mathcal{E}\big(\Phi'(s,u_s), u_s\big)\,ds \\
&\quad = \int_{\mathcal{O}} \Phi\big(0, \xi(x)\big)\,dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}\big(s, u_s(x)\big)\,dx\,ds \\
&\qquad + \int_0^t \big(\Phi'(s,u_s), f_s\big)\,ds - \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi''\big(s, u_s(x)\big)\partial_i u_s(x) g_i(x)\,dx\,ds \\
&\qquad + \sum_{j=1}^{+\infty} \int_0^t \big(\Phi'(s,u_s), h_j\big)\,dB_s^j \\
&\qquad + \frac{1}{2} \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi''\big(s, u_s(x)\big)\big(h_{j,s}(x)\big)^2\,dx\,ds \\
&\qquad + \int_0^t \int_{\mathcal{O}} \Phi'\big(s, \tilde{u}_s(x)\big)\nu(dx\,ds).
\end{aligned}$$

PROOF. We keep the same notation as in the previous subsection and so consider the sequence $(u^n)_n$ approximating u and also $(\hat{u}^n)_n$ the sequence of convex combinations $\hat{u}^n = \sum_{k=1}^{N_n} \alpha_k^n u^{n_k}$ converging strongly to u in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$.

Moreover, by standard arguments such as the Banach–Saks theorem, since $(u^n)_n$ is nondecreasing, we can choose the convex combinations such that $(\hat{u}^n)_n$ is also a nondecreasing sequence. We start by a key lemma:

LEMMA 5. *Let $t \in [0,T]$, then*

$$\lim_{n \to +\infty} E \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^- \sum_{k=1}^{N_n} \alpha_k^n n_k \big(u_s^{n_k} - S_s\big)^-\,dx\,ds = 0.$$

PROOF. We write as above $u^n = v^n + z$ and we denote $\hat{\nu}^n = \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^-$ so that

$$\int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)\hat{\nu}^n(dx\,ds) = \int_0^t \int_{\mathcal{O}} \hat{v}_s^n \hat{\nu}^n(dx\,ds) + \int_0^t \int_{\mathcal{O}} (z_s - S_s)\hat{\nu}^n(dx\,ds).$$

From Lemma 1, we know that

$$\int_0^t \int_{\mathcal{O}} (z_s - S_s)\hat{v}^n(dx\,ds) \to \int_0^t \int_{\mathcal{O}} (z_s - S_s)\nu(dx\,ds).$$

Moreover, by Lemma II.6 in [17], we have for all n

$$\frac{1}{2}\|\hat{v}_T^n\|^2 + \int_0^T \mathcal{E}(\hat{v}_s^n)\,ds = \int_0^T \int_{\mathcal{O}} \hat{v}_s^n \hat{v}^n(dx\,ds)$$

and

$$\frac{1}{2}\|v_T\|^2 + \int_0^T \mathcal{E}(v_s)\,ds = \int_0^T \int_{\mathcal{O}} \tilde{v}_s \nu(dx\,ds).$$

As $(\hat{v}^n)_n$ tends to v in $L^2([0,T], H_0^1(\mathcal{O}))$,

$$\lim_{n\to+\infty} \int_0^T \mathcal{E}(\hat{v}_s^n)\,ds = \int_0^T \mathcal{E}(v_s)\,ds.$$

Let us prove that $(\|\hat{v}_T^n\|)_n$ tends to $\|v_T\|$.

Since $(\hat{v}_T^n)_n$ is nondecreasing and bounded in $L^2(\mathcal{O})$, it converges in $L^2(\mathcal{O})$ to $m = \sup_n \hat{v}_T^n$. Let $\rho \in H_0^1(\mathcal{O})$, then the map defined by $\varphi(t,x) = \rho(x)$ belongs to $\mathcal{W}$, hence, as a consequence of Proposition 4,

$$\int_{[0,T[\times\mathcal{O}} \rho\, d\hat{v}^n = \int_0^T \mathcal{E}(\rho, \hat{v}_s^n)\,ds + (\rho, \hat{v}_T^n)$$

and

$$\int_{[0,T[\times\mathcal{O}} \tilde{\rho}\, d\nu = \int_0^T \mathcal{E}(\rho, v_s)\,ds + (\rho, v_T);$$

making n tend to $+\infty$ and using one more time Lemma 1, we get

$$\lim_{n\to+\infty} (\rho, \hat{v}_T^n) = (\rho, m) = (\rho, v_T).$$

Since ρ is arbitrary, we have $v_T = m$ and so $\lim_{n\to+\infty} \|\hat{v}_T^n\| = \|v_T\|$ and this yields

$$\lim_{n\to+\infty} \int_0^T \int_{\mathcal{O}} \hat{v}_s^n \hat{v}^n(dx\,ds) = \int_0^T \int_{\mathcal{O}} \tilde{v}_s \nu(dx\,ds) = \int_0^T \int_{\mathcal{O}} (S_s - z_s)\nu(dx\,ds).$$

This proves that

$$\lim_{n\to+\infty} \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)\hat{v}^n(dx\,ds) = 0.$$

We conclude by remarking that

$$\begin{aligned}\lim_{n\to+\infty} \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^+ \hat{v}^n(dx\,ds) &\le \lim_{n\to+\infty} \int_0^t \int_{\mathcal{O}} (u_s - S_s)\hat{v}^n(dx\,ds) \\ &= \int_0^t \int_{\mathcal{O}} (\tilde{u}_s - S_s)\nu(dx\,ds) = 0. \qquad \square\end{aligned}$$

We now end the proof of Theorem 5. We consider the penalized solution $(u^n)_n$, and we know that its convex combination $(\hat{u}^n)_n$ converges strongly to u in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$. And $\hat{u}^n$ satisfies the following SPDE:

$$d\hat{u}_t^n + A\hat{u}_t^n\,dt = f_t\,dt + \sum_{i=1}^d \partial_i g_t^i\,dt + \sum_{j=1}^{+\infty} h_t^j\,dB_t^j + \sum_{k=1}^{N_n} \alpha_k^n n_k \big(u_s^{n_k} - S_s\big)^-\,dt.$$

From the Itô formula for the solution of SPDE without obstacle (see Lemma 7 in [7]), we have, almost surely, for all $t \in [0,T]$,

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi\big(t, \hat{u}_t^n(x)\big)\,dx + \int_0^t \mathcal{E}\big(\Phi'(s,\hat{u}_s^n), \hat{u}_s^n\big)\,ds \\
&\quad = \int_{\mathcal{O}} \Phi\big(0, \xi(x)\big)\,dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}\big(s, \hat{u}_s^n\big)\,dx\,ds \\
&\qquad + \int_0^t \big(\Phi'(s,\hat{u}_s^n), f_s\big)\,ds \\
&\qquad - \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi''\big(s, \hat{u}_s^n(x)\big)\partial_i \hat{u}_s^n(x) g_i(x)\,dx\,ds \\
&\qquad + \sum_{j=1}^{+\infty} \int_0^t \big(\Phi'(s,\hat{u}_s^n), h_j\big)\,dB_s^j \\
&\qquad + \frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi''\big(s, \hat{u}_s^n(x)\big)\big(h_j(x)\big)^2\,dx\,ds \\
&\qquad + \int_0^t \int_{\mathcal{O}} \Phi'\big(s, \hat{u}_s^n\big) \sum_{k=1}^{N_n} \alpha_k^n n_k \big(u_s^{n_k} - S_s\big)^-\,dx\,ds.
\end{aligned}$$

Because of the strong convergence of $(\hat{u}^n)_n$, the convergence of all the terms except the last one are clear. To obtain the convergence of the last term, we do as follows:

$$\begin{aligned}
&\int_0^t \int_{\mathcal{O}} \Phi'\big(s, \hat{u}_s^n\big) \sum_{k=1}^{N_n} \alpha_k^n n_k \big(u_s^{n_k} - S_s\big)^-\,dx\,ds \\
&\quad = \int_0^t \int_{\mathcal{O}} \big(\Phi'\big(s,\hat{u}_s^n\big) - \Phi'(s, S_s)\big) \sum_{k=1}^{N_n} \alpha_k^n n_k \big(u_s^{n_k} - S_s\big)^-\,dx\,ds \\
&\qquad + \int_0^t \int_{\mathcal{O}} \Phi'(s, S_s) \sum_{k=1}^{N_n} \alpha_k^n n_k \big(u_s^{n_k} - S_s\big)^-\,dx\,ds.
\end{aligned}$$

For the first term in the right member, we have

$$\begin{aligned}
&\left|\int_0^t \int_{\mathcal{O}} (\Phi'(s, \hat{u}_s^n) - \Phi'(s, S_s)) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds\right| \\
&\quad \le C \int_0^t \int_{\mathcal{O}} |\hat{u}_s^n - S_s| \cdot \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds \\
&\quad = C \int_0^t \int_{\mathcal{O}} ((\hat{u}_s^n - S_s)^+ + (\hat{u}_s^n - S_s)^-) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds \\
&\quad = C \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^+ \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds \\
&\qquad + C \int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^- \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds.
\end{aligned}$$

We have the following inequality because $(\hat{u}^n)_n$ converges to u increasingly:

$$\begin{aligned}
&\int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^- \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds \\
&\qquad \le \int_0^t \int_{\mathcal{O}} (u_s - S_s)^+ \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds \\
&\qquad = \int_0^t \int_{\mathcal{O}} (u_s - S_s) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds.
\end{aligned}$$

With Lemma 1, we know that

$$\lim_{n\to\infty} \int_0^t \int_{\mathcal{O}} (u_s - S_s) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds \to \int_0^t \int_{\mathcal{O}} (\tilde{u}_s - \tilde{S}_s) \nu(dx\, ds) = 0.$$

And from Lemma 5, we have

$$\int_0^t \int_{\mathcal{O}} (\hat{u}_s^n - S_s)^- \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds \to 0.$$

Therefore,

$$\int_0^t \int_{\mathcal{O}} (\Phi'(s, \hat{u}_s^n) - \Phi'(s, S_s)) \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- \, dx \, ds \to 0.$$

Moreover, with Lemma 1, we have

$$\int_0^t \int_{\mathcal{O}} \Phi'(s, S_s) \sum_{k=1}^{N_n} \alpha_k^n n_k \big(u_s^{n_k} - S_s\big)^- \, dx \, ds \to \int_0^t \int_{\mathcal{O}} \Phi'(s, S_s) \nu(dx \, ds)$$

and

$$\begin{aligned}
&\left| \int_0^t \int_{\mathcal{O}} \Phi'(s, u_s) \nu(dx \, ds) - \int_0^t \int_{\mathcal{O}} \Phi'(s, S_s) \nu(dx \, ds) \right| \\
&\quad \leq C \int_0^t \int_{\mathcal{O}} |\tilde{u}_s - S_s| \nu(dx \, ds) \\
&\quad = C \int_0^t \int_{\mathcal{O}} (\tilde{u}_s - S_s) \nu(dx \, ds) = 0.
\end{aligned}$$

Therefore, taking the limit, we get the desired Itô formula. □

5.4. *Itô's formula for the difference of the solutions of two OSPDEs.* We still consider (u, ν) the solution of the linear equation as in Section 5.2,

$$\begin{cases} du_t + Au_t \, dt = f_t \, dt + \displaystyle\sum_{i=1}^{d} \partial_i g_t^i \, dt + \sum_{j=1}^{+\infty} h_t^j \, dB_t^j + \nu(dt, x), \\ u \geq S, \end{cases}$$

and consider another linear equation with adapted coefficients $\bar{f}, \bar{g}, \bar{h}$, respectively, in $L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R})$, $L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$ and $L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}^*})$ and obstacle $\bar{S}$ which satisfies the same hypotheses (O) as S, that is, $\bar{S}_0 \leq \xi$ and $\bar{S}$ is dominated by the solution of an SPDE (not necessarily the same as S). We denote by $(y, \bar{\nu})$ the unique solution to the associated SPDE with obstacle with initial condition $y_0 = u_0 = \xi$:

$$\begin{cases} dy_t + Ay_t \, dt = \bar{f}_t \, dt + \displaystyle\sum_{i=1}^{d} \partial_i \bar{g}_t^i \, dt + \sum_{j=1}^{+\infty} \bar{h}_t^j \, dB_t^j + \bar{\nu}(dt, x), \\ y \geq \bar{S}, \end{cases}$$

THEOREM 6. *Let Φ as in Theorem 5, then the difference of the two solutions satisfy the following Itô formula for all $t \in [0, T]$:*

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi\big(t, u_t(x) - y_t(x)\big) \, dx + \int_0^t \mathcal{E}\big(\Phi'(s, u_s - y_s), u_s - y_s\big) \, ds \\
&\quad = \int_0^t \big(\Phi'(s, u_s - y_s), f_s - \bar{f}_s\big) \, ds \\
&\qquad - \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s - y_s) \partial_i (u_s - y_s) \big(g_s^i - \bar{g}_s^i\big) \, dx \, ds \qquad (17)
\end{aligned}$$

$$+\sum_{j=1}^{+\infty}\int_0^t\big(\Phi'(s,u_s-y_s),h_s^j-\bar h_s^j\big)\,dB_s^j$$
$$+\frac{1}{2}\sum_{j=1}^{+\infty}\int_0^t\int_{\mathcal{O}}\Phi''(s,u_s-y_s)\big(h_s^j-\bar h_s^j\big)^2\,dx\,ds$$
$$+\int_0^t\int_{\mathcal{O}}\frac{\partial\Phi}{\partial s}(s,u_s-y_s)\,dx\,ds$$
$$+\int_0^t\int_{\mathcal{O}}\Phi'(s,\tilde u_s-\tilde y_s)(\nu-\bar\nu)(dx,ds)\qquad a.s.$$

PROOF. We begin with the penalized solutions. The corresponding penalization equations are

$$du_t^n+Au_t^n\,dt=f_t\,dt+\sum_{i=1}^d\partial_i g_t^i\,dt+\sum_{j=1}^{+\infty}h_t^j\,dB_t^j+n\big(u_t^n-S_t\big)^-\,dt$$

and

$$dy_t^m+Ay_t^m\,dt=\bar f_t\,dt+\sum_{i=1}^d\partial_i\bar g_t^i\,dt+\sum_{j=1}^{+\infty}\bar h_t^j\,dB_t^j+m\big(y_t^m-\bar S_t\big)^-\,dt.$$

From the proofs above, we know that the penalized solution converges weakly to the solution and we can take convex combinations $\hat u^n=\sum_{i=1}^{N_n}\alpha_i^n u^{n_i}$ and $\hat y^n=\sum_{i=1}^{N_n'}\beta_i^n y^{n_i'}$ such that $(\hat u^n)_n$ and $(\hat y^n)_n$ are nondecreasing and converge strongly to u and y, respectively, in $L^2(\Omega\times[0,T],H_0^1(\mathcal{O}))$ as n tends to $+\infty$.

As in the proof of Theorem 5, we first establish a key lemma:

LEMMA 6. *For all $t\in[0,T]$,*

$$\lim_{n\to+\infty}E\int_0^t\int_{\mathcal{O}}\hat u_s^n\sum_{k=1}^{N_n'}\beta_k^n n_k'\big(y_s^{n_k'}-\bar S_s\big)^-\,dx\,ds=E\int_0^t\int_{\mathcal{O}}\tilde u\bar\nu(ds,dx)$$

and

$$\lim_{n\to+\infty}E\int_0^t\int_{\mathcal{O}}\hat y_s^n\sum_{k=1}^{N_n}\alpha_k^n n_k\big(u_s^{n_k}-S_s\big)^-\,dx\,ds=E\int_0^t\int_{\mathcal{O}}\tilde y\nu(ds,dx).$$

PROOF. We put for all n,

$$\nu^n(ds,dx)=\sum_{k=1}^{N_n}\alpha_k^n n_k\big(u_s^{n_k}-S_s\big)^-\,dx\,ds\qquad\text{and}$$
$$\bar\nu^n(ds,dx)=\sum_{k=1}^{N_n'}\beta_k^n n_k'\big(y_s^{n_k'}-\bar S_s\big)^-\,dx\,ds.$$

As in the proof of Lemma 5, we write for all $n \in \mathbb{N}^*$: $u^n = z + v^n$.

In the same spirit, we introduce $\bar{z}$ the solution of the linear SPDE:

$$d\bar{z}_t + A\bar{z}_t = \bar{f}_t\,dt + \sum_{i=1}^{d} \partial_i \bar{g}_t^i\,dt + \sum_{j=1}^{+\infty} \bar{h}_t^j\,dB_t^j,$$

with initial condition $\bar{z}_0 = \xi$ and put $\forall n \in \mathbb{N}^*$, $\bar{v}^n = y^n - \bar{z}$, $\hat{\bar{v}}^n = \hat{y}^n - \bar{z}$ and $\bar{v} = y - \bar{z}$.

As a consequence of Lemma II.6 in [18], we have for all $n \in \mathbb{N}^*$, P-almost surely,

$$\frac{1}{2}\|\hat{v}_t^n - \hat{\bar{v}}_t^n\|^2 + \int_0^t \mathcal{E}(\hat{v}_s^n - \hat{\bar{v}}_s^n)\,ds = \int_0^t \int_{\mathcal{O}} (\hat{v}_s^n - \hat{\bar{v}}_s^n)(\nu^n - \bar{\nu}^n)(dx, ds)$$

and

$$\frac{1}{2}\|v_t - \bar{v}_t\|^2 + \int_0^t \mathcal{E}(v_s - \bar{v}_s)\,ds = \int_0^t \int_{\mathcal{O}} (\tilde{v}_s - \tilde{\bar{v}}_s)(\nu - \bar{\nu})(dx, ds).$$

But, as in the proof of Lemma 5, we get that $(\hat{v}_t^n - \hat{\bar{v}}_t^n)_n$ tends to $v_t - \bar{v}_t$ in $L^2(\mathcal{O})$ almost surely and

$$\lim_n \int_0^t \int_{\mathcal{O}} \hat{v}_s^n \nu^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{v}_s \nu(dx, ds),$$

$$\lim_n \int_0^t \int_{\mathcal{O}} \hat{\bar{v}}_s^n \bar{\nu}^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{\bar{v}}_s \bar{\nu}(dx, ds).$$

This yields

$$\lim_n \left(\int_0^t \int_{\mathcal{O}} \hat{v}_s^n \bar{\nu}^n(dx, ds) + \int_0^t \int_{\mathcal{O}} \hat{\bar{v}}_s^n \nu^n(dx, ds) \right) = \int_0^t \int_{\mathcal{O}} \tilde{v}_s \bar{\nu}(dx, ds) + \int_0^t \int_{\mathcal{O}} \tilde{\bar{v}}_s \nu(dx, ds).$$

But, we have

$$\limsup_n \int_0^t \int_{\mathcal{O}} \hat{v}_s^n \bar{\nu}^n(dx, ds) \le \limsup_n \int_0^t \int_{\mathcal{O}} v_s \bar{\nu}^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{v}_s \bar{\nu}(dx, ds),$$

and in the same way

$$\limsup_n \int_0^t \int_{\mathcal{O}} \hat{\bar{v}}_s^n \nu^n(dx, ds) \le \int_0^t \int_{\mathcal{O}} \tilde{\bar{v}}_s \nu(dx, ds).$$

Let us remark that these inequalities also hold for any subsequence. From this, it is easy to deduce that necessarily

$$\lim_n \int_0^t \int_{\mathcal{O}} \hat{v}_s^n \bar{\nu}^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{v}_s \bar{\nu}(dx, ds)$$

and

$$\lim_n \int_0^t \int_{\mathcal{O}} \hat{\bar{v}}_s^n \nu^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \tilde{\bar{v}}_s \nu(dx, ds).$$

We end the proof of this lemma by using similar arguments as in the proof of Lemma 5. □

We now end the proof of Theorem 6. We begin with the equation which $\hat{u}^n - \hat{y}^n$ satisfies

$$\begin{aligned} &d(\hat{u}_t^n - \hat{y}_t^n) + A(\hat{u}_t^n - \hat{y}_t^n)\,dt \\ &\quad = (f_t - \bar{f}_t)\,dt + \sum_{i=1}^d \partial_i(g_t^i - \bar{g}_t^i)\,dt + \sum_{j=1}^{+\infty}(h_t^j - \bar{h}_t^j)\,dB_t^j + (\nu^n - \bar{\nu}^n)(x, dt). \end{aligned}$$

Applying Itô's formula to $\Phi(\hat{u}^n - \hat{y}^n)$, we have

$$\begin{aligned} &\int_{\mathcal{O}} \Phi(t, \hat{u}_t^n(x) - \hat{y}_t^n(x))\,dx + \int_0^t \mathcal{E}(\Phi'(s, \hat{u}_s^n - \hat{y}_s^n), \hat{u}_s^n - \hat{y}_s^n)\,ds \\ &\quad = \int_0^t (\Phi'(s, \hat{u}_s^n - \hat{y}_s^n), f_s - \bar{f}_s)\,ds \\ &\qquad - \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi''(s, \hat{u}_s^n - \hat{y}_s^n)\partial_i(\hat{u}_s^n - \hat{y}_s^n)(g_s^i - \bar{g}_s^i)\,dx\,ds \\ &\qquad + \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s, \hat{u}_s^n - \hat{y}_s^n), h_s^j - \bar{h}_s^j)\,dB_s^j \\ &\qquad + \frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi''(s, \hat{u}_s^n - \hat{y}_s^n)(h_s^j - \bar{h}_s^j)^2\,dx\,ds \\ &\qquad + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, \hat{u}_s^n - \hat{y}_s^n)\,dx\,ds \\ &\qquad + \int_0^t \int_{\mathcal{O}} \Phi'(s, \hat{u}_s^n - \hat{y}_s^n)(\nu^n - \bar{\nu}^n)(dx, dt) \qquad \text{a.s.} \end{aligned}$$

Because $(\hat{u}^n)_n$ and $(\hat{y}^n)_n$ converge strongly to u and y, respectively, the convergence of all the terms except the last term are clear. For the convergence of the last term, we do as follows:

$$\begin{aligned} &\biggl| \int_0^t \int_{\mathcal{O}} [\Phi'(s, \hat{u}_s^n - \hat{y}_s^n) - \Phi'(s, u_s - \hat{y}_s^n)]\nu^n(dx\,ds) \\ &\quad + \int_0^t \int_{\mathcal{O}} [\Phi'(s, u_s - \hat{y}_s^n) - \Phi'(s, u_s - y_s)]\nu^n(dx\,ds) \biggr| \\ &\qquad \le C \int_0^t \int_{\mathcal{O}} |\hat{u}_s^n - u_s|\nu^n(dx\,ds) + \int_0^t \int_{\mathcal{O}} |\hat{y}_s^n - y_s|\nu^n(dx, ds). \end{aligned}$$

As a consequence of Lemma 5 and using the fact that $\hat{u}^n \le u$,

$$\lim_n \int_0^t \int_{\mathcal{O}} |\hat{u}_s^n - u_s| \nu^n(dx\,ds) = \lim_n \int_0^t \int_{\mathcal{O}} (u_s - \hat{u}_s^n) \nu^n(dx\,ds) = 0.$$

By Lemma 6 and the fact that $\hat{y}^n \le y$,

$$\lim_n \int_0^t \int_{\mathcal{O}} |\hat{y}_s^n - y_s| \nu^n(dx, ds) = \lim_n \int_0^t \int_{\mathcal{O}} (y_s - \hat{y}_s^n) \nu^n(dx, ds) = 0.$$

This yields

$$\lim_n \int_0^t \int_{\mathcal{O}} \big(\Phi'(s, \hat{u}_s^n - \hat{y}_s^n) - \Phi'(s, u_s - y_s)\big) \nu^n(dx, dt) = 0,$$

but, by Lemma 1, we know that

$$\lim_n \int_0^t \int_{\mathcal{O}} \Phi'(s, u_s - y_s) \nu^n(dx, dt) = \int_0^t \int_{\mathcal{O}} \Phi'(s, \tilde{u}_s - \tilde{y}_s) \bar{\nu}(dx, dt),$$

so

$$\lim_n \int_0^t \int_{\mathcal{O}} \Phi'(s, \hat{u}_s^n - \hat{y}_s^n) \nu^n(dx, dt) = \int_0^t \int_{\mathcal{O}} \Phi'(s, \tilde{u}_s - \tilde{y}_s) \nu(dx, dt).$$

In the same way, we prove

$$\lim_n \int_0^t \int_{\mathcal{O}} \Phi'(s, \hat{u}_s^n - \hat{y}_s^n) \bar{\nu}^n(dx, dt) = \int_0^t \int_{\mathcal{O}} \Phi'(s, \tilde{u}_s - \tilde{y}_s) \bar{\nu}(dx, dt).$$

The proof is now complete. □

5.5. *Proof of Theorem 4 in the nonlinear case*. Let γ and δ be 2 positive constants. On $L^2(\Omega \times [0, T]; H_0^1(\mathcal{O}))$, we introduce the norm

$$\|u\|_{\gamma,\delta} = E\bigg(\int_0^T e^{-\gamma s} \big(\delta \|u_s\|^2 + \|\nabla u_s\|^2\big)\, ds\bigg),$$

which clearly defines an equivalent norm on $L^2(\Omega \times [0, T]; H_0^1(\mathcal{O}))$.

Let us consider the Picard sequence $(u^n)_n$ defined by $u^0 = \xi$ and for all $n \in \mathbb{N}^*$ we denote by (u^{n+1}, ν^{n+1}) the solution of the linear SPDE with obstacle

$$\big(u^{n+1}, \nu^{n+1}\big) = \mathcal{R}\big(\xi, f\big(u^n, \nabla u^n\big), g\big(u^n, \nabla u^n\big), h\big(u^n, \nabla u^n\big), S\big).$$

Then, by Itô's formula (17), we have almost surely

$$\begin{aligned}
&e^{-\gamma T} \|u_T^{n+1} - u_T^n\|^2 + 2 \int_0^T e^{-\gamma s} \mathcal{E}\big(u_s^{n+1} - u_s^n\big)\, ds \\
&\quad = -\gamma \int_0^T e^{-\gamma s} \|u_s^{n+1} - u_s^n\|^2\, ds \\
&\qquad + 2 \int_0^T e^{-\gamma s} \big(\hat{f}_s, u_s^{n+1} - u_s^n\big)\, ds - 2 \sum_{i=1}^d \int_0^T e^{-\gamma s} \big(\hat{g}_s^i, \partial_i \big(u_s^{n+1} - u_s^n\big)\big)\, ds
\end{aligned}$$

$$+ 2\sum_{j=1}^{+\infty}\int_0^T e^{-\gamma s}\big(\hat{h}_s^j, u_s^{n+1}-u_s^n\big)\,dB_s^j$$
$$+ \int_0^T e^{-\gamma s}\||\hat{h}_s|\|^2\,ds + 2\int_0^T\int_{\mathcal{O}} e^{-\gamma s}\big(u_s^{n+1}-u_s^n\big)\big(\nu^{n+1}-\nu^n\big)(dx\,ds),$$

where $\hat{f} = f(u^n, \nabla u^n) - f(u^{n-1}, \nabla u^{n-1})$, $\hat{g} = g(u^n, \nabla u^n) - g(u^{n-1}, \nabla u^{n-1})$ and $\hat{h} = h(u^n, \nabla u^n) - h(u^{n-1}, \nabla u^{n-1})$. Clearly, the last term is nonpositive, so using Cauchy–Schwarz's inequality and the Lipschitz conditions on f, g and h, we have

$$\begin{aligned}
&2\int_0^T e^{-\gamma s}\big(u_s^{n+1}-u_s^n, \hat{f}_s\big)\,ds\\
&\quad\le \frac{1}{\varepsilon}\int_0^T e^{-\gamma s}\|u_s^{n+1}-u_s^n\|^2\,ds + \varepsilon\int_0^T \|\hat{f}_s\|^2\,ds\\
&\quad\le \frac{1}{\varepsilon}\int_0^T e^{-\gamma s}\|u_s^{n+1}-u_s^n\|^2\,ds + C\varepsilon\int_0^T e^{-\gamma s}\|u_s^n-u_s^{n-1}\|^2\,ds\\
&\qquad + C\varepsilon\int_0^T e^{-\gamma s}\|\nabla\big(u_s^n-u_s^{n-1}\big)\|^2\,ds
\end{aligned}$$

and

$$\begin{aligned}
&2\sum_{i=1}^{d}\int_0^T e^{-\gamma s}\big(\hat{g}_s^i, \partial_i\big(u_s^{n+1}-u_s^n\big)\big)\,ds\\
&\quad\le 2\int_0^T e^{-\gamma s}\|\nabla\big(u_s^{n+1}-u_s^n\big)\|\big(C\|u_s^n-u_s^{n-1}\| + \alpha\|\nabla\big(u_s^n-u_s^{n-1}\big)\|\big)\,ds\\
&\quad\le C\varepsilon\int_0^T e^{-\gamma s}\|\nabla\big(u_s^{n+1}-u_s^n\big)\|^2\,ds + \frac{C}{\varepsilon}\int_0^T e^{-\gamma s}\|u_s^n-u_s^{n-1}\|^2\,ds\\
&\qquad + \alpha\int_0^T e^{-\gamma s}\|\nabla\big(u_s^{n+1}-u_s^n\big)\|^2\,ds + \alpha\int_0^T e^{-\gamma s}\|u_s^n-u_s^{n-1}\|^2\,ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T e^{-\gamma s}\||\hat{h}_s|\|^2\,ds\\
&\quad\le C\Big(1+\frac{1}{\varepsilon}\Big)\int_0^T e^{-\gamma s}\|u_s^n-u_s^{n-1}\|^2\,ds\\
&\qquad + \beta^2(1+\varepsilon)\int_0^T e^{-\gamma s}\|\nabla\big(u_s^n-u_s^{n-1}\big)\|^2\,ds,
\end{aligned}$$

where C, α and β are the constants in the Lipschitz conditions. Using the elliptic condition and taking expectation, we get

$$\Big(\gamma-\frac{1}{\varepsilon}\Big)E\int_0^T e^{-\gamma s}\|u_s^{n+1}-u_s^n\|^2\,ds+(2\lambda-\alpha)E\int_0^T e^{-\gamma s}\|\nabla(u_s^{n+1}-u_s^n)\|^2\,ds$$
$$\le C\Big(1+\varepsilon+\frac{2}{\varepsilon}\Big)\int_0^T e^{-\gamma s}\|u_s^n-u_s^{n-1}\|^2\,ds$$
$$+\big(C\varepsilon+\alpha+\beta^2(1+\varepsilon)\big)E\int_0^T e^{-\gamma s}\|\nabla(u_s^n-u_s^{n-1})\|^2\,ds.$$

We choose ε small enough and then γ such that

$$C\varepsilon+\alpha+\beta^2(1+\varepsilon)<2\lambda-\alpha\quad\text{and}\quad\frac{\gamma-1/\varepsilon}{2\lambda-\alpha}=\frac{C(1+\varepsilon+2/\varepsilon)}{C\varepsilon+\alpha+\beta^2(1+\varepsilon)}.$$

If we set $\delta=\frac{\gamma-1/\varepsilon}{2\lambda-\alpha}$, we have the following inequality:

$$\|u^{n+1}-u^n\|_{\gamma,\delta}\le\frac{C\varepsilon+\alpha+\beta^2(1+\varepsilon)}{2\lambda-\alpha}\|u^n-u^{n-1}\|_{\gamma,\delta}\le\cdots$$
$$\le\Big(\frac{C\varepsilon+\alpha+\beta^2(1+\varepsilon)}{2\lambda-\alpha}\Big)^n\|u^1\|_{\gamma,\delta}$$

when $n\to\infty$, $(\frac{C\varepsilon+\alpha+\beta^2(1+\varepsilon)}{2\lambda-\alpha})^n\to 0$, and we deduce that $(u^n)_n$ converges strongly to u in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$.

Moreover, as $(u^{n+1},\nu^{n+1})=\mathcal{R}(\xi,f(u^n,\nabla u^n),g(u^n,\nabla u^n),h(u^n,\nabla u^n),S)$, we have for any $\varphi\in\mathcal{D}$,

$$\big(u_t^{n+1},\varphi_t\big)-(\xi,\varphi_0)-\int_0^t\big(u_s^n,\partial_s\varphi_s\big)\,ds+\int_0^t\mathcal{E}\big(u_s^{n+1},\varphi_s\big)\,ds$$
$$+\sum_{i=1}^d\int_0^t\big(g_s^i(u_s^n,\nabla u_s^n),\partial_i\varphi_s\big)\,ds$$
$$=\int_0^t\big(f_s(u_s^n,\nabla u_s^n),\varphi_s\big)\,ds+\sum_{j=1}^{+\infty}\int_0^t\big(h_s^j(u_s^n,\nabla u_s^n),\varphi_s\big)\,dB_s^j$$
$$+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\nu^{n+1}(dx\,ds)\quad\text{a.s.}$$

Let v^{n+1} be the random parabolic potential associated to ν^{n+1}:

$$\nu^{n+1}=\partial_t v^{n+1}+Av^{n+1}.$$

We denote $z^{n+1}=u^{n+1}-v^{n+1}$, so

$$z^{n+1}=\mathcal{U}\big(\xi,f(u^n,\nabla u^n),g(u^n,\nabla u^n),h(u^n,\nabla u^n)\big)$$

converges strongly to z in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$. As a consequence of the strong convergence of $(u^{n+1})_n$, we deduce that $(v^{n+1})_n$ converges strongly to v in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$. Therefore, for fixed ω,

$$\begin{aligned}\int_0^t \left(-\frac{\partial_s \varphi_s}{\partial s}, v_s\right) ds + \int_0^t \mathcal{E}(\varphi_s, v_s)\, ds \\ = \lim \int_0^t \left(-\frac{\partial_s \varphi_s}{\partial s}, v_s^{n+1}\right) ds + \int_0^t \mathcal{E}(\varphi_s, v_s^{n+1})\, ds \ge 0,\end{aligned}$$

that is, $v(\omega) \in \mathcal{P}$. Then from Proposition 1, we obtain a regular measure associated with v, and $(\nu^{n+1})_n$ converges vaguely to ν.

Taking the limit, we obtain

$$\begin{aligned}(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s)\, ds + \int_0^t \mathcal{E}(u_s, \varphi_s)\, ds \\ + \sum_{i=1}^d \int_0^t \big(g_s^i(u_s, \nabla u_s), \partial_i \varphi_s\big)\, ds \\ = \int_0^t \big(f_s(u_s, \nabla u_s), \varphi_s\big)\, ds + \sum_{j=1}^{+\infty} \int_0^t \big(h_s^j(u_s, \nabla u_s), \varphi_s\big)\, dB_s^j \\ + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds) \qquad \text{a.s.}\end{aligned}$$

From the fact that u and z are in $\mathcal{H}_T$, we know that v is also in $\mathcal{H}_T$, and by definition, ν is a random regular measure. $\square$

6. Comparison theorem.

6.1. *A comparison theorem in the linear case.* We first establish a comparison theorem for the solutions of linear SPDE with obstacle in the case where the obstacles are the same; this also gives a comparison between the regular measures.

So, for this part only, we consider the same hypotheses as in Section 5.2. So we consider adapted processes f, g, h, respectively, in $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$, $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$ and $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}^*})$, an obstacle S which satisfies assumption (O) and $\xi \in L^2(\Omega \times \mathcal{O})$ is an $\mathcal{F}_0$-measurable random variable such that $\xi \le S_0$. We denote by (u, ν) the solution of $\mathcal{R}(\xi, f, g, h, S)$.

We are given another $\xi' \in L^2(\Omega \times \mathcal{O})$ is $\mathcal{F}_0$-measurable and such that $\xi' \le S_0$ and another adapted process f' in $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$. We denote by (u', ν') the solution of $\mathcal{R}(\xi', f', g, h, S)$. We have the following comparison theorem:

THEOREM 7. *Assume that the following conditions hold*:

(1) $\xi \le \xi'$, $dx \otimes dP$-*a.e.*

(2) $f \le f'$, $dt \otimes dx \otimes dP$*-a.e.*

Then for almost all $\omega \in \Omega$, $u \le u'$, $q.e.$ *and* $\nu \ge \nu'$ *in the sense of distribution.*

PROOF. We consider the following two penalized equations:

$$du_t^n = Au_t^n\,dt + f_t\,dt + \sum_{i=1}^{d} \partial_i g_t^i\,dt + \sum_{j=1}^{+\infty} h_t^j\,dB_t^j + n\big(u_t^n - S_t\big)^-\,dt,$$

$$du_t'^n = Au_t'^n\,dt + f_t'\,dt + \sum_{i=1}^{d} \partial_i g_t^i\,dt + \sum_{j=1}^{+\infty} h_t^j\,dB_t^j + n\big(u_t'^n - S_t\big)^-\,dt,$$

and we denote

$$F_t\big(x, u_t^n\big) = f_t(x) + n\big(u_t^n - S_t\big)^-,$$
$$F_t'\big(x, u_t^n\big) = f_t'(x) + n\big(u_t^n - S_t\big)^-.$$

With assumption (2) we have that $F_t(x, u_t^n) \le F_t'(x, u_t^n)$, $dt \otimes dx \otimes dP$-a.e. Therefore, from the comparison theorem for SPDE (without obstacle, see [6]), we know that $\forall t \in [0, T]$, $u_t^n \le u_t'^n$, $dx \otimes dP$-a.e., thus, $n(u_t^n - S_t)^- \ge n(u_t'^n - S_t)^-$.

The results are an immediate consequence of the construction of (u, ν) and (u', ν') given in Section 5.2. $\square$

6.2. *A comparison theorem in the general case.* We now come back to the general setting and consider $(u^1, \nu^1) = \mathcal{R}(\xi^1, f^1, g, h, S^1)$ the solution of the SPDE with obstacle with null boundary condition:

$$\begin{cases} du_t^1(x) = Lu_t^1(x)\,dt + f^1\big(t, x, u_t^1(x), \nabla u_t^1(x)\big)\,dt \\ \qquad\qquad + \displaystyle\sum_{i=1}^{d} \partial_i g_i\big(t, x, u_t^1(x), \nabla u_t^1(x)\big)\,dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{+\infty} h_j\big(t, x, u_t^1(x), \nabla u_t^1(x)\big)\,dB_t^j + \nu^1(x, dt), \\ u^1 \ge S^1, u_0^1 = \xi^1, \end{cases}$$

where we assume (ξ^1, f^1, g, h) satisfy hypotheses (H), (I) and (O).

We consider another coefficient f^2 which satisfies the same assumptions as f^1, another obstacle S^2 which satisfies (O) and another initial condition ξ^2 belonging to $L^2(\Omega \times \mathcal{O})$ and $\mathcal{F}_0$ adapted such that $\xi^2 \ge S_0^2$. We denote by $(u^2, \nu^2) = \mathcal{R}(\xi^2, f^2, g, h, S^2)$.

THEOREM 8. *Assume that the following conditions hold*:

(1) $\xi^1 \le \xi^2$, $dx \otimes dP$*-a.e.*

(2) $f^1(u^1, \nabla u^1) \leq f^2(u^1, \nabla u^1)$, $dt \otimes dx \otimes dP$*-a.e.*
(3) $S^1 \leq S^2$, $dt \otimes dx \otimes dP$*-a.e.*

Then for almost all $\omega \in \Omega$, $u^1(t,x) \leq u^2(t,x)$, *q.e.*

We put $\hat{u} = u^1 - u^2$, $\hat{\xi} = \xi^1 - \xi^2$, $\hat{f}_t = f^1(t, u_t^1, \nabla u_t^1) - f^2(t, u_t^2, \nabla u_t^2)$, $\hat{g}_t = g(t, u_t^1, \nabla u_t^1) - g(t, u_t^2, \nabla u_t^2)$ and $\hat{h}_t = h(t, u_t^1, \nabla u_t) - h(t, u_t^2, \nabla u_t^2)$. The main idea is to evaluate $E\|\hat{u}_t^+\|^2$, thanks to Itô's formula, and then apply Gronwall's inequality. Therefore, we start by the following lemma:

LEMMA 7. *For all* $t \in [0,T]$, *we have*

$$
\begin{aligned}
&E\|\hat{u}_t^+\|^2 + 2E\int_0^t \mathcal{E}(\hat{u}_s^+)\,ds \\
&\quad = E\|\hat{\xi}^+\|^2 + 2E\int_0^t (\hat{u}_s^+, \hat{f}_s)\,ds - 2E\int_0^t (\nabla\hat{u}_s^+, \hat{g}_s)\,ds \qquad (18)\\
&\qquad + 2E\int_0^t\int_{\mathcal{O}} \hat{u}_s^+(x)(\nu - \nu')(dx\,ds) + E\int_0^t \|I_{\{\hat{u}_s > 0\}}|\hat{h}_s|\|^2\,ds.
\end{aligned}
$$

PROOF. We approximate the function $\psi : y \in R \to (y^+)^2$ by a sequence $(\psi_n)_{n\in\mathbb{N}^*}$ of regular functions: let φ be a C^∞ increasing function such that

$$\forall y \in]-\infty, 1] \qquad \varphi(y) = 0 \quad \text{and} \quad \forall y \in [2, +\infty[\qquad \varphi(y) = 1.$$

We set for all $n \in \mathbb{N}^*$,

$$\forall y \in R \qquad \psi_n(y) = y^2\varphi(ny).$$

It is easy to verify that $(\psi_n)_n$ converges uniformly to the function ψ and that, moreover, we have the estimates

$$\forall y \in R^+, \forall n \qquad 0 \leq \psi_n(y) \leq \psi(y), \qquad 0 \leq \psi_n'(y) \leq Cy, \qquad |\psi_n''(y)| \leq C.$$

Thanks to Theorem 6, for all $n \in \mathbb{N}^*$ and $t \in [0,T]$, we have

$$
\begin{aligned}
&E\int_{\mathcal{O}} \psi_n(\hat{u}_s)\,dx + E\int_0^t \mathcal{E}(\psi_n'(\hat{u}_s), \hat{u}_s)\,ds \\
&\quad = E\int_{\mathcal{O}} \psi_n(\hat{\xi})\,dx + E\int_0^t (\psi_n'(\hat{u}_s), \hat{f}_s)\,ds - E\int_0^t (\nabla\psi_n'(\hat{u}_s), \hat{g}_s)\,ds \qquad (19)\\
&\qquad + E\int_0^t\int_{\mathcal{O}} \psi_n'(\hat{u}_s(x))\hat{\nu}(dx\,ds) + \frac{1}{2}E\int_0^t\int_{\mathcal{O}} \psi_n''(\hat{u}_s(x))\hat{h}_s^2(x)\,dx\,ds.
\end{aligned}
$$

Taking the limit, thanks to the dominated convergence theorem, we obtain the convergences of all the terms except $E\int_0^t\int_{\mathcal{O}} \psi_n'(\hat{u}_s(x))\hat{\nu}(dx\,ds)$.

From (19), we know that

$$-E\int_0^t\int_{\mathcal{O}} \psi_n'(\hat{u}_s(x))\hat{\nu}(dx\,ds) \leq C.$$

Moreover, we have the following relation:

$$
\begin{aligned}
-E&\int_0^t\int_{\mathcal{O}}\psi_n'(\hat{u}_s(x))\hat{\nu}(dx\,ds)\\
&=-E\int_0^t\int_{\mathcal{O}}\psi_n'(S_s^1(x)-u_s^2(x))\nu^1(dx\,ds)\\
&\qquad+E\int_0^t\int_{\mathcal{O}}\psi_n'(u_s^1(x)-S_s^2(x))\nu^2(dx\,ds)\\
&=E\int_0^t\int_{\mathcal{O}}\psi_n'(u_s^2(x)-S_s^1(x))\nu^1(dx\,ds)\\
&\qquad+E\int_0^t\int_{\mathcal{O}}\psi_n'(u_s^1(x)-S_s^2(x))\nu^2(dx\,ds).
\end{aligned}
$$

By Fatou's lemma, we obtain

$$
\begin{aligned}
&2E\int_0^t\int_{\mathcal{O}}(u_s^2(x)-S_s^1(x))^+\nu^1(dx\,ds)+2E\int_0^t\int_{\mathcal{O}}(u_s^1(x)-S_s^2(x))^+\nu^2(dx\,ds)\\
&\quad<+\infty.
\end{aligned}
$$

Hence, the convergence of the term $E\int_0^t\int_{\mathcal{O}}\psi_n'(\hat{u}_s(x))\hat{\nu}(dx\,ds)$ comes from the dominated convergence theorem. $\square$

PROOF OF THEOREM 8. Applying Itô's formula (18) to $(\hat{u}_t^+)^2$, we have

$$
\begin{aligned}
E&\|\hat{u}_t^+\|^2+2E\int_0^t I_{\{\hat{u}_s>0\}}\mathcal{E}(\hat{u}_s)\,ds\\
&=2E\int_0^t(\hat{u}_s^+,\hat{f}_s)\,ds+2E\int_0^t(\hat{u}_s^+,\hat{g}_s)\,ds\\
&\qquad+E\int_0^t\|I_{\{\hat{u}_s>0\}}|\hat{h}_s|\|^2\,ds+2E\int_0^t\int_{\mathcal{O}}(u_s^1-u_s^2)^+(x)(\nu^1-\nu^2)(dx,ds).
\end{aligned}
$$

As we assume that $f^1(u^1,\nabla u^1)\le f^2(u^1,\nabla u^1)$,

$$
\begin{aligned}
\hat{u}_s^+\hat{f}_s&=\hat{u}_s^+\{f^1(s,u_s^1,\nabla u_s^1)-f^2(s,u_s^1,\nabla u_s^1)\}\\
&\qquad+\hat{u}_s^+\{f^2(s,u_s^1,\nabla u_s^1)-f^2(s,u_s^2,\nabla u_s^2)\}\\
&\le\hat{u}_s^+\{f^2(s,u_s^1,\nabla u_s^1)-f^2(s,u_s^2,\nabla u_s^2)\}.
\end{aligned}
$$

Then with the Lipschitz condition, using Cauchy–Schwarz's inequality, we have the following relations:

$$
\begin{aligned}
E\int_0^t(\hat{u}_s^+,\hat{f}_s)\,ds&\le\left(C+\frac{C}{\varepsilon}\right)E\int_0^t\|\hat{u}_s^+\|^2\,ds+\frac{C\varepsilon}{\lambda}E\int_0^t\mathcal{E}(\hat{u}_s^+)\,ds,\\
E\int_0^t(\nabla\hat{u}_s^+,\hat{g}_s)&\le\frac{\varepsilon+\alpha}{\lambda}E\int_0^t\mathcal{E}(\hat{u}_s^+)\,ds+\frac{C}{\varepsilon}E\int_0^t\|\hat{u}_s^+\|^2\,ds,
\end{aligned}
$$

$$E\int_0^t \|I_{\{\hat{u}_s>0\}}|\hat{h}_s|\|^2\,ds \le CE\int_0^t \|\hat{u}_s^+\|^2\,ds + \frac{\beta^2+\varepsilon}{\lambda}E\int_0^t \mathcal{E}(\hat{u}_s^+)\,ds.$$

The last term is equal to $-2E\int_0^t\int_{\mathcal{O}}(u_s^1-u_s^2)^+(x)\nu^2(dx,ds)\le 0$, because on $\{u^1\le u^2\}$, $(u^1-u^2)^+=0$ and on $\{u^1>u^2\}$, $\nu^1(dx,ds)=0$. Thus, we have the following inequality:

$$E\|\hat{u}_t^+\|^2 + \left(2-\frac{2\alpha+2\varepsilon}{\lambda}-\frac{2C\varepsilon}{\lambda}-\frac{\beta^2+\varepsilon}{\lambda}\right)E\int_0^t \mathcal{E}(\hat{u}_s^+)\,ds \le CE\int_0^t \|\hat{u}_s^+\|^2\,ds.$$

We can take ε small enough such that $2-\frac{2\alpha+2\varepsilon}{\lambda}-\frac{2C\varepsilon}{\lambda}-\frac{\beta^2+\varepsilon}{\lambda}>0$, and we have

$$E\|\hat{u}_t^+\|^2 \le CE\int_0^t \|\hat{u}_s^+\|^2\,ds.$$

Then we deduce the result from Gronwall's lemma. □

REMARK 6. Applying the comparison theorem to the same obstacle gives another proof of the uniqueness of the solution.

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L. DENIS
J. ZHANG
LABORATOIRE ANALYSE ET PROBABILITÉS
UNIVERSITÉ D'EVRY-VAL-D'ESSONNE
23 BOULEVARD DE FRANCE
F-91037 EVRY CEDEX
FRANCE
E-MAIL: ldenis@univ-evry.fr
jing.zhang.etu@gmail.com

A. MATOUSSI
LABORATOIRE MANCEAU DE MATHÉMATIQUES
UNIVERSITÉ DU MAINE
FÉDÉRATION DE RECHERCHE 2962 DU CNRS
MATHÉMATIQUES DES PAYS DE LOIRE
AVENUE OLIVIER MESSIAEN
F-72085 LE MANS CEDEX 9
FRANCE
AND
CMAP
ECOLE POLYTECHNIQUE, PALAISEAU
FRANCE
E-MAIL: anis.matoussi@univ-lemans.fr