The Annals of Probability
2013, Vol. 41, No. 5, 3345–3391
DOI: 10.1214/12-AOP793

DISORDER CHAOS IN THE SHERRINGTON–KIRKPATRICK MODEL WITH EXTERNAL FIELD

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We consider a spin system obtained by coupling two distinct Sherrington–Kirkpatrick (SK) models with the same temperature and external field whose Hamiltonians are correlated. The disorder chaos conjecture for the SK model states that the overlap under the corresponding Gibbs measure is essentially concentrated at a single value. In the absence of external field, this statement was first confirmed by Chatterjee [Disorder chaos and multiple valleys in spin glasses (2009) Preprint]. In the present paper, using Guerra's replica symmetry breaking bound, we prove that the SK model is also chaotic in the presence of the external field and the position of the overlap is determined by an equation related to Guerra's bound and the Parisi measure.

1. Introduction and main results. The phenomenon of chaos arose from the discovery that in some models, a slight perturbation on the parameters such as the temperature, external field or disorder will result in a dramatic change to the system. In this paper, we will be concerned with the Sherrington–Kirkpatrick (SK) model [12] and study its chaotic property mainly due to the change of the disorder. Let us begin by recalling the definition of the SK model and the formulation of the Parisi formula. Suppose that $\xi : \mathbb{R} \to \mathbb{R}$ is a convex function satisfying $\xi(x) = \xi(-x)$, $\xi''(x) > 0$ if $x \neq 0$, and $\xi^{(3)} \geq 0$ if $x > 0$. For each N, we consider a centered Gaussian process $H = H_N$ indexed by the configuration space $\Sigma_N = \{-1, +1\}^N$ with covariance

$$EH_N(\boldsymbol{\sigma}^1)H_N(\boldsymbol{\sigma}^2) = N\xi(R_{1,2})$$

for $\boldsymbol{\sigma}^1 = (\sigma_1^1, \dots, \sigma_N^1), \boldsymbol{\sigma}^2 = (\sigma_1^2, \dots, \sigma_N^2) \in \Sigma_N$, where

$$R_{1,2} = R_{1,2}(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \frac{1}{N}\sum_{i \leq N} \sigma_i^1 \sigma_i^2$$

is called the overlap of the configurations $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$. Let h be a random variable and $(h_i)_{i \leq N}$ be i.i.d. copies of h. Then the SK model with external field h possesses the Hamiltonian

$$-H(\boldsymbol{\sigma}) + \sum_{i \leq N} h_i \sigma_i$$

Received September 2011; revised July 2012.
MSC2010 subject classifications. 60K35, 82B44.
Key words and phrases. Disorder chaos, Guerra's replica symmetry breaking bound, Parisi formula, Parisi measure, Sherrington–Kirkpatrick model.

for $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_N) \in \Sigma_N$ and its Gibbs measure is defined as

$$G_N(\boldsymbol{\sigma}) = \frac{1}{Z_N} \exp\Bigl(-H(\boldsymbol{\sigma}) + \sum_{i \le N} h_i \sigma_i\Bigr),$$

where Z_N is a normalizing factor, called the partition function. Let us also define

$$p_N = \frac{1}{N} E \log Z_N = \frac{1}{N} E \log \sum_{\boldsymbol{\sigma} \in \Sigma_N} \exp\Bigl(-H(\boldsymbol{\sigma}) + \sum_{i \le N} h_i \sigma_i\Bigr).$$

This quantity is usually called the free energy for the SK model in physics and its thermodynamic limit $\lim_{N \to \infty} p_N$ can be computed by the Parisi formula described below.

Consider an integer $k \ge 0$ and numbers

$$\begin{aligned} &\mathbf{m}\colon m_0 = 0 \le m_1 \le \cdots \le m_k \le m_{k+1} = 1, \\ &\mathbf{q}\colon q_0 = 0 \le q_1 \le \cdots \le q_{k+1} \le q_{k+2} = 1. \end{aligned} \tag{1.1}$$

It helps to think of the triplet $k, \mathbf{m}, \mathbf{q}$ as a probability measure μ on [0, 1] that has all its mass concentrated at a finite number of points $q_1, \dots, q_{k+1}$ and $\mu([0, q_p]) = m_p$ for $1 \le p \le k+1$. Let $z_0, \dots, z_{k+1}$ be independent Gaussian r.v.'s with $E z_p^2 = \xi'(q_{p+1}) - \xi'(q_p)$ for $0 \le p \le k+1$. Starting with

$$X_{k+2} = \log \cosh\Bigl(h + \sum_{0 \le p \le k+1} z_p\Bigr),$$

we define by decreasing induction for $1 \le p \le k+1$,

$$X_p = \frac{1}{m_p} \log E_p \exp m_p X_{p+1},$$

where E_p means the expectation on the r.v.'s $z_p, z_{p+1}, \dots, z_{k+1}$. If $m_p = 0$ for some p, we define $X_p = E_p X_{p+1}$. Finally, we define $X_0 = E X_1$. Set

$$\mathcal{P}_k(\mathbf{m}, \mathbf{q}) = \log 2 + X_0 - \frac{1}{2} \sum_{p=1}^{k+1} m_p \bigl(\theta(q_{p+1}) - \theta(q_p)\bigr), \tag{1.2}$$

where $\theta(x) = x\xi'(x) - \xi(x)$. This quantity is the famous Guerra replica symmetry breaking bound of the kth level [7] that yields a fundamental inequality, for every $k, \mathbf{m}, \mathbf{q}$,

$$p_N \le \mathcal{P}_k(\mathbf{m}, \mathbf{q}). \tag{1.3}$$

Let us define the Parisi functional on the space of all probability measures on [0, 1] consisting of only a finite number of point masses by $\mathcal{P}(\xi, h, \mu) = \mathcal{P}_k(\mathbf{m}, \mathbf{q})$

if μ corresponds to $(k, \mathbf{m}, \mathbf{q})$. We define $\mathcal{P}(\xi, h) = \inf_{k,\mathbf{m},\mathbf{q}} \mathcal{P}_k(\mathbf{m}, \mathbf{q})$, where the infimum is over all choices of $(k, \mathbf{m}, \mathbf{q})$ as above. Then the Parisi formula says that

$$\lim_{N\to\infty} p_N = \mathcal{P}(\xi, h).$$

This formula was first rigorously proven in Talagrand [13]. It is well known [7] that the Parisi functional is Lipschitz continuous with respect to the metric $d(\mu, \mu') = \int_0^1 |\mu([0, q]) - \mu'([0, q])|\, dq$. Thus, it can be extended continuously to the space of all probability measures defined on [0, 1] and is denoted again by $\mathcal{P}(\xi, h, \cdot)$. Then clearly $\lim_{N\to\infty} p_N = \mathcal{P}(\xi, h) = \min \mathcal{P}(\xi, h, \mu)$, where the minimum is taken over all probability measures defined on [0, 1]. Any measure that achieves the minimum is called a Parisi measure. Heuristically, one may think of the Parisi measure as the limiting distribution of the overlap.

We are now ready to formulate the disorder chaos problem in the SK model. Let $0 \le t \le 1$. Suppose that $H^1 = H_N^1$ and $H^2 = H_N^2$ are two centered Gaussian processes having the same distribution as H and they are correlated in the following way,

$$EH^1(\boldsymbol{\sigma}^1)H^2(\boldsymbol{\sigma}^2) = Nt\xi(R_{1,2}). \tag{1.4}$$

That is, we allow a portion $1 - t$ of independence between two systems. Consider the coupled Hamiltonian

$$-H^1(\boldsymbol{\sigma}^1) - H^2(\boldsymbol{\sigma}^2) + \sum_{i\le N} h_i(\sigma_i^1 + \sigma_i^2)$$

on Σ_N^2. Proceeding as before, we define its Gibbs measure by

$$G'_N(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \frac{1}{Z'_N} \exp\left(-H^1(\boldsymbol{\sigma}^1) - H^2(\boldsymbol{\sigma}^2) + \sum_{i\le N} h_i(\sigma_i^1 + \sigma_i^2)\right),$$

where the normalizing factor Z'_N is the partition function of this model. As we have already mentioned in the beginning of this section, the chaos phenomenon is concerned with the instability occurring in some spin glass models due to the change of some external parameters. In the SK model, one very basic way to measure such instability mainly due to the change of the disorder, or, briefly, chaos in disorder, is to study the behavior of the overlap. A typical statement one is looking for in this case is that if $0 < t < 1$, the overlap takes essentially only one value under G'_N. This is quite different from the typical lack of self-averaging property of the overlap in the low temperature phase when $t = 1$. The phenomenon of chaos itself was first conjectured by Fisher and Huse [6]. Early discussion on the disorder chaos for the SK model can be found in [3] and [9]. For further references in the physics literature, one may refer to [8]. However, the mathematically rigorous results have appeared only lately. In the absence of the external field, Chatterjee [4] recently proved chaos in disorder and discovered that the overlap is concentrated at 0.

In the present work, we aim to prove that the disorder chaos also holds in the presence of the external field, that is, $Eh^2 \neq 0$. Moreover, we find that when there is chaos, the position of the overlap can be described by an equation, which is related to the Parisi measure and can be formulated as follows. Suppose that μ is a Parisi measure. Recall that μ minimizes the Parisi functional. We can approximate μ weakly by a sequence of ε_n-stationary measures (μ_n) satisfying $\mathcal{P}(\xi, h, \mu_n) \to \mathcal{P}(\xi, h)$. Here, by ε_n-stationarity, it means that the measure μ_n minimizes the kth level Guerra replica symmetry breaking bound for some k depending on n and $\mathcal{P}(\xi, h, \mu_n) < \mathcal{P}(\xi, h) + \varepsilon_n$, where $\varepsilon_n \downarrow 0$ (see Definition 3 below). This approximation is for technical purposes that have played a crucial role in Talagrand's proof on the Parisi formula [13] and will also be of great importance in our argument. For a given $(k, \mathbf{m}, \mathbf{q})$ corresponding to μ, recall the definition of X_0 from (1.2). A very nice and useful fact about this quantity is that it can be computed as $E\Phi(h, 0)$, where $\Phi : \mathbb{R} \times [0, 1] \to \mathbb{R}$ is the solution to the following PDE,

$$(1.5) \qquad \frac{\partial \Phi}{\partial q} = -\frac{\xi''(q)}{2}\left(\frac{\partial^2 \Phi}{\partial x^2} + \mu([0, q])\left(\frac{\partial \Phi}{\partial x}\right)^2\right) \qquad \forall (x, q) \in \mathbb{R} \times [0, 1]$$

with $\Phi(x, 1) = \log\cosh x$. For each n, let Φ_n be the PDE solution (1.5) corresponding to μ_n. From [14], we know that (Φ_n) converges uniformly and we denote its limit by Φ. Moreover, [14] yields that the first partial derivative of Φ with respect to x exists. From this, for each fixed $0 < v < 1$, we define

$$(1.6) \qquad \varphi_v(u, t) = E\frac{\partial \Phi}{\partial x}(h + \chi_1, v)\frac{\partial \Phi}{\partial x}(h + \chi_2, v) - u$$

for all $0 \le u \le v$ and $0 \le t \le 1$, where χ_1 and χ_2 are jointly Gaussian with $E\chi_1^2 = E\chi_2^2 = \xi'(v)$ and $E\chi_1\chi_2 = t\xi'(u)$ independent of h. The motivation of φ_v comes from the Guerra replica symmetry breaking bound for the coupled free energy that will be explained in great detail in Section 5 below. An important fact about the Parisi measure μ in the case of $Eh^2 \neq 0$ is that the smallest value c of its support is positive. This is called the positivity of the overlap (see Chapter 14 [17]). When $v = c$ and $0 \le t < 1$, we are able to determine the number of the solutions of $\varphi_c(\cdot, t) = 0$.

PROPOSITION 1. *For each* $0 \le t < 1$, *there exists a unique* u_t *in* $[0, c]$ *such that* $\varphi_c(u_t, t) = 0$. *Moreover,* $\varphi_c(c, t) < 0$ *for* $0 \le t < 1$ *and* $\varphi_c(c, 1) = 0$.

Now, the quantitative result of the disorder chaos in the SK model is stated as follows.

THEOREM 1. *Suppose that* $0 < t < 1$ *and* $Eh^2 > 0$. *Then the SK model has disorder chaos, namely, for any* $\varepsilon > 0$, *the following holds*:

$$(1.7) \qquad EG'_N\left(\{(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : |R_{1,2} - u_t| \ge \varepsilon\}\right) \le K\exp\left(-\frac{N}{K}\right),$$

where K *is a constant depending on* t, ξ, h, ε *and* μ.

A consequence of Theorem 1 is that even though we do not know that the Parisi measure μ is unique, the quantity u_t is independent of the choice of μ. However, the convergence rate K in (1.7) does depend on μ. In [14] and [15], other types of chaos problems in the SK model are also proposed, such as chaos in temperature and chaos in external field. Again, the rigorous results are still scarce. Theorem 1 is the first result in chaos problems of any kind in the SK model with the external field. To the best of our knowledge, the only other two instances of chaos problems in spin glasses are in the work of Chatterjee [4], who proved chaos in disorder in the SK model without the external field, and in the work of Panchenko and Talagrand [10], who established chaos in the external field in the spherical SK model.

The approach of the present paper is motivated by Talagrand's proof on the positivity of the overlap in the SK model; see Section 14.12 [17]. We also refer to a sketch of a possible proof for the disorder chaos problem discussed in Research Problem 15.7.14 [17]. However, it is by no means clear how to implement these approaches properly that contain several technical issues and require some new ideas. Here is our main result.

PROPOSITION 2. *Let* $0 < t < 1$ *and* $Eh^2 > 0$. *For* $\varepsilon > 0$, *there exists some* $\varepsilon^* > 0$ *such that*

$$\begin{aligned} p_{N,u} &:= \frac{1}{N} E \log \sum_{R_{1,2}=u} \exp\Big(-H^1(\boldsymbol{\sigma}^1) - H^2(\boldsymbol{\sigma}^2) + \sum_{i \le N} h_i(\sigma_i^1 + \sigma_i^2)\Big) \\ &\le 2\mathcal{P}(\xi, h) - \varepsilon^* \end{aligned} \tag{1.8}$$

for all u *satisfying* $|u - u_t| \ge \varepsilon$, *where* ε^* *is a constant depending on* t, ξ, h, ε *and* μ.

As an immediate consequence of the Gaussian concentration of measure phenomenon (see Theorem 13.4.3 in [17] and also the argument for the positivity of the overlap on page 449 of [17]), Theorem 1 follows from Proposition 2. Let us continue by giving a brief description of how we proceed to prove Proposition 2. The approach for proving (1.8) is based on the Guerra replica symmetry breaking bound that was first used for the coupled system in [13]. We divide our discussion into three cases: $-1 \le u \le 0$, $0 \le u \le c'$, and $c' < u \le 1$, where c' satisfies $c' > c$ and is very close to c. In the presence of the external field, we adapt a similar argument as Talagrand's proof on the positivity of the overlap (see Section 14.12 in [17]) to conclude (1.8) for $-1 \le u \le 0$. In the case that $0 \le u \le c'$, if there is chaos, the system should exhibit "high temperature behavior" and u_t should be determined by an equation related to the Parisi measure, as is the case of the original SK model in the high temperature regime; see Chapter 2 in [16].

This observation then leads to (1.8). The most difficult part of our study is the case when $c' < u \le 1$. We establish an iterative inequality, which is very sensitive to the parameter t. From the construction of the Parisi measure, we are able to find parameters such that (1.8) holds even in the absence of the external field.

The paper is organized as follows. Throughout the paper, we denote by E the expectation with respect to all randomness and we assume that the external field h satisfies $Eh^2 > 0$ and every Gaussian r.v. is centered. In Section 2 we first give the formulation of an extended version of Guerra's replica symmetry breaking bound and explain why this is applicable to our study. We then continue to carry out the core of the proof of Proposition 2. In Section 3 we state some results that help to control Guerra's bound. Most of their proofs can be found in [17]. Section 4 is devoted to proving (1.8) for $-1 \le u \le 0$ based on the same argument as Section 14.12 in [17]. In Section 5 we study how Guerra's bound relates to the definition of φ_v and give the proof of Proposition 1. Together they imply (1.8) for $0 \le u \le c'$. Finally, we develop an iterative inequality and prove (1.8) for $c' < u < 1$ in Section 6.

2. Methodology. Let us first state an extension of the Guerra replica symmetry breaking bound. Suppose that $-1 \le u \le 1$ and $\eta \in \{-1, +1\}$ satisfies $u = \eta|u|$. For a given integer $\kappa \ge 1$, we consider numbers

$$\begin{aligned} &1 \le \tau \le \kappa, \qquad \tau \in \mathbb{N}, \\ &n_0 = 0 \le n_1 \le \cdots \le n_{\kappa-1} \le n_\kappa = 1, \\ &\rho_0 = 0 \le \rho_1 \le \cdots \le \rho_\tau = |u| \le \rho_{\tau+1} \le \cdots \le \rho_{\kappa+1} = 1. \end{aligned} \tag{2.1}$$

For $0 \le p \le \kappa$, suppose that we are given independent pairs of jointly Gaussian r.v.'s (y_p^1, y_p^2) with

$$E(y_p^1)^2 = E(y_p^2)^2 = \xi'(\rho_{p+1}) - \xi'(\rho_p)$$

such that

$$Ey_p^1 y_p^2 = \eta t(\xi'(\rho_{p+1}) - \xi'(\rho_p)) \qquad \text{if } 0 \le p < \tau$$

and

$$y_p^1 \text{ and } y_p^2 \text{ are independent if } \tau \le p \le \kappa.$$

These r.v.'s are independent of h. For our convenience, from now on, we set $\mathrm{sh}(x) = \sinh x$, $\mathrm{ch}(x) = \cosh x$, and $\mathrm{th}(x) = \tanh x$. Let λ be any real number. Starting with

$$\begin{aligned} Y_{\kappa+1} = \log\Big(&\mathrm{ch}\Big(h + \sum_{0 \le p \le \kappa} y_p^1\Big)\mathrm{ch}\Big(h + \sum_{0 \le p \le \kappa} y_p^2\Big)\mathrm{ch}\,\lambda \\ &+ \mathrm{sh}\Big(h + \sum_{0 \le p \le \kappa} y_p^1\Big)\mathrm{sh}\Big(h + \sum_{0 \le p \le \kappa} y_p^2\Big)\mathrm{sh}\,\lambda\Big), \end{aligned}$$

we define by decreasing induction for $p \geq 1$,

$$Y_p = \frac{1}{n_p} \log E_p \exp n_p Y_{p+1},$$

where E_p denotes expectation in the r.v.'s y_n^j for $n \geq p$. In the case of $n_p = 0$ for some p, we set $Y_p = E_p Y_{p+1}$. Finally, we define $Y_0 = EY_1$.

THEOREM 2. *We have*

$$p_{N,u} \leq 2\log 2 + Y_0 - \lambda u - (1+t) \sum_{0 \leq p < \tau} n_p\bigl(\theta(\rho_{p+1}) - \theta(\rho_p)\bigr) \\ - \sum_{\tau \leq p \leq \kappa} n_p\bigl(\theta(\rho_{p+1}) - \theta(\rho_p)\bigr). \tag{2.2}$$

Recalling Guerra's original bound (1.3), (2.2) is a kind of two-dimensional extension. Its proof is essentially the same as that of Proposition 14.12.4 [17] and a more generalized version can be found in Section 15.7 [17]. One might have already observed that from the definition of $p_{N,u}$ and (1.3), $p_{N,u} \leq 2p_N \leq 2\mathcal{P}_k(\mathbf{m}, \mathbf{q})$ for any $k, \mathbf{m}, \mathbf{q}$. Before we proceed to state our main results in this section, let us illustrate that for any given $k, \mathbf{m}, \mathbf{q}$, we can find parameters (2.1) such that the right-hand side of (2.2) is equal to $2\mathcal{P}_k(\mathbf{m}, \mathbf{q})$. This recovers the inequality $p_{N,u} \leq 2\mathcal{P}_k(\mathbf{m}, \mathbf{q})$. To do this, let $k, \mathbf{m}, \mathbf{q}$ satisfy (1.1) and τ with $1 \leq \tau \leq k+2$ satisfying

$$q_{\tau-1} \leq |u| \leq q_\tau.$$

Without loss of generality, we may assume that $|u|$ is in the list of $\mathbf{q}$. Indeed, we can always consider a new triplet $k+1, \mathbf{m}', \mathbf{q}'$ obtained by inserting $|u|$ into $\mathbf{q}$ and keeping $\mathbf{m}$ fixed in the following way:

$\mathbf{m}'$: $m'_p = m_p$ for $0 \leq p \leq \tau - 1$, m_{p-1} if $p = \tau$, and m_{p-1} if $\tau + 1 \leq p \leq k+2$,

$\mathbf{q}'$: $q'_p = q_p$ for $0 \leq p \leq \tau - 1$, $|u|$ if $p = \tau$, and q_{p-1} for $\tau + 1 \leq p \leq k+3$.

Then $|u|$ is in the list of $\mathbf{q}'$ and from (1.2), one can easily check that $\mathcal{P}_k(\mathbf{m}, \mathbf{q}) = \mathcal{P}_{k+1}(\mathbf{m}', \mathbf{q}')$. Let us notice that this concept, though simple, will simplify many of our future discussions.

We specify the following values for (2.1):

$$\begin{aligned} &\kappa = k+1, \\ &n_p = \frac{m_p}{1+t} \quad \text{if } 0 \leq p < \tau \quad \text{and} \quad m_p \quad \text{if } \tau \leq p \leq \kappa, \\ &\rho_p = q_p \quad \text{for } 0 \leq p \leq \kappa + 1. \end{aligned} \tag{2.3}$$

Let $\lambda = 0$. From Theorem 2, it follows that

$$p_{N,u} \le 2\log 2 + Y_0 - \sum_{0\le p\le k+1} m_p\big(\theta(q_{p+1}) - \theta(q_p)\big).$$

Let $y_0^1, y_0^2, \dots, y_{k+1}^1, y_{k+1}^2$ be jointly Gaussian r.v.'s defined in Theorem 2 and be independent of h. For $j = 1, 2$, we define $(X_p^j)_{0\le p\le k+2}$ in the same way as $(X_p)_{0\le p\le k+2}$ by using $k, \mathbf{m}, \mathbf{q}$, and $(y_p^j)_{0\le p\le k+1}$. Since y_p^1 and y_p^2 are independent of each other for each $\tau \le p \le k+1$, it implies $Y_\tau = X_\tau^1 + X_\tau^2$. To bound Y_0 from above, we need the following lemma, which can be proven by following the same idea as Proposition 12 in Section 6 below and is left to the reader.

LEMMA 1. *Suppose that η is a constant which takes value 1 or -1. Consider two jointly Gaussian r.v.'s y_1 and y_2 such that $Ey_1^2 = Ey_2^2$ and $Ey_1y_2 = \eta t Ey_1^2$. Consider two functions F_1 and F_2 such that their first four derivatives are uniformly bounded. Then for any values of x_1, x_2 and $m > 0$ we have*

$$\begin{aligned}\frac{1+t}{m}\log E\exp\frac{m}{1+t}\big(F_1(x_1+y_1) + F_2(x_2+y_2)\big) \\ \le \sum_{j=1,2}\frac{1}{m}\log E\exp mF_j(x_j+y_j).\end{aligned}\tag{2.4}$$

Since y_p^1 and y_p^2 satisfy $Ey_p^1y_p^2 = \eta t E(y_p^1)^2 = \eta t E(y_p^2)^2$ for $0 \le p < \tau$, using (2.4) and decreasing induction, $Y_0 \le X_0^1 + X_0^2 = 2X_0$. Hence, we conclude that for any given numbers $k, \mathbf{m}, \mathbf{q}$, we can find parameters (2.1) such that $\mathcal{P}_k(\mathbf{m},\mathbf{q})$ can be recovered by the right-hand side of (2.2), that is, $p_{N,u} \le 2\mathcal{P}_k(\mathbf{m},\mathbf{q})$. Now, to prove Proposition 2, we have to find suitable parameters (2.1) for Guerra's bound. It turns out that this can be done and leads to the following three crucial propositions. First, we have the following result.

PROPOSITION 3. *For $0 < t \le 1$, there exists a number $\varepsilon^* < 0$ depending only on t, ξ and h such that for every $u \le 0$, $p_{N,u} \le 2\mathcal{P}(\xi, h) - \varepsilon^*$.*

This proposition means that the overlap takes essentially nonnegative values, which is mainly due to the presence of the external field, that is, $Eh^2 \ne 0$. Let μ be the Parisi measure and c be the smallest value of its support. Recall the definition of $\varphi_v(u,t)$ corresponding to μ from (1.6). Two crucial facts about Y_0 that will be derived in Sections 3 and 5 below are that for arbitrary choice of (2.1), the second partial derivative of Y_0 with respect to λ is bounded by 1 and if we choose (2.1) properly, the first partial derivative of Y_0 at $\lambda = 0$ roughly gives the formulation of φ_c. From Guerra's bound and these facts, they imply our next proposition.

PROPOSITION 4. *For $0 \le u \le c$ and $0 \le t \le 1$ we have*

$$p_{N,u} \le 2\mathcal{P}(\xi, h) - \frac{1}{2}\varphi_c(u, t)^2. \tag{2.5}$$

If $0 \le t < 1$, then there exists a $\gamma > 0$ depending on the Parisi measure μ and t such that

$$p_{N,u} \le 2\mathcal{P}(\xi, h) - \frac{1}{16}\varphi_c(c, t)^2 \tag{2.6}$$

for every $c \le u \le c + \gamma$.

At last, we investigate the upper bound for $p_{N,u}$ when $u > c'$ for some fixed $c' > c$. This strongly relies on the assumption that these two SK models use different disorders, that is, $0 < t < 1$. Our main result is stated as follows.

PROPOSITION 5. *Suppose that $0 < t < 1$ and $c < c' < 1$. Then there exists $\varepsilon^* > 0$ such that $p_{N,u} \le 2\mathcal{P}(\xi, h) - \varepsilon^*$ for every $c' \le u \le 1$, where ε^* depends only on t, ξ, h, c'.*

These propositions are the main ingredients of the proof of Proposition 2 and their proofs are deferred to Sections 4, 5 and 6, respectively. Now, let us proceed to prove Proposition 2.

PROOF OF PROPOSITION 2. Let $0 < t < 1$ be fixed. From Proposition 3, there exists ε_1^* depending only on t, ξ and h such that for every $-1 \le u \le 0$,

$$p_{N,u} \le 2\mathcal{P}(\xi, h) - \varepsilon_1^*. \tag{2.7}$$

Now, for given $\varepsilon > 0$, we set

$$\varepsilon_2^* = \frac{1}{2}\min\{\varphi_c(w, t)^2 : 0 \le w \le c, |w - u_t| \ge \varepsilon\}.$$

Since u_t is the unique solution of $\varphi_c(\cdot, t)$ in $[0, c]$, it follows that $\varepsilon_2^* > 0$ and from (2.5),

$$p_{N,u} \le 2\mathcal{P}(\xi, h) - \varepsilon_2^*, \tag{2.8}$$

whenever $0 \le u \le c$ and $|u - u_t| \ge \varepsilon$. Since we also know $\varphi_c(c, t) < 0$, from (2.6), there exists some $\gamma > 0$ depending only on μ and t such that

$$p_{N,u} \le 2\mathcal{P}(\xi, h) - \varepsilon_3^* \tag{2.9}$$

for every $c \le u \le c + \gamma$, where $\varepsilon_3^* = \varphi_c(c, t)^2/16 > 0$. Let us put $c' = c + \gamma$ in Proposition 5. Then there exists $\varepsilon_4^* > 0$ depending only on t, ξ, h, c' such that

$$p_{N,u} \le 2\mathcal{P}(\xi, h) - \varepsilon_4^*, \tag{2.10}$$

whenever $c' \le u \le 1$. Finally, we obtain (1.8) by combining (2.7), (2.8), (2.9) and (2.10) together and letting $\varepsilon^* = \min(\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*)$. □

3. Preliminary results. Let $k, \mathbf{m}, \mathbf{q}$ be given by (1.1). Suppose that $(z_p)_{0\le p\le k+1}$ are independent Gaussian r.v.'s with $Ez_p^2 = \xi'(q_{p+1}) - \xi'(q_p)$. Starting with $A_{k+2}(x) = \log \operatorname{ch} x$, we define

$$A_p(x) = \frac{1}{m_p} \log E \exp m_p A_{p+1}(x + z_p) \tag{3.1}$$

for $0 \le p \le k+1$. If $m_p = 0$, we define $A_p(x) = EA_{p+1}(x + z_p)$. Recall X_0 from (1.2). It should be clear that

$$X_p = A_p\biggl(h + \sum_{0\le n<p} z_n\biggr)$$

for every $1 \le p \le k+2$ and $X_0 = EA_0(h)$. Since we will be working with $(A_p)_{0\le p\le k+2}$ for much of the remainder of this paper, we summarize some quantitative results in Lemma 2.

LEMMA 2. *For every* $0 \le p \le k+2$, *we have*

$$\begin{gathered} A_p(x) = A_p(-x), \qquad |A_p'| \le 1, \\ \frac{1}{C \operatorname{ch}^2 x} \le A_p''(x) \le \min\biggl(1, \frac{C}{\operatorname{ch}^2 x}\biggr), \\ |A_p^{(3)}| \le 4, \qquad |A_p^{(4)}| \le 8, \end{gathered} \tag{3.2}$$

where C is a constant depending only on ξ.

PROOF. The Poisson–Dirichlet cascade was of great importance in the study of the random energy model and generalized random energy model in [5, 11] and was put forward to the SK model, in particular, in [1, 2]. Following similar ideas in these works, it is known from Theorem 14.2.1 [17] that A_p has a very beautiful representation via the Poisson–Dirichlet cascade [see (3.3) below]. Our argument will be started with such representation and is concentrated on the inequality $A_p''(x) \le C(\operatorname{ch}^2 x)^{-1}$. For the other statements, one may refer to Lemma 14.7.16 [17]. Since A_p is an even function, it suffices to prove that $A_p''(x) \le C \exp(-2x)$ for all $x \ge 0$. Let $\tau_1 \ge 1$ be the smallest integer with $m_{\tau_1} > 0$ and $\tau_2 \le k$ be the largest integer with $m_{\tau_2} < 1$. Suppose for the moment that there exists $C_1 > 0$ such that $A_p''(x) \le C_1 \exp(-2x)$ for all $x \ge 0$ and $\tau_1 \le p \le \tau_2$. By definition of A_p, for all $x \ge 0$ and $0 \le p < \tau_1$, we have that

$$A_p(x) = EA_{\tau_1}\biggl(x + \sum_{p\le n<\tau_1} z_n\biggr)$$

and then

$$A''_p(x) = EA''_{\tau_1}\Big(x + \sum_{p\le n<\tau_1} z_n\Big)$$
$$\le 2C_1 \exp(-2x) E \operatorname{ch}\Big(2 \sum_{p\le n<\tau_1} z_n\Big)$$
$$= 2C_1 \exp\big(2\big(\xi'(q_{\tau_1}) - \xi'(q_p)\big)\big) \exp(-2x).$$

Also for all $x \ge 0$ and $\tau_2 < p \le k+2$, it is easy to see that

$$A_p(x) = \log E \operatorname{ch}\Big(x + \sum_{p\le n<k+2} z_n\Big) = \log \operatorname{ch} x + \frac{1}{2}\big(\xi'(1) - \xi'(q_p)\big)$$

and so $A''_p(x) = (\operatorname{ch}^2 x)^{-1} \le C_2 \exp(-2x)$ for some constant $C_2 > 0$. If we set

$$C = \max\big(C_2, 2C_1 \exp\big(2\xi'(1)\big)\big),$$

then $A''_p(x) \le C \exp(-2x)$ for all $x \ge 0$ and $0 \le p \le k+2$. So in the following, we may assume, without loss of generality, that $0 < m_1, m_k < 1$ and $1 \le p \le k$. Also, from the discussion right below Theorem 2, we may let $0 < m_1 < m_2 < \cdots < m_k < 1$.

For p' with $p \le p' \le k$ and $j_p, j_{p+1}, \dots, j_{p'-1} \in \mathbb{N}$, we consider a nonincreasing rearrangement $(u_{j_p j_{p+1}\cdots j_{p'-1} j})_{j\in\mathbb{N}}$ of a Poisson point process of intensity measure $x^{-m_{p'}-1}\,dx$. All of these are independent of each other. For $\alpha = (j_p, j_{p+1}, \dots, j_k) \in \mathbb{N}^{k+1-p}$, we set

$$u^*_\alpha = u_{j_p} u_{j_p j_{p+1}} \cdots u_{j_p j_{p+1}\cdots j_k}$$

and

$$\nu_\alpha = \frac{u^*_\alpha}{\sum_\gamma u^*_\gamma}.$$

This family of random weights is called the Poisson–Dirichlet cascade associated with the sequence $0 < m_p < m_{p+1} < \cdots < m_k < 1$. For each p' with $p \le p' \le k$, let us consider a sequence of independent copies of $z_{p'}$,

$$(z_{p', j_p, j_{p+1}, \dots, j_{p'}})_{j_p, j_{p+1}, \dots, j_{p'} \in \mathbb{N}}.$$

These sequences are independent of each other and of $(u_{j_p j_{p+1}\cdots j_{p'-1} j})_{j\in\mathbb{N}}$ for $p \le p' \le k$ and $j_p, j_{p+1}, \dots, j_{p'-1} \in \mathbb{N}$. To simplify the notation, for $\alpha = (j_p, j_{p+1}, \dots, j_k) \in \mathbb{N}^{k+1-p}$, we write

$$z_{p',\alpha} = z_{p', j_p, j_{p+1}, \dots, j_{p'}}.$$

Then from Theorem 14.2.1 [17],

$$A_p(x) = E \log \sum_\alpha \nu_\alpha \operatorname{ch}(x + z_\alpha) + \frac{1}{2}\big(\xi'(1) - \xi'(q_{k+1})\big), \tag{3.3}$$

where

$$z_\alpha = \sum_{p \le p' \le k} z_{p',\alpha}.$$

Taking derivatives, we obtain

$$\begin{aligned} A''_p(x) &= 1 - E\left(\frac{\sum_\alpha \nu_\alpha \operatorname{sh}(x+z_\alpha)}{\sum_\alpha \nu_\alpha \operatorname{ch}(x+z_\alpha)}\right)^2 \\ &= E\left(\frac{\sum_\alpha \nu_\alpha (\operatorname{ch}(x+z_\alpha) - \operatorname{sh}(x+z_\alpha))}{\sum_\alpha \nu_\alpha \operatorname{ch}(x+z_\alpha)} \right.\\ &\qquad \left. \times \frac{\sum_\alpha \nu_\alpha (\operatorname{ch}(x+z_\alpha) + \operatorname{sh}(x+z_\alpha))}{\sum_\alpha \nu_\alpha \operatorname{ch}(x+z_\alpha)}\right) \\ &\le 2E\left(\frac{\sum_\alpha \nu_\alpha \exp(-z_\alpha)}{\sum_\alpha \nu_\alpha \operatorname{ch}(x+z_\alpha)}\right)\exp(-x) \\ &\le 4E\left(\frac{\sum_\alpha \nu_\alpha \exp(-z_\alpha)}{\sum_\alpha \nu_\alpha \exp(z_\alpha)}\right)\exp(-2x), \end{aligned}$$

where the first inequality holds since $\operatorname{ch} y - \operatorname{sh} y = \exp(-y)$ and $|\operatorname{sh} y| \le \operatorname{ch} y$, while the second inequality follows from $2\operatorname{ch} y \ge \exp y$. Let us now turn to the computation of this quantity

$$\gamma_p := E\left(\frac{\sum_\alpha \nu_\alpha \exp(-z_\alpha)}{\sum_\alpha \nu_\alpha \exp(z_\alpha)}\right).$$

Set $F(x_p, x_{p+1}, \dots, x_k) = \sum_{p \le p' \le k} x_{p'}$ for $(x_p, x_{p+1}, \dots, x_k) \in \mathbb{R}^{k+1-p}$. For $\alpha \in \mathbb{N}^{k+1-p}$, define the random variables

$$\begin{aligned} F(\alpha) &= F(z_{p,\alpha}, \dots, z_{k,\alpha}), \\ U(\alpha) &= \exp(-2F(\alpha)). \end{aligned}$$

Then we can write

$$\gamma_p = E\frac{\sum_\alpha \nu_\alpha U(\alpha) \exp F(\alpha)}{\sum_\alpha \nu_\alpha \exp F(\alpha)}. \tag{3.4}$$

Starting from

$$F_{k+1} = F(z_p, z_{p+1}, \dots, z_k),$$

we define by decreasing induction for $p \le p' \le k$,

$$F_{p'} = \frac{1}{m_{p'}} \log E_{p'} \exp m_{p'} F_{p'+1},$$

where $E_{p'}$ means the expectation with respect to the r.v.'s $z_{p'}, z_{p'+1}, \dots, z_k$. We also define for $p \le p' \le k$,

$$W_{p'} = \exp m_{p'}(F_{p'+1} - F_{p'}).$$

From formula (14.27) in [17], (3.4) can be computed as

$$\gamma_p = EW_pW_{p+1}\cdots W_k \exp(-2F_{k+1}). \tag{3.5}$$

Using the independence of $z_p, z_{p+1}, \dots, z_k$, it is easy to compute that

$$F_{p'} = \sum_{n=p}^{p'-1} z_n + \frac{1}{2}\sum_{n=p'}^{k} m_n\big(\xi'(q_{n+1}) - \xi'(q_n)\big).$$

Therefore, we obtain

$$W_{p'} = \exp\left(m_{p'}z'_p - \frac{m_{p'}^2}{2}\big(\xi'(q_{p'+1}) - \xi'(q_{p'})\big)\right)$$

and from (3.5), this implies

$$\begin{aligned}\gamma_p &= E\exp\left(\sum_{p'=p}^{k}(m_{p'}-2)z_{p'} - \frac{1}{2}\sum_{p'=p}^{k} m_{p'}^2\big(\xi'(q_{p'+1}) - \xi'(q_{p'})\big)\right)\\ &= \exp\left(\frac{1}{2}\sum_{p'=p}^{k}\big((m_{p'}-2)^2 - m_{p'}^2\big)\big(\xi'(q_{p'+1}) - \xi'(q_{p'})\big)\right)\\ &= \exp\left(2\sum_{p'=p}^{k}(1-m_{p'})\big(\xi'(q_{p'+1}) - \xi'(q_{p'})\big)\right)\\ &\le \exp\big(2\xi'(1)\big).\end{aligned}$$

Finally, we are done by letting $C = 4\exp(2\xi'(1))$. $\square$

As a consequence of Lemma 2, we have the following lemma.

LEMMA 3. *There exists a number M depending only on ξ and h such that for every $0 \le p \le k+2$,*

$$EA'_p\big(h+\chi'_1\big)A'_p\big(h+\chi'_2\big) \ge \frac{1}{M}, \tag{3.6}$$

$$EA''_p\big(h+\chi''_1\big)A''_p\big(h+\chi''_2\big) \ge \frac{1}{M}, \tag{3.7}$$

where $\chi'_1, \chi'_2, \chi''_1, \chi''_2$ are jointly Gaussian r.v.'s with the same variance $\xi'(q_p)$ and $E\chi'_1\chi'_2 = 0$ independent of h.

PROOF. The first inequality is Lemma 14.12.8 [17] and from there a similar argument yields the second inequality. $\square$

Recall that the external field h in this paper is always assumed to satisfy $Eh^2 > 0$. Based on this assumption, we set up the definition for the Parisi measure.

DEFINITION 1. Given $\varepsilon > 0$, we say that $k, \mathbf{m}, \mathbf{q}$ satisfy condition MIN(ε) if the following occurs. First, the sequences

$$\mathbf{m} = (m_0, m_1, \dots, m_k, m_{k+1}),$$
$$\mathbf{q} = (q_0, q_1, \dots, q_{k+1}, q_{k+2})$$

satisfy

$$m_0 = 0 < m_1 < \cdots < m_k < m_{k+1} = 1,$$
$$q_0 = 0 < q_1 < \cdots < q_{k+1} < q_{k+2} = 1.$$

In addition,

$$\mathcal{P}_k(\mathbf{m}, \mathbf{q}) \le \mathcal{P}(\xi, h) + \varepsilon$$

and

$$\mathcal{P}_k(\mathbf{m}, \mathbf{q}) \text{ realizes the minimum of } \mathcal{P}_k \text{ over all choices of } \mathbf{m} \text{ and } \mathbf{q}.$$

DEFINITION 2. Suppose that μ is a probability measure associated to $k, \mathbf{m}, \mathbf{q}$. Then we say that μ is ε-stationary for some $\varepsilon > 0$ if $k, \mathbf{m}, \mathbf{q}$ satisfy condition MIN(ε).

Let us note from Lemma 14.5.5 [17] that for any given $\varepsilon > 0$, we can find an ε-stationary measure μ associated to some $k, \mathbf{m}, \mathbf{q}$.

DEFINITION 3. We say that a probability measure μ is a Parisi measure (corresponding to the function ξ and external field h) if there exist a sequence (ε_n) with $\varepsilon_n \downarrow 0$ and a sequence of probability measures (μ_n) such that the following two conditions hold:

$$\mu_n \text{ is } \varepsilon_n\text{-stationary},$$
$$\mu \text{ is the limit of } (\mu_n).$$

Definition 1 is the same as Definition 14.5.3 [17], while our definition of the stationarity in Definition 2 is stronger than that in Definition 14.11.4 [17]. This is for technical purposes and it should be clear that under these assumptions, our future arguments are still valid.

LEMMA 4. *Suppose $h \neq 0$ and $k, \mathbf{q}, \mathbf{m}$ satisfy condition* MIN(ε). *Then for $1 \le p \le k+1$,*

$$EW_1 \cdots W_{p-1} A_p'(\zeta_p)^2 = q_p, \tag{3.8}$$

$$\xi''(q_p) EW_1 \cdots W_{p-1} A_p''(\zeta_p)^2 \le 1 + M\varepsilon^{1/6}, \tag{3.9}$$

where $\zeta_p = h + \sum_{0 \le n < p} z_n$ and

$$W_p = \exp m_p\bigl(A_{p+1}(\zeta_{p+1}) - A_p(\zeta_p)\bigr) = \exp m_p(X_{p+1} - X_p).$$

Here, M is a constant depending only on ξ and h.

PROOF. These results are (14.222) and (14.461) in [17]. □

At the end of this section we will find a manageable bound for $p_{N,u}$ via Guerra's bound. Recall that the right-hand side of (2.2) depends on (2.1). If we keep every parameter but λ fixed, then it is a quantity depending only on λ and, for clarity, we denote it by $\alpha(\lambda)$. For the same reason, we also think of Y_0 as a function of λ. Recall the r.v.'s $(y_p^j)_{0\le p\le\kappa, j=1,2}$ defined in Theorem 2. Suppose that $(y_p)_{0\le p\le\kappa}$ are independent Gaussian r.v.'s with $E(y_p)^2 = \xi'(\rho_{p+1}) - \xi'(\rho_p)$ for $0 \le p \le \kappa$. Starting with

$$D_{\kappa+1}(x) = \log \operatorname{ch} x,$$

we define D_p for $0 \le p \le \kappa$ by decreasing induction:

$$D_p(x) = \begin{cases} \dfrac{1}{n_p} \log E_p \exp n_p D_{p+1}(x + y_p), & \text{if } \tau \le p \le \kappa, \\ \dfrac{1}{(1+t)n_p} \log E_p \exp(1+t) n_p D_{p+1}(x + y_p), & \text{if } 0 \le p < \tau, \end{cases}$$

where E_p means the expectation with respect to y_n for $p \le n \le \kappa$. If $n_p = 0$ for some p, then we define $D_p(x) = E_p D_{p+1}(x + y_p)$. For $j = 1, 2$ and $1 \le p \le \kappa + 1$, set

$$\zeta_p^j = h + \sum_{0\le n<p} y_n^j.$$

PROPOSITION 6. *If $n_p = 0$ for every $0 \le p < \tau$, then*

$$Y_0(0) = E D_\tau(\zeta_\tau^1) + E D_\tau(\zeta_\tau^2), \tag{3.10}$$

$$Y_0'(0) = E D_\tau'(\zeta_\tau^1) D_\tau'(\zeta_\tau^2). \tag{3.11}$$

For the second derivative of Y_0, we have for every λ,

$$0 \le Y_0''(\lambda) \le 1. \tag{3.12}$$

PROOF. The proofs of (3.10) and (3.11) are essentially the same as that of part (b) of Proposition 14.6.4 [17]. Also, (3.12) and Lemma 14.6.5 [17] have the same proof. □

COROLLARY 1. *We have*

$$p_{N,u} \le \inf_\lambda \alpha(\lambda) \le \alpha(0) - \frac{1}{2}\alpha'(0)^2. \tag{3.13}$$

PROOF. This is an immediate consequence of (3.12). □

Let us remark here that (3.13) helps us in at least two ways: First, it reduces the difficulty of choosing parameters since we do not have to choose λ now. Second, this inequality gives us a reasonable way to choose parameters. Roughly speaking, in many cases, we choose parameters in such a way that the quantity $\alpha(0)$ is very close to $\mathcal{P}(\xi, h)$, while the term $\alpha'(0)^2/2$ is the error that we expect to obtain on the right-hand side of (1.8).

4. Proof of Proposition 3. This section is devoted to proving Proposition 3. Our approach is based on Talagrand's proof of the positivity of the overlap in Section 14.12 [17]. Suppose that $u = -v$ for $0 \le v \le 1$. Proposition 3 relies on the following two results:

PROPOSITION 7. *There exists $\delta > 0$ and $\varepsilon_0 > 0$ depending only on ξ and h with the following property. Whenever we can find $k, \mathbf{m}, \mathbf{q}$ that satisfy condition* MIN(ε_0) *and for an integer s with $1 \le s \le k+1$,*

$$m_{s-1} \le \delta \quad \textit{and} \quad q_s \ge v - \delta,$$

then we can find parameters in (2.2) *such that $p_{N,u} \le 2\mathcal{P}(\xi, h) - 1/M$, where M depends only on ξ and h.*

PROPOSITION 8. *Consider δ as in Proposition 7. Then we can find $\varepsilon_1 > 0$ with the following property. Whenever we can find $k, \mathbf{m}, \mathbf{q}$ such that $\mathcal{P}_k(\mathbf{m}, \mathbf{q}) \le \mathcal{P}(\xi, h) + \varepsilon_1$ and an integer s with $1 \le s \le k+1$,*

$$m_s \ge \delta \quad \textit{and} \quad q_s \le v - \delta,$$

then we can find parameters in (2.2) *such that $p_{N,u} \le 2\mathcal{P}(\xi, h) - 1/M$, where M depends only on ξ and h.*

PROOF OF PROPOSITION 3. Let $v \ge 0$. Consider δ, ε_0 as in Proposition 7 and ε_1 as in Proposition 8. Suppose that $k, \mathbf{m}, \mathbf{q}$ is a triplet satisfying MIN$(\min(\varepsilon_0, \varepsilon_1))$. Here, the existence of such $k, \mathbf{m}, \mathbf{q}$ is ensured by Lemma 14.5.5 [17]. Let $1 \le s \le k+1$ be the largest integer such that $m_{s-1} \le \delta$. If $q_s \ge v - \delta$, we apply Proposition 7. Otherwise we have $q_s \le v - \delta$. If $s = k+1$, then $m_s = m_{k+1} = 1 \ge \delta$. If $s < k+1$, then from the definition of s, $m_s \ge \delta$. In both cases, we conclude Proposition 3 by using Proposition 8 and we are done. □

Note that since the proof of Proposition 8 is essentially the same as that of Proposition 5, we defer it to Section 6. Now we turn to the proof of Proposition 7 and proceed with the following lemma:

LEMMA 5. *Suppose that $A:\mathbb{R}\to\mathbb{R}$ has uniformly bounded first and second derivatives. Consider two independent pairs of jointly Gaussian r.v.'s (χ_1,χ_2) and (χ_1',χ_2'), all of variance a, and a standard Gaussian r.v. χ. These r.v.'s are independent of h. Then we have*

$$\begin{aligned}&\left|EA'(h+\chi_1)A'(h+\chi_2)-EA'(h+\chi_1')A'(h+\chi_2')\right|\\&\quad\le\left|E\chi_1\chi_2-E\chi_1'\chi_2'\right|EA''(h+\chi\sqrt{a})^2.\end{aligned}\tag{4.1}$$

PROOF. This is a typical application of the Gaussian interpolation technique and the Cauchy–Schwarz inequality. For details, one may refer to Lemma 14.9.5 [17]. □

Suppose that k, $\mathbf{m}$, $\mathbf{q}$ is a triplet satisfying MIN(ε). Based on our discussion in Section 2, we may assume, without loss of generality, that $v=q_a$ for some a. The only thing we have to keep in mind is that when using (3.9), we will not be able to use the value $p=a$. From the assumption that $q_s\ge v-\delta$, we divide our discussion into two cases $v-\delta\le q_s\le v$ and $q_s>v$. First, let us proceed with the case that for an integer s with $1\le s\le k+1$,

$$m_{s-1}\le\delta\quad\text{and}\quad v-\delta\le q_s\le v.\tag{4.2}$$

Note that $s\le a$. We consider the following numbers:

$$\begin{aligned}&\tau=1,\\&\kappa=k+2-a,\\&n_0=0,\qquad n_1=m_a,\qquad n_2=m_{a+1},\dots,\qquad n_\kappa=m_{k+1}=1,\\&\rho_0=0,\qquad \rho_1=v=q_a,\qquad \rho_2=q_{a+1},\dots,\qquad \rho_{\kappa+1}=q_{k+2}=1\end{aligned}\tag{4.3}$$

and apply (4.3) to Theorem 2. Recall that we use α to denote the right-hand side of (2.2).

LEMMA 6. *Assuming (4.2) and (4.3), we have*

$$\alpha(0)\le 2\mathcal{P}_k(\mathbf{m},\mathbf{q})+M\delta.\tag{4.4}$$

PROOF. The proof is essentially the same as that of Lemma 14.12.7 in [17]. □

In view of (3.13) and (4.4), our goal is then to bound $\alpha'(0)$ from below. Proposition 6 implies that $D_1(x)=A_a(x)$ and so

$$\alpha'(0)=EA_a'(h+\chi_1)A_a'(h+\chi_2)+v,\tag{4.5}$$

where χ_1 and χ_2 are Gaussian with $E(\chi_1)^2 = E(\chi_2)^2 = \xi'(v)$ and $E\chi_1\chi_2 = -t\xi'(v)$ independent of h. Consider two independent Gaussian r.v.'s χ_1' and χ_2' with $E(\chi_1')^2 = E(\chi_2')^2 = \xi'(v)$ independent of h. By using (4.1),

$$\begin{aligned} &EA_a'(h+\chi_1)A_a'(h+\chi_2) \\ &\quad \geq EA_a'(h+\chi_1')A_a'(h+\chi_2') - t\xi'(v)EA_a''\big(h+\chi\sqrt{\xi'(v)}\big)^2, \end{aligned}$$

where χ is standard Gaussian independent of h. Since $\xi'(v) \leq v\xi''(v)$, it follows that from (4.5),

$$\begin{aligned} \alpha'(0) &\geq EA_a'(h+\chi_1')A_a'(h+\chi_2') + v\big(1 - t\xi''(v)EA_a''\big(h+\chi\sqrt{\xi(v)}\big)^2\big) \\ &= EA_a'(h+\chi_1')A_a'(h+\chi_2') + v(1-t) \\ &\quad + tv\big(1 - \xi''(v)EA_a''\big(h+\chi\sqrt{\xi(v)}\big)^2\big). \end{aligned} \tag{4.6}$$

To use (4.6), we have to bound the quantity

$$\xi''(v)EA_a''\big(h+\chi\sqrt{\xi'(v)}\big)$$

from above. The starting point of the proof is that from (3.9),

$$\xi''(q_s)EW_1\cdots W_{s-1}A_s''(\zeta_s)^2 \leq 1 + M\varepsilon^{1/6}, \tag{4.7}$$

where $\zeta_p = h + \sum_{0\leq n<p} z_n$ and $W_p = \exp m_p(A_{p+1}(\zeta_{p+1}) - A_p(\zeta_p))$.

LEMMA 7. *Assuming* (4.2), *there exists* $\delta_0 > 0$ *depending only on* ξ *and* h *such that when* $\delta \leq \delta_0$, *we have*

$$\xi''(v)EA_a''\big(h+\chi\sqrt{\xi'(v)}\big)^2 \leq \xi''(q_s)EW_1\cdots W_{s-1}A_s''(\zeta_s)^2 + M\sqrt{\delta}. \tag{4.8}$$

PROOF. This is Lemma 14.12.9 in [17]. □

As a conclusion, by assuming (4.2) and using (4.3), we see that (3.6), (4.6), (4.7) and (4.8) together imply

$$\alpha'(0) \geq \frac{1}{M} - M\varepsilon^{1/6} - M\sqrt{\delta} \tag{4.9}$$

for $\delta \leq \delta_0$.

Next, let us consider the other case that for some $1 \leq s \leq k+1$,

$$m_{s-1} \leq \delta \quad \text{and} \quad q_s > v = q_a. \tag{4.10}$$

Since $q_{a+1} \geq q_a \geq v - \delta$ and $m_a \leq m_{s-1} \leq \delta$, we may assume, without loss of generality, that $s = a+1$. Consider the following numbers:

$$\begin{aligned} &\tau = 1, \\ &\kappa = k + 2 - a, \\ &n_0 = 0, \qquad n_1 = 0, \qquad n_2 = m_{a+1}, \ldots, \qquad n_\kappa = m_{k+1} = 1, \\ &\rho_0 = 0, \qquad \rho_1 = v = q_a, \qquad \rho_2 = q_{a+1}, \ldots, \qquad \rho_{\kappa+1} = q_{k+2} = 1 \end{aligned} \tag{4.11}$$

and apply (4.11) to (2.2).

LEMMA 8. *Assuming* (4.10) *and* (4.11), *we have*

$$\alpha(0) \le 2\mathcal{P}_k(\mathbf{m}, \mathbf{q}) + M\delta. \tag{4.12}$$

PROOF. A similar proof as Lemma 6 yields the announced statement. □

Again, our goal is to bound $\alpha'(0)$ from below. From (3.11), we have $D_2(x) = A_{a+1}(x)$ and then

$$\alpha'(0) = EA'_{a+1}(h+\chi_1)A'_{a+1}(h+\chi_2) + v, \tag{4.13}$$

where χ_1 and χ_2 are jointly Gaussian with $E(\chi_1)^2 = E(\chi_2)^2 = \xi'(q_{a+1})$ and $E\chi_1\chi_2 = -t\xi'(v)$ independent of h. Let χ_1' and χ_2' be two independent Gaussian r.v.'s with $E(\chi_1')^2 = E(\chi_2')^2 = \xi'(q_{a+1})$ independent of h. Using (4.1), we obtain

$$\begin{aligned} &EA'_{a+1}(h+\chi_1)A'_{a+1}(h+\chi_2) \\ &\quad \ge EA'_{a+1}(h+\chi_1')A'_{a+1}(h+\chi_2') \\ &\qquad - t\xi'(v)EA''_{a+1}\bigl(h+\chi\sqrt{\xi'(q_{a+1})}\bigr)^2, \end{aligned} \tag{4.14}$$

where χ is standard Gaussian independent of h. Let us apply $p = a+1$ to (3.9) and use the fact $q_{a+1} \ge v$. Then we have

$$\begin{aligned} \xi''(v)EW_1\cdots W_aA''_{a+1}(\zeta_{a+1})^2 &\le \xi''(q_{a+1})EW_1\cdots W_aA''_{a+1}(\zeta_{a+1})^2 \\ &\le 1 + M\varepsilon^{1/6}. \end{aligned} \tag{4.15}$$

LEMMA 9. *Assuming* (4.10), *we have*

$$E|W_1\cdots W_{s-1} - 1| \le M\delta. \tag{4.16}$$

PROOF. One can find the proof from Lemma 14.12.9 [17]. □

Using (4.16) and $EA''_{a+1}(\zeta_{a+1})^2 = EA''_{a+1}(h+\chi\sqrt{\xi'(q_{a+1})})^2$, it follows that from (4.15),

$$\xi''(v)EA''_{a+1}\bigl(h+\chi\sqrt{\xi'(q_{a+1})}\bigr)^2 \le 1 + M\delta + M\varepsilon^{1/6} \tag{4.17}$$

and from (3.6), (4.13), (4.14), (4.17) and $\xi'(v) \le v\xi''(v)$, we then have

$$\begin{aligned} \alpha'(0) &\ge EA'_{a+1}(h+\chi_1')A'_{a+1}(h+\chi_2') + v \\ &\quad - t\xi'(v)EA''_{a+1}\bigl(h+\chi\sqrt{\xi'(q_{a+1})}\bigr)^2 \\ &\ge \frac{1}{M} + v(1-t) + tv\bigl(1 - \xi''(v)EA''_{a+1}\bigl(h+\chi\sqrt{\xi'(q_{a+1})}\bigr)^2\bigr) \\ &\ge \frac{1}{M} - M\delta - M\varepsilon^{1/6}. \end{aligned} \tag{4.18}$$

PROOF OF PROPOSITION 7. First we complete the proof for the case (4.2). Let M_1 be the constant obtained from (4.4) and (4.9) and assume, without loss of generality, that $M_1 \geq 1$ and $1/16M_1^4 \leq \delta_0$. Set $\delta = 1/16M_1^4$. If $\varepsilon \leq \varepsilon_1 = (1/4M_1^2)^6$, (4.9) implies

$$\alpha'(0) \geq \frac{1}{M_1} - \frac{M_1}{4M_1^2} - \frac{M_1}{4M_1^2} = \frac{1}{2M_1}$$

and combining this with (4.4) yields

$$\inf_{\lambda} \alpha(\lambda) \leq 2\mathcal{P}_k(\mathbf{m}, \mathbf{q}) + M_1\delta - \frac{1}{8M_1^2} \leq 2\mathcal{P}(\xi, h) + 2\varepsilon_0 - \frac{1}{16M_1^2}.$$

Letting ε_0 be sufficiently small completes our proof of this case. For the second case (4.10), using (4.18) and Lemma 8, we may argue similarly to obtain the announced result. □

5. Proofs of Propositions 1 and 4. Given $0 \leq v < 1$, recall the definition of φ_v from (1.6). In this section we first study how the Guerra bound relates to φ_v and then study some of its basic properties to conclude Propositions 1 and 4.

Let $k, \mathbf{m}, \mathbf{q}$ be given by (1.1). Suppose that μ is the probability measure associated to $k, \mathbf{m}, \mathbf{q}$ and Φ is the corresponding solution of (1.5). Recall the definition of $(A_p)_{0 \leq p \leq k+2}$ from (3.1). Then Φ and $(A_p)_{0 \leq p \leq k+2}$ can be related in the following way. Let $(g_p)_{0 \leq p \leq k+1}$ be i.i.d. standard Gaussian r.v.'s. For $q \in [0, 1]$, we have that $\Phi(x, 1) = A_{k+2}(x)$ if $q = 1$ and

$$\Phi(x, q) = \frac{1}{m_p} \log E \exp m_p A_{p+1}\big(x + g_p\sqrt{\xi'(q_{p+1}) - \xi'(q)}\big),$$

if $q_p \leq q < q_{p+1}$ for some $0 \leq p \leq k+1$. In particular, for $0 \leq p \leq k+2$,

$$\Phi(x, q_p) = A_p(x). \tag{5.1}$$

For fixed u and v with $0 \leq u \leq v < 1$, we suppose $q_a \leq v < q_{a+1}$ for some $0 \leq a \leq k+1$ and consider numbers

$$\begin{aligned} &\tau = 1, \\ &\kappa = k + 3 - a, \\ &n_0 = 0, \qquad n_1 = 0, \qquad n_2 = m_a, \\ &n_3 = m_{a+1}, \ldots, \qquad n_\kappa = m_{k+1} = 1, \\ &\rho_0 = 0, \qquad \rho_1 = u, \qquad \rho_2 = v, \\ &\rho_3 = q_{a+1}, \ldots, \qquad \rho_{\kappa+1} = q_{k+2} = 1. \end{aligned} \tag{5.2}$$

Let us apply (5.2) to (2.1) and recall that we use $\alpha(\lambda)$ to denote the right-hand side of (2.2). Recall that c is the smallest value of the support of the Parisi measure. Since $Eh^2 \neq 0$, the positivity of the overlap implies $c > 0$.

LEMMA 10. *For* $0 < \delta < c$, *we have*

$$\alpha(0) \leq 2\mathcal{P}_k(\mathbf{m}, \mathbf{q}) + \mu([0, c-\delta])\theta(1) + (\theta(v) - \theta(c-\delta))_+. \tag{5.3}$$

The derivative of α *at* 0 *can be computed as*

$$\alpha'(0) = E\frac{\partial \Phi}{\partial x}(h + \chi_1, v)\frac{\partial \Phi}{\partial x}(h + \chi_2, v) - u, \tag{5.4}$$

where χ_1 *and* χ_2 *are two Gaussian r.v.'s with* $E(\chi_1)^2 = E(\chi_2)^2 = \xi'(v)$ *and* $E\chi_1\chi_2 = t\xi'(u)$ *independent of* h.

PROOF. Without loss of generality, we may assume that $v = q_a$ and $u = q_b$ with $0 \leq b \leq a$. Let us write

$$\begin{aligned}\sum_{1 \leq p \leq \kappa} n_p(\theta(\rho_{p+1}) - \theta(\rho_p)) &= \sum_{a \leq p \leq k+1} m_p(\theta(q_{p+1}) - \theta(q_p)) \\ &= \sum_{1 \leq p \leq k+1} m_p(\theta(q_{p+1}) - \theta(q_p)) - C,\end{aligned} \tag{5.5}$$

where

$$C = \sum_{1 \leq p \leq a-1} m_p(\theta(q_{p+1}) - \theta(q_p)).$$

If $q_a \leq c - \delta$, then

$$\begin{aligned}C &\leq \max\{m_p : q_p \leq c - \delta\} \sum_{0 \leq p \leq a-1} (\theta(q_{p+1}) - \theta(q_p)) \\ &\leq \mu([0, c-\delta])\theta(1);\end{aligned}$$

if $q_a > c - \delta$, then

$$\begin{aligned}C &\leq \max\{m_p : q_p \leq c - \delta\} \sum_{0 \leq p \leq a-1} (\theta(q_{p+1}) - \theta(q_p)) \\ &\quad + \sum_{0 \leq p \leq a-1 : q_p > c-\delta} \theta(q_{p+1}) - \theta(q_p) \\ &\leq \mu([0, c-\delta])\theta(1) + \theta(v) - \theta(c-\delta).\end{aligned}$$

So (5.3) holds. From (3.10), we have $D_2(x) = A_a(x)$ and, consequently, $Y_0 = 2EA_a(h + \chi)$, where χ is Gaussian with $E\chi^2 = \xi'(q_a)$. Since χ has the same distribution as $\sum_{0 \leq p < a} z_p$, from Jensen's inequality, $A_p(x) \geq EA_{p+1}(x + z_p)$ and iterating this inequality implies

$$EA_a\Big(h + \sum_{0 \leq p < a} z_p\Big) \leq EA_0(h).$$

So $Y_0 \le 2EA_0(h) = 2X_0$ and this together with (5.5) yields (5.3). Next, using (3.11) and (5.1), we obtain

$$\begin{aligned} Y_0'(0) &= EA_a'(h+\chi_1)A_a'(h+\chi_2) \\ &= E\frac{\partial\Phi}{\partial x}(h+\chi_1,q_a)\frac{\partial\Phi}{\partial x}(h+\chi_2,q_a), \end{aligned}$$

where χ_1 and χ_2 are jointly Gaussian with $E(\chi_1)^2 = E(\chi_2)^2 = \xi'(q_a)$ and $E\chi_1\chi_2 = t\xi'(q_b)$ independent of h. This completes our proof. $\square$

Now, suppose that μ is a Parisi measure and c is the smallest value of its support. By Definition 3, μ is the limit of a sequence of ε_n-stationary measures (μ_n) such that $\mathcal{P}(\xi,h,\mu_n)\to\mathcal{P}(\xi,h)$. By Definition 2, for each μ_n, there exist k, $\mathbf{m}$, $\mathbf{q}$ satisfying MIN(ε_n). Here, to clarify notation, we keep the dependence of k, $\mathbf{m}$, $\mathbf{q}$, and ε_n on n implicit. For u and v satisfying $0\le u\le v<1$, we consider numbers (5.2) associated to u, v and μ_n, and we use α_n to denote the right-hand side of (2.2). Suppose that Φ_n is the solution of (1.5) associated to μ_n. Recall that we define Φ as the uniform limit of (Φ_n). An argument similar to the proof of Theorem 3.2 [14] implies that in the sense of uniform convergence,

$$\frac{\partial^i\Phi}{\partial x^i} = \lim_{n\to\infty}\frac{\partial^i\Phi_n}{\partial x^i}$$

on $\mathbb{R}\times[0,1]$ for $i=1,2,3$.

PROPOSITION 9. *For any u and v satisfying $0\le u\le v<1$, we have*

$$\limsup_{n\to\infty}\alpha_n(0) \le 2\mathcal{P}(\xi,h) + \big(\theta(v)-\theta(c)\big)_+ \tag{5.6}$$

and

$$\lim_{n\to\infty}\alpha_n'(0) = E\frac{\partial\Phi}{\partial x}(h+\chi_1,v)\frac{\partial\Phi}{\partial x}(h+\chi_2,v) - u, \tag{5.7}$$

where χ_1 and χ_2 are jointly Gaussian with $E(\chi_1)^2 = E(\chi_2)^2 = \xi'(v)$ and $E\chi_1\chi_2 = t\xi'(u)$ independent of h.

PROOF. Using (5.3), we have for $0<\delta<c$,

$$\begin{aligned} &\limsup_{n\to\infty}\alpha_n(0) \\ &\quad\le 2\mathcal{P}(\xi,h) + \limsup_{n\to\infty}\mu_n\big([0,c-\delta]\big)\theta(1) + \big(\theta(v)-\theta(c-\delta)\big)_+ \\ &\quad= 2\mathcal{P}(\xi,h) + \big(\theta(v)-\theta(c-\delta)\big)_+ \end{aligned}$$

and this implies (5.6) by letting δ tend to zero. For (5.7), we use (5.4). $\square$

Let us now turn to the study of some basic properties of φ_c. Recall from (1.6) and (5.7), for fixed $0 < v < 1$, φ_v is defined by

$$\varphi_v(u,t) = \lim_{n\to\infty} \alpha_n'(0)$$

for $0 \le u \le v$ and $0 \le t \le 1$, where χ_1 and χ_2 are jointly Gaussian r.v.'s with $E(\chi_1)^2 = E(\chi_2)^2 = \xi'(v)$ and $E\chi_1\chi_2 = t\xi'(u)$ independent of h. For given $k, \mathbf{m}, \mathbf{q}$, let us recall the definition of $(A_p)_{0\le p\le k+2}$ from (3.1). We also recall the definitions of $(W_p)_{1\le p\le k+1}$ and $(\zeta_p)_{1\le p\le k+1}$ from Lemma 4. Let us proceed with the following lemmas.

LEMMA 11. *Let $\varepsilon > 0$ and $0 < \delta < c$. Suppose that l and l' are fixed integers with $1 \le l < l' \le k+1$. If $m_p \le \varepsilon$ for every $1 \le p \le l-1$, then*

$$E|W_1 W_2 \cdots W_{l-1} - 1| \le M\varepsilon. \tag{5.8}$$

If $c - \delta \le q_p \le q_{l'}$ for every $l \le p \le l'$, then

$$EW_1W_2\cdots W_{l-1}|W_l W_{l+1}\cdots W_{l'-1} - 1| \le M\sqrt{q_{l'} - c + \delta}. \tag{5.9}$$

Here, M depends only on ξ and h.

PROOF. Similar arguments as (14.468) and (14.469) in [17] will yield the announced results immediately. □

LEMMA 12. *We have*

$$E\left(\frac{\partial \Phi}{\partial x}(h+\chi, c)\right)^2 = c, \tag{5.10}$$

$$\xi''(c)E\left(\frac{\partial^2 \Phi}{\partial x^2}(h+\chi, c)\right)^2 \le 1, \tag{5.11}$$

where χ denotes a Gaussian r.v. with $E\chi^2 = \xi'(c)$.

PROOF. Recall that each μ_n corresponds to $k, \mathbf{m}, \mathbf{q}$ and ε. Since $0 < c < 1$, for each n there exists some $0 \le s \le k+1$ such that $q_s \le c < q_{s+1}$. Let us first claim that

$$\lim_{n\to\infty} E|W_1\cdots W_{s-1} - 1| = 0 \tag{5.12}$$

and if $\lim_{n\to\infty} q_{s+1} = c$, then we further have

$$\lim_{n\to\infty} E|W_1\cdots W_s - 1| = 0. \tag{5.13}$$

Let $0 < \delta < c$ be fixed. Suppose that $1 \le l \le s+1$ is the largest integer such that $q_{l-1} \le c - \delta$. Since $\lim_{n\to\infty}\mu_n([0, c-\delta]) = 0$, we have that for large n, $m_p \le \varepsilon$ for every $0 \le p \le l-1$. Using (5.8),

$$E|W_1W_2\cdots W_{l-1} - 1| \le M\varepsilon. \tag{5.14}$$

On the other hand, since $c-\delta \le q_p \le c < q_{s+1}$ for $l \le p \le s$, using (5.9), we also get

$$EW_1W_2\cdots W_{l-1}|W_lW_{l+1}\cdots W_{s-1}-1| \le M\sqrt{q_s-c+\delta} \le M\sqrt{\delta} \tag{5.15}$$

and

$$EW_1W_2\cdots W_{l-1}|W_lW_{l+1}\cdots W_s-1| \le M\sqrt{q_{s+1}-c+\delta}. \tag{5.16}$$

Using the triangle inequality, (5.14) and (5.15), it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} E|W_1W_2\cdots W_{s-1}-1| \\
&\quad \le \limsup_{n\to\infty} EW_1W_2\cdots W_{l-1}|W_lW_{l+1}\cdots W_{s-1}-1| \\
&\qquad + \limsup_{n\to\infty} E|W_1W_2\cdots W_{l-1}-1| \\
&\quad \le \lim_{n\to\infty} M\sqrt{\delta}+M\varepsilon \\
&\quad = M\sqrt{\delta}.
\end{aligned}$$

Similarly, if $\lim_{n\to\infty} q_{s+1}=c$, using the triangle inequality, (5.14) and (5.16), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} E|W_1W_2\cdots W_s-1| \\
&\quad \le \limsup_{n\to\infty} EW_1W_2\cdots W_{l-1}E_l|W_lW_{l+1}\cdots W_s-1| \\
&\qquad + \limsup_{n\to\infty} E|W_1W_2\cdots W_{l-1}-1| \\
&\quad \le \lim_{n\to\infty} M\sqrt{q_{s+1}-c+\delta}+M\varepsilon \\
&\quad = \lim_{n\to\infty} M\sqrt{\delta}+M\varepsilon \\
&\quad = M\sqrt{\delta}.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, our claim follows.

Now, let us assume, without loss of generality, that the following limits exist:

$$\lim_{n\to\infty} q_s, \qquad \lim_{n\to\infty} q_{s+1}, \qquad \lim_{n\to\infty} m_s$$

and denote them by c_-, c_+ and m_c, respectively. If $c_- < c < c_+$, then the first inequality implies $m_c = 0$, which leads to a contradiction since the second inequality implies $m_c > 0$. Thus, we may assume

$$\text{either } c_- = c \quad \text{or} \quad c_+ = c. \tag{5.17}$$

Note that from the stationarity of μ_n, $q_p = 0$ if and only if $p = 0$, and also $q_p = 1$ if and only if $p = k+2$. If $q_s = 0$ for all but finitely many n, then $s+1 = 1 \le k+1$

for large n and so $c_+ = c$. If $q_{s+1} = 1$ for all but finitely many n, then $s = k + 1$ for large n and so $c_- = c$. Finally, if $0 < q_s$ and $q_{s+1} < 1$ for infinitely many n, then these s satisfy $1 \le s \le k$ and (5.17). Hence, in the following argument, we assume further that one of the following cases holds:

(i) $1 \le s \le k + 1$ for all n and $c_- = c$.
(ii) $1 \le s + 1 \le k + 1$ for all n and $c_+ = c$.
(iii) $1 \le s \le k$ for all n and (5.17) holds.

If (i) holds, then from (3.8), (3.9) and (5.12), we have

$$\begin{aligned} E\left(\frac{\partial \Phi}{\partial x}(h + \chi, c)\right)^2 &= \lim_{n\to\infty} E A_s'(h + \chi_s)^2 \\ &= \lim_{n\to\infty} E W_1 \cdots W_{s-1} A_s'(h + \chi_s)^2 \\ &= \lim_{n\to\infty} q_s \\ &= c \end{aligned}$$

and

$$\begin{aligned} \xi''(c) E\left(\frac{\partial^2 \Phi}{\partial x^2}(h + \chi, c)\right)^2 &= \lim_{n\to\infty} \xi''(q_s) E A_s''(h + \chi_s)^2 \\ &\le \limsup_{n\to\infty} \xi''(q_s) E W_1 \cdots W_{s-1} A_s''(h + \chi_s)^2 \\ &\le 1, \end{aligned}$$

where χ_s is Gaussian with $E(\chi_s)^2 = \xi'(q_s)$. If (ii) holds, again from (3.8) and (3.9), we have

$$E W_1 \cdots W_s A_{s+1}'(h + \chi_{s+1})^2 = q_{s+1},$$
$$\xi''(q_{s+1}) E W_1 \cdots W_s A_{s+1}''(h + \chi_{s+1})^2 \le 1 + M\varepsilon^{1/6}.$$

Using (5.13) and proceeding as in (i), we obtain the announced results, where χ_{s+1} is Gaussian with $E(\chi_{s+1})^2 = \xi'(q_{s+1})$. Finally, for the case (iii), the same argument completes our proof. □

PROPOSITION 10. *For each* $0 \le t \le 1$, $\varphi_v(\cdot, t)$ *is a convex function on* $[0, v]$. *For* $0 \le u \le c$ *and* $0 \le t \le 1$,

$$\frac{\partial \varphi_c}{\partial u} \le 0; \tag{5.18}$$

$$\frac{\partial \varphi_c}{\partial t} \ge \frac{\xi'}{M}, \tag{5.19}$$

where M *is a constant depending only on* ξ *and* h.

PROOF. Define for each n,

$$\varphi_{n,v}(u,t)=\alpha_n'(0)=E\frac{\partial\Phi_n}{\partial x}(h+\chi_1,v)\,\frac{\partial\Phi_n}{\partial x}(h+\chi_2,v)-u$$

for $0\le u\le v$ and $0\le t\le 1$, where χ_1 and χ_2 are jointly Gaussian with $E(\chi_1)^2=E(\chi_2)^2=\xi'(v)$ and $E\chi_1\chi_2=t\xi'(u)$ independent of h. Again, without loss of generality, we may assume that $v=q_a$ for some $1\le a\le k+1$. Let $g,g_0^1,g_0^2,g_1^1,g_1^2$ be i.i.d. Gaussian r.v.'s with variance $\xi'(q_a)$ such that for $i=1,2$,

$$\chi_i=(g\sqrt{t}+g_0^i\sqrt{1-t})\sqrt{\frac{\xi'(u)}{\xi'(q_a)}}+g_1^i\sqrt{1-\frac{\xi'(u)}{\xi'(q_a)}}.$$

Then $\varphi_{n,v}(u,t)$ can be written as

$$\varphi_{n,v}(u,t)=\phi_n\biggl(\frac{\xi'(u)}{\xi'(q_a)},t\biggr)-u,$$

where

$$\phi_n(w,t)=EA_a'\bigl(V_1(w,t)\bigr)A_a'\bigl(V_2(w,t)\bigr),$$

and for $i=1,2$,

$$V_i(w,t)=h+\bigl(g\sqrt{t}+g_0^i\sqrt{1-t}\bigr)\sqrt{w}+g_1^i\sqrt{1-w}.$$

So for $0\le u\le v$ and $0\le t\le 1$, by using Gaussian integration by parts,

$$\frac{\partial\varphi_{n,v}}{\partial u}(u,t)=t\xi''(u)\Gamma_1(u,t)-1, \tag{5.20}$$

$$\frac{\partial^2\varphi_{n,v}}{\partial u^2}(u,t)=t\xi^{(3)}(u)\Gamma_1(u,t)+t^2\xi''(u)^2\Gamma_2(u,t), \tag{5.21}$$

$$\frac{\partial\varphi_{n,v}}{\partial t}(u,t)=\xi'(u)\Gamma_1(u,t), \tag{5.22}$$

$$\frac{\partial^2\varphi_{n,v}}{\partial u\,\partial t}(u,t)=t\xi'(u)\xi''(u)\Gamma_2(u,t), \tag{5.23}$$

where

$$\Gamma_1(u,t)=EA_a''(h+\chi_1)A_a''(h+\chi_2),$$
$$\Gamma_2(u,t)=EA_a^{(3)}(h+\chi_1)A_a^{(3)}(h+\chi_2).$$

Since $A_a''>0$, we have $\Gamma_1>0$. Let us also observe that

$$E\bigl(A_a^{(3)}\bigl(V_1(w,t)\bigr)|g,h\bigr)=E\bigl(A_a^{(3)}\bigl(V_2(w,t)\bigr)|g,h\bigr),$$

which implies $\Gamma_2 > 0$. Thus, using these and from (5.21) and (5.23), we obtain

$$\frac{\partial^2 \varphi_v}{\partial u^2} = \lim_{n\to 0} \frac{\partial^2 \varphi_{n,v}}{\partial u^2} \ge 0, \tag{5.24}$$

$$\frac{\partial^2 \varphi_v}{\partial u\, \partial t} = \lim_{n\to \infty} \frac{\partial^2 \varphi_{n,v}}{\partial u\, \partial t} \ge 0. \tag{5.25}$$

Thus, the convexity of $\varphi_v(\cdot, t)$ follows from (5.24). By (5.11) and (5.20), we know $\frac{\partial \varphi_c}{\partial u}(c, 1) \le 0$ and from (5.25), this implies $\frac{\partial \varphi_c}{\partial u}(c, t) \le 0$. So we obtain (5.18) by using (5.24). Finally, (5.19) can be easily obtained from (3.7) and (5.22). □

PROOF OF PROPOSITION 1. Let $0 \le t < 1$. Notice that if $u = 0$, then χ_1 and χ_2 are independent and from (3.6), it implies $\varphi_c(0, t) > 0$. Since $\varphi_c(c, 1) = 0$ by (5.10) and $\frac{\partial \varphi_c}{\partial t}(c, t) \ge \xi'(c)/M > 0$ from (5.19), we conclude that $\varphi_c(c, t) < 0$ and so $\varphi_c(\cdot, t)$ has a solution in $[0, c]$. Suppose that u_1, u_2 with $0 < u_1 < u_2 < c$ are two solutions of $\varphi_c(\cdot, t) = 0$ in $[0, c]$. From Rolle's theorem, there exists some u_3 with $u_1 < u_3 < u_2$ such that $\frac{\partial \varphi_c}{\partial u}(u_3, t) = 0$. Using the convexity of $\varphi_c(\cdot, t)$, it implies $\frac{\partial \varphi_c}{\partial u}(u, t) \ge 0$ for all $u_3 \le u \le c$ and so $\varphi_c(c, t) \ge \varphi_c(u_2, t) = 0$, which contradicts to $\varphi_c(c, t) < 0$. □

PROOF OF PROPOSITION 4. Combining (1.6), (3.13), (5.6) and (5.7), we get that for u, v, t with $0 \le u \le v < 1$ and $0 \le t \le 1$,

$$p_{N,u} \le 2\mathcal{P}(\xi, h) - \tfrac{1}{2}\varphi_v(u, t)^2 + \big(\theta(v) - \theta(c)\big)_+. \tag{5.26}$$

Applying $v = c$ to this inequality, we obtain (2.5). Suppose that $0 \le t < 1$ is fixed. It is easy to see that $(u, v) \mapsto \varphi_v(u, t)$ is continuous on $0 \le u \le v < 1$. Since $\varphi_c(c, t) < 0$, there exists some $\gamma > 0$ such that $\varphi_v(u, t) \le \varphi_c(c, t)/2$ whenever $c \le u \le v \le c + \gamma$. By the continuity of θ, we may also let γ be small enough such that $\theta(v) - \theta(c) < \varphi_c(c, t)^2/16$ whenever $c \le v \le c + \gamma$. Therefore, we obtain (2.6) from (5.26). □

6. Proof of Proposition 5. In this section our main goal is to establish an iterative inequality that is used in the proofs of Propositions 5 and 8. Let us start by stating our main result as follows. Suppose that y_1 and y_2 are jointly Gaussian r.v.'s with $E(y_1)^2 = E(y_2)^2 = 1$ and $Ey_1y_2 = t \ge 0$ independent of h. Define

$$F_1(x_1, x_2, w) = E\big(\mathrm{th}(x_1 + y_1\sqrt{w}) - \mathrm{th}(x_2 + y_2\sqrt{w})\big)^2,$$

$$F_{-1}(x_1, x_2, w) = E\big(\mathrm{th}(x_1 + y_1\sqrt{w}) + \mathrm{th}(x_2 - y_2\sqrt{w})\big)^2$$

for $x_1, x_2 \in \mathbb{R}$ and $w \ge 0$. For convenience, we sometimes simply denote F_1 by F. Recall the constant C stated in Lemma 2. Set $C_0 = t(2(1+t)C^2)^{-1}$. For $0 < |u| \le 1$, let $\eta \in \{-1, +1\}$ satisfy $u = \eta|u|$. Then the following inequality holds.

PROPOSITION 11. *There exists a constant K_1 depending only on C and ξ such that the following statement holds. Suppose that $0 < c_1 < c_2 < 1$ and*

$$0 < \xi'(c_2) - \xi'(c_1) < \min\left(\frac{1}{8}, \frac{1}{2(2C_0\xi'(1) + K_1)}\right); \tag{6.1}$$

and $k, \mathbf{m}, \mathbf{q}$ are such that for some $1 \leq s \leq k+1$,

$$q_s \leq c_1 \quad \textit{and} \quad m_s \geq \delta. \tag{6.2}$$

Then we have

$$p_{N,u} \leq 2\mathcal{P}_k(\mathbf{m}, \mathbf{q}) - C_0\delta K_2 \int_{c_1}^{c_2} EF_\eta\big(h, h, \xi'(q)\big)\xi''(q)\,dq \tag{6.3}$$

for every u with $c_2 \leq |u| \leq 1$, where K_2 is a constant depending only on ξ.

As consequences of Proposition 11, Propositions 5 and 8 now follow.

PROOF OF PROPOSITION 5. Set $c_2 = c'$. Let us choose $c_1 \in (c, c')$ such that (6.1) holds and μ is continuous at c_1. Since c is the minimum of the support of μ, $\mu([0, c_1]) > 0$. From the definition of μ, there exists a sequence of ε_n-stationary measures (μ_n) such that $\mu_n \to \mu$ weakly and $\mathcal{P}(\xi, h, \mu_n) \to \mathcal{P}(\xi, h)$. For each n, μ_n corresponds to some $k, \mathbf{m}, \mathbf{q}$. We assume that c_1 is in the list of $\mathbf{q}$ and $c_1 = q_s$ for some $1 \leq s \leq k+1$. Then for large n,

$$\mu_n\big([0, q_s]\big) = m_s \geq \delta,$$

where $\delta = \mu([0, c_1])/2$. We then apply Proposition 11 to obtain for every $c' \leq u \leq 1$,

$$p_{N,u} \leq 2\mathcal{P}_k(\mathbf{m}, \mathbf{q}) - \varepsilon^*,$$

where $\varepsilon^* = C_0\delta K_2 \int_{c_1}^{c_2} EF_1(h, h, \xi'(q))\xi''(q)\,dq$. Since $0 < t < 1$, we have that $\varepsilon^* > 0$. Letting n tend to infinity completes our proof. □

PROOF OF PROPOSITION 8. Note that from the given condition, we have $v \geq \delta$. Let $c_1 = v - \delta$ and $c_2 = v - \delta/2$. Without loss of generality, we may assume that $\delta > 0$ is small enough such that (6.1) holds. Since (6.2) is satisfied and $|u| = v > c_2$, it follows that from (6.3),

$$p_{N,u} \leq 2\mathcal{P}_k(\mathbf{m}, \mathbf{q}) - \varepsilon^*(v) \leq 2\mathcal{P}(\xi, h) - \big(\varepsilon^*(v) - 2\varepsilon_1\big)$$

for $\varepsilon^*(v) = C_0\delta K_2 \int_{v-\delta}^{v-\delta/2} EF_{-1}(h, h, \xi'(q))\xi''(q)\,dq$. Clearly, $\varepsilon^*(\cdot)$ is a continuous function on $[\delta, 1]$. Since $0 < t \leq 1$ and $Eh^2 \neq 0$, $\varepsilon^*(v) > 0$ for every $v \in [\delta, 1]$. Thus, $\min_{v\in[\delta,1]} \varepsilon^*(v) > 0$ and the announced result follows by letting ε_1 be sufficiently small. □

At this moment, we explain the motivation of the proof of Proposition 11. Let us apply (2.3) to Theorem 2 and recall the definitions of $(Y_p)_{0\le p\le k+2}$ and $(y_p^1, y_p^2)_{0\le p\le k+1}$. Using the independence of y_p^1 and y_p^2 for $\tau \le p \le k+1$ and decreasing induction, one may clearly derive

$$Y_\tau = A_\tau\biggl(h + \sum_{0\le p<\tau} y_p^1\biggr) + A_\tau\biggl(h + \sum_{0\le p<\tau} y_p^2\biggr).$$

For $0 \le p < \tau$, from Lemma 1 and again using decreasing induction, we also have

$$\begin{aligned} Y_p &= \frac{1+t}{m_p}\log E_p \exp\frac{m_p}{1+t}Y_{p+1} \\ &\le \frac{1+t}{m_p}\log E_p \exp\frac{m_p}{1+t}\bigl(A_{p+1}(x_p^1 + y_p^1) + A_{p+1}(x_p^2 + y_p^2)\bigr) \qquad (6.4)\\ &\le \frac{1}{m_p}E_p \exp m_p A_{p+1}(x_p^1 + y_p^1) + \frac{1}{m_p}E_p \exp m_p A_{p+1}(x_p^2 + y_p^2), \end{aligned}$$

where $x_p^j = h + \sum_{0\le r<p-1} y_r^j$ for $j = 1, 2$. In particular, if $p = 0$, $Y_0 \le 2EA_0(h) = 2X_0$. To prove (6.3), we expect that when $0 < t < 1$, equality will not hold in (6.4) and, with the help of the condition (6.2), the small difference between the two sides will keep accumulating over p. Let us emphasize that this should be true even in the absence of the external field. A similar approach is also presented in Section 14.12 of Talagrand's book [17], where he considered the case $t = 1$ and used the Cauchy–Schwarz inequality to quantify the difference. However, in the case $0 < t < 1$, his argument no longer holds. We then resort to another approach using the Gaussian interpolation technique.

Before we state our main estimate, for convenience, let us set up a definition. Let $C_1 > 0$ be a constant and y be a standard Gaussian r.v. Suppose that m and ω are two fixed numbers with $0 \le m \le 1$ and $\omega \ge 0$ and A is a real-valued function defined on $\mathbb{R}$ such that

$$E \exp mA(x + y\sqrt{w}) \quad \text{and} \quad EA(x + y\sqrt{w})$$

exist for $x \in \mathbb{R}$ and $0 \le w \le \omega$. We define

$$T(x, w) = \frac{1}{m}\log E \exp mA(x + y\sqrt{w}), \qquad (6.5)$$

where y is standard Gaussian. Here, if $m = 0$, $T(x, w)$ is defined as $EA(x + y\sqrt{w})$. Then we say that A satisfies condition $\mathcal{A}(m, \omega, C_1)$ if

$$\begin{aligned} &\left|\frac{\partial T}{\partial x}\right| \le 1, \qquad \frac{1}{C_1 \operatorname{ch}^2 x} \le \frac{\partial^2 T}{\partial x^2} \le \min\biggl(1, \frac{C_1}{\operatorname{ch}^2 x}\biggr), \\ &\left|\frac{\partial^3 T}{\partial x^3}\right| \le 4, \qquad \left|\frac{\partial^4 T}{\partial x^4}\right| \le 8 \end{aligned} \qquad (6.6)$$

for all $x \in \mathbb{R}$ and $0 \le w \le \omega$.

PROPOSITION 12. *Suppose that A satisfies $\mathcal{A}(m,\omega,C_1)$. Let y_1, y_2 be jointly Gaussian r.v.'s with $Ey_1^2 = Ey_2^2 = 1$ and $Ey_1y_2 = t \geq 0$. Let $K > 0$ and $L \in \mathbb{N}$ be fixed constants. Suppose that $\alpha_0, \alpha_1, \ldots, \alpha_\ell \geq 0$. Then there exist constants $C_\ell^0, C_\ell^1, \ldots, C_\ell^\ell$ satisfying*

$$0 < C_\ell^0, C_\ell^1, \ldots, C_\ell^\ell \leq 4 \sum_{n=0}^{\ell} \alpha_n + K_1 \tag{6.7}$$

for some constant K_1 depending only on C_1 and L such that for any given numbers $x_1, x_2 \in \mathbb{R}$, $0 < m \leq 1$, $0 \leq w \leq \min(1/8, \omega, 1/2C_\ell^0)$, $w_0 = 0$, and $0 \leq w_1, w_2, \ldots, w_\ell \leq L$, the following inequality holds:

$$\begin{aligned} &\frac{1+t}{m} \log E \exp \frac{m}{1+t} \Biggl(A(x_1 + y_1\sqrt{w}) + A(x_2 + y_2\sqrt{w}) \\ &\qquad\qquad - \sum_{n=0}^{\ell} \alpha_n F(x_1 + y_1\sqrt{w}, x_2 + y_2\sqrt{w}, w_n) \Biggr) \\ &\quad \leq \sum_{j=1}^{2} \frac{1}{m} \log E \exp mA(x_j + y_j\sqrt{w}) \\ &\qquad - \sum_{n=0}^{\ell} \biggl(\alpha_n (1 - wC_\ell^n) + \frac{C_0}{2} mw\delta_0(n) \biggr) F\bigl(x_1, x_2, (1-\delta_0(n))w + w_n\bigr), \end{aligned} \tag{6.8}$$

where $C_0 = t(2(1+t)C_1^2)^{-1}$ and we define $\delta_0(n) = 1$ if $n = 0$ and 0 otherwise.

Let us explain how to use this inequality. Observe that the left-hand side of (6.8) differs from (2.4) by the $\ell + 1$ quantities

$$\bigl(\alpha_n F(x_1, x_2, w_n)\bigr)_{0 \leq n \leq \ell}$$

at the present stage. Most of them will be preserved in the new stage by

$$\bigl(\alpha_n (1 - wC_\ell^n) F\bigl(x_1, x_2, (1 - \delta_0(n))w + w_n\bigr)\bigr)_{0 \leq n \leq \ell}$$

with the additional term

$$\frac{C_0}{2} tmwF(x_1, x_2, 0).$$

So after one step, we obtain $(\ell + 1) + 1$ terms in the new stage. Continued iterations of (6.8) lead to a sum of these small quantities that will converge to some positive number if w is not too small at each iteration. This is the main reason we need the growth control on $C_\ell^0, C_\ell^1, \ldots, C_\ell^\ell$ through (6.7).

Now, we turn to the proof of Proposition 11. Let k, $\mathbf{m}$, $\mathbf{q}$ be a given triplet. Recall the definition of $(A_p)_{0 \leq p \leq k+2}$ from (3.1). We will need the following lemma.

LEMMA 13. *For each* $0 \le p \le k+1$, A_{p+1} *satisfies* $\mathcal{A}(m_p, \xi'(q_{p+1}), C)$.

PROOF. Let $0 \le p \le k+1$ be fixed. Suppose that $0 \le w \le \xi'(q_{p+1})$. Note that ξ' is strictly increasing on $[0, \infty)$ and $\xi'(0) = 0$. Let q satisfy $\xi'(q) = \xi'(q_{p+1}) - w$. Set $k' = k + 1 - p$. Consider

$$\mathbf{m}'\text{: } m'_0 = 0, m'_1 = m_p, \text{ and } m'_n = m_{n+p-1} \text{ for } 2 \le n \le k'+1,$$

$$\mathbf{q}'\text{: } q'_0 = 0, q'_1 = q, \text{ and } q'_n = q_{n+p-1} \text{ for } 2 \le n \le k'+2.$$

Let $(B_n)_{0\le n\le k'+2}$ be defined in the same way as $(A_p)_{0\le p\le k+2}$ by using the triplet $k', \mathbf{m}', \mathbf{q}'$. Then it should be clear that $B_2 = A_p$ and so

$$\begin{aligned} B_1(x) &= \frac{1}{m'_1} \log E \exp m'_1 B_2\bigl(x + y\sqrt{\xi'(q_{p+1}) - \xi'(q)}\bigr) \\ &= \frac{1}{m_p} \log E \exp m_p A_p(x + y\sqrt{w}), \end{aligned}$$

where y is a standard Gaussian r.v. Since $(B_n)_{0\le n\le k'+2}$ satisfies (3.2), this completes our proof. $\square$

PROOF OF PROPOSITION 11. Let C be the constant in Lemma 2 and L be the smallest integer such that $L \ge \xi'(1)$. Suppose that K_1 is obtained from Proposition 12 by using $C_1 = C$ and L. Again, without loss of generality, we may assume that $c_1 = q_{s_1}, c_2 = q_{s_2}$, $u = q_a$ for $1 \le s_1 < s_2 \le a \le k+2$. Moreover, for $s_1 \le p \le s_2 - 1$,

$$0 < \xi'(q_{p+1}) - \xi'(q_p) < \frac{1}{2\gamma(s_2 - s_1)} \tag{6.9}$$

and for $0 \le p < s_1$,

$$0 < \xi'(q_{p+1}) - \xi'(q_p) < \frac{1}{2\gamma}, \tag{6.10}$$

where $\gamma := \max(4, 2C_0\xi'(1) + K_1)$. Let us note that such $(q_p)_{0\le p\le k+2}$ exists by the discussion right below Theorem 2 and using the assumption (6.1). Let us consider the following numbers:

$$\begin{aligned} &\lambda = 0, \\ &\tau = a, \\ &\kappa = k+1, \\ &n_p = \frac{m_p}{t+1} \quad \text{if } 0 \le p < \tau \quad \text{and} \quad m_p \quad \text{if } \tau \le p \le \kappa, \\ &\rho_p = q_p \quad \text{for } 0 \le p \le \kappa + 1. \end{aligned}$$

From (2.1),

$$p_{N,u} \le 2\log 2 + Y_0 - \sum_{0\le p\le k+1} m_p\big(\theta(q_{p+1}) - \theta(q_p)\big).$$

Recall the definition of $(y_p^1, y_p^2)_{0\le p\le\kappa}$ from Theorem 2. We define $Y_a(x_1,x_2) = A_a(x_1) + A_a(x_2)$ and for $1 \le p < a$,

$$Y_p(x_1,x_2) = \frac{1+t}{m_p}\log E\exp\frac{m_p}{1+t}Y_{p+1}\big(x_1+y_p^1, x_2+y_p^2\big).$$

Finally, set $Y_0(x_1,x_2) = EY_1(x_1+y_0^1, x_2+y_0^2)$. It is obvious that from the definition $Y_0 = EY_0(h,h)$. From Proposition 12, we know $Y_{s_2}(x_1,x_2) \le A_{s_2}(x_1) + A_{s_2}(x_2)$. Set $\eta_p = \xi'(q_{p+1}) - \xi'(q_p)$ for $0 \le p \le k+1$. We claim that for $s_1 \le p < s_2$,

$$Y_p(x_1,x_2) \le A_p(x_1) + A_p(x_2) - \sum_{n=p}^{s_2-1}\beta_{n,p}F_\eta\left(x_1, x_2, \sum_{l=p}^{n-1}\eta_l\right), \tag{6.11}$$

where

$$\beta_{n,p} = \frac{C_0}{2}m_n\eta_n\left(1 - \frac{1}{s_2-s_1}\right)^{n-p}. \tag{6.12}$$

Here, we adapt the definition $\sum_{\ell=p}^{p'}u_\ell = 0$ whenever $p > p'$ that remains enforced thereafter. Let $s_1 \le p < s_2$ and consider the following numbers:

$$\begin{aligned}
&\ell = s_2 - p - 1,\\
&m = m_p,\\
&\alpha_0 = 0 \quad\text{and}\quad \alpha_n = \beta_{n+p,p+1} \qquad \text{for } 1 \le n \le \ell,\\
&w_0 = 0 \quad\text{and}\quad w_n = \sum_{l=p+1}^{n+p-1}\eta_l \qquad \text{for } 1 \le n \le \ell.
\end{aligned} \tag{6.13}$$

From the definition of w_n, we know that $0 \le w_n \le \xi'(1) \le L$ for $0 \le n \le \ell$. Since A_{p+1} satisfies $\mathcal{A}(m_p, \xi'(q_{p+1}), C_1)$, applying (6.13) to Proposition 12, we obtain $(C_\ell^n)_{0\le n\le\ell}$ that, from (6.7), satisfies

$$C_\ell^0, C_\ell^1, \dots, C_\ell^\ell \le 4\sum_{n=0}^{\ell}\alpha_n + K_1 \le 2C_0\sum_{n=p+1}^{s_2-1}m_n\eta_n + K_1 \le \gamma. \tag{6.14}$$

Using (6.9) and (6.14), we know for $0 \le n \le \ell$,

$$C_\ell^n\eta_p \le \gamma\eta_p < \frac{1}{2(s_2-s_1)} < \frac{1}{s_2-s_1}. \tag{6.15}$$

Take $w = \eta_p$. Notice that from (6.9) and (6.15), $w \le \min(1/8, \xi'(q_{p+1}), 1/2C_\ell^0)$. If $u > 0$, then from (6.8), (6.13) and (6.15), we obtain (6.11) since

$$\begin{aligned}
Y_p(x_1,x_2) &\le A_p(x_1) + A_p(x_2) \\
&\qquad - \frac{C_0}{2} m w F(x_1,x_2,0) - \sum_{n=1}^{\ell} \alpha_n \bigl(1 - C_\ell^n w\bigr) F(x_1,x_2,w+w_n) \\
&\le A_p(x_1) + A_p(x_2) - \frac{C_0}{2} m_p \eta_p F(x_1,x_2,0) \\
&\qquad - \sum_{n=p+1}^{s_2-1} \beta_{n,p} F\left(x_1,x_2,\sum_{l=p}^{n-1} \eta_l\right).
\end{aligned}$$

If $u < 0$, then

$$\begin{aligned}
E y_p^1\bigl(-y_p^2\bigr) &= t\bigl(\xi'(q_{p+1}) - \xi'(q_p)\bigr), \\
A_{p+1}\bigl(x_2 + y_p^2\bigr) &= A_{p+1}\bigl(-x_2 - y_p^2\bigr), \\
F_{-1}\bigl(x_1 + y_p^1, x_2 + y_p^2, w_n\bigr) &= F\bigl(x_1 + y_p^1, -x_2 - y_p^2, w_n\bigr)
\end{aligned} \tag{6.16}$$

and it follows by applying $(x_1, -x_2)$ instead of (x_1, x_2) and $(y_1, y_2) = (y_p^1, -y_p^2)$ to Proposition 12 that

$$\begin{aligned}
Y_p(x_1,x_2) &\le A_p(x_1) + A_p(-x_2) \\
&\qquad - \frac{C_0}{2} m w F(x_1,-x_2,0) \\
&\qquad - \sum_{n=1}^{\ell} \alpha_n \bigl(1 - C_\ell^n w\bigr) F(x_1,-x_2,w+w_p) \\
&\le A_p(x_1) + A_p(x_2) - \frac{C_0}{2} m_p \eta_p F_{-1}(x_1,x_2,0) \\
&\qquad - \sum_{n=p+1}^{s_2-1} \beta_{n,p} F_{-1}\left(x_1,x_2,\sum_{l=p}^{n-1} \eta_l\right),
\end{aligned}$$

where, again, we use (6.15) for the second inequality. This completes the proof of our claim.

Next, we claim that for $0 \le p \le s_1$,

$$\begin{aligned}
Y_p(x_1,x_2) &\le A_p(x_1) + A_p(x_2) \\
&\qquad - \exp\left(-2\gamma \sum_{l=p}^{s_1-1} \eta_l\right) \sum_{n=s_1}^{s_2-1} \beta_{n,s_1} F_\eta\left(x_1,x_2,\sum_{l=p}^{n-1} \eta_l\right).
\end{aligned} \tag{6.17}$$

If $p = s_1$, then (6.17) holds by (6.11). Suppose $0 \le p < s_1$. Let us consider the following numbers:

$$\begin{aligned} &\ell = s_2 - s_1, \\ &m = m_p, \\ &\alpha_0 = 0 \quad \text{and} \quad \alpha_n = \exp\left(-2\gamma \sum_{l=p+1}^{s_1-1} \eta_l\right)\beta_{n+s_1-1,s_1} \qquad \text{for } 1 \le n \le \ell, \\ &w_0 = 0 \quad \text{and} \quad w_n = \sum_{l=p+1}^{n+s_1-2} \eta_l \qquad \text{for } 1 \le n \le \ell, \end{aligned} \tag{6.18}$$

where $\beta_{n,p}$ is defined in (6.12). As in our first claim, since $0 \le w_n \le \xi'(1) \le L$ for $0 \le n \le k$ and A_{p+1} satisfies $\mathcal{A}(m_p, \xi'(q_{p+1}), C_1)$, we can apply Proposition 12 using (6.18) to obtain $(C_\ell^n)_{n=0}^{\ell}$ that, from (6.7), satisfies

$$C_\ell^0, C_\ell^1, \ldots, C_\ell^\ell \le 4 \sum_{n=0}^{\ell} \alpha_n + K_1 \le 2C_0 \sum_{n=0}^{\ell} m_n \eta_n + K_1 \le \gamma. \tag{6.19}$$

We conclude from (6.10) and (6.19) that

$$C_\ell^n \eta_p \le 1/2 \tag{6.20}$$

for $0 \le n \le \ell$. Note that $1 - x \ge \exp(-2x)$ if $x \le 1/2$. Using this, (6.19), and (6.20) yield

$$1 - C_\ell^n \eta_p \ge \exp(-2C_\ell^n \eta_p) \ge \exp(-2\gamma \eta_p). \tag{6.21}$$

Set $w = \eta_p$. Notice that from (6.10) and (6.20), $w \le \min(1/8, \xi'(q_{p+1}), 1/2C_\ell^0)$. If $u > 0$, using (6.8) and (6.21), we obtain

$$\begin{aligned} Y_p(x_1, x_2) &\le A_p(x_1) + A_p(x_2) - \frac{C_0}{2} m_p \eta_p F(x_1, x_2, 0) \\ &\quad - \exp\left(-2\gamma \sum_{l=p+1}^{s_1-1} \eta_l\right) \sum_{n=s_1}^{s_2-1} \beta_{n,s_1}(1 - C_\ell^n \eta_p) F\left(x_1, x_2, \sum_{l=p}^{n-1} \eta_l\right) \\ &\le A_p(x_1) + A_p(x_2) \\ &\quad - \exp\left(-2\gamma \sum_{l=p}^{s_1-1} \eta_l\right) \sum_{n=s_1}^{s_2-1} \beta_{n,s_1} F\left(x_1, x_2, \sum_{l=p}^{n-1} \eta_l\right). \end{aligned}$$

If $u < 0$, we obtain (6.17) by using (6.16), applying $(x_1, -x_2)$ instead of (x_1, x_2) and $(y_1, y_2) = (y_p^1, -y_p^2)$ to (6.11), and a similar argument as in the case $u > 0$. This completes the proof of our second claim.

Now, let $p=0$ in (6.17) and note that $m_n \geq \delta/2$ for $n \geq s_1$. We then obtain

$$\begin{aligned} Y_0 &= EY_0(h,h) \\ &\leq 2EA_0(h) \\ &\quad - \frac{C_0\delta}{2}\exp(-2\gamma\xi'(1)) \\ &\quad \times \left(1-\frac{1}{s_2-s_1}\right)^{s_2-s_1}\sum_{n=s_1}^{s_2-1}(\xi'(q_{n+1})-\xi'(q_n))EF_\eta(h,h,\xi'(q_n)). \end{aligned}$$

Since we can partition $[c_1,c_2]$ so that $\max_{s_1\leq p\leq s_2-1}\eta_p$ is arbitrarily small, by passing to the limit,

$$Y_0 \leq 2X_0 - \frac{C_0\delta}{2}\exp(-2\gamma\xi'(1)-1)\int_{c_1}^{c_2} EF_\eta(h,h,\xi'(q))\xi''(q)\,dq$$

and we are done. □

At the end of this section, we will prove Proposition 12 and we proceed by two lemmas.

LEMMA 14. *For any* $x_1,x_2\in\mathbb{R}$, $0\leq w\leq \frac{1}{8}$, *and* $w'\geq 0$, *we have*

$$F(x_1,x_2,w'+w) \geq \tfrac{1}{2}F(x_1,x_2,w'). \tag{6.22}$$

PROOF. First we prove that for $x_1,x_2\in\mathbb{R}$ and $0\leq w\leq 1/4$,

$$F(x_1,x_2,w) \geq (1-4w)F(x_1,x_2,0). \tag{6.23}$$

If (6.23) holds, then

$$F(x_1,x_2,w)\geq \tfrac{1}{2}F(x_1,x_2,0),$$

whenever $x_1,x_2\in\mathbb{R}$ and $0\leq w\leq 1/8$ and this implies (6.22) since for $w'\geq 0$,

$$\begin{aligned} F(x_1,x_2,w'+w) &= EF(x_1+y_1\sqrt{w'},x_2+y_2\sqrt{w'},w) \\ &\geq \tfrac{1}{2}EF(x_1+y_1\sqrt{w'},x_2+y_2\sqrt{w'},0) \\ &= \tfrac{1}{2}F(x_1,x_2,w'), \end{aligned}$$

where y_1 and y_2 are jointly Gaussian r.v.'s with $E(y_1)^2=E(y_2)^2=1$ and $Ey_1y_2=t$. To prove (6.23), for fixed x_1,x_2, let us set $\varphi(w)=F(x_1,x_2,w)$. Define $G(x,y)=(\operatorname{th}x-\operatorname{th}y)^2$. Using Gaussian integration by parts, we have

$$\begin{aligned} \varphi'(0) &= \tfrac{1}{2}(G_{11}(x_1,x_2)+G_{22}(x_1,x_2)+2tG_{12}(x_1,x_2)) \\ &= (\operatorname{th}x_1-\operatorname{th}x_2)(\operatorname{th}''x_1-\operatorname{th}''x_2)+(\operatorname{th}'x_1-\operatorname{th}'x_2)^2 \\ &\quad + 2(1-t)\operatorname{th}'x_1\operatorname{th}'x_2 \\ &\geq (\operatorname{th}x_1-\operatorname{th}x_2)(\operatorname{th}''x_1-\operatorname{th}''x_2). \end{aligned} \tag{6.24}$$

Since

$$\mathrm{th}'' x_1 - \mathrm{th}'' x_2 = 2(\mathrm{th}\, x_1 - \mathrm{th}\, x_2)\big((\mathrm{th}\, x_1 + \mathrm{th}\, x_2)^2 - 1 - \mathrm{th}\, x_1 \,\mathrm{th}\, x_2\big),$$

using this equation together with (6.24) leads to

$$\begin{aligned}\varphi'(0) &\geq 2(\mathrm{th}\, x_1 - \mathrm{th}\, x_2)^2\big((\mathrm{th}\, x_1 + \mathrm{th}\, x_2)^2 - 1 - \mathrm{th}\, x_1 \,\mathrm{th}\, x_2\big)\\ &\geq -4(\mathrm{th}\, x_1 - \mathrm{th}\, x_2)^2.\end{aligned}$$

We may also compute the second derivative of φ to see that

$$\max_{0 \leq w \leq 1} \big|\varphi''(w)\big|/2 \leq C,$$

where C is a constant independent of t, w, x_1, x_2. So

$$\begin{aligned}F(x_1, x_2, w) &= \varphi(w)\\ &\geq \varphi(0) + \varphi'(0)w - Cw^2\\ &\geq (1 - 4w)F(x_1, x_2, 0) - Cw^2.\end{aligned}$$

Set $\delta_i = wi/N$. It is easy to see by induction

$$F(x_1, x_2, \delta_i) \geq (1 - 4\delta_1)^i F(x_1, x_2, 0) - Ci\delta_1^2$$

for $1 \leq i \leq N$. In particular, if we put $i = N$ and let N tend to infinity, we obtain $F(x_1, x_2, w) \geq \exp(-4w)F(x_1, x_2, 0) \geq (1 - 4w)F(x_1, x_2, 0)$ and this completes the proof. □

LEMMA 15. *Suppose that A is a function defined on $\mathbb{R}$ satisfying*

$$\begin{aligned}&\big|A'\big| \leq 1, \qquad \frac{1}{C_1 \,\mathrm{ch}\, x^2} \leq A''(x) \leq \min\biggl(1, \frac{C_1}{\mathrm{ch}^2 x}\biggr),\\ &\big|A^{(3)}\big| \leq 4, \qquad \big|A^{(4)}\big| \leq 8\end{aligned}$$

for some constant C_1. Let y_1, y_2 be jointly Gaussian r.v.'s with $Ey_1^2 = Ey_2^2 = 1$ and $Ey_1 y_2 = t \geq 0$. Let $K > 0$ and $L \in \mathbb{N}$ be fixed constants. Suppose that $0 \leq \alpha_0, \alpha_1, \dots, \alpha_\ell \leq K$. Then there exist constants

$$K_1 \text{ depending only on } C_1 \text{ and } L,$$

$$C_\ell^0, C_\ell^1, \dots, C_\ell^\ell \leq \sum_{n=0}^{\ell} \alpha_n + K_1$$

and

$$C_\ell^{\ell+1} \text{ depending only on } \ell \text{ and } K$$

such that for any given numbers $x_1, x_2 \in \mathbb{R}$, $0 < m \le 1$, $0 \le w \le 1/8$, $w_0 = 0$, *and* $0 \le w_1, w_2, \dots, w_\ell \le L$, *the following inequality holds*:

$$
\begin{aligned}
\frac{1+t}{m} \log E \exp \frac{m}{1+t} \Biggl(& A(x_1 + y_1\sqrt{w}) + A(x_2 + y_2\sqrt{w}) \\
& - \sum_{n=0}^{\ell} \alpha_n F(x_1 + y_1\sqrt{w}, x_2 + y_2\sqrt{w}, w_n) \Biggr) \\
\le {} & \sum_{j=1}^{2} \frac{1}{m} \log E \exp m A(x_j + y_j\sqrt{w}) \\
& - \sum_{n=0}^{\ell} \bigl(\alpha_n (1 - C_\ell^n w) + C_0 m w \delta_0(n)\bigr) F\bigl(x_1, x_2, (1 - \delta_0(n)) w + w_n\bigr) \\
& + C_\ell^{\ell+1} w^2,
\end{aligned} \tag{6.25}
$$

where $C_0 = t(2(1+t)C_1^2)^{-1}$ *and* $\delta_0(n) = 1$ *if* $n = 0$ *and* 0 *otherwise.*

PROOF. The proof is based on the Gaussian interpolation technique. Suppose for the moment that (y_1, y_2) are jointly Gaussian with $E(y_1)^2 = E(y_2)^2 \le 1/8$ and $E y_1 y_2 = t E(y_1)^2$. Let (z_1, z_2) be an independent copy of (y_1, y_2). Define $(z_1^0, z_2^0) = (0, 0)$ and for $1 \le n \le \ell$, $(z_1^n, z_2^n) = (z_1, z_2)$. For convenience, we set for $j = 1, 2$,

$$
\begin{aligned}
A_j(x) &= A(x_j + x), \\
\mathrm{th}_j(x) &= \mathrm{th}(x_j + x), \\
U_j(u) &= y_j \sqrt{u}
\end{aligned}
$$

and for $j = 1, 2$ and $n = 0, 1, 2, \dots, \ell$,

$$
\begin{aligned}
V_{n,j}(u) &= y_j \sqrt{u} + z_j^n \sqrt{1-u}, \\
G_n(u) &= F\bigl(x_1 + V_{n,1}(u), x_2 + V_{n,2}(u), w_n\bigr), \\
G_{n,j}(u) &= F_j\bigl(x_1 + V_{n,1}(u), x_2 + V_{n,2}(u), w_n\bigr), \\
G_{n,ij}(u) &= F_{ij}\bigl(x_1 + V_{n,1}(u), x_2 + V_{n,2}(u), w_n\bigr),
\end{aligned}
$$

where F_j is the partial derivative of F with respect to the jth variable and F_{ij} means the second partial derivative of F with respect to ith and then jth variables. Define the interpolation functions

$$
\begin{aligned}
\varphi(u) &= E_z \psi(u), \\
\varphi_j(u) &= \psi_j(u), \qquad j = 1, 2,
\end{aligned}
$$

where

$$\psi(u) = \frac{1+t}{m} \log E_y T(u),$$
$$\psi_j(u) = \frac{1}{m} \log E_y T_j(u)$$

and

$$T(u) = \exp \frac{m}{1+t}\Big(A_1\big(U_1(u)\big) + A_2\big(U_2(u)\big) - \sum_{n=0}^{\ell} \alpha_n G_n(u)\Big),$$
$$T_j(u) = \exp m A_j\big(U_j(u)\big).$$

Then

$$\varphi(1) = \frac{1+t}{m} \log E \exp \frac{m}{1+t}\Big(A_1(y_1) + A_2(y_2) - \sum_{n=0}^{\ell} \alpha_n F(x_1+y_1, x_2+y_2, w_n)\Big),$$
$$\varphi(0) = \varphi_1(0) + \varphi_2(0) - \sum_{n=0}^{\ell} \alpha_n E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big).$$

In the following, we will try to find an upper bound for $\varphi'(0)$. Consider

$$\psi'(u) = \frac{1}{E_y T(u)} E_y\Big[\sum_{j=1}^{2}\Big(U_j'(u) A_j'\big(U_j(u)\big) - \sum_{n=0}^{\ell} \alpha_n V_{n,j}'(u) G_{n,j}(u)\Big) T(u)\Big]$$
$$= \frac{1}{2} J_0(u) - \frac{1}{2}\sum_{n=0}^{\ell} \alpha_n \big(J_1^n(u) + J_2^n(u)\big),$$

where

$$J_0(u) = \frac{1}{E_y T(u)} E_y\Big[\frac{1}{\sqrt{u}}\big(y_1 A_1'\big(U_1(u)\big) + y_2 A_2'\big(U_2(u)\big)\big) T(u)\Big]$$

and for $j = 1, 2$ and $n = 0, \dots, \ell$,

$$J_j^n(u) = \frac{1}{E_y T(u)} E_y\Big[\Big(\frac{y_j}{\sqrt{u}} - \frac{z_j^n}{\sqrt{1-u}}\Big) G_{n,j}(u) T(u)\Big].$$

Using Gaussian integration by parts on y, we have

$$\begin{aligned}
J_0(0) &= E_y(y_1)^2 \sum_{j=1}^{2} \biggl(A_j''(0) + \frac{m}{1+t} A_j'(0) \biggl(A_j'(0) - \sum_{n=0}^{\ell} \alpha_n G_{n,j}(0) \biggr) \biggr) \\
&\qquad + \frac{tm}{1+t} E_y(y_1)^2 \biggl(A_1'(0) \biggl(A_2'(0) - \sum_{n=0}^{\ell} \alpha_n G_{n,2}(0) \biggr) \\
&\qquad\qquad + A_2'(0) \biggl(A_1'(0) - \sum_{n=0}^{\ell} \alpha_n G_{n,1}(0) \biggr) \biggr) \\
&= E_y(y_1)^2 \biggl(J_0^1(0) - \frac{m}{1+t} \bigl(J_0^2(0) + J_0^3(0) \bigr) \biggr),
\end{aligned}$$

where

$$\begin{aligned}
J_0^1(0) &= \sum_{j=1}^{2} \bigl(A_j''(0) + m A_j'(0)^2 \bigr) - \frac{mt}{1+t} \bigl(A_1'(0) - A_2'(0) \bigr)^2, \\
J_0^2(0) &= \bigl(A_1'(0) - A_2'(0) \bigr) \sum_{n=0}^{\ell} \alpha_n \bigl(G_{n,1}(0) + t G_{n,2}(0) \bigr), \\
J_0^3(0) &= (1+t) A_2'(0) \sum_{n=0}^{\ell} \alpha_n \bigl(G_{n,1}(0) + G_{n,2}(0) \bigr).
\end{aligned}$$

Let us try to find an upper bound for $J_0(0)$ first. Since

$$\frac{1}{C_1 \operatorname{ch}^2 x} \le A''(x) \le \frac{C_1}{\operatorname{ch}^2 x},$$

it is easy to see from (6.22) that

$$\begin{aligned}
\frac{1}{C_1^2} F(x_1, x_2, w_0) &\le \bigl(A_1'(0) - A_2'(0) \bigr)^2 \\
&\le C_1^2 F(x_1, x_2, w_0) \qquad (6.26) \\
&\le 2^{8L+1} C_1^2 E_z F\bigl(x_1 + z_1^n, x_2 + z_2^n, w_n \bigr).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial}{\partial x} (\operatorname{th}_1 x - \operatorname{th}_2 y)^2 &= 2\bigl(1 - \operatorname{th}_1^2 x\bigr)(\operatorname{th}_1 x - \operatorname{th}_2 y), \\
\frac{\partial}{\partial y} (\operatorname{th}_1 x - \operatorname{th}_2 y)^2 &= 2\bigl(1 - \operatorname{th}_2^2 y\bigr)(\operatorname{th}_2 y - \operatorname{th}_1 x)
\end{aligned}$$

from the Cauchy–Schwarz inequality, we have

$$\begin{aligned}E_z G_{n,j}(0)^2 &= E_z F_j\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big)^2 \\ &= E_z\big(2E_{y'}\big[\big(1 - \mathrm{th}_j^2\big(z_j^n + y_j'\sqrt{w_n}\big)\big) \\ &\qquad \times \big(\mathrm{th}_1\big(z_1^n + y_1'\sqrt{w_n}\big) - \mathrm{th}_2\big(z_2^n + y_2'\sqrt{w_n}\big)\big)\big]\big)^2 \qquad (6.27)\\ &\le 4E_z E_{y'}\big(\mathrm{th}_1\big(z_1^n + y_1'\sqrt{w_n}\big) - \mathrm{th}_2\big(z_2^n + y_2'\sqrt{w_n}\big)\big)^2 \\ &= 4E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big),\end{aligned}$$

where y_1', y_2' are jointly Gaussian r.v.'s with $E(y_1')^2 = E(y_2')^2 = 1$ and $Ey_1'y_2' = t$. Straightforward computation yields

$$\frac{\partial}{\partial x}(\mathrm{th}_1\, x - \mathrm{th}_2\, y)^2 + \frac{\partial}{\partial y}(\mathrm{th}_1\, x - \mathrm{th}_2\, y)^2 = -2(\mathrm{th}_1\, x - \mathrm{th}_2\, y)^2(\mathrm{th}_1\, x + \mathrm{th}_2\, y)$$

and this implies

$$E_z\big|G_{n,1}(0) + G_{n,2}(0)\big| \le 4E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big). \qquad (6.28)$$

Now, combining (6.26), (6.27), (6.28), and using Jensen's inequality,

$$E_z J_0^1(0) \le \sum_{n=1}^{2}\big(A_j''(0) + mA_j'(0)^2\big) - \frac{mt}{(1+t)C_1^2}F(x_1, x_2, w_0),$$

$$\begin{aligned}E_z\big|J_0^2(0)\big| &\le \big|A_1'(0) - A_2'(0)\big| \sum_{n=0}^{\ell}\alpha_n\big(\big(E_z G_{n,1}(0)^2\big)^{1/2} + t\big(E_z G_{n,2}(0)^2\big)^{1/2}\big) \\ &\le 2(1+t)\big|A_1'(0) - A_2'(0)\big| \sum_{n=0}^{\ell}\alpha_n\big(E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big)\big)^{1/2} \\ &\le 2^{4L+3/2}C_1(1+t)\sum_{n=0}^{\ell}\alpha_n E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big)\end{aligned}$$

and

$$E_z\big|J_0^3(0)\big| \le 4(1+t)\sum_{n=0}^{\ell}\alpha_n E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big).$$

To sum up, we obtain that

$$\begin{aligned}E_z J_0(0) &\le E_y(y_1)^2\sum_{j=1}^{2}\big(A_j''(0) + mA_j'(0)^2\big) \\ &\quad + mE_y(y_1)^2\sum_{n=0}^{\ell}\Big(\alpha_n\big(2^{4L+3/2}C_1 + 4\big) - \frac{t}{(1+t)C_1^2}\delta_0(n)\Big) \qquad (6.29)\\ &\qquad \times E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big).\end{aligned}$$

Next, let us turn to the computation of J_1^n. By using Gaussian integration by parts on y, we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{u}} E_y\big(y_1 G_{n,1}(u) T(u)\big) \\
&\quad = E_y(y_1)^2 E_y\big[G_{n,11}(u) T(u)\big] + E_y(y_1 y_2) E_y\big[G_{n,12}(u) T(u)\big] \\
&\qquad + \frac{m}{1+t} E_y(y_1)^2 E_y\Bigg[G_{n,1}(u)\Bigg(A_1'\big(U_1(u)\big) - \sum_{l=0}^{\ell} \alpha_l G_{l,1}(u)\Bigg) T(u)\Bigg] \\
&\qquad + \frac{m}{1+t} E_y(y_1 y_2) E_y\Bigg[G_{n,1}(u)\Bigg(A_2'\big(U_2(u)\big) - \sum_{l=0}^{\ell} \alpha_l G_{l,2}(u)\Bigg) T(u)\Bigg],
\end{aligned}$$

and this implies that

$$\begin{aligned}
&\lim_{u\to 0} E_z\bigg[\frac{1}{\sqrt{u} E_y T(u)} E_y\big(y_1 G_{n,1}(u) T(u)\big)\bigg] \\
&\quad = E_y(y_1)^2\bigg(E_z G_{n,11}(0) + t E_z G_{n,12}(0) + \frac{m}{1+t}\big(I_1^n + t I_2^n\big)\bigg),
\end{aligned} \tag{6.30}$$

where for $0 \le n \le \ell$,

$$I_1^n = E_z\Bigg[G_{n,1}(0)\Bigg(A_1'(0) - \sum_{l=1}^{\ell} \alpha_l G_{l,1}(0)\Bigg)\Bigg],$$

$$I_2^n = E_z\Bigg[G_{n,1}(0)\Bigg(A_2'(0) - \sum_{l=1}^{\ell} \alpha_l G_{l,2}(0)\Bigg)\Bigg].$$

On the other hand, letting $u \to 0$ and then using Gaussian integration by parts on z, we also have

$$\begin{aligned}
&\lim_{u\to 0} E_z\bigg[\frac{1}{\sqrt{1-u} E_y T(u)} E_y\big(z_1^n G_{n,1}(u) T(u)\big)\bigg] \\
&\quad = E_z\bigg[\frac{1}{E_y T(0)} z_1^n G_{n,1}(0) E_y T(0)\bigg] = E_z\big[z_1^n G_{n,1}(0)\big] \\
&\quad = E_z\big(z_1^n\big)^2\big(E_z G_{n,11}(0) + t E_z G_{n,12}(0)\big).
\end{aligned} \tag{6.31}$$

So from (6.30) and (6.31),

$$\begin{aligned}
&\lim_{u\to 0} E_z J_1^n(u) \\
&\quad = \lim_{u\to 0} E_{y,z}\bigg[\frac{1}{E_y T(u)}\bigg(\frac{y_1}{\sqrt{u}} - \frac{z_1^n}{\sqrt{1-u}}\bigg) G_{n,1}(u) T(u)\bigg] \\
&\quad = E_y(y_1)^2\bigg(\delta_0(n)\big(G_{0,11}(0) + t G_{0,12}(0)\big) + \frac{m}{1+t}\big(I_1^n + t I_2^n\big)\bigg).
\end{aligned} \tag{6.32}$$

We may also compute $\lim_{u\to 0} E_z J_2^n(u)$ and this yields

$$\begin{aligned}(6.33)\qquad &\lim_{u\to 0} E_z J_1^n(u) + E_z J_2^n(u)\\ &= \delta_0(n) E_y(y_1)^2 I_0 + \frac{m E_y(y_1)^2}{1+t}\big((I_1^n + I_3^n) + t(I_2^n + I_4^n)\big),\end{aligned}$$

where $I_0 = G_{0,11}(0) + G_{0,22}(0) + 2t G_{0,12}(0)$ and for $0 \le n \le \ell$,

$$I_3^n = E_z\left[G_{n,2}(0)\left(A_2'(0) - \sum_{l=1}^{\ell} \alpha_l G_{l,2}(0)\right)\right],$$

$$I_4^n = E_z\left[G_{n,2}(0)\left(A_1'(0) - \sum_{l=1}^{\ell} \alpha_l G_{l,1}(0)\right)\right].$$

Let us now try to find a suitable lower bound for (6.33). Observe that

$$\begin{aligned}&\mathrm{th}_1'(0), \mathrm{th}_2'(0) \ge 0,\\ &\mathrm{th}_1'(0) - \mathrm{th}_2'(0) = -\big(\mathrm{th}_1(0) - \mathrm{th}_2(0)\big)\big(\mathrm{th}_1(0) + \mathrm{th}_2(0)\big),\\ &\mathrm{th}_1''(0) - \mathrm{th}_2''(0) = 2\big(\mathrm{th}_1(0) - \mathrm{th}_2(0)\big)\\ &\qquad\qquad \times \big(\big(\mathrm{th}_1(0) + \mathrm{th}_2(0)\big)^2 - 1 - \mathrm{th}_1(0)\,\mathrm{th}_2(0)\big).\end{aligned}$$

This implies

$$\begin{aligned}I_0 &= 2\big(\mathrm{th}_1(0) - \mathrm{th}_2(0)\big)\big(\mathrm{th}_1''(0) - \mathrm{th}_2''(0)\big)\\ &\qquad + 2\big(\mathrm{th}_1'(0) - \mathrm{th}_2'(0)\big)^2 + 4(1-t)\,\mathrm{th}_1'(0)\,\mathrm{th}_2'(0)\\ (6.34)\qquad &\ge 2\big(\mathrm{th}_1(0) - \mathrm{th}_2(0)\big)\big(\mathrm{th}_1''(0) - \mathrm{th}_2''(0)\big)\\ &\ge -4\big(\mathrm{th}_1(0) - \mathrm{th}_2(0)\big)^2\\ &= -4F(x_1, x_2, w_0).\end{aligned}$$

As for the upper bounds for $|I_1^n + I_3^n|$ and $|I_2^n + I_4^n|$, we write

$$I_1^n + I_3^n = I_{11}^n + I_{12}^n + I_{13}^n,$$
$$I_2^n + I_4^n = I_{21}^n + I_{22}^n + I_{23}^n,$$

where

$$\begin{aligned}I_{11}^n &= E_z\big[G_{n,1}(0)\big(A_1'(0) - A_2'(0)\big)\big],\\ I_{21}^n &= E_z\big[G_{n,2}(0)\big(A_1'(0) - A_2'(0)\big)\big],\\ I_{12}^n &= E_z\big[A_2'(0)\big(G_{n,1}(0) + G_{n,2}(0)\big)\big],\\ I_{22}^n &= E_z\big[A_2'(0)\big(G_{n,1}(0) + G_{n,2}(0)\big)\big],\end{aligned}$$

$$I_{13}^n = -\sum_{l=0}^{\ell} \alpha_l E_z\big[\big(G_{n,1}(0)G_{l,1}(0) + G_{n,2}(0)G_{l,2}(0)\big)\big],$$

$$I_{23}^n = -\sum_{l=0}^{\ell} \alpha_l E_z\big[\big(G_{n,1}(0)G_{l,2}(0) + G_{n,2}(0)G_{l,1}(0)\big)\big].$$

Using (6.26), (6.27) and Jensen's inequality, we have

$$\begin{aligned} |I_{j1}^n| &\le \big(E_y G_{n,j}(0)^2\big)^{1/2} |A_1'(0) - A_2'(0)| \\ &\le 2^{4L+3/2} C_1 E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big). \end{aligned} \tag{6.35}$$

From (6.28), we also have

$$|I_{j2}^n| \le 4 E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big). \tag{6.36}$$

To bound $|I_{j3}^n|$, we use $ab \le (a^2 + b^2)/2$ and (6.27). Then this leads to

$$\begin{aligned} |I_{j3}^n| &\le \frac{1}{2} \sum_{l=0}^{\ell} \alpha_l \big(G_{n,1}(0)^2 + G_{n,2}(0)^2 + G_{l,1}(0)^2 + G_{l,2}(0)^2\big) \\ &\le 4\left(\sum_{l=0}^{\ell} \alpha_l\right) E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big) \\ &\quad + 4 \sum_{l=0}^{\ell} \alpha_l E_z F\big(x_1 + z_1^l, x_2 + z_2^l, w_l\big). \end{aligned} \tag{6.37}$$

Now, combining (6.34), (6.35), (6.36) and (6.37) together, we obtain from (6.33),

$$\begin{aligned} &\sum_{n=0}^{\ell} \alpha_n \big(E_z J_1^n(0) + E_z J_2^n(0)\big) \\ &\quad \ge -4\alpha_0 \delta_0(n) E_y (y_1)^2 F(x_1, x_2, w_0) \\ &\qquad - m E_y (y_1)^2 \sum_{n=0}^{\ell} \alpha_n \left(8 \sum_{l=0}^{\ell} \alpha_l + 2^{4L+3/2} C_1 + 4\right) \\ &\qquad\qquad \times E_z F\big(x_1 + z_1^n, x_2 + z_2^n, w_n\big). \end{aligned} \tag{6.38}$$

From now on, we replace (y_1, y_2) by $(y_1\sqrt{w}, y_2\sqrt{w})$ with $E(y_1)^2 = 1$. Combining (6.29) and (6.38), we get

$$\begin{aligned} \varphi'(0) &\le \frac{w}{2} \sum_{j=1}^{2} \big(A_j''(0) + m A_j'(0)^2\big) \\ &\quad + w \sum_{n=0}^{\ell} \big(\alpha_n C_\ell^n - C_0 m \delta_0(n)\big) F\big(x_1, x_2, (1 - \delta_0(n)) w + w_n\big), \end{aligned} \tag{6.39}$$

where $C_0 = t(2(1+t)C_1^2)^{-1}$ and for $0 \le n \le \ell$,

$$C_\ell^n = 4m\sum_{l=0}^{\ell}\alpha_l + 2^{4L+3/2}mC_1 + 4m + 2\delta_0(n).$$

It is easy to compute that

$$\varphi_j'(0) = \frac{w}{2}\big(A_j''(0) + mA_j'(0)^2\big).$$

We may also use Gaussian integration by parts and the given conditions on the first four derivatives to compute the second derivatives of φ_1, φ_2 and φ and this yields

$$\frac{1}{2}\max_{0\le u\le 1}\big(|\varphi_1''(u)| + |\varphi_2''(u)| + |\varphi''(u)|\big) \le C_\ell^{\ell+1}w^2,$$

where $C_\ell^{\ell+1}$ depends only on ℓ and K. Finally, we finish by using the mean value theorem and (6.39),

$$\begin{aligned}\varphi(1) &\le \varphi(0) + \varphi'(0) + \frac{1}{2}\max_{0\le u\le 1}|\varphi''(u)| \\ &\le \varphi_1(1) + \varphi_2(1) \\ &\quad - \sum_{n=0}^{\ell}\big(\alpha_n(1 - C_\ell^n w) + C_0 m w\delta_0(n)\big)F\big(x_1, x_2, (1-\delta_0(n))w + w_n\big) \\ &\quad + C_\ell^{\ell+1}w^2.\end{aligned}$$

□

PROOF OF PROPOSITION 12. Recall that Lemma 15 guarantees the existence of constants $C_0, C_\ell^0, \ldots, C_\ell^\ell$, which satisfy (6.7). From (6.25), we only need to prove that $C_\ell^{\ell+1}$ can be eliminated. To do this, let $\alpha_0, \ldots, \alpha_\ell \ge 0$ and let $C_\ell^{\ell+1}$ be obtained by using $K = C_0\omega + \max(\alpha_0, \ldots, \alpha_\ell)$ in Lemma 15. Let us keep $0 < m \le 1$, $t \ge 0$, $0 \le w \le \min(1/8, \omega, 1/2C_\ell^0)$, $w_0 = 0$, and $0 \le w_1, \ldots, w_\ell \le L$ fixed. We use $\varphi(x_1, x_2, w)$ to denote the left-hand side of (6.8). Recall the definition of $T(x, w)$ from (6.5) using A and m. Set $\delta_i = wi/N$ for $1 \le i \le N$. We claim that for large N, the following inequality holds:

$$\begin{aligned}\varphi(x_1, x_2, \delta_i) &\le T(x_1, \delta_i) + T(x_2, \delta_i) \\ &\quad - \sum_{n=0}^{\ell}\beta_{n,i}F\big(x_1, x_2, (1-\delta_0(n))\delta_i + w_n\big) + iC_\ell^{\ell+1}\delta_1^2\end{aligned} \tag{6.40}$$

for all $1 \le i \le N$, where

$$\beta_{n,i} = \delta_0(n)C_0 m\delta_1\sum_{j=1}^{i}\big(1 - C_\ell^0\delta_1\big)^{j-1} + \alpha_n\big(1 - C_\ell^n\delta_1\big)^i.$$

If $i = 1$, then (6.25) implies (6.40). Suppose that (6.40) holds for some i with $1 \le i < N$. Then by using the induction hypothesis,

$$\begin{aligned}
&\varphi(x_1, x_2, \delta_{i+1}) \\
&\quad = \frac{1+t}{m} \log E \exp \frac{m}{1+t}\big(\varphi(x_1 + y_1\sqrt{\delta_1}, x_2 + y_2\sqrt{\delta_1}, \delta_i)\big) \\
&\quad \le \frac{1+t}{m} \log E \exp \frac{m}{1+t} \\
&\qquad \times \Bigg(T(x_1 + y_1\sqrt{\delta_1}, \delta_i) + T(x_2 + y_2\sqrt{\delta_1}, \delta_i) \\
&\qquad\qquad - \sum_{n=0}^{\ell} \beta_{n,i} F\big(x_1 + y_1\sqrt{\delta_1}, x_2 + y_2\sqrt{\delta_1}, \big(1 - \delta_0(n)\big)\delta_i + w_n\big)\Bigg) \\
&\qquad + iC_\ell^{\ell+1}\delta_1^2.
\end{aligned} \tag{6.41}$$

Observe that from the definition $\beta_{n,i} \le C_0 wi/N + \alpha_n \le K$ for large N and $T(\cdot, \delta_i)$ satisfies $\mathcal{A}(m, \delta_{N-i}, C_1)$ since $0 \le w \le \omega$. Also, notice

$$\delta_1 = \frac{w}{N} \le \frac{1}{N} \min\left(\frac{1}{8}, \omega, \frac{1}{2C_\ell^0}\right) \le \min\left(\frac{1}{8}, \delta_{N-i}, \frac{1}{2C_\ell^0}\right).$$

Applying (6.25) to (6.41), we obtain

$$\begin{aligned}
&\varphi(x_1, x_2, \delta_{i+1}) \\
&\quad \le T(x_1, \delta_{i+1}) + T(x_2, \delta_{i+1}) \\
&\qquad - \sum_{n=0}^{\ell} \big(\delta_0(n)C_0 m\delta_1 + \beta_{n,i}\big(1 - C_\ell^n\delta_1\big)\big) F\big(x_1, x_2, \big(1 - \delta_0(n)\big)\delta_{i+1} + w_n\big) \\
&\qquad + (i+1)C_\ell^{\ell+1}\delta_1^2.
\end{aligned}$$

Since

$$\begin{aligned}
&\delta_0(n)C_0 m\delta_1 + \beta_{n,i}\big(1 - C_\ell^n\delta_1\big) \\
&\quad = \delta_0(n)C_0 m\delta_1 + \delta_0(n)C_0 m\delta_1 \sum_{j=1}^{i}\big(1 - C_\ell^0\delta_1\big)^j + \alpha_n\big(1 - C_\ell^n\delta_1\big)^{i+1} \\
&\quad = \delta_0(n)C_0 m\delta_1 \sum_{j=1}^{i+1}\big(1 - C_\ell^0\delta_1\big)^{j-1} + \alpha_n\big(1 - C_\ell^n\delta_1\big)^{i+1} \\
&\quad = \beta_{n,i+1},
\end{aligned}$$

this completes the proof of our claim. Letting $i = N$ in (6.40) and then $N \to \infty$, we obtain that

$$\varphi(x_1, x_2, w) \le T(x_1, w) + T(x_2, w) - \sum_{n=0}^{\ell} \big(\delta_0(n) C_0 m w \exp(-C_\ell^0 w) + \alpha_n \exp(-C_\ell^n w)\big) \times F\big(x_1, x_2, \big(1 - \delta_0(n)\big) w + w_n\big). \tag{6.42}$$

Since $\exp(-C_\ell^0 w) \ge 1/\sqrt{e} \ge 1/2$ for $0 \le w \le 1/2C_\ell^0$ and also $\exp(-C_\ell^n w) \ge 1 - C_\ell^n w$ using $\exp(-x) \ge 1 - x$ for $x \ge 0$, plugging these results inside (6.42), we are done. □

Acknowledgements. The author would like to thank Michel Talagrand for sharing a preliminary version of his book [17], which motivated the present paper. Also, he would like to thank an anonymous referee and Associate Editor for giving several valuable comments regarding the presentation of the paper.

REFERENCES

[1] AIZENMAN, M., SIMS, R. and STARR, S. (2003). An extended variational principle for the SK spin-glass model. *Phys. Rev. B* **68** 214403.

[2] BOLTHAUSEN, E. and SZNITMAN, A. S. (1998). On Ruelle's probability cascades and an abstract cavity method. *Comm. Math. Phys.* **197** 247–276. MR1652734

[3] BRARY, A. J. and MOORE, M. A. (1987). Chaotic nature of the spin-glass phase. *Phys. Rev. Lett.* **58** 57–60.

[4] CHATTERJEE, S. (2009). Disorder chaos and multiple valleys in spin glasses. Preprint. Available at arXiv:0907.3381.

[5] DERRIDA, B. (1981). Random-energy model: An exactly solvable model of disordered systems. *Phys. Rev. B* (3) **24** 2613–2626. MR0627810

[6] FISHER, D. S. and HUSE, D. A. (1986). Ordered phase of short range Ising spin glasses. *Phys. Rev. Lett.* **56** 1601–1604.

[7] GUERRA, F. (2003). Broken replica symmetry bounds in the mean field spin glass model. *Comm. Math. Phys.* **233** 1–12. MR1957729

[8] KATZGRABER, H. G. and KRZĄKAŁA, F. (2007). Temperature and disorder chaos in three-dimensional Ising spin glasses. *Phys. Rev. Lett.* **98** 017201.

[9] MCKAY, S. R., BERKER, A. N. and KIRKPATRICK, S. (1982). Spin-glass behavior in frustrated Ising models with chaotic renormalization-group trajectories. *Phys. Rev. Lett.* **48** 767–770. MR0647751

[10] PANCHENKO, D. and TALAGRAND, M. (2007). On the overlap in the multiple spherical SK models. *Ann. Probab.* **35** 2321–2355. MR2353390

[11] RUELLE, D. (1987). A mathematical reformulation of Derrida's REM and GREM. *Comm. Math. Phys.* **108** 225–239. MR0875300

[12] SHERRINGTON, D. and KIRKPATRICK, S. (1975). Solvable model of a spin glass. *Phys. Rev. Lett.* **35** 1792–1796.

[13] TALAGRAND, M. (2006). The Parisi formula. *Ann. of Math.* (2) **163** 221–263. MR2195134

[14] TALAGRAND, M. (2006). Parisi measures. *J. Funct. Anal.* **231** 269–286. MR2195333

[15] TALAGRAND, M. (2007). Mean field models for spin glasses: Some obnoxious problems. In *Spin Glasses*. *Lecture Notes in Math*. **1900** 63–80. Springer, Berlin. MR2309598

[16] TALAGRAND, M. (2011). *Mean Field Models for Spin Glasses*. *Volume I*: *Basic Examples*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete*. 3. *Folge*. *A Series of Modern Surveys in Mathematics* [*Results in Mathematics and Related Areas*. 3*rd Series*. *A Series of Modern Surveys in Mathematics*] **54**. Springer, Berlin. MR2731561

[17] TALAGRAND, M. (2011). *Mean Field Models for Spin Glasses*. *Volume II*: *Advanced Replica-Symmetry and Low Temperature*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete*. 3. *Folge*. *A Series of Modern Surveys in Mathematics* [*Results in Mathematics and Related Areas*. 3*rd Series*. *A Series of Modern Surveys in Mathematics*] **55**. Springer, Berlin. MR3024566

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