HITTING DISTRIBUTIONS OF α-STABLE PROCESSES VIA PATH CENSORING AND SELF-SIMILARITY

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We consider two first passage problems for stable processes, not necessarily symmetric, in one dimension. We make use of a novel method of path censoring in order to deduce explicit formulas for hitting probabilities, hitting distributions and a killed potential measure. To do this, we describe in full detail the Wiener–Hopf factorization of a new Lamperti-stable-type Lévy process obtained via the Lamperti transform, in the style of recent work in this area.

1. Introduction. A Lévy process is a stochastic process issued from the origin with stationary and independent increments and càdlàg paths. If $X := (X_t)_{t \geq 0}$ is a one-dimensional Lévy process with law P, then the classical Lévy–Khintchine formula states that for all $t \geq 0$ and $\theta \in \mathbb{R}$, the characteristic exponent $\Psi(\theta) := -t^{-1} \log \mathrm{E}(e^{i\theta X_t})$ satisfies

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbb{1}_{(|x| \leq 1)}) \Pi(dx),$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure (the *Lévy measure*) concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty$.

(X, P) is said to be a (*strictly*) *α-stable process* if it is a Lévy process which also satisfies the *scaling property*: under P, for every $c > 0$, the process $(cX_{tc^{-\alpha}})_{t \geq 0}$ has the same law as X. It is known that $\alpha \in (0, 2]$, and the case $\alpha = 2$ corresponds to Brownian motion, which we exclude. The Lévy–Khintchine representation of such a process is as follows: $\sigma = 0$, and Π is absolutely continuous with density given by

$$c_+ x^{-(\alpha+1)} \mathbb{1}_{(x>0)} + c_- |x|^{-(\alpha+1)} \mathbb{1}_{(x<0)}, \qquad x \in \mathbb{R},$$

where $c_+, c_- \geq 0$, and $c_+ = c_-$ when $\alpha = 1$. It holds that $a = (c_+ - c_-)/(\alpha - 1)$ when $\alpha \neq 1$, and we specify that $a = 0$ when $\alpha = 1$; the latter condition is a restric-

tion which ensures that X is a symmetric process when $\alpha = 1$, so the only 1-stable process we consider is the symmetric Cauchy process.

These choices mean that, up to a multiplicative constant $c > 0$, X has the characteristic exponent

$$\Psi(\theta) = \begin{cases} c|\theta|^{\alpha}\left(1 - i\beta\tan\dfrac{\pi\alpha}{2}\,\mathrm{sgn}\,\theta\right), & \alpha \in (0, 2) \setminus \{1\}, \\ c|\theta|, & \alpha = 1, \end{cases} \qquad \theta \in \mathbb{R},$$

where $\beta = (c_+ - c_-)/(c_+ + c_-)$. For more details, see Sato [31], Section 14.

For consistency with the literature we appeal to in this article, we shall always parameterize our α-stable process such that

$$c_+ = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} \quad \text{and} \quad c_- = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})},$$

where $\rho = P(X_t \geq 0) = P(X_t > 0)$ is the positivity parameter, and $\hat{\rho} = 1 - \rho$.

We take the point of view that the class of stable processes, with this normalization, is parameterized by α and ρ; the reader will note that all the quantities above can be written in terms of these parameters. We shall restrict ourselves a little further within this class by excluding the possibility of having only one-sided jumps. Together with our assumption about the case $\alpha = 1$, this gives us the following set of admissible parameters:

$$\mathcal{A} = \big\{(\alpha, \rho) : \alpha \in (0, 1), \rho \in (0, 1)\big\} \cup \big\{(\alpha, \rho) : \alpha \in (1, 2), \rho \in (1 - 1/\alpha, 1/\alpha)\big\}$$
$$\cup \big\{(\alpha, \rho) = (1, 1/2)\big\}.$$

After Brownian motion, α-stable processes are often considered an exemplary family of processes for which many aspects of the general theory of Lévy processes can be illustrated in closed form. First passage problems, which are relatively straightforward to handle in the case of Brownian motion, become much harder in the setting of a general Lévy process on account of the inclusion of jumps. A collection of articles through the 1960s and early 1970s, appealing largely to potential analytic methods for general Markov processes, were relatively successful in handling a number of first passage problems, in particular for symmetric α-stable processes in one or more dimensions. See, for example, [4, 14, 15, 26, 29] to name but a few.

However, following this cluster of activity, several decades have passed since new results on these problems have appeared. The last few years have seen a number of new, explicit first passage identities for one-dimensional α-stable processes, thanks to a better understanding of the intimate relationship between the aforesaid processes and positive self-similar Markov processes. See, for example, [6, 8, 10, 20, 22].

In this paper we return to the work of Blumenthal, Getoor and Ray [4], published in 1961, which gave the law of the position of first entry of a symmetric

α-stable process into the unit ball. Specifically, we are interested in establishing the same law, but now for all the one-dimensional α-stable processes which fall within the parameter regime $\mathcal{A}$; we remark that Port ([26], Section 3.1, Remark 3) found this law for processes with one-sided jumps, which justifies our exclusion of these processes in this work. Our method is modern in the sense that we appeal to the relationship of α-stable processes with certain positive self-similar Markov processes. However, there are two notable additional innovations. First, we make use of a type of path censoring. Second, we are able to describe in explicit analytical detail a nontrivial Wiener–Hopf factorization of an auxiliary Lévy process from which the desired solution can be sourced. Moreover, as a consequence of this approach, we are able to deliver a number of additional, related identities in explicit form for α-stable processes.

We now state the main results of the paper. Let P_x refer to the law of $X + x$ under P, for each $x \in \mathbb{R}$. We introduce the first hitting time of the interval $(-1, 1)$,

$$\tau^1_{-1} = \inf\{t > 0 : X_t \in (-1, 1)\}.$$

Note that, for $x \notin \{-1, 1\}$, $P_x(X_{\tau^1_{-1}} \in (-1, 1)) = 1$ so long as X is not spectrally one-sided. However, in Proposition 1.3, we will consider a spectrally negative α-stable process, for which $X_{\tau^1_{-1}}$ may take the value -1 with positive probability.

THEOREM 1.1. *Let $x > 1$. Then, when $\alpha \in (0, 1]$,*

$$P_x(X_{\tau^1_{-1}} \in dy, \tau^1_{-1} < \infty)/dy$$

$$= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x + 1)^{\alpha\rho}(x - 1)^{\alpha\hat{\rho}}(1 + y)^{-\alpha\rho}(1 - y)^{-\alpha\hat{\rho}}(x - y)^{-1}$$

for $y \in (-1, 1)$. When $\alpha \in (1, 2)$,

$$P_x(X_{\tau^1_{-1}} \in dy)/dy$$

$$= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x + 1)^{\alpha\rho}(x - 1)^{\alpha\hat{\rho}}(1 + y)^{-\alpha\rho}(1 - y)^{-\alpha\hat{\rho}}(x - y)^{-1}$$

$$- (\alpha - 1)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}(1 + y)^{-\alpha\rho}(1 - y)^{-\alpha\hat{\rho}}\int_1^x (t - 1)^{\alpha\hat{\rho}-1}(t + 1)^{\alpha\rho-1}\,dt$$

for $y \in (-1, 1)$.

When X is symmetric, Theorem 1.1 reduces immediately to Theorems B and C of [4]. Moreover, the following hitting probability can be obtained.

COROLLARY 1.2. *When $\alpha \in (0, 1)$, for $x > 1$,*

$$P_x(\tau^1_{-1} = \infty) = \frac{\Gamma(1 - \alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha)}\int_0^{(x-1)/(x+1)} t^{\alpha\hat{\rho}-1}(1 - t)^{-\alpha}\,dt.$$

This extends Corollary 2 of [4], as can be seen by differentiating and using the doubling formula ([17], 8.335.2) for the gamma function.

The spectrally one-sided case can be found as the limit of Theorem 1.1, as we now explain. The first part of the coming proposition is due to Port [26], but we re-state it for the sake of clarity.

PROPOSITION 1.3. *Let $\alpha \in (1, 2)$, and suppose that X is spectrally negative, that is, $\rho = 1/\alpha$. Then, the hitting distribution of $[-1, 1]$ is given by*

$$P_x(X_{\tau^1_{-1}} \in dy) = \frac{\sin \pi (\alpha - 1)}{\pi} (x - 1)^{\alpha - 1} (1 - y)^{1-\alpha} (x - y)^{-1} dy$$

$$+ \frac{\sin \pi (\alpha - 1)}{\pi} \int_0^{(x-1)/(x+1)} t^{\alpha - 2} (1 - t)^{1-\alpha} dt \, \delta_{-1}(dy)$$

for $x > 1$, $y \in [-1, 1]$, where δ_{-1} is the unit point mass at -1. Furthermore, for each $x > 1$ the measures on $[-1, 1]$ given in Theorem 1.1 converge weakly, as $\rho \to 1/\alpha$, to the limit above.

The following killed potential is also available.

THEOREM 1.4. *Let $\alpha \in (0, 1]$, $x > 1$ and $y > 1$. Then*

$$\mathrm{E}_x \int_0^{\tau^1_{-1}} \mathbb{1}_{(X_t \in dy)} \, dt / dy$$

$$= \begin{cases} \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \left(\dfrac{x - y}{2}\right)^{\alpha - 1} \displaystyle\int_1^{(1-xy)/(y-x)} (t - 1)^{\alpha\rho - 1} (t + 1)^{\alpha\hat{\rho} - 1} \, dt, \\ \quad 1 < y < x, \\ \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \left(\dfrac{y - x}{2}\right)^{\alpha - 1} \displaystyle\int_1^{(1-xy)/(x-y)} (t - 1)^{\alpha\hat{\rho} - 1} (t + 1)^{\alpha\rho - 1} \, dt, \\ \quad y > x. \end{cases}$$

To obtain the potential of the previous theorem for $x < -1$ and $y < -1$, one may easily appeal to duality. In the case that $x < -1$ and $y > 1$, one notes that

(1) $$\mathrm{E}_x \int_0^{\tau^1_{-1}} \mathbb{1}_{(X_t \in dy)} \, dt = \mathrm{E}_x \mathrm{E}_\Delta \int_0^{\tau^1_{-1}} \mathbb{1}_{(X_t \in dy)} \, dt,$$

where the quantity Δ is randomized according to the distribution of $X_{\tau^+_{-1}} \mathbb{1}_{(X_{\tau^+_{-1}} > 1)}$, with

$$\tau^+_{-1} = \inf\{t > 0 : X_t > -1\}.$$

Although the distribution of $X_{\tau^+_{-1}}$ is available from [30], and hence the right-hand side of (1) can be written down explicitly, it does not seem to be easy to find a convenient closed form expression for the corresponding potential density.

Regarding this potential, let us finally remark that our methods give an explicit expression for this potential even when $\alpha \in (1, 2)$, but again, there does not seem to be a compact expression for the density.

A further result concerns the first passage of X into the half-line $(1, \infty)$ before hitting zero. Let

$$\tau_1^+ = \inf\{t > 0 : X_t > 1\} \quad \text{and} \quad \tau_0 = \inf\{t > 0 : X_t = 0\}.$$

Recall that when $\alpha \in (0, 1]$, $P_x(\tau_0 = \infty) = 1$, while when $\alpha \in (1, 2)$, $P_x(\tau_0 < \infty) = 1$, for $x \neq 0$. In the latter case, we can obtain a hitting probability as follows.

THEOREM 1.5.   *Let $\alpha \in (1, 2)$. When $0 < x < 1$,*

$$P_x(\tau_0 < \tau_1^+) = (\alpha - 1)x^{\alpha-1} \int_1^{1/x} (t - 1)^{\alpha\rho - 1} t^{\alpha\hat\rho - 1} \, dt.$$

When $x < 0$,

$$P_x(\tau_0 < \tau_1^+) = (\alpha - 1)(-x)^{\alpha-1} \int_1^{1-1/x} (t - 1)^{\alpha\hat\rho - 1} t^{\alpha\rho - 1} \, dt.$$

It is not difficult to push Theorem 1.5 a little further to give the law of the position of first entry into $(1, \infty)$ on the event $\{\tau_1^+ < \tau_0\}$. Indeed, by the Markov property, for $x < 1$,

$$(2) \qquad \begin{aligned} P_x(X_{\tau_1^+} \in dy, \tau_1^+ < \tau_0) &= P_x(X_{\tau_1^+} \in dy) - P_x(X_{\tau_1^+} \in dy, \tau_0 < \tau_1^+) \\ &= P_x(X_{\tau_1^+} \in dy) - P_x(\tau_0 < \tau_1^+)P_0(X_{\tau_1^+} \in dy). \end{aligned}$$

Moreover, Rogozin [30] found that, for $x < 1$ and $y > 1$,

$$(3) \qquad P_x(X_{\tau_1^+} \in dy) = \frac{\sin(\pi\alpha\rho)}{\pi}(1 - x)^{\alpha\rho}(y - 1)^{-\alpha\rho}(y - x)^{-1} \, dy.$$

Hence substituting (3) together with the hitting probability from Theorem 1.5 into (2) yields the following corollary.

COROLLARY 1.6.   *Let $\alpha \in (1, 2)$. Then, when $0 < x < 1$,*

$$\begin{aligned} P_x(X_{\tau_1^+} &\in dy, \tau_1^+ < \tau_0)/du \\ &= \frac{\sin(\pi\alpha\rho)}{\pi}(1 - x)^{\alpha\rho}(y - 1)^{-\alpha\rho}(y - x)^{-1} \\ &\quad - (\alpha - 1)\frac{\sin(\pi\alpha\rho)}{\pi}x^{\alpha-1}(y - 1)^{-\alpha\rho}y^{-1}\int_1^{1/x}(t - 1)^{\alpha\rho - 1}t^{\alpha\hat\rho - 1} \, dt \end{aligned}$$

for $y > 1$. When $x < 0$,

$$P_x\big(X_{\tau_1^+} \in dy, \tau_1^+ < \tau_0\big)/dy$$

$$= \frac{\sin(\pi\alpha\rho)}{\pi}(1 - x)^{\alpha\rho}(y - 1)^{-\alpha\rho}(y - x)^{-1}$$

$$- (\alpha - 1)\frac{\sin(\pi\alpha\rho)}{\pi}(-x)^{\alpha - 1}(y - 1)^{-\alpha\rho}y^{-1}\int_1^{1 - 1/x}(t - 1)^{\alpha\hat{\rho} - 1}t^{\alpha\rho - 1}\,dt$$

for $y > 1$.

We conclude this section by giving an overview of the rest of the paper. In Section 2, we recall the Lamperti transform and discuss its relation to α-stable processes. In Section 3, we explain the operation which gives us the path-censored α-stable process Y, that is to say the α-stable process with the negative components of its path removed. We show that Y is a positive self-similar Markov process, and can therefore be written as the exponential of a time-changed Lévy process, say ξ. We show that the Lévy process ξ can be decomposed into the sum of a compound Poisson process and a so-called Lamperti-stable process. Section 4 is dedicated to finding the distribution of the jumps of this compound Poisson component, which we then use in Section 5 to compute in explicit detail the Wiener–Hopf factorization of ξ. Finally, we make use of the explicit nature of the Wiener–Hopf factorization in Section 6 to prove Theorems 1.1 and 1.4. There we also prove Theorem 1.5 via a connection with the process conditioned to stay positive.

2. Lamperti transform and Lamperti-stable processes.

A *positive self-similar Markov process (pssMp)* with *self-similarity index* $\alpha > 0$ is a standard Markov process $Y = (Y_t)_{t \geq 0}$ with filtration $(\mathscr{G}_t)_{t \geq 0}$ and probability laws $(P_x)_{x > 0}$, on $[0, \infty)$, which has 0 as an absorbing state and which satisfies the *scaling property*, that for every $x, c > 0$,

(4) $\qquad$ the law of $(cY_{tc^{-\alpha}})_{t \geq 0}$ under P_x is P_{cx}.

Here, we mean "standard" in the sense of [3], which is to say, $(\mathscr{G}_t)_{t \geq 0}$ is a complete, right-continuous filtration, and Y has càdlàg paths and is strong Markov and quasi-left-continuous.

In the seminal paper [25], Lamperti describes a one-to-one correspondence between pssMps and Lévy processes, which we now outline. It may be worth noting that we have presented a slightly different definition of pssMp from Lamperti; for the connection, see [34], Section 0.

Let $S(t) = \int_0^t (Y_u)^{-\alpha}\,du$. This process is continuous and strictly increasing until Y reaches zero. Let $(T(s))_{s \geq 0}$ be its inverse, and define

$$\xi_s = \log Y_{T(s)}, \qquad s \geq 0.$$

Then $\xi := (\xi_s)_{s \geq 0}$ is a Lévy process started at $\log x$, possibly killed at an independent exponential time; the law of the Lévy process and the rate of killing do not depend on the value of x. The real-valued process ξ with probability laws $(\mathbb{P}_y)_{y \in \mathbb{R}}$ is called the *Lévy process associated to Y*, or the *Lamperti transform of Y*.

An equivalent definition of S and T, in terms of ξ instead of Y, is given by taking $T(s) = \int_0^s \exp(\alpha \xi_u) \, du$ and S as its inverse. Then

$$(5) \qquad\qquad Y_t = \exp(\xi_{S(t)})$$

for all $t \geq 0$, and this shows that the Lamperti transform is a bijection.

Let $T_0 = \inf\{t > 0 : Y_t = 0\}$ be the first hitting time of the absorbing state zero. Then the large-time behavior of ξ can be described by the behavior of Y at T_0, as follows:

  (i) If $T_0 = \infty$ a.s., then ξ is unkilled and either oscillates or drifts to $+\infty$.
  (ii) If $T_0 < \infty$ and $Y_{T_0-} = 0$ a.s., then ξ is unkilled and drifts to $-\infty$.
  (iii) If $T_0 < \infty$ and $Y_{T_0-} > 0$ a.s., then ξ is killed.

It is proved in [25] that the events mentioned above satisfy a zero-one law independently of x, and so the three possibilites above are an exhaustive classification of pssMps.

Three concrete examples of positive self-similar Markov processes related to α-stable processes are treated in Caballero and Chaumont [6]. We present here the simplest case, namely that of the α-stable process absorbed at zero. To this end, let X be the α-stable process as defined in the Introduction, and let

$$\tau_0^- = \inf\{t > 0 : X_t \leq 0\}.$$

Denote by ξ^* the Lamperti transform of the pssMp $(X_t \mathbb{1}_{(t < \tau_0^-)})_{t \geq 0}$. Then ξ^* has Lévy density

$$(6) \qquad\qquad c_+ \frac{e^x}{(e^x - 1)^{\alpha+1}} \mathbb{1}_{(x>0)} + c_- \frac{e^x}{(1 - e^x)^{\alpha+1}} \mathbb{1}_{(x<0)}$$

and is killed at rate $c_-/\alpha = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}$.

We note here that in [6] the authors assume that X is symmetric when $\alpha = 1$, which motivates the same assumption in this paper.

3. The path-censored process and its Lamperti transform. We now describe the construction of the path-censored α-stable process that will lie at the heart of our analysis, show that it is a pssMp and discuss its Lamperti transform.

Henceforth, X, with probability laws $(P_x)_{x \in \mathbb{R}}$, will denote the α-stable process defined in the Introduction. Define the occupation time of $(0, \infty)$,

$$A_t = \int_0^t \mathbb{1}_{(X_s > 0)} \, ds,$$

and let $\gamma(t) = \inf\{s \geq 0 : A_s > t\}$ be its right-continuous inverse. Define a process $(\check{Y}_t)_{t \geq 0}$ by setting $\check{Y}_t = X_{\gamma(t)}$, $t \geq 0$. This is the process formed by erasing the negative components of X and joining up the gaps.

Write $(\mathscr{F}_t)_{t \geq 0}$ for the augmented natural filtration of X, and $\mathscr{G}_t = \mathscr{F}_{\gamma(t)}$, $t \geq 0$.

PROPOSITION 3.1. *The process $\check{Y}$ is strong Markov with respect to the filtration $(\mathscr{G}_t)_{t \geq 0}$ and satisfies the scaling property with self-similarity index α.*

PROOF. The strong Markov property follows directly from Rogers and Williams [28], Section III.21. Establishing the scaling property is a straightforward exercise. $\square$

We now make zero into an absorbing state. Define the stopping time

$$T_0 = \inf\{t > 0 : \check{Y}_t = 0\}$$

and the process

$$Y_t = \check{Y}_t \mathbb{1}_{(t < T_0)}, \qquad t \geq 0,$$

so that $Y := (Y_t)_{t \geq 0}$ is $\check{Y}$ absorbed at zero. We call the process Y with probability laws $(\mathrm{P}_x)_{x > 0}$ the *path-censored α-stable process*.

PROPOSITION 3.2. *The process Y is a pssMp with respect to the filtration $(\mathscr{G}_t)_{t \geq 0}$.*

PROOF. The scaling property follows from Proposition 3.1, and zero is evidently an absorbing state. It remains to show that Y is a standard process, and the only point which may be in doubt here is quasi-left-continuity. This follows from the Feller property, which in turn follows from scaling and the Feller property of X. $\square$

REMARK 3.3. The definition of Y via time-change and stopping at zero bears some resemblance to a number of other constructions:

(a) Bertoin's construction ([1], Section 3.1) of the Lévy process conditioned to stay positive. The key difference here is that, when a negative excursion is encountered, instead of simply erasing it, Bertion [1] patches the last jump from negative to positive onto the final value of the previous positive excursion.

(b) Bogdan, Burdzy and Chen's "censored stable process" for the domain $D = (0, \infty)$; see [5], in particular Theorem 2.1 and the preceding discussion. Here the authors suppress any jumps of a symmetric α-stable process X by which the process attempts to escape the domain, and kill the process if it reaches the boundary continuously.
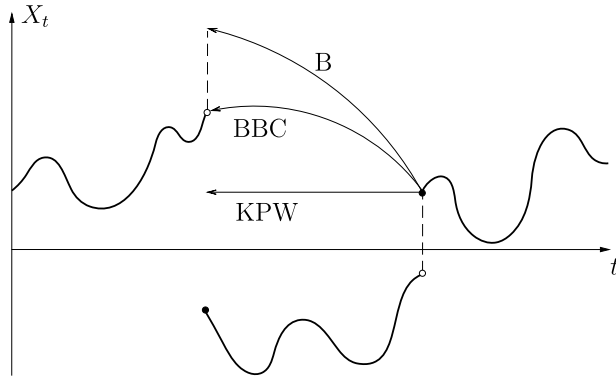
FIG. 1.   *The construction of three related processes from X, the stable process: "B" is the stable process conditioned to stay positive* [1]; *"BBC" is the censored stable process* [5]; *and "KPW" is the process Y in this work.*

Both processes (a) and (b) are also pssMps with index α. These processes, together with the process Y just described, therefore represent three choices of how to restart an α-stable process in a self-similar way after it leaves the positive half-line. We illustrate this in Figure 1.

We now consider the pssMp Y more closely for different values of $\alpha \in (0, 2)$. Taking account of Bertoin ([2], Proposition VIII.8) and the discussion immediately above it we know that for $\alpha \in (0, 1]$, points are polar for X. That is, $T_0 = \infty$ a.s., and so in this case $Y = \check{Y}$. Meanwhile, for $\alpha \in (1, 2)$, every point is recurrent, so $T_0 < \infty$ a.s. However, the process X makes infinitely many jumps across zero before hitting it. Therefore, in this case Y approaches zero continuously. In fact, it can be shown that, in this case, $\check{Y}$ is the *recurrent extension* of Y in the spirit of [27] and [13].

Now, let $\xi = (\xi_s)_{s \geq 0}$ be the Lamperti transform of Y. That is,

$$(7) \qquad\qquad\qquad \xi_s = \log Y_{T(s)}, \qquad s \geq 0,$$

where T is a time-change. As in Section 2, we will write $\mathbb{P}_y$ for the law of ξ started at $y \in \mathbb{R}$; note that $\mathbb{P}_y$ corresponds to $P_{\exp(y)}$. The space transformation (7), together with the above comments and, for instance, the remark on page 34 of [2], allows us to make the following distinction based on the value of α:

   (i) If $\alpha \in (0, 1)$, $T_0 = \infty$ and X (and hence Y) is transient a.s. Therefore, ξ is unkilled and drifts to $+\infty$.
   (ii) If $\alpha = 1$, $T_0 = \infty$ and every neighborhood of zero is an a.s. recurrent set for X, and hence also for Y. Therefore, ξ is unkilled and oscillates.
   (iii) If $\alpha \in (1, 2)$, $T_0 < \infty$ and Y hits zero continuously. Therefore, ξ is unkilled and drifts to $-\infty$.

Furthermore, we have the following result.

PROPOSITION 3.4.   *The Lévy process ξ is the sum of two independent Lévy processes ξ^L and ξ^C, which are characterized as follows*:

(i) *The Lévy process ξ^L has characteristic exponent*

$$\Psi^*(\theta) - c_-/\alpha, \qquad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of the process ξ^* defined in Section 2. That is, ξ^L is formed by removing the independent killing from ξ^*.

(ii) *The process ξ^C is a compound Poisson process whose jumps occur at rate c_-/α.*

Before beginning the proof, let us make some preparatory remarks. Let

$$\tau = \inf\{t > 0 : X_t < 0\} \quad \text{and} \quad \sigma = \inf\{t > \tau : X_t > 0\}$$

be hitting and return times of $(-\infty, 0)$ and $(0, \infty)$ for X. Note that, due to the time-change γ, $Y_\tau = X_\sigma$, while $Y_{\tau-} = X_{\tau-}$. We require the following lemma.

LEMMA 3.5.   *The joint law of $(X_\tau, X_{\tau-}, X_\sigma)$ under P_x is equal to that of $(x X_\tau, x X_{\tau-}, x X_\sigma)$ under P_1.*

PROOF.   This can be shown in a straightforward way using the scaling property.   □

PROOF OF PROPOSITION 3.4.   First we note that, applying the strong Markov property to the $(\mathcal{G}_t)_{t \geq 0}$-stopping time τ, it is sufficient to study the process $(Y_t)_{t \leq \tau}$.

It is clear that the path section $(Y_t)_{t < \tau}$ agrees with $(X_t)_{t < \tau_0^-}$; however, rather than being killed at time τ, the process Y jumps to a positive state. Recall now that the effect of the Lamperti transform on the time τ is to turn it into an exponential time of rate c_-/α which is independent of $(\xi_s)_{s < S(\tau)}$. This immediately yields the decomposition of ξ into the sum of $\xi^L := (\xi_s^L)_{s \geq 0}$ and $\xi^C := (\xi_s^C)_{s \geq 0}$, where ξ^C is a process which jumps at the times of a Poisson process with rate c_-/α, but whose jumps may depend on the position of ξ prior to this jump. What remains to be shown is that the values of the jumps of ξ^C are also independent of ξ^L.

By the remark at the beginning of the proof, it is sufficient to show that the first jump of ξ^C is independent of the previous path of ξ^L. Now, using only the independence of the jump times of ξ^L and ξ^C, we can compute

$$\begin{aligned}
\Delta Y_\tau := Y_\tau - Y_{\tau-} &= \exp(\xi_{S(\tau)}^L + \xi_{S(\tau)}^C) - \exp(\xi_{S(\tau)-}^L + \xi_{S(\tau)-}^C) \\
&= \exp(\xi_{S(\tau)-})[\exp(\Delta \xi_{S(\tau)}^C) - 1] \\
&= X_{\tau-}[\exp(\Delta \xi_{S(\tau)}^C) - 1],
\end{aligned}$$

where S is the Lamperti time change for Y, and $\Delta \xi_s^C = \xi_s^C - \xi_{s-}^C$. Now,

$$\exp(\Delta \xi_{S(\tau)}^C) = 1 + \frac{\Delta Y_\tau}{X_{\tau-}} = 1 + \frac{X_\sigma - X_{\tau-}}{X_{\tau-}} = \frac{X_\sigma}{X_{\tau-}}.$$

Hence, it is sufficient to show that $\frac{X_\sigma}{X_{\tau-}}$ is independent of $(X_t, t < \tau)$. The proof of this is essentially the same as that of part (iii) in Theorem 4 from Chaumont, Panti and Rivero [11], which we reproduce here for clarity.

First, observe that one consequence of Lemma 3.5 is that, for g a Borel function and $x > 0$,

$$\mathrm{E}_x \left[g\left(\frac{X_\sigma}{X_{\tau-}} \right) \right] = \mathrm{E}_1 \left[g\left(\frac{X_\sigma}{X_{\tau-}} \right) \right].$$

Now, fix $n \in \mathbb{N}$, f and g Borel functions and $s_1 < s_2 < \cdots < s_n = t$. Then, using the Markov property and the above equality,

$$\mathrm{E}_1 \left[f(X_{s_1}, \ldots, X_t) g\left(\frac{X_\sigma}{X_{\tau-}} \right) \mathbb{1}_{(t < \tau)} \right] = \mathrm{E}_1 \left[f(X_{s_1}, \ldots, X_t) \mathbb{1}_{(t < \tau)} \mathrm{E}_{X_t} \left[g\left(\frac{X_\sigma}{X_{\tau-}} \right) \right] \right]$$

$$= \mathrm{E}_1 \left[f(X_{s_1}, \ldots, X_t) \mathbb{1}_{(t < \tau)} \right] \mathrm{E}_1 \left[g\left(\frac{X_\sigma}{X_{\tau-}} \right) \right].$$

We have now shown that ξ^L and ξ^C are independent, and this completes the proof.
□

REMARK 3.6.   Let us consider the effect of the Lamperti transform on each of the pssMps in Remark 3.3. For the process conditioned to stay positive, the associated Lévy process is the process $\xi^\uparrow$ defined in Caballero and Chaumont [6]. As regards the censored stable process in $(0, \infty)$, we can reason as in the above proposition to deduce that its Lamperti transform is simply the process ξ^L which we have just defined.

4. Jump distribution of the compound Poisson component.   In this section, we express the jump distribution of ξ^C in terms of known quantities, and hence derive its characteristic function and density.

Before stating a necessary lemma, we establish some more notation. Let $\hat{X}$ be an independent copy of the dual process $-X$ and denote its probability laws by $(\hat{\mathrm{P}}_x)_{x \in \mathbb{R}}$, and let

$$\hat{\tau} = \inf\{t > 0 : \hat{X}_t < 0\}.$$

Furthermore, we shall denote by $\Delta \xi^C$ the random variable whose law is the same as the jump distribution of ξ^C.

LEMMA 4.1.   *The random variable* $\exp(\Delta \xi^C)$ *is equal in distribution to*

$$\left(-\frac{X_\tau}{X_{\tau-}} \right)(-\hat{X}_{\hat{\tau}}),$$

where X and $\hat{X}$ are taken to be independent with respective laws P_1 and $\hat{P}_1$.

PROOF. In the proof of Proposition 3.4, we saw that

$$\exp(\Delta \xi^C_{S(\tau)}) = \frac{X_\sigma}{X_{\tau-}}.$$
(8)

Applying the Markov property, and then using Lemma 3.5 with the α-self-similar process $\hat{X}$, we obtain

$$P_1(X_\sigma \in \cdot | \mathscr{F}_\tau) = \hat{P}_{-y}(-\hat{X}_{\hat{\tau}} \in \cdot)|_{y=X_\tau}$$
$$= \hat{P}_1(y\hat{X}_{\hat{\tau}} \in \cdot)|_{y=X_\tau}.$$

Then, by disintegration,

$$E_1\left[f\left(\frac{X_\sigma}{X_{\tau-}}\right)\right] = E_1\left[E_1\left[f\left(\frac{X_\sigma}{X_{\tau-}}\right)\Big|\mathscr{F}_\tau\right]\right] = E_1\left[\int f\left(\frac{x}{X_{\tau-}}\right)P_1[X_\sigma \in dx | \mathscr{F}_\tau]\right]$$
$$= E_1\left[\int f\left(\frac{x}{X_{\tau-}}\right)\hat{P}_1[y\hat{X}_{\hat{\tau}} \in dx]\Big|_{y=X_\tau}\right]$$
$$= E_1\left[\hat{E}_1\left[f\left(\frac{y\hat{X}_{\hat{\tau}}}{z}\right)\right]\Big|_{y=X_\tau, z=X_{\tau-}}\right]$$
$$= E_1 \otimes \hat{E}_1\left[f\left(\frac{X_\tau \hat{X}_{\hat{\tau}}}{X_{\tau-}}\right)\right].$$

Combining this with (8), we obtain that the law under P_1 of $\exp(\Delta \xi^C_{S(\tau)})$ is equal to that of $\frac{X_\tau \hat{X}_{\hat{\tau}}}{X_{\tau-}}$ under $P_1 \otimes \hat{P}_1$, which establishes the claim. □

The characteristic function of $\Delta \xi^C$ can now be found by rewriting the expression in Lemma 4.1 in terms of overshoots and undershoots of stable Lévy processes, whose marginal and joint laws are given in Rogozin [30] and Doney and Kyprianou [12]. Following the notation of [12], let

$$\tau^+_a = \inf\{t > 0 : X_t > a\},$$

and let $\hat{\tau}^+_a$ be defined similarly for $\hat{X}$.

PROPOSITION 4.2. *The characteristic function of the jump distribution of ξ^C is given by*

(9)
$$\mathbb{E}_0\left[\exp(i\theta \Delta \xi^C)\right]$$
$$= \frac{\sin(\pi \alpha \rho)}{\pi \Gamma(\alpha)} \Gamma(1 - \alpha\rho + i\theta)\Gamma(\alpha\rho - i\theta)\Gamma(1 + i\theta)\Gamma(\alpha - i\theta).$$

PROOF.   In the course of the coming computations, we will make use several times of the beta integral

$$\int_0^1 s^{x-1}(1-s)^{y-1}\,\mathrm{d}s = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \mathrm{Re}\,x,\,\mathrm{Re}\,y>0.$$

See, for example, [17], formulas 8.830.1–3.

Now, for $\theta\in\mathbb{R}$,

$$\hat{\mathrm{E}}_1(-\hat{X}_{\hat\tau})^{\mathrm{i}\theta} = \mathrm{E}_0(X_{\tau_1^+}-1)^{\mathrm{i}\theta} = \frac{\sin(\pi\alpha\rho)}{\pi}\int_0^\infty t^{\mathrm{i}\theta-\alpha\rho}(1+t)^{-1}\,\mathrm{d}t$$

(10)

$$= \frac{\sin(\pi\alpha\rho)}{\pi}\Gamma(1-\alpha\rho+\mathrm{i}\theta)\Gamma(\alpha\rho-\mathrm{i}\theta).$$

Furthermore,

$$\mathrm{E}_1\left(-\frac{X_\tau}{X_{\tau-}}\right)^{\mathrm{i}\theta} = \hat{\mathrm{E}}_0\left(\frac{\hat{X}_{\hat\tau_1^+}-1}{1-\hat{X}_{\hat\tau_1^+-}}\right)^{\mathrm{i}\theta}$$

(11)

$$= K\int_0^1\int_y^\infty\int_0^\infty \frac{u^{\mathrm{i}\theta}(1-y)^{\alpha\hat\rho-1}(v-y)^{\alpha\rho-1}}{v^{\mathrm{i}\theta}(v+u)^{1+\alpha}}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y,$$

where $K=\frac{\sin(\pi\alpha\hat\rho)}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}$. For the innermost integral above we have

$$\int_0^\infty \frac{u^{\mathrm{i}\theta}}{(u+v)^{1+\alpha}}\,\mathrm{d}u \overset{w=u/v}{=} v^{\mathrm{i}\theta-\alpha}\int_0^\infty \frac{w^{\mathrm{i}\theta}}{(1+w)^{1+\alpha}}\,\mathrm{d}w$$

$$= v^{\mathrm{i}\theta-\alpha}\frac{\Gamma(\mathrm{i}\theta+1)\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\alpha+1)}.$$

The next iterated integral in (11) becomes, substituting $z=v/y-1$,

$$\int_y^\infty v^{-\alpha}(v-y)^{\alpha\rho-1}\,\mathrm{d}v = y^{-\alpha\hat\rho}\int_0^\infty \frac{z^{\alpha\rho-1}}{(1+z)^\alpha}\,\mathrm{d}z = y^{-\alpha\hat\rho}\frac{\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)}{\Gamma(\alpha)},$$

and finally it remains to calculate

$$\int_0^1 y^{-\alpha\hat\rho}(1-y)^{\alpha\hat\rho-1}\,\mathrm{d}y = \Gamma(1-\alpha\hat\rho)\Gamma(\alpha\hat\rho).$$

Multiplying together these expressions and using the reflection identity $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$, we obtain

(12)
$$\mathrm{E}_1\left(-\frac{X_\tau}{X_{\tau-}}\right)^{\mathrm{i}\theta} = \frac{\Gamma(\mathrm{i}\theta+1)\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\alpha)}.$$

The result now follows from Lemma 4.1 by multiplying (10) and (12) together.
□

REMARK 4.3. The recent work of Chaumont, Panti and Rivero [11] on the so-called Lamperti–Kiu processes can be applied to give the same result. The quantity $\Delta \xi^C$ in the present work corresponds to the independent sum $\xi_\zeta^- + U^+ + U^-$ in that paper, where U^+ and U^- are "log-Pareto" random variables, and ξ^- is the Lamperti-stable process corresponding to $\hat{X}$ absorbed below zero; see [11], Corollary 11, for details. It is straightforward to show that the characteristic function of this sum is equal to the right-hand side of (9).

It is now possible to deduce the density of the jump distribution from its characteristic function. By substituting on the left and using the beta integral, it can be shown that

$$\int_{-\infty}^{\infty} e^{i\theta x} \alpha e^x (1 + e^x)^{-(\alpha+1)} \, dx = \frac{\Gamma(1 + i\theta)\Gamma(\alpha - i\theta)}{\Gamma(\alpha)},$$

$$\int_{-\infty}^{\infty} e^{i\theta x} \frac{\sin(\pi\alpha\rho)}{\pi} e^{(1-\alpha\rho)x} (1 + e^x)^{-1} \, dx = \frac{\sin(\pi\alpha\rho)}{\pi} \Gamma(\alpha\rho - i\theta)\Gamma(1 - \alpha\rho + i\theta),$$

and so the density of $\Delta \xi^C$ can be seen as the convolution of these two functions. Moreover, it is even possible to calculate this convolution directly,

$$
\begin{aligned}
&\mathbb{P}_0\big(\Delta \xi^C \in dx\big)/dx \\
(13) \quad &= \frac{\alpha}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} \int_{-\infty}^{\infty} e^u (1 + e^u)^{-(\alpha+1)} e^{(1-\alpha\rho)(x-u)} (1 + e^{x-u})^{-1} \, du \\
&= \frac{\alpha}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} e^{-\alpha\rho x} \int_0^{\infty} t^{\alpha\rho} (1 + t)^{-(\alpha+1)} (t e^{-x} + 1)^{-1} \, dt \\
&= \frac{\alpha\Gamma(\alpha\rho + 1)\Gamma(\alpha\hat{\rho} + 1)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)\Gamma(\alpha + 2)} e^{-\alpha\rho x} {}_2\mathcal{F}_1\big(1, \alpha\rho + 1; \alpha + 2; 1 - e^{-x}\big),
\end{aligned}
$$

where the final line follows from [17], formula 3.197.5, and is to be understood in the sense of analytic continuation when $x < 0$.

5. Wiener–Hopf factorization.

We begin with a brief sketch of the Wiener–Hopf factorization for Lévy processes, and refer the reader to [21], Chapter 6, or [2], Section VI.2, for further details, including proofs.

The Wiener–Hopf factorization describes the characteristic exponent of a Lévy process in terms of the Laplace exponents of two subordinators. For our purposes, a *subordinator* is defined as an increasing Lévy process, possibly killed at an independent exponentially distributed time and sent to the cemetery state $+\infty$. If H is a subordinator with expectation operator $\mathbb{E}$, we define its *Laplace exponent* ϕ by the equation

$$\mathbb{E}\big[\exp(-\lambda H_1)\big] = \exp(-\phi(\lambda)), \qquad \lambda \geq 0.$$

Standard theory allows us to analytically extend $\phi(\lambda)$ to $\{\lambda \in \mathbb{C} : \text{Re } \lambda \geq 0\}$. Similarly, let ξ be a Lévy process, again with expectation $\mathbb{E}$, and denote its characteristic exponent by Ψ, so that

$$\mathbb{E}\big[\exp(i\theta\xi_1)\big] = \exp(-\Psi(\theta)), \qquad \theta \in \mathbb{R}.$$

The *Wiener–Hopf factorization* of ξ consists of the decomposition

(14) $$k\Psi(\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta), \qquad \theta \in \mathbb{R},$$

where $k > 0$ is a constant which may, without loss of generality, be taken equal to unity, and the functions κ and $\hat{\kappa}$ are the Laplace exponents of certain subordinators which we denote H and $\hat{H}$.

Any decomposition of the form (14) is unique, up to the constant k, provided that the functions κ and $\hat{\kappa}$ are Laplace exponents of subordinators. The exponents κ and $\hat{\kappa}$ are termed the *Wiener–Hopf factors* of ξ.

The subordinator H can be identified in law as an appropriate time change of the running maximum process $\bar{\xi} := (\bar{\xi}_t)_{t \geq 0}$, where $\bar{\xi}_t = \sup\{\xi_s, s \leq t\}$. In particular, the range of H and $\bar{\xi}$ are the same. Similarly, $\hat{H}$ is equal in law to an appropriate time-change of $-\underline{\xi} := (-\underline{\xi})_{t \geq 0}$, with $\underline{\xi}_t = \inf\{\xi_s, s \leq t\}$, and they have the same range. Intuitively speaking, H and $\hat{H}$ keep track of how ξ reaches its new maxima and minima, and they are therefore termed the *ascending* and *descending ladder height processes* associated to ξ.

In Sections 5.4 and 5.5 we shall deduce in explicit form the Wiener–Hopf factors of ξ from its characteristic exponent. Analytically, we will need to distinguish the cases $\alpha \in (0, 1]$ and $\alpha \in (1, 2)$; in probabilistic terms, these correspond to the regimes where X cannot and can hit zero, respectively.

Accordingly, the outline of this section is as follows. We first introduce two classes of Lévy processes and two transformations of subordinators which will be used to identify the process ξ and the ladder processes $H, \hat{H}$. We then present two subsections with the same structure: first a theorem identifying the factorization and the ladder processes, and then a proposition collecting some further details of important characteristics of the ladder height processes, which will be used in the applications.

5.1. *Hypergeometric Lévy processes.* A process is said to be a *hypergeometric Lévy process* with parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ if it has characteristic exponent

$$\frac{\Gamma(1 - \beta + \gamma - i\theta)}{\Gamma(1 - \beta - i\theta)} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + i\theta)}{\Gamma(\hat{\beta} + i\theta)}, \qquad \theta \in \mathbb{R},$$

and the parameters lie in the admissible set

$$\{\beta \leq 1, \gamma \in (0, 1), \hat{\beta} \geq 0, \hat{\gamma} \in (0, 1)\}.$$

In Kuznetsov and Pardo [20] the authors derive the Lévy measure and Wiener–Hopf factorization of such a process, and show that the processes ξ^*, $\xi^{\uparrow}$ and $\xi^{\downarrow}$ of

Caballero and Chaumont [6] belong to this class; these are, respectively, the Lévy processes appearing in the Lamperti transform of the α-stable process absorbed at zero, conditioned to stay positive and conditioned to hit zero continuously.

5.2. *Lamperti-stable subordinators.* A *Lamperti-stable subordinator* is characterized by parameters in the admissible set

$$\{(q, \mathrm{a}, \beta, c, \mathrm{d}) : \mathrm{a} \in (0, 1), \beta \leq 1 + \mathrm{a}, q, c, \mathrm{d} \geq 0\},$$

and it is defined as the (possibly killed) increasing Lévy process with killing rate q, drift d and Lévy density

$$c \frac{e^{\beta x}}{(e^x - 1)^{\mathrm{a}+1}}, \qquad x > 0.$$

It is simple to see from [7], Theorem 3.1, that the Laplace exponent of such a process is given, for $\lambda \geq 0$, by

$$(15) \qquad \Phi(\lambda) = q + \mathrm{d}\lambda - c\Gamma(-\mathrm{a})\left(\frac{\Gamma(\lambda + 1 - \beta + \mathrm{a})}{\Gamma(\lambda + 1 - \beta)} - \frac{\Gamma(1 - \beta + \mathrm{a})}{\Gamma(1 - \beta)}\right).$$

5.3. *Esscher and $\mathcal{T}_\beta$ transformations and special Bernstein functions.* The Lamperti-stable subordinators just introduced will not be sufficient to identify the ladder processes associated to ξ in the case $\alpha \in (1, 2)$. We therefore introduce two transformations of subordinators in order to expand our repertoire of processes.

The first of these is the classical Esscher transformation, a generalization of the Cameron–Girsanov–Martin transformation of Brownian motion. The second, the $\mathcal{T}_\beta$ transformation, is more recent, but we will see that, in the cases we are concerned with, it is closely connected to the Esscher transform. We refer the reader to [21], Section 3.3, and [23], Section 2, respectively, for details.

The following result is classical.

LEMMA 5.1. *Let H be a subordinator with Laplace exponent ϕ, and let $\beta > 0$. Define the function*

$$\mathcal{E}_\beta \phi(\lambda) = \phi(\lambda + \beta) - \phi(\beta), \qquad \lambda \geq 0.$$

Then, $\mathcal{E}_\beta \phi$ is the Laplace exponent of a subordinator, known as the Esscher transform of H (or of ϕ).

The Esscher transform of H has no killing and the same drift coefficient as H, and if the Lévy measure of H is Π, then its Esscher transform has Lévy measure $e^{-\beta x} \Pi(\mathrm{d}x)$.

Before giving the next theorem, we need to introduce the notions of special Bernstein function and conjugate subordinators, first defined by Song and Vondraček [33]. Consider a function $\phi : [0, \infty) \to \mathbb{R}$, and define $\phi^* : [0, \infty) \to \mathbb{R}$ by

$$\phi^*(\lambda) = \lambda/\phi(\lambda).$$

The function ϕ is called a *special Bernstein function* if both ϕ and ϕ^* are the Laplace exponents of subordinators. In this case, ϕ and ϕ^* are said to be *conjugate* to one another, as are their corresponding subordinators.

PROPOSITION 5.2.   *Let H be a subordinator with Laplace exponent ϕ, and let $\beta > 0$. Define*

$$(16) \qquad \mathcal{T}_\beta \phi(\lambda) = \frac{\lambda}{\lambda + \beta} \phi(\lambda + \beta), \qquad \lambda \geq 0.$$

Then $\mathcal{T}_\beta \phi$ is the Laplace exponent of a subordinator with no killing and the same drift coefficient as H.

Furthermore, if ϕ is a special Bernstein function conjugate to ϕ^*, then $\mathcal{T}_\beta \phi$ is a special Bernstein function conjugate to

$$\mathcal{E}_\beta \phi^* + \phi^*(\beta).$$

PROOF.   The first assertion is proved in Gnedin ([16], page 124) as the result of a path transformation, and directly, for spectrally negative Lévy processes (from which the case of subordinators is easily extracted) in Kyprianou and Patie [23]. The killing rate and drift coefficient can be read off as $\mathcal{T}_\beta \phi(0)$ and $\lim_{\lambda \to \infty} \mathcal{T}_\beta \phi(\lambda)/\lambda$.

The second claim can be seen immediately by rewriting (16) as

$$\mathcal{T}_\beta \phi(\lambda) = \frac{\lambda}{\phi^*(\lambda + \beta)}$$

and observing that $\phi^*(\lambda + \beta) = \mathcal{E}_\beta \phi^*(\lambda) + \phi^*(\beta)$ for $\lambda \geq 0$. $\square$

5.4. *Wiener–Hopf factorization for $\alpha \in (0, 1]$.*

THEOREM 5.3 (Wiener–Hopf factorization).   (i) *When $\alpha \in (0, 1]$, the Wiener–Hopf factorization of ξ has components*

$$\kappa(\lambda) = \frac{\Gamma(\alpha\rho + \lambda)}{\Gamma(\lambda)}, \qquad \hat{\kappa}(\lambda) = \frac{\Gamma(1 - \alpha\rho + \lambda)}{\Gamma(1 - \alpha + \lambda)}, \qquad \lambda \geq 0.$$

Hence, ξ is a hypergeometric Lévy process with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1, \alpha\rho, 1 - \alpha, \alpha\hat{\rho}).$$

(ii) *The ascending ladder height process is a Lamperti-stable subordinator with parameters*

$$(q, \mathtt{a}, \beta, c, \mathtt{d}) = \left(0, \alpha\rho, 1, -\frac{1}{\Gamma(-\alpha\rho)}, 0\right).$$

(iii) *The descending ladder height process is a Lamperti-stable subordinator with parameters*

$$(q, \mathtt{a}, \beta, c, \mathtt{d}) = \left(\frac{\Gamma(1 - \alpha\rho)}{\Gamma(1 - \alpha)}, \alpha\hat{\rho}, \alpha, -\frac{1}{\Gamma(-\alpha\hat{\rho})}, 0 \right),$$

when $\alpha < 1$, *and*

$$(q, \mathtt{a}, \beta, c, \mathtt{d}) = \left(0, \alpha\hat{\rho}, \alpha, -\frac{1}{\Gamma(-\alpha\hat{\rho})}, 0 \right),$$

when $\alpha = 1$.

PROOF. First we compute Ψ^C and Ψ^L, the characteristic exponents of ξ^C and ξ^L. As Ψ^C is a compound Poisson process with jump rate c_-/α and jump distribution given by (9), we obtain, after using the reflection formula $\Gamma(x)\Gamma(1 - x) = \pi/\sin(\pi x)$, for $\theta \in \mathbb{R}$,

$$\Psi^C(\theta) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})}$$
$$\times \left(1 - \frac{\Gamma(1 - \alpha\rho + i\theta)\Gamma(\alpha\rho - i\theta)\Gamma(1 + i\theta)\Gamma(\alpha - i\theta)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)\Gamma(\alpha)} \right).$$

On the other hand, Kuznetsov and Pardo ([20], Theorem 1) provide an expression for the characteristic exponent Ψ^* of the Lamperti-stable process ξ^* from Section 2, and removing the killing from this gives us

$$\Psi^L(\theta) = \frac{\Gamma(\alpha - i\theta)}{\Gamma(\alpha\hat{\rho} - i\theta)} \frac{\Gamma(1 + i\theta)}{\Gamma(1 - \alpha\hat{\rho} + i\theta)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})}.$$

We can now compute, applying the reflection formula twice,

$$\Psi(\theta) = \Psi^L(\theta) + \Psi^C(\theta)$$
$$= \Gamma(\alpha - i\theta)\Gamma(1 + i\theta)$$
$$\times \left(\frac{1}{\Gamma(\alpha\hat{\rho} - i\theta)\Gamma(1 - \alpha\hat{\rho} + i\theta)} - \frac{\Gamma(1 - \alpha\rho + i\theta)\Gamma(\alpha\rho - i\theta)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \right)$$
$$= \Gamma(\alpha - i\theta)\Gamma(1 + i\theta)\Gamma(1 - \alpha\rho + i\theta)\Gamma(\alpha\rho - i\theta)$$
$$\times \left(\frac{\sin(\pi(\alpha\hat{\rho} - i\theta))\sin(\pi(\alpha\rho - i\theta))}{\pi^2} - \frac{\sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho)}{\pi^2} \right).$$

It may be proved, using product and sum identities for trigonometric functions, that

$$\sin\big(\pi(\alpha\hat{\rho} - i\theta)\big)\sin\big(\pi(\alpha\rho - i\theta)\big) + \sin(\pi i\theta)\sin\big(\pi(\alpha - i\theta)\big) = \sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho).$$

Again using the reflection formula twice, this leads to

$$\Psi(\theta) = \frac{\Gamma(\alpha - i\theta)\Gamma(1 + i\theta)}{\Gamma(1 + i\theta)\Gamma(-i\theta)} \frac{\Gamma(\alpha\rho - i\theta)\Gamma(1 - \alpha\rho + i\theta)}{\Gamma(\alpha - i\theta)\Gamma(1 - \alpha + i\theta)}$$

(17)

$$= \frac{\Gamma(\alpha\rho - i\theta)}{\Gamma(-i\theta)} \times \frac{\Gamma(1 - \alpha\rho + i\theta)}{\Gamma(1 - \alpha + i\theta)}.$$

Part (i) now follows by the uniqueness of the Wiener–Hopf factorization, once we have identified κ and $\hat{\kappa}$ as Laplace exponents of subordinators. Substituting the parameters in parts (ii) and (iii) into formula (15) for the Laplace exponent of a Lamperti-stable subordinator, and adding killing in the case of part (iii), completes the proof. □

PROPOSITION 5.4.   (i) *The process ξ has Lévy density*

$$\pi(x) = \begin{cases} -\dfrac{1}{\Gamma(1 - \alpha\hat{\rho})\Gamma(-\alpha\rho)} e^{-\alpha\rho x} {}_2\mathcal{F}_1\big(1 + \alpha\rho, 1; 1 - \alpha\hat{\rho}; e^{-x}\big), & x > 0, \\[4mm] -\dfrac{1}{\Gamma(1 - \alpha\rho)\Gamma(-\alpha\hat{\rho})} e^{(1-\alpha\rho)x} {}_2\mathcal{F}_1\big(1 + \alpha\hat{\rho}, 1; 1 - \alpha\rho; e^{x}\big), & x < 0. \end{cases}$$

(ii) *The ascending ladder height has Lévy density*

$$\pi_H(x) = -\frac{1}{\Gamma(-\alpha\rho)} e^{x}\big(e^{x} - 1\big)^{-(\alpha\rho+1)}, \qquad x > 0.$$

The ascending renewal measure $U(dx) = \mathbb{E}\int_0^\infty \mathbb{1}_{(H_t \in dx)}\, dt$ is given by

$$U(dx)/dx = \frac{1}{\Gamma(\alpha\rho)}\big(1 - e^{-x}\big)^{\alpha\rho - 1}, \qquad x > 0.$$

(iii) *The descending ladder height has Lévy density*

$$\pi_{\hat{H}}(x) = -\frac{1}{\Gamma(-\alpha\hat{\rho})} e^{\alpha x}\big(e^{x} - 1\big)^{-(\alpha\hat{\rho}+1)}, \qquad x > 0.$$

The descending renewal measure is given by

$$\hat{U}(dx)/dx = \frac{1}{\Gamma(\alpha\hat{\rho})}\big(1 - e^{-x}\big)^{\alpha\hat{\rho} - 1} e^{-(1-\alpha)x}, \qquad x > 0.$$

PROOF.   The Lévy density of ξ follows from [20], Proposition 1, and the expressions for π_H and $\pi_{\hat{H}}$ are obtained by substituting in Section 5.2. The renewal measures can be verified using the Laplace transform identity

$$\int_0^\infty e^{-\lambda x} U(dx) = 1/\kappa(\lambda), \qquad \lambda \geq 0,$$

and the corresponding identity for the descending ladder height. □

5.5. *Wiener–Hopf factorization for $\alpha \in (1, 2)$.*

THEOREM 5.5 (Wiener–Hopf factorization).   (i) *When $\alpha \in (1, 2)$, the Wiener–Hopf factorization of ξ has components*

$$\kappa(\lambda) = (\alpha - 1 + \lambda)\frac{\Gamma(\alpha\rho + \lambda)}{\Gamma(1 + \lambda)}, \qquad \hat{\kappa}(\lambda) = \lambda\frac{\Gamma(1 - \alpha\rho + \lambda)}{\Gamma(2 - \alpha + \lambda)}, \qquad \lambda \geq 0.$$

(ii) *The ascending ladder height process can be identified as the conjugate subordinator (see Section* 5.3*) to $\mathcal{T}_{\alpha-1}\psi^*$, where*

$$\psi^*(\lambda) = \frac{\Gamma(2 - \alpha + \lambda)}{\Gamma(1 - \alpha\hat{\rho} + \lambda)}, \qquad \lambda \geq 0,$$

is the Laplace exponent of a Lamperti-stable process. This Lamperti-stable process has parameters

$$(q, \mathtt{a}, \beta, c, \mathtt{d}) = \left(\frac{\Gamma(2 - \alpha)}{\Gamma(1 - \alpha\hat{\rho})}, 1 - \alpha\rho, \alpha\hat{\rho}, -\frac{1}{\Gamma(\alpha\rho - 1)}, 0\right).$$

(iii) *The descending ladder process is the conjugate subordinator to a Lamperti-stable process with Laplace exponent*

$$\phi^*(\lambda) = \frac{\Gamma(2 - \alpha + \lambda)}{\Gamma(1 - \alpha\rho + \lambda)}, \qquad \lambda \geq 0,$$

which has parameters

$$(q, \mathtt{a}, \beta, c, \mathtt{d}) = \left(\frac{\Gamma(2 - \alpha)}{\Gamma(1 - \alpha\rho)}, 1 - \alpha\hat{\rho}, \alpha\rho, -\frac{1}{\Gamma(\alpha\hat{\rho} - 1)}, 0\right).$$

PROOF.   Returning to the proof of Theorem 5.3(i), observe that the derivation of (17) does not depend on the value of α. However, the factorization for $\alpha \in (0, 1]$ does not apply when $\alpha \in (1, 2)$ because, for example, the expression for $\hat{\kappa}$ is equal to zero at $\alpha - 1 > 0$ which contradicts the requirement that it be the Laplace exponent of a subordinator.

Now, applying the identity $x\Gamma(x) = \Gamma(x + 1)$ to each denominator in that expression, we obtain for $\theta \in \mathbb{R}$,

$$\Psi(\theta) = (\alpha - 1 - \mathrm{i}\theta)\frac{\Gamma(\alpha\rho - \mathrm{i}\theta)}{\Gamma(1 - \mathrm{i}\theta)} \times \mathrm{i}\theta\frac{\Gamma(1 - \alpha\rho + \mathrm{i}\theta)}{\Gamma(2 - \alpha + \mathrm{i}\theta)}.$$

Once again, the uniqueness of the Wiener–Hopf factorization is sufficient to prove part (i) once we know that κ and $\hat{\kappa}$ are Laplace exponents of subordinators, and so we now prove (iii) and (ii), in that order.

To prove (iii), note that Example 2 in Kyprianou and Rivero [24] shows that ϕ^* is a special Bernstein function, conjugate to $\hat{\kappa}$. The fact that ϕ^* is the Laplace exponent of the given Lamperti-stable process follows, as before, by substituting the parameters in (iii) into (15).

For (ii), first observe that

$$\kappa(\lambda) = \lambda \frac{\alpha - 1 + \lambda}{\lambda} \frac{\Gamma(\alpha\rho + \lambda)}{\Gamma(1 + \lambda)} = \frac{\lambda}{\mathcal{T}_{\alpha-1}\psi^*(\lambda)}.$$

It follows again from [24], Example 2, that ψ^* is a special Bernstein function, and then Proposition 5.2 implies that $\mathcal{T}_{\alpha-1}\psi^*$ is also a special Bernstein function, conjugate to κ. The rest of the claim about ψ^* follows as for part (iii). □

REMARK 5.6. There is another way to view the ascending ladder height, which is often more convenient for calculation. Applying the second part of Proposition 5.2, we find that

$$\kappa(\lambda) = \mathcal{E}_{\alpha-1}\psi(\lambda) + \psi(\alpha - 1),$$

where ψ is conjugate to ψ^*. Hence, H can be seen as the Esscher transform of the subordinator conjugate to ψ^*, with additional killing.

PROPOSITION 5.7.   (i) *The process ξ has Lévy density*

$$\pi(x) = \begin{cases} -\dfrac{1}{\Gamma(1 - \alpha\hat{\rho})\Gamma(-\alpha\rho)} e^{-\alpha\rho x} {}_2\mathcal{F}_1\big(1 + \alpha\rho, 1; 1 - \alpha\hat{\rho}; e^{-x}\big), & x > 0, \\[2ex] -\dfrac{1}{\Gamma(1 - \alpha\rho)\Gamma(-\alpha\hat{\rho})} e^{(1-\alpha\rho)x} {}_2\mathcal{F}_1\big(1 + \alpha\hat{\rho}, 1; 1 - \alpha\rho; e^{x}\big), & x < 0. \end{cases}$$

(ii) *The ascending ladder height has Lévy density*

$$\pi_H(x) = \frac{(e^x - 1)^{-(\alpha\rho+1)}}{\Gamma(1 - \alpha\rho)} (\alpha - 1 + (1 - \alpha\hat{\rho})e^x), \qquad x > 0.$$

The ascending renewal measure $U(dx) = \mathbb{E}\int_0^\infty \mathbb{1}_{(H_t \in dx)}\, dt$ is given by

$$U(dx)/dx = e^{-(\alpha-1)x}\left[\frac{\Gamma(2 - \alpha)}{\Gamma(1 - \alpha\hat{\rho})} + \frac{1 - \alpha\rho}{\Gamma(\alpha\rho)}\int_x^\infty e^{\alpha\hat{\rho}z}(e^z - 1)^{\alpha\rho-2}\, dz\right]$$

for $x > 0$.

(iii) *The descending ladder height has Lévy density*

$$\pi_{\hat{H}}(x) = \frac{e^{(\alpha-1)x}(e^x - 1)^{-(\alpha\hat{\rho}+1)}}{\Gamma(1 - \alpha\hat{\rho})} (\alpha - 1 + (1 - \alpha\rho)e^x), \qquad x > 0.$$

The descending renewal measure is given by

$$\hat{U}(dx)/dx = \frac{\Gamma(2 - \alpha)}{\Gamma(1 - \alpha\rho)} + \frac{1 - \alpha\hat{\rho}}{\Gamma(\alpha\hat{\rho})}\int_x^\infty e^{\alpha\rho z}(e^z - 1)^{\alpha\hat{\rho}-2}\, dz, \qquad x > 0.$$

PROOF.   As before, we will prove (i), and then (iii) and (ii) in that order.

(i) When $\alpha \in (1, 2)$, the process ξ no longer falls in the class of hypergeometric Lévy processes. Therefore, although the characteristic exponent Ψ is the same as

it was in Proposition 5.4, we can no longer rely on [20], and need to calculate the Lévy density ourselves.

Multiplying the jump density (13) of ξ^C by c_-/α, we can obtain an expression for its Lévy density π^C in terms of a $_2\mathcal{F}_1$ function. When we apply the relations ([17], formulas 9.131.1–2), we obtain

$$\pi^C(x) = \begin{cases} -\dfrac{1}{\Gamma(1-\alpha\hat{\rho})\Gamma(-\alpha\rho)}e^{-\alpha\rho x}{}_2\mathcal{F}_1\big(1+\alpha\rho, 1; 1-\alpha\hat{\rho}; e^{-x}\big) \\ \quad + \dfrac{\Gamma(\alpha+1)}{\Gamma(1+\alpha\rho)\Gamma(-\alpha\rho)}e^{-\alpha x}{}_2\mathcal{F}_1\big(1+\alpha\hat{\rho}, \alpha+1; 1+\alpha\hat{\rho}; e^{-x}\big), \\ \quad x > 0, \\ -\dfrac{1}{\Gamma(1-\alpha\rho)\Gamma(-\alpha\hat{\rho})}e^{(1-\alpha\rho)x}{}_2\mathcal{F}_1\big(1+\alpha\hat{\rho}, 1; 1-\alpha\rho; e^{x}\big) \\ \quad - \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}e^{x}{}_2\mathcal{F}_1\big(1+\alpha\rho, \alpha+1; 1+\alpha\rho; e^{x}\big), \\ \quad x < 0. \end{cases}$$

Recall that $_2\mathcal{F}_1(a, b; a; z) = (1-z)^{-b}$. Then, comparing with (6), the equation reads

$$\pi^C(x) = \pi(x) - \pi^L(x), \qquad x \neq 0,$$

where π^L is the Lévy density of ξ^L. The claim then follows by the independence of ξ^C and ξ^L.

(iii) In [24], Example 2, the authors give the tail of the Lévy measure $\Pi_{\hat{H}}$, and show that it is absolutely continuous. The density $\pi_{\hat{H}}$ is obtained by differentiation.

In order to obtain the renewal measure, start with the following standard observation. For $\lambda \geq 0$,

$$(18) \qquad \int_0^\infty e^{-\lambda x}\hat{U}(\mathrm{d}x) = \frac{1}{\hat{\kappa}(\lambda)} = \frac{\phi^*(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda x}\overline{\Pi}_{\phi^*}(x)\,\mathrm{d}x,$$

where $\overline{\Pi}_{\phi^*}(x) = q_{\phi^*} + \Pi_{\phi^*}(x, \infty)$, and q_{ϕ^*} and Π_{ϕ^*} are, respectively, the killing rate and Lévy measure of the subordinator corresponding to ϕ^*. Comparing with Section 5.2, we have

$$q_{\phi^*} = \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\rho)},$$

$$\Pi_{\phi^*}(\mathrm{d}x)/\mathrm{d}x = -\frac{1}{\Gamma(\alpha\hat{\rho}-1)}e^{\alpha\rho x}(e^x - 1)^{\alpha\hat{\rho}-2}, \qquad x > 0,$$

and substituting these back into (18) leads immediately to the desired expression for $\hat{U}$.

(ii) To obtain the Lévy density, it is perhaps easier to use the representation of H as corresponding to a killed Esscher transform, noted in Remark 5.6. As in

part (iii), applying [24], Example 2, gives

$$\pi_\psi(x) = \frac{e^{(\alpha-1)x}(e^x - 1)^{-(\alpha\rho+1)}}{\Gamma(1 - \alpha\rho)}(\alpha - 1 + (1 - \alpha\hat{\rho})e^x), \qquad x > 0,$$

where π_ψ is the Lévy density corresponding to $\psi(\lambda) = \lambda/\psi^*(\lambda)$. The effect of the Esscher transform on the Lévy measure gives

$$\pi_H(x) = e^{-(\alpha-1)x}\pi_\psi(x), \qquad x > 0,$$

and putting everything together we obtain the required expression.

Emulating the proof of (iii), we calculate

$$\int_0^\infty e^{-\lambda x} U(\mathrm{d}x) = \frac{1}{\kappa(\lambda)} = \frac{\psi^*(\alpha - 1 + \lambda)}{\alpha - 1 + \lambda} = \int_0^\infty e^{-\lambda x} e^{-(\alpha-1)x} \overline{\Pi}_{\psi^*}(x)\,\mathrm{d}x,$$

using similar notation to previously, and the density of $\hat{U}$ follows. □

6. Proofs of main results. In this section, we use the Wiener–Hopf factorization of ξ to prove Theorems 1.1 and 1.4 and deduce Corollary 1.2. We then make use of a connection with the process conditioned to stay positive in order to prove Theorem 1.5.

Our method for proving each theorem will be to prove a corresponding result for the Lévy process ξ, and to relate this to the α-stable process X by means of the Lamperti transform and censoring. In this respect, the following observation is elementary but crucial. Let

$$\tau_0^b = \inf\{t > 0 : X_t \in (0, b)\}$$

be the first time at which X enters the interval $(0, b)$, where $b < 1$, and

$$S_a^- = \inf\{s > 0 : \xi_s < a\}$$

the first passage of ξ below the negative level a. Notice that if $e^a = b$, then

$$S_a^- < \infty \quad \text{and} \quad \xi_{S_a^-} \leq x \quad \Longleftrightarrow \quad \tau_0^b < \infty \quad \text{and} \quad X_{\tau_0^b} \leq e^x.$$

We will use this relationship several times.

Our first task is to prove Theorem 1.1. We split the proof into two parts, based on the value of α. In principle, the method which we use for $\alpha \in (0, 1]$ extends to the $\alpha \in (1, 2)$ regime; however, it requires the evaluation of an integral including the descending renewal measure. For $\alpha \in (1, 2)$ we have been unable to calculate this in closed form, and have instead used a method based on the Laplace transform. Conversely, the second method could be applied in the case $\alpha \in (0, 1]$; however, it is less transparent.

PROOF OF THEOREM 1.1, $\alpha \in (0, 1]$. We begin by finding a related law for ξ. By [2], Proposition III.2, for $a < 0$,

$$\mathbb{P}_0(\xi_{S_a^-} \in dw) = \mathbb{P}_0(-\hat{H}_{S_{-a}^+} \in dw)$$

$$= \int_{[0,-a]} \hat{U}(dz) \pi_{\hat{H}}(-w-z)\, dw.$$

Using the expressions obtained in Section 5 and changing variables,

$$\mathbb{P}_0(\xi_{S_a^-} \in dw)$$

$$= \frac{\alpha \hat{\rho} e^{-\alpha w}\, dw}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \int_0^{1-e^a} t^{\alpha\hat{\rho}-1}(e^{-w}-1-e^{-w}t)^{-\alpha\hat{\rho}-1}\, dt$$

$$(19) \qquad = \frac{\alpha\hat{\rho}\, dw}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} e^{-\alpha\rho w}(e^{-w}-1)^{-1}$$

$$\times \int_0^{(1-e^a)/(1-e^w)} s^{\alpha\hat{\rho}-1}(1-s)^{-\alpha\hat{\rho}-1}\, ds$$

$$= \frac{1}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}(1-e^a)^{\alpha\hat{\rho}} e^{(1-\alpha\rho)w}(1-e^w)^{-1}(e^a-e^w)^{-\alpha\hat{\rho}}\, dw,$$

where the last equality can be reached by [17], formula 8.391, and the formula $_2\mathcal{F}_1(a, b; a; z) = (1-z)^{-b}$.

Denoting by $f(a, w)$ the density on the right-hand side of (19), the relationship between $\xi_{S_a^-}$ and $X_{\tau_0^b}$ yields that

$$g(b, z) := P_1(X_{\tau_0^b} \in dz)/dz = z^{-1} f(\log b, \log z), \qquad b < 1, z \in (0, b).$$

Finally, using the scaling property we obtain

$$\frac{P_x(X_{\tau_{-1}^1} \in dy)}{dy}$$

$$= \frac{1}{x+1} g\left(\frac{2}{x+1}, \frac{y+1}{x+1}\right)$$

$$= \frac{1}{y+1} f\left(\log\left(\frac{2}{x+1}\right), \log\left(\frac{y+1}{x+1}\right)\right)$$

$$= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x+1)^{\alpha\rho}(x-1)^{\alpha\hat{\rho}}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}}(x-y)^{-1}$$

for $y \in (-1, 1)$. $\square$

PROOF OF THEOREM 1.1, $\alpha \in (1, 2)$. We begin with the "second factorization identity" ([21], Exercise 6.7) for the process ξ, adapted to passage below a level,

$$\int_0^\infty \int \exp(qa - \beta y)\mathbb{P}_0(a - \xi_{S_a^-} \in dy)\, da = \frac{\hat{\kappa}(q) - \hat{\kappa}(\beta)}{(q-\beta)\hat{\kappa}(q)}, \qquad a < 0, q, \beta > 0.$$

A lengthy calculation, which we omit, inverts the two Laplace transforms to give the overshoot distribution for ξ,

$$
\begin{aligned}
f(a, w) &:= \frac{\mathbb{P}_0(a - \xi_{S_a^-} \in \mathrm{d}w)}{\mathrm{d}w} \\
&= \frac{\sin(\pi\alpha\hat{\rho})}{\pi} e^{-(1-\alpha\rho)w}(1 - e^{-w})^{-\alpha\hat{\rho}} \\
&\quad \times \left[e^{(1-\alpha)a}(1 - e^a)^{\alpha\hat{\rho}} e^{-w}(e^{-a} - e^{-w})^{-1} \right. \\
&\quad \left. - (\alpha\rho - 1)\int_0^{1-e^a} t^{\alpha\hat{\rho}-1}(1 - t)^{1-\alpha}\,\mathrm{d}t \right]
\end{aligned}
$$

for $a < 0$, $w > 0$. Essentially the same argument as in the $\alpha \in (0, 1]$ case gives the required hitting distribution for X,

(20)
$$
\begin{aligned}
\frac{\mathrm{P}_x(X_{\tau_{-1}^1} \in \mathrm{d}y)}{\mathrm{d}y} &= \frac{1}{y+1} f\left(\log\left(\frac{2}{x+1}\right), \log\left(\frac{2}{y+1}\right)\right) \\
&= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(1 + y)^{-\alpha\rho}(1 - y)^{-\alpha\hat{\rho}} \\
&\quad \times \left[(y + 1)(x - 1)^{\alpha\hat{\rho}}(x + 1)^{\alpha\rho-1}(x - y)^{-1} \right. \\
&\quad \left. - (\alpha\rho - 1)2^{\alpha-1}\int_0^{(x-1)/(x+1)} t^{\alpha\hat{\rho}-1}(1 - t)^{1-\alpha}\,\mathrm{d}t \right]
\end{aligned}
$$

for $x > 1$, $y \in (-1, 1)$.

By the substitution $t = \frac{s-1}{s+1}$,

$$
\begin{aligned}
2^{\alpha-1}&\int_0^{(x-1)/(x+1)} t^{\alpha\hat{\rho}-1}(1 - t)^{1-\alpha}\,\mathrm{d}t \\
&= 2\int_1^x (s - 1)^{\alpha\hat{\rho}-1}(s + 1)^{\alpha\rho-2}\,\mathrm{d}s \\
&= \int_1^x (s - 1)^{\alpha\hat{\rho}-1}(s + 1)^{\alpha\rho-1}\,\mathrm{d}s - \int_1^x (s - 1)^{\alpha\hat{\rho}}(s + 1)^{\alpha\rho-2}\,\mathrm{d}s.
\end{aligned}
$$

Now evaluating the second term on the right-hand side above via integration by parts and substituting back into (20) yields the required law. $\square$

REMARK 6.1. It is worth noting that in recent work of Kuznetsov, Kyprianou and Pardo [19], the law of the position of first entry of a so-called Meromorphic Lévy process into an interval was computed as a convergent series of exponential densities by solving a pair of simultaneous nonlinear equations; see Rogozin [29] for the original use of this method in the context of first passage of α-stable

processes when exiting a finite interval. In principle the method of solving two simultaneous nonlinear equations [i.e., writing the law of first entry in $(-1, 1)$ from $x > 1$ in terms of the law of first entry in $(-1, 1)$ from $x < -1$ and vice-versa] may provide a way of proving Theorem 1.1. However it is unlikely that this would present a more convenient approach because of the complexity of the two nonlinear equations involved and because of the issue of proving uniqueness of their solution. Finally we note that Kadankova and Veraverbeke [18] also consider the formalities of this approach when dealing with first entry into a finite interval for Lévy processes.

PROOF OF COROLLARY 1.2.   This will follow by integrating out Theorem 1.1. First making the substitutions $z = (y + 1)/2$ and $w = \frac{1-z}{1-2z/(x+1)}$, we obtain

$$P_x(\tau^1_{-1} < \infty)$$

$$= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x + 1)^{\alpha\rho}(x - 1)^{\alpha\hat{\rho}} \int_{-1}^1 (1 + u)^{-\alpha\rho}(1 - u)^{-\alpha\hat{\rho}}(x - u)^{-1} \, du$$

$$= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x + 1)^{\alpha\rho}(x - 1)^{\alpha\hat{\rho}}2^{1-\alpha} \int_0^1 z^{-\alpha\rho}(1 - z)^{-\alpha\hat{\rho}}\left(1 - \frac{2}{x+1}z\right)^{-1} \, dz$$

$$= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}\left(\frac{2}{x+1}\right)^{1-\alpha} \int_0^1 w^{-\alpha\hat{\rho}}(1 - w)^{-\alpha\rho}\left(1 - \frac{2}{x+1}w\right)^{\alpha-1} \, dw$$

$$= \frac{\Gamma(1 - \alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha)} \int_0^{2/(x+1)} s^{-\alpha}(1 - s)^{\alpha\hat{\rho}-1} \, ds,$$

where the last line follows by [17], formulas 3.197.3, 8.391. Finally, substituting $t = 1 - s$, it follows that

$$P_x(\tau^1_{-1} = \infty) = \frac{\Gamma(1 - \alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha)} \int_0^{(x-1)/(x+1)} t^{\alpha\hat{\rho}-1}(1 - t)^{-\alpha} \, dt,$$

and this was our aim. $\square$

PROOF OF PROPOSITION 1.3.   In Port ([26], Section 3, Remark 3), the author establishes, for $s > 0$, the hitting distribution of $[0, s]$ for a spectrally positive α-stable process started at $x < 0$. In our situation, we have a spectrally negative α-stable process X, and so the dual process $\hat{X}$ is spectrally positive,

$$P_x(X_{\tau^1_{-1}} \in dy) = \hat{P}_{1-x}(\hat{X}_{\tau^2_0} \in 1 - dy)$$

$$= f_{1-x}(1 - y)\,dy + \gamma(1 - x)\delta_{-1}(dy),$$

using the notation from [26] in the final line. Port gives expressions for f_{1-x} and γ which differ somewhat from the density and atom seen in our Proposition 1.3; our expression

$$f_{1-x}(1 - y) = \frac{\sin(\pi(\alpha - 1))}{\pi}(x - 1)^{\alpha-1}(1 - y)^{1-\alpha}(x - y)^{-1}\mathbb{1}_{(-1,1)}(y)$$

is obtained from Port's by evaluating an integral, and one may compute $\gamma(1 - x)$ similarly.

We now prove weak convergence. For this purpose, identity (20) is more convenient than the final expression in Theorem 1.1. Let us denote the right-hand side of (20), treated as the density of a measure on $[-1, 1]$, by the function $g_\rho: [-1, 1] \to \mathbb{R}$, so that

$$
\begin{aligned}
g_\rho(y) = {} & \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x - 1)^{\alpha\hat{\rho}}(x + 1)^{\alpha\rho-1}(1 + y)^{1-\alpha\rho}(1 - y)^{-\alpha\hat{\rho}} \\
& + (1 - \alpha\rho)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}2^{\alpha-1}(1 + y)^{-\alpha\rho}(1 - y)^{-\alpha\hat{\rho}} \\
& \times \int_0^{(x-1)/(x+1)} t^{\alpha\hat{\rho}-1}(1 - t)^{1-\alpha}\,\mathrm{d}t
\end{aligned}
$$

for $y \in (-1, 1)$, and we set $g_\rho(-1) = g_\rho(1) = 0$ for definiteness.

As we take the limit $\rho \to 1/\alpha$, $g_\rho(y)$ converges pointwise to $f_{1-x}(1 - y)$. Furthermore, the functions g_ρ are dominated by a function $h: [-1, 1] \to \mathbb{R}$ of the form

$$
h(y) = C(1 - y)^{1-\alpha}(x - y)^{-1} + D(1 + y)^{-1}(1 - y)^{1-\alpha}, \qquad y \in (-1, 1),
$$

for some $C, D \geq 0$ depending only on x and α; again we set $h(-1) = h(1) = 0$.

Let $z > -1$. The function h is integrable on $[z, 1]$, and therefore dominated convergence yields

$$
\int_{[z,1]} g_\rho(y)\,\mathrm{d}y \to \int_{[z,1]} f_{1-x}(1 - y)\,\mathrm{d}y = \mathrm{P}_x(X_{\tau_{-1}^1} \geq z),
$$

while

$$
\int_{[-1,1]} g_\rho(y)\,\mathrm{d}y = 1 = \mathrm{P}_x(X_{\tau_{-1}^1} \geq -1),
$$

and this is sufficient for weak convergence. $\square$

PROOF OF THEOREM 1.4.   We begin by determining a killed potential for ξ. Let

$$
u(p, w)\,\mathrm{d}w = \mathbb{E}_p \int_0^{S_0^-} \mathbb{1}_{(\xi_s \in \mathrm{d}w)}\,\mathrm{d}s, \qquad p, w > 0,
$$

if this density exists. Using an identity of Silverstein (see Bertoin [2], Theorem VI.20, or Silverstein [32], Theorem 6), and the fact that the renewal measures of ξ are absolutely continuous, we find that the density $u(p, \cdot)$ does exist, and

$$
u(p, w) = \begin{cases} \displaystyle\int_{p-w}^p \hat{v}(z)v(w + z - p)\,\mathrm{d}z, & 0 < w < p, \\ \displaystyle\int_0^p \hat{v}(z)v(w + z - p)\,\mathrm{d}z, & w > p, \end{cases}
$$

where v and $\hat{v}$ are the ascending and descending renewal densities from Proposition 5.4. For $w > p$,

$$u(p, w) = \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_0^p (1 - e^{-z})^{\alpha\hat{\rho}-1} e^{(1-\alpha)z} (1 - e^{p-w}e^{-z})^{\alpha\rho-1} \, dz$$

$$= \frac{(1 - e^{p-w})^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \left(\frac{1 - e^{-p}}{1 - e^{-w}}\right)^{\alpha\hat{\rho}} \int_0^1 s^{\alpha\hat{\rho}-1} \left(1 - \frac{1 - e^{-p}}{1 - e^{-w}}s\right)^{-\alpha} \, ds,$$

where we have used the substitution $\frac{1-e^{-z}}{1-e^{-p}} = s(1 - q + qs)^{-1}$ with $q = \frac{e^{-p}-1}{e^{w-p}-1}$. Finally we conclude that

$$u(p, w) = \frac{(e^{p-w} - 1)^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_0^{(1-e^{-w})/(1-e^{-p})} t^{\alpha\hat{\rho}-1}(1 - t)^{-\alpha} \, dt, \qquad w > p.$$

The calculation for $0 < w < p$ is very similar, and in summary we have

$$u(p, w) = \begin{cases} \dfrac{(e^{p-w} - 1)^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \displaystyle\int_0^{(1-e^{-w})/(1-e^{-p})} t^{\alpha\rho-1}(1 - t)^{-\alpha} \, dt, \\ \qquad 0 < w < p, \\ \dfrac{(1 - e^{p-w})^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \displaystyle\int_0^{(1-e^{-p})/(1-e^{-w})} t^{\alpha\hat{\rho}-1}(1 - t)^{-\alpha} \, dt, \\ \qquad w > p. \end{cases}$$

We can now start to calculate the killed potential for X. Let

$$\bar{u}(b, z) \, dz = E_1 \int_0^{\tau_0^b} \mathbb{1}_{(X_t \in dz)} \, dt, \qquad 0 < b < 1, z > b.$$

Let us recall now the censoring method and the Lamperti transform described in Section 3. We defined $dA_t = \mathbb{1}_{(X_t > 0)} \, dt$, denoted by γ the right-inverse of A, and defined $Y_t = X_{\gamma(t)}\mathbb{1}_{(t < T_0)}$ for $t \geq 0$. Furthermore, from the Lamperti transform, $dt = \exp(\alpha\xi_{S(t)}) \, dS(t)$, where S is the Lamperti time change. As before, we write T for the inverse time-change to S. Finally, the measure P_x for the stable process X (and the pssMp Y) corresponds under the Lamperti transform to the measure $\mathbb{P}_{\log x}$; in particular, P_1 corresponds to $\mathbb{P}_0$, and E_1 to $\mathbb{E}_0$.

With this in mind, we make the following calculation:

$$\bar{u}(b, z) \, dz = E_1 \int_0^{\tau_0^b(X)} \mathbb{1}_{(X_t \in dz)} \, dA_t = E_1 \int_0^{\tau_0^b(Y)} \mathbb{1}_{(Y_t \in dz)} \, dt$$

$$= \mathbb{E}_0 \int_0^{T(S_a^-)} \mathbb{1}_{(\exp(\xi_{S(t)}) \in dz)} \exp(\alpha\xi_{S(t)}) \, dS(t)$$

$$= z^\alpha \mathbb{E}_0 \int_0^{S_a^-} \mathbb{1}_{(\exp(\xi_s) \in dz)} \, ds = z^\alpha \mathbb{E}_{-a} \int_0^{S_0^-} \mathbb{1}_{(\exp(\xi_s+a) \in dz)} \, ds,$$

where $a = \log b$, and, for clarity, we have written $\tau_0^b(Z)$ for the hitting time of $(0, b)$ calculated for a process Z. Hence,

$$\bar{u}(b, z) = z^{\alpha-1} u\big(\log b^{-1}, \log b^{-1} z\big), \qquad 0 < b < 1, z > b.$$

Finally, a scaling argument yields the following. For $x \in (0, 1)$ and $y > 1$,

$$\mathrm{E}_x \int_0^{\tau_{-1}^1} \mathbb{1}_{(X_t \in \mathrm{d}y)} \, \mathrm{d}t / \mathrm{d}y$$

$$= (x + 1)^{\alpha-1} \bar{u}\left(\frac{2}{x+1}, \frac{y+1}{x+1}\right)$$

$$= (y + 1)^{\alpha-1} u\left(\log\frac{x+1}{2}, \log\frac{y+1}{2}\right)$$

$$= \begin{cases} \dfrac{(x-y)^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \displaystyle\int_0^{((y-1)/(y+1))((x+1)/(x-1))} t^{\alpha\rho-1}(1-t)^{-\alpha} \, \mathrm{d}t, \\ \qquad 1 < y < x, \\ \dfrac{(y-x)^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \displaystyle\int_0^{((y+1)/(y-1))((x-1)/(x+1))} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha} \, \mathrm{d}t, \\ \qquad y > x. \end{cases}$$

The integral substitution $t = \frac{s-1}{s+1}$ gives the form in the theorem. $\square$

We now turn to the problem of first passage upward before hitting a point. To tackle this problem, we will use the *stable process conditioned to stay positive*. This process has been studied by a number of authors; for a general account of conditioning to stay positive, see, for example, Chaumont and Doney [9]. If X is the standard α-stable process defined in the Introduction and

$$\tau_0^- = \inf(t \geq 0 : X_t < 0)$$

is the first passage time below zero, then the process conditioned to stay positive, denoted $X^\uparrow$, with probability laws $(\mathrm{P}_x^\uparrow)_{x>0}$, is defined as the Doob h-transform of the killed process $(X_t \mathbb{1}_{(t < \tau_0^-)}, t \geq 0)$ under the invariant function

$$h(x) = x^{\alpha\hat{\rho}}.$$

That is, if T is any a.s. finite stopping time, Z an $\mathscr{F}_T$-measurable random variable, and $x > 0$, then

$$\mathrm{E}_x^\uparrow(Z) = \mathrm{E}_x\left[Z\frac{h(X_T)}{h(x)}, T < \tau_0^-\right].$$

In fact we will make use of this construction for the dual process $\hat{X}$, with invariant function $\hat{h}(x) = x^{\alpha\rho}$, and accordingly we will denote the conditioned process

by $\hat{X}^{\uparrow}$ and use $(\hat{P}_x^{\uparrow})_{x>0}$ for its probability laws. It is known that the process $\hat{X}^{\uparrow}$ is a strong Markov process which drifts to $+\infty$.

Caballero and Chaumont [6] show that the process $\hat{X}^{\uparrow}$ is a pssMp, and so we can apply the Lamperti transform to it. We will denote the Lévy process associated to $\hat{X}^{\uparrow}$ by $\hat{\xi}^{\uparrow}$ with probability laws $(\hat{\mathbb{P}}_y^{\uparrow})_{y>0}$. The crucial observation here is that $\hat{X}^{\uparrow}$ hits the point 1 if and only if its Lamperti transform, $\hat{\xi}^{\uparrow}$, hits the point 0.

We now have all the apparatus in place to begin the proof.

PROOF OF THEOREM 1.5. For each $y \in \mathbb{R}$, let τ_y be the first hitting time of the point y, and let τ_y^+ and τ_y^- be the first hitting times of the sets (y, ∞) and $(-\infty, y)$, respectively. When $\alpha \in (1, 2)$, these are all a.s. finite stopping times for the α-stable process X and its dual $\hat{X}$. Then, when $x \in (-\infty, 1)$,

$$P_x(\tau_0 < \tau_1^+) = P_{x-1}(\tau_{-1} < \tau_0^+) = \hat{P}_{1-x}(\tau_1 < \tau_0^-)$$

(21)
$$= \hat{h}(1-x)\hat{E}_{1-x}\left[\mathbb{1}_{(\tau_1<\infty)}\frac{\hat{h}(\hat{X}_{\tau_1})}{\hat{h}(1-x)}, \tau_1 < \tau_0^-\right]$$

$$= (1-x)^{\alpha\rho}\hat{P}_{1-x}^{\uparrow}(\tau_1 < \infty),$$

where we have used the definition of $\hat{P}_{\cdot}^{\uparrow}$ at τ_1. (Note that, to unify notation, the various stopping times refer to the canonical process for each measure.)

We now use facts coming from Bertoin [2], Proposition II.18 and Theorem II.19. Provided that the potential measure $U = \hat{E}_0^{\uparrow} \int_0^{\infty} \mathbb{1}_{(\hat{\xi}^{\uparrow} \in \cdot)} \, dt$ is absolutely continuous and there is a bounded continuous version of its density, say u, then the following holds:

(22)
$$\hat{P}_{1-x}^{\uparrow}(\tau_1 < \infty) = \hat{\mathbb{P}}_{\log(1-x)}^{\uparrow}(\tau_0 < \infty) = Cu(-\log(1-x)),$$

where C is the capacity of $\{0\}$ for the process $\hat{\xi}^{\uparrow}$.

Therefore, we have reduced our problem to that of finding a bounded, continuous version of the potential density of $\hat{\xi}^{\uparrow}$ under $\hat{\mathbb{P}}_0^{\uparrow}$. Provided the renewal measures of $\hat{\xi}^{\uparrow}$ are absolutely continuous, it is readily deduced from Silverstein's identity ([2], Theorem VI.20) that a potential density u exists and is given by

$$u(y) = \begin{cases} k \displaystyle\int_0^{\infty} v(y+z)\hat{v}(z) \, dz, & y > 0, \\ k \displaystyle\int_{-y}^{\infty} v(y+z)\hat{v}(z) \, dz, & y < 0, \end{cases}$$

where v and $\hat{v}$ are the ascending and descending renewal densities of the process $\hat{\xi}^{\uparrow}$, and k is the constant in the Wiener–Hopf factorization (14) of $\hat{\xi}^{\uparrow}$.

The work of Kyprianou, Pardo and Rivero [22] gives the Wiener–Hopf factorization of $\hat{\xi}^{\uparrow}$, shows that the renewal measures are absolutely continuous and computes their densities, albeit for a different normalization of the α-stable process X.

In our normalization, the renewal densities are given by

$$v(z) = \frac{1}{\Gamma(\alpha\hat{\rho})}(1 - e^{-z})^{\alpha\hat{\rho}-1}, \qquad \hat{v}(z) = \frac{1}{\Gamma(\alpha\rho)}e^{-z}(1 - e^{-z})^{\alpha\rho-1}$$

and $k = 1$. See, for example, the computations in [20], where the normalisation of the α-stable process agrees with ours. It then follows, with similar calculations to those in the proof of Theorem 1.4,

$$u(y) = \begin{cases} \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(1 - e^{-y})^{\alpha-1}e^{\alpha\rho y}\displaystyle\int_0^{e^{-y}} t^{\alpha\rho-1}(1-t)^{-\alpha}\,dt, & y > 0, \\[4mm] \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(1 - e^y)^{\alpha-1}e^{(1-\alpha\hat{\rho})y}\displaystyle\int_0^{e^y} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\,dt, & y < 0. \end{cases}$$

This u is the bounded continuous density which we seek, so by substituting into (22) and (21), we arrive at the hitting probability

$$(23) \quad \mathrm{P}_x\big(\tau_0 < \tau_1^+\big) = \begin{cases} C'x^{\alpha-1}\displaystyle\int_0^{1-x} t^{\alpha\rho-1}(1-t)^{-\alpha}\,dt, & 0 < x < 1, \\[4mm] C'(-x)^{\alpha-1}\displaystyle\int_0^{(1-x)^{-1}} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\,dt, & x < 0, \end{cases}$$

where $C' = \frac{C}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}$. It only remains to determine the unknown constant here, which we will do by taking the limit $x \uparrow 0$ in (23). First we manipulate the second expression above, by recognizing that $1 = t + (1 - t)$ and integrating by parts. For $x < 0$,

$$\mathrm{P}_x\big(\tau_0 < \tau_1^+\big)$$

$$= C'(-x)^{\alpha-1}\left[\int_0^{(1-x)^{-1}} t^{\alpha\hat{\rho}}(1-t)^{-\alpha}\,dt + \int_0^{(1-x)^{-1}} t^{\alpha\hat{\rho}-1}(1-t)^{1-\alpha}\,dt\right]$$

$$= \frac{C'}{\alpha-1}\left[(1-x)^{\alpha\rho-1} - (1-\alpha\rho)(-x)^{\alpha-1}\int_0^{(1-x)^{-1}} t^{\alpha\hat{\rho}-1}(1-t)^{1-\alpha}\,dt\right].$$

Now taking $x \uparrow 0$, we find that $C' = \alpha - 1$.

Finally, we obtain the expression required by performing the integral substitution $s = 1/(1 - t)$ in (23). $\square$

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