HARMONIC MAPS ON AMENABLE GROUPS AND A DIFFUSIVE LOWER BOUND FOR RANDOM WALKS

BY JAMES R. LEE[1] AND YUVAL PERES

University of Washington and Microsoft Research

We prove diffusive lower bounds on the rate of escape of the random walk on infinite transitive graphs. Similar estimates hold for finite graphs, up to the relaxation time of the walk. Our approach uses nonconstant equivariant harmonic mappings taking values in a Hilbert space. For the special case of discrete, amenable groups, we present a more explicit proof of the Mok–Korevaar–Schoen theorem on the existence of such harmonic maps by constructing them from the heat flow on a Følner set.

1. Introduction. Let G be a d-regular, transitive graph (i.e., with transitive automorphism group), let $\{X_t\}$ denote the symmetric simple random walk on G with X_0 arbitrary and let dist be the path metric on G. In the case when G is a Cayley graph of a finitely-generated, amenable group, Èrshler [7] showed that $\mathbb{E}[\text{dist}(X_0, X_t)^2] \geq Ct/d$ for all times $t \geq 1$, where $C > 0$ is some absolute constant.

Our first theorem concerns a more precise analysis of the random walk behavior, as well as an extension to general transitive, amenable graphs. Recall that a graph G is *amenable* if there exists a sequence of finite subsets $\{S_j\}$ of the vertices such that $|S_j \triangle N(S_j)|/|S_j| \to 0$, where $N(S_j)$ denotes the neighborhood of S_j in G.

THEOREM 1.1. *Suppose G is an infinite, connected, and amenable transitive d-regular graph. Then the simple random walk on G satisfies the estimate*

$$\mathbb{E}\big[\text{dist}(X_0, X_t)^2\big] \geq t/d.$$

Moreover, for some universal constants $C > 0$ and $C' \geq 1$, and $t \geq d$, we have the estimates

$$\mathbb{E}\big[\text{dist}(X_0, X_t)\big] \geq C\sqrt{t/d},$$

and for every $\varepsilon \geq 1/\sqrt{t}$,

$$\frac{1}{t}\sum_{s=0}^{t}\mathbb{P}\big[\text{dist}(X_0, X_s) \leq \varepsilon\sqrt{t/d}\big] \leq C'\varepsilon.$$

In Section 4.3, we prove a version of the preceding theorem for the Cayley graph of any group without property (T).

In various senses, Theorem 1.1 shows that among infinite transitive graphs, the random walk disperses slowest for the standard random walk on $\mathbb{Z}$; see Corollary 2.6 and Remark 2.13. We also prove a version for finite graphs which holds up to the relaxation time of the random walk. In this case, the bound is matched (up to constant factors) for the finite cycle graphs; see Remark 2.3.

THEOREM 1.2. *Suppose G is a finite, connected, transitive d-regular graph, and λ denotes the second-largest eigenvalue of the transition matrix P of the random walk on G. Then for every $t \leq (1 - \lambda)^{-1}$,*

$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq t/(2d).$$

Moreover, for some universal constants $C > 0$ and $C' \geq 1$, and all t such that $(1 - \lambda)^{-1} \geq t \geq d$, we have the estimates

$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)\big] \geq C\sqrt{t/d},$$

and for every $\varepsilon \geq 1/\sqrt{t}$,

$$\frac{1}{t} \sum_{s=0}^{t} \mathbb{P}\big[\mathsf{dist}(X_0, X_s) \leq \varepsilon\sqrt{t/d}\big] \leq C'\varepsilon.$$

We remark that, in both cases, the dependence on d is necessary; see Remark 2.3.

The proof of Theorem 1.1 is based on the existence of nonconstant, equivariant harmonic maps on transitive, amenable graphs. For the simplicity of presentation, we first restrict ourselves to the setting of groups. Let Γ be a group with finite generating set $S \subseteq \Gamma$, and let G be the corresponding Cayley graph. Here and throughout the paper, all Cayley graphs will be defined using multiplication by the generators on the right.

Suppose that $\mathcal{H}$ is some Hilbert space on which Γ acts by isometries, and we have a nonconstant equivariant harmonic map $\Psi : \Gamma \to \mathcal{H}$, that is, such that $g\Psi(h) = \Psi(gh)$ and $\Psi(h) = |S|^{-1} \sum_{s \in S} \Psi(hs)$ hold for every $h \in \Gamma$. Èrshler [7] observed that this can be used to lower bound $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2]$, as follows.

We may normalize Ψ so that, if $e \in \Gamma$ is the identity,

$$(1) \qquad \frac{1}{|S|} \sum_{s \in S} \big\|\Psi(e) - \Psi(s)\big\|^2 = 1.$$

By equivariance, this implies that Ψ is $\sqrt{|S|}$-Lipschitz, hence

$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq \frac{1}{|S|}\mathbb{E}\big\|\Psi(X_0) - \Psi(X_t)\big\|^2.$$

But since Ψ is harmonic, $\Psi(X_t)$ is a martingale, thus

$$\mathbb{E}\big\|\Psi(X_0) - \Psi(X_t)\big\|^2 = \sum_{j=0}^{t-1} \mathbb{E}\big\|\Psi(X_j) - \Psi(X_{j+1})\big\|^2 = t,$$

where in the final line we have used equivariance and (1).

By results of Mok [19] and Korevaar and Schoen [14], if Γ is amenable, then it always admits such an equivariant harmonic map. On the other hand, if Γ is not amenable, then G has spectral radius $\rho < 1$ [12], hence $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2] \geq Ct^2$, for some constant $C = C(\rho) > 0$; see, for example, [29], Proposition 8.2. Thus the preceding discussion shows that $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2]$ grows at least linearly in t, for any infinite group Γ.

In Section 2, we exhibit a general method for proving escape lower bounds. For any function $\psi \in \ell^2(\Gamma)$, we have

$$(2) \qquad \mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq \frac{1}{d}\left(t - t^2 \frac{\|(I - P)\psi\|^2}{2\langle \psi, (I - P)\psi \rangle}\right),$$

where P is the transition kernel of the random walk on G. For finite groups, we choose ψ to be the eigenfunction corresponding to the second-largest eigenvalue of P. For infinite amenable groups, one can obtain ψ directly from spectral projection.

For a more explicit approach in the infinite, amenable case, we show that one can obtain the $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2] \geq t/|S|$ bound by taking a sequence of functions $\{\psi_n\}$ to be a truncated heat flow from some sets $A_n \subseteq \Gamma$, that is, $\psi_n = \sum_{i=0}^{n} P^i \mathbf{1}_{A_n}$, where $\{A_n\}$ forms an appropriate Følner sequence in G. These lower bounds, and indeed all the results in our paper, are proved for amenable, transitive graphs (and even quasi-transitive graphs), and more general forms of stochastic processes.

The existence of nonconstant, equivariant, harmonic maps on groups without property (T) is proved in [14, 19]; see also [13], Appendix A, for an exposition in the case of discrete groups, based on [8]. In Section 3, inspired by the preceding escape lower bounds, we give an explicit construction of these harmonic maps, simple enough to describe here. We focus now on the amenable case; in Theorem 3.8, we show that this approach generalizes to all discrete groups without property (T).

Define $\Psi_n : \Gamma \to \ell^2(\Gamma)$ by

$$\Psi_n(x) : g \mapsto \frac{\psi_n(gx)}{\sqrt{2\langle \psi_n, (I - P)\psi_n \rangle}},$$

where ψ_n is as before.

We argue that, after applying an appropriate affine isometry to each Ψ_n, there is a subsequence of $\{\Psi_n\}$ which converges pointwise to a nonconstant, equivariant, harmonic map. Our construction works for any infinite, transitive, amenable graph; see Theorem 3.1.

THEOREM 1.3. *Let $G = (V, E)$ be any infinite, connected, amenable, transitive graph. Then there exists a Hilbert space $\mathcal{H}$ and an $\mathcal{H}$-valued, nonconstant equivariant harmonic mapping on G, where equivariance is understood with respect to the transitive action on G.*

In Section 3.2, we show that our approach also proves the preceding theorem for the Cayley graph of any group without property (T). It is known [14, 19] that a group admits such an equivariant harmonic mapping if and only if it does not have property (T); see also [13], Lemma A.6.

One can use such mappings to obtain more detailed information on the random walk. Virág [28] showed that, in the setting of Cayley graphs of amenable groups, one has $\mathbb{E}[\mathsf{dist}(X_0, X_t)] \geq C\sqrt{t}/|S|^{3/2}$ for some $C > 0$. This can be proved by analyzing the process $\Psi(X_t)$ via the BDG martingale inequalities; see, for example, [15], Theorem 5.6.1.[2]

In Section 4, we show how a stronger bound can be derived directly from hitting time estimates, which can themselves be easily derived for martingales, then transferred to the group setting via harmonic maps. More generally, we study some finer properties of the escape behavior of the random walk.

1.1. *Related work.* We recall some previous results on the rate of escape of random walks on groups. A large amount of work has been devoted to classifying situations where the rate of escape $\mathbb{E}[\mathsf{dist}(X_0, X_t)]$ is linear; we refer to the survey of Vershik [27]. Èrshler has given examples where the rate can be asymptotic to $t^{1-2^{-k}}$ for any $k \in \mathbb{N}$ [6]. Following seminal work of Varopoulos [26], Hebisch and Saloff-Coste [11] obtained precise heat kernel estimates for symmetric bounded-range random walks on groups of polynomial growth. In particular, Theorem 5.1 in [11] implies our Theorem 1.1 for groups of polynomial growth. However, for groups of super-polynomial growth, it seems that existing heat-kernel bounds (see, e.g., Theorem 4.1 in [11]) are not powerful enough to imply Theorem 1.1. Diaconis and Saloff-Coste [5] show that on finite groups satisfying a certain "moderate growth" condition, the random walk mixes in $O(D^2)$ steps, where D is the diameter of the group in the word metric. A sequence of works [1, 20, 21] have related the rate of escape of random walks to questions in geometric group theory, notably to estimates of Hilbert compression exponents of groups. Our argument for finite groups was motivated by the work of the first author with Y. Makarychev [16] on effective, finitary versions of Gromov's polynomial growth theorem. Another substantial work in this direction is the recent preprint of Shalom and Tao [24], written independently of the present paper. Constructions of nearly harmonic functions play a key role there as well.

---

[2]In fact, Virag proceeds by explicitly bounding $\mathbb{E}[\|M_0 - M_t\|^4] \leq O(|S|^2 t^2)$ when $\{M_t\}$ is any Hilbert space-valued martingale with $\mathbb{E}[\|M_{t+1} - M_t\|^2 | \mathcal{F}_t] \leq 1$ and $\mathbb{E}[\|M_{t+1} - M_t\|^4 | \mathcal{F}_t] \leq |S|^2$, for all $t \geq 0$.

2. Escape rate of random walks. In the present section, we will consider a finite or infinite symmetric, stochastic matrix $\{P(x, y)\}_{x,y \in V}$ for some index set V. We write $\mathsf{Aut}(P)$ for the set of all bijections $\sigma : V \to V$ whose diagonal action preserves P, that is, $P(x, y) = P(\sigma x, \sigma y)$ for all $x, y \in V$. For the most part, we will be concerned with matrices P for which $\mathsf{Aut}(P)$ acts transitively on V. A primary example is given by taking P to be the transition matrix of the simple random walk on a finite or infinite vertex-transitive graph G.

THEOREM 2.1. *Let V be an at most countable index set, and consider any symmetric, stochastic matrix $\{P(x, y)\}_{x,y \in V}$. Suppose that $\Gamma \leq \mathsf{Aut}(P)$ is a closed, unimodular subgroup which acts transitively on V, and let $G = (V, E)$ be any graph on which Γ acts by automorphisms. If dist is the path metric on G, and $\psi \in \ell^2(V)$, then*

$$(3) \quad \mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq p_* \frac{\langle \psi, (I - P^t)\psi \rangle}{\langle \psi, (I - P)\psi \rangle} \geq p_* \left(t - t^2 \frac{\|(I - P)\psi\|^2}{2\langle \psi, (I - P)\psi \rangle} \right),$$

where $\{X_t\}$ denotes the random walk with transition kernel P started at any $X_0 = x_0 \in V$, and

$$p_* = \min\big\{P(x, y) : \{x, y\} \in E\big\}.$$

PROOF. Since Γ is unimodular, we can choose the Haar measure μ on Γ to be normalized so that $\mu(\Gamma_x) = 1$ for every $x \in V$, where Γ_x is the stabilizer subgroup of x. (Note that the stabilizer Γ_x is compact since Γ acts by automorphisms on G, which has all its vertex degrees bounded by $1/p_*$.) Define $\Psi : V \to L^2(\Gamma, \mu)$ by $\Psi(x) : \sigma \mapsto \psi(\sigma x)$.

In this case, for every $z \in V$,

$$\sum_{y \in V} P(z, y) \|\Psi(y) - \Psi(z)\|^2_{L^2(\Gamma, \mu)}$$

$$(4) \qquad = \sum_{y \in V} P(z, y) \int |\psi(\sigma z) - \psi(\sigma y)|^2 \, d\mu(\sigma)$$

$$= \sum_{y \in V} \int P(\sigma z, \sigma y) |\psi(\sigma z) - \psi(\sigma y)|^2 \, d\mu(\sigma)$$

$$= \mu(\Gamma_z) \sum_{x, y \in V} P(x, y) |\psi(x) - \psi(y)|^2$$

$$= 2\langle \psi, (I - P)\psi \rangle.$$

Thus for $\{x, y\} \in E$, we have $\|\Psi(x) - \Psi(y)\|^2_{L^2(\Gamma, \mu)} \leq \frac{2\langle \psi, (I-P)\psi \rangle}{p_*}$, which implies that

$$(5) \qquad \|\Psi\|_{\mathrm{Lip}} \leq \sqrt{\frac{2\langle \psi, (I - P)\psi \rangle}{p_*}},$$

where Ψ is considered as a map from (V, dist) to $L^2(\Gamma, \mu)$, and we use $\|\Psi\|_{\mathrm{Lip}}$ to denote the infimal number L such that Ψ is L-Lipschitz.

So, for any $x_0 \in V$, we have

$$
\begin{aligned}
\|\Psi\|_{\mathrm{Lip}}^2 &\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2 | X_0 = x_0\big] \\
&\geq \mathbb{E}\big[\|\Psi(X_0) - \Psi(X_t)\|_{L^2(\Gamma,\mu)}^2 | X_0 = x_0\big] \\
&= \int \mathbb{E}\big[|\psi(\sigma X_0) - \psi(\sigma X_t)|^2 | X_0 = x_0\big] d\mu(\sigma) \\
&= \sum_{x \in V} \mathbb{E}\big[|\psi(X_0) - \psi(X_t)|^2 | X_0 = x\big] \\
&= 2\langle \psi, (I - P^t)\psi \rangle \\
&= 2\sum_{i=0}^{t-1} \langle \psi, (I - P)P^i \psi \rangle,
\end{aligned}
$$
(6)

where in the third line, we have used the fact that the action of σ preserves P.

To finish, we use the fact that $I - P$ is self-adjoint to compare adjacent terms via

$$
\begin{aligned}
\big|\langle \psi, (I - P)P^i \psi \rangle &- \langle \psi, (I - P)P^{i+1}\psi \rangle\big| \\
&= \big|\langle (I - P)\psi, P^i(I - P)\psi \rangle\big| \leq \|(I - P)\psi\|^2,
\end{aligned}
$$

where the final inequality follows because P is stochastic, and hence a contraction. From this, we infer that $\langle \psi, (I - P)P^i \psi \rangle \geq \langle \psi, (I - P)\psi \rangle - i\|(I - P)\psi\|^2$, whence

$$
\sum_{i=0}^{t-1} \langle \psi, (I - P)P^i \psi \rangle \geq t\langle \psi, (I - P)\psi \rangle - \frac{t^2}{2}\|(I - P)\psi\|^2.
$$

Combining the preceding line with (5) and (6) yields

$$
\frac{1}{p_*}\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq \frac{\langle \psi, (I - P^t)\psi \rangle}{\langle \psi, (I - P)\psi \rangle} \geq t - t^2 \frac{\|(I - P)\psi\|^2}{2\langle \psi, (I - P)\psi \rangle}. \qquad \square
$$

We now demonstrate circumstances in which an appropriate $\psi \in \ell^2(V)$ exists. Corollaries 2.2, 2.11 and Conjecture 2.5 all assume the notation of Theorem 2.1.

COROLLARY 2.2 (The finite case). *Let V be a finite index set, and suppose that* $\mathsf{Aut}(P)$ *acts transitively on V. If $\lambda < 1$ is the second-largest eigenvalue of P, then*

(7) $$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq p_*\big(1 + \lambda + \lambda^2 + \cdots + \lambda^{t-1}\big).$$

In particular,

$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq p_* t / 2$$

for $t \leq (1 - \lambda)^{-1}$.

PROOF.   Let $\psi : V \to \mathbb{R}$ satisfy $P\psi = \lambda \psi$. By Theorem 2.1,

$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq p_* \frac{\langle \psi, (I - P^t)\psi \rangle}{\langle \psi, (I - P)\psi \rangle} = p_* \frac{1 - \lambda^t}{1 - \lambda} = p_*\big(1 + \lambda + \lambda^2 + \cdots + \lambda^{t-1}\big).$$

To complete the proof, use the inequality $\lambda^j \geq (1 - t^{-1})^j \geq 1 - j/t$.   □

REMARK 2.3 (Weighted graphs).   In particular, if P is irreducible and $p_* = \min\{P(x, y) : P(x, y) > 0\}$, then the conclusion is that $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2] \geq p_* t / 2$ for $t \leq (1 - \lambda)^{-1}$. Thus if P is the simple random walk on a d-regular graph, the conclusion is $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2] \geq t / (2d)$.

To see that the asymptotic dependence on d is tight, one can consider a cycle of length n, together with $d - 2$ self loops at each vertex, for $d \geq 2$. In this case, $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2] \leq 2t/d$ for all $t \geq 0$.

REMARK 2.4 (After the relaxation time).   The quantity $(1 - \lambda)^{-1}$ is called the *relaxation time* of the random walk specified by P, and bound (7) degrades after this time. It is interesting to consider what happens between the relaxation time and the mixing time which is always at most $O(\log |V|)(1 - \lambda)^{-1}$. One might conjecture that $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2]$ continues to have a linear lower bound until the mixing time. Toward this end, we pose the following conjecture.

CONJECTURE 2.5.   *There exists a constant $\varepsilon_0 > 0$ such that the following holds. For every finite, connected, d-regular transitive graph $G = (V, E)$ with diameter D,*

$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq \varepsilon_0 t / d$$

for $t \leq \varepsilon_0 D^2$, where $\{X_t\}$ is the simple random walk on G.

We remark that the results of [5] imply that the conjecture is correct for a wide class of groups of "moderate growth."

COROLLARY 2.6 (Infinite amenable graphs).   *If $G = (V, E)$ is an infinite, transitive, connected, amenable graph with degree d, and $\{X_t\}$ is the simple random walk, then*

$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq t / d.$$

PROOF. If P is the transition kernel of the simple random walk, it is a standard fact [12] that when G is infinite, connected and amenable, the spectrum of P has an accumulation point at 1, but does not contain 1. Therefore, for every $\delta > 0$ and $\varepsilon > 0$, there exists a $\delta' \in (0, \delta]$ so that, by considering the spectral projection of P onto the interval $[1 - \delta' - \varepsilon, 1 - \delta')$, we obtain a unit vector $\psi \in \ell^2(V)$ (an approximate eigenvector) for which $\langle \psi, P^t \psi \rangle \leq (1 - \delta')^t$, while $\langle \psi, P\psi \rangle \geq 1 - \delta' - \varepsilon$.

Plugging this into Theorem 2.1, we conclude that

$$\mathbb{E}\big[\operatorname{dist}(X_0, X_t)^2\big] \geq \frac{1}{d} \cdot \frac{1 - (1 - \delta')^t}{\delta' + \varepsilon}.$$

Sending $\varepsilon \to 0$ and then $\delta \to 0$ (and hence $\delta' \to 0$) yields the desired claim. $\quad\square$

QUESTION 2.7. *The preceding corollary yields a uniform lower bound of the form* $\mathbb{E}[\operatorname{dist}(X_0, X_t)^2] \geq Ct/d$ *for all d-regular infinite, connected, amenable graphs. In fact, one can take $C = 1$. Certainly for every d-regular infinite, connected graph G, there exists a constant C_G such that* $\mathbb{E}[\operatorname{dist}(X_0, X_t)^2] \geq C_G t/d$, *since in the nonamenable case* $\operatorname{dist}(X_0, X_t)$ *grows linearly with t, but with some constant depending on G. It is natural to ask whether one can take $C_G \geq \Omega(1)$, that is, whether a uniform lower bound holds without the amenability assumption. This seems closely related to Conjecture 2.5.*

2.1. *Infinite amenable graphs.* While Corollary 2.6 gives satisfactory results for infinite, amenable graphs, we take some time in this section to further analyze the amenable case; in particular, the explicit construction of Lemma 2.9 serves as a connection between random walks and our construction of harmonic functions in Section 3.

The following theorem will play a role in a number of arguments. The transitive version is due to Soardi and Woess [25], and the extension to quasi-transitive actions is from [23]. See also a different proof in [4], Theorem 3.4.

We recall that for a graph $G = (V, E)$, we say that a group $\Gamma \leq \operatorname{Aut}(G)$ is *quasi-transitive* if $|\Gamma \setminus V| < \infty$, where $\Gamma \setminus V$ denotes the set of Γ-orbits of V.

THEOREM 2.8. *Let G be a graph and $\Gamma \leq \operatorname{Aut}(G)$ a closed, quasi-transitive subgroup. Then G is amenable if and only if Γ is amenable and unimodular.*

We begin with the following general construction. Gromov [10], Sections 3.6–3.7, uses a similar analysis in the setting of the continuous heat flow on manifolds (see, in particular, Remark 3.7(B) in [10]). We remark that, in this setting, the result itself follows rather directly from spectral projection as in the proof of Corollary 2.6. We present the following proof because it is quite elementary and explicit, and directly relates our harmonic function construction (Section 3) to random walks.

LEMMA 2.9.   *Let $\mathcal{H}$ be a Hilbert space, and let $Q : \mathcal{H} \to \mathcal{H}$ be a self-adjoint linear operator which is contractive, that is, with $\|Q\|_{\mathcal{H} \to \mathcal{H}} \leq 1$. Suppose that for some $\theta \in (0, \frac{1}{2})$, there exists an $f \in \mathcal{H}$ which satisfies $\|f\|_{\mathcal{H}} = 1$, $\|Qf - f\|_{\mathcal{H}} \leq \theta$, and*

$$
\text{(8)} \qquad \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \langle Q^i f, f \rangle = 0.
$$

Then there exists an element $\varphi \in \mathcal{H}$ with

$$
\frac{\|(I - Q)\varphi\|_{\mathcal{H}}^2}{\langle \varphi, (I - Q)\varphi \rangle_{\mathcal{H}}} \leq 32\theta.
$$

Before venturing into the proof, we mention a natural approach for a special case. Consider the situation where Γ is a finitely-generated, amenable group, and G is the corresponding Cayley graph. Let P be the transition kernel of the corresponding random walk, and let $\{F_k\}$ be a Følner sequence in G which satisfies $\langle \mathbf{1}_{F_k}, P^i \mathbf{1}_{F_k} \rangle \geq \frac{1}{2} |F_k|$ for $i = 0, 1, 2, \ldots, k$.

In that case, one might choose to put

$$
\text{(9)} \qquad \psi_k = \sum_{i=0}^{\infty} P^i \mathbf{1}_{F_k},
$$

where $\mathbf{1}_{F_k}$ is the characteristic function of F_k. Assume, for the moment, that $\psi_k \in \ell^2(V)$. In this case, $(I - P)\psi_k = \mathbf{1}_{F_k}$, so

$$
\|(I - P)\psi_k\|^2 = |F_k|,
$$

while

$$
\langle \psi_k, (I - P)\psi_k \rangle = \sum_{i=0}^{\infty} \langle \mathbf{1}_{F_k}, P^i \mathbf{1}_{F_k} \rangle \geq \frac{k}{2} |F_k|,
$$

by our assumption on $\{F_k\}$. This implies that

$$
\lim_{k \to \infty} \frac{\|(I - P)\psi_k\|^2}{\langle \psi_k, (I - P)\psi_k \rangle} \to 0,
$$

yielding an analog to the conclusion of Lemma 2.9.

The only remaining issue is whether $\psi_k \in \ell^2(V)$, which requires some assumptions on the group Γ. To get around this, we truncate the sum in (9). The somewhat delicate issue that arises is where to truncate. The following proof shows that a good truncation point always exists.

PROOF OF LEMMA 2.9.   Given $f \in \mathcal{H}$ and $k \in \mathbb{N}$, we define $\varphi_k \in \mathcal{H}$ by

$$
\varphi_k = \sum_{i=0}^{k-1} Q^i f.
$$

First, using $(I - Q)\varphi_k = (I - Q^k)f$ and the fact that Q is a contraction, we have

(10) $$\|(I - Q)\varphi_k\|_{\mathcal{H}}^2 \le 4\|f\|_{\mathcal{H}}^2.$$

On the other hand,

$$
\begin{aligned}
\langle \varphi_k, (I - Q)\varphi_k \rangle_{\mathcal{H}} &= \langle \varphi_k, (I - Q^k)f \rangle_{\mathcal{H}} \\
&= \left\langle (I - Q^k) \sum_{i=0}^{k-1} Q^i f, f \right\rangle_{\mathcal{H}} \\
&= \langle 2\varphi_k - \varphi_{2k}, f \rangle_{\mathcal{H}},
\end{aligned}
$$

where in the second line we have used the fact that $I - Q^k$ is self-adjoint. Combining this with (10) yields

(11) $$\frac{\|(I - Q)\varphi_k\|_{\mathcal{H}}^2}{\langle \varphi_k, (I - Q)\varphi_k \rangle_{\mathcal{H}}} \le \frac{4\|f\|_{\mathcal{H}}^2}{\langle 2\varphi_k - \varphi_{2k}, f \rangle_{\mathcal{H}}}.$$

The following claim will conclude the proof.

CLAIM. *There exists a $k \in \mathbb{N}$ such that*

(12) $$\langle 2\varphi_k - \varphi_{2k}, f \rangle_{\mathcal{H}} \ge \frac{1}{8\theta}.$$

It remains to prove the claim. By assumption, f satisfies $\|f\|_{\mathcal{H}} = 1$, and $\|Qf - f\|_{\mathcal{H}} \le \theta$. Since Q is a contraction, we have $\|Q^j f - Q^{j-1} f\|_{\mathcal{H}} \le \theta$ for every $j \ge 1$, and thus by the triangle inequality, $\|Q^j f - f\|_{\mathcal{H}} \le j\theta$ for every $j \ge 1$. It follows by Cauchy–Schwarz that $\langle f, (I - Q^j)f \rangle_{\mathcal{H}} \le j\theta$, therefore

$$\langle f, Q^j f \rangle_{\mathcal{H}} \ge 1 - j\theta.$$

Thus for every $j \ge 1$,

$$\langle \varphi_{2^j}, f \rangle_{\mathcal{H}} \ge 2^j (1 - 2^j \theta).$$

Fix $\ell \in \mathbb{N}$ so that $2^\ell \theta \le \frac{1}{2} \le 2^{\ell+1}\theta$, yielding

(13) $$\langle \varphi_{2^\ell}, f \rangle_{\mathcal{H}} \ge \frac{1}{8\theta}.$$

Now, let $a_m = \langle \varphi_{2^m}, f \rangle_{\mathcal{H}}$, and write, for some $N \ge 1$,

$$a_\ell - \frac{a_N}{2^{N-\ell}} = \sum_{m=\ell}^{N-1} \frac{2a_m - a_{m+1}}{2^{m-\ell+1}}.$$

By (8), we have

(14) $$\lim_{N \to \infty} \frac{a_N}{2^N} = 0.$$

Using (13) and taking $N \to \infty$ on both sides above yields

$$\frac{1}{8\theta} \le a_\ell = \sum_{m=\ell}^{\infty} \frac{2a_m - a_{m+1}}{2^{m-\ell+1}}.$$

Since $\sum_{m=\ell}^{\infty} \frac{1}{2^{m-\ell+1}} = 1$, there must exist some $m \ge \ell$ with $2a_m - a_{m+1} \ge \frac{1}{8\theta}$. This establishes claim (12) for $k = 2^m$, and in view of (11), completes the proof of the lemma. $\square$

We now arrive at the following corollaries. Recall that if P is transient or null-recurrent, then we have the pointwise limit,

(15) $P^i f \to 0 \qquad$ for every $f \in \ell^2(V)$.

(This is usually proved for finitely supported f; see, e.g., [9], Theorem 6.4.17 or [17], Theorem 21.17. The general case follows by approximation using the contraction property of P.)

COROLLARY 2.10. *If V is infinite, P satisfies (15), and $\Gamma \le \mathsf{Aut}(P)$ is a closed, amenable, unimodular subgroup, which acts transitively on V, then*

(16) $$\inf_{\varphi \in \ell^2(V)} \frac{\|(I - P)\varphi\|^2}{\langle \varphi, (I - P)\varphi \rangle} = 0.$$

PROOF. This follows from Lemma 2.9 using the fact that, under the stated assumptions, for every $\theta > 0$, there exists an $f \in \ell^2(V)$ with $\|f\| = 1$ and $\|Pf - f\| \le \theta$.

This is a standard fact that can be proved, as in [29], Theorem 12.10. In general, for every $\theta > 0$, one considers, for some $\varepsilon = \varepsilon(\theta) > 0$, the graph G_ε with vertices V and an edge $\{x, y\}$ whenever $P(x, y) \ge \varepsilon$. Since $\Gamma \le \mathsf{Aut}(G_\varepsilon)$, Theorem 2.8 implies that G_ε is amenable, and then one can take $f = |F|^{-1/2}\mathbf{1}_F$ to be the (normalized) indicator of a suitable Følner set $F \subseteq V$ in G_ε. The idea is that for $\varepsilon > 0$ chosen small enough, $\|P\mathbf{1}_F - \mathbf{1}_F\|^2$ is close to the size of the outer vertex boundary of F in G_ε. $\square$

The following is an immediate consequence of Theorem 2.1 combined with the preceding result.

COROLLARY 2.11 (The amenable case). *Under the assumptions of Theorem 2.1, the following holds. If V is a countably infinite index set, P satisfies (15), and $\Gamma \le \mathsf{Aut}(P)$ is a closed, amenable, unimodular subgroup which acts transitively on V, then*

(17) $$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \ge p_* t.$$

COROLLARY 2.12 (The nearly amenable case, for small times). *Under the assumptions of Theorem* 2.1, *the following holds. If* $\rho = \rho(P)$ *is the spectral radius of* P, *then for all times* $t \leq (32(1 - \rho))^{-1}$,

$$\mathbb{E}\big[\text{dist}(X_0, X_t)^2\big] \geq \frac{p_* t}{2}.$$

PROOF.   Since P is self-adjoint and positive, by standard variational principles, we have $\rho = \|P\|_{2 \to 2} = \sup_{\|f\|=1} \langle Pf, f \rangle$. It follows that

$$\inf_{\|f\|=1} \|f - Pf\|^2 \leq \Big(\inf_{\|f\|=1} 1 + \rho^2 - 2\langle f, Pf \rangle \Big) = (1 - \rho)^2.$$

Combining this with Lemma 2.9 yields the claimed result. □

Compare the preceding bound with the finite case (Corollary 2.2).

REMARK 2.13 (Asymptotic rate of escape).   The constant p_* in (3) is not tight. To do slightly better, one can argue as follows. Let $\Psi : V \to L^2(\Gamma, \mu)$ be as in the proof of Theorem 2.1. Fix $x, y \in V$ with $L = \text{dist}(x, y)$, and let $x = v_0, v_1, \ldots, v_L = y$ be a shortest path from x to y in G. In this case, the triangle inequality yields

$$2\|\Psi(x) - \Psi(y)\|$$
$$\leq \|\Psi(v_0) - \Psi(v_1)\|$$
$$+ \sum_{i=1}^{L-1} \big(\|\Psi(v_{i-1}) - \Psi(v_i)\| + \|\Psi(v_i) - \Psi(v_{i+1})\| \big)$$
$$+ \|\Psi(v_{L-1}) - \Psi(v_L)\|.$$

But for every $i \in \{1, 2, \ldots, L-1\}$, there are two terms involving v_i, and for such i, we can bound

$$\big\|\Psi(v_i) - \Psi(v_{i-1})\big\|^2 + \big\|\Psi(v_i) - \Psi(v_{i+1})\big\|^2 \leq \frac{2\langle \psi, (I - P)\psi \rangle}{p_*}$$

as in (4). In this way, we gain a factor of 2 for such terms. Letting α denote the right-hand side of the preceding inequality, we have

$$\|\Psi(x) - \Psi(y)\| \leq \sqrt{\alpha}\Big(1 + \frac{L-1}{\sqrt{2}}\Big) \leq \sqrt{\frac{\alpha}{2}}(L + 1).$$

Thus for all $x, y \in V$, we have $\|\Psi(x) - \Psi(y)\|^2 \leq [\text{dist}(x, y) + 1]^2 \frac{\langle \psi, (I-P)\psi \rangle}{p_*}$. Plugging this improvement into the proof of Theorem 2.1 yields

$$(18) \qquad \mathbb{E}\big[(\text{dist}(X_0, X_t) + 1)^2\big] \geq 2p_* \frac{\langle \psi, (I - P^t)\psi \rangle}{\langle \psi, (I - P)\psi \rangle},$$

which is asymptotically tight since, on the one hand, the simple random walk on $\mathbb{Z}$ satisfies $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2] = t$, while plugging (18) into Corollary 2.11 yields $\mathbb{E}[(\mathsf{dist}(X_0, X_t) + 1)^2] \geq t$.

The dependence on p_* is easily seen to be tight for the simple random walk on $\mathbb{Z}$ with a $1 - 2p_*$ holding probability added to every vertex, as in Remark 2.3.

3. Equivariant harmonic maps. Let V be a countably infinite index set, and let $\{P(x, y)\}_{x,y \in V}$ be a stochastic, symmetric matrix. If $\mathcal{H}$ is a Hilbert space, a mapping $\Psi : V \to \mathcal{H}$ is called *P-harmonic* if, for all $x \in V$,

$$\Psi(x) = \sum_{y \in V} P(x, y)\Psi(y).$$

Suppose furthermore that we have a group Γ acting on V. We say that Ψ is Γ-*equivariant* if there exists an affine isometric action π of Γ on $\mathcal{H}$, such that for every $g \in \Gamma$, $\pi(g)\Psi(x) = \Psi(gx)$ for all $x \in V$. If we wish to emphasize the particular action π, we will say that Ψ is Γ-*equivariant with respect to π*.

We remark in passing that there do exist amenable groups admitting a nonunimodular action; see, for example, [22], Example 2.2.

THEOREM 3.1. *For P as above, let $\Gamma \leq \mathsf{Aut}(P)$ be a closed, amenable, unimodular subgroup which acts transitively on V. Suppose there exists a connected graph $G = (V, E)$ on which Γ acts by automorphisms, and that for $x \in V$,*

$$(19) \qquad \sum_{y \in V} P(x, y)\,\mathsf{dist}(x, y)^2 < \infty,$$

where dist *is the path metric on G. Suppose also that*

$$p_* = \min\{P(x, y) : \{x, y\} \in E\} > 0.$$

Then there exists a Hilbert space $\mathcal{H}$, and a nonconstant Γ-equivariant P-harmonic mapping from V into $\mathcal{H}$.

PROOF. It is a standard result that since G is connected, P satisfies (15). Let $\{\psi_j\} \subseteq \ell^2(V)$ be a sequence of functions satisfying

$$(20) \qquad \frac{\langle (I - P)\psi_j, (I - P)\psi_j \rangle}{\langle \psi_j, (I - P)\psi_j \rangle} \to 0.$$

The existence of such a sequence is the content of Corollary 2.10.

Define $\Psi_j : V \to L^2(\Gamma, \mu)$ by

$$\Psi_j(x) : \sigma \mapsto \frac{\psi_j(\sigma^{-1}x)}{\sqrt{2\langle \psi_j, (I - P)\psi_j \rangle}}.$$

Since Γ is unimodular, we can choose the Haar measure μ on Γ to be normalized $\mu(\Gamma_x) = 1$ for all $x \in V$, where Γ_x is the stabilizer subgroup of x.

Now, observe that for every $x \in V$,

$$\sum_{y \in V} P(x, y) \|\Psi_j(x) - \Psi_j(y)\|_{L^2(\Gamma, \mu)}^2$$

$$= \frac{\sum_{y \in V} P(x, y) \int |\psi_j(\sigma^{-1} x) - \psi_j(\sigma^{-1} y)|^2 \, d\mu(\sigma)}{2 \langle \psi_j, (I - P) \psi_j \rangle}$$

(21)

$$= \mu(\Gamma_x) \frac{\sum_{u, y \in V} P(u, y) |\psi_j(u) - \psi_j(y)|^2}{2 \langle \psi_j, (I - P) \psi_j \rangle}$$

$$= 1.$$

Next, for every $x \in V$, we have

$$\left\| \Psi_j(x) - \sum_{y \in V} P(x, y) \Psi_j(y) \right\|_{L^2(\Gamma, \mu)}^2$$

$$= \frac{\int |\psi_j(\sigma^{-1} x) - \sum_{y \in V} P(x, y) \psi_j(\sigma^{-1} y)|^2 \, d\mu(\sigma)}{2 \langle \psi_j, (I - P) \psi_j \rangle}$$

$$= \mu(\Gamma_x) \frac{\sum_{u \in V} |\psi_j(u) - \sum_{y \in V} P(u, y) \psi_j(y)|^2}{2 \langle \psi_j, (I - P) \psi_j \rangle}$$

$$= \frac{\langle (I - P) \psi_j, (I - P) \psi_j \rangle}{2 \langle \psi_j, (I - P) \psi_j \rangle} \, .$$

In particular, from (20),

(22)
$$\lim_{j \to \infty} \left\| \Psi_j(x) - \sum_{y \in V} P(x, y) \Psi_j(y) \right\|_{L^2(\Gamma, \mu)}^2 = 0,$$

where the limit is uniform in $x \in V$.

Define a unitary action π_0 of Γ on $L^2(\Gamma, \mu)$ as follows: For $\gamma \in \Gamma, h \in L^2(\Gamma, \mu)$, $[\pi_0(\gamma) h](\sigma) = h(\gamma^{-1} \sigma)$ for all $\sigma \in \Gamma$. Notice that each Ψ_j is Γ-equivariant since for $\gamma \in \Gamma, x \in V$, we have

$$(\pi_0(\gamma) [\Psi_j(x)])(\sigma) = [\Psi_j(x)](\gamma^{-1} \sigma) = \frac{\psi_j(\sigma^{-1} \gamma x)}{\sqrt{2 \langle \psi_j, (I - P) \psi_j \rangle}} = [\Psi_j(\gamma x)](\sigma).$$

We state the next lemma in slightly more generality than we need presently, since we will use it also in Section 3.2.

LEMMA 3.2. *Suppose that $\mathfrak{H}$ is a Hilbert space, Γ is a group, and π_0 is an affine isometric action of Γ on $\mathfrak{H}$. Let (V, dist) be a countable metric space, and consider a sequence of functions $\{\Psi_j : V \to \mathfrak{H}\}_{j=1}^{\infty}$, where the Ψ_j's are uniformly Lipschitz and Γ-equivariant with respect to π_0. Then there is a sequence of affine*

isometries $T_j : \mathfrak{H} \to \mathfrak{H}$ and a subsequence $\{\alpha_j\}$ such that $T_{\alpha_j}\Psi_{\alpha_j}$ converges point-wise to a map $\widehat{\Psi} : V \to \mathfrak{H}$ which is Γ-equivariant with respect to an affine isometric action π.

Before proving the lemma, let us see that it can be used to finish the proof of Theorem 3.1. Using (21), one observes that for all $j \in \mathbb{N}$, the map Ψ_j is $\sqrt{1/p_*}$-Lipschitz on (V, dist). Thus we are in position to apply the preceding lemma and arrive at a map $\widehat{\Psi} : V \to L^2(\Gamma, \mu)$ which is Γ-equivariant with respect to an affine isometric action.

From (22), we see that $\widehat{\Psi}$ is P-harmonic. Furthermore, since the Ψ_j's are uniformly Lipschitz, and we have the estimate (19), we see that (21) holds for $\widehat{\Psi}$ as well, showing that $\widehat{\Psi}$ is nonconstant, and completing the proof.

PROOF OF LEMMA 3.2.   Arbitrarily order the points of $V = \{x_1, x_2, \ldots\}$ and fix a sequence of subspaces $\{W_j\}_{j=1}^{\infty}$ of $\mathfrak{H}$ with $W_j \subseteq W_{j+1}$ for each $j = 1, 2, \ldots$, and $\dim(W_j) = j$. For each such j, define an affine isometry $T_j : \mathfrak{H} \to \mathfrak{H}$ which satisfies $T_j \Psi_j(x_1) = 0$ and, for every $r = 1, 2, \ldots, j$, $\{T_j \Psi_j(x_k)\}_{k=1}^{r} \subseteq W_r$. Put $\widehat{\Psi}_j = T_j \Psi_j$, and define an affine isometric action π_j of Γ on $\mathfrak{H}$ by $\pi_j = T_j \pi_0 T_j^{-1}$. It is straightforward to check that each $\widehat{\Psi}_j$ is Γ-equivariant with respect π_j.

Since the maps $\{\Psi_j\}$ are uniformly Lipschitz, the same holds for the family $\{\widehat{\Psi}_j\}$. We now pass to a subsequence $\{\alpha_j\}$ along which $\widehat{\Psi}_{\alpha_j}(x)$ converges point-wise for every $x \in V$. To see that this is possible, notice that by construction, for every fixed $x \in V$, there is a finite-dimensional subspace $W \subseteq \mathfrak{H}$ such that $\widehat{\Psi}_j(x) \subseteq W$ for every $j \in \mathbb{N}$. Hence by the uniform Lipschitz property of $\widehat{\Psi}_j$, the sequence $\{\widehat{\Psi}_j(x)\}_{j=1}^{\infty}$ lies in a compact set.

We are thus left to show that $\widehat{\Psi}$ is Γ-equivariant. Toward this end, we define an action π of Γ on $\mathfrak{H}$ as follows: On the image of $\widehat{\Psi}$, set $\pi(\gamma)\widehat{\Psi}(x) = \widehat{\Psi}(\gamma x)$. For $g \in \mathfrak{H}$ lying in the orthogonal complement of the span of $\{\widehat{\Psi}(x)\}_{x \in V}$, we put $\pi(\gamma)g = \pi(\gamma)0$, and then extend $\pi(\gamma)$ affine linearly to the whole space. To see that such an affine linear extension exists, observe that

$$\pi(\gamma)\widehat{\Psi}(x) = \widehat{\Psi}(\gamma x) = \lim_{j \to \infty} \pi_{\alpha_j}(\gamma)\widehat{\Psi}_{\alpha_j}(x).$$

From this expression, it follows immediately that π acts by affine isometries, since each π_{α_j} does. Thus $\widehat{\Psi}$ is Γ-equivariant with respect to π, completing our construction. $\square$

REMARK 3.3.   Note that, in the case where P is simply the kernel of the simple random walk on the Cayley graph of a finitely-generated amenable group, one can take the Hilbert space $\mathcal{H}$ in Theorem 3.1 to be simply $\ell^2(V)$.

3.1. *Quasi-transitive graphs.* Only for the present section, we allow P to be a nonsymmetric kernel on the state space V. We recall that Γ is said to *act quasi-transitively on a set V* if $|\Gamma \setminus V| < \infty$, where $\Gamma \setminus V$ denotes the set of Γ-orbits of V. We prove an analog of Theorem 3.1 in the quasi-transitive setting, under the assumption that the kernel P is reversible.

COROLLARY 3.4 (Quasi-transitive actions). *Let $\Gamma \leq \mathsf{Aut}(P)$ be a closed, amenable, unimodular subgroup which acts quasi-transitively on V. Suppose also that P is the kernel of a reversible Markov chain, and there exists a connected graph $G = (V, E)$ on which Γ acts by automorphisms, and that for $x \in V$,*

$$\tag{23} \sum_{y \in V} P(x, y) \, \mathsf{dist}(x, y)^2 < \infty,$$

where dist is the path metric on G. Suppose also that

$$p_* = \min\{P(x, y) : \{x, y\} \in E\} > 0.$$

Then there exists a Hilbert space $\mathcal{H}$, and a nonconstant Γ-equivariant P-harmonic mapping from V into $\mathcal{H}$.

PROOF. Let $x_0, x_1, \ldots, x_L \in V$ be a complete set of representatives of the orbits of Γ. Let $V_0 = \Gamma x_0$, and let P_0 be the induced transition kernel of the P-random walk watched on V_0, that is, $P_0(x, y) = \mathbb{P}[X_\tau = y \mid X_0 = x]$ for $x, y \in V_0$, where $\tau = \min\{t \geq 1 : X_t \in V_0\}$. Since P is reversible and Γ acts transitively on V_0, we see that P_0 is symmetric.

Letting $D = \max_{i \neq j} \mathsf{dist}(x_i, x_j)$, we define a new graph $G_0 = (V_0, E_0)$ by having an edge $\{x, y\} \in E_0$ whenever:

(1) $\{x, y\} \in E$ and $x, y \in V_0$ or
(2) there exists a path $x = v_0, v_1, \ldots, v_k = y$ in G with $v_1, \ldots, v_{k-1} \notin V_0$ and $k \leq 2D$.

Let dist_0 denote the path metric on G_0. It is clear that Γ acts on G_0 by automorphisms, and also that $p_*(G_0) = \min\{P_0(x, y) : \{x, y\} \in E_0\} \geq (p_*)^{2D} > 0$.

Now, since every point $x \notin V_0$ has $\mathsf{dist}(x, V_0) \leq D$, we see that actually $\mathsf{dist}(x, y) \asymp \mathsf{dist}_0(x, y)$ for all $x, y \in V_0$ (up to a multiplicative constant depending on D). Furthermore, this implies that for any $x \in V$ there exists $y \in V_0$ with $\sum_{i=0}^{D} P^i(x, y) \geq (p_*)^D$, and hence (23) implies that for every $x \in V_0$,

$$\sum_{y \in V_0} P_0(x, y) \, \mathsf{dist}(x, y)^2 < \infty,$$

since number of P-steps taking before returning to V_0 is dominated by a geometric random variable. This implies the same for dist_0.

Thus we can apply Theorem 3.1 to obtain a Hilbert space $\mathcal{H}$ and a nonconstant Γ-equivariant P_0-harmonic map $\Psi_0 : V_0 \to \mathcal{H}$. We extend this to a mapping Ψ :

$V \to \mathcal{H}$ by defining $\Psi(x) = \mathbb{E}[\Psi_0(W_0(x))]$ where $W_0(x)$ is the first element of V_0 encountered in the P-random walk started at x. Note that $\Psi|_{V_0} = \Psi_0$, and Ψ is again Γ-equivariant. To finish the proof, it suffices to check that Ψ is P-harmonic.

From the definition of Ψ, this is immediately clear for $x \notin V_0$. Since Ψ_0 is P_0-harmonic, it suffices to check that for $x \in V_0$,

$$\sum_{y \in V} P(x, y)\Psi(y) = \sum_{y \in V_0} P_0(x, y)\Psi_0(y),$$

but both sides are precisely $\mathbb{E}[\Psi_0(W_0(Z))]$, where Z is the random vertex arising from one step of the P-walk started at x. $\square$

COROLLARY 3.5 (Harmonic functions on quasi-transitive graphs). *If $G = (V, E)$ is an infinite, connected, amenable graph, and $\Gamma \leq \mathsf{Aut}(G)$ is a quasi-transitive subgroup, then G admits a nonconstant Γ-equivariant harmonic mapping into some Hilbert space.*

Now let $G = (V, E)$ be an infinite, connected, quasi-transitive, amenable graph. The preceding construction of harmonic functions also gives escape lower bounds for the random walk on G. By Theorem 2.8, when G is amenable, $\Gamma = \mathsf{Aut}(G)$ is amenable and unimodular. Let $R \subseteq V$ be a complete set of representatives from $\Gamma \backslash V$. Let μ be the Haar measure on Γ. For $r \in R$, let $\mu_r = \mu(\Gamma_r)$, and normalize μ so that $\sum_{r \in R} \deg(r)/\mu_r = 1$.

COROLLARY 3.6 (Random walks on quasi-transitive graphs). *Let dist be the path metric on G, and let X_0 have the distribution $\mathbb{P}[X_0 = r] = \deg(r)/\mu_r$ for $r \in R$. Then*

$$(24) \qquad \mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq \frac{t}{\max\{\mu_r : r \in R\}},$$

where $\{X_t\}$ denotes the simple random walk on G.

PROOF.   Let $\Psi : V \to \mathcal{H}$ be the harmonic map guaranteed by Corollary 3.5 normalized so that

$$(25) \qquad \sum_{r \in R} \frac{1}{\mu_r} \sum_{x:\{x,r\} \in E} \big\|\Psi(x) - \Psi(r)\big\|^2 = 1.$$

We have

$$\|\Psi\|_{\mathrm{Lip}} \leq \max_{r \in R} \sqrt{\mu_r}.$$

For every $r, \hat{r} \in R$, the mass transport principle [4], Corollary 3.5, implies that

$$\frac{1}{\mu_r}\#\{x \in \Gamma\hat{r} : \{r, x\} \in E\} = \frac{1}{\mu_{\hat{r}}}\#\{x \in \Gamma r : \{\hat{r}, x\} \in E\}.$$

Thus if we use the notation $[x]$ to denote the unique $r \in R$ such that $x \in \Gamma r$, then $[X_i]$ and $[X_0]$ are identically distributed for every $i \geq 0$. (This is also a special case of [18], Theorem 3.1.) It follows that

$$
\begin{aligned}
\|\Psi\|_{\text{Lip}}^2 \mathbb{E}[\text{dist}(X_0, X_t)^2] \geq \mathbb{E}\|\Psi(X_0) - \Psi(X_t)\|^2 \\
= \sum_{i=0}^{t-1} \mathbb{E}\|\Psi(X_{i+1}) - \Psi(X_i)\|^2 \\
= \sum_{i=0}^{t-1} \mathbb{E}\frac{1}{\deg(X_i)} \sum_{x:\{x,X_i\}\in E} \|\Psi(x) - \Psi(X_i)\|^2 \\
= t \cdot \sum_{r\in R} \frac{\deg(r)}{\mu_r} \frac{1}{\deg(r)} \sum_{x:\{x,r\}\in E} \|\Psi(x) - \Psi(r)\|^2 \\
= t,
\end{aligned}
$$

where in the second line we have used the fact that $\{\Psi(X_t)\}$ is a martingale, in the fourth line we have used equivariance of Ψ and the fact that each $[X_i]$ has the same distribution, and in the final line we have used (25). $\quad\square$

3.2. *Groups without property* (T). We now state a version of Theorem 3.1 that applies to the simple random walk on Cayley graphs of groups without property (T). (We refer to [3] for a thorough discussion of Kazhdan's property (T).) To this end, let Γ be a finitely-generated group, with finite, symmetric generating set $S \subseteq \Gamma$. Let P be the transition kernel of the simple random walk on Γ (with steps from S).

THEOREM 3.7. *Under the preceding assumptions, if Γ does not have property* (T), *there exists a Hilbert space $\mathcal{H}$ and a unitary action π of Γ on $\mathcal{H}$ such that*

$$
\inf_{\varphi\in\mathcal{H}} \frac{\|(I - P_\dagger)\varphi\|_{\mathcal{H}}^2}{\langle\varphi, (I - P_\dagger)\varphi\rangle_{\mathcal{H}}} = 0,
$$

where $P_\dagger : \mathcal{H} \to \mathcal{H}$ is defined by

$$
\tag{26} P_\dagger f = \frac{1}{|S|} \sum_{\gamma\in S} \pi(\gamma)f.
$$

PROOF. Since Γ does not have property (T), it admits a unitary action π on some Hilbert space $\mathcal{H}$ without fixed points such that we can find, for every $\theta > 0$, an $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = 1$ and $\|P_\dagger f - f\|_{\mathcal{H}} \leq \theta$. Now, $P_\dagger$ is self-adjoint and contractive by construction, thus to apply Lemma 2.9 (with $Q = P_\dagger$) and reach our desired conclusion, we are left to show that $\lim \frac{1}{k} \sum_{i=0}^{k-1} \langle P_\dagger^i f, f \rangle = 0$.

Fix some nonzero $f \in \mathcal{H}$, and let $\varphi_k = \frac{1}{k} \sum_{i=0}^{k-1} P_\dagger^i f$. If

(27) $$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \langle P_\dagger^i f, f \rangle \neq 0,$$

then there exists a subsequence $\{k_\alpha\}$ and a nonzero $\varphi \in \mathcal{H}$ such that φ is a weak limit of $\{\varphi_{k_\alpha}\}$.

But in this case, we claim that

(28) $$P_\dagger \varphi = \varphi,$$

since for any $g \in \mathcal{H}$, we have

$$\langle P_\dagger \varphi, g \rangle_\mathcal{H} = \lim_{\alpha \to \infty} \left\langle \frac{1}{k_\alpha} \sum_{i=0}^{k_\alpha - 1} P_\dagger^{i+1} f, g \right\rangle_\mathcal{H}$$

$$= \lim_{\alpha \to \infty} \left\langle \frac{1}{k_\alpha} \sum_{i=0}^{k_\alpha - 1} P_\dagger^i f, g \right\rangle_\mathcal{H}$$

$$= \langle \varphi, g \rangle_\mathcal{H},$$

where we have used the fact that $\lim_{\alpha \to \infty} \frac{1}{k_\alpha}(P_\dagger^{k_\alpha} f - f) = 0$, since $P_\dagger$ is contractive. On the other hand, if (28) holds, then we must have $\pi(\Gamma)\varphi = \{\varphi\}$. This follows by strict convexity since $P_\dagger f$ is an average of elements of $\mathcal{H}$, all with norm $\|f\|_\mathcal{H}$. Since π does not have fixed points, we have reached a contradiction, and (27) cannot hold, completing the proof. $\square$

THEOREM 3.8. *Let Γ be a group with finite, symmetric generating set $S \subseteq \Gamma$, and let P be the transition kernel of the simple random walk on the Cayley graph* $\mathsf{Cay}(G; S)$. *If Γ does not have property* (T), *then there exists a Hilbert space $\mathcal{H}$ and a nonconstant Γ-equivariant P-harmonic mapping from Γ into $\mathcal{H}$.*

PROOF. We write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the inner product and norm on $\mathcal{H}$. Let $\{\psi_j\}_{j=0}^\infty$ be a sequence in $\mathcal{H}$ with

$$\frac{\|(I - P_\dagger)\psi_j\|^2}{\langle \psi_j, (I - P_\dagger)\psi_j \rangle} \to 0.$$

The existence of such a sequence is the content of Theorem 3.7.

Define $\Psi_j : \Gamma \to \mathcal{H}$ by

$$\Psi_j(g) = \frac{\pi(g)\psi_j}{2\langle \psi_j, (I - P_\dagger)\psi_j \rangle},$$

where we recall the definition of $P_\dagger$ from (26). Then, for every $j = 0, 1, \ldots,$ and for any $g \in \Gamma$,

$$\frac{1}{|S|} \sum_{s \in S} \|\Psi_j(g) - \Psi_j(gs)\|^2 = \frac{1}{|S|} \sum_{s \in S} \frac{\|\pi(g)\psi_j - \pi(gs)\psi_j\|^2}{2\langle \psi_j, (I - P_\dagger)\psi_j \rangle}$$

$$= \frac{1}{|S|} \sum_{s \in S} \frac{\|\psi_j - \pi(s)\psi_j\|^2}{2\langle \psi_j, (I - P_\dagger)\psi_j \rangle} = 1,$$

where we have used the fact that π acts by isometries.

By the same token,

$$\left\| \Psi_j(g) - \frac{1}{|S|} \sum_{s \in S} \Psi_j(gs) \right\|^2 = \frac{\|(I - P_\dagger)\psi_j\|^2}{2\langle \psi_j, (I - P_\dagger)\psi_j \rangle} \to 0.$$

Equipping Γ with the word metric on $\mathsf{Cay}(G; S)$, an application of Lemma 3.2 finishes the proof, just as in Theorem 3.1. $\square$

4. The rate of escape. We now show how simple estimates derived from harmonic functions lead to more detailed information about the random walk. In fact, we will show that in general situations, a hitting time bound alone leads to some finer estimates.

4.1. *Graph estimates.* Consider again a symmetric, stochastic matrix $\{P(x, y)\}_{x,y \in V}$ for some at most countable index set V. Let $\Gamma \leq \mathsf{Aut}(P)$ be a subgroup that acts transitively on V, and let dist be a Γ-invariant metric on V. Finally, let $\{X_t\}$ denote the random walk with transition kernel P started at some fixed point $x_0 \in V$.

For any $k \in \mathbb{N}$, let H_k denote the first time t at which $\mathsf{dist}(X_0, X_t) \geq k$, and define the function $h : \mathbb{N} \to \mathbb{R}$ by $h(k) = \mathbb{E}[H_k]$. We start with the following simple lemma which employs reversibility, transitivity and the triangle inequality. It is based on an observation due to Mark Braverman; see also the closely related inequalities of Babai [2].

LEMMA 4.1. *For any $T \geq 0$, we have*

$$\mathbb{E}\,\mathsf{dist}(X_0, X_T) \geq \frac{1}{2} \max_{0 \leq t \leq T} \mathbb{E}\big[\mathsf{dist}(X_0, X_t) - \mathsf{dist}(X_0, X_1)\big].$$

PROOF. Let $s' \leq T$ be such that

$$\mathbb{E}\,\mathsf{dist}(X_0, X_{s'}) = \max_{0 \leq t \leq T} \mathbb{E}\,\mathsf{dist}(X_0, X_t).$$

Then there exists an even time $s \in \{s', s' - 1\}$ such that $\mathbb{E}\,\mathsf{dist}(X_0, X_s) \geq \mathbb{E}[\mathsf{dist}(X_0, X_{s'}) - d(X_0, X_1)]$.

Consider $\{X_t\}$ and an identically distributed walk $\{\tilde{X}_t\}$ such that $\tilde{X}_t = X_t$ for $t \leq s/2$, and $\tilde{X}_t$ evolves independently after time $s/2$. By the triangle inequality, we have

$$\mathsf{dist}(X_0, \tilde{X}_T) + \mathsf{dist}(\tilde{X}_T, X_s) \geq \mathsf{dist}(X_0, X_s).$$

But by reversibility and transitivity, each of $\mathsf{dist}(X_0, \tilde{X}_T)$ and $\mathsf{dist}(\tilde{X}_T, X_s)$ are distributed as $\mathsf{dist}(X_0, X_T)$. Taking expectations, the claimed result follows. □

LEMMA 4.2. *If* $h(k) \leq T$, *then*

$$\mathbb{E}\,\mathsf{dist}(X_0, X_{2T}) \geq \frac{k}{4} - \frac{1}{4}\mathbb{E}\,\mathsf{dist}(X_0, X_1).$$

PROOF. Let $\alpha = \frac{1}{k} \max_{0 \leq t \leq 2T} \mathbb{E}\,\mathsf{dist}(X_0, X_t)$. First, observe that the triangle inequality implies

(29) $$\mathsf{dist}(X_0, X_{2T}) \geq \mathbf{1}_{\{H_k \leq 2T\}}\big(k - \mathsf{dist}(X_{H_k}, X_{2T})\big).$$

By transitivity, we also have

$$\mathbb{E}\big[\mathbf{1}_{\{H_k \leq 2T\}} \cdot \mathsf{dist}(X_{H_k}, X_{2T})\big] = \mathbb{E}\big[\mathbf{1}_{\{H_k \leq 2T\}} \cdot \mathsf{dist}(X_0, X_{2T-H_k})\big] \leq \mathbb{P}(H_k \leq 2T)\alpha k.$$

Thus taking expectations in (29) yields

$$\mathbb{E}\,\mathsf{dist}(X_0, X_{2T}) \geq \mathbb{P}(H_k \leq 2T)(1-\alpha)k \geq \tfrac{1}{2}(1-\alpha)k,$$

recalling our assumption that $\mathbb{E}[H_k] \leq T$. On the other hand, Lemma 4.1 shows that

$$\mathbb{E}\,\mathsf{dist}(X_0, X_{2T}) \geq \tfrac{1}{2}\big(\alpha k - \mathbb{E}\big[\mathsf{dist}(X_0, X_1)\big]\big).$$

Averaging these two inequalities yields the desired result. □

Using transitivity, one can also prove a small-ball occupation estimate, directly from information on the hitting times.

THEOREM 4.3. *Assume that* $\mathsf{dist}(X_0, X_1) \leq B$ *almost surely. Consider any* $k \in \mathbb{N}$. *If* $h(k) \leq T$, *then for any* $\varepsilon > 0$, *we have*

$$\frac{1}{T}\sum_{t=0}^{T}\mathbb{P}\big[\mathsf{dist}(X_0, X_t) \leq \varepsilon k\big] \leq O(\varepsilon + B/k).$$

PROOF. Let $\alpha = 3\varepsilon k$. We define a sequence of random times $\{t_i\}_{i=0}^{\infty}$ as follows. First, $t_0 = 0$. We then define t_{i+1} as the smallest time $t > t_i$ such that $\mathsf{dist}(X_t, X_{t_j}) \geq \alpha$ for all $j \leq i$. We put $t_{i+1} = \infty$ if no such t exists. Observe that the set $\{X_{t_i} : t_i < \infty\}$ is α-separated in the metric dist.

We then define, for each $i \geq 0$, the quantity

$$\tau_i = \begin{cases} 0, & \text{if } t_i > 2T, \\ \#\{t \in [t_i, t_i + 2T] : \text{dist}(X_t, X_{t_i}) \leq \alpha/3\}, & \text{otherwise.} \end{cases}$$

Since the set $\{X_{t_i} : t_i < \infty\}$ is α-separated, the $(\alpha/3)$-balls about the centers X_{t_i} are disjoint, and thus we have the inequality

$$(30) \qquad\qquad 4T \geq \sum_{i=0}^{\infty} \tau_i,$$

where the latter sum is over only finitely many terms.

We can also calculate for any $i \geq 0$,

$$\mathbb{E}[\tau_i] \geq \mathbb{P}(t_i \leq 2T) \cdot \mathbb{E}[\tau_0],$$

using transitivity. Now, we have $t_i \leq 2T$ if $\text{dist}(X_0, X_T) \geq (B + \alpha)i$, and thus for $i \leq k/(B + \alpha)$, we have

$$\mathbb{P}(t_i > 2T) \leq \mathbb{P}(H_k > 2T) \leq \frac{\mathbb{E}[H_k]}{2T} \leq \frac{1}{2}.$$

We conclude that for $i \leq k/(B + \alpha)$, we have $\mathbb{E}[\tau_i] \geq \frac{1}{2}\mathbb{E}[\tau_0]$. Combining this with (30) yields

$$\mathbb{E}[\tau_0] \leq O\big((\varepsilon + B/k)T\big). \qquad\qquad \square$$

REMARK 4.4. In the next section, we prove analogs of the preceding statements for Hilbert space-valued martingales. One can then use harmonic functions to obtain such results in the graph setting. The results in this section are somewhat more general though, since they give general connections between the function $h(k)$ and other properties of the chain. For instance, for every $j \in \mathbb{N}$, there are groups where $h(k) \asymp k^{1/(1-2^{-j})}$ as $k \to \infty$ [6].

4.2. *Martingale estimates.* We now prove analogs of Lemma 4.2 and Theorem 4.3 in the setting of martingales.

Let $\{M_t\}$ be a martingale taking values in some Hilbert space $\mathcal{H}$, with respect to the filtration $\{\mathcal{F}_t\}$. Assume that $\mathbb{E}[\|M_{t+1} - M_t\|^2 \mid \mathcal{F}_t] \geq 1$ for every $t \geq 0$, and there exists a $B \geq 1$ such that for every $t \geq 0$, we have $|M_{t+1} - M_t| \leq B$ almost surely.

LEMMA 4.5 (Martingale hitting times). *For $R \geq 0$, let τ be the first time t such that $\|M_t - M_0\| \geq R$. Then, $\mathbb{E}(\tau) \leq (R + B)^2$.*

PROOF. Applying the optional stopping theorem (see, e.g., [17], Corollary 17.7) to the submartingale $\|M_t - M_0\|^2 - t$, with the stopping time τ, we see that $\mathbb{E}(\tau) \leq \mathbb{E}(\|M_\tau - M_0\|^2)$, and $\mathbb{E}(\|M_\tau - M_0\|^2) \leq (R + B)^2$. $\square$

The following simple estimate gives a lower bound on the L^1 rate of escape for a martingale.

LEMMA 4.6 (L^1 estimate). *For every $T \geq 0$, we have $\mathbb{E}\|M_0 - M_T\| \geq \sqrt{T/8} - B/2$.*

PROOF. Let $\tau \geq 0$ be the first time such that $\|M_0 - M_\tau\| \geq \sqrt{T/2} - B$, and let $\tau' = \min(\tau, T)$. First, by Lemma 4.5 and Markov's inequality, we have

$$\mathbb{E}\|M_0 - M_{\tau'}\| \geq \mathbb{P}(\tau \leq T) \cdot (\sqrt{T/2} - B) \geq \sqrt{\frac{T}{8}} - \frac{B}{2}.$$

Then, since $\|M_0 - M_t\|$ is a submartingale and τ' and T are stopping times with $\tau' \leq T$, we have

$$\mathbb{E}\|M_0 - M_T\| \geq \mathbb{E}\|M_0 - M_{\tau'}\|. \qquad \square$$

Now we prove an analog of Theorem 4.3 in the martingale setting, beginning with a preliminary lemma.

LEMMA 4.7. *For $R \geq R' \geq 0$, let p_R denote the probability that $\|M_t\| \geq \|M_0\| + R$ occurs before $\|M_t\| \leq \|M_0\| - R'$. Then $p_R \geq \frac{R'}{2R+B}$.*

PROOF. Let $\tau \geq 0$ be the first time at which $\|M_\tau\| \geq \|M_0\| + R$ or $\|M_\tau\| \leq \|M_0\| - R'$. Since $\|M_t\| - \|M_0\|$ is a submartingale, the optional stopping theorem implies

$$0 \leq \mathbb{E}(\|M_\tau\| - \|M_0\|) \leq p_R(R + B) - (1 - p_R)R' \leq p_R(2R + B) - R'.$$

Rearranging yields the desired result. $\square$

From this, we can prove a general occupation time estimate.

LEMMA 4.8 (Martingale occupation times). *Suppose that $M_0 = 0$. Then for every $\varepsilon \geq B/\sqrt{T}$ and $T \geq 1$, we have*

$$\frac{1}{T} \sum_{t=0}^{T} \mathbb{P}[\|M_t\| \leq \varepsilon\sqrt{T}] \leq O(\varepsilon).$$

PROOF. Let $h = \lceil 2(3\varepsilon\sqrt{T} + B)^2 \rceil$. Let $\mathcal{B}$ denote the ball of radius $\varepsilon\sqrt{T}$ about 0 in $\mathcal{H}$. For $t \leq T - h$, let p_t denote the probability that $M_t \in \mathcal{B}$, but $M_{t+h}, M_{t+h+1}, \ldots, M_T \notin \mathcal{B}$. We first show that for every such t,

$$(31) \qquad\qquad p_t \geq \frac{\varepsilon}{40} \cdot \mathbb{P}(\|M_t\| \leq \varepsilon\sqrt{T}).$$

To this end, we prove three bounds. First,

$$
(32) \qquad
\begin{aligned}
&\mathbb{P}\big(\exists i \leq h \text{ such that } \|M_{t+i}\| \geq 2\varepsilon\sqrt{T}\,|M_t \in \mathcal{B}\big) \\
&\qquad \geq \mathbb{P}\big(\exists i \leq h \text{ such that } \|M_{t+i} - M_t\| \geq 3\varepsilon\sqrt{T}\,|M_t \in \mathcal{B}\big) \geq \tfrac{1}{2},
\end{aligned}
$$

where the latter bound follows from Markov's inequality and Lemma 4.5.

Next, observe that for $R \geq \varepsilon\sqrt{T}$, Lemma 4.7 gives

$$
(33) \qquad
\begin{aligned}
&\mathbb{P}\big(\|M_s\| \geq R \text{ occurs before } \|M_s\| \leq \varepsilon\sqrt{T} \text{ for } s \geq t+i\,|\,\|M_{t+i}\| \geq 2\varepsilon\sqrt{T}\big) \\
&\qquad \geq \frac{\varepsilon\sqrt{T}}{2R + B}.
\end{aligned}
$$

Finally, the Doob–Kolmogorov maximal inequality implies that

$$
(34) \quad \mathbb{P}\Big(\max_{0 \leq r \leq T} \|M_s - M_{s+r}\| > R|\mathcal{F}_s\Big) \leq \frac{\mathbb{E}[\|M_s - M_{s+T}\|^2 \mid \mathcal{F}_s]}{R^2} = \frac{T}{R^2}.
$$

Setting $R = 2\sqrt{T}$, (33) and (34) imply that for any time $t \geq 0$, we have

$$
\mathbb{P}\big(M_{t+i}, M_{t+i+1}, \ldots, M_T \notin \mathcal{B}\,|\,\|M_{t+i}\| \geq 2\varepsilon\sqrt{T}\big) \geq \frac{\varepsilon}{20}.
$$

Combining this with (32) yields (31).

But we must have

$$
\sum_{t=0}^{T} p_t \leq h = O(\varepsilon^2 T),
$$

by construction. Thus (31) yields

$$
\sum_{t=0}^{T} \mathbb{P}\big[\|M_t\| \leq \varepsilon\sqrt{T}\big] \leq O(\varepsilon T),
$$

completing the proof. □

4.3. *Applications.* Combining the observations of the preceding section, together with the existence of harmonic functions, yields our claimed results on transitive graphs. In particular, the following result, combined with Theorem 3.1, proves Theorem 1.1.

THEOREM 4.9. *Let V be a countably infinite index set, and let $\{P(x, y)\}_{x,y \in V}$ be a stochastic, symmetric matrix. Suppose that $\Gamma \leq \mathsf{Aut}(P)$ is a closed, amenable, unimodular subgroup that acts transitively on V, and there exists a connected graph $G = (V, E)$ on which Γ acts by automorphisms. Suppose further that for some $B > 0$, for all $x, y \in V$, we have*

$$
(35) \qquad\qquad P(x, y) \qquad \text{implies } \mathsf{dist}(x, y) \leq B,
$$

where dist *is the path metric on G. Suppose also that*

$$p_* = \min\{P(x, y) : \{x, y\} \in E\} > 0.$$

If there exists a Hilbert space $\mathcal{H}$ and a nonconstant Γ-equivariant $\mathcal{H}$-valued P-harmonic mapping, then the following holds.

Let $\{X_t\}$ denote the random walk with transition kernel P. For every $t \geq 0$, we have the estimates

$$(36) \qquad\qquad \mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq p_* t,$$

$$(37) \qquad\qquad \mathbb{E}\big[\mathsf{dist}(X_0, X_t)\big] \geq \frac{\sqrt{p_* t}}{24} - \frac{3}{2} B,$$

and, for every $\varepsilon \geq 1/\sqrt{T}$ and $T \geq 4/p_*$,

$$(38) \qquad\qquad \frac{1}{T} \sum_{t=0}^{T} \mathbb{P}\big[\mathsf{dist}(X_0, X_t) \leq \varepsilon \sqrt{p_* T/B}\big] \leq O(\varepsilon).$$

PROOF.   Let $\mathcal{H}$ and $\Psi : V \to \mathcal{H}$ be the Hilbert space and nonconstant Γ-equivariant P-harmonic mapping. Let $\|\cdot\| = \|\cdot\|_{\mathcal{H}}$, and normalize Ψ so that for every $x \in V$,

$$(39) \qquad\qquad \sum_{y \in V} P(x, y)\big\|\Psi(x) - \Psi(y)\big\|^2 = 1.$$

Then $M_t = \Psi(X_t)$ is an $\mathcal{H}$-valued martingale with $\mathbb{E}[\|M_{t+1} - M_t\|^2 | \mathcal{F}_t] = 1$ for every $t \geq 0$.

Furthermore, from (39), we see that Ψ is $\sqrt{1/p_*}$-Lipschitz as a mapping from (V, dist) to $\mathcal{H}$. Thus one has immediately the estimate

$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq p_* \mathbb{E}\big[\|M_t - M_0\|^2\big] = p_* t.$$

Now, for any $k \in \mathbb{N}$, let H_k denote the first time t at which $\mathsf{dist}(X_0, X_t) = k$, and define the function $h : \mathbb{N} \to \mathbb{R}$ by $h(k) = \mathbb{E}[H_k]$. Since Ψ is $\sqrt{1/p_*}$-Lipschitz, Lemma 4.5 applied to $\{M_t\}$ shows that for every $k \in \mathbb{N}$,

$$h(k) \leq \frac{(k + B)^2}{p_*}.$$

Combining this with Lemma 4.2 yields (36). Combining it with Theorem 4.3 yields (38). $\square$

Although we have proved a result about occupation times, we conjecture that a stronger bound holds.

CONJECTURE 4.10. *Suppose that G is an infinite, transitive, connected, amenable graph with degree d, and $\{X_t\}$ is the simple random walk on G. Theorem 4.9 shows that for every $\varepsilon > 1/\sqrt{T}$ and $T \geq 4d$, we have*

$$\frac{1}{T} \sum_{t=0}^{T} \mathbb{P}\big(\mathsf{dist}(X_0, X_t) \leq \varepsilon \sqrt{T/d}\big) \leq O(\varepsilon).$$

We conjecture that this holds pointwise; that is, for every large enough time t, we have

$$\mathbb{P}\big(\mathsf{dist}(X_0, X_t) \leq \varepsilon \sqrt{t/d}\big) \leq O(\varepsilon).$$

Finally, we conclude with a theorem about finite graphs which, in particular, yields Theorem 1.2.

THEOREM 4.11. *Let V be a finite index set and suppose that $\mathsf{Aut}(P)$ acts transitively on V, and on the graph $G = (V, E)$ by automorphisms. If*

$$p_* = \min\big\{P(x, y) : \{x, y\} \in E\big\} > 0,$$

and $\lambda < 1$ is the second-largest eigenvalue of P, then for every $t \leq (1 - \lambda)^{-1}$, we have

$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)^2\big] \geq p_* t/2,$$
$$\mathbb{E}\big[\mathsf{dist}(X_0, X_t)\big] \geq \Omega(\sqrt{p_* t}) - B,$$

and, for every $\varepsilon > 0$ and $(1 - \lambda)^{-1} \geq T \geq 4/p_*$,

(40) $$\frac{1}{T} \sum_{t=0}^{T} \mathbb{P}\big[\mathsf{dist}(X_0, X_t) \leq \varepsilon \sqrt{p_* T/B}\big] \leq O(\varepsilon).$$

PROOF. Let $\psi : V \to \mathbb{R}$ be such that $P\psi = \lambda\psi$, and define $\Psi : V \to \ell^2(\mathsf{Aut}(P))$ by

$$\Psi(x) = \frac{(\psi(\sigma x))_{\sigma \in \mathsf{Aut}(P)}}{\sqrt{2\langle \psi, (I - P)\psi \rangle}}.$$

An argument as in (5) shows that $\|\Psi\|_{\mathrm{Lip}} \leq \sqrt{1/p_*}$.

Now, observe that $\{\lambda^{-t}\Psi(X_t)\}$ is a martingale. This follows from the fact that $\lambda^{-t}\psi(X_t)$ is a martingale, which one easily checks.

$$\mathbb{E}\big[\lambda^{-t-1}\psi(X_{t+1})|X_t\big] = \lambda^{-t-1}(P\psi)(X_t) = \lambda^{-t}\psi(X_t).$$

Note that $t \leq (1 - \lambda)^{-1}$, hence the mapping $x \mapsto \lambda^{-t}\Psi(x)$ is $O(\sqrt{1/p_*})$-Lipschitz, and the same argument as in Theorem 4.9 applies. $\square$

Acknowledgments. We thank Tonci Antunovic for detailed comments on earlier drafts of this manuscript, and Tim Austin for his suggestions toward obtaining equivariance in Theorem 3.1. We also thank Anna Èrshler, David Fisher, Subhroshekhar Gosh, Gady Kozma, Gábor Pete and Bálint Virág for useful discussions. Finally, we are grateful to an anonymous referee for many detailed suggestions.

REFERENCES

[1] AUSTIN, T., NAOR, A. and PERES, Y. (2009). The wreath product of $\mathbb{Z}$ with $\mathbb{Z}$ has Hilbert compression exponent $\frac{2}{3}$. *Proc. Amer. Math. Soc.* **137** 85–90. MR2439428

[2] BABAI, L. (1991). Local expansion of vertex-transitive graphs and random generation in finite groups. In *Proceedings of the Twenty Third Annual ACM Symposium on Theory of Computing* 164–174. ACM, New York.

[3] BEKKA, B., DE LA HARPE, P. and VALETTE, A. (2008). *Kazhdan's Property* (*T*), *Volume* 11 *of New Mathematical Monographs*. Cambridge Univ. Press, Cambridge.

[4] BENJAMINI, I., LYONS, R., PERES, Y. and SCHRAMM, O. (1999). Group-invariant percolation on graphs. *Geom. Funct. Anal.* **9** 29–66. MR1675890

[5] DIACONIS, P. and SALOFF-COSTE, L. (1994). Moderate growth and random walk on finite groups. *Geom. Funct. Anal.* **4** 1–36. MR1254308

[6] ÈRSHLER, A. G. (2001). On the asymptotics of the rate of departure to infinity. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **283** 251–257, 263. MR1879073

[7] ÈRSHLER, A. G. (2005). Personal communication.

[8] FISHER, D. and MARGULIS, G. (2005). Almost isometric actions, property (T), and local rigidity. *Invent. Math.* **162** 19–80. MR2198325

[9] GRIMMETT, G. R. and STIRZAKER, D. R. (1992). *Probability and Random Processes*, 2nd ed. Clarendon Press Oxford Univ. Press, New York. MR1199812

[10] GROMOV, M. (2003). Random walk in random groups. *Geom. Funct. Anal.* **13** 73–146. MR1978492

[11] HEBISCH, W. and SALOFF-COSTE, L. (1993). Gaussian estimates for Markov chains and random walks on groups. *Ann. Probab.* **21** 673–709. MR1217561

[12] KESTEN, H. (1959). Symmetric random walks on groups. *Trans. Amer. Math. Soc.* **92** 336–354. MR0109367

[13] KLEINER, B. (2010). A new proof of Gromov's theorem on groups of polynomial growth. *J. Amer. Math. Soc.* **23** 815–829. MR2629989

[14] KOREVAAR, N. J. and SCHOEN, R. M. (1997). Global existence theorems for harmonic maps to non-locally compact spaces. *Comm. Anal. Geom.* **5** 333–387. MR1483983

[15] KWAPIEŃ, S. and WOYCZYŃSKI, W. A. (1992). *Random Series and Stochastic Integrals*: *Single and Multiple*. Birkhäuser, Boston, MA. MR1167198

[16] LEE, J. R. and MAKARYCHEV, Y. (2009). Eigenvalue multiplicity and volume growth. Available at arXiv:0806.1745 [math.MG].

[17] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2009). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. MR2466937

[18] LYONS, R. and SCHRAMM, O. (1999). Stationary measures for random walks in a random environment with random scenery. *New York J. Math.* **5** 107–113 (electronic). MR1703207

[19] MOK, N. (1995). Harmonic forms with values in locally constant Hilbert bundles. In *Proceedings of the Conference in Honor of Jean-Pierre Kahane* (*Orsay*, 1993). *J. Fourier Anal. Appl. Special Issue* 433–453. MR1364901

[20] NAOR, A. and PERES, Y. (2008). Embeddings of discrete groups and the speed of random walks. *Int. Math. Res. Not. IMRN* Art. ID rnn 076, 34. MR2439557

[21] NAOR, A. and PERES, Y. (2011). L_p compression, traveling salesmen, and stable walks. *Duke Math. J.* **157** 53–108. MR2783928

[22] PERES, Y., PETE, G. and SCOLNICOV, A. (2006). Critical percolation on certain nonunimodular graphs. *New York J. Math.* **12** 1–18 (electronic). MR2217160

[23] SALVATORI, M. (1992). On the norms of group-invariant transition operators on graphs. *J. Theoret. Probab.* **5** 563–576. MR1176438

[24] SHALOM, Y. and TAO, T. (2010). A finitary version of Gromov's polynomial growth theorem. *Geom. Funct. Anal.* **20** 1502–1547. MR2739001

[25] SOARDI, P. M. and WOESS, W. (1990). Amenability, unimodularity, and the spectral radius of random walks on infinite graphs. *Math. Z.* **205** 471–486. MR1082868

[26] VAROPOULOS, N. T. (1985). Isoperimetric inequalities and Markov chains. *J. Funct. Anal.* **63** 215–239. MR0803093

[27] VERSHIK, A. M. (2000). Dynamic theory of growth in groups: Entropy, boundaries, examples. *Uspekhi Mat. Nauk* **55** 59–128. MR1786730

[28] VIRAG, B. (2005). Personal communication.

[29] WOESS, W. (2000). *Random Walks on Infinite Graphs and Groups. Cambridge Tracts in Mathematics* **138**. Cambridge Univ. Press, Cambridge. MR1743100

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING
BOX 352350
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195-2350
USA
E-MAIL: jrl@cs.washington.edu

MICROSOFT RESEARCH
ONE MICROSOFT WAY
WASHINGTON, 98052
USA
E-MAIL: peres@microsoft.com