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CONDITIONING SUPER-BROWNIAN MOTION ON ITS BOUNDARY STATISTICS, AND FRAGMENTATION

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We condition super-Brownian motion on "boundary statistics" of the exit measure X_D from a bounded domain D. These are random variables defined on an auxiliary probability space generated by sampling from the exit measure X_D. Two particular examples are: conditioning on a Poisson random measure with intensity βX_D and conditioning on X_D itself. We find the conditional laws as h-transforms of the original SBM law using Dynkin's formulation of X-harmonic functions. We give explicit expression for the (extended) X-harmonic functions considered. We also obtain explicit constructions of these conditional laws in terms of branching particle systems. For example, we give a fragmentation system description of the law of SBM conditioned on $X_D = \nu$, in terms of a particle system, called the backbone. Each particle in the backbone is labeled by a measure $\tilde{\nu}$, representing its descendants' total contribution to the exit measure. The particle's spatial motion is an h-transform of Brownian motion, where h depends on $\tilde{\nu}$. At the particle's death two new particles are born, and $\tilde{\nu}$ is passed to the newborns by fragmentation.

1. Introduction. Studying conditioned Markov processes is a kind of inverse problem—given information about how the process ends up, one tries to infer how it got there, at least in terms of probabilities. In the context of Brownian motion and finite-dimensional Markov processes, one can make very explicit calculations, starting with the work of Doob (1959). Attempts to make similar calculations for super Brownian motion are more recent. These studies typically aim to recover the conditional law of a superprocess as the law of a distinct probabilistic object. Several authors have succeeded in coming up with such descriptions for certain conditionings, and produced models with remarkably rich structure. The first of these models was the immortal particle system of Evans and Perkins (1990) and Evans (1993), for super-Brownian motion in $\mathbb{R}^d$ conditioned on survival. In this model, an immortal particle moves according to a Brownian motion, and throws off mass at

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a uniform rate, and then this mass evolves in the space as an unconditioned super-Brownian motion. Following Evans and Perkins, Salisbury and Verzani (1999) considered a super Brownian motion X in a domain D, with conditioning based on the *exit measure* X_D from D. More specifically, they conditioned X on the event that the support of X_D contains certain points $z_1, \dots, z_k$, and recovered the resulting conditional law in terms of a branching backbone system. The branching backbone is a random tree with k leaves reaching the points $z_1, \dots, z_k$. Similar to Evans and Perkins's model, there is mass uniformly created along the branching backbone which follows the law of an unconditioned super-process independent of the points $z_1, \dots, z_k$. Giving such an explicit characterization of a conditioned process is an interesting problem from a probabilistic modeling point of view. For example, in population dynamics, one can view it as an analogue of a host of biological problems in which one has information about the state of the population at certain times or locations, and one wishes to infer the genealogical structure of the populations of the ancestors (e.g., the "out of Africa" problem of human origins). For explicit representations of other related conditioned processes, see Roelly-Coppoletta and Rouault (1989), Overbeck (1993, 1994), Etheridge (1993) or Etheridge and Williams (2003).

It turns out that there is more to the conditioning problem than described above. A conditioned process represents a special case of a Girsanov transformation, or a martingale change of measure. For concreteness, let us consider the following example: let ξ be Brownian motion in a domain E. Let τ_E be the exit time from E. We compute the conditional law, Π_x^z of ξ given $\xi_{\tau_E} = z$, by a martingale change of measure from Π_x, the law of ξ. This martingale change of measure is given in terms of a certain harmonic function $h^z(\cdot)$. More precisely, for any domain D compactly contained in E, and any Y measurable with respect to $\mathcal{F}_{\tau_D}$, we have $\Pi_x^z(Y) = \Pi_x(Y h^z(\xi_{\tau_D})/h^z(x))$. A Girsanov transformation defined in terms of a harmonic function is called an h-transform, and typically conditional laws of Markov processes are formulated as h-transforms of their original laws. This classical relationship between harmonic functions and conditioning a Markov process leads to an elegant probabilistic formulation of the Martin boundary theory for elliptic differential operators.

In the context of super-processes, the analogue of harmonic functions are X-harmonic functions. Following the definition of Dynkin (2002), let us consider a super-Brownian motion $X = (X_D, P_\mu)$, a family of random measures (*exit measures*) and their associated probability laws where D is an open subset of a given domain E in $\mathbb{R}^d$, and μ is a finite measure on E. Write $D \Subset E$ if D is open and its closure is a compact subset of E. A nonnegative function H is *X-harmonic* if for any $D \Subset E$ and any finite measure μ with support in D,

$$P_\mu\big(H(X_D)\big) = H(\mu). \tag{1}$$

Note that this property resembles the mean value property of a harmonic function, hence the name X-harmonic. Moreover, the X-harmonic functions are related to conditioning super-Brownian motion in the same way as harmonic functions are related to conditioning Brownian motion; they give us the explicit Girsanov transformation to switch from the unconditioned probability law to the conditioned probability law. An H-transform P_μ^H is obtained from P_μ by setting

$$P_\mu^H(Y) = \frac{1}{H(\mu)} P_\mu\big(H(X_D)Y\big)$$

for Y nonnegative and $\mathcal{F}_{\subset D}$-measurable, where $\mathcal{F}_{\subset D}$ is the σ-algebra generated by $X_{D'}$, $D' \subset D$. In his book, Dynkin (2002) suggests a new direction for investigating the solutions of the p.d.e. $\frac{1}{2}\Delta = 2u^2$, namely to explore X-harmonic functions (thinking them as the analogue of harmonic functions) and ultimately, to build a Martin boundary theory for this nonlinear p.d.e. In this case, the Martin boundary is defined as the set of extreme elements of the convex set of all X-harmonic functions. Dynkin points out that a concrete understanding of extreme X-harmonic functions might yield further insights about the solutions of the p.d.e. $\frac{1}{2}\Delta u = 2u^2$. Since then major progress has been made on the study of the solutions of this p.d.e. using other approaches. For example, Mselati (2004) classified the solutions as σ-moderate in the case of a smooth domain (solving a conjecture of Dynkin and Kuznetsov). However, the relationship between X-harmonic functions and solutions remains largely unexplored. Dynkin, in a series of papers, has taken concrete steps to better formulate and understand extreme X-harmonic functions. Dynkin (2006b) obtains the extreme X-harmonic functions by a limiting procedure from the Radon–Nykodym densities $H_D^\nu(\mu) = \frac{dP_{\mu,X_D}}{dP_{c,X_D}}(\nu)$ of $P_{\mu,X_D}(d\nu) = P_\mu(X_D \in d\nu)$ with respect to $P_{c,X_D}(d\nu) = P_c(X_D \in d\nu)$. Dynkin (2006a) derives a formula for $H_D^\nu(\mu)$ using diagram description of moments. The functions H_D^ν will be central to our analysis as well; we will run into them while studying conditionings of SBM, and use them to derive results about the structure of conditioned SBM.

Our goal in this paper is to explore various ways of conditioning a super-Brownian motion. We are motivated by the rich structure of the underlying probabilistic objects as well as its potential connection to Dynkin's research program on Martin boundary theory of SBM. Here is a summary of our contributions: we develop a way of conditioning a super-Brownian motion, which we call "conditioning on boundary statistics." The random variables which we condition on are defined on an auxiliary probability space, and generated by sampling from the exit measure X_D. We find representations of these conditionings as H-transforms of unconditioned SBM. In general, we identify the functions H as "extended" X-harmonic. The term "extended" is used because, even though we show that these functions satisfy the mean value property (1), in general, we do not know whether they are finite for all μ. An example of a boundary statistic is a Poisson random measure with intensity βX_D, where $\beta > 0$. It turns out that for this kind of conditioning the resulting H-transform is through an X-harmonic family of functions studied earlier in the literature. Another important and more complex example we studied is conditioning SBM on its exit measure X_D, which is what most of our paper is devoted to. We find that the corresponding H-transform uses the family $H_D^{\nu}(\mu) = \frac{dP_{\mu,X_D}}{dP_{c,X_D}}(\nu)$, densities first introduced in Dynkin (2006b). This paper shows that one can choose a version of this family such that for each ν, $H^{\nu}(\mu)$ will satisfy the mean value property (1) for each μ. Although this family is claimed to be X-harmonic in Dynkin (2006b), we are not aware of any results that actually show that H^{ν} will be a finite function of μ. In this paper we shall classify H_D^{ν} as an extended X-harmonic function and leave the question of finiteness to be resolved in a different paper, as this will require us to develop analytical bounds on the densities of moment measures of SBM. In this paper we will not go beyond describing the probabilistic structure of conditioned SBM on its exit measure; as such we are not going to lose much generality by stating and proving our results without assuming that H^{ν} is finite. Indeed, for our purposes it will suffice that $H_D^{X_D}(\mu)$ is finite P_μ almost surely, which is true. The family $H_D^{\nu}(\mu)$ is of special interest because it can be considered as the analogue of the Poisson kernel of D. Also, Dynkin (2006b) showed that if H is an extreme X-harmonic function in E, then for every μ, and for every sequence D_k exhausting E

$$H(\mu) = \lim_{k\to\infty} H_{D_k}^{X_{D_k}}(\mu)$$

P_μ^H almost surely.

The heart of the paper is Theorem 8, which gives a new formula for H_D^{ν}. From this we deduce an infinite particle fragmentation system description of $P_\mu^{\nu} = P_\mu^{H^{\nu}}$, the conditional law of X in D given $X_D = \nu$ (Theorem 11). This is carried out in terms of a particle system, called the backbone in Salisbury and Verzani (1999), along which a mass is created uniformly. In the backbone, each particle is assigned a measure $\tilde{\nu}$ at its birth. The spatial motion of the particle is an h-transform of Brownian motion, where h is a potential that depends on $\tilde{\nu}$. The measure $\tilde{\nu}$ represents the particle's contribution to the exit measure. At the particle's death two new particles are born, and $\tilde{\nu}$ is passed to the newborns by fragmentation into two bits. Here, we use the techniques of Salisbury and Verzani (1999) applied to a more general setting. This description connects the theory of conditioned super-processes to the growing literature on infinite fragmentation and coalescent processes; see, for example, Bertoin (2006) for a comprehensive exposition.

2. Preliminaries.

2.1. *Super-Brownian motion.* We will follow Dynkin's definition of super-Brownian motion (SBM). Let E be a domain of $\mathbb{R}^d$, and let $\mathcal{M}_E$ be the positive finite measures on E. A super-Brownian motion, (X_D, P_μ), is a family of random measures (*exit measures*) and their associated probability laws where D is an open subset of a given domain E in $\mathbb{R}^d$, and μ is a finite measure on E with the following properties:

(a) Exit property: $P_\mu(X_D(D)=0)=1$ for every μ, and if $\mu(D)=0$, then $P_\mu(X_D=\mu)=1$.

(b) Markov property: if $Y\ge 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\subset D}$ generated by $X_{D'}$, $D'\subset D$ and $Z\ge 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\supset D}$ generated by $X_{D''}$, $D''\supset D$, then

$$P_\mu(YZ)=P_\mu(YP_{X_D}Z).$$

(c) Branching property: for any nonnegative Borel f,

$$P_\mu(e^{-\langle X_D,f\rangle})=e^{-\langle \mu,V_Df\rangle} \qquad \text{where } V_Df(y)=-\log P_y(e^{-\langle X_D,f\rangle})$$

and $P_y=P_{\delta_y}$.

(d) Integral equation for the log-Laplace functional: V_Df solves the integral equation

$$u+G_D(2u^2)=K_Df,$$

where G_D and K_D are, respectively, Green and Poisson operators for Brownian motion in D. In other words, if ξ_t is a Brownian motion starting from x, under a probability measure Π_x, then $K_Df(x)=\Pi_xf(\xi_{\tau_D})$, where τ_D is the exit time from D. Likewise, $G_Df(x)=\Pi_x(\int_0^{\tau_D}f(\xi_t)\,dt)$.

Under certain regularity conditions on D and f [see, e.g., Dynkin (2002)], the integral equation in (d) is equivalent to the boundary value problem

$$\tfrac{1}{2}\Delta u=2u^2,$$
$$u(x)=f(x), \qquad x\in\partial D.$$

X_D represents the exit measure from D, and the first property simply means that X_D is concentrated on D^c, so that exiting is instantaneous if we start outside D. The third property means that distinct clumps of initial mass evolve independently. It follows from continuity of Brownian motion that X^D is supported on ∂D, if the initial measure is supported on D; see Property 2.2.A of Dynkin (2002). The fourth property restricts attention to finite variance branching, and normalizes the branching rate. The normalizing factor 2 in front of u^2 is chosen to be consistent with Le Gall (1999) and Salisbury and Verzani (1999).

2.2. *Infinite divisibility and Poisson representation.* It is well known that X_D has an infinitely divisible distribution for each D. This property leads to the construction of a new measure, $\mathbb{N}_x$, called the super-Brownian excursion law starting from x. Under $\mathbb{N}_x$, X evolves as a super-Brownian motion, but $\mathbb{N}_x$ will be σ-finite, not a probability. Thus, it is technically more complicated than P_μ. But $\mathbb{N}_x$ is actually a more basic object heuristically, under which the genealogies are simpler, because all the mass starts from a single particle located initially at x.

In fact P_μ can be built up from a Poisson random measure with intensity $\theta(d\chi) = \int \mathbb{N}_x(d\chi)\mu(dx)$. More precisely, let

$$\Pi(d\chi) = \sum_i \delta_{\chi^i}$$

be such a Poisson random measure, where the χ^i are random measure valued paths. Then $X = \sum \chi^i = \int \chi \Pi(d\chi)$ is a super-Brownian motion with initial state μ. In terms of X_D, this yields the following formula; see Theorem 5.3.4 of Dynkin (2004): let F be a nonnegative measurable function defined on $\mathcal{M}_E$. Then

$$\begin{aligned} P_\mu\big(F(X_D)\big) &= e^{-\mathcal{R}_\mu(\mathcal{M}_E)} F(0) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n!} e^{-\mathcal{R}_\mu(\mathcal{M}_E)} \int \mathcal{R}_\mu(d\nu_1) \cdots \mathcal{R}_\mu(d\nu_n) F(\nu_1 + \cdots + \nu_n), \end{aligned} \tag{2}$$

where $\mathcal{R}_\mu$ is the canonical measure of X_D with respect to P_μ and can be derived from $\mathbb{N}_x$ by $\mathcal{R}_\mu(A) = \langle \mu, \mathbb{N}_{(\cdot)}(X_D \in A, X_D \neq 0)\rangle$. In other words, we obtain X_D as a superposition of a Poisson number of more "basic" exit measures, each descended from a single initial individual. These "basic" exit measures arise as the atoms of a Poisson random measure whose characteristic measure is $\mathcal{R}_\mu$.

Note that a special case of the above representation gives us $V_D f(x) = \mathbb{N}_x(1 - e^{-\langle X_D, f\rangle})$. Note further that we will, in the future, write this as

$$P_\mu\big(F(X_D)\big) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{-\mathcal{R}_\mu(\mathcal{M}_E)} \int \mathcal{R}_\mu(d\nu_1) \cdots \mathcal{R}_\mu(d\nu_n) F(\nu_1 + \cdots + \nu_n)$$

by taking the convention that the $n = 0$ term is the first expression in (2).

The measure $\mathbb{N}_x$ was first considered by Le Gall (1999) in his random snake formulation of super-Brownian motion. As we follow Dynkin's framework to study our problem, we refer the reader to Dynkin (2004) for a systematic account of the theory of the measures $\mathbb{N}_x$ and their applications. Note that the latter has a general branching function ψ, but for us, this is taken to be $\psi(u) = 2u^2$.

2.3. *Moment measures of super Brownian motion.* Among the key tools in our analysis are the recursive moment formulas of SBM. The moment measures of SBM are the following measures:

Let $\phi, f_1, \ldots, f_n$ be positive Borel functions and write $f = (f_1, \ldots, f_n)$. For $C \subset \{1, \ldots, n\}$, let

$$n_C(\phi, f, x) = \mathbb{N}_x e^{-\langle X_D, \phi\rangle} \Pi_{i \in C} \langle X_D, f_i\rangle, \tag{3}$$

$$p_C(\phi, f, \mu) = P_\mu e^{-\langle X_D, \phi\rangle} \Pi_{i \in C} \langle X_D, f_i\rangle. \tag{4}$$

Let K_D^l and G_D^l be the Poisson and Green operator for the operator

$$\mathcal{L}^l = \tfrac{1}{2}\triangle - l,$$

where $l(x) = 4V_D\phi(x)$. In other words, let ξ_t be a diffusion starting from x, with generator $\mathcal{L}^l$ under a probability measure Π_x^l. Then $K_D^l f(x) = \Pi_x^l f(\xi_{\tau_D})$ and $G_D^l f(x) = \Pi_x^l(\int_0^{\tau_D} f(\xi_t)\,dt)$, where τ_D is the exit time from D.

For $C = \{i\}$ we have the Palm formula

$$n_C(\phi, f, x) = K_D^l f_i(x), \tag{5}$$

and for general C we have the following recursive formulas; see, for example, Theorem 5.1.1 of Dynkin (2004), or Lemma 2.6 of Salisbury and Verzani (1999):

$$n_C(\phi, f, \cdot) = \frac{1}{2}\sum_{A\subset C, A\neq\varnothing, C} G_D^l\big(4n_A(\phi, f, \cdot)n_{C\setminus A}(\phi, f, \cdot)\big), \tag{6}$$

$$p_C(\phi, f, \mu) = e^{-\langle\mu, V_D(f_0)\rangle}\sum_{\pi(C)}\big\langle\mu, n_{C_1}(\phi, f, \cdot)\big\rangle\cdots\big\langle\mu, n_{C_r}(\phi, f, \cdot)\big\rangle. \tag{7}$$

Here $\pi(C)$ is the set of partitions of C. These formulas will allow us to construct a variety of extended X-harmonic functions of polynomial type.

We will also need extensions of these formulas for

$$\mathbb{N}_x e^{-(\langle X_{D_1}, \phi_1\rangle + \cdots + \langle X_{D_k}, \phi_k\rangle)}\Pi_{i\in C}\langle X_D, f_i\rangle$$

and

$$P_\mu e^{-(\langle X_{D_1}, \phi_1\rangle + \cdots + \langle X_{D_k}, \phi_k\rangle)}\Pi_{i\in C}\langle X_D, f_i\rangle,$$

where $D_i \subset D_k = D$. Formulas (5), (6) and (7) give us these quantities when $k = 1$ and $x \in D$. For $k \geq 2$ we find them recursively as follows. Let us put $I = \{D_1, D_2, \ldots, D_k\}$, $\phi_I = (\phi_1, \ldots, \phi_k)$, $u^I(x) = \mathbb{N}_x(1 - \exp -(\langle X_{D_1}, \phi_1\rangle + \cdots + \langle X_{D_k}, \phi_k\rangle))$, $l^I = 4u^I$, $D_I = D_1 \cap \cdots \cap D_k$ and $I_j = I - \{D_j\}$. We define an operator $n_C^I(\phi_I, f, x)$ as follows. For $\operatorname{card}(I) = 1$,

$$n_C^I(\phi, f, x) = \begin{cases} \mathbb{N}_x e^{-(\langle X_D, \phi\rangle)}\Pi_{i\in C}\langle X_D, f_i\rangle, & x \in D, \\ f_i(x), & x \notin D, C = \{i\}, \\ 0, & x \notin D, \operatorname{card}(C) > 1, \end{cases} \tag{8}$$

and for $\operatorname{card}(I) > 1$,

$$\begin{aligned} &n_C^I(\phi_I, f, x) \\ &\quad = \begin{cases} \mathbb{N}_x e^{-(\langle X_{D_1}, \phi_1\rangle + \cdots + \langle X_{D_k}, \phi_k\rangle)}\Pi_{i\in C}\langle X_D, f_i\rangle, & x \in D_I, \\ n_C^{I_j}(\phi_{I_j}, f, x), & x \notin D_j, j \in I. \end{cases} \end{aligned} \tag{9}$$

We let

$$p_C^I(\phi_1, \phi_2, \ldots, \phi_k, f, \mu) = P_\mu e^{-(\langle X_{D_1}, \phi_1\rangle + \cdots + \langle X_{D_k}, \phi_k\rangle)}\Pi_{i\in C}\langle X_{D_k}, f_i\rangle. \tag{10}$$

Fix an f and write $n_C^I(x)$ for $n_C^I(\phi_I, f, x)$ and $p_C^I(\mu)$ for $p_C^I(\phi_I, f, \mu)$. The following formulas tell us that the values of $n_C^I(x)$ on D_I can be recovered from its values at the boundary of D_I and the values of the functions $n_A^I(x)$ on D_I, with $A \subset C$ and $A \neq \varnothing, C$, using the Poisson and Green operators of $\mathcal{L}^{l^I}$ for the domain D_I. For $C = \{i\}$ and $\operatorname{card}(I) > 1$,

$$n_C^I(x) = K_{D_I}^{l^I}(n_C^I)(x), \qquad x \in D_I, \tag{11}$$

and for $\operatorname{card}(C) > 1$ and $\operatorname{card}(I) > 1$,

$$n_C^I(x) = \frac{1}{2} \sum_{A \subset C, A \neq \varnothing, C} G_{D_I}^{l^I}(4 n_A^I n_{C \setminus A}^I)(x) + K_{D_I}^{l^I}(n_C^I)(x), \qquad x \in D_I. \tag{12}$$

Then for μ compactly supported in D_I

$$p_C^I(\mu) = e^{-\langle \mu, l^I \rangle} \sum_{\pi(C)} \langle \mu, n_{C_1}^I \rangle \cdots \langle \mu, n_{C_r}^I \rangle. \tag{13}$$

Since at the boundary of D_I, $n_C^I(x)$ is recursively defined in terms of $n_C^{I_j}(x)$, we have now a complete recursive algorithm to compute $n_C^I(x)$, starting with $I = \{D\}$, and $C = \{i\}$, recursively first building $n_C^I(x)$ for all C keeping I the same, and then increasing the cardinality of I by 1 and repeating the same procedure until the desired cardinality of I is achieved.

We omit the proof of these formulas and refer the reader to the proof of Theorem 5.1.1 of Dynkin (2004). The reader will realize that the argument in that proof works also for the functions $u_I^\lambda(x)$ which are defined as follows. Set $X_I = (X_{D_1}, \ldots, X_{D_k})$, so $\langle X_I, \phi_I \rangle = \sum_j \langle X_{D_j}, \phi_j \rangle$. For for $I = \{D\}$, set

$$u_I^\lambda(x) = \begin{cases} \mathbb{N}_x\big(1 - \exp -(\langle X_I, \phi_I \rangle + \lambda_1 \langle X_{D_k}, f_1 \rangle + \cdots + \lambda_n \langle X_{D_k}, f_n \rangle)\big), \\ \qquad\qquad\qquad\qquad\qquad\qquad x \in D, \\ \phi_k(x) + \lambda_1 f_1(x) + \cdots + \lambda_n f_n(x), \qquad x \notin D, \end{cases}$$

and recursively for $\operatorname{card}(I) > 1$ as

$$u_I^\lambda(x) = \begin{cases} \mathbb{N}_x\big(1 - \exp -(\langle X_I, \phi_I \rangle + \lambda_1 \langle X_{D_k}, f_1 \rangle + \cdots + \lambda_n \langle X_{D_k}, f_n \rangle)\big), \\ \qquad\qquad\qquad x \in D_I, \\ u_{I_j}^\lambda(x) + \phi_j(x), \qquad x \notin D_j, j \in I, j \neq k. \end{cases}$$

We note that u_I^λ satisfies for $x \in D_I$,

$$u_I^\lambda + G_{D_I}^{l^I}\big(2(u_I^\lambda)^2\big) = K_{D_I}^{l^I}(u_I^\lambda),$$

following formula 2.11 of Dynkin (2002), Chapter 3, which in turn yields the formulas for n_C^I by differentiating u_I^λ with respect to λ.

2.4. *Absolute continuity.* The moment formulas together with the Markov property and Poisson representation yield an important theorem taken in this form from Theorem 5.3.2 of Dynkin (2004). See also Proposition 2.18 of Mselati (2004). Let $\mathcal{M}_D^c$ be the space of finite measures compactly supported in D.

THEOREM 1. *Suppose $A \in \mathcal{F}_{\supset D}$. Then either $P_\mu(A) = 0$ for all $\mu \in \mathcal{M}_D^c$ or $P_\mu(A) > 0$ for all $\mu \in \mathcal{M}_D^c$.*

3. Extended X-harmonic functions and conditioning. In the rest of the paper we fix $D \Subset E$. Following Dynkin (2006b), a nonnegative function $H : \mathcal{M}_D^c \to [0, \infty)$ is called X*-harmonic* in D, if for any $D' \Subset D$ and any finite measure $\mu \in \mathcal{M}_{D'}^c$,

$$P_\mu\big(H(X_{D'})\big) = H(\mu). \tag{14}$$

We will call a nonnegative H extended X-harmonic if it satisfies (14) but is not necessarily everywhere finite.

We are going to touch upon three different kinds of extended X-harmonic functions, which are derived from conditioning SBM on its various boundary statistics. These boundary statistics are:

(a) a Poisson random measure with characteristic measure βX_D;

(b) a random variable Z drawn from the probability distribution $\frac{X_D}{\langle X_D, 1\rangle}$ if $X_D \neq 0$, and set equal to some given $\Delta \notin \partial D$ if $X_D = 0$;

(c) $L(X_D)$, where L is a linear map from $\mathcal{M}_{\partial D}$ to a vector space V [e.g., $L(\mu) = \mu$ or $L(\mu) = \langle \mu, 1\rangle$ means we condition on X_D or on its total mass].

Let S be any one of the above statistics. Let Σ be the state space of S. We will assume that Σ is endowed with a countably generated σ-algebra $\mathcal{S}$ such that $(\Sigma, \mathcal{S})$ is a measurable Luzin space; see Dynkin (2006b) for a definition. For example, when $S = X_D$, Σ is $\mathcal{M}_{\partial D}$, the space of finite measures on ∂D and $\mathcal{S}$ is the σ-algebra in $\mathcal{M}_{\partial D}$ generated by the functions $f(\nu) = \nu(B)$, where B is a Borel subset of ∂D. Given $X_D = \nu$, we let P_S^ν denote the conditional distribution of S. For example, $P_S^\nu(f)$ equals $\langle \nu, f\rangle / \langle \nu, 1\rangle$ in the second case (provided $\nu \neq 0$), and $f(L(\nu))$ in the third.

P_μ denotes a probability measure in which X is an SBM started from $\mu \in \mathcal{M}_D^c$, and in which S is then drawn (if necessary) by further sampling. In other words, P_μ is a probability defined on the σ-field $\mathcal{G} = \mathcal{F}_{\subset D} \vee \sigma\{S\}$. When $\mu = \delta_x$ we set $P_\mu = P_x$. By construction, $P_\mu(f(S) \mid \mathcal{F}_{\subset D}) = P_S^{X_D}(f)$. In other words, for any $\mathcal{F}_{\subset D}$-measurable Y we have that

$$P_\mu\big(f(S)Y\big) = P_\mu\big(P_S^{X_D}(f)Y\big).$$

Likewise, we let $P_{\mu,S}$ and $P_{x,S}$ denote the marginal distribution of S under P_μ and P_x, so $P_{\mu,S}(f) = P_\mu(f(S))$.

Let $\mathcal{F}_{\subset D-} = \sigma\{X_{D'}, D' \Subset D\}$. What we want is the conditional law of $\{X_{D'}, D' \Subset D\}$ given $S = s$, which should therefore be a transition kernel P^s_μ from Σ, the state space of S, to the $\mathcal{F}_{\subset D-}$ measurable functions. More precisely, we will have $P^S_\mu(Y) = P_\mu(Y|S)$, P_μ a.s. for all $\mathcal{F}_{\subset D-}$ measurable Y. The following theorem tells us how we can construct this transition kernel [part (d)]. The first three statements [(a), (b) and (c)] of this theorem are equivalent to Theorem 1.1 of Dynkin (2006b) in the case $S = X_D$. For a general S, we follow Dynkin's proof, with some modifications.

Let us fix a point $x \in D$.

THEOREM 2. *There exists a family of nonnegative functions* $\{H^s_x : \mathcal{M}^c_D \mapsto R_+, s \in \Sigma\}$ *with the following properties*:

(a) $H^{(\cdot)}_x(\cdot) : (s, \mu) \mapsto H^s_x(\mu)$ *is measurable and strictly positive*;
(b) *For all* μ, $H^{(\cdot)}_x(\mu) : s \mapsto H^s_x(\mu)$ *is a version of* $\frac{dP_{\mu,S}}{dP_{x,S}}$;
(c) *For all* s, $H^s_x(\cdot)$ *is extended* X*-harmonic in* D;
(d) *Define a probability* P^s_μ *on* $\mathcal{F}_{\subset D-}$ *by setting*

$$P^s_\mu(Y) = \frac{1}{H^s_x(\mu)} P_\mu\big(Y H^s_x(X_{D'})\big) \tag{15}$$

for all $D' \Subset D$ *containing the support of* μ, *and* $\mathcal{F}_{\subset D'}$*-measurable* Y [*whenever* $H^s_x(\mu) < \infty$, *and otherwise setting* P^s_μ *to an arbitrary probability measure*]. *Then* P^S_μ *is a version of the conditional law of* X *given* S *with respect to* P_μ *for all* μ.

Any two families satisfying the above properties will coincide for $P_{x,S}$*-a.e.* $s \in \Sigma$.

PROOF. Existence of a family $\{\bar{H}^s_x, s \in \Sigma\}$ with the first two properties follows from Theorem A.1 of Dynkin (2006b) and the absolute continuity of the family $\{P_{\mu,S}, \mu \in \mathcal{M}^c_D\}$ with respect to $P_{x,S}$. Let O be a subdomain compactly contained in D. Then

$$P_\mu \bar{H}^s_x(X_O) = \bar{H}^s_x(\mu) \tag{16}$$

for $P_{x,S}$-a.e. s, $\forall \mu \in \mathcal{M}^c_O$. Dynkin (2006b) proves this when $S = X_D$, and the proofs for the other cases are almost identical to his. Next, we want to construct an extended X-harmonic function H^s for all $s \in \Sigma$. To do this, we choose a countable base O_n (w.l.o.g. closed under finite unions), and probability measures $\mu_n \in \mathcal{M}^c_{O_n}$, and we let

$$R(d\eta) = \sum 2^{-n} P_{\mu_n}(X_{O_n} \in d\eta).$$

Note that (16) implies

$$P_\mu \bar{H}^s_x(X_{O_n}) = \bar{H}^s_x(\mu)$$

for $R \times P_{x,S}$-a.e. (μ, s). By Fubini's theorem we deduce that there exists a $P_{x,S}$-null set $\mathcal{N}$ s.t.

$$P_\mu \bar{H}_x^s(X_{O_n}) = \bar{H}_x^s(\mu) \qquad \forall n, \text{ for } R\text{-a.e. } \mu \in \mathcal{M}_D^c, \forall s \in \mathcal{N}^c. \tag{17}$$

For $s \in \mathcal{N}^c$ and $\mu \in \mathcal{M}_D^c$, we choose O_n containing the support of μ and define

$$H_x^s(\mu) = P_\mu \bar{H}_x^s(X_{O_n}).$$

$H_x^s(\mu) > 0$ since this is true of $\bar{H}_x^s$, but we cannot rule out $H_x^s(\mu) = \infty$. We set $H_x^s(\mu)$ to some arbitrary positive constant for $s \in \mathcal{N}$. The definition of $H_x^s(\mu)$ is independent of the choice of O_n since, if $O_k \supset O_n$, then

$$\begin{aligned} P_\mu \bar{H}_x^s(X_{O_k}) &= P_\mu P_{X_{O_n}} \bar{H}_x^s(X_{O_k}) \\ &= P_\mu \bar{H}_x^s(X_{O_n}). \end{aligned}$$

The first equality is due to the Markov property. The second equality is due to (17) and the fact that $P_\mu(X_{O_n} \in (\cdot))$ is absolutely continuous with respect to R, by Theorem 1.

Clearly, $H_x^s(\mu)$ is measurable and by (16), is a version of $\frac{dP_{\mu,S}}{dP_{x,S}}(s)$ for each $\mu \in \mathcal{M}_D^c$. To show that $H_x^s(\mu)$ satisfies property (c), we need to show that $H_x^s(\mu)$ is extended X-harmonic for each $s \in \mathcal{N}^c$. Let μ and O be s.t. $\mu \in \mathcal{M}_O^c$ and pick O_n s.t. O is compactly contained in O_n. Then, by definition,

$$H_x^s(\mu) = P_\mu\big(\bar{H}_x^s(X_{O_n})\big)$$

and

$$P_\mu H_x^s(X_O) = P_\mu P_{X_O}\big(\bar{H}_x^s(X_{O_n})\big).$$

By the Markov property these two are equal. Now the family $\{H_x^s, s \in S\}$ satisfies properties (a), (b) and (c).

Let us define P_μ^s as in (15). It remains to prove that P_μ^s is the desired transition kernel. Let $D' \Subset D$ and $Y \in \mathcal{F}_{\subset D'}$. Then

$$\begin{aligned} P_\mu\big(f(S) P_\mu^S(Y)\big) &= \int f(s) P_\mu^s(Y) P_{\mu,S}(ds) \\ &= \int f(s) \frac{1}{H_x^s(\mu)} \int Y(\omega) H_x^s\big(X_{D'}(\omega)\big) P_\mu(d\omega) P_{\mu,S}(ds) \\ &= \int Y(\omega) \int \frac{1}{H_x^s(\mu)} f(s) H_x^s\big(X_{D'}(\omega)\big) P_{\mu,S}(ds) P_\mu(d\omega) \\ &= \int Y(\omega) \int f(s) H_x^s\big(X_{D'}(\omega)\big) P_{x,S}(ds) P_\mu(d\omega) \\ &= \int Y(\omega) \int f(s) P_{X_{D'}(\omega),S}(ds) P_\mu(d\omega) \\ &= P_\mu\big(Y P_{X_{D'}}\big(f(S)\big)\big) \\ &= P_\mu\big(f(S) Y\big). \end{aligned}$$

Here we are using the definition of $P_{\mu,S}$, the definition of P_μ^s, Fubini's theorem, the definition of H_x^s, the definition of $P_{\nu,S}$, and the Markov property of X.

Uniqueness follows by a similar argument. Suppose $\{H_x^s\}_{s\in\Sigma}$ and $\{\tilde{H}_x^s\}_{s\in\Sigma}$ be any two families with the properties (a), (b), (c) and (d). Then

$$H_x^s(\mu) = \tilde{H}_x^s(\mu) \qquad \text{for } P_{x,S}\text{-a.e. } s, \forall \mu$$

because of property (b). With $R(d\mu)$ as before, there is therefore a $P_{x,S}$-null set $\mathcal{N}$ such that

$$H_x^s(\mu) = \tilde{H}_x^s(\mu) \qquad \text{for } R\text{-a.e. } \mu \text{ and for } s \notin \mathcal{N}.$$

Let $\mu \in \mathcal{M}_D^c$ and $s \notin \mathcal{N}$. Choose O_n such that $\mu \in \mathcal{M}_{O_n}^c$. Then by absolute continuity and the property (c),

$$H_x^s(\mu) = P_\mu H_x^s(X_{O_n}) = P_\mu \tilde{H}_x^s(X_{O_n}) = \tilde{H}_x^s(\mu). \qquad \square$$

In the remainder of Section 3, we consider the three special cases described above. Our goal is to obtain relatively explicit formulas for H_x^s in each case.

3.1. *Conditioning on a Poisson random measure with characteristic measure βX_D.* Let N be a Poisson random variable with mean $\langle X_D, \beta\rangle$. Let $Z = \{Z_1, Z_2, \dots\}$ be an i.i.d. sequence of random variables from $X_D/\langle X_D, 1\rangle$. Let

$$Y_\beta = \sum_{i=1}^N \delta_{Z_i}.$$

Note that conditioned on X_D, Y_β is a Poisson random measure with characteristic measure βX_D, and that the construction makes sense even if $X_D = 0$, because then both N and Y equal 0.

Taking $S = Y_\beta$, Theorem 2 gives an extended X-harmonic function (which we denote $H_x^{\beta,\nu}$ to make explicit the dependence on β) for conditioning on $Y_\beta = \nu$. Here ν is an atomic measure. We let $P_\mu^{\beta,\nu}$ denote the law of the corresponding conditioned process. In principle this is only uniquely defined for a.e. ν, but we will find an explicit form that is valid more generally.

It will be convenient to also define variants of these objects. For any positive integer k, take $S_k = (Z_1, \dots, Z_k)$ if $N = k$, and $S_k = \Delta \notin \partial D$ otherwise. Set $X_D^k(dz_1, \dots, dz_k)$ to be the product measure $X_D(dz_1) \times \cdots \times X_D(dz_k)$, and let P_{μ,S_k}^β be the distribution of S_k with respect to P_μ. In other words, for $k \ge 1$ and $f : (\partial D)^k \cup \{\Delta\} \to \mathbb{R}$ such that $f(\Delta) = 0$, we have

$$\begin{aligned}
P_{\mu,S_k}^\beta(f) &= P_\mu\bigl(f(S_k)\bigr) \\
&= \frac{1}{k!} P_\mu\biggl(\biggl(\int_{(\partial D)^k} \frac{f}{\langle X_D, 1\rangle^k}\, dX_D^k\biggr) \langle X_D, \beta\rangle^k e^{-\langle X_D, \beta\rangle} 1_{\{X_D \neq 0\}}\biggr) \\
&= \frac{\beta^k}{k!} P_\mu\biggl(\biggl(\int_{(\partial D)^k} f\, dX_D^k\biggr) e^{-\langle X_D, \beta\rangle}\biggr).
\end{aligned}$$

So for any $\beta > 0$, positive integer k, and any k-tuple $z = \{z_i\}$ of elements of ∂D, Theorem 2 gives us a family of extended X-harmonic functions $\{H^{\beta,k,z} : z = (z_1, \dots, z_k) \in (\partial D)^k\}$ such that

$$H_x^{\beta,k,z}(\mu) = \frac{dP^{\beta}_{\mu,S_k}}{dP^{\beta}_{x,S_k}}(z_1, z_2, \dots, z_k). \tag{18}$$

Let $l_\beta \doteq 4V_D(\beta)$. We have $P_{\mu,Y_\beta}\{0\} = P_\mu(e^{-\langle X_D,\beta\rangle}) = e^{-\langle \mu,l_\beta\rangle}$ and $P_{x,Y_\beta}\{0\} = e^{-l_\beta(x)}$, and therefore we find $H^{\beta,\nu}$ for $\nu = 0$ simply by the ratio

$$H_x^{\beta,0}(\mu) = \frac{e^{-\langle \mu,l_\beta\rangle}}{e^{-l_\beta(x)}} \tag{19}$$

by Theorem 2.

Let $l \geq 0$ be a bounded Borel function on D. For $x \in D$, we let $m_x^l(dz) = \Pi_x^l(\xi_{\tau_D} \in dz)$ denote harmonic measure on ∂D for the operator L^l. Then m_x^l and m_y^l are mutually absolutely continuous, for $x, y \in D$. [This is a well-known fact; however, for the curious reader, here is a quick argument for why it is true. Let D' be a smooth domain, compactly contained in D, and $x, y \in D'$. Let $m^l_{x,D'}$ be the harmonic measure on $\partial D'$. If A is a Borel subset of ∂D and $m_x^l(A) = 0$, because of the strong Markov property and the fact that $m^l_{x,D'}$ is equivalent to the surface measure $\gamma_{D'}$ on $\partial D'$, we have that $m_z^l(A) = 0$ for $\gamma_{D'}$ almost all z. This implies $m_y^l(A) = 0$, again due to the strong Markov property and the fact that $m^l_{y,D'} \sim \gamma_{D'}$.]

Let

$$k_x^l(y,z) = \frac{dm_y^l}{dm_x^l}(z)$$

denote the density. If D were sufficiently regular, this would be a version of the Martin kernel for the operator L^l, but we make no such regularity assumptions at this point. We take k^l to be a jointly measurable version of this density that is harmonic in y, for each $z \in \partial D$. One can construct k^l in a similar way as in Theorem 2. That is, we start with a family $\{\tilde{k}((\cdot), z), z \in \partial D\}$ such that $\tilde{k}$ is measurable as a function of (y, z), and for fixed y, $\tilde{k}(y, (\cdot))$ is a version of $\frac{dm_y^l}{dm_x^l}(z)$. The existence of such a family follows from Theorem A.1 of Dynkin (2006b) and the absolute continuity of the family $\{m_y^l, y \in D\}$ with respect to m_x^l. Then we take a sequence $D_n \Subset D$ exhausting D, and let $k^l(y,z) = \Pi_y^l(\tilde{k}(\xi_{\tau_{D_n}}, z))$ for $y \in D_n$. Then we prove that $k^l((\cdot), z)$ is well defined, and harmonic for all z except on an m_x^l-null set $\mathcal{N}$, on which we set k^l to be an arbitrary constant. We omit the details as the arguments are very similar to those in the proof of Theorem 2.

In the case $l = 0$ we write $m_x(dz) = m_x^0(dz)$ and $k_x(y,z) = k_x^0(y,z)$.

The particular case of interest is $l = l_\beta = 4V_D(\beta)$. Suppose that $k \ge 1$ and that $z_1, \dots, z_k \in \partial D$. For $C \subset K = \{1, \dots, k\}$, recursively define

$$\rho_C^\beta = \begin{cases} k_x^{l_\beta}(\cdot, z_i), & \text{for } C = \{i\}, \\ \dfrac{1}{2} \displaystyle\sum_{A \subset C, \varnothing \ne A \ne C} G_D^{l_\beta}(4\rho_A^\beta \rho_{C\setminus A}^\beta), & \text{for } |C| > 1. \end{cases}$$

Finally, set

$$\rho_\mu^{\beta,k}(z_1, \dots, z_k) = e^{-\langle \mu, l_\beta \rangle} \sum \langle \mu, \rho_{C_1}^\beta \rangle \cdots \langle \mu, \rho_{C_r}^\beta \rangle,$$

where the sum ranges over all partitions $\{C_1, \dots, C_r\}$ of K.

In the following theorem, we use the convention that $H^{\beta,0,z} = H^{\beta,0}$.

THEOREM 3. *Let $D \Subset E$, and $\beta \ge 0$ and $x \in D$. Then*:

(a) $H_x^{\beta,\nu} = H_x^{\beta,k,z}$ *for P_{x,Y_β}-almost all finite atomic measures ν, where k and z are such that*

$$\nu(dx) = \sum_1^k \delta_{z_i}(dx). \tag{20}$$

(b) *For $(m_x^{l_\beta})^k$-a.e. $(z_1, \dots, z_k)$, for all $\mu \in \mathcal{M}_D^c$,*

$$H_x^{\beta,k,z}(\mu) = \frac{\rho_\mu^{\beta,k}(z_1, \dots, z_k)}{\rho_x^{\beta,k}(z_1, \dots, z_k)}. \tag{21}$$

(c) *If D is smooth, then in fact $\rho_\mu^{\beta,k}(z_1, \dots, z_k) < \infty$ for all $\mu \in \mathcal{M}_D^c$ whenever $z_1, \dots, z_k$ are distinct.*

PROOF. (a) $P_{\mu,S_k}^\beta(f)$ remains unchanged if we permute the arguments of f. Thus we can choose the densities $H_x^{\beta,k,z}(\mu)$ to be both X-harmonic and invariant under permutations of the z_i. A simple way to confirm this is to replace an X-harmonic choice of $H_x^{\beta,k,z}(\mu)$ by $\frac{1}{k!}\sum_\sigma H_x^{\beta,k,\sigma(z)}(\mu)$, where the sum is over permutations σ. The latter is still X-harmonic, and a version of the density $dP_{\mu,S_k}^\beta / dP_{x,S_k}^\beta$, but is also clearly invariant under permutations.

For a finite atomic measure ν, all of whose atoms have mass 1, find k and $z_1, \dots, z_k$ such that (20) holds. Then define

$$\tilde{H}_x^{\beta,\nu}(\mu) := H_x^{\beta,k,z}(\mu).$$

Note that $\tilde{H}_x^{\beta,\nu}(\mu)$ is well defined, since $H_x^{\beta,k,z}$ depends only on $z^k := (z_1, \dots, z_k)$ and is invariant under permuting z^k. [Note, if two sequences z and $\tilde{z}$ satisfy (20), then z^k and $\tilde{z}^k$ must be permutations of each other.]

Let f be a function defined on the space of finite atomic measures. If $\nu = \sum_{i=1}^k \delta_{z_i}$, write $f_k(z)$ for $f(\nu)$. To finish the proof it is enough to observe

$$\begin{aligned}
&P_{x,Y_\beta}\big(\tilde{H}_x^{\beta,(\cdot)}(\mu) f(\cdot)\big)\\
&\quad = P_x\big(\tilde{H}_x^{\beta,Y_\beta}(\mu) f(Y_\beta)\big)\\
&\quad = P_{x,Y_\beta}\{0\}\tilde{H}^{\beta,0}(\mu) f(0)\\
&\qquad + \sum_{k=1}^\infty P_x\bigg(e^{-\langle X_D,\beta\rangle}\frac{\langle X_D,\beta\rangle^k}{k!}1_{\{X_D\neq 0\}}\int H_x^{\beta,k,z}(\mu) f_k(z)\frac{X_D^k(dz)}{\langle X_D,1\rangle^k}\bigg)\\
&\quad = P_{\mu,Y_\beta}\{0\} f(0) + \sum_{k=1}^\infty \frac{\beta^k}{k!} P_x\bigg(e^{-\langle X_D,\beta\rangle}\int H_x^{\beta,k,z}(\mu) f_k(z) X_D^k(dz)\bigg)\\
&\quad = P_{\mu,Y_\beta}\{0\} f(0) + \sum_{k=1}^\infty P_{x,S_k}^\beta\big(H_x^{\beta,k,(\cdot)}(\mu) f_k(\cdot)\big)\\
&\quad = P_{\mu,Y_\beta}\{0\} f(0) + \sum_{k=1}^\infty P_{\mu,S_k}^\beta(f_k)\\
&\quad = P_{\mu,Y_\beta}\{0\} f(0) + \sum_{k=1}^\infty P_\mu\bigg(e^{-\langle X_D,\beta\rangle}\frac{\langle X_D,\beta\rangle^k}{k!}1_{\{X_D\neq 0\}}\int f_k(z)\frac{X_D^k(dz)}{\langle X_D,1\rangle^k}\bigg)\\
&\quad = P_\mu\big(f(Y_\beta)\big)\\
&\quad = P_{\mu,Y_\beta}(f).
\end{aligned}$$

(b) Define $\tilde{H}_x^{\beta,k,z}(\mu)$ to be the right-hand side of (21). Following an argument of Dynkin (2004), Chapter 5, one can show that $\rho_\mu^{\beta,k}$ is the density of P_{μ,S_k}^β with respect to $(m_x^{l_\beta})^k$. The argument uses the moment formulas (5), (6), (7) and then pulls k factors of harmonic measure out of the resulting expressions, leaving the densities $k_x^{l_\beta}$ behind. It follows that $\tilde{H}_x^{\beta,k,z}(\mu)$ is a version of the Radon–Nikodym derivative in (18). The finiteness condition for $\rho_\mu^{\beta,k}$ follows immediately.

Furthermore, by Theorem 3.1 of Salisbury and Verzani (1999), $\tilde{H}_x^{\beta,k,z}$ is X-harmonic; see remark (iv) below. Thus $\tilde{H}_x^{\beta,k,z} = H_x^{\beta,k,z}$ for $(m_x^{l_\beta})^k$-a.e. z, which is the sense up to which $H_x^{\beta,k,z}$ is well defined.

(c) The argument for (c) is a straightforward modification of the estimates used in Theorem 5.3 of Salisbury and Verzani (1999). □

REMARKS. (i) The conclusion is that we have obtained an explicit formula for $H_x^{\beta,\nu}(\mu)$. The abstract definition of this X-harmonic function was valid only up to an unspecified null set of ν's, whereas the canonical expression we have obtained is well defined as long as ν is a finite atomic measure, all of whose atoms have mass 1 (assuming that D is smooth).

(ii) The arguments of this section would work equally well for conditioning on the value of a Poisson random measure with characteristic measure $\beta(x)X_D(dx)$, where $\beta(x)$ is now a bounded measurable function on ∂D.

(iii) If D is smooth, then instead of taking $k_x^l(y,z)$ to be the density of m_y^l with respect to m_x^l, we could use the Poisson kernel in its place, and get a similar result. In other words, we could take the density of m_y^l with respect to the surface measure γ on ∂D, rather than the density with respect to m_x^l.

(iv) $\tilde{H}^{\beta,k,z}$ falls in the family of X-harmonic functions considered in Salisbury and Verzani (1999). This family of X-harmonic functions are characterized by a function g, and $\mathcal{L}^{4g}$-harmonic functions $v_1,\dots,v_k$. In our example the function g is $u_\beta = V_D\beta$, and the harmonic functions v_i are the functions $k_x^{l_\beta}(\cdot,z_i)$. In Salisbury and Verzani (1999) it is shown that for D Lipschitz of dimension $d \geq 4$, $g = 0$ and $v_i = k_x(\cdot,z_i)$ where $z_1,\dots,z_k$ are distinct points chosen on the boundary, the resulting X-harmonic function corresponds to conditioning SBM to hit the points z_i. The same argument would work in dimension $d = 3$, at least when D is smooth.

3.2. *Conditioning on a r.v. Z sampled from measure $\frac{X_D}{\langle X_D,1\rangle}$.* Recall that the random variable Z is drawn from the probability distribution $\frac{X_D}{\langle X_D,1\rangle}$ if $X_D \neq 0$, and set equal to some given $\Delta \notin \partial D$ if $X_D = 0$. Applying Theorem 2 gives us a family of extended X-harmonic functions

$$H_x^z = \frac{dP_{\mu,Z}}{dP_{x,Z}}(z)$$

indexed by points z of $\{\Delta\} \cup \partial D$. We denote the law of the corresponding conditional process by P_μ^z.

Recall that ξ_t is a Brownian motion under Π_y. For $z \in \partial D$, we let Π_y^z be a probability under which ξ_t is a $k_x(\cdot,z)$-transform of Brownian motion. [Recall $k_x((\cdot),z) \doteq k_x^0((\cdot),z)$.] In other words,

$$\Pi_y^z\big(f(\xi_t), t < \tau_D\big) = \frac{1}{k_x(y,z)}\Pi_y\big(f(\xi_t)k_x(\xi_t,z), t < \tau_D\big)$$

for every bounded measurable f.

The following result establishes a concrete formula for H_x^z that is defined for m_x-a.e. $z \in \partial D$ when D is a general domain. When D is smooth, the same argument as in the previous section gives a canonical version, defined for all $z \in \partial D$.

THEOREM 4. *Let $D \Subset E$ and $x \in D$. Then for $m_x(dz)$-almost all $z \in \partial D$, $H_x^z(\mu) < \infty$, and*

$$H_x^z(\mu) = \frac{\int_0^\infty \langle \mu, k_x(\cdot,z)\Pi_{(\cdot)}^z(e^{-\phi(u_\beta)})\rangle e^{-\langle \mu,u_\beta\rangle}\,d\beta}{\int_0^\infty \Pi_x^z(e^{-\phi(u_\beta)})e^{-u_\beta(x)}\,d\beta} \tag{22}$$

for every μ, where $u_\beta = V_D\beta$ and the random variable $\phi(u_\beta)$ is defined as

$$\phi(u_\beta) = 4\int_0^{\tau_D} u_\beta(\xi_t)\,dt. \tag{23}$$

PROOF. We first find the Radon–Nikodym derivative of $P_{\mu,Z}$ w.r.t. the harmonic measure $m_x(dz)$ on the boundary of D. We observe that

$$\begin{aligned} P_{\mu,Z}(f) &= P_\mu\left(\frac{\langle X_D, f\rangle}{\langle X_D, 1\rangle} 1_{\{X_D\neq 0\}}\right) \\ &= -\int_0^\infty \frac{d}{d\lambda} P_\mu\big(e^{-\lambda\langle X_D,f\rangle-\beta\langle X_D,1\rangle}\big)\Big|_{\lambda=0}\,d\beta. \end{aligned} \tag{24}$$

Note that the above derivative equals 0 when $X_D = 0$. By the branching property,

$$P_\mu\big(e^{-\lambda\langle X_D,f\rangle-\beta\langle X_D,1\rangle}\big) = e^{-\langle\mu,u_{\lambda f+\beta}\rangle}, \tag{25}$$

where

$$u_{\lambda f+\beta} = V_D(\lambda f+\beta) = \mathbb{N}_{(\cdot)}\big(1 - e^{-\langle X_D,\lambda f+\beta\rangle}\big).$$

Taking the derivative of the right-hand side of (25), and evaluating at $\lambda = 0$ we get

$$P_\mu\left(\frac{\langle X_D, f\rangle}{\langle X_D, 1\rangle} 1_{\{X_D\neq 0\}}\right) = \int_0^\infty \big\langle\mu, \mathbb{N}_{(\cdot)}\big(\langle X_D, f\rangle e^{-\beta\langle X_D,1\rangle}\big)\big\rangle e^{-\langle\mu,u_\beta\rangle}\,d\beta. \tag{26}$$

Differentiation under the integral sign is easily justified. By the Palm formula,

$$\begin{aligned} \mathbb{N}_y\big(\langle X_D, f\rangle e^{-\beta\langle X_D,1\rangle}\big) &= \Pi_y\big(f(\xi_{\tau_D})e^{-\phi(u_\beta)}\big) \\ &= \int_{\partial D} \Pi_y^z\big(e^{-\phi(u_\beta)}\big) f(z) m_y(dz) \\ &= \int_{\partial D} \Pi_y^z\big(e^{-\phi(u_\beta)}\big) k_x(y,z) f(z) m_x(dz). \end{aligned}$$

So,

$$\big\langle\mu, \mathbb{N}_{(\cdot)}\big(\langle X_D, f\rangle e^{-\beta\langle X_D,1\rangle}\big)\big\rangle = \int_{\partial D} \big\langle\mu, k_x(\cdot,z)\Pi_{(\cdot)}^z\big(e^{-\phi(u_\beta)}\big)\big\rangle f(z) m_x(dz).$$

Hence

$$\begin{aligned} &\int_0^\infty \big\langle\mu, \mathbb{N}_{(\cdot)}\big(\langle X_D, f\rangle e^{-\beta\langle X_D,1\rangle}\big)\big\rangle e^{-\langle\mu,u_\beta\rangle}\,d\beta \\ &\quad = \int_{\partial D} f(z)\left(\int_0^\infty \big\langle\mu, k_x(\cdot,z)\Pi_{(\cdot)}^z\big(e^{-\phi(u_\beta)}\big)\big\rangle e^{-\langle\mu,u_\beta\rangle}\,d\beta\right) m_x(dz). \end{aligned}$$

Therefore, both $P_{\mu,Z}$ and $P_{x,Z}$ have densities with respect to $m_x(dz)$, given by

$$\int_0^\infty \big\langle\mu, k_x(\cdot,z)\Pi_{(\cdot)}^z\big(e^{-\phi(u_\beta)}\big)\big\rangle e^{-\langle\mu,u_\beta\rangle}\,d\beta$$

and

$$\int_0^\infty \Pi_x^z(e^{-\phi(u_\beta)})e^{-u_\beta(x)}\,d\beta,$$

respectively. The ratio of these two is a version of the desired Radon–Nikodym derivative. To show that it equals H_x^z for almost all z, it simply remains to show that it is extended X-harmonic.

The denominator is simply a normalizing factor, so consider the numerator. For $l_\beta = 4u_\beta$, it is known [see Theorem 1.1 of Salisbury and Verzani (1999)] that $\mu \mapsto \langle\mu, v\rangle e^{-\langle\mu,u_\beta\rangle}$ is X-harmonic whenever v is $\mathcal{L}^{l_\beta}$-harmonic. And in our case, $\langle\mu, k_x(\cdot,z)\Pi_{(\cdot)}^z(e^{-\phi(u_\beta)})\rangle e^{-\langle\mu,u_\beta\rangle} = \langle\mu, k_x^{l_\beta}(\cdot,z)\rangle e^{-\langle\mu,u_\beta\rangle}$, as required.

Now we show that $H_x^z(\mu) < \infty$ for all μ, for m_x-almost all z. Let $\mu_0 \in \mathcal{M}$ be fixed. Since $H_x^z(\mu_0)$ is a density, there exist a m_x null set $B \subset \partial D$ s.t. for $z \in B$, $H_x^z(\mu_0)$ is finite all $z \in B^c$. Let $\mu \in \mathcal{M}_D^c$ and choose $D' \Subset D$ such that both μ and μ_0 are compactly supported in D', and assume that D' is smooth. Let p_μ be the measure defined on $\partial D'$ by $p_\mu(f) = P_\mu(e^{-\langle X_{D'},u_\beta\rangle}\langle X_{D'}, f\rangle)$. Then our analysis in Section 3.1 gives us that p_μ and p_{μ_0} are equivalent, and the Radon–Nikodym density is

$$\frac{dp_\mu}{dp_{\mu_0}}(y) = \frac{\langle\mu, \tilde{k}^{l_\beta}((\cdot),y)\rangle}{\langle\mu_0, \tilde{k}^{l_\beta}((\cdot),y)\rangle},$$

where $\tilde{k}^{l_\beta}(u,y)$ is the Poisson kernel of D' for the operator $\mathcal{L}^{l_\beta}$. Note that this density is bounded by a constant $C(\mu,\mu_0)$ since $\tilde{k}^{l_\beta}(u,y)$ is harmonic in u on the support of μ and μ_0. Hence

$$\begin{aligned}\langle\mu, k_x^{l_\beta}(\cdot,z)\rangle e^{-\langle\mu,u_\beta\rangle} &= P_\mu\langle X_{D'}, k_x^{l_\beta}(\cdot,z)\rangle e^{-\langle X_{D'},u_\beta\rangle}\\ &\le C(\mu,\mu_0)P_{\mu_0}\langle X_{D'}, k_x^{l_\beta}(\cdot,z)\rangle e^{-\langle X_{D'},u_\beta\rangle}\\ &= C(\mu,\mu_0)\langle\mu_0, k_x^{l_\beta}(\cdot,z)\rangle e^{-\langle\mu,u_\beta\rangle}.\end{aligned}$$

It follows that $H_x^z(\mu) \le C(\mu,\mu_0)H_x^z(\mu_0) < \infty$ for all $z \in B$, and hence the proof is complete. □

3.3. *Conditioning on a linear function of X_D.* Let L be a linear and measurable map from the linear cone of positive finite measures $\mathcal{M}_{\partial D}$ on ∂D to a Luzin measurable space $(V, \mathcal{V})$ where V is a vector space, and $\mathcal{V}$ is countably generated. Let V_+ be the image of $\mathcal{M}_{\partial D}$, and write $V^* = V_+ \setminus \{0\}$. Assume that $L\mu = 0$ implies $\mu = 0$.

Let T_n be the map $V_+^n \to V_+^n$ defined by

$$T_n(v_1,\dots,v_n) \mapsto (v_1 + v_2 + \cdots + v_n, v_1,\dots,v_{n-1}),$$

and for $A \in \mathcal{V}$ and $A \subset V^*$, let

$$N_{x,L(X_D)}(A) = \mathbb{N}_x\big(L(X_D) \in A, X_D \ne 0\big).$$

We fix $x \in D$ as before and define a reference measure R_x on V^* by

$$R_x(A) = P_{x,L(X_D)}(A, X_D \neq 0).$$

Its total mass is $r_{x,0} = P_{x,L(X_D)}(V^*) = 1 - e^{-u(x)}$, where $u(x) = -\log P_x(X_D = 0)$. Note that $u = \lim_{\beta \to \infty} V_D(\beta)$. Throughout this section we will assume that

(27) D is a bounded domain, all of whose boundary points are regular.

This holds, for example, if the boundary of D is smooth. Under assumption (27), $V_D(\beta)$ is the unique solution of

$$\tfrac{1}{2}\Delta u = 2u^2$$

on D with $u = \beta$ on ∂D [Proposition 8.2.1.B of Dynkin (2002)]. Because of this and the comparison principle [Proposition 8.2.1.H of Dynkin (2002)], $V_D(\beta)$ is also the maximal solution of $\frac{1}{2}\Delta u = 2u^2$ on D bounded by β. We will need this in the proof of Lemma 5.

By Theorem 2 we know the existence of a family of extended X-harmonic functions $\{H_x^v, v \in V\}$ such that $H_x^v(\mu) = dP_{\mu,L(X_D)}/dP_{x,L(X_D)}(v)$. In this section we are going to find a more explicit formula for this family.

LEMMA 5. *Assume* (27). *There exists a family of functions* $\{\gamma_{x,v} : D \mapsto (0,\infty), v \in V^*\}$ *such that the mapping* $(v, y) \mapsto \gamma_{x,v}(y)$ *is measurable and for all* $y \in D$

$$N_{y,L(X_D)}(dv) = \gamma_{x,v}(y) R_x(dv). \tag{28}$$

In addition, there exists a measurable kernel $K_{x,n}(v; dv_1, dv_2, \ldots, dv_{n-1})$ *from* V^* *to* $(V^*)^{n-1}$, *such that*

$$\begin{aligned} &R_x^n \circ T_n^{-1}(dv, dv_1, dv_2, \ldots, dv_{n-1}) \\ &\quad = K_{x,n}(v; dv_1, dv_2, \ldots, dv_{n-1}) R(dv). \end{aligned} \tag{29}$$

Moreover $K_{x,n}(v,\cdot)$ *is a strictly positive measure, for* R*-a.e.* v.

PROOF. Recall that

$$P_{\mu,L(X_D)}(A) = P_\mu\bigl(L(X_D) \in A\bigr).$$

Because all $P_\mu(X_D \in \cdot)$ are equivalent, so are the $P_{\mu,L(X_D)}$, as are their restriction to V^*. Thus all $P_{\mu,L(X_D)}$ (when restricted to V^*) are equivalent to R_x. Moreover, since

$$N_{y,L(X_D)}(A) = \mathbb{N}_y\bigl(L(X_D) \in A, X_D \neq 0\bigr) = \mathbb{N}_y\bigl(P_{X_{D'}}\bigl(L(X_D) \in A, X_D \neq 0\bigr)\bigr)$$

for all $y \in D$ and $D' \Subset D$ such that $y \in D'$, $N_{y,L(X_D)}$ is also equivalent to R_x for all $y \in D$. By Theorem A.1 of Dynkin (2006b) we get a family of functions

$\{\gamma_{x,v} : D \mapsto [0,\infty), v \in V^*\}$ such that the mapping $(v, y) \mapsto \gamma_{x,v}(y)$ is measurable, and (28) holds. Clearly such a $\gamma_{x,v}(y)$ can be chosen strictly positive since $N_{y,L(X_D)}$ and R_x are equivalent.

It will be convenient for the proof to write $R_x(dv) = r_{x,0}\tilde{R}_x(dv)$, where $\tilde{R}_x$ is a probability measure. In other words, $\tilde{R}_x(dv) = P_x(L(X_D) \in dv \mid X_D \neq 0)$. Note that with this choice of $\tilde{R}_x$, $\tilde{R}_x^n \circ T_n^{-1}(dv, dv_1, dv_2, \ldots, dv_{n-1})$ is the joint distribution of $(V_1 + \cdots + V_n, V_1, \ldots, V_{n-1})$ where V_i are independent random variables with distribution $\tilde{R}_x$. Let $R_{x,n}$ be the marginal distribution of $V_1 + \cdots + V_n$, where the V_i are as above. The following decomposition is then immediate:

$$\tilde{R}_x^n \circ T_n^{-1}(dv, dv_1, dv_2, \ldots, dv_{n-1}) = \tilde{K}_{x,n}(v; dv_1, dv_2, \ldots, dv_{n-1}) R_{x,n}(dv),$$

where $\tilde{K}_{x,n}(v; dv_1, dv_2, \ldots, dv_{n-1})$ is the conditional probability kernel for $(V_1, \ldots, V_{n-1})$ given $V_1 + \cdots + V_n$.

We now show that $R_{x,n}$ is absolutely continuous with respect to R_x. Let $X_D^1, \ldots, X_D^n$ be n independent realizations of the exit measure under the law P_x. Then the distribution of $X_D^1 + \cdots + X_D^n$ is given by the $P_{n\delta_x}$ distribution of X_D.

Let F be s.t. $R_x(F) = 0$, that is,

$$P_x\big(F\big(L(X_D)\big)1_{\{X_D \neq 0\}}\big) = 0. \tag{30}$$

Because P_x and $P_{n\delta_x}$ are absolutely continuous, (30) implies

$$P_{n\delta_x}\big(F\big(L(X_D)\big)1_{\{X_D \neq 0\}}\big) = 0. \tag{31}$$

Since

$$\begin{aligned} P_{n\delta_x}&\big(F\big(L(X_D)\big)1_{\{X_D \neq 0\}}\big) \\ &= (P_x)^n\big(F\big(L\big(X_D^1 + \cdots + X_D^n\big)\big)1_{\{X_D^1 + \cdots + X_D^n \neq 0\}}\big) \\ &\geq (P_x)^n\big(F\big(L\big(X_D^1\big) + \cdots + L\big(X_D^n\big)\big)1_{\{X_D^1 \neq 0\}} \cdots 1_{\{X_D^n \neq 0\}}\big) \\ &= r_{x,0}^n R_{x,n}(F), \end{aligned}$$

this implies that $R_{x,n}(F) = 0$, so indeed, $R_{x,n}$ is absolutely continuous with respect to R.

If $h_x^n(v)$ is the Radon–Nikodym derivative of $R_{x,n}$ with respect to R_x, we get (29) with

$$K_{x,n}(v; dv_1, dv_2, \ldots, dv_{n-1}) = \tilde{K}_{x,n}(v; dv_1, dv_2, \ldots, dv_{n-1}) h_x^n(v).$$

It remains only to show that $K_{x,n}(v, \cdot)$ is strictly positive. Because $\tilde{K}_{x,n}(v, \cdot)$ is, this amounts to showing the converse to the absolute continuity result above, namely that R_x is absolutely continuous with respect to $R_{x,n}$.

Our approach is to use the Poisson representation, as in the absolute continuity argument in Dynkin (2004). Suppose $R_{x,n}(F)=0$ and $0\le F\le 1$. The Poisson representation gives that

$$\begin{aligned}
&P_\mu\bigl(F\bigl(L(X_D)\bigr), X_D\neq 0\bigr)\\
&\quad=\sum_{k=1}^{\infty}\frac{e^{-\langle\mu,u\rangle}}{k!}\iint F\bigl(L(\nu_1)+\cdots+L(\nu_k)\bigr)\mathbb{N}_{x_1}(X_D\in d\nu_1, X_D\neq 0)\cdots\\
&\qquad\qquad\times\mathbb{N}_{x_n}(X_D\in d\nu_k, X_D\neq 0)\mu(dx_1)\cdots\mu(dx_k)\\
&\quad=\sum_{k=1}^{\infty}\frac{e^{-\langle\mu,u\rangle}}{k!}\int f_k(x_1,\dots,x_k)\mu(dx_1)\cdots\mu(dx_k),
\end{aligned}$$

where

$$\begin{aligned}
f_k(x_1,\dots,x_k)=\int F\bigl(L(\nu_1)+\cdots+L(\nu_k)\bigr)\mathbb{N}_{x_1}(X_D\in d\nu_1, X_D\neq 0)\cdots\\
\times\mathbb{N}_{x_k}(X_D\in d\nu_k, X_D\neq 0).
\end{aligned}$$

Let $D_m\Subset D$ such that $x\in D_m$ and $D_m\uparrow D$. Then

$$\begin{aligned}
&P_x\bigl(F\bigl(L(X_D)\bigr), X_D\neq 0\bigr)\\
&\quad=P_x\bigl(P_{X_{D_m}}\bigl(F\bigl(L(X_D)\bigr), X_D\neq 0\bigr)\bigr)\\
&\quad=P_x\Biggl(\sum_{k=1}^{\infty}\frac{e^{-\langle X_{D_m},u\rangle}}{k!}\int f_k(x_1,\dots,x_k)X_{D_m}(dx_1)\cdots X_{D_m}(dx_k)\Biggr).
\end{aligned}$$

There is a similar Poisson representation for $R_{x,n}(F)$, involving a sum of integrals of the f_k for $k\ge n$. Since $R_{x,n}(F)=0$, we conclude that for each $k\ge n$ there are $x_1,\dots,x_k\in D$ such that $f_k(x_1,\dots,x_k)=0$. By absolute continuity, we conclude that $f_k(x_1,\dots,x_k)=0$ for every $x_1,\dots,x_k$.

Since $F\le 1$ we obtain the bound

$$f(x_1,\dots,x_k)\le\prod_{j=1}^{k}\mathbb{N}_{x_j}(X_D\neq 0)=u(x_1)\cdots u(x_k).$$

Therefore

$$P_x\bigl(F\bigl(L(X_D)\bigr), X_D\neq 0\bigr)\le\sum_{k=1}^{n-1}P_x\biggl(\frac{e^{-\langle X_{D_m},u\rangle}}{k!}\langle X_{D_m},u\rangle^k\biggr).$$

The result will follow once we argue that all terms $e^{-\langle X_{D_m},u\rangle}\langle X_{D_m},u\rangle^k$ converge to 0 P_x-a.s. as we let $m\to\infty$, by the dominated convergence theorem since these terms are bounded. To show that $e^{-\langle X_{D_m},u\rangle}\langle X_{D_m},u\rangle^k\to 0$, it is enough to show that the stochastic boundary value of u (i.e., $\lim_{m\to\infty}\langle X_{D_m},u\rangle$) is 0 or ∞, P_x-

a.s. Let Z_β be the stochastic boundary value of the constant function β. The sequence $\langle X_{D_m}, \beta\rangle$ is a uniformly integrable martingale with respect to P_y for all $y \in D$. That it is a martingale follows because the constant function β is harmonic. The uniform integrability follows because this martingale is square integrable. Indeed, by the moment formula (7), $P_y(\langle X_{D_m}, \beta\rangle)^2 = G_{D_m}(4\beta^2)(y) + \beta \le G_D(4\beta^2)(y) + \beta < \infty$, since D is bounded. It follows that the log-potential of Z is the maximal solution of $\frac{1}{2}\Delta u = 2u^2$ on D bounded by β, which is $V_D\beta$, as we argued at the beginning of this section; see Sections 9.2.1 and 9.2.2 of Dynkin (2002). So $V_D\beta(x) = -\log P_x e^{-Z_\beta}$. Also note that $Z_\beta = \beta Z_1$, and $Z_\beta \uparrow Z$ where

$$Z = \begin{cases} 0, & \text{if } Z_1 = 0, \\ \infty, & \text{if } Z_1 > 0. \end{cases} \tag{32}$$

By the dominated convergence theorem

$$P_x e^{-Z} = \lim_{\beta\to\infty} P_x e^{-Z_\beta} = \lim_{\beta\to\infty} e^{-V_D\beta(x)} = e^{-u(x)},$$

so u is the log potential of Z, and therefore Z is the stochastic boundary value of u. Since Z is 0 or ∞ P_x-a.s., the proof is complete. □

LEMMA 6. *Assume* (27). *Let* $(K_{x,n})_{n\ge 2}$ *be a sequence of transition kernels satisfying* (29) *for* $n \ge 2$. *Define*

$$\bar{K}_{x,n}(v; dv_1, \dots, dv_n) := K_n(v; dv_1, \dots, dv_{n-1}) \times \delta_{v-(v_1+\cdots+v_{n-1})}(dv_n). \tag{33}$$

Then for R_x*-almost all* $v \in V^*$, *the following holds for all* $n \ge 2$, $1 \le r \le n$, *and any partition* $C_1, \dots, C_r$ *of* $\{1, \dots, n\}$, *where* $n_i = |C_i|$:

$$\bar{K}_{x,n}(v; dv_1, \dots, dv_n) = \int \bar{K}_{x,r}(v; dv_1, \dots, dv_r) \prod_{i=1}^r \bar{K}_{x,n_i}(\tilde{v}_i; dv_{C_i}).$$

PROOF. Since $\mathcal{V}$ is countably generated, so is $\mathcal{V}^n$ (the product σ-field), and therefore for each $n \ge 1$, there exists a sequence of nonnegative Borel measurable functions $\{f_i^n\}_{i=1}^\infty$ generating $\mathcal{V}^n$.

It will suffice to show for any $k, j \ge 1$, $n \ge 2$, $1 \le r \le n$, and any given partition $C_1, \dots, C_r$ of $\{1, \dots, n\}$ that

$$\begin{aligned} &\int f_j^1(v)\biggl[\int f_k^n(v_1, \dots, v_n)\bar{K}_{x,n}(v, dv_1, \dots, dv_n)\biggr] R_x(dv) \\ &\quad = \int f_j^1(v)\biggl[\int \bar{K}_{x,r}(v, d\tilde{v}_1, \dots, d\tilde{v}_r) \\ &\qquad\qquad \times \int \prod_{i=1}^r \bar{K}_{x,n_i}(\tilde{v}_i, dv_{C_i}) f_k^n(v_1, \dots, v_n)\biggr] R_x(dv). \end{aligned}$$

Let

$$\tilde{f}_k^n(\tilde{v}_1,\dots,\tilde{v}_r)=\int\prod_{i=1}^r \bar{K}_{x,n_i}(\tilde{v}_i,dv_{C_i})f_k^n(v_1,\dots,v_n).$$

Then

$$\begin{aligned}
&\int f_j^1(v)\int \bar{K}_{x,r}(v,d\tilde{v}_1,\dots,d\tilde{v}_r)\int\prod_{i=1}^r \bar{K}_{x,n_i}(\tilde{v}_i,dv_{C_i})f_k^n(v_1,\dots,v_n)R_x(dv)\\
&\quad=\int f_j^1(\tilde{v}_1+\cdots+\tilde{v}_r)\tilde{f}_k^n(\tilde{v}_1,\dots,\tilde{v}_r)R_x^r(d\tilde{v}_1,\dots,d\tilde{v}_r)\\
&\quad=\int f_j^1(\tilde{v}_1+\cdots+\tilde{v}_r)f_k^n(v_1,\dots,v_n)\Biggl[\prod_{i=1}^r \bar{K}_{x,n_i}(\tilde{v}_i,dv_{C_i})R_x(d\tilde{v}_i)\Biggr]\\
&\quad=\int f_j^1(\Sigma_{C_1}v_m+\cdots+\Sigma_{C_r}v_m)f_k^n(v_1,\dots,v_n)\prod_{i=1}^r\prod_{C_i}R_x(dv_m)\\
&\quad=\int f_j^1(v_1+\cdots+v_n)f_k^n(v_1,\dots,v_n)R_x^n(dv_1,\dots,dv_n)\\
&\quad=\int f_j^1(v)\biggl[\int f_k^n(v_1,\dots,v_n)\bar{K}_{x,n}(v,dv_1,\dots,dv_n)\biggr]R_x(dv). \qquad\square
\end{aligned}$$

LEMMA 7. *Assume* (27). *For any* $B_0\subset V^*$ *with* $R_x(B_0^c)=0$, *there exists* $B\subset V^*$ *with* $R_x(B^c)=0$, *such that* $B\subset B_0$ *and for all* $n\ge 2$ *and* $v\in B$, *we will have* $(v_1,\dots,v_n)\in B^n$, $\bar{K}_{x,n}(v,dv_1,\dots,dv_n)$ *a.s.*

PROOF. Define recursively $B_m, m\ge 1$ as

$$B_m=\Biggl\{v\in B_{m-1}:\sum_{n=2}^\infty\int\sum_{i=1}^n 1_{B_{m-1}^c}(v_i)\bar{K}_{x,n}(v,dv_1,\dots,dv_n)=0\Biggr\}.$$

Then $R_x(B_m^c)=0$. Because $R_x(B_0^c)=0$, and assuming $R_x(B_{m-1}^c)=0$, we have that

$$\begin{aligned}
&\int\sum_{n=2}^\infty\int\sum_{i=1}^n 1_{B_{m-1}^c}(v_i)\bar{K}_{x,n}(v,dv_1,\dots,dv_n)R_x(dv)\\
&\qquad=\sum_{n=2}^\infty\int\sum_{i=1}^n 1_{B_{m-1}^c}(v_i)R_x^n(dv_1,\dots,dv_n)\\
&\qquad=\sum_{n=2}^\infty nR_x(B_{m-1}^c)r_{x,0}^{n-1}=0,
\end{aligned}$$

which implies $R_x\{v : \sum_{n=2}^{\infty} \int \sum_{i=1}^{n} 1_{B_{m-1}^c}(v_i)\bar{K}_{x,n}(v, dv_1, \ldots, dv_n) \neq 0\} = 0$. This implies that $R_x(B_m^c) = 0$ since

$$B_m^c \subset \left\{ v : \int \sum_{i=1}^{n} 1_{B_{m-1}^c}(v_i)\bar{K}_{x,n}(v, dv_1, \ldots, dv_n) \neq 0 \right\}.$$

Let $B = \bigcap_{m=0}^{\infty} B_m$. Clearly $R_x(B^c) = 0$ and $B \subset B_0$.

Now, if $v \in B$, then for any m, $\sum_{n=2}^{\infty} \int \sum_{i=1}^{n} 1_{B_m^c}(v_i)\bar{K}_{x,n}(v, dv_1, \ldots, dv_n) = 0$. By the monotone convergence theorem, this implies that

$$\sum_{n=2}^{\infty} \int \sum_{i=1}^{n} 1_{B^c}(v_i)\bar{K}_{x,n}(v, dv_1, \ldots, dv_n) = 0.$$

Hence for all n, $\bar{K}_{x,n}(v, dv_1, \ldots, dv_n)$ almost all $(v_1, \ldots, v_n)$ is in B. □

So far $\gamma_{x,v}(y)$ is any jointly measurable version of the Radon–Nikodym derivative of N_{y,X_D} with respect to R_x. In the following theorem, we refine this choice and find a formula for the family of extended X harmonic functions $\{H_x^v, v \in V^*\}$ corresponding to $dP_{\mu,L(X_D)}/dP_{x,L(X_D)}(v)$.

THEOREM 8. *Assume* (27). *There exists a version of* $\{\gamma_{x,v}, v \in V^*\}$ *of Lemma* 5 *such that for* R_x *almost every* $v \in V^*$

$$\gamma_{x,v}(y) = \mathbb{N}_y\big(H_x^v(X_{D'})\big) \tag{34}$$

for every y, *and* $D' \Subset D$ *such that* $y \in D'$, *where*

$$H_x^v(\mu) = \begin{cases} e^{-\langle \mu, u\rangle + u(x)}, & \text{if } v = 0, \\ e^{-\langle \mu, u\rangle}\langle \mu, \gamma_{x,v}\rangle & \\ \qquad + \displaystyle\sum_{n=2}^{\infty} \int \frac{1}{n!} e^{-\langle \mu, u\rangle} K_{x,n}(v; dv_1, \ldots, dv_{n-1}) & \\ \qquad\qquad \times \langle \mu, \gamma_{v_1}\rangle \cdots \langle \mu, \gamma_{x,v_{n-1}}\rangle & \\ \qquad\qquad \times \langle \mu, \gamma_{x,v-(v_1+\cdots+v_{n-1})}\rangle, & \text{if } v \neq 0. \end{cases} \tag{35}$$

Moreover, with such a version of $\gamma_{x,v}$, $\{H_x^v, v \in V^*\}$ *defined by* (35) *is extended* X*-harmonic for* R_x*-almost all* v. *For fixed* μ, $H_x^v(\mu)$ *is a version of* $dP_{\mu,L(X_D)}/dP_{x,L(X_D)}(v)$.

PROOF. Let $\{\bar{\gamma}_{x,v}, v \in V^*\}$ be a family of functions satisfying the properties of Lemma 5. Define $\bar{H}_x^v$ by formula (35) with this $\bar{\gamma}_{x,v}$ in place of $\gamma_{x,v}$. Since $\bar{\gamma}_{x,v} > 0$, it follows that $\bar{H}_x^v > 0$. We will first show that for each μ, $\bar{H}_x^v(\mu)$ is a version of $dP_{\mu,L(X_D)}/dP_{x,L(X_D)}(v)$. Let F be a nonnegative Borel function on V_+.

$$\begin{aligned}
&\int P_{x,L(X_D)}(dv)F(v)\bar{H}_x^v(\mu)\\
&\quad= P_{x,L(X_D)}(\{0\})F(0)\bar{H}_x^0(\mu)+\int F(v)\bar{H}_x^v(\mu)R_x(dv)\\
&\quad= e^{-u(x)}F(0)e^{-\langle\mu,u\rangle+u(x)}+e^{-\langle\mu,u\rangle}\int F(v)\langle\mu,\bar{\gamma}_{x,v}\rangle R_x(dv)\\
&\qquad+\sum_{n=2}^{\infty}\frac{e^{-\langle\mu,u\rangle}}{n!}\iint\langle\mu,\bar{\gamma}_{x,v_1}\rangle\cdots\langle\mu,\bar{\gamma}_{x,v_{n-1}}\rangle\langle\mu,\bar{\gamma}_{x,v-(v_1+\cdots+v_{n-1})}\rangle\\
&\qquad\qquad\times F(v)K_{x,n}(v;dv_1,\dots,dv_{n-1})R_x(dv)\\
&\quad= e^{-\langle\mu,u\rangle}F(0)+\sum_{n=1}^{\infty}\frac{e^{-\langle\mu,u\rangle}}{n!}\int F(v_1+\cdots+v_n)\langle\mu,\bar{\gamma}_{x,v_1}\rangle\cdots\langle\mu,\bar{\gamma}_{x,v_n}\rangle\\
&\qquad\qquad\times R_x(dv_1)\cdots R_x(dv_n)\\
&\quad= e^{-\langle\mu,u\rangle}F(0)\\
&\qquad+\sum_{n=1}^{\infty}\frac{e^{-\langle\mu,u\rangle}}{n!}\int F(v_1+\cdots+v_n)\\
&\qquad\qquad\times\big\langle\mu,N_{(\cdot),L(X_D)}(dv_1)\big\rangle\cdots\big\langle\mu,N_{(\cdot),L(X_D)}(dv_n)\big\rangle\\
&\quad=\sum_{n=0}^{\infty}\frac{e^{-\langle\mu,u\rangle}}{n!}\int F\big(L(\nu_1)+\cdots+L(\nu_n)\big)\mathcal{R}_\mu(d\nu_1)\cdots\mathcal{R}_\mu(d\nu_n)\\
&\quad= P_\mu\big(F\big(L(X_D)\big)\big)=P_{\mu,L(X_D)}(F),
\end{aligned}$$

where we are using the Poisson representation. We choose countable base O_n (w.l.o.g. closed under finite unions) and consider the measure R on $\mathcal{M}_D^c$ defined as in the proof of Theorem 2. The argument in that proof tells us that there exists a $P_{x,L(X_D)}$-null set $\mathcal{N}$ s.t.

$$P_\mu\bar{H}_x^v(X_{O_n})=\bar{H}_x^v(\mu)\qquad\forall n,\text{ for }R\text{-a.e. }\mu\in\mathcal{M}_D^c,\forall v\in\mathcal{N}^c. \tag{36}$$

Without loss of generality we may assume that for $v\in\mathcal{N}^c$, the statement of Lemma 6 holds, and moreover by Lemma 7, $\bar{K}_n(v,dv_1,\dots,dv_n)$ almost surely $(v_1,\dots,v_n)\in(\mathcal{N}^c)^n$ for all n. For $v\in\mathcal{N}^c$ and $y\in D$, we choose O_n containing the support of y and define

$$\gamma_{x,v}(y)=\mathbb{N}_y\big(\bar{H}_x^v(X_{O_n})\big).$$

Since $\bar{H}_x^v>0$ it follows that $\gamma_{x,v}(y)\in(0,\infty]$ for every y. We set $\gamma_{x,v}$ to some arbitrary constant >0 for $v\in\mathcal{N}$. The definition of $\gamma_{x,v}$ is independent of the choice of O_n since, if $O_k \supset O_n$, then

$$\begin{aligned}\mathbb{N}_y \bar{H}_x^v(X_{O_k}) &= \mathbb{N}_y P_{X_{O_n}} \bar{H}_x^s(X_{O_k})\\ &= \mathbb{N}_y \bar{H}_x^s(X_{O_n}).\end{aligned}$$

The first equality is due to Markov property. The second equality is due to (36) and the fact that $\mathbb{N}_y(X_{O_n} \in (\cdot))$ is absolutely continuous with respect to R.

We now show that for fixed y, this gives a version of $\frac{dN_{y,L(X_D)}}{dR_x}$. If $y \in O_n$, then

$$\begin{aligned}\int_{\{v\neq 0\}} F(v) N_{y,L(X_D)}(dv) &= \mathbb{N}_y\big(F\big(L(X_D)\big)1_{\{X_D\neq 0\}}\big)\\ &= \mathbb{N}_y\big(P_{X_{O_n}}\big(F\big(L(X_D)\big)1_{\{X_D\neq 0\}}\big)\big)\\ &= \mathbb{N}_y\left(\int_{\{v\neq 0\}} \bar{H}_x^v(X_{O_n}) F(v) R_x(dv)\right)\\ &= \int_{\{v\neq 0\}} F(v) R_x(dv) \mathbb{N}_y\big(H_x^v(X_{O_n})\big)\\ &= \int_{\{v\neq 0\}} F(v)\gamma_{x,v}(y) R_x(dv).\end{aligned}$$

Let H_x^v be defined by formula (35). Then we know that for fixed μ, $H_x^v(\mu)$ is a version of $dP_{\mu,L(X_D)}/dP_{x,L(X_D)}(v)$. Now we are going to show that for $v \in \mathcal{N}^c$, $H_x^v(\mu) = P_\mu(\bar{H}_x^v(X_{O_k}))$. To simplify the notation, we drop the basepoint x from $\bar{K}_{x,n}$, $\bar{\gamma}_{x,v}$ and $\gamma_{x,v}$ for the remainder of the proof,

$$\begin{aligned}&P_\mu \bar{H}_x^v(X_{O_k})\\ &\quad= P_\mu \sum_{n=1}^{\infty} \int \frac{e^{-\langle X_{O_k}, u\rangle}}{n!} \prod_{i=1}^{n} \langle X_{O_k}, \bar{\gamma}_{v_i}\rangle \bar{K}_n(v; dv_1, \dots, dv_n)\\ &\quad= \sum_{n=1}^{\infty} \frac{1}{n!} \int P_\mu\left(e^{-\langle X_{O_k}, u\rangle} \prod_{i=1}^{n} \langle X_{O_k}, \bar{\gamma}_{v_i}\rangle\right) \bar{K}_n(v; dv_1, \dots, dv_n)\\ &\quad= \sum_{n=1}^{\infty} \frac{1}{n!} \int e^{-\langle \mu, V_{O_k}(u)\rangle} \left[\sum_{\pi(n)} \prod_{i=1}^{r} \left\langle \mu, \mathbb{N}_{(\cdot)}\left(e^{-\langle X_{O_k}, u\rangle} \prod_{j\in C_i} \langle X_{O_k}, \bar{\gamma}_{v_j}\rangle\right)\right\rangle\right]\\ &\qquad\qquad \times \bar{K}_n(v; dv_1, \dots, dv_n)\\ &\quad= e^{-\langle \mu, V_{O_k}(u)\rangle}\\ &\qquad \times \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\pi(n)} \int \bar{K}_r(v; d\tilde{v}_1, \dots, \tilde{v}_r)\\ &\qquad\qquad \times \prod_{i=1}^{r} \left\langle \mu, \mathbb{N}_{(\cdot)}\left(e^{-\langle X_{O_n}, u\rangle}\left[\int \prod_{j\in C_i} \langle X_{O_n}, \bar{\gamma}_{v_j}\rangle \bar{K}_{n_i}(\tilde{v}_i; dv_{C_i})\right]\right)\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= e^{-\langle \mu, V_{O_k}(u)\rangle} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{r=1}^{n} \int \bar{K}_r(v; \tilde{v}_1, \dots, \tilde{v}_r) \sum_{n_1,\dots,n_r} \frac{n!}{n_1! \cdots n_r! r!} \\
&\quad \times \prod_{i=1}^{r} \left\langle \mu, \mathbb{N}_{(\cdot)}\left(e^{-\langle X_{O_k}, u\rangle} \left[\int \prod_{j \in C_i} \langle X_{O_k}, \bar{\gamma}_{v_j}\rangle \bar{K}_{n_i}(\tilde{v}_i; dv_{C_i}) \right]\right)\right\rangle \\
&= e^{-\langle \mu, V_{O_k}(u)\rangle} \sum_{r=1}^{\infty} \int \frac{1}{r!} \bar{K}_r(v; d\tilde{v}_1, \dots, d\tilde{v}_r) \\
&\quad \times \prod_{i=1}^{r} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} \left\langle \mu, \mathbb{N}_{(\cdot)}\left(e^{-\langle X_{O_k}, u\rangle} \left[\int \prod_{j \in C_i} \langle X_{O_k}, \bar{\gamma}_{v_j}\rangle \bar{K}_{n_i}(\tilde{v}_i; dv_{C_i}) \right]\right)\right\rangle \\
&= e^{-\langle \mu, V_{O_k}(u)\rangle} \sum_{r=1}^{\infty} \int \frac{1}{r!} \bar{K}_r(v; d\tilde{v}_1, \dots, d\tilde{v}_r) \\
&\quad \times \prod_{i=1}^{r} \left\langle \mu, \mathbb{N}_{(\cdot)}\left(\sum_{n_i=1}^{\infty} \frac{1}{n_i!} e^{-\langle X_{O_k}, u\rangle} \left[\int \prod_{j \in C_i} \langle X_{O_k}, \bar{\gamma}_{v_j}\rangle \bar{K}_{n_i}(\tilde{v}_i; dv_{C_i}) \right]\right)\right\rangle \\
&= e^{-\langle \mu, u\rangle} \sum_{r=1}^{\infty} \int \frac{1}{r!} \bar{K}_r(v; d\tilde{v}_1, \dots, d\tilde{v}_r) \prod_{i=1}^{r} \langle \mu, \gamma_{x,\tilde{v}_i}\rangle \\
&= H_x^v(\mu),
\end{aligned}
$$

where the last line follows because by our assumption that if $v \in \mathcal{N}^c$, then $\bar{K}_r(v, d\tilde{v}_1, \dots, d\tilde{v}_r)$ almost surely $(\tilde{v}_1, \dots, \tilde{v}_r) \in (\mathcal{N}^c)^n$ and by construction for each $\tilde{v}_i$,

$$
\gamma_{\tilde{v}_i}(y) = \mathbb{N}_y \sum_{n_i=1}^{\infty} \frac{1}{n_i!} e^{-\langle X_{O_k}, u\rangle} \int \prod_{j \in C_i} \langle X_{O_k}, \bar{\gamma}_{v_j}\rangle \bar{K}_{n_i}(\tilde{v}_i; dv_{C_i}).
$$

Now we show that $\gamma_{x,v}$ satisfies equation (34). Let $y \in \tilde{D} \Subset D$. Then for some n, $\tilde{D} \Subset O_n$. By the definition of $\gamma_{x,v}$ and the Markov property,

$$
\begin{aligned}
\mathbb{N}_y\big(H_x^v(X_{\tilde{D}})\big) &= \mathbb{N}_y\big(P_{X_{\tilde{D}}} \bar{H}_{x,v}(X_{O_n})\big) \\
&= \mathbb{N}_y\big(\bar{H}_{x,v}(X_{O_n})\big) = \gamma_{x,v}(y)
\end{aligned}
$$

as desired.

Replacing $\mathbb{N}_y$ with P_μ in the above argument we have that H_x^v is extended X-harmonic for all $v \in \mathcal{N}^c$. It is also clear that H_x^v is X-harmonic for $v = 0$. Hence the proof is complete. $\square$

REMARKS. (i) Although Theorem 8 gives a workable form for $H_x^v(\mu)$, it does not remove all ambiguity in the choice of H_x^v, since $\gamma_{x,v}$ and $K_x(v; \cdot)$ are only well defined for a.e. v. Ideally we would like to prove continuity properties of these objects in v as well, that would then specify them uniquely. But we have not succeeded in doing that. In subsequent sections we will, however, be able to clarify the structure of these objects, and show how they determine the behavior of the H_x^v-transformed super-Brownian motion. We do have regularity in y. In particular, we will soon see that the version of $\gamma_{x,v}(y)$ given by Theorem 8 is already lower semi continuous in y.

(ii) Note that in Sections 3.1 and 3.2 without much difficulty we were able to show that the corresponding extended X-harmonic functions are finite, therefore X-harmonic. Finiteness is harder to prove for the extended X-harmonic functions of this section because it requires analytical bounds on the Radon–Nikodym densities of n-order moment measures of SBM for all $n \geq 1$. We hope to pursue such bounds in a subsequent paper.

(iii) An important case is when $L(X_D) = X_D$. The corresponding extended X-harmonic function H_x^ν can be thought as the analogue of the Martin kernel.

(iv) A more tractable application should be the case $L(X_D) = \langle X_D, 1\rangle$, where we condition on the total mass. In that case, $\gamma_{x,v}(y)$ is a function of only finite-dimensional variables $v \in [0,\infty)$ and $y \in D$. We hope to explore this example further in a subsequent paper.

(v) An interesting direction is to explore the relationship between $H_x^{\beta,\nu}$ of Section 3.1 and H_x^ν. In particular, with $\beta = n$, what happens to H^{n,ν_n} as $n \to \infty$, if $\{\nu_n\}_{n\geq 1}$ is a sequence of finite atomic measures such that $n^{-1}\nu_n$ converges to a finite measure ν on the boundary? One can show that

$$P_\mu^{n,Y_n} \to P_\mu^{X_D}$$

weakly almost surely, in a sense, and use this to investigate whether H_x^ν is extreme. We will pursue this direction in a subsequent paper.

4. Fragmentation system description of P_μ^ν.

The results of this section apply in general to the conditional law P_μ^v given $L(X_D) = v$. For simplicity, however, we will carry out the computations for the case $L(\nu) = \nu$. So in this section V^* is the set of positive finite measures on ∂D.

As in Section 3.3, we assume (27), that is, that D is regular. From Section 3.3, recall that $u(x) = -\log P_x(X_D = 0) = V_D(\infty)$, and u is a solution of the boundary value problem

$$\tfrac{1}{2}\Delta u = 2u^2$$

on D and $u = \infty$ on ∂D. We also have

$$u(x) = \mathbb{N}_x(X_D > 0) = \lim_{\beta\to\infty} V_D\beta(x). \tag{37}$$

Let $\gamma_{x,\nu}$ and H_x^ν be as constructed in Section 3.3. Let $\mathcal{N} \subset V^*$ be the R_x-null set such that for $\nu \in \mathcal{N}^c$, $\gamma_{x,\nu}$ and H_x^ν satisfy the system of equations in (34) and (35), H_x^ν is extended X-harmonic and the kernels $\bar{K}_{x,n}$ are strictly positive and satisfy the decomposition property of Lemma 6. By Lemma 7 we may assume that if $\nu \in \mathcal{N}^c$, then $\bar{K}_{x,n}(\nu, d\nu_1, \dots, d\nu_n)$-almost all $(\nu_1, \dots, \nu_n)$ are in $(\mathcal{N}^c)^n$. We fix $\nu \in \mathcal{N}^c$. Let y be a point in D such that $\gamma_{x,\nu}(y) < \infty$. Recall that $\gamma_{x,\nu}(y) > 0$ by construction. If $y \in D' \Subset D$, we may then define a change of measure by

$$\mathbb{N}_y^\nu(Z) = \frac{1}{\gamma_{x,\nu}(y)} \mathbb{N}_y\big(Z H_x^\nu(X_{D'})\big)$$

for positive $\mathcal{F}^y_{\subset D'}$-measurable Z. ($\mathcal{F}^y_{\subset D'}$ is defined as the σ-algebra generated by $\{X_O, y \in O \subset D'\}$.) Since H_x^ν is extended X-harmonic, $\mathbb{N}_y^\nu$ is defined consistently on $\mathcal{F}^y_{\subset D-}$, and we have that $\mathbb{N}_y^\nu$ is a probability law because of equation (34).

In the remainder of this section we turn to the problem of giving an explicit probabilistic construction of $\mathbb{N}_y^\nu$ in terms of a backbone along which unconditioned mass is created. We do this in two steps. Let $H_{x,1}^\nu(\mu) = e^{-\langle \mu, u\rangle}\langle \mu, \gamma_{x,\nu}\rangle$, and for $n \geq 2$ let

$$H_{x,n}^\nu(\mu) = \int \frac{e^{-\langle \mu, u\rangle}}{n!} \bar{K}_{x,n}(\nu; d\nu_1, \dots, d\nu_n)\langle \mu, \gamma_{x,\nu_1}\rangle \cdots \langle \mu, \gamma_{x,\nu_n}\rangle. \tag{38}$$

Then $H_x^\nu = \sum_{n\geq 1} H_{x,n}^\nu$. We will first use the recursive moment formula to establish an inductive relationship for $H_{x,n}^\nu$, and then compare this to the inductive relationship coming from the first branch of the backbone.

We will make use of a stochastic process ξ_t under various measures Π_y or Π_y^{4u}. In either case we use the shorthand

$$\mathcal{N}_t^{D'}(\phi) = e^{-\int_0^t 4\mathbb{N}_{\xi_s}(1 - e^{-\langle X_{D'}, \phi\rangle})\, ds},$$

where $D' \Subset D$ and let $\tau_{D'}$ be the exit time of ξ from D'. We similarly let

$$\mathcal{N}_t^{I, D_I}(\phi_I) = e^{-\int_0^{t\wedge\tau_{D_I}} 4\mathbb{N}_{\xi_s}(1 - e^{-\langle X_I, \phi_I\rangle})\, ds},$$

where $I = \{D_1, \dots, D_k\}$ is such that each $D_j \Subset D$, $\phi_I = (\phi_1, \dots, \phi_k)$, $X_I = (X_{D_1}, \dots, X_{D_k})$, so $\langle X_I, \phi_I\rangle = \sum_j \langle X_{D_j}, \phi_j\rangle$. Also write $D_I = D_1 \cap \cdots \cap D_k$.

In the rest of the paper we will denote $K_{x,2}(\nu, d\nu')$ by $K_x(\nu, d\nu')$. Let $I = \{D_1, \dots, D_k\}$ where $D_i \subset D_k = D' \Subset D$. Define a family of operators $N_y^{I,\nu,n}$ as follows:

For $\operatorname{card}(I) = 1$,

$$N_y^{I,\nu,n}(\phi) = \begin{cases} \mathbb{N}_y\big(e^{-\langle X_{D'}, \phi\rangle} H_{x,1}^\nu(X_{D'})\big), & y \in D', \\ \gamma_{x,\nu}(y), & y \notin D', n = 1, \\ 0, & y \notin D', n > 1. \end{cases} \tag{39}$$

For card$(I) > 1$,

$$N_y^{I,\nu,n}(\phi_I) = \begin{cases} \mathbb{N}_y\big(e^{-\langle X_I,\phi_I\rangle} H_{x,n}^{\nu}(X_{D_k})\big), & y \in D_I, \\ N_y^{I^j,\nu,n}, & y \notin D_j, j \neq k, \end{cases} \tag{40}$$

where $I_j = I - \{D_j\}$. Note that we have

$$\begin{aligned} N_y^{I,\nu,n}(\phi_I) &= \frac{1}{n!}\int n_{C_n}^I\big(\phi_1,\ldots,\phi_k+u,\gamma^{\nu_1},\ldots,\gamma^{\nu_n}\big)(y) \\ &\qquad \times \bar{K}_{x,n}(\nu,d\nu_1,\ldots,d\nu_n), \end{aligned}$$

where $C_n = \{1,\ldots,n\}$, and the n_C^I are the operators defined by equations (8) and (9).

LEMMA 9. *Assume* (27). *Let* $\phi_I^u = (\phi_1,\ldots,\phi_{k-1},\phi_k+u)$. *For* $n=1$ *and* $y \in D_I$,

$$N_y^{I,\nu,1}(\phi_I) = \Pi_y\big(N_{\xi_{\tau_{D_I}}}^{I,\nu,1}(\phi_I)\mathcal{N}_{\tau_{D_I}}^{I,D_I}(\phi_I^u)\big). \tag{41}$$

For $n \geq 2$ *and* $y \in D_I$,

$$\begin{aligned} &N_y^{I,\nu,n}(\phi_I) \\ &\quad = \sum_{m=1}^{n-1}\int \Pi_y\bigg(\int_0^{\tau_{D_I}} 2N_{\xi_t}^{I,\nu',m}(\phi_I)N_{\xi_t}^{I,\nu-\nu',n-m}(\phi_I)\mathcal{N}_t^{I,D_I}(\phi_I^u)\,dt\bigg) \\ &\qquad\qquad \times K_x\big(\nu,d\nu'\big) \\ &\qquad + \Pi_y N_{\xi_{\tau_{D_I}}}^{I,\nu,n}(\phi_I)\mathcal{N}_{\tau_{D_I}}^{I,D_I}(\phi_I^u). \end{aligned} \tag{42}$$

PROOF. If $n=1$, then $H_{x,1}^{\nu}(\mu) = e^{-\langle\mu,u\rangle}\langle\mu,\gamma_{x,\nu}\rangle$ and the result is an immediate consequence of the basic Palm formula (5) [for card$(I)=1$] and the extended Palm formula (11) [for card$(I)>1$].

If $n \geq 2$, by the recursive moment formulas (6),

$$\begin{aligned} &\mathbb{N}_y\big(e^{-\langle X_I,\phi_I\rangle}H_{x,n}^{\nu}(X_{D_k})\big) \\ &\quad = \frac{1}{n!}\int \mathbb{N}_y\big(e^{-\langle X_I,\phi_I\rangle}e^{-\langle X_{D_k},u\rangle}\Pi_i\langle X_{D_k},\gamma_{x,\nu_i}\rangle\big) \\ &\qquad\qquad \times\bar{K}_{x,n}(\nu;d\nu_1,\ldots,d\nu_{n-1}) \\ &\quad = A+B, \end{aligned}$$

where

$$A = \frac{1}{2\cdot n!}\int \sum_{\substack{M\subset N\\ \varnothing, N\neq M}} \Pi_y\biggl(\int_0^{\tau_{D_I}} 4\mathcal{N}_t^{I,D_I}(\phi_I^u) \times \mathbb{N}_{\xi_t}\bigl(e^{-\langle X_I,\phi_I^u\rangle}\Pi_{i\in M}\langle X_{D'},\gamma_{x,\nu_i}\rangle\bigr) \times \mathbb{N}_{\xi_t}\bigl(e^{-\langle X_I,\phi_I^u\rangle}\Pi_{i\notin M}\langle X_{D'},\gamma_{x,\nu_i}\rangle\bigr)\,dt\biggr) \times \bar{K}_{x,n}(\nu; d\nu_1,\dots,d\nu_n).$$

There are $\binom{n}{m}$ possible choices of M in the above expression, with cardinality m. Therefore, by rearranging the indices, and using Lemma 6 with $r=2$, we get

$$\begin{aligned} A = \sum_{m=1}^{n-1}\int K_x(\nu; d\nu') &\times \Pi_y\biggl(\int_0^{\tau_{D_I}} 2\mathcal{N}_t^{I,D_I}(\phi_I^u) \\ &\times\biggl[\frac{1}{m!}\int \mathbb{N}_{\xi_t}\biggl(e^{-\langle X_I,\phi_I^u\rangle}\prod_{i=1}^{m}\langle X_{D_k},\gamma_{x,\nu_i}\rangle\biggr) \times \bar{K}_{x,m}(\nu'; d\nu_1,\dots,d\nu_m)\biggr] \\ &\times\biggl[\frac{1}{(n-m)!}\int \mathbb{N}_{\xi_t}\biggl(e^{-\langle X_I,\phi_I^u\rangle}\prod_{i=1}^{n-m}\langle X_{D_k},\gamma_{x,\nu_i}\rangle\biggr) \times \bar{K}_{x,n-m}(\nu-\nu'; d\nu_1,\dots,d\nu_{n-m})\biggr]\,dt\biggr). \end{aligned}$$

The term B is 0 for $\mathrm{card}(I)=1$ and for $\mathrm{card}(I)>1$ is found using the extended moment formula (12) as

$$\begin{aligned} B &= \frac{1}{n!}\int \Pi_y\bigl(n_{C_n}^I(\phi_1,\dots,\phi_k+u,\gamma_{x,\nu_1},\dots,\gamma_{x,\nu_n},\xi_{\tau_{D_I}})\mathcal{N}_{\tau_{D_I}}^{D_I}(\phi_I^u)\bigr) \times \bar{K}_{x,n}(\nu; d\nu_1,\dots,d\nu_n) \\ &= \Pi_y\bigl(N_{\xi_{\tau_{D_I}}}^{I,\nu,n}(\phi_I)\mathcal{N}_{\tau_{D_I}}^{I,D_I}(\phi_I^u)\bigr). \qquad\square \end{aligned}$$

Now set $\Gamma_{x,\nu} = 2\int \gamma_{x,\nu'}\gamma_{x,\nu-\nu'}K_x(\nu; d\nu')$.

THEOREM 10. *Assume* (27). *The function* $\gamma_{x,\nu}$ *is* $\mathcal{L}^{4u}$*-superharmonic, and hence lower-semi-continuous in* y. *For* R_x*-almost all* $\nu\in\mathcal{N}^c$ *it is in fact an* $\mathcal{L}^{4u}$*-*

potential, *and satisfies*

$$\gamma_{x,\nu} = G_D^{4u}\Big[2\int \gamma_{x,\nu}\gamma_{x,\nu-\nu'} K_x(\nu; d\nu')\Big]. \tag{43}$$

PROOF. Let D_k be a sequence of domains exhausting D. Since $V_{D_k}u = u$, we have that $\mathcal{N}_t^{D_k}(u) = e^{-\int_0^t 4u(\xi_s)\,ds}$ for $t < \tau_{D_k}$. Thus

$$\begin{aligned}
\gamma_{x,\nu}(y) &= \mathbb{N}_y(H_x^\nu(X_{D_k})) = \sum_{n=1}^{\infty} \mathbb{N}_y(H_{x,n}^\nu(X_{D_k})) \\
&= \mathbb{N}_y(H_{x,1}^\nu(X_{D_k})) \\
&\quad + \sum_{n=2}^{\infty}\sum_{m=1}^{n-1}\int K_x(\nu; d\nu') \\
&\qquad \times \Pi_y\Big(\int_0^{\tau_{D_k}} 2\mathcal{N}_t^{D_k}(u)\mathbb{N}_{\xi_t}(H_{x,m}^{\nu'})\mathbb{N}_{\xi_t}(H_{x,n-m}^{\nu-\nu'})\,dt\Big) \\
&= \mathbb{N}_y(H_{x,1}^\nu(X_{D_k})) + \sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\int K_x(\nu; d\nu') \\
&\qquad \times \Pi_y^{4u}\Big(\int_0^{\tau_{D_k}} 2\mathbb{N}_{\xi_t}(H_{x,m}^{\nu'})\mathbb{N}_{\xi_t}(H_{x,j}^{\nu-\nu'})\,dt\Big) \\
&= \mathbb{N}_y(H_{x,1}^\nu(X_{D_k})) + 2\int \Pi_y^{4u}\Big(\int_0^{\tau_{D_k}} \gamma_{x,\nu'}(\xi_t)\gamma_{x,\nu-\nu'}(\xi_t)\,dt\Big)K_x(\nu; d\nu') \\
&= \mathbb{N}_y(H_{x,1}^\nu(X_{D_k})) + \Pi_y^{4u}\Big(\int_0^{\tau_{D_k}} \Gamma_\nu(\xi_t)\,dt\Big).
\end{aligned}$$

The first term is $\mathcal{L}^{4u}$-harmonic on D_k by Lemma 9, and the second term is an $\mathcal{L}^{4u}$-potential, so $\gamma_{x,\nu}$ is $\mathcal{L}^{4u}$-superharmonic on each D_k. Thus it is so on D as well.

Moreover,

$$\begin{aligned}
\int \mathbb{N}_y(H_{x,1}^\nu(X_{D_k}))R_x(d\nu) &= \mathbb{N}_y\Big(e^{-\langle X_{D_k},u\rangle}\int \langle X_{D_k}, \gamma_{x,\nu}\rangle R_x(d\nu)\Big) \\
&= \mathbb{N}_y\Big(e^{-\langle X_{D_k},u\rangle}\iint X_{D_k}(dw)\gamma_{x,\nu}(w)R_x(d\nu)\Big) \\
&= \mathbb{N}_y\Big(e^{-\langle X_{D_k},u\rangle}\int X_{D_k}(dw)\mathbb{N}_w(X_D \neq 0)\Big) \\
&= \mathbb{N}_y(e^{-\langle X_{D_k},u\rangle}\langle X_{D_k}, u\rangle) \\
&= e^{u(y)}P_y(e^{-\langle X_{D_k},u\rangle}\langle X_{D_k}, u\rangle),
\end{aligned}$$

where the last equality follows from the recursive moment formula (7). As we argued in the proof of Lemma 5, the stochastic boundary value of u is 0 or ∞ P_y-a.s.; therefore $P_y(e^{-\langle X_{D_k},u\rangle}\langle X_{D_k},u\rangle)$ converge to 0 as $k\to\infty$ by the dominated convergence theorem. Thus

$$\int \mathbb{N}_y\big(H_1^\nu(X_{D_k})\big)R_x(d\nu)\to 0$$

as $k\to\infty$. By Fatou's lemma,

$$\liminf_{k\to\infty}\mathbb{N}_y\big(H_1^\nu(X_{D_k})\big)=0 \tag{44}$$

for R_x-a.e. ν. But the second term in our expression for $\gamma_{x,\nu}(y)$ is monotone in k, and therefore $\lim_{k\to\infty}\mathbb{N}_y(H_1^\nu(X_{D_k}))$ must exist and thus equal to 0 for R_x-a.e. ν by (44). So, we get that $\gamma_{x,\nu}(y)=\Pi_y^{4u}(\int_0^{\tau_{D_k}}\Gamma_{x,\nu}(\xi_t)\,dt)$ for R_x-a.e. ν, for every y.

Choosing a countable dense set $y_1,y_2,\dots,$ we therefore have a set $\mathcal{N}_0\supset\mathcal{N}$ such that $R_x(\mathcal{N}_0^c)=0$, and the above equality holds for every y_i and for every $\nu\in(\mathcal{N}_0)^c$. Since both functions are lower-semi-continuous, and agree on a countable dense set, it follows that (43) holds for every $\nu\in(\mathcal{N}_0)^c$. $\square$

Suppose now that $\nu\in\mathcal{N}_0^c$ where $\mathcal{N}_0$ is the R_x-null set described in Theorem 10. Again we may assume that if $\nu\in\mathcal{N}_0^c$, then $\bar{K}_{x,n}(\nu,d\nu_1,\dots,d\nu_n)$-almost all $(\nu_1,\dots,\nu_n)$ are in $(\mathcal{N}^c)^n$. Let $\hat{\mathbb{N}}_y$ be the excursion measure of a super-process whose spatial motion is killed at rate u. In other words, for any $D'\Subset D$, we have

$$\hat{\mathbb{N}}_y\big(F(X_{D'})\big)=\mathbb{N}_y\big(e^{-\langle X_{D'},u\rangle}F(X_{D'})\big).$$

Let $y\in D$ be such that $\gamma_{x,\nu}(y)<\infty$. We define a probability Q_y^ν on an auxiliary probability space $\tilde{\Omega}$ where there is a branching diffusion on D, and conditional on this branching diffusion, a Poisson random measure is generated on the infinite product space $\mathcal{M}^{\mathcal{O}_{\mathcal{D}-}}$. We endow $\mathcal{M}^{\mathcal{O}_{\mathcal{D}-}}$ with the σ-algebra $\tilde{\mathcal{F}}_{\subset D-}$ generated by the coordinate maps $\tilde{x}_{D'}$, $D'\Subset D$ [i.e., $\tilde{x}_{D'}(\omega)=\omega_{D'}$ for $\omega\in\mathcal{M}^{\mathcal{O}_{\mathcal{D}-}}$]. Our goal is to construct an $\mathcal{M}^{\mathcal{O}_{\mathcal{D}-}}$-valued process $\tilde{X}=(\tilde{X}_{D'})_{D'\Subset D}$ on $\tilde{\Omega}$ such that the law of $\tilde{X}$ with respect to Q_y^ν will be the same as the $\mathbb{N}_y^\nu$ law of the exit measures $(X_{D'})_{D'\Subset D}$ of a SBM. First we describe how the branching diffusion evolves: we start a $\gamma_{x,\nu}$-transform of a $\mathcal{L}^{4u}$ process off at y. Since $\gamma_{x,\nu}$ is a potential, this process dies before reaching ∂D. Say it dies at w. Then almost surely $\Gamma_{x,\nu}(w)$ is finite. Because

$$\Pi_y^{u,\gamma_{x,\nu}}(1_{\Gamma_{x,\nu}(\xi_\zeta)=\infty})=\frac{1}{\gamma_{x,\nu}(y)}\int_0^\infty \Pi_y^u\big(1_{\Gamma_{x,\nu}(\xi_t)=\infty}\Gamma_{x,\nu}(\xi_t)1_{\zeta>t}\big)\,dt,$$

which must be equal to 0 since $\Pi_y^u(1_{\Gamma_{x,\nu}(\xi_t)=\infty}\Gamma_{x,\nu}(\xi_t)1_{\zeta>t})$ is 0 or ∞ for any t, and the assumption $\gamma_{x,\nu}(y)=\Pi_y^u(\Gamma_{x,\nu}(\xi_t)1_{\zeta>t})<\infty$ implies $\Pi_y^u(1_{\Gamma_{x,\nu}(\xi_t)=\infty}\times\Gamma_{x,\nu}(\xi_t)1_{\zeta>t})=0$ for Lebesgue-almost all t, making the right-hand side of this equation equal to 0. Note also $\Gamma_{x,\nu}>0$ since $\nu\in\mathcal{N}_0^c$, and therefore $K_x(\nu,d\nu')$

is a strictly positive measure. Now we can choose ν' at random with distribution density

$$\frac{2}{\Gamma_{x,\nu}(w)}\gamma_{x,\nu'}(w)\gamma_{x,\nu-\nu'}(w)K_x\big(\nu; d\nu'\big). \tag{45}$$

Almost surely, both ν' and $\nu - \nu'$ are in $\mathcal{N}_0^c$, and $\gamma_{x,\nu}(w)$ and $\gamma_{x,\nu'}(w)$ are finite, so we may start two new processes at w, following $\gamma_{x,\nu}$ and $\gamma_{x,\nu-\nu'}$ transforms of $\mathcal{L}^{4u}$. Note that we may repeat this process infinitely often. This defines a branching particle system. Let Υ_t denote the measure-valued process putting a unit point mass at the historical paths of each particle alive at time t. (A historical path of a given particle at time t is the path describing for any $s < t$ the location of the particle or whichever ancestor that is alive at time s.) We then create mass uniformly along this set of particle paths, which then evolves according to the law $\hat{\mathbb{N}}_{(\cdot)}$. Loosely speaking, we want to generate a Poisson random measure with intensity

$$\int_0^\infty 4\Upsilon_t(dz)\hat{\mathbb{N}}_{z_t}(X \in \cdot)\,dt$$

and add up the resulting measure-valued processes to form $\tilde{X}$. But we must be careful to represent $\tilde{X}_{D'}$ using only the portions of Υ corresponding to particles whose historical paths have never left D'. To formulate this, for each t and historical path z, we define the following map $X^{t,z} : \Omega \mapsto \mathcal{M}^{\mathcal{O}_{D-}}$

$$X^{t,z}(\omega)_{D'} = \begin{cases} X_{D'}(\omega), & \text{if } \tau_{D'}(z) > t, \\ 0, & \text{otherwise.} \end{cases}$$

Now we generate a Poisson random measure on $\mathcal{M}^{\mathcal{O}_{D-}}$ with intensity

$$\lambda(A) = \int_0^\infty 4\Upsilon_t(dz)\hat{\mathbb{N}}_{z_t}\big((X^{t,z})^{-1}(A)\big)\,dt,$$

which is now well defined for any $A \in \tilde{\mathcal{F}}_{\subset D-}$, since $(X^{t,z})^{-1}(A)$ is $\mathcal{F}^{z(t)}_{\subset D-}$ measurable. (Recall, for any $c \in E$, $\mathcal{F}^c_{\subset D-}$ is defined as the σ-algebra generated by $\{X_{D'}, D' \in \mathcal{O}, c \in D'\}$, which is the domain where the measure $\mathbb{N}_c$ is defined.) Adding up the resulting measure-valued processes gives us $\tilde{X} = (\tilde{X}_{D'})_{D' \in \mathcal{O}_{D-}}$.

More precisely, the n-dimensional transition operators for $\tilde{X}$ with respect to Q_y^ν is given by the formula

$$Q_y^{I,\nu}(\phi_I) := Q_y^\nu\big(e^{-\langle \tilde{X}_I, \phi_I\rangle}\big) = Q_y^\nu\big(e^{-\int_0^\infty 4\langle \Upsilon_t, \hat{\mathbb{N}}.(1-e^{-\langle X_I^{t,z}, \phi_I\rangle})\rangle\,dt}\big),$$

where $I = \{D_1, \dots, D_k\} \subset \mathcal{O}$, and $y \in D$.

Note that for $y \notin D_I$, $Q_y^{I,\nu}(\phi_I) = Q_y^{I_j,\nu}(\phi_{I_j})$ for some j such that $y \notin D_j$ (since all paths in the backbone start from y, $X^{t,z}_{D_j} = 0$, Υ_t almost all z for all t, Q_y-almost surely).

The main result of this section is:

THEOREM 11. *Assume* (27), *that* $\nu \in \mathcal{N}_0^c$ *and that* $\gamma_{x,\nu}(y) < \infty$. *Then* $\mathbb{N}_y^\nu$-*law of* $(X_{D'})_{y\in D'\Subset D}$ *is the same as the* Q_y^ν-*law of* $(\tilde{X}_{D'})_{y\in D'\Subset D}$.

PROOF. Let $y \in U_1 \Subset U_2 \Subset \cdots \Subset U_i \cdots$ s.t. $D = \bigcup_{i=1}^\infty U_i$. It will suffice to show that $Q_y^{I,\nu}(\phi_I) = N_y^{I,\nu}(\phi_I)$ for $I = \{D_1, \dots, D_k\}$ where $D_j \subset D_k = U_i$ for a fixed i.

Let $D' = U_i$. Define $\Upsilon^{D'}(d\tilde{z})$ as the random measure on D'-valued paths defined by $\Upsilon^{D'}(d\tilde{z}) = \lim_{t\to\infty} \int \Upsilon_t(dz) 1_{\{z^{\tau_{D'}} \in d\tilde{z}\}}$ where z^{τ_D} is the path z stopped at $\tau_{D'}(z)$. We will write $\Upsilon^{D'} \sim n$ if the support of $\Upsilon^{D'}$ consists of exactly n paths (i.e., exactly n particles of Υ exit D'). Let

$$Q_y^{I,\nu,n}(\phi_I) := Q_y^\nu\big(e^{-\langle \tilde{X}_I, \phi_I\rangle}, \Upsilon^{D'} \sim n\big).$$

We will show by induction on n and card(I) that

$$N_y^{I,\nu,n}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,n}(\phi_I) \tag{46}$$

for all $y \in D_I$. The theorem then follows by summing on n.

First observe that, for card$(I) = 1$,

$$\gamma_{x,\nu}(y) Q_y^{I,\nu,n}(\phi) = \begin{cases} \gamma_{x,\nu}(y), & y \notin D', n = 1, \\ 0, & y \notin D', n > 1, \end{cases} \tag{47}$$

and for card$(I) > 1$,

$$\gamma_{x,\nu}(y) Q_y^{I,\nu,n}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I^j,\nu,n}, \qquad y \notin D_j, j \neq k. \tag{48}$$

Comparing these equations to equations (39) and (40), we see that equation (46) holds on $\partial D'$.

Let $(\xi_t)_{t\geq 0}$ be an $\mathcal{L}$-diffusion under the law Π_y. Π_y^{4u} and $\Pi_y^{4u,\gamma_{x,\nu}}$ will denote the laws under which ξ is, respectively, an $\mathcal{L}^{4u}$ diffusion, and $\gamma_{x,\nu}$-transform of an $\mathcal{L}^{4u}$ diffusion. In what follows ξ will represent the first particle of the branching backbone which is by construction following a $\gamma_{x,\nu}$-transform of an $\mathcal{L}^{4u}$-diffusion. We let ζ be the lifetime of ξ. Note first that

$$\begin{aligned}
&e^{-\int_0^{t\wedge\tau_{D_I}} 4u(\xi_s)\,ds} e^{-\int_0^{t\wedge\tau_{D_I}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_I,\phi_I\rangle})} \\
&\quad = \exp\biggl(-4\int_0^{t\wedge\tau_{D_I}} \bigl[u(\xi_s) + \mathbb{N}_{\xi_s}\bigl(e^{-\langle X_{D'},u\rangle} - e^{-\langle X_I,\phi_I^u\rangle}\bigr)\bigr]\,ds\biggr) \\
&\quad = \exp -4\int_0^{t\wedge\tau_{D_I}} \bigl[u(\xi_s) - \mathbb{N}_{\xi_s}\bigl(1 - e^{-\langle X_{D'},u\rangle}\bigr) + \mathbb{N}_{\xi_s}\bigl(1 - e^{-\langle X_I,\phi_I^u\rangle}\bigr)\bigr]\,ds \\
&\quad = \mathcal{N}_t^{I,D_I}(\phi_I^u).
\end{aligned}$$

If $n = 1$, then

$$
\begin{aligned}
&\gamma_{x,\nu}(y) Q_y^{I,\nu,n}(\phi_I) \\
&\quad = \gamma_{x,\nu}(y) \Pi_y^{4u,\gamma_{x,\nu}}\bigl(e^{-\int_0^{\tau_{D_I}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_I,\phi_I\rangle})\,ds} Q^{I,\nu,1}_{\xi_{\tau_{D_I}}}(\phi_I) 1_{\tau_{D_I}<\zeta}\bigr) \\
&\quad = \Pi_y^{4u}\bigl(\gamma_\nu(\xi_{\tau_{D_I}}) e^{-\int_0^{\tau_{D_I}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_I,\phi_I\rangle})\,ds} Q^{I,\nu,1}_{\xi_{\tau_{D_I}}}(\phi_I)\bigr) \qquad (49)\\
&\quad = \Pi_y\bigl(\gamma_\nu(\xi_{\tau_{D_I}}) e^{-\int_0^{\tau_{D_I}} 4u(\xi_s)\,ds} e^{-\int_0^{\tau_{D_I}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_I,\phi_I\rangle})\,ds} Q^{I,\nu,1}_{\xi_{\tau_{D_I}}}(\phi_I)\bigr) \\
&\quad = \Pi_y\bigl(\gamma_\nu(\xi_{\tau_{D_I}}) Q^{I,\nu,1}_{\xi_{\tau_{D_I}}}(\phi_I) \mathcal{N}^{I,D_I}_{\tau_{D'}}(\phi_I^u)\bigr).
\end{aligned}
$$

The first equation is true, because of the strong Markov property of Υ_t at the first exit time of D_I of its first branch ξ. The second and third equations follow, respectively, from the definition of $\gamma_{x,\nu}$-transform, and L^{4u} diffusion. If $\mathrm{card}(I) = 1$, equations (49) and (41) and the fact that $N_y^{I,\nu,1}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,1}(\phi_I)$ on the boundary of $D_I = D'$ implies that

$$N_y^{I,\nu,1}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,1}(\phi_I)$$

holds for $y \in \bar{D}_I$. If we assume $N_y^{I,\nu,1}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,1}(\phi_I)$ holds for $y \in \bar{D}_I$ when $\mathrm{card}(I) = k-1$, then equations (48) and (40) and (49) and (41) imply that $N_y^{I,\nu,1}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,1}(\phi_I)$ for $y \in \bar{D}_I$ when $\mathrm{card}(I) = k$ as well.

If $n \geq 2$, then

$$\gamma_{x,\nu}(y) Q_y^\nu\bigl(e^{-\langle X_I,\phi_I\rangle}, \Upsilon^{D'} \sim n\bigr) = A + B,$$

where

$$
\begin{aligned}
A &= \gamma_{x,\nu}(y) Q_y^\nu\bigl(e^{-\langle X_I,\phi_I\rangle}, \Upsilon^{D'} \sim n, \text{the first branch of } \Upsilon \text{ dies inside } D_I\bigr),\\
B &= \gamma_{x,\nu}(y) Q_y^\nu\bigl(e^{-\langle X_I,\phi_I\rangle}, \Upsilon^{D'} \sim n, \text{the first branch of } \Upsilon \text{ exits } D_I\bigr).
\end{aligned}
$$

We compute A as follows:

$$
\begin{aligned}
A &= \gamma_{x,\nu}(y) \\
&\quad \times \Pi_y^{4u,\gamma_{x,\nu}}\Biggl(1_{\zeta<\tau_{D_I}} e^{-\int_0^\zeta 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_I,\phi_I\rangle})\,ds} \\
&\qquad\qquad \times \sum_{m=1}^n \int \frac{2\gamma_{x,\nu'}(\xi_\zeta)\gamma_{x,\nu-\nu'}(\xi_\zeta)}{\Gamma_{x,\nu}(\xi_\zeta)} Q_{\xi_\zeta}^{\nu'}\bigl(e^{-\langle \tilde{X}_I,\phi_I\rangle}, \Upsilon^{D'} \sim m\bigr) \\
&\qquad\qquad\qquad \times Q_{\xi_\zeta}^{\nu-\nu'}\bigl(e^{-\langle \tilde{X}_I,\phi_I\rangle}, \Upsilon^{D'} \sim n-m\bigr) K_x(\nu; d\nu')\Biggr) \qquad (50)
\end{aligned}
$$

$$
\begin{aligned}
&= \Pi_y^{4u}\biggl(\int_0^{\tau_{D_I}} dt\, \Gamma_{x,\nu}(\xi_t) 1_{t<\zeta} e^{-\int_0^t 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle \tilde{X}_I, \phi_I\rangle})\, ds} \\
&\qquad \times \sum_{m=1}^n \int K_x(\nu; d\nu') \frac{2\gamma_{x,\nu'}(\xi_t)\gamma_{x,\nu-\nu'}(\xi_t)}{\Gamma_{x,\nu}(\xi_t)} \\
&\qquad\qquad \times Q_{\xi_t}^{\nu'}(e^{-\langle \tilde{X}_I, \phi_I\rangle}, \Upsilon^{D'} \sim m) Q_{\xi_t}^{\nu-\nu'}(e^{-\langle \tilde{X}_I, \phi_I\rangle}, \Upsilon^{D'} \sim n-m)\biggr) \\
&= \Pi_y\biggl(\int_0^{\tau_{D_I}} dt\, e^{-\int_0^t 4u(\xi_s)\, ds} e^{-\int_0^t 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_I, \phi_I\rangle})\, ds} \\
&\qquad \times \sum_{m=1}^n \int K_x(\nu; d\nu') 2\gamma_{x,\nu'}(\xi_t)\gamma_{x,\nu-\nu'}(\xi_t) \\
&\qquad\qquad \times Q_{\xi_t}^{\nu'}(e^{-\langle \tilde{X}_I, \phi_I\rangle}, \Upsilon^{D'} \sim m) Q_{\xi_t}^{\nu-\nu'}(e^{-\langle \tilde{X}_I, \phi_I\rangle}, \Upsilon^{D'} \sim n-m)\biggr) \\
&= \sum_{m=1}^n \int \Pi_y\biggl(\int_0^{\tau_{D'}} 2\mathcal{N}_t^{I,D_I}(\phi_I^u) \cdot \gamma_{x,\nu'}(\xi_t) Q_{\xi_t}^{\nu'}(e^{-\langle \tilde{X}_I, \phi_I\rangle}, \Upsilon^{D'} \sim m) \\
&\qquad\qquad \times \gamma_{x,\nu-\nu'}(\xi_t) Q_{\xi_t}^{\nu-\nu'}(e^{-\langle \tilde{X}_I, \phi_I\rangle}, \Upsilon^{D'} \sim n-m)\, dt\biggr) \\
&\quad \times K_x(\nu; d\nu').
\end{aligned}
$$

The first equation is true, because at the lifetime ζ of the first particle Υ branches into two new branching diffusions with joint conditional law $Q_{\xi_\zeta}^{\nu-\nu'} \times Q_{\xi_\zeta}^{\nu-\nu'}$ given $(\xi_\zeta)_{t\le\zeta}$, and ν_1 whose conditional distribution given $(\xi_\zeta)_{t\le\zeta}$ has density equal to (45). The second equation follows from a well-known fact on h-transforms when h is a potential; see, for example, Salisbury and Verzani (1999), formula 2.2. The third equation follows from the definition of a killed diffusion.

We compute B as follows:

$$
\begin{aligned}
B &= \gamma_{x,\nu}(y)\Pi_y^{4u,\gamma_{x,\nu}}\bigl(1_{\zeta>\tau_{D_I}} e^{-\int_0^{\tau_{D_I}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_I, \phi_I\rangle})\, ds} Q_{\xi_{\tau_{D_I}}}^{I,\nu,n}(\phi_I)\bigr) \\
&= \Pi_y^{4u}\bigl(\gamma_{x,\nu}(\xi_{\tau_{D_I}}) 1_{\zeta>\tau_{D_I}} e^{-\int_0^{\tau_{D_I}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_I, \phi_I\rangle})\, ds} Q_{\xi_{\tau_{D_I}}}^{I,\nu,n}(\phi_I)\bigr) \\
&= \Pi_y\bigl(e^{-\int_0^{\tau_{D_I}} 4u(\xi_s)\, ds} \gamma_{x,\nu}(\xi_{\tau_{D_I}}) e^{-\int_0^{\tau_{D_I}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_I, \phi_I\rangle})\, ds} Q_{\xi_{\tau_{D_I}}}^{I,\nu,n}(\phi_I)\bigr) \\
&= \Pi_y\bigl(\gamma_\nu(\xi_{\tau_{D_I}}) Q_{\xi_{\tau_{D_I}}}^{I,\nu,1}(\phi_I) \mathcal{N}_{\tau_{D_I}}^{I,D_I}(\phi_I^u)\bigr)
\end{aligned}
\tag{51}
$$

by first applying the strong Markov property of Υ at the first exit of D_I of its first branch, and then again using the definition of $\gamma_{x,\nu}$ transform and killed diffusion.

We have shown previously that for all I, $N_y^{I,\nu,1}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,1}(\phi_I)$, $y \in \bar{D}_I$. Let us assume for all I, $N_y^{I,\nu,m}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,m}(\phi_I)$, $y \in \bar{D}_I$, for all $m \le n-1$. If $\mathrm{card}(I) = 1$, comparing equations (50) and (51) with (42) and using the fact that $N_y^{I,\nu,n}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,n}(\phi_I)$ on the boundary of $D_I = D'$, we get that $N_y^{I,\nu,n}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,n}(\phi_I)$ for all $y \in \bar{D}_I$. If now in addition we assume $N_y^{I,\nu,n}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,n}(\phi_I)$ holds for $y \in \bar{D}_I$ when $\mathrm{card}(I) = k-1$, our induction hypothesis and equations (48), (40) combined with equations (50), (51) and (42) imply that $N_y^{I,\nu,1}(\phi_I) = \gamma_{x,\nu}(y) Q_y^{I,\nu,1}(\phi_I)$ for $y \in \bar{D}_I$ when $\mathrm{card}(I) = k$ as well. Hence equation (46) holds for all I and $y \in \bar{D}_I$, and therefore the proof is complete. □

Above we described the conditional law $\mathbb{N}_y^\nu$. Now we move to an arbitrary initial measure $\mu \in \mathcal{M}_D^c$ such that $H_x^\nu(\mu) < \infty$, and so need to handle multiple lines of descent starting from time 0. In other words, we are going to describe the distribution of $(X_{D'})_{\mu \in \mathcal{M}_{D'}, D' \Subset D}$, P_μ^ν. Let $\tilde{P}_\mu^\nu$ be a probability defined on an axillary probability space $\tilde{\Omega}$ under which a random cluster of points $(x_i, \nu_i)_{i=1}^n$ in $D \times V^*$ is generated according to a distribution proportional to

$$\frac{1}{n!} \bar{K}_{x,n}(\nu; d\nu_1, \dots, d\nu_n) \gamma_{x,\nu_1}(x_1) \cdots \gamma_{x,\nu_n}(x_n) \mu(dx_1) \cdots \mu(dx_n). \tag{52}$$

Note that this makes sense since

$$\begin{aligned}
&\sum_{n=1}^\infty \frac{1}{n!} \int_{x_1,\dots,x_n} \mu(dx_1) \cdots \mu(dx_n) \\
&\qquad \times \int_{\nu_1,\dots,\nu_n} \bar{K}_{x,n}(\nu; d\nu_1, \dots, d\nu_n) \gamma_{x,\nu_1}(x_1) \cdots \gamma_{x,\nu_n}(x_n) \\
&\qquad = H_x^\nu(\mu) < \infty.
\end{aligned}$$

Once the random cluster is generated, corresponding to each point (y_i, ν_i) in the cluster, a measure valued process X^i begins to evolve following a $Q_{y_i}^{\nu_i}$ law independent of everything else. This is consistent with our construction of the law $Q_{y_i}^{\nu_i}$ since almost surely, each (y_i, ν_i) will satisfy $\gamma_{x,\nu_i}(y_i) < \infty$ and $\nu_i \in \mathcal{N}_0^c$. In addition, independent from all this, another measure valued process $\tilde{X}^0$ evolves following SBM law with spatial motion killed at rate u and initial measure μ. Let $\tilde{X} = \sum_{i=1}^n \tilde{X}^i + X^0$.

THEOREM 12. *Assume* (27). *Let* $H_x^\nu(\mu) < \infty$. *Then* P_μ^ν*-law of*

$$(X_{D'})_{\mu \in \mathcal{M}_{D'}, D' \Subset D}$$

is the same as $\tilde{P}_\mu^\nu$*-law of* $(\tilde{X}_{D'})_{\mu \in \mathcal{M}_{D'}, D' \Subset D}$.

PROOF. Let $P_\mu^{I,\nu}$ and $\tilde{P}_\mu^{I,\nu}$ denote the transition operators of X and $\tilde{X}$. That is,

$$P_\mu^{I,\nu}(\phi_I) = P_\mu\big(e^{-\langle X_I,\phi_I\rangle}\big),$$

where $I = (D_1, \dots, D_k)$, $X_I = (X_{D_1}, \dots, X_{D_k})$ and $\phi_I = (\phi_1, \dots, \phi_k)$. $\tilde{P}_\mu^{I,\nu}(\phi_I)$ is defined similarly.

Let μ be compactly supported in $U_1 \Subset U_2 \Subset \cdots \Subset U_i \cdots$ s.t. $D = \bigcup_{i=1}^\infty U_i$. It will suffice to show that $P_\mu^{I,\nu}(\phi_I) = \tilde{P}_\mu^{I,\nu}(\phi_I)$ for $I = \{D_1, \dots, D_k\}$ where $D_j \in D_k = U_i$ for a fixed i.

Recall $P_\mu^{I,\nu} = H^\nu(\mu)^{-1} P_\mu(e^{-\langle X_I,\phi_I\rangle} H^\nu(X_{D_k}))$. Note

$$\begin{aligned}
&P_\mu\big(e^{-\langle X_I,\phi_I\rangle} H^\nu(X_{D'})\big)\\
&\quad = P_\mu\left(e^{-\langle X_I,\phi_I\rangle}\sum_{n=1}^\infty \int \frac{e^{-\langle X_{D_k},u\rangle}}{n!}\langle X_{D_k},\gamma_{x,\nu_1}\rangle\cdots\langle X_{D_k},\gamma_{x,\nu_n}\rangle\right)\\
&\qquad \times \bar{K}_{x,n}(\nu; d\nu_1,\dots,d\nu_n)\\
&\quad = \sum_{n=1}^\infty \frac{1}{n!}\int P_\mu\big(e^{-\langle X_{D_k},\phi_I^u\rangle}\langle X_{D_k},\gamma_{x,\nu_1}\rangle\cdots\langle X_{D_k},\gamma_{x,\nu_n}\rangle\big)\\
&\qquad\qquad \times \bar{K}_{x,n}(\nu; d\nu_1,\dots,d\nu_n).
\end{aligned}$$

Using the extended moment formula with

$$l^{I,u}(x) = 4\mathbb{N}_x\big(1 - \exp -(\langle X_{D_1},\phi_1\rangle + \cdots + \langle X_{D_k},\phi_k + u\rangle)\big),$$

we rewrite the right-hand side as

$$\begin{aligned}
&= \sum_{n=1}^\infty \frac{1}{n!}\int e^{-\langle \mu, l^{I,u}\rangle}\sum_{\pi(n)}\prod_{i=1}^r\left\langle \mu, \mathbb{N}_{(\cdot)}\left(e^{-\langle X_I,\phi_I^u\rangle}\prod_{j\in C_i}\langle X_{D_k},\gamma_{x,\nu_j}\rangle\right)\right\rangle\\
&\qquad \times \bar{K}_{x,n}(\nu; d\nu_1,\dots,d\nu_n).
\end{aligned}$$

By Lemma 6 we expand the above expression as

$$\begin{aligned}
&= e^{-\langle \mu, l^{I,u}\rangle}\sum_{n=1}^\infty \frac{1}{n!}\sum_{\pi(n)}\int \bar{K}_{x,r}(\nu; d\nu_1,\dots,d\nu_r)\\
&\quad \times \prod_{i=1}^r\left\langle \mu, \mathbb{N}_{(\cdot)}\left(e^{-\langle X_I,\phi_I^u\rangle}\left[\int\prod_{j\in C_i}\langle X_{D_k},\gamma_{x,\nu_{C_i}}\rangle \bar{K}_{x,n_i}(\tilde{\nu}_i; d\nu_{C_i})\right]\right)\right\rangle\\
&= e^{-\langle \mu, l^{I,u}\rangle}\sum_{n=1}^\infty \frac{1}{n!}\sum_{r=1}^n\int \bar{K}_{x,r}(\nu; d\nu_1,\dots,d\nu_r)\sum_{n_1,\dots,n_r}\frac{n!}{n_1!\cdots n_r!r!}\\
&\quad \times \prod_{i=1}^r\left\langle \mu, \mathbb{N}_{(\cdot)}\left(e^{-\langle X_{D'},u+\phi\rangle}\left[\int\prod_{j\in C_i}\langle X_{D'},\gamma_{x,\nu_j}\rangle \bar{K}_{x,n_i}(\tilde{\nu}_i; d\nu_{C_i})\right]\right)\right\rangle
\end{aligned}$$

$$= e^{-\langle \mu, l^{I,u} \rangle} \sum_{r=1}^{\infty} \frac{1}{r!} \int \bar{K}_{x,r}(\nu; d\nu_1, \ldots, d\nu_r) \prod_{i=1}^{r} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} \langle \mu, \gamma_{x,\tilde{\nu}_i} \mathbb{N}_{(\cdot)}^{\tilde{\nu}_i}(e^{-\langle X_I, \phi_I \rangle}) \rangle$$

$$= e^{-\langle \mu, l^{I,u} \rangle} \sum_{r=1}^{\infty} \frac{1}{r!} \int \bar{K}_{x,r}(\nu; d\nu_1, \ldots, d\nu_r) \prod_{i=1}^{r} \langle \mu, \gamma_{x,\tilde{\nu}_i} Q_{(\cdot)}^{\tilde{\nu}_i}(e^{-\langle X_I, \phi_I \rangle}) \rangle.$$

Note that $e^{-\langle \mu, l^{I,u} \rangle}$ is the transition operator of a SBM whose spatial motion killed at rate u, and the rest of the expression is the transition operator of $\Sigma_{i=1}^{n} \tilde{X}_i$, where each $\tilde{X}_i$ evolves according to $Q_{y_i}^{\tilde{\nu}_i}$, where the random cluster of points $(y_i, \nu_i)_{i=1}^{n}$ is selected according to the density (52). Hence the proof is complete. □

Note that the extended X-harmonic functions H_x^{ν} were defined by Dynkin in Dynkin (2006b), and we have followed this approach throughout. The results of this section should be viewed as an attempt to clarify the structure of these extended X-harmonic functions, as well as the structure of the conditioned superprocesses that are obtained from them.

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