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ROOTS OF RANDOM POLYNOMIALS WHOSE COEFFICIENTS HAVE LOGARITHMIC TAILS

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It has been shown by Ibragimov and Zaporozhets [In *Prokhorov and Contemporary Probability Theory* (2013) Springer] that the complex roots of a random polynomial $G_n(z) = \sum_{k=0}^n \xi_k z^k$ with i.i.d. coefficients $\xi_0, \dots, \xi_n$ concentrate a.s. near the unit circle as $n \to \infty$ if and only if $\mathbb{E}\log_+|\xi_0| < \infty$. We study the transition from concentration to deconcentration of roots by considering coefficients with tails behaving like $L(\log|t|)(\log|t|)^{-\alpha}$ as $t \to \infty$, where $\alpha \geq 0$, and L is a slowly varying function. Under this assumption, the structure of complex and real roots of G_n is described in terms of the least concave majorant of the Poisson point process on $[0, 1] \times (0, \infty)$ with intensity $\alpha v^{-(\alpha+1)}\, du\, dv$.

1. Introduction and statement of results.

1.1. *Introduction.* Let $\xi_0, \xi_1, \dots$ be i.i.d. nondegenerate random variables with values in $\mathbb{C}$. Let $\mathcal{Z}_n$ be the collection of complex roots (counted according to their multiplicities) of the random polynomial

$$G_n(z) = \sum_{k=0}^n \xi_k z^k. \tag{1}$$

For $0 \leq a \leq b$ denote by $R_n(a, b)$ the number of roots of G_n in the ring $\{z \in \mathbb{C} : a \leq |z| \leq b\}$. Improving on a result of Šparo and Šur [18], Ibragimov and Zaporozhets [10] show that

$$\frac{1}{n} R_n(1 - \varepsilon, 1 + \varepsilon) \xrightarrow[n\to\infty]{\text{a.s.}} 1 \tag{2}$$

for every $\varepsilon \in (0, 1)$, if and only if

$$\mathbb{E}\log_+|\xi_0| < \infty. \tag{3}$$

Here, $\log_+ x = \max(\log x, 0)$. Without any assumptions on the distribution of ξ_0, Ibragimov and Zaporozhets [10] also prove that for every α, β such that $0 \leq \alpha < \beta \leq 2\pi$,

$$\frac{1}{n} \sum_{z\in\mathcal{Z}_n} \mathbb{1}_{\{\alpha \leq \arg z \leq \beta\}} \xrightarrow[n\to\infty]{\text{a.s.}} \frac{\beta - \alpha}{2\pi}. \tag{4}$$

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Thus, under a very mild moment condition, the complex roots of G_n concentrate near the unit circle uniformly by the argument as $n \to \infty$.

Imposing additional conditions on the distribution of ξ_0 it is possible to obtain more precise information about the asymptotic concentration of the roots near the unit circle. In the case when ξ_0 belongs to the domain of attraction of an α-stable law, $\alpha \in (0, 2]$, Ibragimov and Zeitouni [9] show that for every $t > 0$,

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}R_n\left(1 - \frac{t}{n}, 1 + \frac{t}{n}\right) = \frac{1 + e^{-\alpha t}}{1 - e^{-\alpha t}} - \frac{2}{\alpha t}. \tag{5}$$

This is a generalization of the result of Shepp and Vanderbei [16] who consider real-valued standard Gaussian coefficients.

On the other hand, if $\mathbb{E}\log_+|\xi_0| = \infty$ and thus there is no concentration near the unit circle, it is also possible to describe the asymptotic behavior of the roots when the tail of $|\xi_0|$ is extremely heavy. Götze and Zaporozhets [7] prove that if the distribution of $\log_+ \log_+|\xi_0|$ has a slowly varying tail, then the complex roots of G_n concentrate in probability on two circles centered at the origin whose radii tend to zero and infinity, respectively. See also [19, 20] for more results in the case of extremely heavy tails.

Up to now, the behavior of the roots has been unknown when the tail of ξ_0 is somewhere between the two cases described above. The aim of this paper is to consider a class of distributions which in some sense continuously links the above cases. We will consider coefficients with logarithmic power-law tails. More precisely, we make the following assumption: for some $\alpha \geq 0$,

$$\bar{F}(t) := \mathbb{P}\big[\log|\xi_0| > t\big] \qquad \text{is regularly varying at } +\infty \text{ with index } -\alpha. \tag{6}$$

This class of distributions includes distributions with both finite ($\alpha > 1$) and infinite ($\alpha < 1$) logarithmic moments. We will obtain a precise information on how the concentration of the roots near the unit circle becomes destroyed as α approaches 1 from above and how the roots behave when there is no concentration ($\alpha < 1$).

The case $\alpha = +\infty$ corresponds formally to the light or power-law tails studied in [9, 16]. The roots are concentrated near the unit circle and, apart from this, no global organization is apparent. We will prove that as α becomes finite, the distribution of roots becomes highly organized; see Figure 1. The roots "freeze" on a random set of circles centered at the origin. Both the radii of the circles and the distribution of the roots among the circles are random; however, the distribution of the roots on each circle is uniform by the argument. As long as α stays above 1, the logarithmic moment is finite, and the circles approach the unit circle at rate $n^{1/\alpha-1}$ (ignoring a slowly varying term), in full agreement with the result of [10]. Note also that for α close to $+\infty$, this rate is close to the rate $1/n$ appearing in (5). As α becomes equal to 1, we have a transition from finite to infinite logarithmic moment. We will show that if $\bar{F}(t) \sim c/t$ as $t \to +\infty$, then the empirical measure formed by the roots of G_n converges weakly (without normalization) to a random probability measure concentrated on an infinite number of circles with random

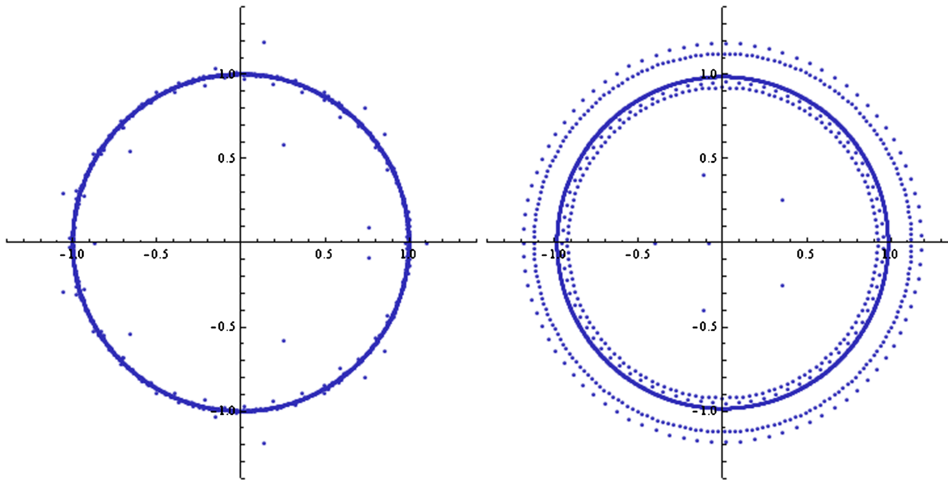

FIG. 1. *Roots of a random polynomial of degree* $n = 2000$ *whose* (*real*) *coefficients are* (*left*) *standard normal*, (*right*) *such that* $\mathbb{P}[\log \xi_0 > t] = 1/t^2$ *for* $t \geq 1$.

radii. For the first time, the roots are not concentrated near the unit circle. As α becomes smaller than 1, the circles divide into two groups approaching 0 and ∞ at the rates $\pm n^{1/\alpha - 1}$, on the logarithmic scale. The number of circles, which was infinite for $\alpha \geq 1$, becomes finite for $\alpha < 1$ and decreases to 2 as $\alpha \to 0$. At $\alpha = 0$ the roots freeze on just 2 circles located very close to 0 and ∞, in accordance with Götze and Zaporozhets [7], whose results we will strengthen. At $\alpha = 0$, the empirical measure formed by the roots becomes almost deterministic: the only parameter which remains random after taking the limit $n \to \infty$ is the proportion of the roots close to 0 (or to ∞), which is uniform on [0, 1].

An interesting phenomenon we will encounter is the appearance of the long-range dependence between the roots under condition (6). Consider a random polynomial G_n of high degree, and suppose that we know that it has a root at some point $z_0 \in \mathbb{C}$. In the case of coefficients from the domain of attraction of a stable law, this information has almost no influence on the other roots of G_n, except for the roots located in an infinitesimal neighborhood of z_0. However, for coefficients with logarithmic power-law tails, the knowledge about the existence of a root at z_0 implies that there exists (with high probability) a circle of roots containing z_0. Moreover, the radii of the other circles of roots are influenced by the existence of the root at z_0. We observe a long-range dependence between the roots: the conditional distribution of roots, given that there is a root at z_0, differs, even on the global scale, from the unconditional distribution of roots.

If the random variables ξ_i are real-valued, we will also analyze the real roots of G_n. For a particular family of distributions satisfying (6) with $\alpha > 1$, Shepp and Farahmand [15] show that the expected number of real roots of G_n is asymptotically $c(\alpha) \log n$ with $c(\alpha) = \frac{2\alpha - 2}{2\alpha - 1}$. As α decreases from $+\infty$ to 1 the function

$c(\alpha)$ decreases from 1 to 0. We will complement this result by showing that for $\alpha \in (0, 1)$, the number of real roots of G_n has two subsequential distributional limits as $n \to \infty$ along the subsequence of even/odd integers. This means that for $\alpha \in (0, 1)$ the polynomial G_n has, roughly speaking, $O(1)$ real roots. Finally, we will prove that for $\alpha = 0$, the number of real roots of G_n can take asymptotically only the values $0, \dots, 4$ and compute the probabilities of these values.

1.2. *Complex roots.* Given a complex number $z = |z|e^{i \arg z}$ and $a \in \mathbb{R}$, we write

$$z^{\langle a \rangle} = |z|^a e^{i \arg z}.$$

The next theorem describes the structure of complex roots of G_n. Let $\delta(z)$ be the unit point mass at z. Denote by $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. We need normalizing sequences a_n, b_n such that

$$\bar{F}(a_n) \sim \frac{1}{n} \qquad \text{as } n \to \infty, \qquad b_n = \frac{n}{a_n}. \tag{7}$$

THEOREM 1.1. *If the tail condition* (6) *is satisfied with some* $\alpha > 0$, *then we have the following weak convergence of random probability measures on* $\bar{\mathbb{C}}$:

$$\frac{1}{n} \sum_{z \in \mathcal{Z}_n} \delta\big(z^{\langle b_n \rangle}\big) \xrightarrow[n\to\infty]{w} \Pi_\alpha.$$

The limiting random probability measure Π_α *is a.s. a convex combination of at most countably many uniform measures concentrated on circles centered at the origin.*

REMARK 1.2 (On convergence of random measures). Let E be a locally compact metric space. Denote by $\mathbb{M}(E)$ the space of locally finite Borel measures on E. Endowed with the topology of vague convergence, $\mathbb{M}(E)$ becomes a Polish space; see [13], Section 3.4. A *random measure* on E is a random element with values in $\mathbb{M}(E)$. A sequence μ_n of random measures *converges weakly* to a random measure μ if $\lim_{n\to\infty} \mathbb{E}F(\mu_n) = \mathbb{E}F(\mu)$ for every continuous, bounded function $F : \mathbb{M}(E) \to \mathbb{R}$. Equivalently, $\int_E f\, d\mu_n$ converges in distribution to $\int_E f\, d\mu$ for every compactly supported continuous function $f : E \to \mathbb{R}$; see [13], Section 3.5.

For $\alpha \geq 1$ the logarithmic moment condition (3) is satisfied, which by [10] means that the roots should concentrate near the unit circle. In the next corollary we compute the rate of convergence of the roots to the unit circle.

COROLLARY 1.3. *Let* $\alpha \geq 1$. *As* $n \to \infty$, *the random probability measure*

$$\frac{1}{n} \sum_{z \in \mathcal{Z}_n} \delta\big(b_n(|z| - 1)\big)$$

converges weakly to a random, a.s. purely atomic probability measure on $\mathbb{R}$.

In the case $\bar{F}(t) \sim c/t$ as $t \to +\infty$, where $c > 0$, the logarithmic moment condition (3) just fails. We have no concentration of the roots near the unit circle for the first time. In this case, Theorem 1.1 simplifies as follows.

COROLLARY 1.4. *Suppose that $\bar{F}(t) \sim c/t$ as $t \to +\infty$. Then, the empirical measure $\frac{1}{n}\sum_{z \in \mathcal{Z}_n} \delta(z)$ converges weakly to some nontrivial limiting random probability measure on $\mathbb{C}$.*

We proceed to the description of the random probability measure Π_α. Let $\rho = \sum_{k=1}^{\infty} \delta(U_k, V_k)$ be a Poisson point process on $[0,1] \times (0, \infty)$ with intensity measure $\alpha v^{-(\alpha+1)}\,du\,dv$. Equivalently, U_k, $k \in \mathbb{N}$, are i.i.d. random variables with a uniform distribution on $[0,1]$ and, independently, $V_k = W_k^{-1/\alpha}$, where $W_1, W_2, \ldots$ are the arrival times of a homogeneous Poisson point process on $(0,\infty)$ with intensity 1. Of major importance for the sequel is the *least concave majorant* (called simply *majorant*) of ρ (see Figure 2) which is a function $\mathfrak{C}_\rho : [0,1] \to [0,\infty)$ defined by

$$\mathfrak{C}_\rho(t) := \inf_f f(t), \qquad t \in [0,1],$$

where the infimum is taken over the set of all concave functions $f : [0,1] \to [0,\infty)$ satisfying $f(U_k) \geq V_k$ for all $k \in \mathbb{N}$. From a constructive viewpoint, the least concave majorant $\mathfrak{C}_\rho$ may be defined as follows. Let (X_0, Y_0) be the a.s. unique atom of ρ having a maximal second coordinate Y_0 among all atoms of ρ. Consider a horizontal line passing through (X_0, Y_0). Rotate this line around (X_0, Y_0) in a *clock-wise* direction until it hits some atom of ρ, denoted by (X_1, Y_1), other than (X_0, Y_0). Continue to rotate the line in the clock-wise direction, this time around (X_1, Y_1), until it hits some atom of ρ, denoted by (X_2, Y_2), other than (X_1, Y_1). Continue to rotate the line around (X_2, Y_2), and so on. The procedure

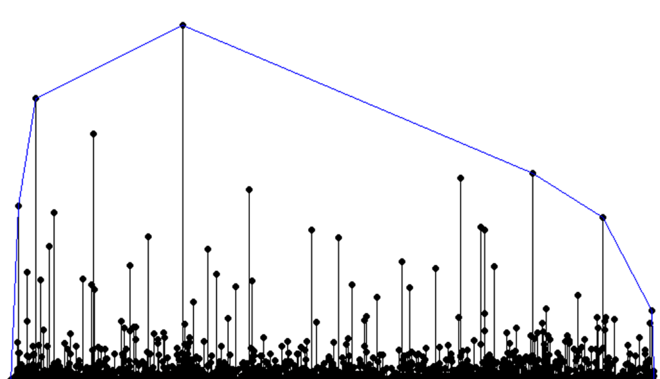

FIG. 2. *The least concave majorant of the Poisson point process ρ on $[0,1] \times (0,\infty)$ with intensity $\alpha v^{-(\alpha+1)}\,du\,dv$, where $\alpha = 2$.*

is terminated if at some time the line hits the point $(1,0)$. [As we will see later, this happens a.s. if and only if $\alpha \in (0,1)$.] Otherwise, the procedure is repeated indefinitely. Analogously, we can start with a horizontal line passing through (X_0, Y_0) and rotate it in an *anti-clockwise* direction obtaining a sequence of points $(X_{-1}, Y_{-1}), (X_{-2}, Y_{-2}), \dots$. The sequence may eventually terminate at $(0,0)$. [We will see that this happens a.s. if and only if $\alpha \in (0,1)$.] Now, join any point (X_k, Y_k) to the next point (X_{k+1}, Y_{k+1}) by a line segment. The polygonal path constructed in this way is the graph of the majorant $\mathfrak{C}_\rho$. The points (X_k, Y_k) are called the *vertices* of the majorant, the intervals $[X_k, X_{k+1}]$ are called the *linearity intervals* of the majorant. The least concave majorant $\mathfrak{C}_\rho$ is thus a piecewise linear function with at most countably many linearity intervals. We write $\mathfrak{C}_\rho$ in the form

$$\mathfrak{C}_\rho(t) = S_k - R_k t, \qquad t \in [X_k, X_{k+1}]. \tag{8}$$

The limiting random probability measure Π_α in Theorem 1.1 can be constructed as follows. For $r > 0$ let Λ_r be the length measure (normalized to have total mass 1) on the circle $\{z \in \mathbb{C} : |z| = r\}$. Then

$$\Pi_\alpha = \sum_k (X_{k+1} - X_k) \Lambda_{\exp(R_k)},$$

where the (finite or infinite) sum is taken over all linearity intervals $[X_k, X_{k+1}]$ of the majorant $\mathfrak{C}_\rho$. Thus Theorem 1.1 states that the roots of G_n asymptotically concentrate on random circles which correspond to the linearity intervals of the majorant. The radii of these random circles are $\exp(R_k)$, where the R_k's are the negatives of the slopes of the majorant. The proportion of roots on any circle is the length of the corresponding linearity interval.

Our next result describes the distribution of the complex roots of G_n in the case $\alpha = 0$. We assume that

$$\bar{F}(t) := \mathbb{P}[\log|\xi_0| > t] \qquad \text{is slowly varying at } +\infty. \tag{9}$$

We will show that under (9), with probability close to 1, the complex roots of G_n are located on just 2 concentric circles, one of them with a radius close to 0 and the other one with a radius close to ∞. A weaker result was obtained by Götze and Zaporozhets [7] under a more restrictive assumption on the tails. Let τ_n be the index of the maximal (in the sense of absolute value) coefficient of G_n, that is, $\tau_n \in \{0, \dots, n\}$ is such that $|\xi_{\tau_n}| = \max_{k=0,\dots,n} |\xi_k|$. Denote by $w_{1n}, \dots, w_{\tau_n n}$ the roots of the equation $\xi_{\tau_n} z^{\tau_n} + \xi_0 = 0$ and by $w_{(\tau_n+1)n}, \dots, w_{nn}$ the roots of the equation $\xi_n z^{n-\tau_n} + \xi_{\tau_n} = 0$.

THEOREM 1.5. *Suppose that* (9) *is satisfied and* $\xi_0 \neq 0$ *a.s. Fix some* $A > 0$. *Then the probability that the following three statements hold simultaneously goes to* 1 *as* $n \to \infty$:

(1) τ_n *is uniquely defined*;

(2) *it is possible to renumber the roots* $z_{1n}, \dots, z_{nn}$ *of* G_n *so that*

$$|z_{kn} - w_{kn}| < e^{-n^A} |w_{kn}|, \qquad 1 \le k \le n;$$

(3) *we have* $|w_{kn}| < e^{-n^A}$ *for* $1 \le k \le \tau_n$ *and* $|w_{kn}| > e^{n^A}$ *for* $\tau_n < k \le n$.

COROLLARY 1.6. *Under* (9), *the empirical measure* $\frac{1}{n}\sum_{z \in \mathcal{Z}_n} \delta(z)$ *converges weakly, as a random probability measure on the Riemann sphere* $\bar{\mathbb{C}}$, *to* $U\delta(0) + (1-U)\delta(\infty)$, *where* U *is a random variable with a uniform distribution on* $[0, 1]$.

1.3. *Properties of the majorant.* In this section we study some of the properties of the least concave majorant $\mathfrak{C}_\rho$. Note that random convex hulls similar to $\mathfrak{C}_\rho$ appeared in the literature; see [12] and the references therein. The next proposition will be used frequently.

PROPOSITION 1.7. *Let* L_α *be the number of linearity intervals of the majorant* $\mathfrak{C}_\rho$. *If* $\alpha \in (0, 1)$, *then* $L_\alpha < \infty$ *a.s. If* $\alpha \ge 1$, *then* $L_\alpha = \infty$ *a.s. Moreover, in this case any neighborhood of* 0 (*as well as any neighborhood of* 1) *contains infinitely many linearity intervals of* $\mathfrak{C}_\rho$ *a.s., and we have* $\lim_{k\to-\infty} R_k = -\infty$ *and* $\lim_{k\to+\infty} R_k = +\infty$ *a.s.*

PROOF. Take any $\varepsilon > 0$ and consider the set D_ε of all pairs $(x, y) \in [0, 1] \times (0, \infty)$ such that $y > \varepsilon x$. Integrating the intensity of ρ over D_ε we see that $\rho(D_\varepsilon) = \infty$ a.s. if and only if $\alpha \ge 1$. If $\alpha \in (0, 1)$, we have only finitely many points above any line $y > \varepsilon x$ and hence, the majorant $\mathfrak{C}_\rho$ has a well-defined first segment starting at $(0, 0)$. On the other hand, if $\alpha \ge 1$, then no such first segment exists and consequently, we have infinitely many linearity intervals of ρ in any neighborhood of 0. By symmetry, the same is true for the point 1. □

The distribution of L_α in the case $\alpha \in (0, 1)$ seems difficult to characterize. In the next theorem, we compute the expectation of L_α in terms of the modular constant $C(\beta)$ introduced by Barnes [1] in his theory of the double Gamma function. Let $\psi(z) = \Gamma'(z)/\Gamma(z)$ be the logarithmic derivative of the Gamma function. Barnes [1] showed that the following limit exists for $\beta > 0$:

$$C(\beta) := \lim_{n\to\infty} \left\{ \sum_{m=1}^{n} \psi(m\beta) - \left(n + \frac{1}{2} - \frac{1}{2\beta}\right) \log(n\beta) + n \right\}. \tag{10}$$

The role of the constant $C(\beta)$ in the theory of the double Gamma function is similar to the role of the Euler–Mascheroni constant $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n} \frac{1}{k} - \log n)$ in the theory of the usual Gamma function.

THEOREM 1.8. *For $\alpha \in (0,1)$, $\alpha \neq 1/2$, we have*

$$\mathbb{E}L_\alpha = 2 + \frac{2-2\alpha}{2\alpha-1}\left(1 - 2C(1-\alpha) + \frac{\log(1-\alpha) - \alpha\gamma}{1-\alpha}\right). \tag{11}$$

For $\alpha = 1/2$ the result should be interpreted by continuity.

We will provide a representation of $\mathbb{E}L_\alpha$ as a definite integral in equation (73) below. Using this representation it is possible to compute the value of $\mathbb{E}L_\alpha$ in closed form for any rational α. Here are some examples:

α	0	1/4	1/3	1/2	2/3	3/4	1
$\mathbb{E}L_\alpha$	2	$(\frac{3}{2} - \frac{4}{3\sqrt{3}})\pi$	$\frac{4\pi}{3\sqrt{3}}$	$\frac{3}{2} + \frac{\pi^2}{8}$	$2 + \frac{2\pi}{3\sqrt{3}}$	$2 + \frac{\pi}{2}$	$+\infty$

The values at $\alpha = 0$ and $\alpha = 1$ should be understood as one-sided limits. As a corollary, we have $L_\alpha \to 2$ in distribution as $\alpha \downarrow 0$. Another way to see this is the following theorem.

THEOREM 1.9. *For $\alpha \in (0,1)$ we have $\mathbb{P}[L_\alpha = 2] = 1 - \alpha$.*

1.4. *Real roots*. Suppose now that the coefficients of the polynomial $G_n(z) = \sum_{k=0}^n \xi_k z^k$ are i.i.d. *real-valued* random variables. Denote by $\mathcal{R}_n$ the collection of real roots of G_n, and let $N_n = |\mathcal{R}_n|$ be the number of real roots. For a special family of distributions satisfying (6) with $\alpha > 1$ Shepp and Farahmand [15] showed that $\mathbb{E}N_n \sim \frac{2\alpha-2}{2\alpha-1}\log n$ as $n \to \infty$. In the next theorem we describe the positions of the real roots of G_n in the limit $n \to \infty$ for every $\alpha > 0$. Recall the notation $z^{\langle a \rangle} = |z|^a \operatorname{sgn}(z)$, where $z, a \in \mathbb{R}$. Define a point process Υ_n on $\mathbb{R}$ by

$$\Upsilon_n = \sum_{z \in \mathcal{R}_n} \delta(z^{\langle b_n \rangle}).$$

Recall that a random measure μ is called a point process if the random variable $\mu(K)$ is integer-valued for every compact set K; see [13], Section 3.1. In addition to (6) we assume that the following limit exists:

$$c := \lim_{t \to +\infty} \frac{\mathbb{P}[\xi_0 > t]}{\mathbb{P}[|\xi_0| > t]} \in [0,1]. \tag{12}$$

THEOREM 1.10. *Suppose that (6) and (12) hold with some $\alpha > 0$. Write $p = \mathbb{P}[\xi_0 > 0]$ and suppose that $\xi_0 \neq 0$ a.s.*

(1) *For $\alpha \geq 1$ the point process Υ_n converges weakly to some point process $\Upsilon_{\alpha,c}$ on $\mathbb{R} \setminus \{0\}$.*

(2) *For $\alpha \in (0,1)$ the point process Υ_{2n} (resp., Υ_{2n+1}) converges weakly to some point process $\Upsilon^+_{\alpha,c,p}$ (resp., $\Upsilon^-_{\alpha,c,p}$) on $[-\infty,+\infty]$ and on $\mathbb{R}$.*

The somewhat technical description of the point processes $\Upsilon_{\alpha,c}$, $\Upsilon^{\pm}_{\alpha,c,p}$ is postponed to Section 6.1. Recall that by Theorem 1.1 the complex roots of G_n are located asymptotically on a set of random circles. Each circle crosses the real line at 2 points. We will show that any of these points may or may not be a real root of G_n with some probabilities. For $\alpha \in (0,1)$ the point processes $\Upsilon^{\pm}_{\alpha,c,p}$ have a.s. finitely many atoms, whereas for $\alpha \geq 1$ the atoms of the point process $\Upsilon_{\alpha,c}$ accumulate a.s. at ± 0 and $\pm\infty$. (Of course, this is related to Proposition 1.7.) Since the map assigning to a finite counting measure on $[-\infty,\infty]$ its total mass is continuous (locally constant) in the weak topology, we obtain the following statement on the number of real roots of G_n.

COROLLARY 1.11. *Suppose that* (6) *and* (12) *hold with* $\alpha \in (0,1)$ *and let* $\xi_0 \neq 0$ *a.s. Then the sequence* $\{N_{2n}\}_{n\in\mathbb{N}}$ *(resp.,* $\{N_{2n+1}\}_{n\in\mathbb{N}}$*) converges in distribution to a random variable* $N^+_{\alpha,c,p}$ *(resp.,* $N^-_{\alpha,c,p}$*).*

REMARK 1.12. The expectations $\mathbb{E}N^+_{\alpha,c,p}$ and $\mathbb{E}N^-_{\alpha,c,p}$ can be computed using the representation of the point processes $\Upsilon^{\pm}_{\alpha,c,p}$ given in Section 6.1.

$$\mathbb{E}N^+_{\alpha,c,p} = \mathbb{E}N^-_{\alpha,c,p} = \bigl(2c(1-c)+\tfrac{1}{2}\bigr)(\mathbb{E}L_\alpha - 2) + 2(p+c-2pc) + 1.$$

For instance, if the distribution of ξ_0 is symmetric with respect to the origin, then both expectation are equal to $\mathbb{E}L_\alpha$. We conjecture that the convergence in Corollary 1.11 holds in the L^1-sense.

REMARK 1.13. The behavior of N_n in the case $\alpha = 1$ remains open. For $\alpha = 1$ the result of [15] turns formally into $\mathbb{E}N_n = o(\log n)$, whereas the fact that $\Upsilon_{1,c}$ has infinitely many atoms a.s. suggests that $\mathbb{E}N_n$ should be infinite. It is natural to conjecture that for $\alpha = 1$, we should have $\mathbb{E}N_n \sim K \log\log n$ for some $K > 0$.

Finally, we investigate the number of real roots of G_n in the case $\alpha = 0$.

THEOREM 1.14. *Suppose that* (9) *and* (12) *hold,* $\xi_0 \neq 0$ *a.s., and write* $p = \mathbb{P}[\xi_0 > 0]$. *Then, the sequence* $\{N_{2n}\}_{n\in\mathbb{N}}$ *converges weakly to a random variable* $N^+_{0,c,p}$ *and the sequence* $\{N_{2n+1}\}_{n\in\mathbb{N}}$ *converges weakly to a random variable* $N^-_{0,c,p}$ *such that*

$$\mathbb{P}[N^+_{0,c,p} = m] = \begin{cases} \frac{1}{2}\bigl(cp^2 + (1-c)(1-p)^2\bigr), & m = 0, \\ \frac{1}{2} + p(1-p), & m = 2, \\ \frac{1}{2}\bigl(c(1-p)^2 + (1-c)p^2\bigr), & m = 4; \end{cases} \tag{13}$$

$$\mathbb{P}[N^-_{0,c,p} = m] = \begin{cases} 1 - p - c + 2pc, & m = 1, \\ p + c - 2pc, & m = 3. \end{cases} \tag{14}$$

REMARK 1.15. If the distribution of ξ_0 is symmetric with respect to the origin, we obtain the following results: $N^+_{0,1/2,1/2}$ takes the values $0, 2, 4$ with probabilities $1/8, 3/4, 1/8$, and $N^-_{0,1/2,1/2}$ takes the values $1, 3$ with probabilities $1/2, 1/2$.

REMARK 1.16. For fixed $p \in [0, 1]$, both

$$\min_{c\in[0,1]} \mathbb{E}N^+_{0,c,p} \quad \text{and} \quad \min_{c\in[0,1]} \mathbb{E}N^-_{0,c,p}$$

are equal to $1 + 2\min(p, 1 - p)$. The same number appeared in [20] as the minimal expected number of real roots of a random polynomial.

1.5. *Emergence of the majorant.* The least concave majorant which we encountered above is reminiscent of the Newton polygons appearing when solving polynomial equations with non-Archimedian (e.g., p-adic) coefficients; see [11], Chapter IV. Of course, our random polynomial G_n has complex (Archimedian) coefficients. However, non-Archimedian effects will appear in the following way. Consider the sum $c_1e^{nx_1} + \cdots + c_de^{nx_d}$, where $x_i > 0$ and $c_i \in \mathbb{C}$. If n is large, then the most easy way such sum may become zero is if two terms, say $c_ke^{nx_k}$ and $c_le^{nx_l}$, cancel each other and the other terms are much smaller than these two. We will show that under (6) similar considerations apply to the polynomial $G_n(z) = \sum_{j=0}^{n} \xi_j z^j$ with high probability: $z \in \mathbb{C}$ is a root of G_n essentially only if two of the terms, $\xi_k z^k$ and $\xi_l z^l$, cancel each other, and all other terms are of smaller order. Geometrically, this means that the points $(k, \log|\xi_k|)$ and $(l, \log|\xi_l|)$ are neighboring vertices of the least concave majorant of the set $\{(j, \log|\xi_j|) : j = 0, \dots, n\}$. The nonzero roots of $\xi_k z^k + \xi_l z^l = 0$ form a regular polygon inscribed into the circle whose radius is the exponential of minus the slope of the line joining the points $(k, \log|\xi_k|)$ and $(l, \log|\xi_l|)$. Taking the union of such circles over all segments of the majorant we obtain essentially all the roots of G_n. To complete the argument, we need to find the limiting form of the majorant as $n \to \infty$. This is done using the following proposition which is known in the extreme-value theory; see [13], Corollary 4.19(ii).

PROPOSITION 1.17. *Let $\xi_0, \xi_1, \dots$ be i.i.d. random variables satisfying* (6). *Then the following convergence holds weakly on the space of locally finite counting measures on* $[0, 1] \times (0, \infty]$:

$$\rho_n := \sum_{k=0}^{n} \delta\left(\frac{k}{n}, \frac{\log|\xi_k|}{a_n}\right) \xrightarrow[n\to\infty]{w} \sum_{i=1}^{\infty} \delta(U_i, V_i) =: \rho.$$

Here, ρ is a Poisson point process on $[0, 1] \times (0, \infty)$ *with intensity* $\alpha v^{-(\alpha+1)}\, du\, dv$. *We agree that the points for which* $\log|\xi_k| \le 0$ *are not counted in* ρ_n.

The paper of Shepp and Farahmand [15] seems to be the only work where random polynomials with coefficients satisfying (6) have been considered. The method used there (characteristic functions) is very different from our approach based on majorants. Whether the results of [15] can be recovered (or strengthened) using our approach remains open. Let us also mention that the least concave majorant appeared in the theory of entire functions; see [17], page 28. For example, Hardy [8] showed that the zeros of the deterministic entire function $\sum_{k=0}^{\infty} z^{k^3}/(k^3)!$ have a circle structure similar to the structure of zeros of G_n under (6). Eigenvalues of random matrices with i.i.d. heavy-tailed entries have been studied in [5].

2. The main lemma. The next lemma is the key step in the proof. Let $g(z) = \sum_{j=0}^{n} a_j z^j$ be a (deterministic) polynomial with complex coefficients. Suppose that the points $(k, \log|a_k|)$ and $(l, \log|a_l|)$, where $0 \le k < l \le n$, are neighboring vertices on the least concave majorant of the set $\{(j, \log|a_j|) : j = 0, \dots, n\}$. That is to say, for some $s, r \in \mathbb{R}$, we have

$$\log|a_k| = s - kr, \qquad \log|a_l| = s - lr,$$
$$h := \min_{\substack{0 \le j \le n \\ j \ne k,l}} \bigl(s - jr - \log|a_j|\bigr) > 0. \tag{15}$$

Here, we have assumed that no three points of the majorant are on the same line. Note that h measures the gap between the line passing through the points $(k, \log|a_k|)$, $(l, \log|a_l|)$ and the points lying below this line.

LEMMA 2.1. *If $\delta > 0$ is such that $ne^{\delta n - h} < 1 - e^{-\delta}$, then in the ring $e^{r-\delta} < |z| < e^{r+\delta}$ there are exactly $l - k$ roots of g. Moreover, if ζ is such that $2ne^{2\delta n - h} < \zeta < \frac{\pi}{l-k}$, then the set*

$$\left\{ z \in \mathbb{C} : e^{r-\delta} < |z| < e^{r+\delta}, \left| \arg z - \frac{\varphi + 2\pi m}{l-k} \right| \le \zeta \right\}, \tag{16}$$

where $\varphi = \arg(-a_k/a_l)$, contains exactly one root of g for every $m = 1, \dots, l-k$.

Here, we agree to understand the distance between the arguments of complex numbers as the geodesic distance on the unit circle. Also, let the index j be always restricted to $0 \le j \le n$.

PROOF OF LEMMA 2.1. We will prove a stronger version of the lemma. Namely, we will show that the statement holds for the family of polynomials

$$g_t(z) = a_k z^k + a_l z^l + t \sum_{j \ne k,l} a_j z^j, \qquad 0 \le t \le 1.$$

Note that in particular, $g_0(z) = a_k z^k + a_l z^l$ and $g_1(z) = g(z)$. Let $z \in \mathbb{C}$ be such that $|z| = e^{r-\delta}$. It follows from (15) that

$$t\left|\sum_{j\neq k,l} a_j z^j\right| \le \sum_{j\neq k,l} e^{s-jr-h} e^{j(r-\delta)} < n e^{s-h}.$$

On the other hand, again by (15),

$$|a_k z^k + a_l z^l| \ge |a_k z^k| - |a_l z^l| = e^s e^{-\delta k}(1 - e^{-\delta(l-k)}) > e^{s-\delta n}(1 - e^{-\delta}).$$

Since $ne^{\delta n - h} < 1 - e^{-\delta}$ holds, everywhere on the circle $|z| = e^{r-\delta}$ we have

$$(17)\qquad |a_k z^k + a_l z^l| > t\left|\sum_{j\neq k,l} a_j z^j\right|.$$

Hence, by Rouché's theorem, the polynomial g_t has exactly k roots in the circle $|z| \le e^{r-\delta}$.

Let now $z \in \mathbb{C}$ be such that $|z| = e^{r+\delta}$. Then

$$(18)\qquad t\left|\sum_{j\neq k,l} a_j z^j\right| \le \sum_{j\neq k,l} e^{s-jr-h} e^{j(r+\delta)} < n e^{s-h+\delta n}.$$

On the other hand,

$$|a_k z^k + a_l z^l| \ge |a_l z^l| - |a_k z^k| = e^s e^{\delta l}(1 - e^{\delta(k-l)}) > e^s(1 - e^{-\delta}).$$

Therefore, inequality (17) also holds everywhere on the circle $|z| = e^{r+\delta}$. It follows from Rouché's theorem that the polynomial g_t has exactly l roots in the circle $|z| \le e^{r+\delta}$. Hence, the polynomial g_t has exactly $l-k$ roots in the ring $e^{r-\delta} \le |z| \le e^{r+\delta}$.

Let us now show that these $l-k$ roots are located approximately at the same positions as the nonzero roots of the equation $a_l z^l + a_k z^k = 0$. Let z_0 be some root of g_t satisfying $e^{r-\delta} \le |z_0| \le e^{r+\delta}$. Then, repeating the argument of (18), we obtain that

$$(19)\qquad |a_l z_0^l + a_k z_0^k| = \left|\sum_{j\neq k,l} a_j z_0^j\right| < n e^{s-h+\delta n}.$$

Recall that $\varphi = \arg(-a_k/a_l)$. The arguments of the nonzero roots of the equation $a_l z^l + a_k z^k = 0$ are given by $\frac{\varphi + 2\pi m}{l-k}$, where $m = 1, \dots, l-k$, and their moduli are equal to e^r. Let

$$\varsigma = \min_{m=1,\dots,l-k}\left|\arg z_0 - \frac{\varphi + 2\pi m}{l-k}\right|.$$

Note that $\varsigma \in [0, \frac{\pi}{l-k}]$ by definition. Then

$$|\arg(a_l z_0^l) - \arg(-a_k z_0^k)| = |\arg z_0^{l-k} - \varphi| = (l-k)\varsigma.$$

By the inequality $|z_1 - z_2| \geq 2|z_1| \sin(|\arg z_1 - \arg z_2|/2)$ valid for $|z_1| \leq |z_2|$ and the inequality $\sin x \geq \frac{2}{\pi} x$ valid for $x \in [0, \frac{\pi}{2}]$, we obtain

$$|a_l z_0^l + a_k z_0^k| \geq 2e^{s-\delta l} \sin\left(\frac{(l-k)\varsigma}{2}\right) \geq \frac{1}{2} e^{s-\delta n} \varsigma. \tag{20}$$

It follows from (19) and (20) that $\varsigma < 2ne^{2\delta n - h}$ and hence $\varsigma < \zeta$. Therefore, every root z_0 of g_t such that $e^{r-\delta} \leq |z_0| \leq e^{r+\delta}$ is contained in a set of the form (16) for some $m = 1, \dots, l-k$. To complete the proof, it remains to show that every set (16) contains exactly one root of g_t. Since $\zeta < \frac{\pi}{l-k}$, all these sets are disjoint. By the above, g_t does not vanish on their boundaries. It follows from this and the argument principle that the number of roots of g_t in any set (16) is continuous as a function of $t \in [0, 1]$ and hence, constant. Obviously, every set (16) contains exactly one root of g_0 and hence, exactly one root of g_t. □

3. Least concave majorants and weak convergence. Proposition 1.17 states the convergence of the point process ρ_n formed by the logarithms of the coefficients of the random polynomial G_n to the limiting Poisson process ρ. We will need to deduce from this the weak convergence of certain functionals of ρ_n to the same functionals of ρ. This will be done using the following well-known continuous mapping theorem; see [13], page 152, or [3], page 30.

PROPOSITION 3.1. *Let $F : M_1 \to M_2$ be a map between two metric spaces (M_1, d_1) and (M_2, d_2). Let X_n be a sequence of M_1-valued random variables converging weakly to some M_1-valued random variable X. If F satisfies*

$$\mathbb{P}[F \text{ is discontinuous at } X] = 0,$$

then $F(X_n)$ converges weakly to $F(X)$ on (M_2, d_2).

In order to apply Proposition 3.1 we need to prove the a.s. continuity of the functionals under consideration. This is the aim of the present section. First we introduce some notation. Let $\mathfrak{M}$ be the set of locally finite counting measures μ on $[0, 1] \times (0, \infty]$ such that $\mu([0, 1] \times \{\infty\}) = 0$. We endow $\mathfrak{M}$ with the topology of vague convergence. Every $\mu \in \mathfrak{M}$ can be written in the form $\mu = \sum_i \delta(u_i, v_i)$, where i ranges in some at most countable index set and $u_i \in [0, 1]$, $v_i \in (0, \infty)$. The number of atoms of μ in a set of the form $[0, 1] \times [\varepsilon, \infty)$ is finite for every $\varepsilon > 0$, but the atoms of μ may (and often will) have accumulation points in the set $[0, 1] \times \{0\}$.

The least concave majorant of $\mu \in \mathfrak{M}$ is a function $\mathfrak{C}_\mu : [0, 1] \to [0, \infty)$ defined by $\mathfrak{C}_\mu(t) = \inf_f f(t)$, where the infimum is taken over all concave functions $f : [0, 1] \to [0, \infty)$ such that $f(u_i) \geq v_i$ for all i. We write the piecewise linear function $\mathfrak{C}_\mu$ in the form

$$\mathfrak{C}_\mu(t) = s_k - r_k t, \qquad t \in [x_k, x_{k+1}], \tag{21}$$

where k ranges over a finite or infinite discrete subinterval of $\mathbb{Z}$. We set $y_k = \mathfrak{C}_\mu(x_k)$. The intervals $[x_k, x_{k+1}]$ (called the linearity intervals of the majorant) are always supposed to be chosen in such a way that the points (x_k, y_k) and (x_{k+1}, y_{k+1}) are atoms of μ, and there are no further atoms of μ on the segment joining these two points. Fix some small $\kappa \in (0, 1/2)$. Given a counting measure $\mu \in \mathfrak{M}$, we define the indices $q' = q'_\kappa(\mu)$ and $q'' = q''_\kappa(\mu)$ by the conditions $x_{q'} \le \kappa < x_{q'+1}$ and $x_{q''-1} < 1 - \kappa \le x_{q''}$.

Let $\mathfrak{M}_1$ be the set of all counting measures $\mu \in \mathfrak{M}$ with the following properties:

(1) both 0 and 1 are accumulation points for the linearity intervals of $\mathfrak{C}_\mu$;
(2) $\mu(L) \le 2$ for every line $L \subset \mathbb{R}^2$;
(3) no atom of μ has first coordinate κ or $1 - \kappa$.

Note that every $\mu \in \mathfrak{M}_1$ has only simple atoms. Denote by $\mathfrak{N}$ the space of finite measures on $\mathbb{R}$ endowed with the weak topology. Let $V_k(\mu)$ be the subset of $[0, 1] \times [0, \infty)$ consisting of $[0, 1] \times \{0\}$ together with all atoms of μ, except for (x_k, y_k) and (x_{k+1}, y_{k+1}).

LEMMA 3.2. *The following mappings are continuous on* $\mathfrak{M}_1$:

(1) $\Psi_1 : \mathfrak{M} \to \mathfrak{N}$ *defined by* $\Psi_1(\mu) = \sum_{k=q'}^{q''-1} (x_{k+1} - x_k)\delta(r_k)$;
(2) $H_1 : \mathfrak{M} \to \mathbb{R}$ *defined by* $H_1(\mu) = \min\{s_k - r_k u - v\}$, *where the minimum is over* $q' \le k < q''$ *and* $(u, v) \in V_k(\mu)$;
(3) $L_1 : \mathfrak{M} \to \mathbb{R}$ *defined by* $L_1(\mu) = \min_{q' \le k < q''}(x_{k+1} - x_k)$.

PROOF. Let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ be a sequence converging to $\mu \in \mathfrak{M}_1$ in the vague topology on $[0, 1] \times (0, \infty]$. Let $\varepsilon > 0$ be such that $2\varepsilon < \min_{q' \le k < q''}\{s_k, s_k - r_k\}$. Note that the minimum is strictly positive by the definition of $\mathfrak{M}_1$. Denote by (u_l, v_l), where $1 \le l \le m$, all atoms of μ (excluding those which are vertices of $\mathfrak{C}_\mu$) with the property that $v_l > \varepsilon$. Since $\mu_n \to \mu$ vaguely, we can find (see [13], Proposition 3.13) atoms of μ_n denoted by (x_{kn}, y_{kn}) (where $q' \le k \le q''$) and (u_{ln}, v_{ln}) (where $1 \le l \le m$) such that

$$\lim_{n\to\infty} (x_{kn}, y_{kn}) = (x_k, y_k), \qquad q' \le k \le q'', \tag{22}$$

$$\lim_{n\to\infty} (u_{ln}, v_{ln}) = (u_l, v_l), \qquad 1 \le l \le m. \tag{23}$$

Moreover, since the vague convergence was required to hold on $[0, 1] \times (0, \infty]$, there are no other atoms of μ_n having a second coordinate exceeding 2ε, provided that n is sufficiently large. It follows that as $n \to \infty$ and for all $q' \le k < q''$,

$$r_{kn} := -\frac{y_{(k+1)n} - y_{kn}}{x_{(k+1)n} - x_{kn}} \to r_k, \qquad s_{kn} := y_{kn} + r_{kn}x_{kn} \to s_k. \tag{24}$$

In particular, for sufficiently large n, all $q' \le k < q''$ and all $1 \le l \le m$,

$$s_{kn} - r_{kn}u_{ln} > v_{ln}, \qquad \inf_{u\in[0,1]}(s_{kn} - r_{kn}u) > 2\varepsilon.$$

It follows that for sufficiently large n, the segment joining the points (x_{kn}, y_{kn}) and $(x_{(k+1)n}, y_{(k+1)n})$ belongs to the majorant of μ_n for every $q' \le k < q''$. Also, $x_{q'n} < \kappa < x_{(q'+1)n}$ and $x_{(q''-1)n} < 1 - \kappa < x_{q''n}$.

By using (22), (23), (24) and letting $\varepsilon \downarrow 0$, we obtain that $H_1(\mu_n) \to H_1(\mu)$ and $L_1(\mu_n) \to L_1(\mu)$ as $n \to \infty$. This proves the continuity of H_1 and L_1 on $\mathfrak{M}_1$. To prove the continuity of Ψ_1, note that for every continuous, bounded function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} f\, d\Psi_1(\mu_n) = \sum_{k=q'}^{q''-1}(x_{(k+1)n} - x_{kn}) f(r_{kn}) \underset{n\to\infty}{\longrightarrow} \sum_{k=q'}^{q''-1}(x_{k+1} - x_k) f(r_k).$$

Thus, $\Psi_1(\mu_n) \to \Psi_1(\mu)$ weakly, which proves the continuity of Ψ_1. □

The next lemma will be needed to prove our main results for $\alpha \in (0, 1)$. Let $\mathfrak{M}_0$ be the set of all nonzero counting measures $\mu \in \mathfrak{M}$ with the following properties:

(1) the number of linearity intervals of $\mathfrak{C}_\mu$ is finite and $\mathfrak{C}_\mu(0) = \mathfrak{C}_\mu(1) = 0$;
(2) $\bar{\mu}(L) \le 2$ for every line $L \subset \mathbb{R}^2$, where $\bar{\mu} = \mu + \delta(0, 0) + \delta(1, 0)$;
(3) no atom of μ has first coordinate κ or $1 - \kappa$.

LEMMA 3.3. *The following mappings are continuous on* $\mathfrak{M}_0$:

(1) $\Psi_0 : \mathfrak{M} \to \mathfrak{N}$ *defined by* $\Psi_0(\mu) = \sum_k (x_{k+1} - x_k)\delta(r_k)$, *where the sum is over all linearity intervals* $[x_k, x_{k+1}]$ *of the majorant* $\mathfrak{C}_\mu$;

(2) $H_0 : \mathfrak{M} \to [0, \infty]$ *defined by* $H_0(\mu) = \min\{s_k - r_k u - v\}$, *where the minimum is over* $q' < k < q'' - 1$ *and* $(u, v) \in V_k(\mu)$;

(3) $L_0 : \mathfrak{M} \to [0, \infty]$ *defined by* $L_0(\mu) = \min_{q'<k<q''-1}(x_{k+1} - x_k)$.

REMARK 3.4. In fact, Ψ_0 is continuous on the whole of $\mathfrak{M}$, but we will not need this. The minimum over an empty set is $+\infty$.

PROOF OF LEMMA 3.3. Let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ be a sequence converging vaguely to $\mu \in \mathfrak{M}_0$. The majorant $\mathfrak{C}_\mu$ is a piecewise linear function whose graph is a broken line connecting the points denoted by (x_k, y_k), where $p' \le k \le p''$ and $(x_{p'}, y_{p'}) = (0, 0)$, $(x_{p''}, y_{p''}) = (1, 0)$. For $p' < k < p''$, the point (x_k, y_k) is an atom of μ. Denote by (u_l, v_l), where $1 \le l \le m$, all atoms of μ (excluding those which are vertices of the majorant) with the property that $v_l > \varepsilon$, where $\varepsilon > 0$ is a number such that $2\varepsilon < \min_{p'<k<p''-1}\{s_k, s_k - r_k\}$. Note that the minimum is taken over the set of linearity intervals of the majorant excluding the first and the last interval. If the majorant consists of just two segments, then the minimum is $+\infty$. The vague

convergence $\mu_n \to \mu$ implies (see [13], Proposition 3.13) that we can find atoms of μ_n denoted by (x_{kn}, y_{kn}) (where $p' < k < p''$) and (u_{ln}, v_{ln}) (where $1 \le l \le m$) such that

$$\lim_{n\to\infty}(x_{kn}, y_{kn}) = (x_k, y_k), \qquad p' < k < p'', \tag{25}$$

$$\lim_{n\to\infty}(u_{ln}, v_{ln}) = (u_l, v_l), \qquad 1 \le l \le m. \tag{26}$$

Moreover, if n is sufficiently large, then there are no other atoms of μ_n having a second coordinate exceeding 2ε. It follows that as $n \to \infty$ and for all $p' < k < p'' - 1$,

$$r_{kn} := -\frac{y_{(k+1)n} - y_{kn}}{x_{(k+1)n} - x_{kn}} \to r_k, \qquad s_{kn} := y_{kn} + r_{kn}x_{kn} \to s_k. \tag{27}$$

Note that by concavity $s_k - r_k u_l > v_l$ for all $p' < k < p'' - 1$ and $1 \le l \le m$. Thus, for sufficiently large n,

$$s_{kn} - r_{kn}u_{ln} > v_{ln}, \qquad \inf_{u\in[0,1]}(s_{kn} - r_{kn}u) > 2\varepsilon.$$

This means that for sufficiently large n the segment joining the points (x_{kn}, y_{kn}) and $(x_{(k+1)n}, y_{(k+1)n})$ belongs to the majorant of μ_n for every $p' < k < p'' - 1$. Also, $x_{q'n} < \kappa < x_{(q'+1)n}$ and $x_{(q''-1)n} < 1 - \kappa < x_{q''n}$.

From (25), (26), (27) we obtain that $H_0(\mu_n) \to H_0(\mu)$ and $L_0(\mu_n) \to L_0(\mu)$ as $n \to \infty$. This proves the continuity of H_0 and L_0 on $\mathfrak{M}_0$. To prove the continuity of Ψ_0 we need to show that for every continuous, bounded function $f : \mathbb{R} \to [0, \infty)$,

$$\lim_{n\to\infty}\int_{\mathbb{R}} f\, d\Psi_0(\mu_n) = \int_{\mathbb{R}} f\, d\Psi_0(\mu). \tag{28}$$

By (25) and (27), we have

$$\lim_{n\to\infty}\sum_{p'<k<p''-1}(x_{(k+1)n} - x_{kn})f(r_{kn}) = \sum_{p'<k<p''-1}(x_{k+1} - x_k)f(r_k). \tag{29}$$

However, we have to be more careful about approximating the first and the last segments of $\mathfrak{C}_\mu$. Denote by (x_{kn}, y_{kn}), where $k \le p' + 1$, the vertices of the majorant of μ_n (counted from left to right) with the property $x_{kn} \le x_{(p'+1)n}$. Note that the number of such vertices is, in general, arbitrary and may be infinite. Since the first segment of the majorant of μ joins $(0, 0)$ and $(x_{p'+1}, y_{p'+1})$, all points (u_{ln}, v_{ln}), where $1 \le l \le m$, are located below the line joining $(0, 0)$ and $(x_{(p'+1)n}, y_{(p'+1)n})$ for large n. Therefore, for large n there are no atoms of μ_n above the line joining $(0, 2\varepsilon)$ and $(x_{(p'+1)n}, y_{(p'+1)n})$. Hence,

$$\begin{aligned} r_{p'n} &:= -\frac{y_{(p'+1)n} - y_{p'n}}{x_{(p'+1)n} - x_{p'n}} \\ &\in \left[-\frac{y_{(p'+1)n} - 2\varepsilon}{x_{(p'+1)n}}, -\frac{y_{(p'+1)n}}{x_{(p'+1)n}}\right], \qquad x_{p'n} < 2\varepsilon\frac{y_{(p'+1)n}}{x_{(p'+1)n}}. \end{aligned}$$

It follows that $r_{p'n} \to r_{p'}$ as $n \to \infty$. The contribution of linearity intervals to the left of $x_{p'n}$ can be estimated as follows: for large n,

$$\sum_{k<p'} (x_{(k+1)n} - x_{kn}) f(r_{kn}) \le x_{p'n} \|f\|_\infty \le 4\varepsilon \frac{y_{p'+1}}{x_{p'+1}} \|f\|_\infty.$$

Since $\varepsilon > 0$ can be made as small as we like, we have

$$\lim_{n\to\infty} \sum_{k\le p'} (x_{(k+1)n} - x_{kn}) f(r_{kn}) = x_{p'+1} f(r_{p'}). \tag{30}$$

Similar arguments can be applied to the part of the majorant of μ_n located to the right of $(x_{(p''-1)n}, y_{(p''-1)n})$: with straightforward notation,

$$\lim_{n\to\infty} \sum_{k\ge p''-1} (x_{(k+1)n} - x_{kn}) f(r_{kn}) = (1 - x_{p''-1}) f(r_{p''-1}). \tag{31}$$

Bringing (29), (30), (31) together we obtain (28). □

In our proofs we will often consider some "good" random event $E_n(\kappa)$ under which we will be able to localize the roots of G_n. The next lemma will be useful.

LEMMA 3.5. *Let $\{S_n\}_{n\in\mathbb{N}}$ and S be random variables defined on a common probability space. Suppose that for each $\kappa > 0$ we have random events $\{E_n(\kappa)\}_{n\in\mathbb{N}}$ and random variables $\{S_n(\kappa)\}_{n\in\mathbb{N}}$, $S(\kappa)$ such that the following conditions hold*:

(1) *for every $\kappa > 0$, $S_n(\kappa) \to S(\kappa)$ in distribution as $n \to \infty$;*
(2) *$S(\kappa) \to S$ in distribution as $\kappa \downarrow 0$;*
(3) $\lim_{\kappa\downarrow 0} \liminf_{n\to\infty} \mathbb{P}[E_n(\kappa)] = 1$;
(4) $|S_n(\kappa) - S_n| < m_n(\kappa)$ *on $E_n(\kappa)$, where* $\lim_{\kappa\downarrow 0} \limsup_{n\to\infty} m_n(\kappa) = 0$.

Then, $S_n \to S$ in distribution as $n \to \infty$.

PROOF. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. Write $C = \|f\|_\infty$. Take some $\varepsilon > 0$. We can choose $\kappa = \kappa(\varepsilon) > 0$ such that

$$\begin{aligned} &\left|\mathbb{E}f\left(S(\kappa)\right) - \mathbb{E}f(S)\right| < \varepsilon, \qquad \limsup_{n\to\infty} \mathbb{P}\left[E_n^c(\kappa)\right] < \varepsilon, \\ &\limsup_{n\to\infty} m_n(\kappa) < \varepsilon. \end{aligned} \tag{32}$$

Here, $E_n^c(\kappa)$ denotes the complement of $E_n(\kappa)$. After having fixed κ we choose $n_0 = n_0(\varepsilon)$ such that for all $n > n_0$,

$$\left|\mathbb{E}f\left(S_n(\kappa)\right) - \mathbb{E}f\left(S(\kappa)\right)\right| < \varepsilon, \qquad \mathbb{P}\left[E_n^c(\kappa)\right] < 2\varepsilon, \qquad m_n(\kappa) < 2\varepsilon. \tag{33}$$

Denoting by $\omega_f(\delta) = \sup_{|z_1 - z_2|\le\delta} |f(z_1) - f(z_2)|$ the continuity modulus of f, we have

$$\left|\mathbb{E}f(S_n) - \mathbb{E}f\left(S_n(\kappa)\right)\right| \le \omega_f\left(m_n(\kappa)\right) + 2C\mathbb{P}\left[E_n^c(\kappa)\right] \le \omega_f(2\varepsilon) + 4C\varepsilon. \tag{34}$$

Taking $\varepsilon \downarrow 0$ in (32), (33), (34), we obtain $\lim_{n\to\infty} \mathbb{E}f(S_n) = \mathbb{E}f(S)$. □

4. Proof of Theorem 1.1.

4.1. *Notation.* Let $\xi_0, \xi_1, \dots$ be i.i.d. random variables satisfying (6). Consider the least concave majorant $\mathfrak{C}_n$ of the set $\{(k, \log|\xi_k|) : k = 0, \dots, n\}$, where we agree to exclude points with $\log|\xi_k| \le 0$ from consideration. By definition, $\mathfrak{C}_n(t) = \inf_f f(t)$ for all $t \in [0, n]$, where the infimum is taken over all concave functions $f : [0, n] \to [0, \infty)$ satisfying $f(k) \ge \log|\xi_k|$ for all $k = 0, \dots, n$. For simplicity, we will call $\mathfrak{C}_n$ the majorant of the polynomial G_n. Denote the vertices of $\mathfrak{C}_n$ (from left to right) by $(k_{in}, \log_+ |\xi_{k_{in}}|)$, where $0 \le i \le d_n$ and $k_{0n} = 0$, $k_{d_n n} = n$. On the interval $[k_{in}, k_{(i+1)n}]$ the majorant is a linear function which we write in the form

$$\mathfrak{C}_n(t) = S_{in} - R_{in} t, \qquad t \in [k_{in}, k_{(i+1)n}], \qquad 0 \le i < d_n. \tag{35}$$

Further, denote by ρ a Poisson point process on $[0, 1] \times (0, \infty)$ with intensity $\alpha v^{-(\alpha+1)}\, du\, dv$. The majorant of ρ is denoted by $\mathfrak{C}_\rho$. As in Section 1.2, we denote the vertices of $\mathfrak{C}_\rho$, counted from left to right, by (X_k, Y_k). In the case $\alpha \ge 1$ the index k ranges (with probability 1) in $\mathbb{Z}$ by Proposition 1.7. In the case $\alpha \in (0, 1)$ the index k ranges in $p' \le k \le p''$, where p', p'' are a.s. finite random variables and $(X_{p'}, Y_{p'}) = (0, 0)$, $(X_{p''}, Y_{p''}) = (1, 0)$. On each interval $[X_k, X_{k+1}]$ the majorant $\mathfrak{C}_\rho$ is a linear function written in the form

$$\mathfrak{C}_\rho(t) = S_k - R_k t, \qquad t \in [X_k, X_{k+1}]. \tag{36}$$

We will be mostly interested in the "main" parts of the majorants $\mathfrak{C}_n$ and $\mathfrak{C}_\rho$. To make this precise, we take some small $\kappa \in (0, 1/2)$ and let $0 \le q_n' < q_n'' \le d_n$ and $q' < q''$ be indices (depending on κ) defined by the conditions

$$k_{q_n' n} \le \kappa n < k_{(q_n'+1)n}, \qquad k_{(q_n''-1)n} < (1-\kappa)n \le k_{q_n'' n}, \tag{37}$$

$$X_{q'} \le \kappa < X_{q'+1}, \qquad X_{q''-1} < 1-\kappa \le X_{q''}. \tag{38}$$

In our proof of Theorem 1.1 it will be convenient to consider the logarithms of the roots of G_n rather than the roots themselves. We will prove the following weak convergence of random probability measures on the space $E = [-\infty, \infty] \times [0, 2\pi]$:

$$\frac{1}{n} \sum_{z \in \mathcal{Z}_n} \delta\big(b_n \log|z|, \arg z\big) \xrightarrow[n\to\infty]{w} \sum_k (X_{k+1} - X_k) \lambda_{R_k}, \tag{39}$$

where λ_r is the Lebesgue measure on $\{r\} \times [0, 2\pi]$ normalized to have total mass 1. The sum on the right-hand side is over all linearity intervals $[X_k, X_{k+1}]$ of the majorant $\mathfrak{C}_\rho$. To see that (39) implies the statement of Theorem 1.1 note that the map $F : E \to \bar{\mathbb{C}}$ given by $F(r, \varphi) = e^{r+i\varphi}$ is continuous, and hence it induces a weakly continuous map between the corresponding spaces of probability measures; see [13], Proposition 3.18. By Proposition 3.1 we can apply F to the both sides of (39) which yields Theorem 1.1. So, let $f : E \to [0, \infty)$ be a continuous function. To prove Theorem 1.1 it suffices to show that

$$S_n := \frac{1}{n} \sum_{z \in \mathcal{Z}_n} f\big(b_n \log|z|, \arg z\big) \xrightarrow[n\to\infty]{d} \sum_k (X_{k+1} - X_k) \bar{f}(R_k) =: S, \tag{40}$$

where $\bar{f} : [-\infty, \infty] \to \mathbb{R}$ is defined by $\bar{f}(r) = \int_E f \, d\lambda_r = \frac{1}{2\pi} \int_0^{2\pi} f(r, \varphi)\, d\varphi$.

We will need to consider the cases $\alpha \ge 1$ and $\alpha \in (0, 1)$ separately. The main difference is that in the former case the linearity intervals of the majorant $\mathfrak{C}_\rho$ cluster at 0 and 1, whereas in the latter case we have a well-defined first and a well-defined last linearity interval of $\mathfrak{C}_\rho$. These intervals cannot be ignored and have to be considered separately. This makes the case $\alpha \in (0, 1)$ somewhat more difficult.

4.2. *Proof in the case $\alpha \ge 1$.* The next lemma shows that with probability approaching 1 the majorant of G_n has some "good" properties. In particular, there is a gap between the majorant and the points lying below the majorant. Let $W_{in} \subset [0, n] \times [0, \infty)$ be the set consisting of $[0, n] \times \{0\}$ together with the points $(k, \log_+ |\xi_k|)$ for all $0 \le k \le n$ such that $k \ne k_{in}, k_{(i+1)n}$.

LEMMA 4.1. *Fix sufficiently small $\varepsilon > 0$, and consider a random event $E_n := E_n^1 \cap E_n^2$, where*

$$E_n^1 = \Big\{ \min_{q_n' \le i < q_n''} \min_{(u,v) \in W_{in}} (S_{in} - R_{in} u - v) > n^{1/\alpha - \varepsilon} \Big\}, \tag{41}$$

$$E_n^2 = \Big\{ \min_{q_n' \le i < q_n''} (k_{(i+1)n} - k_{in}) > \sqrt{n} \Big\}. \tag{42}$$

Then, $\lim_{n\to\infty} \mathbb{P}[E_n] = 1$.

PROOF. By Proposition 1.17 the point process $\rho_n = \sum_{k=0}^n \delta(\frac{k}{n}, \frac{\log|\xi_k|}{a_n})$ converges to ρ weakly on $\mathfrak{M}$, where the points with $\log|\xi_k| \le 0$ are ignored. Recall the definition of the functionals H_1 and L_1 in Lemma 3.2. By scaling,

$$H_1(\rho_n) = \frac{1}{a_n} \min_{q_n' \le i < q_n''} \min_{(u,v) \in W_{in}} (S_{in} - R_{in} u - v),$$

$$L_1(\rho_n) = \frac{1}{n} \min_{q_n' \le i < q_n''} (k_{(i+1)n} - k_{in}).$$

It follows that

$$\mathbb{P}[E_n^1] = \mathbb{P}[H_1(\rho_n) > a_n^{-1} n^{1/\alpha - \varepsilon}], \qquad \mathbb{P}[E_n^2] = \mathbb{P}[L_1(\rho_n) > n^{-1/2}].$$

By Lemma 3.2 and Proposition 3.1 (which is applicable since $\mathbb{P}[\rho \in \mathfrak{M}_1] = 1$ for $\alpha \ge 1$), we have $H_1(\rho_n) \to H_1(\rho)$ and $L_1(\rho_n) \to L_1(\rho)$ in distribution as $n \to \infty$. Note that $H_1(\rho) > 0$ and $L_1(\rho) > 0$ a.s. Also, $a_n > n^{1/\alpha - \varepsilon/2}$ for large n by (6) and (7). It follows that $\lim_{n\to\infty} \mathbb{P}[E_n] = 1$. □

In the next lemma we localize most complex roots of G_n under the event E_n.

LEMMA 4.2. *On the random event E_n the following holds*: *for every $q'_n \le i < q''_n$ and $1 \le m \le k_{(i+1)n} - k_{in}$ there is exactly one root of G_n in the set*

$$Z_{i,m}(n) := \left\{z \in \mathbb{C} : \big|\log|z| - R_{in}\big| < \delta_n, \left|\arg z - \frac{\varphi_{in} + 2\pi m}{k_{(i+1)n} - k_{in}}\right| < \delta_n\right\},$$

where $\delta_n = \exp(-n^{1/\alpha - 2\varepsilon})$ and $\varphi_{in} = \arg(-\xi_{k_{in}}/\xi_{k_{(i+1)n}})$. The above sets are disjoint, and there are no other roots in the ring $R_{q'_n n} - \delta_n \le \log|z| < R_{(q''_n - 1)n} + \delta_n$.

PROOF. First note that on E_n it is impossible that $q'_n = 0$ and $\log|\xi_0| \le 0$. Similarly, on E_n it is impossible that $q''_n = d_n$ and $\log|\xi_n| \le 0$. It follows from (41) that on the event E_n the conditions of Lemma 2.1 are fulfilled for the polynomial G_n with $k = k_{in}$, $l = k_{(i+1)n}$, $\delta = \zeta = \delta_n$ for every $q'_n \le i < q''_n$. Hence, every set $Z_{i,m}(n)$ contains exactly one root of G_n. Also, it follows from the proof of Lemma 2.1 that there are exactly $k_{q'_n n}$ roots of G_n in the disk $\log|z| < R_{q'_n n} - \delta_n$ and exactly $k_{q''_n n}$ roots in the disc $\log|z| < R_{(q''_n - 1)n} + \delta_n$. Hence, there are exactly $k_{q''_n n} - k_{q'_n n}$ roots in the ring $R_{q'_n n} - \delta_n \le \log|z| < R_{(q''_n - 1)n} + \delta_n$, which coincides with the number of different sets $Z_{i,m}(n)$. It remains to show that the sets $Z_{i,m}(n)$ are disjoint on E_n. To this end, it suffices to show that on E_n it holds that $R_{(i+1)n} - R_{in} > 3\delta_n$ for every $q'_n \le i < q''_n - 1$. We have

$$(k_{(i+2)n} - k_{(i+1)n})(R_{(i+1)n} - R_{in}) = S_{in} - R_{in} k_{(i+2)n} - \log|\xi_{k_{(i+2)n}}| > n^{1/\alpha - \varepsilon}$$

on E_n. Since $k_{(i+2)n} - k_{(i+1)n} \le n$, this implies that which is required. $\square$

Our aim is to show that $S_n \to S$ in distribution as $n \to \infty$; see (40). Define random variables $S_n(\kappa)$ and $S(\kappa)$ which approximate S_n and S by

$$S_n(\kappa) = \frac{1}{n} \sum_{q'_n \le i < q''_n} (k_{(i+1)n} - k_{in}) \bar{f}(b_n R_{in}),$$

$$S(\kappa) = \sum_{q' \le i < q''} (X_{i+1} - X_i) \bar{f}(R_i).$$

Let $\omega_f(\delta) = \sup_{|z_1 - z_2| \le \delta} |f(z_1) - f(z_2)|$, where $\delta > 0$, be the continuity modulus of the function f.

LEMMA 4.3. *On the random event E_n it holds that*

$$\big|S_n - S_n(\kappa)\big| \le \omega_f\big(10/\sqrt{n}\big) + 2\kappa \|f\|_\infty.$$

PROOF. We always assume that the event E_n occurs. Take some $q'_n \le i < q''_n$. By Lemma 4.2, the polynomial G_n has a unique root, denoted by $z_{i,m}(n)$, in the set $Z_{i,m}(n)$, where $1 \le m \le \Delta_{in}$ and $\Delta_{in} = k_{(i+1)n} - k_{in}$. Denote by $\mathcal{Z}_{in}$ the finite set

$\{z_{i,m}(n) : 1 \le m \le \Delta_{in}\}$. By (42) we have $\Delta_{in} > \sqrt{n}$. By the definition of $Z_{i,m}(n)$ in Lemma 4.2,

$$\left| f\big(b_n \log|z_{i,m}(n)|, \arg z_{i,m}(n)\big) - \frac{\Delta_{in}}{2\pi} \int_{(\varphi_{in}+2\pi m-\pi)/\Delta_{in}}^{(\varphi_{in}+2\pi m+\pi)/\Delta_{in}} f(b_n R_{in}, \varphi)\, d\varphi \right|$$

is smaller than $\omega_f(10/\sqrt{n})$. Taking the sum over $1 \le m \le \Delta_{in}$, we obtain

$$\frac{1}{n}\left| \sum_{z \in \mathcal{Z}_{in}} f\big(b_n \log|z|, \arg z\big) - \Delta_{in} \bar{f}(b_n R_{in}) \right| \le \frac{\Delta_{in}}{n} \omega_f\big(10/\sqrt{n}\big). \tag{43}$$

Let $\mathcal{Z}_n^*$ be the set of roots (counted with multiplicities) of the polynomial G_n not belonging to $\bigcup_{q_n' \le i < q_n''} \mathcal{Z}_{in}$. The number of roots in $\mathcal{Z}_n^*$ is $n - k_{q_n'' n} + k_{q_n' n}$, which is at most $2\kappa n$ by (37). Hence,

$$\frac{1}{n} \sum_{z \in \mathcal{Z}_n^*} f\big(b_n \log|z|, \arg z\big) \le 2\kappa \|f\|_\infty. \tag{44}$$

Taking the sum of (43) over all $q_n' \le i < q_n''$ and applying (44), we obtain the required inequality. □

LEMMA 4.4. *We have $S_n(\kappa) \to S(\kappa)$ in distribution as $n \to \infty$.*

PROOF. By Proposition 1.17 the point process $\rho_n = \sum_{k=0}^n \delta(\frac{k}{n}, \frac{\log|\xi_k|}{a_n})$ converges to ρ weakly on $\mathfrak{M}$. By Lemma 3.2 and Proposition 3.1 (which is applicable since $\mathbb{P}[\rho \in \mathfrak{M}_1] = 1$ for $\alpha \ge 1$), we obtain that $\Psi_1(\rho_n)$ converges weakly (as a random finite measure on $\mathbb{R}$) to $\Psi_1(\rho)$. This implies that $\int_{\mathbb{R}} \bar{f}\, d\Psi_1(\rho_n)$ converges in distribution to $\int_{\mathbb{R}} \bar{f}\, d\Psi_1(\rho)$, which is exactly what is stated in the lemma. □

The proof of Theorem 1.1 in the case $\alpha \ge 1$ can be completed as follows. Recall that $\lim_{n\to\infty} \mathbb{P}[E_n] = 1$ by Lemma 4.1. Trivially, $S(\kappa) \to S$ as $\kappa \downarrow 0$ a.s. and hence, in distribution. By Lemma 3.5 (whose conditions have been verified above) we obtain that $S_n \to S$ in distribution as $n \to \infty$. This proves (40).

4.3. *Proof in the case $\alpha \in (0, 1)$.* This case is somewhat more difficult since we have to analyze the first and the last segment of the majorant of G_n separately. In our proof we will assume that $\xi_0 \ne 0$ a.s. This assumption will be removed afterward. Let $0 < \tau_n \le n$, $0 \le \theta_n < n$ be indices (for concreteness, we choose the smallest possible values) such that

$$\frac{\log|\xi_{\tau_n}|}{\tau_n} = \max_{k=1,\dots,n} \frac{\log|\xi_k|}{k}, \qquad \frac{\log|\xi_{\theta_n}|}{n - \theta_n} = \max_{k=0,\dots,n-1} \frac{\log|\xi_k|}{n-k}.$$

Recall that $W_{in} \subset [0, n] \times [0, +\infty)$ denotes the set consisting of $[0, n] \times \{0\}$ together with the points $(k, \log_+ |\xi_k|)$ for all $0 \le k \le n$ such that $k \ne k_{in}, k_{(i+1)n}$.

LEMMA 4.5. *For sufficiently small* $\varepsilon > 0$ *and* $\kappa \in (0, 1/2)$, *consider a random event* $E_n := \bigcap_{i=1}^{6} E_n^i$, *where*

$$E_n^1 = \Big\{ \min_{0<i<d_n-1} \min_{(u,v)\in W_{in}} (S_{in} - R_{in}u - v) > n^{1/\alpha-\varepsilon} \Big\}, \tag{45}$$

$$E_n^2 = \Big\{ \min_{0\le i<d_n} (k_{(i+1)n} - k_{in}) > \sqrt{n} \Big\}, \tag{46}$$

$$E_n^3 = \left\{ \min_{j\neq 0,\tau_n} \left(\frac{\log|\xi_{\tau_n}|}{\tau_n} - \frac{\log_+|\xi_j|}{j} \right) > n^{1/\alpha-1-\varepsilon} \right\}, \tag{47}$$

$$E_n^4 = \left\{ \min_{j\neq n,\theta_n} \left(\frac{\log|\xi_{\theta_n}|}{n-\theta_n} - \frac{\log_+|\xi_j|}{n-j} \right) > n^{1/\alpha-1-\varepsilon} \right\}, \tag{48}$$

$$E_n^5 = \{ \tau_n > \kappa n, \theta_n < (1-\kappa)n \}, \tag{49}$$

$$E_n^6 = \{ |\log|\xi_0|| < n^{\varepsilon}, |\log|\xi_n|| < n^{\varepsilon} \}. \tag{50}$$

Then $\lim_{\kappa\downarrow 0} \liminf_{n\to\infty} \mathbb{P}[E_n] = 1$ *for every* $\varepsilon > 0$.

REMARK 4.6. Note that E_n^1 states that all segments of the majorant, except for the first and the last one, are well separated from the points below the majorant. For the first and the last segment the well-separation property is stated in random events E_n^3 and E_n^4.

REMARK 4.7. We will see that on $E_n^3 \cap E_n^6$ the segment joining the points $(0, \log_+|\xi_0|)$ and $(\tau_n, \log|\xi_{\tau_n}|)$ is the first segment of the majorant of G_n. In general, this segment need not be the first one, for example, if $\log_+|\xi_0|$ is very large. Similarly, on $E_n^4 \cap E_n^6$ the segment joining $(\theta_n, \log|\xi_{\theta_n}|)$ and $(n, \log_+|\xi_n|)$ is the last segment of the majorant of G_n. It follows that $q_n' = 0$ and $q_n'' = d_n$ on the event $\bigcap_{i=3}^{6} E_n^i$.

PROOF OF LEMMA 4.5. We start by considering E_n^3. Let $\tilde{\rho}$ be a Poisson point process on $(0, \infty)$ with intensity $\frac{\alpha}{1-\alpha} v^{-(\alpha+1)}\, dv$. We will show that the following weak convergence of point processes on $(0, \infty]$ holds:

$$\tilde{\rho}_n := \sum_{k=1}^{n} \delta\left(\frac{b_n \log|\xi_k|}{k} \right) \xrightarrow[n\to\infty]{w} \tilde{\rho}. \tag{51}$$

Again, we agree that the terms with $\log|\xi_k| \le 0$ are ignored. Recall from (7) that $\bar{F}(a_n) \sim 1/n$ as $n \to \infty$. Take some $t > 0$. By (6) and a well-known uniform convergence theorem for regularly varying functions we have, uniformly in $\kappa n \le k \le n$,

$$\mathbb{P}\left[\frac{b_n \log|\xi_k|}{k} > t \right] = \bar{F}\left(\frac{kta_n}{n} \right) \sim n^{\alpha-1} k^{-\alpha} t^{-\alpha}, \qquad n \to \infty. \tag{52}$$

To estimate the terms with $1 \le k \le \kappa n$ recall the following Potter bound: for every small $\delta > 0$ we have $\bar{F}(x)/\bar{F}(y) \le 2(x/y)^{-\alpha-\delta}$ as long as $x < y$ are sufficiently large; see [4], Theorem 1.5.6. We have

$$\begin{aligned} \sum_{k=1}^{[\kappa n]} \mathbb{P}\left[\frac{b_n \log|\xi_k|}{k} > t\right] &= \sum_{k=1}^{[\kappa n]} \bar{F}\left(\frac{kta_n}{n}\right) \\ &\le 2\bar{F}(\kappa t a_n) \sum_{k=1}^{[\kappa n]} \left(\frac{\kappa n}{k}\right)^{\alpha+\delta} \qquad (53) \\ &< C\kappa^{1-\alpha} t^{-\alpha}. \end{aligned}$$

From (52) and (53) with $\kappa \downarrow 0$, we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}\left[\frac{b_n \log|\xi_k|}{k} > t\right] = \frac{1}{(1-\alpha)} t^{-\alpha}. \qquad (54)$$

By a standard argument this implies (51). Since the weak convergence of point processes in (51) implies (via Proposition 3.1) the weak convergence of the corresponding upper order statistics, we have

$$\min_{j \ne 0, \tau_n} \left\{ b_n \left(\frac{\log|\xi_{\tau_n}|}{\tau_n} - \frac{\log_+|\xi_j|}{j} \right) \right\} \xrightarrow[n\to\infty]{d} \tilde{V}_1 - \tilde{V}_2,$$

where $\tilde{V}_1, \tilde{V}_2$ are the largest and the second largest points of $\tilde{\rho}$. Since $b_n^{-1} > n^{1/\alpha - 1 - \varepsilon/2}$ for large n and since $\tilde{V}_1 > \tilde{V}_2$ a.s., we have $\lim_{n\to\infty} \mathbb{P}[E_n^3] = 1$. By symmetry, $\lim_{n\to\infty} \mathbb{P}[E_n^4] = 1$.

Let us consider E_n^5. By (51) and (53) we have, for every $t > 0$ and sufficiently large n,

$$\begin{aligned} \mathbb{P}[\tau_n \le \kappa n] &\le \mathbb{P}\left[\max_{k=1,\dots,n} \frac{b_n \log|\xi_k|}{k} \le t\right] + \mathbb{P}\left[\max_{k=1,\dots,[\kappa n]} \frac{b_n \log|\xi_k|}{k} > t\right] \\ &< 2\exp\left\{-\frac{1}{1-\alpha} t^{-\alpha}\right\} + C\kappa^{1-\alpha} t^{-\alpha}. \end{aligned}$$

Taking $t^\alpha = \kappa^{(1-\alpha)/2}$ and letting $\kappa \downarrow 0$, we obtain $\lim_{\kappa\downarrow 0} \limsup_{n\to\infty} \mathbb{P}[\tau_n \le \kappa n] = 0$. By symmetry, $\lim_{\kappa\downarrow 0} \liminf_{n\to\infty} \mathbb{P}[E_n^5] = 1$. Since we assume that $\xi_0 \ne 0$ a.s., we have $\lim_{n\to\infty} \mathbb{P}[E_n^6] = 1$.

To proceed further we need to prove Remark 4.7. Let $s, r \in \mathbb{R}$ be such that $s = \log_+|\xi_0|$ and $s - \tau_n r = \log|\xi_{\tau_n}|$. On the random event $E_n^3 \cap E_n^6$ we have that for every $1 \le j \le n$, $j \ne \tau_n$,

$$s - jr - \log|\xi_j| = j\left(\frac{\log|\xi_{\tau_n}|}{\tau_n} - \frac{\log|\xi_j|}{j} - s\left(\frac{1}{\tau_n} - \frac{1}{j}\right)\right) > n^{1/\alpha - 1 - \varepsilon} - 2n^\varepsilon > 0.$$

This proves what is required.

Let us turn our attention to E_n^1 and E_n^2. By Proposition 1.17 the point process $\rho_n = \sum_{k=0}^n \delta(\frac{k}{n}, \frac{\log|\xi_k|}{a_n})$ converges weakly to ρ. Recall the definition of the functionals H_0 and L_0 in Lemma 3.3. By a scaling argument,

$$H_0(\rho_n) = \frac{1}{a_n} \min_{q_n' < i < q_n'' - 1} \min_{(u,v) \in W_{in}} (S_{in} - R_{in} u - v),$$

$$L_0(\rho_n) = \frac{1}{n} \min_{q_n' < i < q_n'' - 1} (k_{(i+1)n} - k_{in}).$$

As observed in Remark 4.7, on the event $\bigcap_{i=3}^6 E_n^i$ we have $q_n' = 0$ and $q_n'' = d_n$. Hence,

$$\mathbb{P}[E_n^1] \ge \mathbb{P}[H_0(\rho_n) > a_n^{-1} n^{1/\alpha - \varepsilon}] - \left(1 - \mathbb{P}\left[\bigcap_{i=3}^{6} E_n^i\right]\right),$$

$$\mathbb{P}[E_n^2] \ge \mathbb{P}[L_0(\rho_n) > n^{-1/2}] - \left(1 - \mathbb{P}\left[\bigcap_{i=3}^{6} E_n^i\right]\right).$$

By Lemma 3.3 and Proposition 3.1 (which is applicable since $\mathbb{P}[\rho \in \mathfrak{M}_0] = 1$ for $\alpha \in (0, 1)$), we have $H_0(\rho_n) \to H_0(\rho)$ and $L_0(\rho_n) \to L_0(\rho)$ weakly on $[0, \infty]$ as $n \to \infty$. Note that $H_0(\rho) > 0$ and $L_0(\rho) > 0$ a.s. and $a_n > n^{1/\alpha - \varepsilon/2}$ for large n. Also, we have already shown that the probability of the event $\bigcap_{i=3}^6 E_n^i$ can be made arbitrary close to 1 by choosing κ small and n large. It follows that $\lim_{n\to\infty} \mathbb{P}[E_n^1] = \lim_{n\to\infty} \mathbb{P}[E_n^2] = 1$, as required. □

In the next lemma we isolate all roots of G_n under the event E_n. It will be convenient to modify the definition of the slopes of the majorant of G_n. Let R_{0n}' be such that $\log|\xi_0| - R_{0n}' k_{1n} = \log|\xi_{k_{1n}}|$. This is well-defined since $\xi_0 \ne 0$ a.s. Note that if $\log|\xi_0| < 0$, then R_{0n}' is not the same as R_{0n}. On E_n we have the estimate

$$|R_{0n} - R_{0n}'| \le \tau_n^{-1} |\log|\xi_0|| < n^{2\varepsilon - 1}. \tag{55}$$

In a similar way, we can define $R_{(d_n-1)n}'$. For all $0 < i < d_n - 1$, set $R_{in}' = R_{in}$.

LEMMA 4.8. *On the random event E_n the following holds: for every $0 \le i < d_n$ and $1 \le m \le k_{(i+1)n} - k_{in}$, there is exactly one root of G_n in the set*

$$Z_{i,m}(n) := \left\{ z \in \mathbb{C} : |\log|z| - R_{in}'| < \delta_n, \left| \arg z - \frac{\varphi_{in} + 2\pi m}{k_{(i+1)n} - k_{in}} \right| < \delta_n \right\},$$

where $\varphi_{in} = \arg(-\xi_{k_{in}}/\xi_{k_{(i+1)n}})$ and $\delta_n = \exp(-n^{1/\alpha - 1 - 3\varepsilon})$. The above sets are disjoint, and there are no other roots of G_n.

PROOF. Consider the case $i = 0$ first. Let $s = \log|\xi_0|$ (well defined since $\xi_0 \ne 0$ a.s.) and $r = R_{0n}'$. Note that $\tau_n = k_{1n}$ on E_n by Remark 4.7. In order to apply Lemma 2.1 with $k=0$, $l=\tau_n$ we need to estimate $h := \min_{j\neq 0,\tau_n}(s - jr - \log|\xi_j|)$. On the event E_n we have

$$\min_{j\neq 0,\tau_n} \frac{s - jr - \log|\xi_j|}{j} = \min_{j\neq 0,\tau_n}\left(\frac{\log|\xi_{\tau_n}|}{\tau_n} - \frac{\log|\xi_j|}{j} - s\left(\frac{1}{\tau_n} - \frac{1}{j}\right)\right)$$
$$> n^{1/\alpha-1-2\varepsilon},$$

which implies that $h > n^{1/\alpha-1-2\varepsilon}$. To prove the lemma for $i=0$, apply Lemma 2.1 with $k=0$, $l=\tau_n$ and $\delta=\zeta=\delta_n$. The case $i=d_n-1$ is similar. Let us now consider the case $0<i<d_n-1$. On the event E_n, the conditions of Lemma 2.1 are fulfilled for the polynomial G_n with $k=k_{in}$, $l=k_{(i+1)n}$ and $\delta=\zeta=\delta_n$; see (45). The statement follows by Lemma 2.1.

It remains to prove that the sets $Z_{i,m}(n)$ are disjoint. It suffices to show that on E_n it holds that $R'_{(i+1)n} - R'_{in} > 3\delta_n$ for every $0 \le i < d_n$. We have

$$(k_{(i+2)n} - k_{(i+1)n})(R_{(i+1)n} - R_{in}) = S_{in} - R_{in}k_{(i+2)n} - \log_+|\xi_{k_{(i+2)n}}|. \tag{56}$$

For $i \neq 0, d_n - 1$ it follows from (45) that the right-hand side can be estimated below by $n^{1/\alpha-\varepsilon}$ on E_n. The required follows since $k_{(i+2)n} - k_{(i+1)n} \le n$. Using (56) we obtain that for $i=0$ on the event E_n it holds that

$$\frac{k_{2n}-k_{1n}}{k_{2n}}(R_{1n} - R_{0n}) = \frac{\log|\xi_{\tau_n}|}{\tau_n} - \frac{\log|\xi_{k_{2n}}|}{k_{2n}} - \log_+|\xi_0|\left(\frac{1}{\tau_n} - \frac{1}{k_{2n}}\right)$$
$$> n^{1/\alpha-1-2\varepsilon},$$

where the last inequality follows from (47), (50). It follows that $R_{1n} - R_{0n} > n^{1/\alpha-1-2\varepsilon}$. Recalling (55) we obtain $R'_{1n} - R'_{0n} > 3\delta_n$. The case $i = d_n - 1$ is similar. □

Recall from (40) that we need to prove that $S_n \to S$ in distribution as $n\to\infty$. Define a random variable S_n^* which approximates S_n by

$$S_n^* = \frac{1}{n}\sum_{0\le i<d_n-1}(k_{(i+1)n} - k_{in})\bar{f}(b_n R_{in}).$$

LEMMA 4.9. *On the random event E_n it holds that $|S_n^* - S_n| < \omega_f(n^{-\varepsilon})$.*

PROOF. Assume that the event E_n occurs. Take some $0 \le i < d_n$. Write $\Delta_{in} = k_{(i+1)n} - k_{in}$. By Lemma 4.8, the polynomial G_n has a unique root, denoted by $z_{i,m}(n)$, in the set $Z_{i,m}(n)$ for every $1 \le m \le \Delta_{in}$. Denote by $\mathcal{Z}_{in}$ the finite set $\{z_{i,m}(n) : 1 \le m \le \Delta_{in}\}$. Recall from (46) that $\Delta_{in} > \sqrt{n}$. It follows from the definition of the set $Z_{i,m}(n)$ that for every $1 \le m \le \Delta_{in}$,

$$\left| f\left(b_n \log|z_{i,m}(n)|, \arg z_{i,m}(n)\right) - \frac{\Delta_{in}}{2\pi}\int_{(\varphi_{in}+2\pi m-\pi)/\Delta_{in}}^{(\varphi_{in}+2\pi m+\pi)/\Delta_{in}} f(b_n R_{in}, \varphi)\,d\varphi \right|$$

is smaller than $\omega_f(n^{-\varepsilon})$. Note that for $i = 0$ and $i = d_n - 1$, we need to use (55) to prove this estimate. Taking the sum over $1 \le m \le \Delta_{in}$, we obtain

$$\frac{1}{n}\left|\sum_{z\in\mathcal{Z}_{in}} f(b_n \log|z|, \arg z) - \Delta_{in}\bar{f}(b_n R_{in})\right| \le \frac{\Delta_{in}}{n}\omega_f(n^{-\varepsilon}).$$

Taking the sum over $0 \le i < d_n$, we obtain what is required. □

LEMMA 4.10. *We have $S_n^* \to S$ in distribution as $n \to \infty$.*

PROOF. By Proposition 1.17 the point process $\rho_n = \sum_{k=0}^n \delta(\frac{k}{n}, \frac{\log|\xi_k|}{a_n})$ converges weakly to ρ. By Lemma 3.3 and Proposition 3.1 (which is applicable since $\mathbb{P}[\rho \in \mathfrak{M}_0] = 1$ for $\alpha \in (0, 1)$) we have that $\Psi_0(\rho_n)$ converges weakly (as a random probability measure on $\mathbb{R}$) to $\Psi_0(\rho)$. It follows that $\int_{\mathbb{R}} \bar{f}\, d\Psi_0(\rho_n)$ converges in distribution to $\int_{\mathbb{R}} \bar{f}\, d\Psi_0(\rho)$, which is exactly what is stated in the lemma. □

The proof of Theorem 1.1 in the case $\alpha \in (0, 1)$ can be completed as follows. By Lemma 3.5 with $S_n(\kappa) = S_n^*$ and $S(\kappa) = S$, we obtain $S_n \to S$ in distribution as $n \to \infty$. This proves (40).

The following explains how to get rid of the assumption $\xi_0 \ne 0$ a.s. Let $\mathbb{P}[\xi_0 \ne 0]$ be strictly positive. Denote the first (resp., last) nonzero coefficient of G_n by ξ_{l_n} (resp., ξ_{n-m_n}). For fixed $l, m \in \mathbb{N}_0$, consider the conditional distribution $\mathbb{P}_{l,m}^n$ of the random variables ξ_k, $l \le k \le n - m$, given that $l_n = l$, $m_n = m$. Under $\mathbb{P}_{l,m}^n$, these variables are independent and, apart from the first and the last variable, identically distributed. It is easily seen that the above proof applies to the polynomial $\sum_{k=l}^{n-m} \xi_k z^k$ under $\mathbb{P}_{l,m}^n$. Since this holds for all $l, m \in \mathbb{N}_0$, the proof is complete.

5. Proof of Theorem 1.5. Recall that $\tau_n \in \{0, \dots, n\}$ is such that $M_n := \max_{k=0,\dots,n} \log|\xi_k| = \log|\xi_{\tau_n}|$. Intuitively, under the slow variation condition (9), the maximum M_n is, with probability close to 1, much larger than all the other terms $\log|\xi_k|$, $1 \le k \le n$. The majorant of the set $\{(j, \log|\xi_j|) : j = 0, \dots, n\}$ consists, with high probability, of two segments joining the endpoints $(0, \log_+ |\xi_0|)$ and $(n, \log_+ |\xi_n|)$ to the maximum $(\tau_n, \log|\xi_{\tau_n}|)$. The roots of G_n group around two circles corresponding to these segments. Our aim is to make this precise. Let the index k be always restricted to $0 \le k \le n$. We may always assume that the index τ_n is defined uniquely, since this event has probability converging to 1 as $n \to \infty$; see [6].

LEMMA 5.1. *For $\kappa \in (0, 1/2)$, $A > 0$ define a random event $E_n = \bigcap_{i=1}^4 E_n^i$, where*

$$E_n^1 = \left\{\min_{k\ne 0,\tau_n}\left(\frac{M_n}{\tau_n} - \frac{\log|\xi_k|}{k}\right) > n^{2A}\right\},$$

$$E_n^2 = \left\{ \min_{k \neq \tau_n, n} \left(\frac{M_n}{\tau_n} - \frac{\log|\xi_k|}{n-k} \right) > n^{2A} \right\},$$

$$E_n^3 = \{\kappa n < \tau_n < (1-\kappa)n\},$$

$$E_n^4 = \{|\log|\xi_0|| < n^A, M_n > n^{2A+1}, |\log|\xi_n|| < n^A\}.$$

Then, for every $A > 0$, $\lim_{\kappa \downarrow 0} \liminf_{n\to\infty} \mathbb{P}[E_n] = 1$.

PROOF. By symmetry, τ_n/n converges as $n \to \infty$ to the uniform distribution, which implies that $\lim_{\kappa\downarrow 0} \liminf_{n\to\infty} \mathbb{P}[E_n^3] = 1$. By [6], Theorem 3.2, the slow variation condition (9) implies that

$$\frac{1}{M_n} \max_{\substack{0\le k\le n\\ k\neq \tau_n}} \log|\xi_k| \xrightarrow[n\to\infty]{P} 0.$$

It follows that

$$\mathbb{P}\left[\max_{\substack{\kappa n\le k<n\\ k\neq\tau_n}} \frac{\log|\xi_k|}{k} > \frac{M_n}{2n}\right] \le \mathbb{P}\left[\max_{\substack{0\le k\le n\\ k\neq\tau_n}} \log|\xi_k| > \frac{\kappa}{2} M_n\right] \xrightarrow[n\to\infty]{} 0. \tag{57}$$

Put $c_n = \inf\{s : \bar{F}(s) \le 1/(\sqrt{\kappa} n)\}$. Then $\bar{F}(c_n) \sim 1/(\sqrt{\kappa} n)$ by [13], pages 15 and 16, and $\lim_{n\to\infty} c_n/n = \infty$. Recall the Potter bound for slowly varying functions: for every $\delta > 0$, we have $\bar{F}(y)/\bar{F}(x) < 2(x/y)^\delta$, provided that $x > y$ are sufficiently large; see [4], Theorem 1.5.6. We have

$$\begin{aligned}
\mathbb{P}\left[\max_{1\le k\le \kappa n} \frac{\log|\xi_k|}{k} > \frac{M_n}{2n}\right] &\le \sum_{1\le k\le\kappa n} \bar{F}\left(\frac{k}{2n} c_n\right) + \mathbb{P}[M_n < c_n] \\
&< \frac{3}{\sqrt{\kappa} n} \sum_{1\le k\le \kappa n} \left(\frac{2n}{k}\right)^{1/4} + \left(1 - \frac{1}{2\sqrt{\kappa} n}\right)^{n+1} \qquad (58)\\
&< C\left(\kappa^{1/4} + e^{-1/(2\sqrt{\kappa})}\right).
\end{aligned}$$

Since $\bar{F}$ decays more slowly than any negative power of n,

$$\mathbb{P}\left[\frac{M_n}{2n} > n^{2A}\right] = 1 - (1 - \bar{F}(2n^{2A+1}))^{n+1} > 1 - \left(1 - \frac{1}{n^2}\right)^{n+1} \xrightarrow[n\to\infty]{} 1. \tag{59}$$

Putting (57), (58) and (59) together and letting $\kappa \downarrow 0$, we obtain $\lim_{n\to\infty} \mathbb{P}[E_n^1] = 1$. By symmetry, we also have $\lim_{n\to\infty} \mathbb{P}[E_n^2] = 1$. From (59) it also follows that $\lim_{n\to\infty} \mathbb{P}[E_n^4] = 1$. □

PROOF OF THEOREM 1.5. In the sequel, we always suppose that the event E_n occurs. The roots of the equation $\xi_{\tau_n} z^{\tau_n} + \xi_0 = 0$, denoted by $w_{1n}, \dots, w_{\tau_n n}$, satisfy

$$|w_{kn}| = (|\xi_0|/|\xi_{\tau_n}|)^{1/\tau_n} = e^{(\log|\xi_0| - M_n)/\tau_n} < e^{-n^A}, \qquad 1 \le k \le \tau_n.$$

Similarly, the roots of the equation $\xi_n z^{n-\tau_n} + \xi_{\tau_n} = 0$, denoted by $w_{(\tau_n+1)n}, \dots, w_{nn}$, satisfy

$$|w_{kn}| = \left(|\xi_{\tau_n}|/|\xi_n|\right)^{1/(n-\tau_n)} = e^{(M_n - \log|\xi_n|)/(n-\tau_n)} > e^{n^A}, \qquad \tau_n < k \le n.$$

Choose $s, r \in \mathbb{R}$ so that $s = \log|\xi_0|$ and $s - r\tau_n = \log|\xi_{\tau_n}| = M_n$. To apply Lemma 2.1 with $k = 0$, $l = \tau_n$ we need to estimate $h := \min_{k \ne 0, \tau_n}(s - rk - \log|\xi_k|)$. We have, by definition of E_n,

$$\min_{k\ne 0,\tau_n} \frac{s - rk - \log|\xi_k|}{k} = \min_{k \ne 0, \tau_n}\left(\frac{M_n}{\tau_n} - \frac{\log|\xi_k|}{k} + s\left(\frac{1}{k} - \frac{1}{\tau_n}\right)\right) > n^{3A/2}.$$

Hence, $h > n^{3A/2}$. It follows that on the event E_n the conditions of Lemma 2.1 are fulfilled for $k = 0$, $l = \tau_n$ and $\delta = \zeta = e^{-n^A}$. Then, for every $1 \le k \le \tau_n$, the set

$$\left\{z \in \mathbb{C} : \left|\log|z| - r\right| \le e^{-n^A}, |\arg z - \arg w_{kn}| \le e^{-n^A}\right\}$$

contains exactly one root, say z_{kn}, of the polynomial G_n. It follows that

$$|z_{kn} - w_{kn}| < 10\delta e^r = 10 e^{-n^A}|w_{kn}|, \qquad 1 \le k \le \tau_n.$$

By symmetry, a similar inequality holds for $\tau_n < k \le n$. □

6. Proofs of Theorems 1.10 and 1.14.

6.1. *Limiting point processes*. First of all, we describe the limiting point processes $\Upsilon_{\alpha,c}$ and $\Upsilon^{\pm}_{\alpha,c,p}$. Let ρ be a Poisson point process on $[0,1] \times (0, \infty)$ with intensity $\alpha v^{-(\alpha+1)}\,du\,dv$ and majorant $\mathfrak{C}_\rho$ as in Section 1.2. Recall that the vertices of the majorant $\mathfrak{C}_\rho$ are denoted by (X_k, Y_k). For $\alpha \ge 1$ the index k ranges in $\mathbb{Z}$, whereas for $\alpha \in (0,1)$ we have $p' \le k \le p''$ and $(X_{p'}, Y_{p'}) = (0,0)$, $(X_{p''}, Y_{p''}) = (1, 0)$. Let σ_k, π_k be independent $\{-1, 1\}$-valued random variables [attached to the *vertices* (X_k, Y_k) of $\mathfrak{C}_\rho$ except for the boundary vertices $(0,0)$ and $(1,0)$ in the case $\alpha \in (0,1)$] such that

$$\mathbb{P}[\sigma_k = 1] = c, \qquad \mathbb{P}[\pi_k = 1] = 1/2.$$

In the case $\alpha \in (0,1)$, we have to add the following boundary conditions:

(1) $\pi_{p'} = 1$;
(2) $\pi_{p''} = 1$ in the definition of $\Upsilon^{+}_{\alpha,c,p}$ and $\pi_{p''} = -1$ in the definition of $\Upsilon^{-}_{\alpha,c,p}$;
(3) $\mathbb{P}[\sigma_{p'} = 1] = \mathbb{P}[\sigma_{p''} = 1] = p$.

Define random variables ε^+_k and ε^-_k attached to the *linearity intervals* $[X_k, X_{k+1}]$ of the majorant $\mathfrak{C}_\rho$ by

$$\varepsilon^+_k = \mathbb{1}_{\{\sigma_k \ne \sigma_{k+1}\}}, \qquad \varepsilon^-_k = \mathbb{1}_{\{\sigma_k\pi_k \ne \sigma_{k+1}\pi_{k+1}\}}. \tag{60}$$

With this notation, the limiting point processes $\Upsilon_{\alpha,c}$ and $\Upsilon^{\pm}_{\alpha,c,p}$ are defined by

$$\Upsilon^{(\pm)}_{\alpha,c(,p)} = \sum_k \varepsilon_k^+ \delta(e^{R_k}) + \sum_k \varepsilon_k^- \delta(-e^{R_k}), \tag{61}$$

where the sum is over all linearity intervals of the majorant $\mathfrak{C}_\rho$, and R_k is the negative of the slope of the kth segment of $\mathfrak{C}_\rho$ as in (8). We proceed to the proof of Theorem 1.10.

6.2. *Proof in the case* $\alpha \geq 1$. We will show that the following weak convergence of point processes on $E = \mathbb{R} \times \{-1, 1\}$ holds true:

$$\sum_{z \in \mathcal{R}_n} \delta(b_n \log|z|, \operatorname{sgn} z) \underset{n\to\infty}{\overset{w}{\longrightarrow}} \sum_k \varepsilon_k^+ \delta(R_k, 1) + \sum_k \varepsilon_k^- \delta(R_k, -1), \tag{62}$$

where the sum on the right-hand side is over all linearity intervals of the majorant $\mathfrak{C}_\rho$. To see that (62) implies Theorem 1.10 for $\alpha \geq 1$ note that the mapping $F : E \to \mathbb{R} \setminus \{0\}$ given by $F(r, \sigma) = \sigma e^r$ is continuous and proper (preimages of compact sets are compact). By [13], Proposition 3.18, it induces a vaguely continuous mapping between the spaces of locally finite counting measures on E and $\mathbb{R} \setminus \{0\}$. By Proposition 3.1 we may apply this mapping to the both sides of (62), which implies the statement of Theorem 1.10 for $\alpha \geq 1$. Denote by $\mathcal{R}_n^+$ (resp., $\mathcal{R}_n^-$) the set of positive (resp., negative) real roots of G_n, counted with multiplicities. Let $f^+, f^- : \mathbb{R} \to [0, \infty)$ be two continuous functions supported on an interval $[-A, A]$. Define random variables S_n and S by

$$S_n = \sum_{z \in \mathcal{R}_n^+} f^+(b_n \log z) + \sum_{z \in \mathcal{R}_n^-} f^-(b_n \log|z|), \tag{63}$$

$$S = \sum_k \varepsilon_k^+ f^+(R_k) + \sum_k \varepsilon_k^- f^-(R_k), \tag{64}$$

where the sum in (64) is over all linearity intervals of $\mathfrak{C}_\rho$. To prove (62) it suffices to show that $S_n \to S$ in distribution as $n \to \infty$. In fact, we may even suppose additionally that f^+ and f^- are Lipschitz, that is $|f^\pm(z_1) - f^\pm(z_2)| < L|z_1 - z_2|$ for some $L > 0$ and all $z_1, z_2 \in \mathbb{R}$. The first step is to localize the real roots of G_n under some "good" event. We use the same notation as in Section 4.1. Take $\kappa \in (0, 1/2)$ and recall that the random indices q_n' and q_n'' have been defined in (37). Define a random event E_n as in Lemma 4.1. Additionally, we will need another "good" event F_n. The next lemma states that it has probability close to 1.

LEMMA 6.1. *Consider a random event* $F_n = \{b_n R_{q_n' n} < -2A\} \cap \{b_n \times R_{(q_n''-1)n} > 2A\}$. *Then*, $\lim_{\kappa\downarrow 0} \liminf_{n\to\infty} \mathbb{P}[F_n] = 1$.

PROOF. Recall from Section 3 that $\mathfrak{M}$ is the space of locally finite counting measures on $[0, 1] \times (0, \infty]$ which do not charge the set $[0, 1] \times \{\infty\}$. Given $\mu \in \mathfrak{M}$

we denote by $[x_{q'}, x_{q'+1}]$ the unique linearity interval of the majorant $\mathfrak{C}_\mu$ such that $x_{q'} \le \kappa < x_{q'+1}$. Denote by $r_{q'}$ the negative of the slope of the corresponding segment of $\mathfrak{C}_\mu$. Define a map $T_\kappa : \mathfrak{M} \to \mathbb{R}$ by $T_\kappa(\mu) = r_{q'}$. Then, the same argument as in Lemma 3.2 shows that T_κ continuous on $\mathfrak{M}_1$; see (24). Applying Proposition 1.17 together with Proposition 3.1 and noting that $T_\kappa(\rho_n) = b_n R_{q'_n n}$ we obtain that for every $\kappa > 0$, $b_n R_{q'_n n} \to T_\kappa(\rho)$ in distribution as $n \to \infty$. By Proposition 1.7 we have $T_\kappa(\rho) \to -\infty$ a.s. as $\kappa \downarrow 0$. It follows easily that $\lim_{\kappa \downarrow 0} \liminf_{n\to\infty} \mathbb{P}[b_n R_{q'_n n} < -2A] = 1$. The statement of the lemma follows by symmetry. □

In the next lemma we will localize, under the event $E_n \cap F_n$, those real roots of G_n which are contained in $[-A, A]$. Recall that the vertices of the majorant of G_n are denoted (from left to right) by $(k_{in}, \log_+|\xi_{k_{in}}|)$, where $0 \le i \le d_n$ and $k_{0n} = 0$, $k_{d_n n} = n$. We already know that any linearity interval $[k_{in}, k_{(i+1)n}]$ of the majorant corresponds to a "circle" of *complex* roots of G_n located approximately at the same positions as the nonzero roots of the polynomial $\xi_{k_{in}} z^{k_{in}} + \xi_{k_{(i+1)n}} z^{k_{(i+1)n}}$. In order to localize the *real* roots of G_n we have to keep track of two things: the signs of the coefficients $\xi_{k_{in}}, \xi_{k_{(i+1)n}}$ and the parities of the indices $k_{in}, k_{(i+1)n}$. Write

$$\varepsilon^+_{in} = \mathbb{1}\{\operatorname{sgn}(\xi_{k_{in}}) \ne \operatorname{sgn}(\xi_{k_{(i+1)n}})\}, \tag{65}$$

$$\varepsilon^-_{in} = \mathbb{1}\{(-1)^{k_{in}} \operatorname{sgn}(\xi_{k_{in}}) \ne (-1)^{k_{(i+1)n}} \operatorname{sgn}(\xi_{k_{(i+1)n}})\}. \tag{66}$$

The next lemma shows that ε^+_{in} (resp., ε^-_{in}) is the indicator of the presence of a real root of G_n near $e^{R_{in}}$ (resp., $-e^{R_{in}}$).

LEMMA 6.2. *On the random event E_n the following holds: for every $q'_n \le i < q''_n$ such that $\varepsilon^+_{in} = 1$ (resp., $\varepsilon^-_{in} = 1$) there is exactly one positive (resp., negative) real root of G_n satisfying $|\log|z| - R_{in}| \le \exp(-n^{1/\alpha - 2\varepsilon})$. Moreover, if additionally F_n occurs, then all real roots of G_n satisfying $b_n \log|z| \in [-A, A]$ are among those described above.*

PROOF. We will use the notation of Lemma 4.2. Recall that on the event E_n for every $q'_n \le i < q''_n$ and every $1 \le m \le k_{(i+1)n} - k_{in}$ there is a unique complex root of G_n, denoted by $z_{i,m}(n)$, in the set $Z_{i,m}(n)$. Let $\varepsilon^+_{in} = 1$ for some $q'_n \le i < q''_n$. Then, $\varphi_{in} = 0$ in Lemma 4.2. Setting $m = k_{(i+1)n} - k_{in}$ we have that $z := z_{i,m}(n)$ satisfies $|\log|z| - R_{in}| < \delta_n$ and $|\arg z| < \delta_n$. Since the coefficients of G_n are real, the root z must in fact be real (and positive). Indeed, otherwise, we would have a pair complex conjugate roots (rather than a single root) in the set $Z_{i,m}(n)$. Similarly, if $\varepsilon^-_{in} = 1$ for some $q'_n \le i < q''_n$, then we have a real negative root of the form $z_{i,m}(n)$ for a suitable m. By Lemma 4.2 all real roots in the set $R_{q'_n n} - \delta_n \le \log|z| \le R_{(q''_n - 1)n} + \delta_n$ are of the above form. To complete the proof note that this set contains the set $-A \le b_n \log|z| \le A$ on the event F_n. □

The random variables S_n and S will be approximated by the random variables $S_n(\kappa)$ and $S(\kappa)$, defined by

$$S_n(\kappa) = \sum_{q'_n < i < q''_n - 1} \big(\varepsilon^+_{in} f^+(b_n R_{in}) + \varepsilon^-_{in} f^-(b_n R_{in})\big), \tag{67}$$

$$S(\kappa) = \sum_{q' < i < q'' - 1} \big(\varepsilon^+_{i} f^+(R_{i}) + \varepsilon^-_{i} f^-(R_{i})\big). \tag{68}$$

LEMMA 6.3. *On the random event $E_n \cap F_n$, we have $|S_n - S_n(\kappa)| < 1/n$.*

PROOF. Recall that f^+ and f^- are functions supported on $[-A, A]$ with Lipschitz constant at most L. By Lemma 6.2 and the definition of F_n, we have, on $E_n \cap F_n$,

$$\left|\sum_{z \in \mathcal{R}_n^+} f^+(b_n \log z) - \sum_{i=0}^{d_n - 1} \varepsilon^+_{in} f^+(b_n R_{in})\right| \le L d_n b_n \exp\big(-n^{1/\alpha - 2\varepsilon}\big) \le \frac{1}{2n}.$$

A similar inequality holds for the negative roots, and the statement follows. □

The next proposition determines the limiting structure of the coefficients of G_n together with attached signs and parities. Let $\tilde{\mathfrak{M}}$ be the space of locally finite counting measures on $[0, 1] \times (0, \infty] \times \{-1, 1\}^2$ which do not charge the set $[0, 1] \times \{\infty\} \times \{-1, 1\}^2$. We endow $\tilde{\mathfrak{M}}$ with the topology of vague convergence. Every element $\tilde{\mu} \in \tilde{\mathfrak{M}}$ can be written in the form $\tilde{\mu} = \sum_i \delta(u_i, v_i, \varsigma_i, \varpi_i)$, where $\mu = \sum_i \delta(u_i, v_i) \in \mathfrak{M}$ is the projection of $\tilde{\mu}$ on $\mathfrak{M}$ and $(\varsigma_i, \varpi_i) \in \{-1, 1\}^2$ is considered as a mark attached to the point (u_i, v_i). In the marks (ς_i, ϖ_i) we will record the signs of the coefficients of G_n and the parities of the corresponding indices.

PROPOSITION 6.4. *Let $\xi_0, \xi_1, \dots$ be i.i.d. random variables satisfying (6) and (12). Then the following convergence holds weakly on the space $\tilde{\mathfrak{M}}$:*

$$\tilde{\rho}_n := \sum_{k=0}^{n} \delta\left(\frac{k}{n}, \frac{\log|\xi_k|}{a_n}, \operatorname{sgn} \xi_k, (-1)^k\right) \xrightarrow[n\to\infty]{w} \sum_{i=1}^{\infty} \delta(U_i, V_i, \varsigma_i, \varpi_i) =: \tilde{\rho}. \tag{69}$$

Here, $\rho = \sum_{i=1}^{\infty} \delta(U_i, V_i)$ is a Poisson point process on $[0, 1] \times (0, \infty)$ with intensity $\alpha v^{-(\alpha+1)}\, du\, dv$ and independently, ς_i, ϖ_i are $\{-1, 1\}$-valued random variables with $\mathbb{P}[\varsigma_i = 1] = c$ and $\mathbb{P}[\varpi_i = 1] = 1/2$. Terms with $\log|\xi_k| \le 0$ are ignored.

PROOF. Write $\xi_k^+ = \xi_k \mathbb{1}_{\xi_k > 0}$ and $\xi_k^- = |\xi_k| \mathbb{1}_{\xi_k \le 0}$. Note that by (6), (7) and (12),

$$\mathbb{P}\left[\frac{\log \xi_k^+}{a_n} > t\right] \sim \frac{c}{n t^\alpha}, \qquad \mathbb{P}\left[\frac{\log \xi_k^-}{a_n} > t\right] \sim \frac{1 - c}{n t^\alpha}, \qquad n \to \infty.$$

Fix some $(\varsigma, \varpi) \in \{-1, 1\}^2$. We will consider only coefficients ξ_k with sign ς and parity ϖ. By Proposition 1.17 the point process

$$\tilde{\rho}_n(\varsigma, \varpi) := \sum_{k=0}^{n} \delta\left(\frac{k}{n}, \frac{\log|\xi_k|}{a_n}\right) \mathbb{1}\{\operatorname{sgn}(\xi_k) = \varsigma, (-1)^k = \varpi\}$$

converges weakly to the Poisson point process with intensity $(\alpha/2)cv^{-(\alpha+1)}\,du\,dv$ if $\varsigma = 1$ and $(\alpha/2)(1-c)v^{-(\alpha+1)}\,du\,dv$ if $\varsigma = -1$. Taking the union over all 4 choices of (ς, ϖ), we obtain the statement. $\square$

In order to pass from the convergence of the coefficients to the convergence of the point process of real roots we need a continuity argument. Consider $\tilde{\mu} \in \tilde{\mathfrak{M}}$ with a projection $\mu \in \mathfrak{M}$. We denote the vertices of the majorant of μ counted from left to right by (x_k, y_k). Denote by r_k the negative of the slope of the majorant of μ on the interval $[x_k, x_{k+1}]$. Let $\kappa \in (0, 1/2)$ be fixed and define indices q' and q'' by the conditions $x_{q'} \le \kappa < x_{q'+1}$ and $x_{q''-1} < 1 - \kappa \le x_{q''}$. For $q' < k < q''$ we denote by $(\sigma_k, \pi_k) \in \{-1, 1\}^2$ the mark attached to the vertex (x_k, y_k). Let $\tilde{\mathfrak{M}}_1$ be the set of all $\tilde{\mu} \in \tilde{\mathfrak{M}}$ such that $\mu \in \mathfrak{M}_1$, where $\mathfrak{M}_1 \subset \mathfrak{M}$ is defined as in Section 3. Let $\mathfrak{P}$ be the space of locally finite counting measures on $\mathbb{R}$ endowed with the topology of vague convergence. Define a map $\Phi_1 : \tilde{\mathfrak{M}} \to \mathfrak{P} \times \mathfrak{P}$ by

$$\Phi_1(\tilde{\mu}) = \left(\sum_{q'<k<q''-1} \mathbb{1}_{\sigma_k \ne \sigma_{k+1}} \delta(r_k), \sum_{q'<k<q''-1} \mathbb{1}_{\sigma_k \pi_k \ne \sigma_{k+1} \pi_{k+1}} \delta(r_k) \right).$$

LEMMA 6.5. *The map Φ_1 is continuous on $\tilde{\mathfrak{M}}_1$.*

PROOF. Let $\{\tilde{\mu}_n\}_{n\in\mathbb{N}} \subset \tilde{\mathfrak{M}}$ be a sequence converging vaguely to $\tilde{\mu} \in \tilde{\mathfrak{M}}_1$. This implies the vague convergence of the corresponding projections: $\mu_n \to \mu \in \mathfrak{M}_1$. Arguing as in the proof of Lemma 3.2 (and using the same notation) we arrive at the following conclusions. There exist points (x_{kn}, y_{kn}), $q' \le k \le q''$, which are vertices of the majorant of μ_n, such that $(x_{kn}, y_{kn}) \to (x_k, y_k)$ as $n \to \infty$. Further, $x_{q'n} < \kappa < x_{(q'+1)n}$ and $x_{(q''-1)n} < 1 - \kappa < x_{q''n}$ for sufficiently large n. Also, with the same notation as in (24), $r_{kn} \to r_k$ as $n \to \infty$. Finally, $\tilde{\mu}_n \to \tilde{\mu}$ implies that for sufficiently large n the mark (σ_{kn}, π_{kn}) attached to (x_{kn}, y_{kn}) is the same as the mark (σ_k, π_k) attached to (x_k, y_k), for all $q' \le k \le q''$. This implies that $\Phi_1(\tilde{\mu}_n) \to \Phi_1(\tilde{\mu})$ as $n \to \infty$, whence the continuity. $\square$

LEMMA 6.6. *We have $S_n(\kappa) \to S(\kappa)$ in distribution as $n \to \infty$.*

PROOF. By Proposition 6.4 we have $\tilde{\rho}_n \to \tilde{\rho}$ weakly on $\tilde{\mathfrak{M}}$. Define a map $I : \mathfrak{P} \times \mathfrak{P} \to \mathbb{R}$ by $I(\nu^+, \nu^-) = \int_{\mathbb{R}} f^+\,d\nu^+ + \int_{\mathbb{R}} f^-\,d\nu^-$. Clearly, I is continuous on $\mathfrak{M}_1$. By Lemma 6.5 the map $I \circ \Phi_1 : \tilde{\mathfrak{M}} \to \mathbb{R}$ is continuous. By Proposition 3.1

(which is applicable since $\mathbb{P}[\tilde{\rho} \in \tilde{\mathfrak{M}}_1] = 1$ for $\alpha \geq 1$) we have that $I(\Phi_1(\tilde{\rho}_n)) \to I(\Phi_1(\tilde{\rho}))$ in distribution. This is exactly what is stated in the lemma. □

The proof of Theorem 1.10 in the case $\alpha \geq 1$ can be completed as follows. Trivially, we have $S(\kappa) \to S$ a.s. as $\kappa \downarrow 0$. All the other assumptions of Lemma 3.5 have been verified above. Applying Lemma 3.5 we obtain $S_n \to S$ in distribution as $n \to \infty$.

6.3. *Proof in the case $\alpha \in (0, 1)$.* We will show that the weak convergence of point processes in (62) holds, this time on the space $E = [-\infty, +\infty] \times \{-1, 1\}$ with the restriction that n stays either even or odd and $\varepsilon_k^+, \varepsilon_k^-$ on the right-hand side of (62) is defined accordingly to this choice (see the boundary conditions in Section 6.1). Let $f^+, f^- : [-\infty, \infty] \to [0, \infty)$ be two continuous functions such that $|f^{\pm}(z_1) - f^{\pm}(z_2)| < L|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{R}$. With the same notation as in (63) and (64) it suffices to prove that $S_n \to S$ in distribution as $n \to \infty$. The next lemma localizes all real roots of G_n under a "good" event.

LEMMA 6.7. *On the random event E_n defined as in Lemma 4.5 the following holds: For every $0 \leq i < d_n$ such that $\varepsilon_{in}^+ = 1$ (resp., $\varepsilon_{in}^- = 1$) there is exactly one positive (resp., negative) real root z of G_n satisfying $|\log|z| - R'_{in}| \leq \exp(-n^{1/\alpha - 1 - 3\varepsilon})$. Moreover, there are no other real roots of G_n.*

PROOF. Follows from Lemma 4.8; see the proof of Lemma 6.2. □

Take $\kappa \in (0, 1/2)$, and define random variables $S_n(\kappa)$ and $S(\kappa)$ as in (67) and (68), but with summation over $q'_n \leq k < q''_n$ and $q' \leq k < q''$.

LEMMA 6.8. *On the random event E_n, we have $|S_n - S_n(\kappa)| < 1/\sqrt{n}$.*

PROOF. By Remark 4.7 we have $q'_n = 0$ and $q''_n = d_n$ on E_n. The rest follows from Lemma 6.7, the Lipschitz property of f^+ and f^- and (55). □

Again, we need a continuity argument to transform the convergence of the coefficients in Proposition 6.4 into the convergence of real roots. This time, we have to take care of the first and the last coefficients of the random polynomial G_n. Write $\mathfrak{K} = \tilde{\mathfrak{M}} \times \{-1, 1\}^2$. Every element of $\mathfrak{K}$ can be written in the form $(\tilde{\mu}, \sigma', \sigma'')$, where $\tilde{\mu} \in \tilde{\mathfrak{M}}$ and $(\sigma', \sigma'') \in \{-1, 1\}^2$. In σ' and σ'' we will record the signs of the first and the last coefficients of G_n. As above, the vertices of the majorant of μ counted from left to right are denoted by (x_k, y_k) and the indices q' and q'' are defined by the conditions $x_{q'} \leq \kappa < x_{q'+1}$ and $x_{q''-1} < 1 - \kappa \leq x_{q''}$. For $q' < k < q''$ (note the strict inequalities) we denote by $(\sigma_k, \pi_k) \in \{-1, 1\}^2$ the mark attached to the vertex (x_k, y_k). We will need the following boundary conditions: Define

$(\sigma_{q'}, \pi_{q'}) = (\sigma', 1)$ and put $(\sigma_{q''}, \pi_{q''}) = (\sigma'', 1)$ (if we are proving the convergence of Υ_{2n}) or $(\sigma_{q''}, \pi_{q''}) = (\sigma'', -1)$ (if we are proving the convergence of Υ_{2n+1}). Let $\mathfrak{K}_0$ be the set of all $(\tilde{\mu}, \sigma', \sigma'') \in \mathfrak{K}$ such that the projection μ of $\tilde{\mu}$ satisfies $\mu \in \mathfrak{M}_0$. Here, $\mathfrak{M}_0 \subset \mathfrak{M}$ is defined as in Section 3. Let $\mathfrak{Q}$ be the space of finite counting measures on $[-\infty, \infty]$ endowed with the topology of weak convergence. Define a map $\Phi_0 : \mathfrak{K} \to \mathfrak{Q} \times \mathfrak{Q}$ by

$$\Phi_0(\tilde{\mu}, \sigma', \sigma'') = \left(\sum_{k=q'}^{q''-1} \mathbb{1}_{\sigma_k \neq \sigma_{k+1}} \delta(r_k), \sum_{k=q'}^{q''-1} \mathbb{1}_{\sigma_k \pi_k \neq \sigma_{k+1} \pi_{k+1}} \delta(r_k) \right).$$

LEMMA 6.9. *The map Φ_0 is continuous on $\mathfrak{K}_0$.*

PROOF. Let $\{(\tilde{\mu}_n, \sigma'_n, \sigma''_n)\}_{n \in \mathbb{N}} \subset \mathfrak{K}$ be a sequence converging vaguely to $(\tilde{\mu}, \sigma', \sigma'') \in \mathfrak{K}_0$. This implies that for sufficiently large n, $\sigma'_n = \sigma'$ and $\sigma''_n = \sigma''$. Also, $\tilde{\mu}_n \to \tilde{\mu}$ vaguely. Consequently, we have the vague convergence of the corresponding projections: $\mu_n \to \mu$. As in the proof of Lemma 3.3 we obtain the following results. There exist points (x_{kn}, y_{kn}), $q' < k < q''$, which are vertices of the majorant of μ_n, such that $(x_{kn}, y_{kn}) \to (x_k, y_k)$ as $n \to \infty$. Also, $x_{q'n} < \kappa < x_{(q'+1)n}$ and $x_{(q''-1)n} < 1 - \kappa < x_{q''n}$ for sufficiently large n. Furthermore, with the same notation as in (24), $r_{kn} \to r_k$ as $n \to \infty$. It follows from $\tilde{\mu}_n \to \tilde{\mu}$ that for sufficiently large n the mark (σ_{kn}, π_{kn}) attached to (x_{kn}, y_{kn}) is the same as the mark (σ_k, π_k) attached to (x_k, y_k) for all $q' < k < q''$. The same statement holds for $k = q'$ and $k = q''$ by the boundary conditions. This implies that $\Phi_0(\tilde{\mu}_n, \sigma'_n, \sigma''_n) \to \Phi_0(\tilde{\mu}, \sigma', \sigma'')$ as $n \to \infty$. □

LEMMA 6.10. *We have $S_n(\kappa) \to S(\kappa)$ in distribution as $n \to \infty$.*

PROOF. By Proposition 6.4 we have $\tilde{\rho}_n \to \tilde{\rho}$ weakly on $\tilde{\mathfrak{M}}$. The sum in (69) can be taken from 1 to $n - 1$. Consequently, $(\tilde{\rho}_n, \operatorname{sgn} \xi_0, \operatorname{sgn} \xi_n)$ converges weakly, as a random element in $\mathfrak{K}$, to $(\tilde{\rho}, \sigma', \sigma'')$, where σ' and σ'' are independent (and independent of $\tilde{\rho}$) $\{-1, 1\}$-valued random variables with the same distribution as $\operatorname{sgn} \xi_0$. By Lemma 6.9 and Proposition 3.1 (which is applicable since $\mathbb{P}[(\tilde{\rho}, \sigma', \sigma'') \in \mathfrak{K}_0] = 1$ for $\alpha \in (0, 1)$) we have that $\Phi_0(\tilde{\rho}_n, \operatorname{sgn} \xi_0, \operatorname{sgn} \xi_n)$ converges, as a random element in $\mathfrak{Q} \times \mathfrak{Q}$, to $\Phi_0(\tilde{\rho}, \sigma', \sigma'')$ as $n \to \infty$. Taking the integrals of f^+ and f^- over the components of $\Phi_0(\tilde{\rho}_n, \operatorname{sgn} \xi_0, \operatorname{sgn} \xi_n)$ and $\Phi_0(\tilde{\rho}, \sigma', \sigma'')$, we arrive at the statement of the lemma. □

The proof of Theorem 1.10 in the case $\alpha \in (0, 1)$ can be completed as follows. Trivially, we have $S(\kappa) \to S$ a.s. as $\kappa \downarrow 0$. All the other assumptions of Lemma 3.5 have been verified above. Applying Lemma 3.5, we obtain $S_n \to S$ in distribution as $n \to \infty$. The proof is complete.

6.4. *Proof of Theorem* 1.14. It follows from the proof of Theorem 1.5 that on the event E_n defined as in Lemma 5.1, the number of real roots of G_n is the same as the number of real solution of the equation

$$(\xi_{\tau_n} z^{\tau_n} + \xi_0)(\xi_n z^{n-\tau_n} + \xi_{\tau_n}) = 0. \tag{70}$$

The number of real solutions of (70) depends on whether the numbers $0, \tau_n, n$ are even or odd and on whether the coefficients $\xi_0, \xi_{\tau_n}, \xi_n$ are positive or negative. It is not difficult to show that $(-1)^{\tau_n}$ and $\operatorname{sgn} \xi_{\tau_n}$ become asymptotically independent and that $\mathbb{P}[(-1)^{\tau_n} = 1] \to 1/2$ and $\mathbb{P}[\operatorname{sgn} \xi_{\tau_n} = 1] \to c$ as $n \to \infty$. Considering all possible cases leads to (13) and (14).

7. Proofs of Theorems 1.8 and 1.9.

7.1. *Proof of Theorem* 1.8. Let ρ be a Poisson point process with intensity $\nu(du\,dv) = \alpha v^{-(\alpha+1)}\,du\,dv$ on $E = [0, 1] \times (0, \infty)$, where $\alpha \in (0, 1)$. We are going to compute the expectation of L_α, the number of segments of the least concave majorant of ρ. Denote by $\rho^2_{\neq}$ the set of all ordered pairs of distinct atoms of the point process ρ. For $P_1, P_2 \in E$ consider an indicator function $f_\rho(P_1, P_2)$ taking value 1 if and only if there are no points of the Poisson process ρ lying above the line passing through P_1 and P_2. Counting the first and the last segments of the majorant of ρ separately, we have $\mathbb{E}L_\alpha = 2 + I_\alpha/2$, where

$$I_\alpha = \mathbb{E}\Bigg[\sum_{(P_1,P_2)\in\rho^2_{\neq}} f_\rho(P_1, P_2)\Bigg].$$

In the sequel we compute I_α. Applying the Slyvnyack–Mecke formula (see, e.g., [14], Corollary 3.2.3), we obtain

$$I_\alpha = \int_{E^2} \mathbb{E}[f_\rho(P_1, P_2)]\nu(dP_1)\nu(dP_2).$$

Denoting $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, we have

$$I_\alpha = \alpha^2 \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \mathbb{E}[f_\rho(P_1, P_2)] y_1^{-\alpha-1} y_2^{-\alpha-1}\,dx_1\,dx_2\,dy_1\,dy_2.$$

The probability of the event that there are no points of ρ lying above the line P_1P_2 is nonzero only if the line P_1P_2 intersects both vertical sides of the boundary of E. Therefore,

$$I_\alpha = 2\alpha^2 \int_X \int_Y \mathbb{E}[f_\rho(P_1, P_2)] y_1^{-\alpha-1} y_2^{-\alpha-1}\,dy_1\,dy_2\,dx_1\,dx_2,$$

where $X = \{(x_1, x_2) : 0 < x_1 < x_2 < 1\}$, and $Y = Y_{x_1,x_2}$ is a set defined by

$$Y = \{(y_1, y_2) \in (0, \infty)^2 : y_1x_2 - y_2x_1 > 0, y_2 - y_1 + y_1x_2 - y_2x_1 > 0\}.$$

Let us replace the variables y_1, y_2 by

$$r = -\frac{y_2 - y_1}{x_2 - x_1}, \qquad u = 1 + \frac{y_2 - y_1}{y_1 x_2 - y_2 x_1}.$$

Then, $(y_1, y_2) \in Y$ if and only if $(r, u) \in (-\infty, 0) \times (1, \infty)$ or $(r, u) \in (0, \infty) \times (0, 1)$. The inverse transformation is given by

$$y_1 = r\left(\frac{1}{1-u} - x_1\right), \qquad y_2 = r\left(\frac{1}{1-u} - x_2\right).$$

The Jacobian determinant of the transformation $(r, u) \mapsto (y_1, y_2)$ is equal to $r(x_2 - x_1)/(1-u)^2$. Write $\tilde{f}_\rho(u, r) = f_\rho((x_1, y_1(u, r)), (x_2, y_2(u, r)))$. By symmetry, we can consider only the case $r > 0$, $u \in (0, 1)$. Indeed, considering the case $r > 0$ means that we restrict ourselves to segments of the majorant with positive slope. By a change of variables formula,

$$\begin{aligned} I_\alpha = 4\alpha^2 \int_0^\infty \int_0^1 \int_X \mathbb{E}[\tilde{f}_\rho(u, r)] \\ \times r^{-2\alpha-1}\left(\frac{1}{1-u} - x_1\right)^{-\alpha-1}\left(\frac{1}{1-u} - x_2\right)^{-\alpha-1} \\ \times \frac{x_2 - x_1}{(1-u)^2}\, dx_1\, dx_2\, du\, dr. \end{aligned}$$

Further, by definition of the Poisson process,

$$\begin{aligned} \mathbb{E}[\tilde{f}_\rho(u, r)] &= \exp\left(-\int_{\{(x,y)\in E:\, y \ge -rx + r/(1-u)\}} \alpha y^{-(\alpha+1)}\, dy\, dx\right) \\ &= \exp\left(-\int_0^1 \left(-rx + \frac{r}{1-u}\right)^{-\alpha} dx\right) \\ &= \exp\left(-\frac{r^{-\alpha}}{(1-\alpha)} \frac{1 - u^{1-\alpha}}{(1-u)^{1-\alpha}}\right). \end{aligned} \tag{71}$$

The integral $J := \int_X (c - x_1)^\beta (c - x_2)^\beta (x_2 - x_1)\, dx_1\, dx_2$, where $c > 1$, can be evaluated by writing $(x_2 - x_1) = (c - x_1) - (c - x_2)$. We obtain

$$J = \begin{cases} \dfrac{c^{2\beta+3} - (c-1)^{2\beta+3} - (2\beta+3)c^{\beta+1}(c-1)^{\beta+1}}{(\beta+1)(\beta+2)(2\beta+3)}, & \text{if } \beta \ne -1, -3/2, -2, \\ -4\ln\left(\dfrac{c}{c-1}\right) + \dfrac{4}{\sqrt{c(c-1)}}, & \text{if } \beta = -3/2. \end{cases} \tag{72}$$

In the case $\alpha \neq 1/2$, we apply (71) and (72) to obtain

$$I_\alpha = \frac{4\alpha}{(1-\alpha)(2\alpha-1)} \times \int_0^\infty \int_0^1 r^{-2\alpha-1} \exp\left(-\frac{r^{-\alpha}}{(1-\alpha)} \frac{1-u^{1-\alpha}}{(1-u)^{1-\alpha}}\right) \times (1-u)^{2\alpha-3}\left[1-u^{1-2\alpha}-(1-2\alpha)u^{-\alpha}(1-u)\right] du\, dr.$$

In the case $\alpha = 1/2$ we get, combining (71) with (72),

$$I_\alpha = 4\int_0^\infty \int_0^1 r^{-2} \exp\left(-2r^{-1/2}\frac{1-u^{1/2}}{(1-u)^{1/2}}\right)(1-u)^{-2}\left[u^{-1/2}(1-u)+\ln u\right] du\, dr.$$

Applying in both cases the formula $\int_0^\infty r^{-2\alpha-1}e^{-cr^{-\alpha}}\, dr = (c^2\alpha)^{-1}$, we arrive at

$$\mathbb{E}L_\alpha = \begin{cases} 2 + \dfrac{2(1-\alpha)}{(2\alpha-1)} \displaystyle\int_0^1 \frac{1-u^{1-2\alpha}-(1-2\alpha)u^{-\alpha}(1-u)}{(1-u)(1-u^{1-\alpha})^2}\, du, & \text{if } \alpha \neq 1/2, \\ 2 + \displaystyle\int_0^1 \frac{u^{-1/2}(1-u)+\ln u}{(1-u)(1-u^{1/2})^2}\, du, & \text{if } \alpha = 1/2. \end{cases} \tag{73}$$

REMARK 7.1. The second line is just the limit of the first line as $\alpha \to 1/2$, so that $\mathbb{E}L_\alpha$ depends on α continuously. If $\alpha = p/q \neq 1/2$ is rational, then the substitution $v = u^{1/q}$ reduces the integral in (73) to an integral of a rational function which can be computed in closed form; see the table in Section 1.3. Numerical computation suggests that $\mathbb{E}L_\alpha$ is increasing in $\alpha \in (0, 1)$.

In the rest of the proof we compute the integral on the right-hand side of (73) in terms of the Barnes modular constant. Let

$$K_\alpha = \int_0^1 \frac{1-u^{1-2\alpha}-(1-2\alpha)u^{-\alpha}(1-u)}{(1-u)(1-u^{1-\alpha})^2}\, du.$$

Write $\beta = 1-\alpha$. Recall that $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Gamma function. Using the geometric series $\frac{1}{1-u} = \sum_{n=0}^\infty u^n$ and the formula $\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^\infty (\frac{1}{n} - \frac{1}{z+n})$ (see [2], Section 1.7) we obtain that for every $m > 0$,

$$\begin{aligned} \int_0^1 u^{m\beta}\frac{1-u^{1-2\alpha}}{1-u}\, du &= \int_0^1 \sum_{n=0}^\infty u^{n+m\beta}\left(1-u^{1-2\alpha}\right) du \\ &= \sum_{n=1}^\infty \left(\frac{1}{n+m\beta} - \frac{1}{n+(m+2)\beta}\right) - \frac{1}{(m+2)\beta} \\ &= \psi\big((m+2)\beta\big) - \psi(m\beta) - \frac{1}{m\beta}. \end{aligned}$$

For $m = 0$ the value of the integral is $\psi(2\beta) + \gamma$, where $\gamma = -\psi(1)$ is the Euler–Mascheroni constant; see [2], Section 1.7.2. Using the expansion $\frac{1}{(1-u)^2} = \sum_{m=0}^{\infty}(m+1)u^m$ we obtain that $K_\alpha = \lim_{N\to\infty} S_N$, where

$$\begin{aligned} S_N &= \sum_{m=1}^{N}(m+1)\biggl(\psi\bigl((m+2)\beta\bigr) - \psi(m\beta) - \frac{1}{m\beta}\biggr) \\ &\quad - (N+1)\frac{1-2\alpha}{1-\alpha} + \psi(2\beta) + \gamma \\ &= -2\sum_{m=1}^{N}\psi(m\beta) + (N+1)\psi\bigl((N+2)\beta\bigr) + N\psi\bigl((N+1)\beta\bigr) - 2N \\ &\quad - \sum_{m=1}^{N}\frac{1}{m\beta} + \gamma - \frac{1-2\alpha}{1-\alpha}. \end{aligned}$$

The second equality follows by an elementary transformation of the telescopic sum. Using the asymptotic expansion $\psi(z) = \log z - \frac{1}{2z} + o(\frac{1}{z})$ as $z \to \infty$, we obtain

$$S_N = -2\sum_{m=1}^{N}\psi(m\beta) + (2N+1)\log(\beta N) - 2N - \frac{1}{\beta}\log N + 1 - \frac{\alpha\gamma}{1-\alpha} + o(1).$$

Comparing this with (10) yields

$$K_\alpha = 1 - 2C(1-\alpha) + \frac{\log(1-\alpha)}{1-\alpha} - \frac{\alpha\gamma}{1-\alpha}.$$

The proof of Theorem 1.8 is completed by inserting this into (73).

7.2. *Proof of Theorem* 1.9. We prove that $\mathbb{P}[L_\alpha = 2] = 1 - \alpha$. For a point $P \in E = [0,1] \times (0,\infty)$ let $g_\rho(P)$ be the indicator of the following event: there are no atoms of ρ above the lines joining P to the points $(0,0)$ and $(1,0)$. Then

$$\mathbb{P}[L_\alpha = 2] = \mathbb{E}\biggl[\sum_{P\in\operatorname{supp}\rho} g_\rho(P)\biggr].$$

By the Slivnyak–Mecke formula [14], Corollary 3.2.3,

$$\mathbb{P}[L_\alpha = 2] = \int_E \mathbb{E}\bigl[g_\rho(P)\bigr]\nu(dP) = \alpha\int_0^1\int_0^\infty \mathbb{E}\bigl[g_\rho(x,y)\bigr]y^{-(\alpha+1)}\,dy\,dx. \tag{74}$$

The intensity of the Poisson process ρ integrated over the set $\{(u,v) \in E : u \in [0,x], v > yu/x\}$ is

$$\int_0^x\int_{yu/x}^\infty \alpha v^{-(\alpha+1)}\,du\,dv = \int_0^x\biggl(\frac{yu}{x}\biggr)^{-\alpha}du = \frac{1}{1-\alpha}xy^{-\alpha}.$$

By symmetry, the intensity of ρ integrated over the set $\{(u,v)\in E : u\in[x,1], v > y(u-1)/(x-1)\}$ is $\frac{1}{1-\alpha}(1-x)y^{-\alpha}$. It follows that

$$\mathbb{E}[g_\rho(x,y)] = \exp\left(-\frac{1}{(1-\alpha)y^\alpha}\right).$$

Inserting this into (74) we obtain $\mathbb{P}[L_\alpha = 2] = 1-\alpha$.

REFERENCES

[1] BARNES, E. W. (1899). The genesis of the double Gamma functions. *Proc. Lond. Math. Soc.* **31** 358–381.

[2] BATEMAN, H. and ERDÉLYI, A. (1981). *Higher Transcendental Functions. Vol. I.* Krieger, Melbourne.

[3] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[4] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR0898871

[5] BORDENAVE, C., CAPUTO, P. and CHAFAÏ, D. (2011). Spectrum of non-Hermitian heavy tailed random matrices. *Comm. Math. Phys.* **307** 513–560. MR2837123

[6] DARLING, D. A. (1952). The influence of the maximum term in the addition of independent random variables. *Trans. Amer. Math. Soc.* **73** 95–107. MR0048726

[7] GÖTZE, F. and ZAPOROZHETS, D. N. (2011). On the distribution of complex roots of random polynomials with heavy-tailed coefficients. *Teor. Veroyatn. Primen.* **56** 812–818.

[8] HARDY, G. H. (1905). On the zeroes certain classes of integral Taylor series. Part I. On the integral function $\sum_{n=0}^{\infty}\frac{x^{\phi(n)}}{\{\phi(n)\}!}$. *Proc. Lond. Math. Soc.* **s2-2** 332–339. MR1577279

[9] IBRAGIMOV, I. and ZEITOUNI, O. (1997). On roots of random polynomials. *Trans. Amer. Math. Soc.* **349** 2427–2441. MR1390040

[10] IBRAGIMOV, I. A. and ZAPOROZHETS, D. N. (2013). On distribution of zeros of random polynomials in complex plane. In *Prokhorov and Contemporary Probability Theory* (A. N. Shiryaev, S. R. S. Varadhan and E. L. Presman, eds.). *Springer Proceedings in Mathematics & Statistics*. **33** 303–324. Springer, Berlin.

[11] KOBLITZ, N. (1984). *p-Adic Numbers, p-Adic Analysis, and Zeta-Functions*, 2nd ed. *Graduate Texts in Mathematics* **58**. Springer, New York. MR0754003

[12] MAJUMDAR, S. N., COMTET, A. and RANDON-FURLING, J. (2010). Random convex hulls and extreme value statistics. *J. Stat. Phys.* **138** 955–1009. MR2601420

[13] RESNICK, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes. Applied Probability. A Series of the Applied Probability Trust* **4**. Springer, New York. MR0900810

[14] SCHNEIDER, R. and WEIL, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin. MR2455326

[15] SHEPP, L. and FARAHMAND, K. (2011). Expected number of real zeros of a random polynomial with independent identically distributed symmetric long-tailed coefficients. *Theory Probab. Appl.* **55** 173–181.

[16] SHEPP, L. A. and VANDERBEI, R. J. (1995). The complex zeros of random polynomials. *Trans. Amer. Math. Soc.* **347** 4365–4384. MR1308023

[17] VALIRON, G. (1923). *Lectures on the General Theory of Integral Functions*. Chelsea, New York.

[18] ŠPARO, D. I. and ŠUR, M. G. (1962). On the distribution of roots of random polynomials. *Vestnik Moskov. Univ. Ser. I Mat. Meh.* **1962** 40–43. MR0139199

[19] ZAPOROZHETS, D. N. (2006). An example of a random polynomial with unusual behavior of the roots. *Theory Probab. Appl.* **50** 529–535.

[20] ZAPOROZHETS, D. N. and NAZAROV, A. I. (2009). What is the least expected number of roots of a random polynomial? *Theory Probab. Appl.* **53** 117–133.

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