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Limit theory for some positive stationary processes with infinite mean

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Abstract. We prove stable limit theorems and one-sided laws of the iterated logarithm for a class of positive, mixing, stationary, stochastic processes which contains those obtained from nonintegrable observables over certain piecewise expanding maps. This is done by extending Darling–Kac theory to a suitable family of infinite measure preserving transformations.

Résumé. Nous prouvons des théorèmes limites et des lois du logarithme itéré unilatérales pour une classe de processus stochastiques positifs, mélangeants et stationnaires. Cette classe contient en particulier les processus obtenus par des observables nonintégrables de certaines applications dilatantes. Ceci est obtenu en généralisant la théorie de Darling–Kac à une famille appropriée de transformations préservant la mesure.

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Overview

We prove limit theorems for positive, stationary, processes with infinite mean satisfying mixing conditions which occur naturally in certain dynamical systems: *stable limit theorems* for certain ϑ_μ-mixing processes and *one-sided laws of the iterated logarithm* for certain ψ^*-mixing processes (definitions below).

The method of proof is by *inversion* which is done by first building a *Kakutani tower* over the generating probability preserving transformation, using the time zero observation as height function.

The mixing properties of the stationary process ensure that the resulting infinite measure preserving transformation is *weakly pointwise dual ergodic*, which allows us to develop a generalized *Darling–Kac theory* for ergodic sums of this system. The results for the original stochastic process then follow by a standard inversion argument.

We illustrate both the finite-measure and the infinite-measure results by applying them to certain one-dimensional dynamical systems.

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1. Definitions and preliminaries

Stationary processes

We are going to consider partial sums of ergodic $\mathbb{R}_+$-valued stationary processes $(\xi_n)_{n\geq 0}$ with $\mathbb{E}(\xi_n) = \infty$. Such a process can always be represented as $\xi_n = \varphi \circ S^n$, where S is a measure preserving transformation on a probability space $(\Omega, \mathcal{A}, P)$, and $\varphi : \Omega \to \mathbb{R}_+$ is measurable with $\mathbb{E}(\varphi) = \infty$ (and w.l.o.g. $\mathcal{A} = \sigma\{\varphi \circ S^n\colon\ n \geq 0\}$). Due to nonintegrability, it will suffice to restrict attention to the $\mathbb{N}$-valued case, as by the ergodic theorem the partial sums of the fractional parts are asymptotically negligible compared to the partial sums of the process. Henceforth, $\varphi : \Omega \to \mathbb{N}$, and we let $\alpha = \alpha_\varphi = \{[\varphi = l]\colon\ l \in \mathbb{N}\}$. This gives a *probability preserving fibred system* $(\Omega, \mathcal{A}, P, S, \alpha)$ in the sense of the following definition.

Nonsingular transformations and fibred systems

A measurable map S on a σ-finite space $(\Omega, \mathcal{A}, m)$ is called *nonsingular* if $m \circ S^{-1} \ll m$. Its *transfer operator* (with respect to m) is the positive linear map $\widehat{S} : L^1(m) \to L^1(m)$ defined by

$$\int_A \widehat{S} f \,\mathrm{d}m = \int_{S^{-1}A} f \,\mathrm{d}m \quad \bigl(f \in L^1(m), A \in \mathcal{A}\bigr).$$

A *fibred* or *piecewise invertible system* is a quintuple $(\Omega, \mathcal{A}, m, S, \alpha)$ where S is a nonsingular transformation on $(\Omega, \mathcal{A}, m)$, and $\alpha \subset \mathcal{A}$ is a countable, unilateral generator so that the restriction $S : a \to Sa$ is invertible, nonsingular on each $a \in \alpha$. In this case, for every $k \geq 1$, $(\Omega, \mathcal{A}, m, S^k, \alpha_k)$ is a fibred system, where $\alpha_k := \bigvee_{j=0}^{k-1} S^{-j}\alpha$.

The transfer operator of $(\Omega, \mathcal{A}, m, S, \alpha)$ can be represented as

$$\widehat{S} f = \sum_{a \in \alpha} 1_{Sa} v_a' (f \circ v_a),$$

where $v_a : Sa \to a$ denotes the inverse of $S : a \to Sa$, and $v_a' := \frac{\mathrm{d}m \circ v_a}{\mathrm{d}m}$.

If m actually is an S-invariant probability measure, the system is called *probability preserving*, and we write $P := m$.

Mixing

Let $\mathcal{P}(\Omega, \mathcal{A})$ denote the collection of probability measures on $(\Omega, \mathcal{A})$, and call $\mu \in \mathcal{P}(\Omega, \mathcal{A})$ *equivalent* to P, $\mu \sim P$, if $\mu \ll P \ll \mu$. Various mixing conditions for a probability preserving fibred system $(\Omega, \mathcal{A}, P, S, \alpha)$ are defined in terms of the asymptotics of certain *mixing coefficients*, all defined for $n \geq 1$ (and only using pairs of sets for which the denominator is nonzero). The system is called

- ϑ_μ*-mixing* (for some $\mu \in \mathcal{P}(\Omega, \mathcal{A})$ with $\mu \sim P$) if $\vartheta_\mu(n) \to 0$, where

$$\vartheta_\mu(n) := \sup\left\{\frac{|P(A \cap S^{-(n+k)}B) - P(A)P(B)|}{\mu(B)}\colon\ k \geq 1, A \in \sigma(\alpha_k), B \in \mathcal{A}\right\};$$

- ψ^**-mixing* if $\psi^*(n) \to 1$, where

$$\psi^*(n) := \sup\left\{\frac{P(A \cap S^{-(n+k)}B)}{P(A)P(B)}\colon\ k \geq 1, A \in \sigma(\alpha_k), B \in \mathcal{A}\right\};$$

- ψ*-mixing* if $\psi(n) \to 0$, where

$$\psi(n) := \sup\left\{\frac{|P(A \cap S^{-(n+k)}B) - P(A)P(B)|}{P(A)P(B)}\colon\ k \geq 1, A \in \sigma(\alpha_k), B \in \mathcal{A}\right\};$$

and *continued fraction mixing* if, in addition to ψ-mixing, $\psi(1) < \infty$.

Remark 1. (a) *The notion of ϑ_μ-mixing generalizes that of* reverse ϕ-mixing *which requires that $\phi_-(n) \to 0$, where*

$$\phi_-(n) := \sup\left\{\frac{|P(A \cap S^{-(n+k)}B) - P(A)P(B)|}{P(B)}: k \geq 1, A \in \sigma(\alpha_k), B \in \mathcal{A}\right\},$$

so that $\vartheta_P(n) = \phi_-(n)$ for all n.

(b) *As shown in* [11], *$\psi^*(1) < \infty$ implies ψ^*-mixing. Elementary computation shows that $\phi_-(n) \leq \psi^*(n) - 1$, hence ψ^*-mixing entails reverse ϕ-mixing, that is, ϑ_P-mixing.*

(c) *It is immediate that ψ-mixing implies ψ^*-mixing. Moreover, note that $\psi^*(1) \leq 1 + \psi(1)$. In view of* (a), *$\psi(1) < \infty$ also implies ψ^*-mixing.*

(d) *For examples with $\psi^*(1) < \infty$ which are not ψ-mixing, see chapter* 5 *in* [12].

(e) *In Section* 7 *below, we consider a class of interval maps* (*weakly mixing Rychlik-maps*) *for which $\vartheta_\mu(n) \to 0$ exponentially fast* (*as shown in* [8]). *We prove that, under some natural extra assumptions, $\psi^*(1) < \infty$ implies continued fraction mixing in this setup.*

Strong distributional convergence and limit laws

For $(X, \mathcal{B}, m)$ a σ-finite measure space, $F_n : X \to [0, \infty]$ measurable, and Y a random variable taking values in $[0, \infty]$, we say that (F_n) *converges strongly in distribution to Y*, written

$$F_n \underset{n\to\infty}{\overset{\mathfrak{d}}{\longrightarrow}} Y,$$

if it converges in law with respect to all absolutely continuous probabilities, that is, if for all continuous (hence bounded) $g : [0, \infty] \to \mathbb{R}$ and all $P \in \mathcal{P}(X, \mathcal{B})$ with $P \ll m$,

$$\int_X g(F_n)\,\mathrm{d}P \underset{n\to\infty}{\longrightarrow} \mathbb{E}\big(g(Y)\big).$$

For $\gamma \in [0, 1]$ we let $Y_\gamma \geq 0$ denote a random variable which has the *normalized Mittag-Leffler distribution of order γ*, that is, $\mathbb{E}(Y_\gamma^p) = \frac{p!(\Gamma(1+\gamma))^p}{\Gamma(1+p\gamma)}$ for $p \geq 0$. Evidently $Y_1 \equiv 1$, and Y_0 has exponential distribution. Also, $Y_{1/2}$ is the absolute value of a centered Gaussian random variable.

For $\gamma \in (0, 1]$, the variable $Z_\gamma := Y^{-1/\gamma}$ then has a *positive γ-stable distribution* with $\mathbb{E}(\mathrm{e}^{-tZ_\gamma}) = \exp(-\Gamma(1+\gamma)t^\gamma)$ for $t > 0$.

2. Results on stationary processes

Recall that a function $a : (L, \infty) \to (0, \infty)$ is *regularly varying of index $\rho \in \mathbb{R}$* if it is measurable and $a(ct)/a(t) \to c^\rho$ as $t \to \infty$ for all $c > 0$. In case $\rho > 0$, such an a has an *asymptotically inverse function b* (uniquely determined up to asymptotic equivalence), meaning that $a(b(t)) \sim b(a(t)) \sim t$ as $t \to \infty$, and b is regularly varying of index $1/\rho$. These concepts are extended to sequences $(a(n))$ by interpreting them as functions on $\mathbb{R}_+$ via $t \longmapsto a(\lfloor t \rfloor)$ (which we shall do without further mention).

In the statements below, $(\Omega, \mathcal{A}, P, S, \alpha)$ is a probability preserving fibred system, and $\varphi : \Omega \to \mathbb{N}$ is α-measurable. We let

$$\varphi_n := \sum_{k=0}^{n-1} \varphi \circ S^k, \quad n \geq 0,$$

denote the partial sums of the stationary process $(\xi_n)_{n\geq 0} = (\varphi \circ S^n)_{n\geq 0}$, and define

$$a_\varphi(n) := \sum_{k=1}^{n} P\big([\varphi_k \leq n]\big), \quad n \geq 0.$$

In order to establish our results, we will need to assume that the growth of $a_\varphi(n)$ is adapted to the decay of the mixing coefficients of the process. The main condition is as follows although we will use a stronger version (2.6) in Theorem 2.3.

Definition of adaptedness

Let $\tau(n) \downarrow 0$. We will say that the sequence $(a(n))_{n\in\mathbb{N}}$ in $(0,\infty)$ is *adapted to* $(\tau(n))_{n\in\mathbb{N}}$ if

$$\frac{n\tau(\delta a(n))}{a(n)} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for all } \delta > 0. \tag{2.1}$$

Observe that in this case any sequence asymptotically equivalent to $(a(n))$ is adapted, too. Note also that if $(a(n))$ is regularly varying with positive index, then it is adapted to $(\tau(n))$ as soon as (2.1) holds for one $\delta > 0$.

Our first result is a distributional limit theorem. In the barely infinite measure case ($\gamma = 1$) it comes with an associated a.e. result. For $\gamma \in (0,1)$, corresponding statements will be established under stronger assumptions in Theorem 2.3 below.

Theorem 2.1. *Suppose that $(\Omega, \mathcal{A}, P, S, \alpha)$ is a ϑ_μ-mixing probability preserving fibred system, and that $\varphi : \Omega \to \mathbb{N}$ is α-measurable.*

(a) (Stable limit theorem) *Let $(a(n)) = (a_\varphi(n))$ be γ-regularly varying for some $\gamma \in (0,1]$, and assume that it is adapted to $(\vartheta_\mu(n))$. Then,*

$$\frac{\varphi_n}{b(n)} \underset{n\to\infty}{\overset{\mathfrak{d}}{\longrightarrow}} Z_\gamma, \tag{2.2}$$

where b is asymptotically inverse to a, $b(a(n)) \sim a(b(n)) \sim n$.

(b) (One-sided law of the iterated logarithm for $\gamma = 1$) *If, in addition, $\gamma = 1$ and $b(n/\log\log n)\log\log n \sim b(n)$, then*

$$\underline{\lim}_{n\to\infty} \frac{\varphi_n}{b(n)} = 1 \quad a.s. \tag{2.3}$$

Remark 2. (a) *The conclusion of Theorem* 2.1(a) *was established for certain ϕ-mixing processes in* ([17], *Corollary* 5.10) *and for continued fraction mixing processes in* [13] (*see also* [3]).

(b) *A functional version of* (a) *is also valid, and can be proved using a straightforward, appropriate adaptation of* [9].

(c) *An analogue of Theorem* 2.1(b) *for ψ-mixing processes was established in* [4].

The results mentioned in Remark 2(a) also compute the $a_\varphi(n)$ from the marginal distributions, for which additional "*close correlation*" assumptions such as $\psi^*(1) < \infty$ are required. We now show how to determine the asymptotics of $a_\varphi(n)$ from the marginal distributions under the weaker close correlation condition (2.4) (but we still use the stronger $\psi^*(1) < \infty$ in Theorem 2.3 below).

Theorem 2.2 (Identifying the normalization). *Let $(\Omega, \mathcal{A}, P, S, \alpha)$ be a ϑ_μ-mixing probability preserving fibred system. Assume that $\varphi : \Omega \to \mathbb{N}$ is α-measurable, and that there exists some $\Phi \in L^1(P)_+$ such that*

$$\widehat{S}(\varphi \wedge n) \le \mathbb{E}(\varphi \wedge n)\Phi \quad \forall n \ge 1. \tag{2.4}$$

Assume that $(\tilde{a}(n))$ is regularly varying with index $\gamma \in (0,1]$, and adapted to $(\vartheta_\mu(n))$, then

$$\mathbb{E}(\varphi \wedge n) \underset{n\to\infty}{\sim} \frac{n}{\Gamma(2-\gamma)\Gamma(1+\gamma)\tilde{a}(n)} \tag{2.5}$$

implies

$$a_\varphi(n) \underset{n\to\infty}{\sim} \tilde{a}(n).$$

Finally, replacing adaptedness (2.1) to $(\vartheta_\mu(n))$ by the stronger assumption (2.6) involving $a(a(n))$ below, we establish the following pointwise result.

Theorem 2.3 (The one-sided law of the iterated logarithm). *Suppose that $(\Omega, \mathcal{A}, P, S, \alpha)$ is a ψ^*-mixing probability preserving fibred system with $\psi^*(1) < \infty$, and that $\varphi : \Omega \to \mathbb{N}$ is α-measurable.*

If $(a(n)) = (a_\varphi(n))$ is γ-regularly varying for some $\gamma \in (0, 1)$ with asymptotic inverse $b(n)$, and if

$$\frac{n\vartheta_P(\delta a(a(n)))}{a(n)} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for some } \delta > 0, \tag{2.6}$$

then, letting $C_\gamma := K_\gamma^{-1/\gamma}$ with $K_\gamma := \frac{\Gamma(1+\gamma)}{\gamma^\gamma(1-\gamma)^{1-\gamma}}$, we have

$$\varliminf_{n\to\infty} \frac{\varphi_n}{b(n/\log\log(n))\log\log(n)} = C_\gamma \quad a.s. \tag{2.7}$$

Moreover, if $(\tau(n))$ is any sequence in $(0, \infty)$ with $\tau(n) \uparrow$ and $\tau(n)/n \downarrow$, let κ_τ be the unique number in $[0, \infty]$ such that $\sum_{n\geq 1} \mathrm{e}^{-\kappa\tau(n)}/n$ converges for $\kappa > \kappa_\tau$ and diverges for $\kappa < \kappa_\tau$. Then

$$\varliminf_{n\to\infty} \frac{\varphi_n}{b(n/\kappa\tau(n))\kappa\tau(n)} \geq C_\gamma \quad a.s. \text{ if } \kappa \in [\kappa_\tau, \infty), \tag{2.8}$$

and

$$\varliminf_{n\to\infty} \frac{\varphi_n}{b(n/\kappa\tau(n))\kappa\tau(n)} \leq C_\gamma \quad a.s. \text{ if } \kappa \in (0, \kappa_\tau]. \tag{2.9}$$

Remark 3. (a) *The conclusion of Theorem 2.3 was established for iid processes in* [21], *and for ψ-mixing processes in* [5]. *Our proof of Theorem 2.3 is by establishing the conditions needed for the methods of* [5]. *Therefore the functional version also follows as in* [6].

(b) *Recall that $\vartheta_P(n) = \phi_-(n)$, see Remark* 1(a).

Inversion: *Kakutani towers and return time processes*

Our results will be established using the well-known technique of "inverting" corresponding results for infinite measure preserving transformations, the connection being established via the following concept. Given a probability preserving transformation $(\Omega, \mathcal{A}, P, S)$ and a measurable function $\varphi : \Omega \to \mathbb{N}$, the *Kakutani tower* of $(\Omega, \mathcal{A}, P, S, \varphi)$ is the object $(X, \mathcal{B}, m, T)$ with $(X, \mathcal{B}, m)$ the σ-finite space defined by

- $X := \bigcup_{n\geq 1}[\varphi \geq n] \times \{n\}$,
- $\mathcal{B} := \{\bigcup_{n\geq 1} B_n \times \{n\}\colon\ B_n \in \mathcal{A} \cap [\varphi \geq n]\, \forall n \geq 1\}$,
- $m(A \times \{n\}) := P(A)$,

and $T : X \to X$ is the map given by

- $T(x, n) := \begin{cases} (x, n+1), & \varphi(x) > n, \\ (Sx, 1), & \varphi(x) = n. \end{cases}$

It follows that $(X, \mathcal{B}, m, T)$ is a conservative, measure preserving transformation which is ergodic iff $(\Omega, \mathcal{A}, P, S)$ is ergodic.

This "tower building process" is reversible. Given a conservative ergodic σ-finite measure preserving system $(X, \mathcal{B}, m, T)$, we define the *return time process of T on $\Omega \in \mathcal{F} := \{B \in \mathcal{B}\colon\ 0 < m(B) < \infty\}$* as the $\mathbb{N}$-valued stationary process $(\varphi_\Omega \circ T_\Omega^n)_{n\geq 0}$ on $(\Omega, \mathcal{B} \cap \Omega, m_\Omega,)$, where

- $\varphi(x) = \varphi_\Omega(x) := \min\{n \geq 1\colon\ T^n x \in \Omega\}$,
- $T_\Omega(x) := T^{\varphi(x)}(x)$, and
- $m_\Omega(A) := m(A \cap \Omega)/m(\Omega)$.

It follows that the Kakutani tower of $(\Omega, \mathcal{B} \cap \Omega, m_\Omega, T_\Omega, \varphi_\Omega)$ is a factor of $(X, \mathcal{B}, m', T)$ where $m' = \frac{1}{m(\Omega)} m$ (and an isomorph in case T is invertible).

Now set $\varphi_j := \sum_{i=0}^{j-1} \varphi_\Omega \circ T_\Omega^i$, which is the time of the nth return to Ω. It is straightforward to check that these are dual to the *occupation times* of Ω, $S_n(1_\Omega) := \sum_{k=0}^{n-1} 1_\Omega \circ T^k$ in that

$$S_n(1_\Omega) \leq j \quad \text{iff} \quad \varphi_j \geq n. \tag{2.10}$$

This entails, via routine arguments, that various properties of $(\varphi_j)_{j \geq 1}$ are equivalent to corresponding properties of $(S_n(1_\Omega))_{n \geq 1}$. Specifically, suppose that $a(n)$ is γ-regularly varying with $\gamma \in (0, 1]$, and let b be asymptotically inverse to a. Then, for Y a $[0, \infty]$-valued random variable,

$$\frac{1}{a(n)} S_n(1_\Omega) \xrightarrow{\mathfrak{d}} m(\Omega) Y \quad \text{iff} \quad \frac{\varphi_n}{b(n)} \xrightarrow{\mathfrak{d}} \left(\frac{1}{m(\Omega) Y} \right)^{1/\gamma}, \tag{2.11}$$

and

$$\overline{\lim_{n \to \infty}} \frac{1}{a(n)} S_n(1_\Omega) \stackrel{\text{a.e.}}{=} m(\Omega) \quad \text{iff} \quad \underline{\lim_{n \to \infty}} \frac{\varphi_n}{b(n)} \stackrel{\text{a.e.}}{=} \left(\frac{1}{m(\Omega)} \right)^{1/\gamma}. \tag{2.12}$$

3. Weak pointwise dual ergodic measure preserving transformations

In this section, we consider the properties of infinite ergodic systems to be used in the proofs of the above results.

Weak pointwise dual ergodicity

Let T be a conservative, ergodic, measure preserving transformation (not necessarily invertible) on the σ-finite space $(X, \mathcal{B}, m)$, and $\widehat{T} : L^1(m) \to L^1(m)$ its transfer operator, which naturally extends to all nonnegative measurable functions. Invariance of m means that $\widehat{T} 1_X = 1_X$, and since T is conservative ergodic, any measurable $g : X \to [0, \infty)$ which is subinvariant, $\widehat{T} g \leq g$, is, in fact, constant. Hurewicz's ratio ergodic theorem (Theorem 2.2.1 of [1]), guarantees that

$$\frac{\sum_{k=0}^{n-1} \widehat{T}^k f}{\sum_{k=0}^{n-1} \widehat{T}^k g} \underset{n \to \infty}{\longrightarrow} \frac{m(f)}{m(g)} \quad \text{a.e. on } X \tag{3.1}$$

for all $f, g \in L^1_+(m) := \{f \in L^1(m) \colon f \geq 0 \text{ and } m(f) > 0\}$. (Due to conservativity, $\sum_{k=0}^{n-1} \widehat{T}^k f \to \infty$ a.e. for such f.)

Throughout, convergence in measure, $\xrightarrow{m}$, for our σ-finite measure m, is understood to mean convergence in measure, $\xrightarrow{\nu}$, for every finite $\nu \ll m$ (or, equivalently, for all $\nu = m_A$ with $A \in \mathcal{F}$).

The conservative ergodic measure preserving transformation T on the σ-finite space $(X, \mathcal{B}, m)$ will be called *weakly pointwise dual ergodic* if there exist constants $a(n) > 0, n \geq 1$, such that

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k f \underset{n \to \infty}{\xrightarrow{m}} \int_X f \, \mathrm{d}m \quad \text{for } f \in L^1_+(m), \tag{3.2}$$

and

$$\overline{\lim_{n \to \infty}} \frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k f = \int_X f \, \mathrm{d}m \quad \text{a.e. on } X \text{ for } f \in L^1_+(m). \tag{3.3}$$

This generalizes the notion of *pointwise dual ergodicity* (cf. Section 3.7 of [1], or [2]), which requires $a(n) > 0$ such that

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k f \underset{n \to \infty}{\longrightarrow} \int_X f \, \mathrm{d}m \quad \text{a.e. on } X \text{ for } f \in L^1_+(m). \tag{3.4}$$

In either case the *return sequence* $(a(n))_{n\geq 1}$, which is determined up to asymptotic equivalence and satisfies $a(n+1)\sim a(n)\to\infty$, can (and will) be taken nondecreasing. It is usually denoted $(a_n(T))_{n\geq 1}$.

Remark 4. *No* invertible *conservative ergodic measure preserving transformation* $(X,\mathcal{B},m,T)$ *with* $m(X)=\infty$ *is pointwise dual ergodic*. (*Since in this case* $\widehat{T}f=f\circ T^{-1}$, *so that* (3.4) *would give a pointwise ergodic theorem with normalizing constants* $a(n)$ *for* T^{-1}, *which is impossible*, *see Section* 2.4 *of* [1].) *However, invertible systems can still be weakly pointwise dual ergodic*.

Example 1. *For a concrete example*, *let* $T:(0,1)\to(0,1)$ *be given by* $Tx:=\frac{x}{1-x}$ *for* $x<\frac{1}{2}$ *and* $Tx:=2x-1$ *for* $x>\frac{1}{2}$, *which is conservative ergodic with invariant measure m having density* $\frac{1}{x}$, *and define* $a(n):=n/\log n$. *By the Darling–Kac theorem for pointwise dual ergodic transformations*, (*see* [2,14], *Section* 3.7 *of* [1], *or* [24]),

$$\frac{1}{a(n)}\sum_{k=0}^{n-1} f\circ T^k \xrightarrow[n\to\infty]{m} \int_X f\,\mathrm{d}m \quad \text{for } f\in L^1_+(m),$$

and according to Proposition 2 *of* [4],

$$\overline{\lim_{n\to\infty}}\,\frac{1}{a(n)}\sum_{k=0}^{n-1} f\circ T^k=\int_X f\,\mathrm{d}m \quad \text{a.e. on } X \text{ for } f\in L^1_+(m).$$

These carry over to the natural extension T_* *of* T: *This is immediate for functions of the form* $f_*=f\circ\pi$, *with* π *denoting the factor map. Then use the ratio ergodic theorem to pass to general* $f_*\in L^1_+(m_*)$. *Therefore, the invertible conservative ergodic measure preserving transformation* T_*^{-1}, *with transfer operator* $f\mapsto f\circ T_*$ *is weakly pointwise dual ergodic*, *and hence so is* T_*, *with* $a_n(T_*)\sim a_n(T_*^{-1})\sim a(n)$.

Conditions of this flavour can be exploited most efficiently if one succeeds in identifying special sets on which there is additional control on the convergence. Recall (cf. [1,19]) that $A\in\mathcal{F}=\{B\in\mathcal{B}\colon\ 0<m(B)<\infty\}$ is called a *uniform set* (*for* $f\in L^1_+(m)$) if there are $a(n)>0$, such that

$$\operatorname*{ess\,sup}_A\left|\frac{1}{a(n)}\sum_{k=0}^{n-1}\widehat{T}^k f-m(f)\right|\xrightarrow[n\to\infty]{}0,$$

and, more specifically, a *Darling–Kac set* if

$$\operatorname*{ess\,sup}_A\left|\frac{1}{a(n)}\sum_{k=0}^{n-1}\widehat{T}^k 1_A-m(A)\right|\xrightarrow[n\to\infty]{}0.$$

We now define $A\in\mathcal{F}$ to be a *limited set* if there exist constants $a(n)>0$ such that

$$\frac{1}{a(n)}\sum_{k=0}^{n-1}\widehat{T}^k 1_A \xrightarrow[n\to\infty]{m_A} m(A), \tag{3.5}$$

and

$$\operatorname*{ess\,sup}_A \frac{1}{a(n)}\sum_{k=0}^{n-1}\widehat{T}^k 1_A \xrightarrow[n\to\infty]{} m(A). \tag{3.6}$$

Due to (3.1) the asymptotics of $(a(n))$ does not depend on f or A. It is not hard to see that defining

$$a_A(n):=\sum_{k=0}^{n-1}\frac{m(A\cap T^{-k}A)}{m(A)^2} \quad \text{for } A\in\mathcal{F},$$

we have

$$a(n) \underset{n\to\infty}{\sim} a_A(n) \quad \text{in (3.5) and (3.6) for every limited set } A. \tag{3.7}$$

The existence of uniform sets is equivalent to pointwise dual ergodicity, and $a(n) \sim a_n(T)$ in this case. (But we do not know if every pointwise dual ergodic transformation has a Darling–Kac set.) Similarly, *weak pointwise dual ergodicity is equivalent to the existence of limited sets*, as we have

Proposition 3.1 (Limited sets and weak pointwise dual ergodicity from local behaviour). *Let T be a conservative ergodic measure preserving transformation on the σ-finite space $(X, \mathcal{B}, m)$.*

(a) *Suppose there are $A \in \mathcal{F}$, $f \in L^1_+(m)$, and constants $a(n) > 0$, $n \geq 1$, such that*

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k f \underset{n\to\infty}{\overset{m_A}{\longrightarrow}} \int_X f \, \mathrm{d}m, \tag{3.8}$$

and

$$\overline{\lim_{n\to\infty}} \frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k f = \int_X f \, \mathrm{d}m \quad \text{a.e. on } A. \tag{3.9}$$

Then, for every $\eta > 0$, T possesses a limited set $A' \in \mathcal{F} \cap A$ with return sequence $(a(n))_{n\geq 1}$ and $m(A') \geq m(A) - \eta$.

(b) *If T has a limited set, then it is weakly pointwise dual ergodic.*

Proof. (a) Let $B_0 := A$. By Hurewicz's ratio ergodic theorem we may assume w.l.o.g. that $f = 1_{B_0}$, and given any set $B_j \in \mathcal{F} \cap A$ with $m(B_j) > m(A) - \eta$, we also have $a(n)^{-1} \sum_{k=0}^{n-1} \widehat{T}^k 1_{B_j} \to m(B_j)$ a.e. on A. Egorov's theorem then provides us with some $B_{j+1} \in \mathcal{F} \cap B_j$ such that $m(B_{j+1}) > m(A) - \eta$, and

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k 1_{B_j} \underset{n\to\infty}{\longrightarrow} m(B_j) \quad \text{uniformly on } B_{j+1}.$$

Using this to inductively define a decreasing sequence $(B_j)_{j\geq 0}$ in $\mathcal{F}$, we obtain a set $A' := \bigcap_{j\geq 0} B_j$ with $m(A') \geq m(A) - \eta$. Given $\varepsilon > 0$ choose j with $m(B_j) < m(A') + \varepsilon/2$. Then, for $n \geq n_j(\varepsilon)$,

$$\operatorname*{ess\,sup}_{A'} \frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k 1_{A'} \leq \operatorname*{ess\,sup}_{B_{j+1}} \frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k 1_{B_j} < m(B_j) + \frac{\varepsilon}{2}.$$

This gives the required control from above. Since, by (3.1), (3.8) also holds with $f = 1_{A'}$, we see that A' is indeed a limited set.

(b) Now start from the assumption that T has a limited set A. Due to Hurewicz's theorem, weak pointwise dual ergodicity follows as soon as we check the defining conditions (3.2) and (3.3) for $f = 1_A$, i.e. we have to prove

$$f_n := \frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k 1_A \underset{n\to\infty}{\overset{m}{\longrightarrow}} m(A), \tag{3.10}$$

and

$$\bar{f} := \overline{\lim_{n\to\infty}} \frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k 1_A = m(A) \quad \text{a.e. on } X. \tag{3.11}$$

Letting $A_n := A \cap [\varphi_A > n]$, $n \geq 0$, we have, by routine arguments,

$$\sum_{n\geq 0} \widehat{T}^n 1_{A_n} = 1_X, \tag{3.12}$$

and decomposing, for any $n \geq 0$, A according to the time of the last return before time n, $A = A_n \cup \bigcup_{k=0}^{n-1} A \cap T^{-(n-k)}(A_k)$ (disjoint), we find that for $N \geq 0$,

$$\sum_{n=0}^{N} \widehat{T}^n 1_A = \sum_{k=0}^{N-1} \widehat{T}^k \left(1_{A_k} \sum_{j=0}^{N-k} \widehat{T}^j 1_A \right) + \sum_{n=0}^{N} \widehat{T}^n 1_{A_n}. \tag{3.13}$$

Since A is a limited set, there is some $M \in (0, \infty)$ such that

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k 1_A \leq M \quad \text{on } A \text{ for } n \geq 1. \tag{3.14}$$

With (3.14) and (3.12) providing bounds for the sums in (3.13), we get

$$\frac{1}{a(N)} \sum_{n=0}^{N-1} \widehat{T}^n 1_A \leq M' \cdot 1_X \quad \text{for } N \geq 1, \tag{3.15}$$

where $M' := M + 1/a(1) \in (0, \infty)$. Consequently, $\bar{f} \leq M' \cdot 1_X$. Now observe that $|\widehat{T} f_n - f_n| \leq 2/a(n) \to 0$. Combined with a variant of Fatou's lemma for $\widehat{T}$, Lemma 3.1 below, this yields

$$\widehat{T} \bar{f} \geq \overline{\lim_{n\to\infty}} \widehat{T} f_n = \bar{f} \quad \text{a.e. on } X.$$

Hence $g := M' \cdot 1_X - \bar{f} \geq 0$ is subinvariant, $\widehat{T} g \leq g$. As T is conservative ergodic, this implies that g is constant a.e., and hence so is $\bar{f}$. In view of (3.5), this yields (3.11).

To finally prove convergence in measure on X, it suffices to check it on each of the sets $T^{-l}A$, $l \geq 0$, since these cover X. But for each l,

$$\int_{T^{-l}A} \left(\frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k 1_A \right) \mathrm{d}m = \frac{a_A(n+l) - a_A(l)}{a(n)} m(A) \underset{n\to\infty}{\longrightarrow} m(A).$$

Together with (3.15) and (3.11) this entails $a(n)^{-1} \sum_{k=0}^{n-1} \widehat{T}^k 1_A \overset{m_{T^{-l}A}}{\longrightarrow} m(A)$, and hence (3.10). □

The preceding proof made use of

Lemma 3.1. *Let T be measure preserving on the σ-finite space $(X, \mathcal{A}, m)$, with transfer operator $\widehat{T}$, and let $M' \in (0, \infty)$.*

(a) *If $g, g_k : X \to [0, M']$ are measurable with $g_k \geq g_{k+1} \searrow g$ a.e., then*

$$\widehat{T} g_k \searrow \widehat{T} g \quad \textit{a.e. on } X. \tag{3.16}$$

(b) *If $f_n : X \to [0, M']$ are measurable functions, then*

$$\widehat{T}(\overline{\lim_{n\to\infty}} f_n) \geq \overline{\lim_{n\to\infty}} \widehat{T} f_n \quad \textit{a.e. on } X. \tag{3.17}$$

Proof. (a) By positivity of $\widehat{T}$, we have $\widehat{T} M' \geq \widehat{T} g_1 \geq \widehat{T} g_2 \geq \cdots \geq \widehat{T} g$ a.e., while $\widehat{T} M' = M'$ by invariance of m. Now take any $A \in \mathcal{A}$, $m(A) < \infty$, then $1_{T^{-1}A} g_k \longrightarrow 1_{T^{-1}A} g$ in $L_1(m)$. Due to L_1-continuity of $\widehat{T}$, this entails

$$1_A \widehat{T} g_k = \widehat{T}(1_{T^{-1}A} g_k) \longrightarrow \widehat{T}(1_{T^{-1}A} g) = 1_A \widehat{T} g \quad \text{in } L_1(m),$$

and since A was arbitrary, the latter implies (3.16).

(b) Let $g_k := \sup_{n \geq k} f_n \geq f_k$, and $g := \overline{\lim}_{n\to\infty} f_n$, then g, g_k satisfy the assumptions of (a). Hence, $\widehat{T} g_k \searrow \widehat{T} g$ a.e. By positivity, however, we have $\widehat{T} g_k \geq \widehat{T} f_k$ a.e. for all $k \geq 1$. Together, these observations give (3.17). □

The notion of return sequences has originally been introduced in the context of rationally ergodic transformations (cf. Section 3.3 of [1]). The present use of the term is justified by

Proposition 3.2 (Weak pointwise dual ergodicity implies rational ergodicity). *Let T be a weakly pointwise dual ergodic measure preserving transformation on the σ-finite space $(X, \mathcal{B}, m)$. Then every limited set A satisfies a Rényi inequality, meaning that there is some $M = M(A) \in (0, \infty)$ such that*

$$\int_A \big(S_n(1_A)\big)^2 \,\mathrm{d}m \leq M\left(\int_A S_n(1_A)\,\mathrm{d}m\right)^2 \quad \textit{for } n \geq 1.$$

In particular, T is rationally ergodic with $(a_n(T))_{n\geq 1}$ a return sequence in the sense of [1].

Proof. Same as the proof of Proposition 3.7.1 in [1], using (3.7) and the existence of limited sets established above. □

(Weak) pointwise dual ergodicity and special sets for Kakutani towers

We are going to show that Kakutani towers above ϑ_μ-mixing systems satisfying our adaptedness conditions are weakly pointwise dual ergodic. Moreover, in the presence of regular variation, ϑ_μ-mixing with sufficiently fast rate implies pointwise dual ergodicity.

Note first that if $(X, \mathcal{B}, m, T)$ is a conservative ergodic measure preserving transformation and $\Omega \in \mathcal{B}, m(\Omega) = 1$, has return time $\varphi = \varphi_\Omega$, and $\varphi_j := \sum_{i=0}^{j-1} \varphi \circ T_\Omega^i$, then, in view of (2.11),

$$a_\Omega(n) \sim \sum_{k=1}^{n} m\big(\Omega \cap T^{-k}\Omega\big) = \sum_{j=1}^{n} m\big(\Omega \cap [\varphi_j \leq n]\big) = a_\varphi(n). \tag{3.18}$$

Via (3.7) this links $(a_\varphi(n))$ to the asymptotic type of a weakly pointwise dual ergodic system.

Theorem 3.1 ((Weak) pointwise dual ergodicity via ϑ_μ-mixing return processes). *Let $(X, \mathcal{B}, m, T)$ be a conservative ergodic measure preserving transformation on a σ-finite space, and suppose that $\Omega \in \mathcal{B}, m(\Omega) = 1$, has a countable partition $\alpha \subset \mathcal{B} \cap \Omega$ such that $\varphi = \varphi_\Omega$ is α-measurable and that $(\Omega, \mathcal{B} \cap \Omega, P, T_\Omega, \alpha)$ is ϑ_μ-mixing for some μ equivalent to $P := m_\Omega$.*

(a) *If $(a(n)) := (a_\Omega(n))$ is adapted to $(\vartheta_\mu(n))$, then T is weakly pointwise dual ergodic with $a_n(T) \sim a(n)$. In addition, $\forall \varepsilon > 0$ $\exists N_\varepsilon$ s.t. $\forall n \geq N_\varepsilon$,*

$$\frac{1}{a(n)} \sum_{k=1}^{n} \widehat{T}^k 1_\Omega - 1 \leq \varepsilon\left(1 + \frac{\mathrm{d}\mu}{\mathrm{d}P}\right) \quad \textit{a.e. on } \Omega. \tag{3.19}$$

In particular, if $\|\frac{\mathrm{d}\mu}{\mathrm{d}P}\|_\infty < \infty$, then Ω is a limited set.

(b) *If $(a(n))_{n\geq 1}$ is γ-regularly varying for some $\gamma \in (0, 1]$ and*

$$\vartheta_\mu(n) = \mathrm{O}\left(\frac{1}{n^r}\right) \quad \textit{for some } r > \frac{1}{\gamma} - 1,$$

then $(a(n))$ is adapted to $(\vartheta_\mu(n))$, and T is pointwise dual ergodic with $a_n(T) \sim a(n)$.

(c) *If $(a(n))_{n\geq 1}$ is γ-regularly varying for some $\gamma \in (0, 1]$, and $\psi^*(1) < \infty$, and*

$$\frac{n\vartheta_\mu(\delta a(a(n)))}{a(n)} \underset{n\to\infty}{\longrightarrow} 0 \quad \textit{for all } \delta > 0, \tag{3.20}$$

then $(a(n))$ is adapted to $(\vartheta_\mu(n))$, and T is pointwise dual ergodic with $a_n(T) \sim a(n)$. Moreover, $\forall \varepsilon > 0$ $\exists N_\varepsilon$ such that $\forall n \geq N_\varepsilon$

$$\left| \frac{1}{a(n)} \sum_{k=1}^{n} \widehat{T}^k 1_\Omega - 1 \right| \leq \varepsilon \left(1 + \frac{\mathrm{d}\mu}{\mathrm{d}P} \right) \quad \text{a.e. on } \Omega. \tag{3.21}$$

In particular, if $\|\frac{\mathrm{d}\mu}{\mathrm{d}P}\|_\infty < \infty$, then Ω is a Darling–Kac set.

Proof. We write $(\Omega, \mathcal{A}, P, S, \alpha) := (\Omega, \mathcal{B} \cap \Omega, m_\Omega, T_\Omega, \alpha)$. Note first that by definition of ϑ_μ, we have for $j, p \geq 1$ and $A \in \alpha_j$,

$$\int_B \widehat{S}^{j+p} 1_A \,\mathrm{d}P \leq \int_B \left(P(A) + \vartheta_\mu(p) \frac{\mathrm{d}\mu}{\mathrm{d}P} \right) \mathrm{d}P$$

for arbitrary $B \in \mathcal{A}$, and hence

$$\widehat{S}^{j+p} 1_A \leq P(A) + \vartheta_\mu(p) \frac{\mathrm{d}\mu}{\mathrm{d}P} \quad \text{a.e. on } \Omega. \tag{3.22}$$

We claim that for $n, p \geq 1$,

$$\widehat{T}_n := \sum_{k=1}^{n} \widehat{T}^k 1_\Omega \leq p + a(n) + n\vartheta_\mu(p) \frac{\mathrm{d}\mu}{\mathrm{d}P} \quad \text{a.e. on } \Omega. \tag{3.23}$$

To see this, observe that $\widehat{T}^k 1_\Omega = \sum_{j \geq 1} \widehat{S}^j 1_{[\varphi_j = k]}$ a.e. on Ω, and hence

$$\begin{aligned}
\widehat{T}_n &= \sum_{j=1}^{n} \widehat{S}^j 1_{[\varphi_j \leq n]} \leq \sum_{j=1}^{n+p} \widehat{S}^j 1_{[\varphi_j \leq n]} \\
&\leq p + \sum_{j=1}^{n} \widehat{S}^{j+p} 1_{[\varphi_{j+p} \leq n]} \leq p + \sum_{j=1}^{n} \widehat{S}^{j+p} 1_{[\varphi_j \leq n]} \\
&\leq p + \sum_{j=1}^{n} \left(P\big([\varphi_j \leq n]\big) + \vartheta_\mu(p) \frac{\mathrm{d}\mu}{\mathrm{d}P} \right) \\
&= p + a(n) + n\vartheta_\mu(p) \frac{\mathrm{d}\mu}{\mathrm{d}P} \quad \text{a.e. on } \Omega,
\end{aligned}$$

since $[\varphi_j \leq n] \in \alpha_j$.

Proof of (a): To establish (3.19), we let $p = p_{n,\varepsilon} := \lfloor \varepsilon a(n) \rfloor$ for $\varepsilon > 0$. By adaptedness we can choose N_ε such that $n\vartheta_\mu(p_{n,\varepsilon}) < \varepsilon a(n)$ whenever $n \geq N_\varepsilon$. Then by (3.23),

$$\widehat{T}_n \leq a(n) \left(1 + \varepsilon \left(1 + \frac{\mathrm{d}\mu}{\mathrm{d}P} \right) \right) \quad \text{a.e. on } \Omega \text{ for } n \geq N_\varepsilon, \tag{3.24}$$

as required.

To prove weak pointwise dual ergodicity, we will use part (a) of Proposition 3.1. Write $R_n := \widehat{T}_n / a(n)$ and observe first that by (3.19),

$$\overline{\lim_{n \to \infty}} R_n \leq 1 \quad \text{a.e. on } \Omega. \tag{3.25}$$

In fact, (3.19) shows that for any $M > 0$, the estimate (3.25) holds uniformly on $\Omega \cap [\frac{\mathrm{d}\mu}{\mathrm{d}P} \leq M]$. Consequently,

$$P\big(\big[R_n \geq 1 + \varepsilon'\big]\big) \underset{n \to \infty}{\longrightarrow} 0 \quad \text{for every } \varepsilon' > 0. \tag{3.26}$$

To obtain the analogous statement for convergence from below, we observe that for $t \in (0,1)$,

$$R_n \le 1 + \varepsilon\left(1 + \frac{\mathrm{d}\mu}{\mathrm{d}P}\right) \quad \text{a.e. on } \Omega \quad \Longrightarrow \quad P\big([R_n \le t]\big) \le \frac{2\varepsilon}{1+\varepsilon - t}, \tag{3.27}$$

since ($\mathbb{E}$ denoting expectation with respect to P)

$$\begin{aligned} 1 = \mathbb{E}(R_n) &= \mathbb{E}(R_n 1_{[R_n > t]}) + \mathbb{E}(R_n 1_{[R_n \le t]}) \\ &\le (1+\varepsilon) P\big([R_n > t]\big) + \varepsilon \mu\big([R_n > t]\big) + t P\big([R_n \le t]\big) \\ &\le 1 + 2\varepsilon - (1 + \varepsilon - t) P\big([R_n \le t]\big). \end{aligned}$$

Now fix $\varepsilon \in (0,1)$ and take N_ε as in (3.24). For $t := 1 - \sqrt{\varepsilon} \in (0,1)$, observation (3.27) then yields

$$P\big([R_n \le 1 - \sqrt{\varepsilon}]\big) \le 2\sqrt{\varepsilon} \quad \text{for } n \ge N_\varepsilon.$$

This readily implies

$$P\big(\big[R_n \le 1 - \varepsilon'\big]\big) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for every } \varepsilon' > 0,$$

which, together with (3.26), gives (3.8). Combined with (3.25) the latter yields (3.9). (There is some subsequence $n_k \nearrow \infty$ such that $R_{n_k} \to 1$ a.e. for P.)

Proof of (b): Computation of the indices of regular variation shows that the assumptions of (b) entail adaptedness. As in part (a) this implies (3.25).

To check pointwise dual ergodicity, it remains to show that

$$\varliminf_{n\to\infty} R_n \ge 1 \quad \text{a.e. on } \Omega. \tag{3.28}$$

(Proposition 3.7.5 of [1] ensures that a.e. convergence on some $A \in \mathcal{F}$ suffices.) To this end, choose $c \ge 1$ such that $\vartheta_\mu(n) \le c/n^r$ for all $n \ge 1$. There exist $s > 0$ and N_0 so that

$$a(n) > n^{1/(r+1)+2s} \quad \forall n > N_0.$$

Choosing $p = n^{1/(r+1)}$ in (3.23) we have

$$R_n = \frac{\widehat{T}_n}{a(n)} \le 1 + \frac{1}{n^{2s}}\left(1 + \frac{\mathrm{d}\mu}{\mathrm{d}P}\right) \quad \text{on } \Omega\ \forall n \ge N_0.$$

Due to (3.27), we then see that for $t \in (0,1)$,

$$q_t(n) := P\big([R_n \le t]\big) \le \frac{2c}{n^s} \cdot \left(1 - t + \frac{c}{n^s}\right)^{-1} \le \frac{2c}{(1-t)n^s} \quad \text{for } n \ge N_0.$$

Since, for all t and $\lambda > 1$, $\sum_{n\ge1} q_t(\lfloor \lambda^n \rfloor) < \infty$, the Borel–Cantelli Lemma now implies

$$\varliminf_{n\to\infty} R_{\lfloor \lambda^n \rfloor} \ge 1 \quad \text{a.s. on } \Omega \text{ for all } \lambda > 1. \tag{3.29}$$

To finally prove convergence (3.28) of the full sequence, fix any $\lambda > 1$ and choose integers $\kappa_n(\lambda) \nearrow \infty$ so that $\lfloor \lambda^{\kappa_n(\lambda)} \rfloor \le n \le \lfloor \lambda^{\kappa_n(\lambda)+1} \rfloor$. Then regular variation of $(a(n))_{n\ge1}$ yields

$$\frac{\widehat{T}_n}{a(n)} \ge \frac{\widehat{T}_{\lfloor \lambda^{\kappa_n(\lambda)} \rfloor}}{a(\lfloor \lambda^{\kappa_n(\lambda)+1} \rfloor)} \underset{n\to\infty}{\sim} \frac{1}{\lambda^\gamma} \frac{\widehat{T}_{\lfloor \lambda^{\kappa_n(\lambda)} \rfloor}}{a(\lfloor \lambda^{\kappa_n(\lambda)} \rfloor)} \quad \text{a.e. on } \Omega.$$

In view of (3.29), we thus have $\varliminf_{n\to\infty} R_n \ge \lambda^{-\gamma}$ a.s. on Ω for all $\lambda > 1$, and (3.28) follows.

Proof of (c): Note first that adaptedness, and hence (3.19) holds. Thus, to prove (3.21), it suffices to check that for $n \geq N_\varepsilon$,

$$\frac{1}{a(n)}\sum_{k=1}^{n}\widehat{T}^k 1_\Omega \geq 1-\varepsilon\left(1+\frac{\mathrm{d}\mu}{\mathrm{d}P}\right) \quad \text{a.e. on } \Omega. \tag{3.30}$$

We show first that $\forall p, q, n \in \mathbb{N}$ with $n \geq q$,

$$\widehat{T}_n \geq a(n) - p - n\vartheta_\mu(p)\frac{\mathrm{d}\mu}{\mathrm{d}P} - \psi^*(1)^2\big(P\big([\varphi_p > q]\big)a(n)+q\big) \tag{3.31}$$

a.e. on Ω. To see this, observe that

$$\begin{aligned}\widehat{T}_n &= \sum_{j=1}^{n}\widehat{S}^j 1_{[\varphi_j \leq n]} \geq \sum_{j=1}^{n}\widehat{S}^{j+p} 1_{[\varphi_{j+p} \leq n]} - p \\ &= \sum_{j=1}^{n}\widehat{S}^{j+p} 1_{[\varphi_j \leq n]} - \sum_{j=1}^{n}\widehat{S}^{j+p} 1_{[\varphi_j \leq n < \varphi_{j+p}]} - p \\ &=: \Sigma_1 - \Sigma_2 - p.\end{aligned}$$

Now, because $[\varphi_j \leq n] \in \alpha_j$, we have

$$\Sigma_1 \geq \sum_{j=1}^{n}\left(P\big([\varphi_j \leq n]\big) - \vartheta_\mu(p)\frac{\mathrm{d}\mu}{\mathrm{d}P}\right) = a(n) - n\vartheta_\mu(p)\frac{\mathrm{d}\mu}{\mathrm{d}P}.$$

On the other hand,

$$\begin{aligned}\Sigma_2 &\leq \psi^*(1)\sum_{j=1}^{n} P\big([\varphi_j \leq n < \varphi_{j+p}]\big) \\ &= \psi^*(1)\sum_{j=1}^{n}\sum_{\ell=j}^{n} P\big(\big[\varphi_j = \ell, \varphi_p \circ S^j > n-\ell\big]\big) \\ &\leq \psi^*(1)^2\sum_{j=1}^{n}\sum_{\ell=j}^{n} P\big([\varphi_j = \ell]\big)P\big([\varphi_p > n-\ell]\big) \\ &= \psi^*(1)^2\big(\Sigma_2' + \Sigma_2''\big)\end{aligned}$$

with

$$\begin{aligned}\Sigma_2' &:= \sum_{j=1}^{n}\sum_{\ell=j}^{n-q} P\big([\varphi_j = \ell]\big)P\big([\varphi_p > n-\ell]\big) \\ &\leq P\big([\varphi_p > q]\big)\sum_{j=1}^{n}\sum_{\ell=j}^{n-q} P\big([\varphi_j = \ell]\big) \\ &\leq P\big([\varphi_p > q]\big)\sum_{j=1}^{n} P\big([\varphi_j \leq n]\big) \\ &= P\big([\varphi_p > q]\big)a(n);\end{aligned}$$

and

$$
\begin{aligned}
\Sigma_2'' &:= \sum_{j=1}^{n} \sum_{\ell=n-q+1}^{n} P\big([\varphi_j = \ell]\big) P\big([\varphi_p > n-\ell]\big) \\
&\le \sum_{j=1}^{n} \sum_{\ell=n-q+1}^{n} P\big([\varphi_j = \ell]\big) \\
&= \sum_{j=1}^{n} P\big([n-q < \varphi_j \le n]\big) \\
&= \sum_{k=n-q+1}^{n} m\big(\Omega \cap T^{-k}\Omega\big) \le q.
\end{aligned}
$$

Putting this together establishes (3.31).

To finally check (3.30) for some given $\varepsilon > 0$, choose

$$
\delta \in \left(0, \frac{\varepsilon}{3\psi^*(1)^2}\right) \quad \text{so small that } \Pr\left(Z_\gamma > \frac{1}{\delta}\right) < \frac{\varepsilon}{3\psi^*(1)^2},
$$

let b be asymptotically inverse to a, and define

$$
p_n = p_{n,\varepsilon} := \big\lfloor \delta^{2\gamma} a\big(a(n)\big) \big\rfloor, \quad \text{and} \quad q_n = q_{n,\varepsilon} := \big\lfloor \delta a(n) \big\rfloor.
$$

Then $b(p_n) \sim \delta^2 a(n)$, and $q_n / b(p_n) \longrightarrow 1/\delta$ as $n \to \infty$. Under the present assumptions, the stable limit theorem, Theorem 2.1, applies to our (φ_n), as its proof below does not depend on part (c) of Theorem 3.1. Therefore,

$$
P\big([\varphi_{p_n} > q_n]\big) \underset{n\to\infty}{\longrightarrow} \Pr\left(Z_\gamma > \frac{1}{\delta}\right) < \frac{\varepsilon}{3\psi^*(1)^2}.
$$

Now choose N_ε so large that, for all $n \ge N_\varepsilon$,

$$
p_n < \frac{\varepsilon}{3} a(n), \qquad n\vartheta_\mu(p_n) < \varepsilon a(n) \quad \text{and} \quad P\big([\varphi_{p_n} > q_n]\big) < \frac{\varepsilon}{3\psi^*(1)^2}.
$$

Then, using (3.31), we find indeed that for $n \ge N_\varepsilon$,

$$
\begin{aligned}
\widehat{T}_n &\ge a(n) - p_n - n\vartheta_\mu(p_n)\frac{\mathrm{d}\mu}{\mathrm{d}P} - \psi^*(1)^2\Big(P\big([\varphi_{p_n} > q_n]\big)a(n) + q_n\Big) \\
&\ge a(n) - \frac{\varepsilon}{3}a(n) - \varepsilon a(n)\frac{\mathrm{d}\mu}{\mathrm{d}P} - \frac{2\varepsilon}{3}a(n) \\
&= \left(1 - \varepsilon\left(1 + \frac{\mathrm{d}\mu}{\mathrm{d}P}\right)\right)a(n)
\end{aligned}
$$

a.e. on Ω, which is (3.30). By (3.19) and (3.30),

$$
\frac{1}{a(n)} \sum_{k=1}^{n} \widehat{T}^k 1_\Omega \underset{n\to\infty}{\longrightarrow} 1 \quad \text{a.e. on } \Omega,
$$

so that T is pointwise dual ergodic (Proposition 3.7.5 of [1] again). This convergence is in fact uniform on each $\Omega \cap [\frac{\mathrm{d}\mu}{\mathrm{d}P} \le M]$, $M > 0$, whence the assertion about Darling–Kac sets. □

4. Moment sets and the stable limit theorem

Darling–Kac theorem and stable limits

The statement of the stable limit theorem announced above is dual to a Darling–Kac type limit theorem for the Kakutani tower, which we now establish in the setup of weakly pointwise dual ergodic systems. It quantifies "return rates" and determines the limit distribution of *occupation times* $S_n(1_A) = \sum_{k=0}^{n-1} 1_A \circ T^k$ of sets A of finite measure:

Theorem 4.1 (Darling–Kac theorem for weakly pointwise dual ergodic systems). *Let T be a weakly pointwise dual ergodic measure preserving transformation on the σ-finite space $(X, \mathcal{B}, m)$. If its return sequence $(a(n))_{n\geq 1}$ is regularly varying of index $\alpha \in [0, 1]$, then*

$$\frac{S_n(f)}{a(n)} \xrightarrow[n\to\infty]{\mathfrak{d}} m(f)Y_\gamma \quad \text{for } f \in L^1_+(m),$$

where Y_γ has the normalised Mittag-Leffler distribution of order γ.

The stable limit theorem follows from this:

Proof of Theorem 2.1(a). Let $(X, \mathcal{B}, m, T)$ be the *Kakutani tower* of $(\Omega, \mathcal{A}, P, S, \varphi)$, then $(X, \mathcal{B}, m, T)$ is weakly pointwise dual ergodic with return sequence $(a(n))$. By the Darling–Kac Theorem,

$$\frac{1}{a(n)} S_n(1_\Omega) \xrightarrow{\mathfrak{d}} Y_\gamma,$$

and we need only recall (2.11) and (3.18). □

Moment sets

The proof of the Darling–Kac theorem identifies sets for which the asymptotics of the moments of the occupation time distributions can be understood. Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, measure preserving transformation. For $A \in \mathcal{F} = \{F \in \mathcal{B}, 0 < m(F) < \infty\}$, recall that $a_A(n) = \sum_{k=0}^{n-1} m(A \cap T^{-k}A)/m(A)^2$, and set

$$u_A(\lambda) := \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \frac{m(A \cap T^{-n}A)}{m(A)^2} \quad \text{for } \lambda > 0.$$

The set $A \in \mathcal{F}$ is called a *moment set* for T if for all $p \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \int_A S_n(1_A)^p \,\mathrm{d}m \underset{\lambda\searrow 0}{\sim} p!m(A)^{p+1} \frac{u_A(\lambda)^p}{\lambda}.$$

Remark 5. (a) *If $m(X) < \infty$ then every $A \in \mathcal{B}$ is a moment set. Indeed, $\sum_{k=0}^{n} m(A \cap T^{-k}A) = \int_A S_n(1_A)\,\mathrm{d}m$, and by the ergodic theorem (and dominated convergence), $\int_A S_n(1_A)^p \,\mathrm{d}m \sim n^p m(A)^{p+1}/m(X)^p$, so that the assertion follows from* Karamata's Tauberian theorem (*cf. p.* 116 *of* [1] *or Theorem* 1.7.1 *of* [10]).

(b) *Any conservative, ergodic, measure preserving transformation with moment sets is rationally ergodic. Thus, for example, a squashable conservative, ergodic, measure preserving transformation (which is not rationally ergodic, see* [1]*) has no moment sets.*

(c) *We extend both result and method from pointwise dual ergodic systems (as in* [1]*) to weakly pointwise dual ergodic situations. A similar approach was used in* [19] *to prove an arcsine-type limit theorem for pointwise dual ergodic maps. We do not know if the latter result generalizes accordingly.*

Theorem 4.2 (Moment set theorem). *Suppose that T is weakly pointwise dual ergodic, and that $A \in \mathcal{F}$ is a limited set. Then A is a moment set for T.*

In view of Theorem 3.6.4 of [1], the existence of limited sets established in Proposition 3.1 above, and Karamata's Tauberian theorem, Theorem 4.1 above is an immediate consequence of this result. To prove the Moment set theorem, we need the following observation:

Lemma 4.1 (Convergence of Laplace transforms). *Suppose that T is a conservative ergodic measure preserving transformation on the σ-finite space $(X, \mathcal{B}, m)$, weakly pointwise dual ergodic with A a limited set. Then*

$$R_A(\lambda) := \frac{1}{u_A(\lambda)} \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \widehat{T}^n 1_A \xrightarrow[\lambda \searrow 0]{m_A} m(A), \tag{4.1}$$

and

$$\varlimsup_{\lambda \searrow 0} R_A(\lambda) = m(A) \quad \text{a.e. on } A, \tag{4.2}$$

as well as

$$\varlimsup_{\lambda \searrow 0} \operatorname*{ess\,sup}_A R_A(\lambda) = m(A). \tag{4.3}$$

Moreover, every $B \in \mathcal{F}$, $B \subset A$, satisfies $a_A(n) \sim a_B(n)$ as $n \to \infty$, and hence, for $\lambda \searrow 0$,

$$u_A(\lambda) \sim u_B(\lambda), \quad \text{and} \quad \frac{R_A(\lambda)}{R_B(\lambda)} \to \frac{m(A)}{m(B)} \quad \text{a.e. on } X. \tag{4.4}$$

Remark 6. *As a consequence, each sequence $\lambda_i \searrow 0$ contains a subsequence $\lambda'_i \searrow 0$ for which*

$$R_A(\lambda'_i) \underset{i \to \infty}{\longrightarrow} m(A) \quad \text{a.e. on } A. \tag{4.5}$$

To see this, use the standard fact from integration theory that sequences which converge in probability contain a.e.-convergent subsequences. Moreover, it is immediate from (4.4), that for any sequence (λ'_i) as in (4.5), we also have

$$R_B(\lambda'_i) \underset{i \to \infty}{\longrightarrow} m(B) \quad \text{a.e. on } A. \tag{4.6}$$

Proof of Lemma 4.1. Multiplying numerator and denominator by $(1 - \mathrm{e}^{-\lambda})$, we get

$$0 \le R_A(\lambda) = \frac{\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \sum_{k=0}^{n-1} \widehat{T}^k 1_A}{\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} a_A(n)}$$

for $\lambda > 0$. As A is a limited set, and $\widehat{T}^k 1_A \le 1_X$, we therefore see that (since $\sum_{k=0}^{\infty} \widehat{T}^k 1_A = \infty$ a.e.),

$$\varlimsup_{\lambda \searrow 0} \operatorname*{ess\,sup}_A R_A(\lambda) \le m(A). \tag{4.7}$$

On the other hand, monotone convergence ensures that $\int R_A(\lambda)\,\mathrm{d}m_A = m(A)$ for all $\lambda > 0$. Together with (4.7) this yields (4.1), and combining (4.7) with (4.1) proves (4.2). Together with (4.7) the latter gives (4.3).

Fix $B \in \mathcal{F}$, $B \subset A$. We have $a_B(n) = m(B)^{-2} a_A(n) \int_B g_n(B)\,\mathrm{d}m$ where $g_n(B) := a_A(n)^{-1} \sum_{k=0}^{n-1} \widehat{T}^k 1_B \xrightarrow{m_B} m(B)$ by weak pointwise dual ergodicity. Since $0 \le T^k 1_B \le T^k 1_A$ with A a limited set, we see that $\sup_n \operatorname{ess\,sup}_A g_n(B) < \infty$. Therefore, $\int_B g_n(B)\,\mathrm{d}m \to m(B)^2$, and hence $a_A(n) \sim a_B(n)$. It is then immediate that $u_A(\lambda) \sim u_B(\lambda)$ since $\sum_{n \ge 0} a_A(n) = \infty$. Now, expanding by $(1 - \mathrm{e}^{-\lambda})$ as above, we get

$$\frac{R_A(\lambda)}{R_B(\lambda)} = \frac{u_B(\lambda)}{u_A(\lambda)} \frac{\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \sum_{k=0}^{n-1} \widehat{T}^k 1_A}{\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \sum_{k=0}^{n-1} \widehat{T}^k 1_B} \to 1 \cdot \frac{m(A)}{m(B)} \quad \text{a.e. on X}$$

by Hurewicz' theorem (3.1) since $\sum_{k=0}^{\infty} \widehat{T}^k 1_A = \infty$ a.e., hence (4.4). $\square$

We are now ready for the

Proof of the Moment set theorem. (i) We amend the argument given in the proof of Theorem 3.7.2 in [1], using the same combinatorial decomposition

$$S_n(1_A)^p = \sum_{q=1}^{p} \gamma_p(q) a(q,n), \tag{4.8}$$

where, for $n, p \in \mathbb{N}$, $a(p,n) : X \to \mathbb{Z}$ is defined by $a(0,n)(x) := 1$, and $a(p+1,n)(x) := \sum_{k=1}^{n} 1_A(T^k x) a(p, n-k)(T^k x)$, while $\gamma_1(q) := \delta_{1,q}$, and $\gamma_{p+1}(q) := q(\gamma_p(q) + \gamma_p(q-1))$. In particular, $\gamma_p(p) = p!$.

Proving that A is a moment set reduces to showing that for $p \geq 0$,

$$u_p(\lambda) := \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \int_A a(p,n)\, \mathrm{d}m \underset{\lambda \searrow 0}{\sim} m(A)^{p+1} \frac{u_A(\lambda)^p}{\lambda}. \tag{4.9}$$

This is because $p! u_p(\lambda)$, the $q = p$ term of the Laplace transform of the sum in (4.8), dominates the $q < p$ terms. Indeed, as we now check by induction on p,

$$u_p(\lambda) = \mathrm{O}\left(\frac{u_A(\lambda)^p}{\lambda} \right) \quad \text{as } \lambda \searrow 0\ \forall p \geq 0. \tag{4.10}$$

For $p = 0$ this is evident. More precisely, we have

$$u_0(\lambda) = \frac{m(A)}{1 - \mathrm{e}^{-\lambda}} \underset{\lambda \searrow 0}{\sim} \frac{m(A)}{\lambda}. \tag{4.11}$$

To pass from $p-1$ to p, use the recursive relation

$$\begin{aligned} u_p(\lambda) &= \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \sum_{k=1}^{n} \int_A \big(1_A a(p-1, n-k)\big) \circ T^k \, \mathrm{d}m \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \sum_{k=1}^{n} \int_A \widehat{T}^k 1_A a(p-1, n-k)\, \mathrm{d}m \\ &= \int_A \left(\sum_{k=1}^{\infty} \mathrm{e}^{-\lambda k} \widehat{T}^k 1_A \right) \left(\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} a(p-1, n) \right) \mathrm{d}m, \end{aligned}$$

and combine it with Lemma 4.1. This proves (4.10) and hence sufficiency of (4.9).

Since the $p = 0$ case of (4.9) is trivially fulfilled, we can establish (4.9) by proving

$$\varliminf_{\lambda \searrow 0} \frac{\lambda u_p(\lambda)}{u_A(\lambda)^p} \geq m(A)^{p+1} \quad \forall p \in \mathbb{N} \tag{4.12}$$

and

$$\varlimsup_{\lambda \searrow 0} \frac{\lambda u_p(\lambda)}{u_A(\lambda)^p} \leq m(A)^{p+1} \quad \forall p \in \mathbb{N}. \tag{4.13}$$

(ii) Fix any $p \geq 1$. To prove (4.12) by contradiction, suppose that it is violated. Then there are some $\varepsilon > 0$ and $\lambda_i \searrow 0$ such that

$$\lambda_i u_p(\lambda_i) < (1-\varepsilon)^{p+2} m(A)^{p+1} u_A(\lambda_i)^p \quad \text{for } i \geq 1. \tag{4.14}$$

Let (λ_i') be a subsequence of (λ_i) as in Remark 6, so that

$$R_B(\lambda_i') \underset{i\to\infty}{\longrightarrow} m(B) \quad \text{a.e. on A} \tag{4.15}$$

for all $B \in \mathcal{F}$, $B \subset A$. We claim that there are measurable sets $A = A_0 \supset A_1 \supset \cdots \supset A_p$ with $m(A_j) > (1-\varepsilon)m(A)$ and

$$\begin{aligned} R_{A_j}(\lambda_i') &= \frac{1}{u_{A_j}(\lambda_i')} \sum_{k\ge 1} \mathrm{e}^{-\lambda_i' k} \widehat{T}^k 1_{A_j} \\ &\ge (1-\varepsilon)m(A_j) \quad \text{on } A_{j+1} \text{ for } i \ge \ell_j, \end{aligned}$$

with $(\ell_j)_{j\ge 1}$ increasing in $\mathbb{N}$.

To see this, start with $A_0 = A$ and consider (4.15) with $B = A_0$. By Egorov's theorem, there is some $A_1 \in \mathcal{B} \cap A_0$ with $m(A_1) > (1-\varepsilon)m(A)$ such that this convergence is uniform on A_1. Therefore, there exists a suitable ℓ_1. If $A = A_0 \supset A_1 \supset \cdots \supset A_j$ have been constructed, consider (4.15) with $B = A_j$. By Egorov's theorem, there is some $A_{j+1} \in \mathcal{B} \cap A_j$ with $m(A_{j+1}) > (1-\varepsilon)m(A)$ such that this convergence is uniform on A_{j+1}. Therefore, there exists a suitable ℓ_{j+1}.

We now find, for any $j \in \{0, 1, \ldots, p-1\}$ and $i > \ell_j$, that

$$\begin{aligned} &\int_{A_j} \left(\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_i' n} a(p-j, n) \right) \mathrm{d}m \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_i' n} \sum_{k=1}^{n} \int_{A_j} \big(1_A a(p-j-1, n-k)\big) \circ T^k \, \mathrm{d}m \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_i' n} \sum_{k=1}^{n} \int_{A} \widehat{T}^k 1_{A_j} a(p-j-1, n-k) \, \mathrm{d}m \\ &= \int_A \left(\sum_{k=1}^{\infty} \mathrm{e}^{-\lambda_i' k} \widehat{T}^k 1_{A_j} \right) \left(\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_i' n} a(p-j-1, n) \right) \mathrm{d}m \\ &\ge \int_{A_{j+1}} \left(\sum_{k=1}^{\infty} \mathrm{e}^{-\lambda_i' k} \widehat{T}^k 1_{A_j} \right) \left(\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_i' n} a(p-j-1, n) \right) \mathrm{d}m \\ &> (1-\varepsilon)m(A) u_{A_j}(\lambda_i') \int_{A_{j+1}} \left(\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_i' n} a(p-j-1, n) \right) \mathrm{d}m. \end{aligned}$$

Putting these together, we obtain that for $i > \ell_{p-1}$,

$$\begin{aligned} u_p(\lambda_i') &> \big((1-\varepsilon)m(A)\big)^p \left(\prod_{j=0}^{p-1} u_{A_j}(\lambda_i') \right) \int_{A_p} \left(\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_i' n} a(0, n) \right) \mathrm{d}m \\ &> \big((1-\varepsilon)m(A)\big)^{p+1} \left(\prod_{j=0}^{p-1} u_{A_j}(\lambda_i') \right) \frac{1}{\lambda_i'}, \end{aligned}$$

where the last step is immediate from $a(0, n) = 1$. Since $u_{A_j}(\lambda) \sim u_A(\lambda)$ by Lemma 4.1, this contradicts (4.14), and thus establishes (4.12).

(iii) To prove (4.13), fix any $p \geq 1$, and $\varepsilon > 0$. In view of (4.3) there is some $\lambda' > 0$ such that $R_A(\lambda) < (1+\varepsilon)m(A)$ on A for $\lambda < \lambda'$. For such λ we therefore find

$$\begin{aligned} u_p(\lambda) &= \int_A u_A(\lambda) R_A(\lambda) \left(\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} a(p-1,n) \right) \mathrm{d}m \\ &\leq (1+\varepsilon) m(A) u_A(\lambda) \cdot u_{p-1}(\lambda) \\ &\vdots \\ &\leq \big((1+\varepsilon) m(A) u_A(\lambda) \big)^p \cdot u_0(\lambda), \end{aligned}$$

and our claim is immediate from (4.11). □

5. Wandering rates, return sequences and tails of marginals

Wandering rates

Suppose that $(X, \mathcal{B}, m, T)$ is a conservative ergodic measure preserving transformation on a σ-finite space. The *wandering rate* of the set $A \in \mathcal{F}$ is the sequence given by $L_A(n) := m(\bigcup_{k=0}^{n} T^{-k} A)$, $n \geq 1$. Evidently,

$$A, B \in \mathcal{F}, \quad A \subset B \quad \Rightarrow \quad L_A(n) \leq L_B(n),$$

and

$$\text{for } N \geq 1 \text{ fixed}, \quad L_{\bigcup_{k=0}^{N} T^{-k} A}(n) = L_A(n+N) \underset{n \to \infty}{\sim} L_A(n).$$

Wandering rates are expectations of truncated return times,

$$L_A(n) = \int_A (\varphi_A \wedge n) \, \mathrm{d}m.$$

Therefore, letting $c_A(\lambda) := \int_A (1 - \mathrm{e}^{-\lambda \varphi_A}) \, \mathrm{d}m$, $\lambda > 0$, for $A \in \mathcal{F}$, we have

$$c_A(\lambda) = \big(1 - \mathrm{e}^{-\lambda}\big) \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} m\big(A \cap [\varphi_A > n]\big) \underset{\lambda \searrow 0}{\sim} \lambda^2 \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} L_A(n).$$

Thus if $L_A(n) \sim L_B(n)$ as $n \to \infty$, then $c_A(\lambda) \sim c_B(\lambda)$ as $\lambda \searrow 0$. In fact, since $L_A(n+1) - L_A(n) \searrow 0$ for all $A \in \mathcal{F}$, Korenblum's ratio Tauberian theorem ([15], see also Theorem 2.10.1 of [10]) shows that the converse is also true, so that

$$\text{for } A, B \in \mathcal{F}\text{:} \quad L_A(n) \underset{n \to \infty}{\sim} L_B(n) \quad \Longleftrightarrow \quad c_A(\lambda) \underset{\lambda \searrow 0}{\sim} c_B(\lambda). \tag{5.1}$$

The set $A \in \mathcal{F}$ is said to have *minimal wandering rate* if $L_B(n) \sim L_A(n)$ for all $B \in \mathcal{F}$, $B \subseteq A$. In this case, $\liminf_{n \to \infty} \frac{L_B(n)}{L_A(n)} \geq 1$ for all $B \in \mathcal{F}$. Thus if $A, B \in \mathcal{F}$ both have minimal wandering rate, then $L_B(n) \sim L_A(n)$. Therefore, if such a set A exists, $L_T(n) := L_A(n)$ defines the *wandering rate of* T, $(L_T(n))_{n \geq 1}$ up to asymptotic equivalence. The latter is a nontrivial structural invariant for large classes of systems, see [18] and [25].

There are sufficient conditions for $A \in \mathcal{F}$ to have minimal wandering rate. By Proposition 3.2, Remark 3.6, and equation (2.3) of [20],

$$\text{if } \left(\frac{\widehat{T}_A(\varphi_A \wedge n)}{L_A(n)} \right)_{n \geq 1} \text{ is uniformly integrable,}$$
$$\text{then } A \text{ has minimal wandering rate.} \tag{5.2}$$

Also, uniform sets are known to have minimal wandering rate, provided that the return sequence is regularly varying (Theorem 3.8.3 of [1]). In Theorem 5.1 below we remove the latter condition.

Minimal wandering rates determine the return sequence $(a(n))_{n\geq 1} = (a_n(T))_{n\geq 1}$ of a weakly pointwise dual ergodic system $(X, \mathcal{B}, m, T)$ by means of the *asymptotic renewal equation*. Assuming w.l.o.g. that $(a(n))$ is increasing, we let

$$u_T(\lambda) := \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n}\big(a(n+1) - a(n)\big) \quad \text{for } \lambda > 0.$$

As a consequence of (3.7) we have $u_T(\lambda) \sim u_A(\lambda)$ as $\lambda \searrow 0$ for all limited sets A (with $u_A(\lambda)$ as in Section 4).

Theorem 5.1 (Minimal wandering rates and the asymptotic renewal equation). *Suppose that T is weakly pointwise dual ergodic.*

(i) *If $A \in \mathcal{F}$ has minimal wandering rate, then it satisfies the asymptotic renewal equation*

$$c_A(\lambda) \underset{\lambda\searrow 0}{\sim} \frac{1}{u_T(\lambda)}.$$

(ii) *Uniform sets have minimal wandering rates.*

Proof. (i) By Proposition 3.1 there is some limited set $B \in \mathcal{F} \cap A$. In view of statement (a) in Lemma 4.1, any sequence decreasing to 0 contains a subsequence $(\lambda_j)_{j\geq 1}$ along which

$$\frac{1}{u_B(\lambda_j)} \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_j n} \widehat{T}^n 1_B \underset{j\to\infty}{\longrightarrow} m(B) \quad \text{a.e. on } B.$$

Egorov's theorem then gives us some $C \in \mathcal{F} \cap B$ on which this convergence is in fact uniform, so that

$$\int_C \big(1 - \mathrm{e}^{-\lambda_j \varphi_C}\big) \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_j n} \widehat{T}^n 1_B \,\mathrm{d}m \underset{j\to\infty}{\sim} m(B) u_B(\lambda_j) c_C(\lambda_j).$$

On the other hand, Lemma 3.8.4 in [1] shows that

$$\int_C \big(1 - \mathrm{e}^{-\lambda \varphi_C}\big) \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \widehat{T}^n 1_B \,\mathrm{d}m = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \int_{C_n} 1_B \,\mathrm{d}m \underset{\lambda\searrow 0}{\longrightarrow} 1,$$

where $C_0 := C$ and $C_n := T^{-n}C \setminus \bigcup_{k=0}^{n-1} T^{-k}C$ for $n \geq 1$. Together, these prove $c_C(\lambda_j) \sim 1/u_B(\lambda_j)$ as $j \to \infty$. But as A has minimal wandering rate, (5.1) ensures that $c_A(\lambda) \sim c_C(\lambda)$, and we end up with $c_A(\lambda_j) \sim 1/u_B(\lambda_j) \sim 1/u_T(\lambda_j)$. Our claim follows since this can be done inside any sequence of λ's decreasing to 0.

(ii) Suppose that A is a uniform set for $f \in L^1_+(m)$, with return sequence $(a(n))_{n\geq 1}$. Then, this is also true for all $B \in \mathcal{F} \cap A$. Thus, by the asymptotic renewal equation for uniform sets (cf. 3.8.6 of [1]),

$$c_B(\lambda) \underset{\lambda\searrow 0}{\sim} \frac{1}{u_T(\lambda)} \quad \text{for } B \in \mathcal{F} \cap A,$$

where $u_T(\lambda)$ does not depend on B. In particular, $c_B(\lambda) \sim c_A(\lambda)$ for all $B \in \mathcal{F} \cap A$, which, due to (5.1), shows that A has minimal wandering rate. □

Proof of Theorem 2.2. Let $(X, \mathcal{B}, m, T)$ be the Kakutani tower of $(\Omega, \mathcal{A}, P, S, \varphi)$, then

$$L_\Omega(n) = \mathbb{E}(\varphi \wedge n) \sim \frac{n}{(1+\gamma)\tilde{a}(n)},$$

whence by Theorem 3.8.1 of [1], for large n,

$$a(n)=\sum_{k=1}^{n} m\big(\Omega\cap Y^{-k}\Omega\big)\geq \frac{n}{2L_\Omega(n)}>\frac{1}{2}\tilde{a}(n).$$

Thus, for all $\varepsilon>0$,

$$\frac{n\vartheta_\mu(\varepsilon a(n))}{a(n)}<\frac{2n\vartheta_\mu((\varepsilon/2)\tilde{a}(n))}{\tilde{a}(n)}\xrightarrow[n\to\infty]{}0.$$

Now, by Theorem 3.1(a), T is weakly pointwise dual ergodic with return sequence $a(n)$, and in view of (2.4) and (5.2), Ω has minimal wandering rate. According to the asymptotic renewal equation of Theorem 5.1, $c_\Omega(\lambda)\sim\frac{1}{u(\lambda)}$ as $\lambda\searrow 0$ whence, by Karamata's theorem, we see that indeed $a_\varphi(n)\sim\tilde{a}(n)$. □

6. The one-sided law of the iterated logarithm

The $\gamma=1$ version of the law of the iterated logarithm follows immediately from the previous results.

Proof of Theorem 2.1(b). It has already been pointed out in [4] that (2.3) holds for positive stationary processes satisfying a weak law of large numbers provided that the corresponding infinite measure preserving Kakutani tower is weakly pointwise dual ergodic. The latter is immediate from Theorem 3.1. □

We now prove Theorem 2.3 by applying [5].

Proof of Theorem 2.3. We first show that under the present assumptions

$$\sum_{n=1}^{\infty}\frac{\vartheta_P(n)}{n}<\infty. \tag{6.1}$$

To see this, note that $\vartheta_P(a(a(n)))\leq\frac{a(n)}{n}$ for large n. Let b be asymptotically inverse to a in that $b(a(n))\sim a(b(n))\sim n$, then b is $\frac{1}{\gamma}$-regularly varying, and for large $N:=a(a(n))$ we have

$$\vartheta_P(N)=\vartheta_P\big(a\big(a(n)\big)\big)\leq\frac{a(n)}{n}\leq\frac{2b(N)}{b(b(N))}=\frac{2}{c(N)},$$

where $c(N):=\frac{b(b(N))}{b(N)}$ is $(\frac{1}{\gamma^2}-\frac{1}{\gamma})$-regularly varying. Since $Nc(N)$ is $(\frac{1}{\gamma^2}-\frac{1}{\gamma}+1)$-regularly varying we indeed get $\sum_{N=1}^{\infty}\frac{1}{Nc(N)}<\infty$.

As an immediate consequence of (6.1), $(\Omega,\mathcal{A},P,S,\alpha)$ is *strongly mixing from below* as defined in [5]. That is, for every $B\in\mathcal{A}$ with $P(B)>0$ there are $\eta(n)\searrow 0$ with $\sum_{n\geq 1}\eta(n)/n<\infty$ (in our case $\eta(n):=\vartheta_P(n)$) such that for all $k\geq 1$ and $A\in\sigma(\alpha_k)$ we have

$$P\big(A\cap S^{-(n+k)}B\big)\geq P(A)P(B)-\eta(n)\quad\text{for } n\geq 1.$$

Let $(X,\mathcal{B},m,T)$ be the Kakutani tower of $(\Omega,\mathcal{A},P,S,\varphi)$. Part (c) of Theorem 3.1, with $\mu=P$, ensures that T is pointwise dual ergodic and that Ω is a Darling–Kac set.

The assumptions of Theorem 4 in [5] are now satisfied. Hence,

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{e}^{-\beta\tau(n)}<\infty\quad\forall\beta>1$$

$$\Longrightarrow\quad\varlimsup_{n\to\infty}\frac{1}{a(n/\tau(n))\tau(n)}S_n(f)\leq K_\gamma\int_X f\,\mathrm{d}\mu\quad\text{a.e. }\forall f\in L^1_+$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{e}^{-\beta\tau(n)} = \infty \quad \forall \beta < 1$$

$$\Longrightarrow \quad \overline{\lim_{n\to\infty}} \frac{1}{a(n/\tau(n))\tau(n)} S_n(f) \geq K_\gamma \int_X f \,\mathrm{d}\mu \quad \text{a.e.}\ \forall f \in L_+^1.$$

Using the inversion technique in Section 5 of [5], statements (2.8) and (2.9) of Theorem 2.3 follow, and (2.7) is a consequence of those. □

7. Interval maps

Throughout this section, m denotes one-dimensional Lebesgue measure. A *piecewise monotonic* (*increasing*) *map of the interval* is a triple (Ω, S, α) where Ω is a bounded interval, α is a finite or countable generating partition $(\bmod\, m)$ of Ω into open intervals, and $S : \Omega \to \Omega$ is a map such that the restriction $S : A \to SA$ is an (increasing) homeomorphism for each $A \in \alpha$ so that both $S|_A : A \to SA$ and its inverse $v_A : SA \to A$ are absolutely continuous.

In this case, each iterate (Ω, S^k, α_k), $k \geq 1$, is also piecewise monotonic (increasing), where $\alpha_k := \bigvee_{i=0}^{k-1} S^{-i}\alpha$. Generalizing the above, we let, for $A \in \alpha_k$, v_A denote the inverse of $S^k|_A : A \to S^k A$, so that the transfer operator $\widehat{S}$ of S (with respect to m) satisfies

$$\widehat{S}^k f = \sum_{A\in\alpha_k} 1_{S^k A} v_A' \cdot (f \circ v_A), \quad \text{where } v_A' := \frac{\mathrm{d}m \circ v_A}{\mathrm{d}m}.$$

Consider the following properties for a piecewise monotonic map of the interval (Ω, S, α):

(A) *Adler's condition*: for all $A \in \alpha$, $S|_A$ extends to a C^2 diffeomorphism on $\overline{A}$, and $S''/(S')^2$ is bounded on $\bigcup_{A\in\alpha} A$.
(B) *Big images*: $\inf_{A\in\alpha} m(SA) > 0$.
(R) *Rychlik's condition*: $\sum_{A\in\alpha} \|1_{SA} v_A'\|_{\widehat{BV}} =: \mathcal{R} < \infty$.
(U) *Uniform expansion*: $\inf |S'| > 1$.

Recall that (A) ensures $v_A''/v_A' \leq M < \infty$, whence $v_A' = \mathrm{e}^{\pm M} m(A)/m(SA)$ on SA for all $A \in \alpha$. In (R), the space $\widehat{BV}$ is the subspace of those functions f in $L^\infty(m)$ with a version f^* in BV, the space of functions of bounded variation. With $\bigvee_\Omega(f^*)$ denoting the variation of $f^* : \Omega \to \mathbb{R}$, the norm $\|\cdot\|_{\widehat{BV}}$ is defined by

$$\|f\|_{\widehat{BV}} := \|f\|_\infty + \widehat{\bigvee_\Omega} f, \quad \text{where } \widehat{\bigvee_\Omega} f := \inf\left\{ \bigvee_\Omega (f^*) : f^* = f\ m\text{-a.e.} \right\}.$$

Piecewise monotonic maps (Ω, S, α) of the interval with properties $(P_1), \ldots, (P_N)$ will be called $P_1 \cdots P_N$-maps (e.g. ABU-maps, RU-maps).

Lemma 7.1. *Any ABU-map is an RU-map.*

Proof. This is similar to Proposition 2 of [22]. For any piecewise monotonic map (Ω, S, α), (A) and (B) imply (R). Indeed,

$$\begin{aligned}
\sum_{A\in\alpha} \|1_{SA} v_A'\|_{\widehat{BV}} &\leq 3\|v_A'\|_\infty + \bigvee_{SA}(v_A') \\
&\overset{\text{(A)}}{\leq} \sum_{A\in\alpha} \left(3\mathrm{e}^M \frac{m(A)}{m(SA)} + \int_{SA} |v_A''| \,\mathrm{d}m \right) \\
&\leq \sum_{A\in\alpha} \left(3\mathrm{e}^M \frac{m(A)}{m(SA)} + M \int_{SA} v_A' \,\mathrm{d}m \right)
\end{aligned}$$

$$\leq \sum_{A\in\alpha}\left(3\mathrm{e}^M \frac{m(A)}{m(SA)} + M\mathrm{e}^M m(A)\right)$$
$$\overset{\text{(B)}}{\leq} M' \sum_{A\in\alpha} m(A) = M' m(\Omega).$$

□

Ergodic properties of Rychlik's maps

Suppose that (Ω, S, α) is an RU-map, then, according to [16],

- $(\Omega, \mathcal{B}, m, S, \alpha)$ is a fibred system, where $\mathcal{B}$ is the Borel σ-field,
- the ergodic decomposition of $(\Omega, \mathcal{B}, m, S)$ is finite, and
- to each ergodic component there corresponds an absolutely continuous invariant probability, with density in BV and with respect to which S is isomorphic to the product of a finite permutation and a mixing RU-map.

Moreover, if S is *weakly mixing* (with respect to m in the sense that $f : \Omega \to \mathbb{S}^1$ measurable, $f \circ S = \lambda f$ a.e. where $\lambda \in \mathbb{S}^1$ implies f constant), then there are constants $K > 0$ and $\theta \in (0, 1)$ such that

$$\left\| \widehat{S}^n f - \left(\int_\Omega f \,\mathrm{d}m\right) h \right\|_{\widehat{BV}} \leq K\theta^n \|f\|_{\widehat{BV}},$$

where h is the unique T-invariant probability density. In this case, let $\mathrm{d}P := h\,\mathrm{d}m$ and $\mu := m|_{[h>0]}$. Then [8] shows that the probability preserving fibred system

$$(\Omega, \mathcal{B}, P, S, \alpha) \text{ is exponentially } \vartheta_\mu\text{-mixing.} \tag{7.1}$$

We next observe that $\psi^*(N) < \infty$ already implies continued fraction mixing in the present context, provided h has a positive lower bound. (This shows that for such ABU-maps, the conclusions of Theorem 2.3 already follow from earlier results for continued fraction mixing systems.)

Proposition 7.1 (Continued fraction mixing ABU-maps). *Let (Ω, S, α) be a weakly mixing ABU-map with invariant density h bounded away from 0. If $\exists N \geq 1$ such that $\psi^*(N) < \infty$, then (Ω, S, α) is continued fraction mixing.*

Proof. Suppose that $\eta \in (0, 1)$ satisfies $h = \eta^{\pm 1}$, which we use to abbreviate $\eta \leq h \leq \eta^{-1}$. A standard argument then shows that $\sup_{n\geq 1} \sup_\Omega |(S^n)''|/((S^n)')^2 < \infty$, and we can also assume that

$$v_A' = \eta^{\pm 1} \frac{m(A)}{m(S^k A)} \quad \text{on } S^k A \text{ for all } A \in \alpha_A, k \geq 1.$$

Let $\widehat{S}_P$ be the transfer operator with respect to the absolutely continuous invariant probability P, then $\widehat{S}_P f = \widehat{S}(hf)/h$, and therefore

$$\widehat{S}_P^n f = \eta^{\pm 2} \widehat{S}^n f \quad \text{for all } n \geq 1, f \in L_+^\infty.$$

We now show that

$$m\big(S^k A\big) \geq \Delta \quad \text{for all } A \in \alpha_k, k \geq 1, \text{ where } \Delta := \frac{\eta^6}{\psi^*(N)}. \tag{7.2}$$

To this end, let $B \subset \Omega$ be measurable with $m(B) > 0$. Then

$$\frac{P(A)}{m(S^k A)} \leq \eta^{-1} \frac{m(A)}{m(S^k A)}$$
$$\leq \eta^{-2} \frac{1}{m(S^{-N} B)} \int_{S^{-N} B} 1_{S^k A} v_A' \,\mathrm{d}m$$

$$
\begin{aligned}
&= \eta^{-2} \frac{1}{m(S^{-N}B)} \int_{S^{-N}B} \widehat{S}^k 1_A \,\mathrm{d}m \\
&\le \eta^{-6} \frac{1}{P(S^{-N})} \int_{S^{-N}B} \widehat{S}_P^k 1_A \,\mathrm{d}P \\
&= \eta^{-6} \frac{1}{P(B)} P\big(A \cap S^{-N+k}B\big) \\
&\le \eta^{-6} \psi^*(N) P(A)
\end{aligned}
$$

whence (7.2), as claimed.

To complete the proof of the proposition, we can then proceed as in the proof of Theorem 1(b) in [8]. □

By exponential ϑ_μ-mixing (7.1), Theorem 2.1 implies the general

Proposition 7.2 (Stable limit theorem for RU-maps). *Suppose that (Ω, S, α) is a weakly mixing RU-map with absolutely continuous invariant probability $\mathrm{d}P = h\,\mathrm{d}m$. Let $\varphi : \Omega \to \mathbb{N}$ be α-measurable, and denote $\varphi_n := \sum_{k=0}^{n-1} \varphi \circ S^k$. If $a(n) := a_\varphi(n) = \sum_{k=1}^{n} P([\varphi_k \le n])$ is γ-regularly varying for $\gamma \in (0, 1]$, with asymptotic inverse b, then*

$$
\frac{\varphi_n}{b(n)} \xrightarrow{\mathfrak{d}} Z_\gamma.
$$

Remark 7. *For the subfamily of those RU-maps S which satisfy* (A) *plus the* finite image condition (F) *which requires $\{SA : A \in \alpha\}$ to be finite, more general stable limit theorems (for observables φ which need not have constant sign) follow from* [7] *(see the end of Section* 5 *there). These* AFU-maps *occur as induced maps of the infinite measure preserving* AFN-maps *studied in* [22,23] *(generalizing* [18]*). The final subsections below illustrate that the present results allow us to analyse infinite measure preserving interval maps more general than those studied in* [23].

The asymptotic type

Next, we turn to the asymptotic identification, via Theorem 2.2, of the normalizing constants $a_\varphi(n)$ in this setup.

Proposition 7.3 (Asymptotic type of ABU-maps). *Suppose that (Ω, S, α) is a weakly mixing ABU-map with absolutely continuous invariant probability $\mathrm{d}P = h\,\mathrm{d}m$.*

Suppose that $\varphi : \Omega \to \mathbb{N}$ is α-measurable and satisfies

$$
\frac{P([\varphi \ge n])}{m([\varphi \ge n])} \underset{n\to\infty}{\longrightarrow} c \in (0, \infty), \tag{7.3}
$$

as well as

$$
\int_\Omega \varphi \wedge n \,\mathrm{d}m \underset{n\to\infty}{\sim} \frac{n}{\Gamma(2-\gamma)\Gamma(1+\gamma)\tilde{a}(n)}, \tag{7.4}
$$

where $\tilde{a}(t)$ is regularly varying with index $\gamma \in (0, 1]$. Then

$$
a_\varphi(n) = \sum_{k=1}^{n} P\big([\varphi_k \le n]\big) \underset{n\to\infty}{\sim} c^{-1} \tilde{a}(n).
$$

The main point is condition (2.4) of Theorem 2.2.

Lemma 7.2. *Suppose that (Ω, S, α) is a weakly mixing ABU-map with absolutely continuous invariant probability $\mathrm{d}P = h\,\mathrm{d}m$. Suppose that $\varphi : \Omega \to \mathbb{N}$ is α-measurable and satisfies*

$$
\int_\Omega \varphi \wedge n \,\mathrm{d}m = \mathrm{O}\bigg(\int_\Omega \varphi \wedge n \,\mathrm{d}P\bigg) \quad \text{as } n \to \infty.
$$

Then there is some $\Phi \in L^1(P)_+$ such that

$$\widehat{S}_P(\varphi \wedge n) \le \left(\int_\Omega \varphi \wedge n \,\mathrm{d}P\right)\Phi \quad \textit{a.e. for all } n \ge 1.$$

Proof. We first record a corresponding statement with respect to Lebesgue measure m,

$$\exists \tilde{M} > 0 \text{ so that } \quad \widehat{S}(\varphi \wedge n) \le \tilde{M} \int_\Omega (\varphi \wedge n)\,\mathrm{d}m \quad \forall n \ge 1. \tag{7.5}$$

Letting $F_n := \sum_{A\in\alpha}(\varphi(A) \wedge n)m(A)1_{SA}$ and $M' := (\inf_{A\in\alpha} m(SA))^{-1}$, we have indeed

$$\begin{aligned}\widehat{S}(\varphi \wedge n) &= \sum_{A\in\alpha}\big(\varphi(A) \wedge n\big)\widehat{S}1_A = \sum_{A\in\alpha}\big(\varphi(A) \wedge n\big)v'_A 1_{SA}\\ &\le \mathrm{e}^M \sum_{A\in\alpha}\big(\varphi(A) \wedge n\big)\frac{m(A)}{m(SA)}1_{SA} \le M'\mathrm{e}^M F_n.\end{aligned}$$

But $\|F_n\|_\infty \le \sum_{A\in\alpha}(\varphi(A) \wedge n)m(A) = \int(\varphi \wedge n)\,\mathrm{d}m$, whence (7.5). To deduce (2.4), note that

$$\begin{aligned}\widehat{S}_P(\varphi \wedge n) &= 1_{[h>0]}\frac{1}{h}\widehat{S}\big(h(\varphi \wedge n)\big)\\ &\le 1_{[h>0]}\|h\|_\infty \frac{1}{h}\widehat{S}(\varphi \wedge n)\\ &\le 1_{[h>0]}\|h\|_\infty \frac{1}{h}\tilde{M}\int_\Omega (\varphi \wedge n)\,\mathrm{d}m \quad \text{by (7.5)}\\ &\sim 1_{[h>0]}\|h\|_\infty \frac{1}{h}c\tilde{M}\int_\Omega (\varphi \wedge n)\,\mathrm{d}P\\ &= \left(\int_\Omega (\varphi \wedge n)\,\mathrm{d}P\right)\Phi,\end{aligned}$$

where $\Phi := 1_{[h>0]}\|h\|_\infty \frac{1}{h}cM \in L^1(P)$ since $h \in L^1(m)$. □

Proof of Proposition 7.3. We are going to verify the conditions of Theorem 2.2. Note first that adaptedness follows from the other two by exponential ϑ_μ-mixing (7.1). Next, condition (2.5) is immediate from (7.3), as

$$\int_\Omega \varphi \wedge n \,\mathrm{d}P = \sum_{k=1}^n P\big([\varphi \ge k]\big) \underset{k\to\infty}{\overset{(7.3)}{\sim}} c\sum_{k=1}^n m\big([\varphi \ge k]\big) = c\int_\Omega \varphi \wedge n \,\mathrm{d}m.$$

To check condition (2.4), use this and the previous lemma. □

The common image property

Typically, for interval maps, one will first obtain information on $[\varphi \ge n]$ in terms of Lebesgue measure m. This needs to be combined with an analysis of h to yield information on $P([\varphi \ge n])$, and hence on $\int \varphi \wedge n \,\mathrm{d}P = \sum_{k=1}^n P([\varphi \ge k])$. Here we discuss simple sufficient conditions which allow us to validate property (7.3) of Proposition 7.3 in this way.

Consider a piecewise increasing map (Ω, S, α), with $\Omega = [\omega_l, \omega_r]$. We shall say that (Ω, S, α) has the *common image property* if $\bigcap_{A\in\alpha} SA = (\omega_l, \omega_l + z_S)$ where $z_S > 0$. Evidently, this entails the big image property (B). Moreover, we find:

Lemma 7.3. *Suppose that (Ω, S, α) is a piecewise increasing AU-map with the common image property and an absolutely continuous invariant probability* $\mathrm{d}P = h\,\mathrm{d}m$ *on* $\Omega = [\omega_l, \omega_r]$. *Then*

$$\operatorname*{essinf}_{[\omega_l, \omega_l + z_S]} h > 0. \tag{7.6}$$

Moreover, S is weakly mixing.

Proof. We assume w.l.o.g. that $\Omega = [0,1]$. Fix a version $h \in BV$ of the invariant density and set

$$\mathcal{J} := \Big\{ J \subset [0,1]\colon\ J \text{ is a nonempty open interval with } \inf_J h > 0 \Big\}.$$

It is clear that $\mathcal{J} \neq \varnothing$. We need to show that $(0, z_S) \in \mathcal{J}$. Observe first that

$$\text{there exist } J \in \mathcal{J} \quad \text{and} \quad A \in \alpha \quad \text{so that } J \cap \partial A \neq \varnothing. \tag{7.7}$$

To see this, suppose otherwise i.e. that $\forall J \in \mathcal{J}, \exists A_J \in \alpha\colon\ J \subset A_J$. Then $J \in \mathcal{J}$ implies $SJ \in \mathcal{J}$ since for $x \in SJ \subset SA_J$,

$$h(x) \geq v'_{A_J}(x) h(v_{A_J} x) \geq \text{const} \cdot m(A_J) \inf_J h > 0.$$

But then, for each $k \geq 1$, $S^k J \subset A_k$ for some $A_k \in \alpha$, an impossibility since this entails $m(S^k J) \geq \lambda^k m(J) \to \infty$.

Due to (7.7), there are $J \in \mathcal{J}$ and $A = (u, v) \in \alpha$ such that $u \in J$. Set $J_0 := A \cap J = (u, w)$ with $u < w$. It follows as above that $I_0 := SJ_0 \in \mathcal{J}$, and the common image property implies $I_0 = (0, c)$ for some $c > 0$.

Note then that there exist some $J' \in \mathcal{J}$ and $A' \in \alpha$ such that $J' \supset A'$: unless I_0 contains some A', it is contained in a specific $A' \in \alpha$, and by the special structure of our map there is some $k \geq 1$ for which $S^k I \subset A' \subset S^{k+1} I$. By the argument proving (7.7) we have $J' := S^{k+1} I \in \mathcal{J}$.

But then $(0, z_S) \subset SA' \in \mathcal{J}$ as required.

Finally, in view of Lemma 7.1 and [16], S has only finitely many ergodic acims, and these have densities $h_i \in BV$, which can be chosen to be lower semicontinuous, so that the sets $[h_i > 0]$ are open and pairwise disjoint. However, by the above, each $[h_i > 0]$ contains $(0, z_S)$. Hence h and P are unique, meaning that S is ergodic. Moreover, the structural results of [16] also show that there is a finite tail decomposition $h = \sum_{j=0}^{p-1} g_j$ with $g_j \in BV$ and the $[g_j > 0]$ open and pairwise disjoint, such that $S[g_j > 0] = [g_l > 0]$ a.e., $l = j + 1 \bmod p$, and S is weakly mixing iff $p = 1$. Bounded variation of the g_j together with (7.6) now implies that (after renumbering the g_j if necessary) there is some $y > 0$ for which $(0, y) \subseteq [g_0 > 0]$. However, there is at least one cylinder $A = (a, b) \in \alpha$ with $a < y$, and then $(a, c) := [g_0 > 0] \cap A$ is nonempty and open. Due to the common image property, $S(a, c) \subseteq [g_1 > 0]$ has nonempty open intersection with $(0, y)$. Hence $[g_0 > 0] = [g_1 > 0]$, as these sets overlap. Therefore $p = 1$. □

This immediately allows us to deal with situations in which φ only diverges at 0.

Example 2. *Suppose that $([0,1], S, \alpha)$ is a piecewise increasing AU-map with the common image property and absolutely continuous invariant probability* $\mathrm{d}P = h\,\mathrm{d}m$. *Suppose that $\varphi\colon [0,1] \to \mathbb{N}$ is α-measurable and satisfies $\overline{[\varphi \geq n]} = [0, y_n]$ for $n \geq n_0$, where*

$$y_n \sim \frac{1}{\Gamma(1-\gamma)\Gamma(1+\gamma)\tilde{a}(n)}$$

with $\tilde{a}$ regularly varying of index $\gamma \in (0,1]$. Then $a_\varphi(n) \sim \tilde{a}(n)$.

Indeed, we need only check condition (7.3) of Proposition 7.3. Fixing a version $h \in BV$ of the invariant density, Lemma 7.3 shows that $\lim_{x \to 0+} h(x) =: h(0^+) > 0$. Whence $P([\varphi \geq n]) = \int_{[\varphi \geq n]} h\,\mathrm{d}m \sim h(0^+) m([\varphi \geq n])$, and our claim follows since, by Karamata's theorem,

$$\int_\Omega (\varphi \wedge n)\,\mathrm{d}m \underset{n \to \infty}{\sim} \frac{n}{\Gamma(2-\gamma)\Gamma(1+\gamma)\tilde{a}(n)}.$$

Next, we record a little preparatory observation which will enable us to also study functions φ which diverge at countably many points. (This will be the case for the return time functions of the null-recurrent maps studied in the final subsection below.)

Lemma 7.4. *Let $h:\Omega \to [0,\infty)$ have right-hand limits $h(x^+)$ everywhere. Let $x_j, y_{j,n} \in \Omega$, $j,n \geq 0$, be such that for each n the sets $(x_j, x_j + y_{j,n})$ are pairwise disjoint, and suppose that there are $s_j \in [0,\infty)$ with $\sum_{j\geq 0} s_j < \infty$, and $q_n \searrow 0$ for which $y_{j,n}/q_n \to s_j$ as $n \to \infty$, uniformly in j. If $s_j h(x_j^+) > 0$ for some j, then*

$$\sum_{j\geq 0} \int_{(x_j, x_j+y_{j,n})} h(x)\,\mathrm{d}x \underset{n\to\infty}{\sim} \Big(\sum_{j\geq 0} s_j h\big(x_j^+\big)\Big) q_n. \tag{7.8}$$

Proof. Assume w.l.o.g. that $s_0 h(x_0^+) > 0$, and take any $\varepsilon > 0$. Choose n_1 so large that $y_{j,n} \leq \mathrm{e}^{\varepsilon} s_j q_n$ for $n \geq n_1$ and all j. Take $J \geq 1$ so large that $(\sup h)\sum_{j>J} s_j < \varepsilon \sum_{j\geq 0} s_j h(x_j^+)$. Next, there is some $n_2 \geq n_1$ such that for all $n \geq n_2$ and all $j \leq J$,

$$\sup_{(x_j, x_j+\mathrm{e}^{\varepsilon} s_j q_n)} h \leq \mathrm{e}^{\varepsilon} h\big(x_j^+\big) + \frac{\varepsilon s_0 h(x_0^+)}{J+1}$$

(here $h(x_j^+)$ need not be positive, but $h(x_0^+)$ is). Then, for all $n \geq n_2$,

$$\begin{aligned}\sum_{j\geq 0} \int_{(x_j, x_j+y_{j,n})} h(x)\,\mathrm{d}x &\leq \mathrm{e}^{\varepsilon} q_n \sum_{j\geq 0} s_j \sup_{(x_j, x_j+\mathrm{e}^{\varepsilon} s_j q_n)} h \\ &\leq \mathrm{e}^{\varepsilon} q_n \left(\mathrm{e}^{\varepsilon} \sum_{j=0}^{J} s_j h\big(x_j^+\big) + \varepsilon s_0 h\big(x_0^+\big) + \sup h \sum_{j>J} s_j\right) \\ &\leq \mathrm{e}^{\varepsilon}\big(\mathrm{e}^{\varepsilon} + 2\varepsilon\big)\Big(\sum_{j\geq 0} s_j h\big(x_j^+\big)\Big) q_n.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, this proves one half of our claim. The corresponding estimate from below follows by similar but even simpler arguments which we omit. □

We can now go beyond the scenario of Example 2 above. Situations of the following type naturally occur in the study of interval maps with neutral fixed points, see below.

Proposition 7.4 (**φ with countably many singularities**). *Let (Ω, S, α) be a weakly mixing piecewise increasing ABU-map with absolutely continuous invariant probability $\mathrm{d}P = h\,\mathrm{d}m$. Suppose that $\varphi:\Omega \to \mathbb{N}$ is α-measurable and such that for n sufficiently large, $[\varphi \geq n]$ is a countable disjoint union of intervals $(x_j, x_j + y_{j,n})$ satisfying the assumptions of Lemma 7.4, where*

$$q_n \underset{n\to\infty}{\sim} \frac{1}{\Gamma(1-\gamma)\Gamma(1+\gamma)\tilde{a}(n)}$$

with $\tilde{a}$ regularly varying of index $\gamma \in (0,1]$. Then

$$a_{\varphi}(n) = \sum_{k=1}^{n} P\big([\varphi_k \leq n]\big) \underset{n\to\infty}{\sim} \Big(\sum_{j\geq 0} s_j h\big(x_j^+\big)\Big) \tilde{a}(n).$$

Proof. It is clear that $m([\varphi \geq n]) \sim (\sum_{j\geq 0} s_j) q_n$. According to Lemma 7.4,

$$P\big([\varphi \geq n]\big) = \sum_{j\geq 0} \int_{(x_j, x_j+y_{j,n})} h(x)\,\mathrm{d}x \sim \Big(\sum_{j\geq 0} s_j h\big(x_j^+\big)\Big) q_n,$$

so that by Karamata's theorem

$$\int_{\Omega} (\varphi \wedge n)\,\mathrm{d}P \underset{n\to\infty}{\sim} \frac{(\sum_{j\geq 0} s_j h(x_j^+))n}{\Gamma(2-\gamma)\Gamma(1+\gamma)\tilde{a}(n)},$$

and Proposition 7.3 applies. □

Some infinite measure preserving interval maps

We conclude with a class of infinite measure preserving interval maps T with indifferent fixed point, which induce probability preserving maps S of the above type. These T do not, in general, belong to the family of AFN-maps studied in [23], see Example 3 below.

Proposition 7.5 (Maps with indifferent fixed points). *Let (X, T, β) be a piecewise increasing A-map on $X = [0, 1]$ with the common image property which satisfies $\inf_{(\varepsilon,1)} T' > 1$ for every $\varepsilon \in (0, 1)$. Assume that T possesses a leftmost cylinder $B_* = (0, \xi)$, and that $z_T := \inf_{B\in\beta} m(TB)$ satisfies $z_T > \xi$. Suppose that T is convex near 0, and satisfies, for some $\gamma \in (0, 1)$,*

$$T(x) \sim x + \kappa x^{1+1/\gamma} + \mathrm{o}\big(x^{1+1/\gamma}\big) \quad \text{as } x \searrow 0. \tag{7.9}$$

Then T is conservative ergodic with an infinite invariant measure $m_T = h_T\,\mathrm{d}m$, with h_T bounded on each $(\varepsilon, 1)$. Moreover, T is weakly pointwise dual ergodic and exhibits Darling–Kac asymptotics,

$$\frac{S_n(f)}{a(n)} \underset{n\to\infty}{\overset{\mathfrak{d}}{\longrightarrow}} m_T(f)Y_\gamma \quad \text{for } f \in L^1_+(m_T),$$

with return sequence satisfying $a(n) \sim c/n^\gamma$ for some $c > 0$.

Proof. Let $\Omega := [\xi, 1]$, and consider the induced map $S = T_\Omega$ and the corresponding return time function $\varphi = \varphi_\Omega$. We are going to show that the induced system naturally comes as a piecewise increasing map (Ω, S, α), which together with φ satisfies the assumptions of Proposition 7.4. Therefore Proposition 7.2 applies, which via (2.11) entails the Darling–Kac limit. Weak pointwise dual ergodicity is implicit in the application of these propositions.

Note first that $B_* = \bigcup_{n\geq 1}(\tau_{n+1}, \tau_n)$, where $\tau_1 := \xi$ and $\tau_{n+1} := w(\tau_n)$ with $w := (T|_{B_*})^{-1}$ denoting the inverse of the leftmost branch of T. As a consequence of (7.9) we have (see Corollary on p. 82 of [18])

$$q_n := \tau_n \underset{n\to\infty}{\sim} (\kappa n/\gamma)^\gamma. \tag{7.10}$$

Fix any $B \in \beta \setminus \{B_*\}$, and let $B(k) := B \cap [\varphi = k]$, $k \geq 1$, which defines the cylinders of S inside B. The induced map S is trivially piecewise increasing and satisfies (U). It also satisfies (A), which is checked by the same argument as in [18] or [22], which we do not reproduce here.

Now $SB(1) = TB(1) \cap \Omega \supset (\xi, z_T - \xi)$. For $k \geq 2$, $TB(k) = (\tau_{k-1}, \tau_{k-2})$, and hence $SB(k) = T^k B(k) = T(\tau_1, \tau_0) \supset (\xi, z_T - \xi)$. Therefore S has the common image property.

Enumerating $\beta \setminus \{B_*\} = \{B_0, B_1, \dots\}$, we get

$$[\varphi > n] \cap B_j = v_{B_j}\big((0, \tau_n)\big) =: (x_j, x_j + y_{j,n})$$

for suitable x_j, $y_{j,n}$, where $v_{B_j} := (T|_{B_j})^{-1}$. This collection of intervals now satisfies the assumptions of Lemma 7.4 with $s_j := v'_{B_j}(\beta_j)$ where β_j is the left endpoint of B_j. Uniformity of $y_{j,n}/q_n \to s_j$ in j is a consequence of the distortion control for (the first iterate of) T provided by condition (A), which also implies $\sum s_j < \infty$. To see that $s_j h(x_j^+) > 0$ for some j, use the common image property of S and Lemma 7.3 to obtain some $z_S > 0$ such that $\inf_{(\xi,\xi+z_S)} h > 0$. Now choose some j for which $\beta_j \in (\xi, \xi + z_S)$. □

Example 3. *Fix $\gamma \in (0,1)$ and define $F(x) := x(1+x^{1/\gamma})/(1-x^{1/\gamma})$, $x \in X := (0,1)$. Let $T : X \to X$ be of the form $Tx = F(x) - F(\xi_n)$ for $x \in (\xi_n, \xi_{n+1})$, where $0 = \xi_0 < \xi_1 < \cdots < \xi_n \nearrow 1$ are such that $T\xi_n^- \le T\xi_{n+1}^- \nearrow 1$. Then T satisfies the assumptions of the preceding proposition, but the finite image property (F) is only fulfilled in those exceptional cases where $T\xi_n^- = 1$ for $n \ge n_0$.*

Note that whenever $\xi_0, \ldots, \xi_n$ satisfying the above requirements have been chosen, then the range of admissible values for ξ_{n+1} is the nonempty interval $[F^{-1}(2F(\xi_n) - F(\xi_{n-1})), F^{-1}(F(\xi_n) + 1)] =: J_n$, which is nondegenerate iff $F(\xi_n) - F(\xi_{n-1}) < 1$, that is, iff $T\xi_n^- \neq 1$. Therefore, if one never chooses ξ_{n+1} to be the right-hand endpoint of J_n, all these intervals are nondegenerate, and no T constructed this way belongs to the AFN-maps of [23].

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