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LIMIT THEORY FOR POINT PROCESSES IN MANIFOLDS

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Let $Y_i, i \geq 1$, be i.i.d. random variables having values in an m-dimensional manifold $\mathcal{M} \subset \mathbb{R}^d$ and consider sums $\sum_{i=1}^n \xi(n^{1/m} Y_i, \{n^{1/m} Y_j\}_{j=1}^n)$, where ξ is a real valued function defined on pairs $(y, \mathcal{Y})$, with $y \in \mathbb{R}^d$ and $\mathcal{Y} \subset \mathbb{R}^d$ locally finite. Subject to ξ satisfying a weak spatial dependence and continuity condition, we show that such sums satisfy weak laws of large numbers, variance asymptotics and central limit theorems. We show that the limit behavior is controlled by the value of ξ on homogeneous Poisson point processes on m-dimensional hyperplanes tangent to $\mathcal{M}$. We apply the general results to establish the limit theory of dimension and volume content estimators, Rényi and Shannon entropy estimators and clique counts in the Vietoris–Rips complex on $\{Y_i\}_{i=1}^n$.

1. Introduction. There has been recent interest in the statistical, topological and geometric properties of high-dimensional nonlinear data sets. Typically the data sets may be modeled as realizations of i.i.d. random variables $\{Y_i\}_{i=1}^n$ having support on an unknown nonlinear manifold $\mathcal{M}$ embedded in $\mathbb{R}^d$. Given a sample $\{Y_i\}_{i=1}^n$, whose pairwise distances are given, but whose coordinate representation is not, can one determine geometric characteristics of the manifold, including its intrinsic dimension and volume content? Can one recover global properties of the distribution of $\{Y_i\}_{i=1}^n$ such as its intrinsic entropy? These properties, as well as graph theoretic functionals such as clique counts in the Vietoris–Rips complex may be studied via the statistics of the form

$$\sum_{i=1}^n \xi\big(Y_i, \{Y_j\}_{j=1}^n\big), \tag{1.1}$$

where $\xi(\cdot,\cdot)$ is a real-valued measurable function defined on pairs $(y, \mathcal{Y})$, where $y \in \mathcal{Y}$ and $\mathcal{Y} \subset \mathbb{R}^d$ is locally finite, with $\xi(y, \mathcal{Y})$ locally determined in some sense.

Our goal is to establish the dependency between the large-n behavior of the statistics (1.1), the underlying point density κ of the $\{Y_i\}_{i=1}^n$ and the manifold $\mathcal{M}$.

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For $\mathcal{M} = \mathbb{R}^d$ there is a large literature describing limit theorems for (1.1) [5, 35, 36, 38, 39, 45] whereas when $\mathcal{M} \neq \mathbb{R}^d$, there is a relative dearth of results, although spatial data generated by a curved surface embedded in three dimensional space are arguably more natural than those involving data generated by flat surfaces. This paper partly redresses this situation under reasonably general conditions on κ and $\mathcal{M}$.

In Section 2 we present laws of large numbers and central limit theorems for (i) the Levina–Bickel dimension estimator for data $\{Y_i\}_{i=1}^n$ supported on a manifold, (ii) Rényi and Shannon entropy estimators for $\{Y_i\}_{i=1}^n$, (iii) volume estimators for the support of $\{Y_i\}_{i=1}^n$ and (iv) the order k clique count in the Vietoris–Rips complex on $\{Y_i\}_{i=1}^n$. The asymptotic normality results for dimension and entropy estimators appear to be new even in the setting of linear manifolds. The mean and variance asymptotics for the Levina–Bickel dimension estimator depend only on the dimension of $\mathcal{M}$ and are invariant with respect to κ, whereas for most of the other functionals considered here, the mean and variance asymptotics explicitly depend on κ via the integral $\int_{\mathcal{M}} (\kappa(y))^p \, dy$ for some $p \in \mathbb{R}$.

We shall derive these results from general theorems governing the limit theory of $\sum_{i=1}^n \xi(n^{1/m} Y_i, \{n^{1/m} Y_j\}_{j=1}^n)$, where m is the intrinsic dimension of $\mathcal{M}$, $n^{1/m}$ is a dilation factor and ξ belongs to a general class of translation invariant functionals that are determined by the locations of either the k nearest neighbors of y for some fixed $k \in \mathbb{N}$, or the points of $\mathcal{Y}$ within some fixed distance of y. For ξ locally determined in this way, then since the manifolds are themselves local, one might expect as the number of sample points increases, that the local contribution of each ξ at $y \in \mathcal{M}$ converges in distribution to its linearized version on the tangent space to y. In other words, for locally determined and translation invariant ξ, one might expect that the large n behavior of $\xi(n^{1/m} y, \{n^{1/m} Y_j\}_{j=1}^n) = \xi(\mathbf{0}, \{n^{1/m}(Y_j - y)\}_{j=1}^n)$ is controlled by the behavior of $\xi(\mathbf{0}, \mathcal{H}_{\kappa(y)})$, with $\mathcal{H}_{\kappa(y)}$ a homogeneous Poisson point process of intensity $\kappa(y)$ on a *tangent hyperplane* of Euclidean dimension m (here and elsewhere $\mathbf{0}$ denotes a point at the origin of $\mathbb{R}^k$). Subject to moment conditions on ξ, this is indeed the case, as shown by the general results of Section 3. The locally defined behavior of ξ, quantified in terms of dependency graphs involving radii of stabilization of ξ, yields central limit theorems via Stein's method.

In fact, our methods should work in still greater generality. Most of the examples considered in [5, 35, 36, 38, 39, 45] have ξ stabilizing, that is, they are locally determined in some sense, and it should be possible to adapt our methods to most of these examples. For example, we anticipate that our methods can be extended to establish the limit theory for the total edge length and other stabilizing functionals of the Delaunay and Voronoi graphs on random point sets in manifolds. We also expect that our methods extend to give the limit theory of statistics of germ-grain models, coverage processes and random sequential adsorption models generated by data on manifolds. Moreover, we anticipate that the theory presented here can be modified to establish limit theorems for generalized spacing statistics based on k nearest neighbor distances for random points in a manifold, including estimators of relative entropy such as those considered in [3], although these involve consideration of nontranslation invariant ξ so they do not automatically fall within the scope of this paper.

In many examples the functional $\xi(y, \mathcal{Y})$ is determined by inter-point distances in the vicinity of y. For $\mathcal{M}$ an arbitrary Riemannian manifold, it may be possible to derive similar limit results for such ξ using the geodesic distance rather than the extrinsic distance in $\mathbb{R}^d$ (by the Nash embedding theorem, such $\mathcal{M}$ can always be embedded into some $\mathbb{R}^d$). However, this lies beyond the scope of the present paper.

2. Stochastic functionals on manifolds.

2.1. *Terminology and definitions.* For $k \in \mathbb{N}$, let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^k$. Recall that $\mathbf{0}$ denotes a point at the origin of $\mathbb{R}^k$. For $r \in (0, \infty)$ and $z \in \mathbb{R}^k$, let $B_r(z) := \{y \in \mathbb{R}^k : \|y - z\| \le r\}$. Given $F \subset \mathbb{R}^k$, and $y \in \mathbb{R}^k$, $a > 0$, set $y + F := \{y + z : z \in F\}$ and $aF := \{az : z \in F\}$. If F is locally finite, let $\mathrm{card}(F)$ denote the cardinality (number of elements) of F. If also $y \in \mathbb{R}^k$ and $j \in \mathbb{Z}^+ := \{0, 1, 2, \ldots\}$, then let $N_j(y, F)$ be the Euclidean distance between y and its jth nearest neighbor in $F \setminus \{y\}$, that is,

$$N_j(y, F) := \inf\{r \ge 0 : \mathrm{card}(F \cap B_r(y) \setminus \{y\}) \ge j\} \tag{2.1}$$

with the infimum of the empty set taken to be $+\infty$. In particular, $N_0(y, F) = 0$. Let Φ be the distribution function for the standard normal random variable $\mathcal{N}(0, 1)$, and let $\xrightarrow{P}$ denote convergence in probability. For $\sigma > 0$, let $\mathcal{N}(0, \sigma^2)$ denote the random variable $\sigma\mathcal{N}(0, 1)$.

Let $m \in \mathbb{N}$ and $d \in \mathbb{N}$ with $m \le d$. A nonempty subset $\mathcal{M}$ of $\mathbb{R}^d$, endowed with the subset topology, is called an m-dimensional C^1 submanifold of $\mathbb{R}^d$ if for each $y \in \mathcal{M}$ there exists an open subset U of $\mathbb{R}^m$ and a continuously differentiable injection g from U to $\mathbb{R}^d$, such that (i) $y \in g(U) \subseteq \mathcal{M}$, and (ii) g is an open map from U to $\mathcal{M}$, and (iii) the linear map $g'(u)$ has full rank for all $u \in U$; see, for example, Theorem 2.1.2(v) of [7]. The pair (U, g) is called a *chart*. Let $\mathbb{M} := \mathbb{M}(m, d)$ denote the class of all m-dimensional C^1 submanifolds of $\mathbb{R}^d$ which are also closed subsets of $\mathbb{R}^d$.

Given $\mathcal{M} \in \mathbb{M}$, using a routine compactness argument we can choose an index set $\mathcal{I} \subset \mathbb{N}$, and a set $\{(y_i, \delta_i, U_i, g_i), i \in \mathcal{I}\}$ of ordered quadruples with $y_i \in \mathcal{M}$, $\delta_i \in (0, \infty)$ and (U_i, g_i) a chart for each i, such that (i) $\mathcal{M} \cap B_{3\delta_i}(y_i) \subset g_{y_i}(U_i)$ for each i, and (ii) $\mathcal{M} \subset \bigcup_{i \in \mathcal{I}} B_{\delta_i}(y_i)$.

We refer to $((U_i, g_i), i \in \mathcal{I})$ as an *atlas* for $\mathcal{M}$. Given such an atlas, we can find a *partition of unity* $\{\psi_i\}$ subordinate to the atlas, that is, a collection of functions $(\psi_i, i \in \mathcal{I})$ from $\mathcal{M}$ to $[0, 1]$, such that $\sum_{i \in \mathcal{I}} \psi_i(y) = 1$ for all $y \in \mathcal{M}$, and such that for each i, $\psi_i(y) = 0$ for $y \notin g_i(U_i)$, and $\psi_i \circ g_i$ is a measurable function on U_i. The more common definition of a partition of unity has some extra differentiability conditions on ψ_i but these are not needed here. With our more relaxed definition, the existence of a partition of unity is completely elementary to prove.

Given $i \in \mathcal{I}$ and $x \in U_i$, let $D_{g_i}(x) := \sqrt{\det(J_{g_i}(x))^T (J_{g_i}(x))}$, with J_{g_i} standing for the Jacobian of g_i. For bounded measurable $h : \mathcal{M} \to \mathbb{R}$, the integral $\int_{\mathcal{M}} h(y)\,dy$ is defined by

$$\int_{\mathcal{M}} h(y)\,dy = \sum_{i \in \mathcal{I}} \int_{U_i} h\big(g_i(x)\big)\psi_i\big(g_i(x)\big) D_{g_i}(x)\,dx, \tag{2.2}$$

which is well-defined in the sense that it does not depend on the choice of atlas or the partition of unity. Equation (2.2) is discussed further at (5.18) below.

Given any manifold $\mathcal{M} \in \mathbb{M}$ and nonempty $\mathcal{K} \subset \mathcal{M}$, the relative interior of $\mathcal{K}$ consists of all those $y \in \mathcal{K}$ such that y has a neighborhood in $\mathcal{M}$ that is contained in $\mathcal{K}$. The *boundary* of $\mathcal{K}$ is the set of all other $y \in \mathcal{K}$ (possibly empty). Also, we set $\operatorname{diam}(\mathcal{K}) := \sup\{\|x - y\| : x, y \in \mathcal{K}\}$ (possibly infinite). We say $\mathcal{K}$ is *locally conic* if $0 < \operatorname{diam}(\mathcal{K}) < \infty$ and

$$\inf\left\{r^{-m} \int_{B_r(w)\cap\mathcal{K}} dy : r \in \big(0, \operatorname{diam}(K)\big], w \in \mathcal{K}\right\} > 0. \tag{2.3}$$

We say $\mathcal{K}$ is an m-dimensional C^1 *submanifold-with-boundary* of $\mathcal{M}$ if for all y in the boundary of $\mathcal{K}$, there exists a choice of chart (U, g) for $\mathcal{M}$ such that $\mathbf{0} \in U$, and $g(\mathbf{0}) = y$, and $g([0,\infty) \times \mathbb{R}^{m-1}) = g(U) \cap \mathcal{K}$. This includes the possibility that $\mathcal{K}$ has empty boundary. If $\mathcal{K}$ is a compact m-dimensional C^1 submanifold-with-boundary of $\mathcal{M}$ then it is locally conic; see Remark 4.1.

Given $\mathcal{M} \in \mathbb{M}$, a *probability density function* on $\mathcal{M}$ is a nonnegative scalar field κ on $\mathcal{M}$ satisfying $\int_{\mathcal{M}} \kappa(y)\,dy = 1$. Let $\mathbb{P}(\mathcal{M})$ denote the class of probability density functions on $\mathcal{M}$. Given $\kappa \in \mathbb{P}(\mathcal{M})$, let $\mathcal{K}(\kappa)$ denote the support of κ, that is, the smallest closed set $\mathcal{K} \subset \mathcal{M}$ such that $\int_{\mathcal{K}} \kappa(y)\,dy = 1$. Given also $\rho \in \mathbb{R}$, define the integral

$$I_\rho(\kappa) := \int_{\mathcal{K}(\kappa)} \big(\kappa(y)\big)^\rho\,dy := \int_{\mathcal{M}} \big(\kappa(y)\big)^\rho \mathbf{1}\big\{\mathcal{K}(\kappa)\big\}(y)\,dy. \tag{2.4}$$

Let $\mathbb{P}_b(\mathcal{M})$ denote the class of bounded probability density functions $\kappa \in \mathbb{P}(\mathcal{M})$, such that $\mathcal{K}(\kappa)$ is compact. Let $\mathbb{P}_c(\mathcal{M})$ denote those probability density functions $\kappa \in \mathbb{P}_b(\mathcal{M})$ whose support $\mathcal{K}(\kappa)$ is locally conic and which are bounded away from zero and infinity on their support. The motivation for considering these classes of probability densities appears in Remark 3 following Theorem 3.2.

Suppose $\mathcal{M} \in \mathbb{M}$ and $\kappa \in \mathbb{P}(\mathcal{M})$ are given. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with probability density function κ with respect to the Riemannian volume element dy. Define the binomial point process $\mathcal{Y}_n := \{Y_i\}_{i=1}^n$. Let $\mathcal{P}_\lambda$ denote the

Poisson point process on $\mathcal{M}$ (and also the associated counting measure) having intensity density $\lambda\kappa(\cdot)$, that is,

$$\mathbb{E}\mathcal{P}_\lambda(dy) = \lambda\kappa(y)\,dy. \tag{2.5}$$

Recall that $\xi(\cdot,\cdot)$ denotes a measurable function defined on pairs $(y, \mathcal{Y})$, where $y \in \mathcal{Y} \subset \mathbb{R}^d$ and $\mathcal{Y}$ is locally finite. When $y \notin \mathcal{Y}$, we write $\xi(y, \mathcal{Y})$ instead of $\xi(y, \mathcal{Y} \cup \{y\})$; also, we sometimes write $\mathcal{Y}^y$ for $\mathcal{Y} \cup \{y\}$.

Let $\mathcal{H}$ denote a homogeneous Poisson process of unit intensity in $\mathbb{R}^m$ with $\mathbb{R}^m$ embedded in $\mathbb{R}^d$ (since $m \le d$) so that the random variable $\xi(\mathbf{0}, \mathcal{H})$ is well defined. In keeping with notation from [5, 35], for all such functionals ξ we set

$$V^\xi := \mathbb{E}\xi(\mathbf{0}, \mathcal{H})^2 + \int_{\mathbb{R}^m} \{\mathbb{E}\xi(\mathbf{0}, \mathcal{H}^u)\xi(u, \mathcal{H}^{\mathbf{0}}) - (\mathbb{E}\xi(\mathbf{0}, \mathcal{H}))^2\}\,du \tag{2.6}$$

and

$$\delta^\xi := \mathbb{E}\xi(\mathbf{0}, \mathcal{H}) + \int_{\mathbb{R}^m} \mathbb{E}[\xi(\mathbf{0}, \mathcal{H}^u) - \xi(\mathbf{0}, \mathcal{H})]\,du, \tag{2.7}$$

whenever these integrals are defined.

2.2. *Estimators of intrinsic dimension*, *manifold learning*. Given data embedded in a high-dimensional vector space, a natural problem in manifold learning, signal processing and statistics is to discover the low-dimensional structure of the data, namely the intrinsic dimension of the hypersurface containing the data. Levina and Bickel [30] propose a dimension estimator making use of nearest neighbor statistics. Their estimator, which uses distances between a given sample point and its k nearest neighbors, estimates the dimension of random variables lying on a manifold $\mathcal{M}$ of unknown dimension m embedded in $\mathbb{R}^d$, $d \ge m$. Specifically, for all $k = 3, 4, \dots,$ the Levina and Bickel estimator of the dimension of a finite data cloud $\mathcal{Y} \subset \mathcal{M} \in \mathbb{M}$, is given by

$$\hat{m}_k := \hat{m}_k(\mathcal{Y}) := (\text{card}(\mathcal{Y}))^{-1} \sum_{y \in \mathcal{Y}} \zeta_k(y, \mathcal{Y}), \tag{2.8}$$

where for all $y \in \mathcal{Y}$ we have

$$\zeta_k(y, \mathcal{Y}) := \begin{cases} (k-2)\left(\displaystyle\sum_{j=1}^{k-1} \log \frac{N_k(y)}{N_j(y)}\right)^{-1}, & \text{if card}(\mathcal{Y}) \ge k+1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.9}$$

where $N_j(y) := N_j(y, \mathcal{Y})$, as given by (2.1). For all $\rho > 0$, we also define

$$\zeta_{k,\rho}(y, \mathcal{Y}) := \zeta_k(y, \mathcal{Y})\mathbf{1}\{N_k(y) \le \rho\}, \tag{2.10}$$

using the convention $0 \times \infty = 0$ if necessary, and we put

$$\hat{m}_{k,\rho} := \hat{m}_{k,\rho}(\mathcal{Y}) := (\text{card}(\mathcal{Y}))^{-1} \sum_{y \in \mathcal{Y}} \zeta_{k,\rho}(y, \mathcal{Y}). \tag{2.11}$$

Given $\mathcal{Y}_n := \{Y_i\}_{i=1}^n$ as in Section 2.1, Levina and Bickel [30] argue that $\hat{m}_k(\mathcal{Y}_n)$ estimates the intrinsic dimension of $\mathcal{M}$. Our purpose here is to substantiate this claim and to provide further distributional results. The following shows (i) consistency of the dimension estimator $\hat{m}_k$ over Poisson and binomial samples, and (ii) a central limit theorem for $\hat{m}_{k,\rho}(\mathcal{Y}_n)$, ρ fixed and small, addressing a question raised by Peter Bickel. Here δ^{ζ_k} is given by taking $\xi \equiv \zeta_k$ in (2.6) and (2.7).

THEOREM 2.1. *Let $\mathcal{M} \in \mathbb{M}(m,d)$ and let $\kappa \in \mathbb{P}_c(\mathcal{M})$. For all $k \geq 3$ and $\rho > 0$, we have*

$$\lim_{n\to\infty} \mathbf{1}\{\hat{m}_{k,\rho}(\mathcal{Y}_n) \neq \hat{m}_k(\mathcal{Y}_n)\} = 0 \qquad a.s., \tag{2.12}$$

and if $k \geq 4$, then

$$\hat{m}_k(\mathcal{Y}_n) \xrightarrow{P} m \qquad as\ n \to \infty, \tag{2.13}$$

while if $k \geq 11$, then (2.13) holds a.s. If κ is a.e. continuous, then there exists $\rho_1 > 0$ such that if $\rho \in (0, \rho_1)$ and $k \geq 7$, then

$$\lim_{n\to\infty} n \operatorname{Var}[\hat{m}_{k,\rho}(\mathcal{Y}_n)] = \sigma^2(\zeta_k) := \frac{m^2}{k-3} - (\delta^{\zeta_k})^2 > 0, \tag{2.14}$$

and also we have as $n \to \infty$ that

$$n^{1/2}(\hat{m}_{k,\rho}(\mathcal{Y}_n) - \mathbb{E}\hat{m}_{k,\rho}(\mathcal{Y}_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(\zeta_k)). \tag{2.15}$$

REMARKS. (i) (*Counterexamples.*) Without further conditions on $\mathcal{M} \in \mathbb{M}$, in general $\hat{m}_k$ (as opposed to $\hat{m}_{k,\rho}$) might not satisfy the variance asymptotics (2.14) or the central limit theorem (2.15). To see this, suppose $m = 1$ and $d = 3$, and suppose $\mathcal{M}$ is a compact 1-manifold which includes a segment S on the z-axis, say from $z = 0$ to $z = 1$, as well as an arc of the unit circle in the (x, y) plane. If Y_i are i.i.d. with the uniform measure on $\mathcal{M}$, then there is a positive probability that $Y_1 \in S$ and that its k nearest neighbors are all on the arc. In this case the $N_j(Y_1, \mathcal{Y}_n)$, $j = 1, \ldots, k-1$, all coincide and none of the moments of $\hat{m}_k(\mathcal{Y}_n)$ exist.

For a more general counterexample along similar lines, fix m and put $d = 2(m+1)$. Let S be the set of unit vectors in $\mathbb{R}^d$ and let S_1 be those $x \in S$ such that the first $m+1$ coordinates are zero and let S_2 be those $x \in S$ such that the last $m+1$ coordinates are zero. Then $\mathcal{M} := S_1 \cup S_2$ is an m-dimensional manifold in $\mathbb{M}$. If Y_i are i.i.d. with the uniform volume measure on $\mathcal{M}$, then there is a positive probability that $Y_1 \in S_1$ and that its k nearest neighbors all belong to S_2. In this case $N_j(Y_1, \mathcal{Y}_n) = 2$ for all $j = 1, \ldots, k-1$, showing that $\zeta_k(Y_1, \mathcal{Y}_n)$ is infinite with positive probability, and therefore none of the moments of $\hat{m}_k(\mathcal{Y}_n)$ exist.

(ii) (*On the constant* ρ_1.) The constant ρ_1, loosely speaking, reflects the maximum amount of curvature of the manifold $\mathcal{M}$. The larger this is, the smaller ρ_1 has to be taken. See Lemma 4.3 below.

(iii) (*Relation to previous work*.) Levina and Bickel (Section 3.1 of [9]) argue on heuristic grounds that $\mathrm{Var}[\hat{m}_k(\mathcal{Y}_n)] = O(n^{-1})$ whenever the Y_i are the image under a sufficiently smooth map g of random variables having a smooth density, but the last counterexample shows that this bound is not true in general.

Chatterjee [13] provides a rate of normal approximation for $\hat{m}_k(\mathcal{Y}_n)$ for all $k > 9$ whenever $\mathcal{M}$ is a "nice" manifold and under minimal assumptions on the distribution of Y_i. His rates are with respect to the Kantorovich–Wasserstein distance and are of order $n^{(9-k)/(2k-2)}$, subject to the validity of $\mathrm{Var}[\hat{m}_k(\mathcal{Y}_n)] = \Theta(n^{-1})$.

Bickel and Yan (Theorems 1 and 3 of Section 4 of [10]) establish a central limit theorem for $\hat{m}_k(\mathcal{Y}_n)$ for $\mathcal{M} = \mathbb{R}^d$. The methods of Bickel and Breiman [9] or [5, 35, 38] could be used to establish the asymptotic normality of $\hat{m}_k(\mathcal{Y}_n)$ if the Y_i had a density with respect to Lebesgue measure on $\mathbb{R}^d$. These methods do not appear applicable in the present situation. Under strong assumptions on $\mathcal{M}$, Yukich [46] outlines an approach giving a rate of normal approximation for $\hat{m}_k(\mathcal{P}_\lambda)$, but does not provide consistency results or variance asymptotics for either $\hat{m}_k(\mathcal{P}_\lambda)$ or $\hat{m}_k(\mathcal{Y}_n)$.

(iv) (*Limits are dimension dependent only*.) The mean and variance asymptotics (2.13) and (2.14) are invariant with respect to κ and depend only on $\dim(\mathcal{M})$, and (in the case of variance) the parameter k. These results appear to be new even for $\mathcal{M} = \mathbb{R}^d$.

(v) (*Variance asymptotics*.) For Poisson samples, a similar result to Theorem 2.1 holds (see Theorem 3.3 below) but with the limiting variance $\sigma^2(\zeta_k)$ modified to simply $m^2/(k-3)$. Thus at least for Poisson samples, the limiting variance of the dimension estimator ζ_k decreases with increasing k.

2.3. *Estimators of intrinsic entropy and volume content*. *Rényi entropies*. Let Y_1 be as in Section 2.1. Given $\rho > 0$ with $\rho \neq 1$, the Rényi ρ-entropy [42] of Y_1, denoted $H^*_\rho(\kappa)$, and the closely related Tsallis entropy (or Havrda and Charvát entropy [23]), denoted $H_\rho(\kappa)$, are given, respectively, by

$$H^*_\rho(\kappa) := (1-\rho)^{-1} \log I_\rho(\kappa); \qquad H_\rho(\kappa) := (\rho - 1)^{-1}\bigl(1 - I_\rho(\kappa)\bigr),$$

where $I_\rho(\kappa) := \int_{\mathcal{K}(\kappa)} (\kappa(y))^\rho \, dy$ is at (2.4). When ρ tends to 1, $H^*_\rho(\kappa)$ and $H_\rho(\kappa)$ tend to the Shannon differential entropy

$$H_1(\kappa) := -\int_{\mathcal{M}} \kappa(y) \log\bigl(\kappa(y)\bigr) \, dy. \tag{2.16}$$

Rényi and Tsallis entropies are used in the study of nonlinear Fokker–Planck equations, fractal random walks, parameter estimation in semi-parametric modeling, and data compression; see [16] and the introduction of [29] for details and references. The gradient $\lim_{\rho \to 1}(dH^*/d\rho)$ equals $(-1/2)\,\mathrm{Var}[\log \kappa(Y_1)]$, a measure of the shape of the distribution [29, 43] which also appears in the statement of Theorem 2.4 below.

A problem of interest is to estimate the Rényi and Tsallis entropies given only the sample $\{Y_i\}_{i=1}^n$ and their pairwise distances. Here we show consistency results, variance asymptotics and central limit theorems for nonparametric estimators of $I_\rho(\kappa)$. The authors of the papers [16, 24, 29, 41, 44] consider estimators of $I_\rho(\kappa)$ in terms of the k nearest-neighbor graph; here we restrict to $k = 1$, but this is for presentational purposes only. The approach taken here also yields consistent estimators of volume content.

Recall that $N_1(y, \mathcal{Y})$ is the distance between y and its nearest neighbor in $\mathcal{Y}$. For all $\alpha \in (-\infty, \infty)$ and finite $\mathcal{Y} \subset \mathbb{R}^d$ put

$$R^\alpha(\mathcal{Y}) := \sum_{y \in \mathcal{Y}} N_1(y, \mathcal{Y})^\alpha.$$

For $r \in (0, \infty)$, define the critical moment $r_c(\kappa) \in [0, \infty]$ by

$$r_c(\kappa) := \sup\{r \geq 0 : \mathbb{E}\|Y_1\|^r < \infty\}. \tag{2.17}$$

The next result spells out conditions under which $n^{-1} R^\alpha(n^{1/m}\mathcal{Y}_n)$ consistently estimates a scalar multiple of $I_{1-\alpha/m}(\kappa)$ in the L^q sense. Let $\omega_m := \pi^{m/2}[\Gamma(1 + m/2)]^{-1}$ be the volume of the unit radius m-dimensional ball.

THEOREM 2.2. *If $\kappa \in \mathbb{P}_c(\mathcal{M})$ and $\alpha \in (0, \infty)$, then*

$$n^{-1} R^\alpha(n^{1/m}\mathcal{Y}_n) \to \omega_m^{-\alpha/m} \Gamma\left(1 + \frac{\alpha}{m}\right) I_{1-\alpha/m}(\kappa) \qquad \text{as } n \to \infty \tag{2.18}$$

with both L^2 and a.s. convergence. If instead $\kappa \in \mathbb{P}(\mathcal{M})$ is bounded, and $\alpha \in (-m/q, 0)$ for $q = 1$ or $q = 2$, then (2.18) holds with L^q convergence.

Putting $m = \alpha$, we obtain consistent estimators of $I_0(\kappa)$, that is, the m-dimensional content of the support of κ.

COROLLARY 2.1. *If $\kappa \in \mathbb{P}_c(\mathcal{M})$, then $\omega_m R^m(\mathcal{Y}_n) \to I_0(\kappa)$ as $n \to \infty$, with both L^2 and a.s. convergence.*

We now state variance asymptotics and a central limit theorem for $R^\alpha(n^{1/m}\mathcal{Y}_n)$. Define $V^{N_1^\alpha}$ and $\delta^{N_1^\alpha}$ by taking $\xi \equiv N_1^\alpha$ in (2.6) and (2.7).

THEOREM 2.3. *Suppose $\kappa \in \mathbb{P}_c(\mathcal{M})$ is a.e. continuous, and $\alpha \in (-m/2, 0) \cup (0, \infty)$. Then*

$$\begin{aligned} \lim_{n\to\infty} n^{-1} \operatorname{Var}[R^\alpha(n^{1/m}\mathcal{Y}_n)] &= \sigma^2(N_1^\alpha, \kappa) \\ &:= V^{N_1^\alpha} I_{1-2\alpha/m}(\kappa) - \left(\delta^{N_1^\alpha} I_{1-\alpha/m}(\kappa)\right)^2 \end{aligned} \tag{2.19}$$

and, as $n \to \infty$

$$n^{-1/2}(R^\alpha(n^{1/m}\mathcal{Y}_n) - \mathbb{E}R^\alpha(n^{1/m}\mathcal{Y}_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(N_1^\alpha, \kappa)). \tag{2.20}$$

Also, $\sigma^2(N_1^\alpha, \kappa) > 0$ *and*

$$\delta^{N_1^\alpha} = (1 - \alpha/m)\omega_m^{-\alpha/m}\Gamma\left(1 + \frac{\alpha}{m}\right). \tag{2.21}$$

REMARKS. (i) [*Comparison of* (2.18). *with previous work.*] Limit (2.18) extends the law of large numbers limit theory for entropy estimators developed by Costa and Hero on manifolds [16], who restrict to $\alpha \in (0, m)$ and to compact manifolds. They extend the consistency results of Leonenko et al. (Theorem 3.2 of [29]), Theorem 2.1 of Wade [44], Theorems 2.1–2.3 of [41] and Theorem 2.4 of [39], all of which restrict to $\mathcal{M} = \mathbb{R}^m$. In [41] it is shown that if $\mathcal{M} = \mathbb{R}^m$, then (2.18) holds whenever $\alpha \in (0, m/q)$, $I_{1-\alpha/m}(\kappa) < \infty$ and $r_c(\kappa) > q\alpha m/(m - q\alpha)$. For $\mathcal{M} = \mathbb{R}^m$ and κ supported by an m-dimensional submanifold-with-boundary of $\mathcal{M}$, under a Lipschitz assumption on κ (along with an assumption on its gradient), Liitiäinen et al. [31] develop a closed form expansion for the moments of R^α, $\alpha > 0$.

(ii) [*Comparison of* (2.19) *and* (2.20) *with previous work.*] Existing central limit theorems and variance asymptotics for the entropy estimators R^α and the volume content estimator $\omega_m R^m$ (e.g., Theorem 6.1 of [38]) assume that $\mathcal{M} = \mathbb{R}^m$ and $\alpha > 0$. Theorem 2.3 allows us to relax both assumptions. Subject to $\mathrm{Var}[R^\alpha(n^{1/m}\mathcal{Y}_n)] = \Theta(n)$, [13] yields a rate of convergence in (2.20) with respect to the Kantorovich–Wasserstein distance under minimal assumptions on the distribution of the Y_i.

Shannon entropy. The Shannon entropy $H_1(\kappa)$ defined at (2.16) is an information theoretic measure of how the data $\{Y_i\}_{i=1}^n$ is "spread out;" low entropy implies that the data is confined to a small volume whereas high entropy indicates the data is widely dispersed. Accurate estimation of differential entropy is widely used in pattern recognition, source coding, quantization, parameter estimation, and goodness-of-fit tests; cf. the survey [6].

Shannon differential entropy is commonly estimated by first estimating the density κ and then evaluating $H_1(\kappa_0)$ where κ_0 is the estimated density; such methods involve technical complications involving bin-width selection for histogram methods and window width for kernel methods and are usually restricted to $\mathcal{M} = \mathbb{R}^d$. We bypass these technical issues and use only inter-point data distances to estimate entropy; the methods are thus applicable to general nonlinear manifolds. This extends [28, 29], which restricts to $\mathcal{M} = \mathbb{R}^d$, and it lends rigor to the arguments in [32, 33].

As in [28, 29], we shall consider estimators of $H_1(\kappa)$ in terms of nearest neighbor distances. Put $\psi(y, \mathcal{Y}) := \log(e^\gamma \omega_m N_1^m(y, \mathcal{Y}))$ where $\gamma := 0.57721\ldots$ is Euler's constant; see, for example, [22].

For finite $\mathcal{Y}$ put $S(\mathcal{Y}) := \sum_{y \in \mathcal{Y}} \psi(y, \mathcal{Y})$. Define V^ψ by taking $\xi \equiv \psi$ in (2.6). The next result provides the limit theory for the *Shannon entropy estimators* $S(n^{1/m}\mathcal{Y}_n)$.

THEOREM 2.4. *Suppose $\kappa \in \mathbb{P}(\mathcal{M})$ and that either* (i) $\kappa \in \mathbb{P}_b(\mathcal{M})$ *or* (ii) $\mathcal{M} = \mathbb{R}^m$ *and $r_c(\kappa) > 0$. Then as $n \to \infty$,*

$$n^{-1} S(n^{1/m}\mathcal{Y}_n) \to H_1(\kappa) \qquad \text{in } L^2. \tag{2.22}$$

If $\kappa \in \mathbb{P}_c(\mathcal{M})$ is a.e. continuous, then

$$\begin{aligned}\lim_{n\to\infty} n^{-1} \operatorname{Var}[S(n^{1/m}\mathcal{Y}_n)] &= \sigma^2(\psi, \kappa) \\ &:= V^{\psi} - m^{-2} + \operatorname{Var}[\log \kappa(Y_1)] > 0\end{aligned} \tag{2.23}$$

and

$$n^{-1/2}(S(n^{1/m}\mathcal{Y}_n) - \mathbb{E}S(n^{1/m}\mathcal{Y}_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(\psi, \kappa)). \tag{2.24}$$

REMARKS. The papers [28, 29] show that $n^{-1}S(n^{1/m}\mathcal{Y}_n)$ consistently estimates Shannon entropy when $\mathcal{M} = \mathbb{R}^m$, but they do not treat variance asymptotics, distributional results or general manifolds. Theorem 2.4 redresses this. Subject to $\operatorname{Var}[S(n^{1/m}\mathcal{Y}_n)] = \Theta(n)$, [13] yields (2.24) under minimal assumptions on the distribution of the Y_i.

2.4. *Vietoris–Rips clique counts.* Let $\mathcal{Y} \subset \mathbb{R}^d$ be locally finite, and let $\beta \in (0, \infty)$ be a scale parameter. The Vietoris–Rips complex $\mathcal{R}^{\beta}(\mathcal{Y})$, also called the Vietoris complex or Rips complex, is the simplicial complex whose k-simplices correspond to unordered $(k+1)$ tuples of points of $\mathcal{Y}$ which are pairwise within Euclidean distance β of each other. Thus, if there is a subset S of $\mathcal{Y}$ of size $k+1$ with all points of S distant at most β from each other, then S is a k-simplex in the complex. The Vietoris–Rips complex has received attention in connection with the statistical analysis of high-dimensional data sets [14], manifold reconstruction [15] and gaps in communication coverage and sensor networks [19, 20], because of its close relation to the Čech complex of a set of balls (which contains a simplex for every finite subset of balls with nonempty intersection). It has also received attention amongst topologists [12], who, given a data cloud $\mathcal{Y}$, allow the scale parameter β to vary to obtain homological signatures of the Vietoris–Rips complexes, which when taken together, yield clustering and connectivity information about $\mathcal{Y}$. The limit theory for the kth Betti number of Vietoris–Rips complexes generated by random points in $\mathbb{R}^d$ is given in [25, 26]. The general methods of this paper may be useful in extending these results to the setting of manifolds.

Given $\mathcal{Y}, \beta$ and k, let $\mathcal{C}_k^{(\beta)}(\mathcal{Y})$ be the number of k-simplices (i.e., cliques of order $k+1$) in $\mathcal{R}^{\beta}(\mathcal{Y})$. For example,

$$\mathcal{C}_1^{(\beta)}(\mathcal{Y}_n) = \sum_{1 \le i < j \le n} \mathbf{1}\{\|Y_i - Y_j\| < \beta\},$$

which is the empirical version of the so-called correlation integral

$$\int\int \mathbf{1}\{(x, y) : \|x - y\| < \beta\} \kappa(x)\kappa(y)\, dx\, dy.$$

The quantity $n^{-1}\mathcal{C}_1^{(\beta)}(\mathcal{Y}_n)$, with $\mathcal{Y}_n := \{Y_i\}_{i=1}^n$ and with Y_i having unknown distribution μ, is the widely used sample correlation integral of Grassberger and Procaccia [21]. Grassberger and Procaccia use least squares linear regression of $\log n^{-1}\mathcal{C}_1^{(\beta)}(\mathcal{Y}_n)$ versus $\log\beta$ to estimate the "correlation dimension" of μ, that is, the exponent when the correlation integral is assumed to follow a power law as $\beta \downarrow 0$. The quantity $n^{-1}\mathcal{C}_1^{(\beta)}(\mathcal{Y}_n)$ also features in estimators of the K function, as discussed in, for example, Chapter 8.2.6 of Cressie [17].

The next result provides a law of large numbers and central limit theorem for the clique count $n^{-1}\mathcal{C}_k^{(\beta)}(\mathcal{Y}_n)$ for any $k \in \mathbb{N}$. Define $h_k : (\mathbb{R}^m)^{k+1} \to \mathbb{R}$ by $h_k(x_1,\dots,x_{k+1}) := \prod_{1\le i<j\le k+1} \mathbf{1}\{\|x_i - x_j\| \le 1\}$, that is, the indicator of the event that $x_1,\dots,x_{k+1}$ are all within unit distance of each other. Given $\beta \in (0,\infty)$, put

$$J_{k,j} := \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} \left(\frac{h(\mathbf{0}, x_1,\dots,x_k) h(\mathbf{0}, x_1,\dots,x_{j-1}, x_{k+1},\dots,x_{2k+1-j})}{j!(k+1-j)!^2} \right) dx_1 \cdots dx_{2k+1-j}, \tag{2.25}$$

so in particular $J_{k,k+1} := \int\cdots\int h(\mathbf{0}, x_1,\dots,x_k)\, dx_1 \cdots dx_k/(k+1)!$. Set

$$\sigma_k^2 := \sigma_k^2(\beta,\kappa) := \left(\sum_{j=1}^{k+1} J_{k,j} \beta^{m(2k+1-j)} I_{2k+2-j}(\kappa) \right) - \big((k+1)\beta^{mk} J_{k,k+1} I_k(\kappa)\big)^2. \tag{2.26}$$

THEOREM 2.5. *Let κ be bounded on $\mathcal{M} \in \mathbb{M}$. For all $k = 1, 2, \dots$ and all $\beta \in (0,\infty)$ we have*

$$\lim_{n\to\infty} n^{-1}\mathcal{C}_k^{(\beta)}(n^{1/m}\mathcal{Y}_n) = \beta^{mk} J_{k,k+1} I_{k+1}(\kappa) \qquad \text{in } L^2 \text{ and a.s.} \tag{2.27}$$

If κ is a.e. continuous and if $\kappa \in \mathbb{P}_b(\mathcal{M})$, then

$$\lim_{n\to\infty} n^{-1} \operatorname{Var}[\mathcal{C}_k^{(\beta)}(n^{1/m}\mathcal{Y}_n)] = \sigma_k^2(\beta,\kappa) > 0, \tag{2.28}$$

and as $n \to \infty$,

$$n^{-1/2}\big(\mathcal{C}_k^{(\beta)}(n^{1/m}\mathcal{Y}_n) - \mathbb{E}\mathcal{C}_k^{(\beta)}(n^{1/m}\mathcal{Y}_n)\big) \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, \sigma_k^2(\beta,\kappa)\big). \tag{2.29}$$

REMARK (Related work). Bhattacharya and Ghosh [8] used the limit theory of U-statistics to obtain a central limit theorem similar to (2.29) for Poisson input, in the case where $\mathcal{M} = \mathbb{R}^m$ and κ is uniform on the unit cube. The limit (2.27) extends the results in Penrose [34] (Proposition 3.1, Theorem 3.17) to nonlinear manifolds whereas (2.28) and (2.29) extend Theorem 3.13 of [34] to nonlinear manifolds.

3. General limit theorems. As in Section 2.1, $\xi(y, \mathcal{Y})$ is a real-valued functional defined on locally finite $\mathcal{Y} \subset \mathbb{R}^d$ and $y \in \mathcal{Y}$, and Y_i, $i \geq 1$ are i.i.d. with density κ. We shall say that ξ is *translation invariant* if $\xi(y, \mathcal{Y}) = \xi(z + y, z + \mathcal{Y})$ for all $z \in \mathbb{R}^d$ and all $(y, \mathcal{Y})$. We say ξ is *rotation invariant* if $\xi(\mathbf{0}, \mathcal{Y})$ is invariant under rotations of $\mathcal{Y}$, for all $\mathcal{Y}$. That is, we say ξ is rotation invariant if $\xi(\mathbf{0}, \mathcal{Y}) = \xi(\mathbf{0}, A\mathcal{Y})$ for all orthogonal $m \times m$ matrices A, where $A\mathcal{Y} := \{Az : \in \mathcal{Y}\}$. In this section we provide a general limit theory for the sums $\sum_{i=1}^n \xi(n^{1/m} Y_i, \{n^{1/m} Y_i\}_{i=1}^n)$, namely Theorems 3.1, 3.2 and 3.3. We shall use these general results to prove the results in Sections 2.2–2.4.

We introduce a scaled version of ξ, dilating the pair $(y, \mathcal{Y})$ by the factor $\lambda^{1/m}$; this scaling is natural when $\mathcal{Y} \subset \mathcal{M}$ has cardinality approximately λ, and $\mathcal{M}$ is an m-dimensional manifold in $\mathbb{R}^d$. Thus, given $k \in \mathbb{N}$, for $\lambda, \rho \in (0, \infty)$ set

$$\xi_{\lambda,k,\rho}(y, \mathcal{Y}) := \xi(\lambda^{1/m} y, \lambda^{1/m} \mathcal{Y}) \mathbf{1}\{N_k(y, \mathcal{Y}) < \rho\}; \tag{3.1}$$

$$\xi_\lambda(y, \mathcal{Y}) := \xi_{\lambda,k,\infty}(y, \mathcal{Y}) := \xi(\lambda^{1/m} y, \lambda^{1/m} \mathcal{Y}). \tag{3.2}$$

Here we are allowing for a finite macroscopic cutoff parameter ρ because for some manifolds $\xi_n(y, \mathcal{Y}_n)$ may suffer from nonlocal effects even when n becomes large; see the counterexamples in remark (i) of Section 2.2.

Given $k \in \mathbb{Z}^+$ and $r > 0$, let $\Xi(k, r)$ be the class of translation and rotation invariant functionals ξ such that (i) for all $y, \mathcal{Y}$ with $\mathrm{card}(\mathcal{Y} \setminus \{y\}) \geq k$ we have

$$\xi(y, \mathcal{Y}) = \xi(y, \mathcal{Y} \cap B_{\max(r, N_k(y,\mathcal{Y}))}(y))$$

and (ii) for all n, Lebesgue-almost every $(y_1, \dots, y_n) \in (\mathbb{R}^m)^n$ (with $\mathbb{R}^m$ embedded in $\mathbb{R}^d$) is at a continuity point of the mapping from $(\mathbb{R}^d)^n \to \mathbb{R}$ given by

$$(y_1, \dots, y_n) \mapsto \xi(\mathbf{0}, \{y_1, \dots, y_n\}).$$

For finite $\mathcal{Y} \subset \mathcal{M}$, and for $k \in \mathbb{Z}^+$, $\lambda \in (0, \infty)$ and $\rho \in (0, \infty]$ define

$$H^\xi_{\lambda,k,\rho}(\mathcal{Y}) := \sum_{y \in \mathcal{Y}} \xi_{\lambda,k,\rho}(y, \mathcal{Y}); \qquad H^\xi_\lambda(\mathcal{Y}) := H_{\lambda,k,\infty}(\mathcal{Y}). \tag{3.3}$$

Recalling that $\mathcal{Y}_n := \{Y_i\}_{i=1}^n$ and that $\mathcal{P}_\lambda$ is a Poisson point process on $\mathcal{M}$ of intensity $\lambda\kappa(y)\, dy$ defined at (2.5), we now give a general law of large numbers for scaled versions of the linear statistics (1.1) when $\xi \in \Xi(k, r)$. For $i \in \mathbb{Z}^+$, let $\mathcal{S}_i$ be the collection of all subsets of $\mathcal{K}(\kappa)$ of cardinality at most i (including the empty set). Consider the following moment conditions on ξ:

$$\sup_n \mathbb{E}|\xi_{n,k,\rho}(Y_1, \mathcal{Y}_n)|^p < \infty, \tag{3.4}$$

$$\sup_{n \geq 1, y \in \mathcal{K}(\kappa), \mathcal{A} \in \mathcal{S}_3} \; \sup_{(n/2) \leq \ell \leq (3n/2)} \mathbb{E}|\xi_{n,k,\rho}(y, \mathcal{Y}_\ell \cup \mathcal{A})|^p < \infty \tag{3.5}$$

[noting that (3.5) implies (3.4)], and

$$\sup_{\lambda \geq 1, y \in \mathcal{K}(\kappa), \mathcal{A} \in \mathcal{S}_1} \mathbb{E}|\xi_{\lambda,k,\rho}(y, \mathcal{P}_\lambda \cup \mathcal{A})|^p < \infty. \tag{3.6}$$

THEOREM 3.1 [Laws of large numbers for $\xi \in \Xi(k,r)$]. *Let $\mathcal{M} \in \mathbb{M}$, $\kappa \in \mathbb{P}(\mathcal{M})$, $k \in \mathbb{Z}^+$ and $\rho \in (0,\infty]$, and put $q = 1$ or $q = 2$. Let $\xi \in \Xi(k,r)$, and suppose there exists $p > q$ such that* (3.4) *holds. Then as $n \to \infty$ we have L^q convergence*

$$n^{-1} H^{\xi}_{n,k,\rho}(\mathcal{Y}_n) \to \int_{\mathcal{M}} \mathbb{E}[\xi(\mathbf{0}, \mathcal{H}_{\kappa(y)})]\kappa(y)\,dy. \tag{3.7}$$

If also (3.5) *holds for some $p > 5$, and also $\kappa \in \mathbb{P}_b(\mathcal{M})$ and moreover either $k = 0$ or $\kappa \in \mathbb{P}_c(\mathcal{M})$, then* (3.7) *holds a.s.*

Given $a > 0$, let $\mathcal{H}_a$ denote a homogeneous Poisson process of intensity a in $\mathbb{R}^m$ (embedded in $\mathbb{R}^d$). Extending the earlier definitions (2.6) and (2.7), we set

$$V^{\xi}(a) := \mathbb{E}\xi(\mathbf{0}, \mathcal{H}_a)^2 + a\int_{\mathbb{R}^m} \{\mathbb{E}\xi(\mathbf{0}, \mathcal{H}^u_a)\xi(u, \mathcal{H}^{\mathbf{0}}_a) - (\mathbb{E}\xi(\mathbf{0}, \mathcal{H}_a))^2\}\,du \tag{3.8}$$

and

$$\delta^{\xi}(a) := \mathbb{E}\xi(\mathbf{0}, \mathcal{H}_a) + a\int_{\mathbb{R}^m} \mathbb{E}[\xi(\mathbf{0}, \mathcal{H}^u_a) - \xi(\mathbf{0}, \mathcal{H}_a)]\,du, \tag{3.9}$$

so in particular $V^{\xi}(1) = V^{\xi}$ and $\delta^{\xi}(1) = \delta^{\xi}$. For all $\kappa \in \mathbb{P}(\mathcal{M})$ and ξ, we define

$$\sigma^2(\xi,\kappa) := \int_{\mathcal{M}} V^{\xi}(\kappa(y))\kappa(y)\,dy - \left(\int_{\mathcal{M}} \delta^{\xi}(\kappa(y))\kappa(y)\,dy\right)^2, \tag{3.10}$$

provided that both integrals in (3.10) exist and are finite.

THEOREM 3.2 [Variance asymptotics and CLT for $\xi \in \Xi(k,r)$]. *Let $\mathcal{M} \in \mathbb{M}$ and let $\kappa \in \mathbb{P}_b(\mathcal{M})$ be a.e. continuous. Let $k \in \mathbb{Z}^+$, $r \geq 0$, and $\rho \in (0,\infty]$. Assume $k = 0$ or $\kappa \in \mathbb{P}_c(\mathcal{M})$. Let $\xi \in \Xi(k,r)$ and suppose that ξ satisfies* (3.5) *and* (3.6) *for some $p > 2$. Then $\sigma^2(\xi,\kappa) < \infty$ and*

$$\lim_{n\to\infty} n^{-1}\operatorname{Var}[H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)] = \sigma^2(\xi,\kappa) \tag{3.11}$$

and as $n \to \infty$,

$$n^{-1/2}(H^{\xi}_{n,k,\rho}(\mathcal{Y}_n) - \mathbb{E}H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(\xi,\kappa)). \tag{3.12}$$

REMARKS. (i) (*Related work.*) Under a slightly different set of assumptions, Chatterjee [13] provides estimates for the Kantorovich–Wasserstein distance between the distribution of $H^{\xi}_n(\mathcal{Y}_n)$ and the normal, which imply a central limit theorem subject to the validity of $\operatorname{Var}[H^{\xi}_n(\mathcal{Y}_n)] = \Theta(n)$. Indeed, if in Theorem 3.2 we take $\rho = \infty$, $r = 0$ and $p > 8$, and if also $\sigma^2(\xi,\kappa) > 0$, then combining (3.11) with Theorem 3.4 of [13] shows that the convergence (3.12) is at rate $O(n^{4/p-1/2})$ with respect to the Kantorovich–Wasserstein distance.

(ii) (*Simplification of mean and variance asymptotics for homogeneous ξ.*) The limits (3.7) and (3.11) take a simpler form when there is some $\beta \in \mathbb{R}$ such that

ξ is *homogeneous of order* β, meaning that for all $a \in (0, \infty)$ and all $y \in \mathcal{Y} \subset \mathbb{R}^m$, we have $\xi(ay, a\mathcal{Y}) = a^\beta \xi(y, \mathcal{Y})$. In this case $\xi(\mathbf{0}, \mathcal{H}_a) \stackrel{\mathcal{D}}{=} \xi(\mathbf{0}, a^{-1/m}\mathcal{H}) = a^{-\beta/m}\xi(\mathbf{0}, \mathcal{H})$, so (3.7) becomes

$$n^{-1} H_n^\xi(\mathcal{Y}_n) \to I_{1-\beta/m}(\kappa)\mathbb{E}\xi(\mathbf{0}, \mathcal{H}) \tag{3.13}$$

and similarly, by definitions (2.6), (2.7), we can show that (3.8) and (3.9) simplify to $V^\xi(a) = a^{-2\beta/m} V^\xi$ and $\delta^\xi(a) = a^{-\beta/m}\delta^\xi$. Hence, in this case (3.10) becomes

$$\sigma^2(\xi, \kappa) = V^\xi I_{1-2\beta/m}(\kappa) - \big(\delta^\xi I_{1-\beta/m}(\kappa)\big)^2. \tag{3.14}$$

If ξ is homogeneous of order 0, we say it is *scale invariant.*

(iii) (*Assumptions on* κ.) While some of the general results allow for $\kappa \in \mathbb{P}_b(\mathcal{M})$, it is often necessary to require $\kappa \in \mathbb{P}_c(\mathcal{M})$ in order to guarantee that functionals $\xi \in \Xi(k, r)$ satisfy the spatial localization property termed exponential stabilization, described in Section 6. Additionally, in Section 2, the requirement $\kappa \in \mathbb{P}_c(\mathcal{M})$ is needed when verifying that the estimators of that section satisfy the general results given in Theorems 3.1 and 3.2.

(iv) (*Limit theory for random measures on manifolds*.) Consider the point measures

$$\mu^\xi_{\lambda,k,\rho} := \sum_{i=1}^n \xi_{n,k,\rho}(Y_i, \mathcal{Y}_n)\delta_{Y_i}, \tag{3.15}$$

where δ_y denotes the unit point mass at y. As in [5, 35, 36], Theorem 3.1 admits an extension to the random measures (3.15) as follows. Let $B(\mathcal{M})$ be the space of bounded, measurable, real-valued functions on $\mathcal{M}$ and for $f \in B(\mathcal{M})$, and μ a measure on $\mathcal{M}$, let $\langle f, \mu\rangle$ denote the integral of f with respect to μ. Put $q = 1$ or $q = 2$. Let $\xi \in \Xi(k, r)$ and suppose that there is a $p > q$ such that (3.4) and (3.6) are satisfied. It can be shown that for all $f \in B(\mathcal{M})$,

$$\lim_{n\to\infty} n^{-1}\langle f, \mu^\xi_{n,k,\rho}\rangle = \int_{\mathcal{M}} f(y)\mathbb{E}[\xi(\mathbf{0}, \mathcal{H}_{\kappa(y)})]\kappa(y)\,dy \qquad \text{in } L^q. \tag{3.16}$$

Similarly, if $\xi \in \Xi(k, r)$ satisfies the moment assumptions of Theorem 3.2, it can be shown that $n^{-1/2}(\langle f, \mu^\xi_{n,k,\rho}\rangle - \mathbb{E}\langle f, \mu^\xi_{n,k,\rho}\rangle)$ converges in distribution to a mean zero normal random variable with variance

$$\int_{\mathcal{M}} f(y)^2 V^\xi\big(\kappa(y)\big)\kappa(y)\,dy - \left(\int_{\mathcal{M}} \delta^\xi\big(\kappa(y)\big) f(y)\kappa(y)\,dy\right)^2.$$

We refer to [5, 35] for details.

(v) (*Other functionals*.) The approach also works for more general functionals than $\xi \in \Xi(k, r)$. Along with the moment conditions already discussed, the key properties ξ needs to satisfy are *exponential stabilization* and *continuity*, which we discuss in the proof and which represent ξ being locally determined in some sense; cf. the remark at the end of Section 6. Also, the approach works for marked point processes, where the points carry independent identically distributed marks. We would expect other functionals to satisfy this, as has been considered for a variety of functionals in the case $\mathcal{M} = \mathbb{R}^m$ [5, 35]. Finally, the assumption that ξ is rotation invariant could be relaxed; one would need to change definition (3.1) to

$$\xi_\lambda(y, \mathcal{Y}) = \xi\big(y, y + \lambda^{1/m}(-y + \mathcal{Y})\big),$$

modify (3.1) similarly, and in (3.7)–(3.10) take $\mathcal{H}_{\kappa(y)}$ to be a homogeneous Poisson process in the hyperplane tangent to $-y + \mathcal{M}$ at $\mathbf{0}$, and take the integrals in (3.8) and (3.9) to be over this tangent hyperplane rather than over $\mathbb{R}^m$.

(vi) (*Noisy input*.) It is arguably more realistic to consider input having a d-dimensional noise component. Consider the situation where data $Y_i, i \geq 1$, is corrupted with a noise component $n^{-1/m} Z_i$, with $Z_i, i \geq 1$, being i.i.d. $\mathbb{R}^d$-valued random variables which are independent of $Y_i, i \geq 1$, and which are assumed rotation invariant, for example, with i.i.d. mean zero normal components.

For all $x \in \mathbb{R}^d$, let Z_x denote a copy of Z_1. For all $u \in \mathbb{R}^m, a \in (0, \infty)$, put $\mathcal{H}_a^{u,\mathcal{Z}} := \{x + Z_x : x \in \mathcal{H}_a \cup \{u\}\}$, and put $\mathcal{H}_a^{\mathcal{Z}} := \{x + Z_x : x \in \mathcal{H}_a\}$.

Then it can be shown that the law of large numbers (Theorem 3.1) takes the form

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \xi_n\big(Y_i + n^{-1/m} Z_i, \{Y_j + n^{-1/m} Z_j\}_{j=1}^n\big) \\ = \int_{\mathcal{M}} \mathbb{E}[\xi(\mathbf{0} + Z_{\mathbf{0}}, \mathcal{H}_{\kappa(y)}^{\mathcal{Z}})]\kappa(y)\,dy.\end{aligned}$$

Moreover, it can be shown that Theorem 3.2 still holds if in the statement of that result we replace $\mathcal{Y}_n$ by $\{Y_j + n^{-1/m} Z_j\}_{j=1}^n$ and replace definitions (3.8) and (3.9), respectively, by

$$\begin{aligned}V^\xi(a) &:= \mathbb{E}\xi(\mathbf{0}, \mathcal{H}_a^{\mathcal{Z}})^2 \\ &\quad + a \int_{\mathbb{R}^m} \{\mathbb{E}\xi(\mathbf{0} + Z_{\mathbf{0}}, \mathcal{H}_a^{u,\mathcal{Z}})\xi(u + Z_u, \mathcal{H}_a^{\mathbf{0},\mathcal{Z}}) - (\mathbb{E}\xi(\mathbf{0} + Z_{\mathbf{0}}, \mathcal{H}_a^{\mathcal{Z}}))^2\}\,du\end{aligned}$$

and

$$\delta^\xi(a) := \mathbb{E}\xi(\mathbf{0} + Z_{\mathbf{0}}, \mathcal{H}_a^{\mathcal{Z}}) + a \int_{\mathbb{R}^m} \mathbb{E}[\xi(\mathbf{0}, \mathcal{H}_a^{u,\mathcal{Z}}) - \xi(\mathbf{0}, \mathcal{H}_a^{\mathcal{Z}})]\,du.$$

(vii) (*Poisson input*.) In (3.7) we have presented the law of large numbers for binomial samples, but the same L^q limit holds for functionals of the form $\lambda^{-1} H_{\lambda,k,\rho}^\xi(\mathcal{P}_\lambda)$, with $\mathcal{P}_\lambda$ as in (2.5). Likewise, there is a Poisson analog to (3.16).

We also have variance asymptotics and a central limit theorem for $H_{\lambda,k,\rho}^\xi(\mathcal{P}_\lambda)$, similar to Theorem 3.2 but with a different limiting variance. Moreover, in the Poisson setting, we have a bound on the rate of convergence to the normal, using the Kolmogorov distance. The result goes as follows.

THEOREM 3.3. *Let $\mathcal{M} \in \mathbb{M}$ and let $\kappa \in \mathbb{P}_b(\mathcal{M})$ be a.e. continuous. Let $k \in \mathbb{Z}^+$, $r > 0$ and $\rho \in (0, \infty]$, and suppose $k = 0$ or $\kappa \in \mathbb{P}_c(\mathcal{M})$. Let $\xi \in \Xi(k, r)$ and suppose that ξ satisfies (3.6) for some $p > 2$. Then*

$$\lim_{\lambda \to \infty} \lambda^{-1} \operatorname{Var}\big[H^{\xi}_{\lambda,k,\rho}(\mathcal{P}_\lambda)\big] = \tau^2(\xi, \kappa) := \int_{\mathcal{M}} V^{\xi}\big(y, \kappa(y)\big)\kappa(y)\, dy < \infty, \tag{3.17}$$

and as $\lambda \to \infty$,

$$\lambda^{-1/2}\big(H^{\xi}_{\lambda,k,\rho}(\mathcal{P}_\lambda) - \mathbb{E}H^{\xi}_{\lambda,k,\rho}(\mathcal{P}_\lambda)\big) \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, \tau^2(\xi, \kappa)\big). \tag{3.18}$$

Additionally, if ξ satisfies (3.6) for some $p > 3$ and if $\sigma^2(\xi, \kappa) > 0$, then there exists a finite constant C depending on d, k, ξ, κ, ρ and p such that for all $\lambda \geq 2$,

$$\sup_{t \in \mathbb{R}} \left| P\left[\frac{H^{\xi}_{\lambda,k,\rho}(\mathcal{P}_\lambda) - \mathbb{E}H^{\xi}_{\lambda,k,\rho}(\mathcal{P}_\lambda)}{\sqrt{\operatorname{Var}[H^{\xi}_{\lambda,k,\rho}(\mathcal{P}_\lambda)]}} \leq t\right] - \Phi(t)\right| \leq C(\log \lambda)^{3m}\lambda^{-1/2}. \tag{3.19}$$

As well as being of independent interest, Theorem 3.3 is used in our proof of Theorem 3.2. Equation (3.19) is the counterpoint for manifolds to the rate of normal convergence result in [40]. Theorem 3.3 could be used to provide Poisson analogs to the results presented in Sections 2.2–2.4.

4. Geometrical preliminaries. The lemmas in this section, concerned with properties of manifolds, have no probabilistic content. The first of these relates distances in the manifold to distances in a chart.

LEMMA 4.1. *Suppose (U, g) is a chart for $\mathcal{M} \in \mathbb{M}(m, d)$. Suppose $F \subset g(U)$ is a compact subset of $\mathcal{M}$. Then*

$$0 < \inf_{y,z \in F : y \neq z} \frac{\|g^{-1}(z) - g^{-1}(y)\|}{\|z - y\|} \leq \sup_{y,z \in F : y \neq z} \frac{\|g^{-1}(z) - g^{-1}(y)\|}{\|z - y\|} < \infty. \tag{4.1}$$

PROOF. Suppose (4.1) fails. Then by compactness, we can find a sequence (y_n, z_n), $n \in \mathbb{N}$ with $y_n \in F$, $z_n \in F \setminus \{y_n\}$, and $y_n \to y$ for some $y \in F$, such that setting $u_n := g^{-1}(y_n)$ and $v_n := g^{-1}(z_n)$, we have either

$$\|v_n - u_n\|/\|z_n - y_n\| \to \infty \qquad \text{as } n \to \infty \tag{4.2}$$

or

$$\|v_n - u_n\|/\|z_n - y_n\| \to 0 \qquad \text{as } n \to \infty. \tag{4.3}$$

Since g is an open map, the set $g^{-1}(F)$ is compact. Hence $\|v_n - u_n\|$ remains bounded.

Suppose (4.2) holds. Then $\|z_n - y_n\| \to 0$, and hence $z_n \to y$ as $n \to \infty$. Setting $u := g^{-1}(y)$, by continuity of g^{-1} we have $u_n \to u$ and $v_n \to u$. Hence, arguing componentwise using the mean value theorem and the continuity of g', we have that

$$\|z_n - y_n - g'(u)(v_n - u_n)\| = o(\|v_n - u_n\|) \tag{4.4}$$

and therefore since $g'(u)$ has full rank,

$$\liminf_{n\to\infty} \frac{\|z_n - y_n\|}{\|v_n - u_n\|} = \liminf_{n\to\infty} \frac{\|g'(u)(v_n - u_n)\|}{\|v_n - u_n\|} > 0,$$

which contradicts (4.2). On the other hand, if (4.3) holds we can show by a similar argument that $\limsup_{n\to\infty}(\|z_n - y_n\|/\|v_n - u_n\|) < \infty$, again giving a contradiction. $\square$

For $w \in E \subseteq \mathbb{R}^d$ and $r > 0$, let $B_r^E(w) := B_r(w) \cap E$. Recall that $\omega_m := \pi^{m/2}[\Gamma(1 + m/2)]^{-1}$ is the volume of the ball $B_1(\mathbf{0})$ in $\mathbb{R}^m$.

LEMMA 4.2. *Let $\mathcal{M} \in \mathbb{M}$. Suppose $y_\infty \in \mathcal{M}$, and suppose for $n \in \mathbb{N}$ we are given $y_n \in \mathcal{M}$, $r_n > 0$, $a_n > 0$ with $y_n \to y_\infty$, $r_n \to 0$ and $a_n \to 0$ as $n \to \infty$. Then*

$$\limsup_{n\to\infty}\left(r_n^{-m} \int_{B_{r_n}^{\mathcal{M}}(y_n)} dy\right) \le \omega_m, \tag{4.5}$$

and putting $s_n = r_n(1 - a_n)$, we have

$$\limsup_{n\to\infty}\left((r_n^m - s_n^m)^{-1} \int_{B_{r_n}^{\mathcal{M}}(y_n)\setminus B_{s_n}^{\mathcal{M}}(y_n)} dy\right) < \infty. \tag{4.6}$$

PROOF. Let (U, g) be a chart such that $\mathbf{0} \in U$ and $g(\mathbf{0}) = y_\infty$. Set $U_n = g^{-1}(B_{r_n}^{\mathcal{M}}(y_n)) \subseteq U$, and set $V_n = g^{-1}(B_{s_n}^{\mathcal{M}}(y_n))$. Let $\mathcal{L}$ denote Lebesgue measure, and note that by continuity $\sup_{u\in U_n} |(D_g(u)/D_g(\mathbf{0})) - 1|$ vanishes as $n \to \infty$. Thus there exists n_0 such that for $n \ge n_0$, we have $B_{r_n}^{\mathcal{M}}(y_n) \subset g(U)$, and by (2.2),

$$\int_{B_{r_n}^{\mathcal{M}}(y_n)} dy = D_g(\mathbf{0}) \int_{U_n} (D_g(u)/D_g(\mathbf{0}))\, du \sim D_g(\mathbf{0})\mathcal{L}(U_n) \tag{4.7}$$

and

$$\begin{aligned} &\int_{B_{r_n}^{\mathcal{M}}(y_n)\setminus B_{s_n}^{\mathcal{M}}(y_n)} dy \\ &\quad = D_g(\mathbf{0}) \int_{U_n\setminus V_n} (D_g(u)/D_g(\mathbf{0}))\, du \sim D_g(\mathbf{0})\mathcal{L}(U_n \setminus V_n) \end{aligned} \tag{4.8}$$

where the asymptotics are as $n \to \infty$. Given $n \ge n_0$, set $u_n := g^{-1}(y_n) \in U$. We claim that

$$\limsup_{n\to\infty} \sup_{v\in U_n} r_n^{-1} \|g'(\mathbf{0})(v - u_n)\| \le 1. \tag{4.9}$$

To see (4.9), take $v_n \in U_n$ for $n \in \mathbb{N}$. By continuity of g^{-1}, we have $v_n \to \mathbf{0}$ as $n \to \infty$ so by applying the mean value theorem and using continuity of g', as in (4.4) we have that $\|g(v_n) - y_n - g'(\mathbf{0})(v_n - u_n)\| = o(\|v_n - u_n\|)$ as $n \to \infty$. Therefore since $g'(\mathbf{0})$ has full rank,

$$\|g(v_n) - y_n\| \sim g'(\mathbf{0})(v_n - u_n) \qquad \text{as } n \to \infty. \tag{4.10}$$

Then (4.9) follows because $\|g(v_n) - y_n\| \le r_n$ and the choice of $v_n \in U_n$ was arbitrary.

By (4.9), given $\varepsilon > 0$ we have for large enough n that

$$\mathcal{L}(U_n) \le \mathcal{L}(\{v \in \mathbb{R}^m : \|g'(\mathbf{0})(v - u_n)\| \le r_n(1+\varepsilon)\}) = (1+\varepsilon)^m \omega_m r_n^m / D_g(\mathbf{0}),$$

and (4.5) follows by (4.7).

Finally we prove (4.6). Since $(1-a)^m \le 1 - a$ for all $a \in [0, 1]$, for large n we have

$$r_n^m - s_n^m = r_n^m(1 - (1 - a_n)^m) \ge r_n^m a_n. \tag{4.11}$$

We claim that if $(v_n)_{n\in\mathbb{N}}$ is a sequence in U with $\|g(v_n) - y_n\| = r_n$, then setting $w_n := u_n + (1 - 3a_n)(v_n - u_n)$, we have $\|g(w_n) - y_n\| < s_n$ for large enough n. Indeed, for any such sequence by the mean value theorem we have $r_n = \|g(v_n) - y_n\| \sim \|g'(\mathbf{0})(v_n - u_n)\|$ as in (4.10), and also

$$\|g(w_n) - g(v_n)\| \sim 3a_n \|g'(\mathbf{0})(v_n - u_n)\| \sim 3a_n r_n.$$

Moreover $g(w_n) - y_n$ and $g(v_n) - y_n$ are almost in the same direction, so that for large n, $\|g(w_n) - y_n\| \le r_n(1 - 2a_n)$, and the claim follows.

By the preceding claim, there is a constant C such that the thickness of the deformed annulus $U_n \setminus V_n$ in all directions is bounded by $Ca_n r_n$, so by using polar coordinates we have $\mathcal{L}(U_n \setminus V_n) = O(r_n^m a_n)$, and then using (4.8) and (4.11) we have that

$$\int_{B_{r_n}^{\mathcal{M}}(y_n) \setminus B_{s_n}^{\mathcal{M}}(y_n)} dy = O(r_n^m a_n) = O(r_n^m - s_n^m),$$

demonstrating (4.6). □

Recall that $\mathcal{K}(\kappa)$ denotes the support of κ, and if $\kappa \in \mathbb{P}_c(\mathcal{M})$, then $\mathcal{K}(\kappa)$ is locally conic and satisfies (2.3). Given $\mathcal{M} \in \mathbb{M}$ and $\kappa \in \mathbb{P}_c(\mathcal{M})$, set $\Delta(\kappa) := \operatorname{diam}(\mathcal{K}(\kappa))$.

LEMMA 4.3. *Suppose $\mathbb{M} \in \mathcal{M}$ and $\kappa \in \mathbb{P}_c(\mathcal{M})$. Then there is a constant $C_0 \in (0, \infty)$ such that for all $r \in (0, \Delta(\kappa)]$ and $w \in \mathcal{K}(\kappa)$, we have*

$$C_0^{-1} r^m \le \int_{B_r^{\mathcal{K}(\kappa)}(w)} dy \le \int_{B_r^{\mathcal{M}}(w)} dy \le C_0 r^m. \tag{4.12}$$

There are also positive finite constants C_1 *and* ρ_1 *such that if* $0 < s < r < \rho_1$ *and* $w \in \mathcal{K}(\kappa)$, *then*

$$\int_{B_r^{\mathcal{M}}(w)\setminus B_s^{\mathcal{M}}(w)} dy \le C_1(r^m - s^m). \tag{4.13}$$

PROOF. In the proof, set $\mathcal{K} := \mathcal{K}(\kappa)$ and $\Delta := \Delta(\kappa)$. The first inequality in (4.12) (for large enough C_0) follows from the assumption that $\mathcal{K}$ is locally conic (2.3). Suppose the last inequality of (4.12) fails; then there must be a $(\mathcal{K} \times (0, \Delta])$-valued sequence $\{(y_n, r_n), n \in \mathbb{N}\}$ such that

$$\lim_{n\to\infty} r_n^{-m} \int_{B_{r_n}^{\mathcal{M}}(y_n)} dz = \infty. \tag{4.14}$$

Since $\mathcal{K} \times [0, \Delta]$ is compact, by taking a subsequence we may assume without loss of generality that $y_n \to y$ and $r_n \to r$ for some $y \in \mathcal{K}$ and $r \in [0, \Delta]$. If $r = 0$, then (4.14) would contradict (4.5). If $r > 0$ and (4.14) holds, then since $B_{r_n}(y_n) \subset B_{2r}(y)$ for large n we have $\int_{B_{2r}(y)} dz = \infty$, which is impossible: indeed by compactness $B_{2r}(y) \cap \mathcal{M}$ is covered by finitely many of the regions $g_i(U_i)$, and $\int_{g_i(U_i)} \kappa_i(x)\,dx$ is finite for all i (we may assume the charts were chosen so all the regions U_i are bounded). Therefore we have a contradiction so (4.12) must hold.

It remains to prove there exists positive ρ_1 such that (4.13) holds for all $w \in \mathcal{K}$ and $0 < s < r < \rho_1$. Suppose this is not the case. Then there is a sequence $\{(y_n, r_n, a_n), n \in \mathbb{N}\}$ taking values in $\mathcal{K} \times (0, \Delta] \times (0, 1)$ such that $r_n \to 0$, and setting $s_n = r_n(1 - a_n)$ we have

$$\lim_{n\to\infty} (r_n^m - s_n^m)^{-1} \int_{B_{r_n}^{\mathcal{M}}(y_n)\setminus B_{s_n}^{\mathcal{M}}(y_n)} dy = \infty. \tag{4.15}$$

By taking a subsequence, we may assume that $y_n \to y$ for some $y \in \mathcal{K}$, and either a_n is bounded away from zero or $a_n \to 0$ as $n \to \infty$.

If $\inf_{n\in\mathbb{N}}\{a_n\} > 0$, then $(r_n^m - s_n^m)^{-1} = O(r_n^{-m})$ so (4.15) would give a contradiction of (4.12). If $a_n \to 0$ and $r_n \to 0$, then (4.15) would give a contradiction of (4.6). □

REMARK 4.1. A sufficient condition for $\mathcal{K} \subset \mathcal{M}$ to be locally conic is that $\mathcal{K}$ be a compact m-dimensional C^1 submanifold-with-boundary of $\mathcal{M}$.

This can be proved by similar arguments to the proof of Lemmas 4.2 and 4.3; for details see the proof of these lemmas in the earlier version of this paper [37] (where the definition of $\mathbb{P}_c$ is different from here).

5. Weak convergence lemmas. For all $d \in \mathbb{N}$, we put a topology $\mathcal{T} := \mathcal{T}_d$ on locally finite point sets in $\mathbb{R}^d$. As in Aldous and Steele ([1], page 250), we adopt a topology whereby a sequence of locally finite point sets $(\mathbf{y}_n)_{n\ge1}$ converges to a locally finite $\mathbf{y}$, if and only if (i) it is possible to list the elements of $\mathbf{y}$ as a possibly terminating sequence $(y_i, i \geq 1)$ and the elements of $\mathbf{y}_n$ as a possibly terminating sequence $(y_{n,i}, i \geq 1)$ in such a way that

$$\lim_{n\to\infty} y_{n,i} = y_i \qquad \forall i \tag{5.1}$$

and (ii) for any L with no point of $\mathbf{y}$ on the boundary of $B_L(\mathbf{0})$, we have

$$\lim_{n\to\infty} \operatorname{card}(\mathbf{y}_n \cap B_L(\mathbf{0})) = \operatorname{card}(\mathbf{y} \cap B_L(\mathbf{0})). \tag{5.2}$$

We would like to know that whenever point sets $\mathbf{y}_n \subset \mathbb{R}^d$ are close to the point set $\mathbf{y}$ in the topology $\mathcal{T}_d$, then $\xi(y, \mathbf{y}_n)$ is close to $\xi(y, \mathbf{y})$, $y \in \mathbb{R}^d$. This motivates Definition 5.1 below. Recall that $\mathcal{H}$ denotes a homogeneous Poisson point process of unit intensity on $\mathbb{R}^m$.

DEFINITION 5.1. ξ is *continuous* if for any linear $F : \mathbb{R}^m \to \mathbb{R}^d$ of full rank, for almost all $z \in \mathbb{R}^m$ both $F(\mathcal{H})$ and $F(\mathcal{H}^z)$ lie a.s. at continuity points of $\xi(\mathbf{0}, \cdot)$ with respect to $\mathcal{T}_d$.

If $\mathcal{U}_n$ and $\mathcal{U}$ are simple point processes (i.e., random locally finite point sets in $\mathbb{R}^d$), then, following the discussion in [1] page 251, we shall say that $\mathcal{U}_n$ converges in distribution to $\mathcal{U}$ if the law of $\mathcal{U}_n$ converges weakly to that of $\mathcal{U}$ under the (metrisable) topology $\mathcal{T}_d$, which is the same as the notion of weak convergence of point processes discussed in Daley and Vere-Jones [18], Section 11.1.

Given the atlas $((U_i, g_i), i \in \mathcal{I})$, for $i \in \mathcal{I}$ define the function $\tilde{\kappa}_i : U_i \to [0, \infty)$ by

$$\tilde{\kappa}_i(x) = \kappa(g_i(x)) D_{g_i}(x), \qquad x \in U_i. \tag{5.3}$$

By using (2.2) with a partition of unity for which $\psi_i \equiv 1$ on $g_i(U_i)$, we see that for Borel $B \subseteq U_i$,

$$\int_{g_i(B)} \kappa(y)\, dy = \int_B \kappa(g_i(x)) D_{g_i}(x)\, dx = \int_B \tilde{\kappa}_i(x)\, dx. \tag{5.4}$$

Given $a > 0$ and $x \in U_i$, by $g_i'(x)(\mathcal{H}_a)$ we mean the point process in $\mathbb{R}^d$ obtained by applying to $\mathcal{H}_a$ the linear map $g_i'(x)$. Similarly, $g_i'(x)(z)$ is the image of z under the map $g_i'(x)$. If $\mathcal{M} \in \mathbb{M}(m, d)$ or $\mathcal{M}$ is an open subset of $\mathbb{R}^m$, and $f : \mathcal{M} \to \mathbb{R}$ is measurable, then we say $w \in \mathcal{M}$ is a *Lebesgue point* of f if $\varepsilon^{-m} \int_{B_\varepsilon(w)\cap\mathcal{M}} |f(y) - f(w)|\, dy$ tends to zero as $\varepsilon \downarrow 0$.

LEMMA 5.1. *Suppose $i \in \mathcal{I}$, U_i is bounded, and $u \in U_i$ is a Lebesgue point of $\tilde{\kappa}_i$. Suppose $\ell(n), n \in \mathbb{N}$ is a sequence of integers with $\ell(n) \sim n$ as $n \to \infty$. Set $y_0 = g_i(u)$. Then as $n \to \infty$ we have* (*in the above sense of convergence of point processes in* $\mathbb{R}^d$)

$$n^{1/m}(-y_0 + \mathcal{Y}_{\ell(n)}) \xrightarrow{\mathcal{D}} g_i'(u)(\mathcal{H}_{\tilde{\kappa}_i(u)}). \tag{5.5}$$

PROOF. By taking $B = U_i$ in (5.4), we see that $\int_{U_i} \tilde{\kappa}_i(x)\,dx \le 1$, so that $\tilde{\kappa}_i$ is a (possibly) defective density function on U_i. Extend $\tilde{\kappa}_i$ in an arbitrary manner to a probability density function on $\mathbb{R}^m$.

Let $\mathcal{X}_n$ be a point process in $\mathbb{R}^m$ consisting of $\ell(n)$ independent identically distributed random m-vectors $X_{i,1}, \dots, X_{i,\ell(n)}$ with density $\tilde{\kappa}_i$. Then

$$g_i^{-1}\big(\mathcal{Y}_{\ell(n)} \cap g_i(U_i)\big) \overset{\mathcal{D}}{=} \mathcal{X}_n \cap U_i \tag{5.6}$$

because for Borel $B \subseteq U_i$, (5.4) shows that $P[Y_1 \in g_i(B)] = P[X_{i,1} \in B]$.

By Lemma 3.2 of [36] (restated as Lemma 3.2 of [35]), the distribution of the point process $n^{1/m}(-u + \mathcal{X}_{i,n})$ converges weakly to that of $\mathcal{H}_{\tilde{\kappa}_i(u)}$, in the metric of [36], which is *not* the same as the one we are using here (as discussed in [36], the metric in [36] is complete but not separable). We claim that if $f : \mathbb{R}^m \to \mathbb{R}$ is measurable with bounded support, then

$$\lim_{n\to\infty} \mathbb{E}\exp\Bigg(\sqrt{-1} \sum_{x \in n^{1/m}(-u+\mathcal{X}_{i,n})} f(x)\Bigg) = \mathbb{E}\exp\Bigg(\sqrt{-1} \sum_{x \in \mathcal{H}_{\tilde{\kappa}_i(u)}} f(x)\Bigg). \tag{5.7}$$

Indeed, by the proof of Lemma 3.2 of [36] there is a coupling in which the random variables under the expectations on the left and right-hand sides of (5.7) are equal with probability tending to 1.

It follows from (5.7) that for any finite collection of bounded Borel sets A_j $(1 \le j \le k)$ in $\mathbb{R}^m$, the joint distributions of the variables $\mathrm{card}(n^{1/m}(-u + \mathcal{X}_n) \cap A_j)$, $1 \le j \le k$, converge to those of the variables $\mathrm{card}(\mathcal{H}_{\tilde{\kappa}_i(u)} \cap A_j)$, $1 \le j \le k$. Hence, since U_i is a neighborhood of u, the joint distributions of the variables $\mathrm{card}(n^{1/m}(-u + (\mathcal{X}_n \cap U_i)) \cap A_j)$, $1 \le j \le k$, converge to those of the variables $\mathrm{card}(\mathcal{H}_{\tilde{\kappa}(u)} \cap A_j)$, $1 \le j \le k$.

Therefore the point processes $n^{1/m}(-u + (\mathcal{X}_n \cap U_i))$ converge weakly to $\mathcal{H}_{\tilde{\kappa}_i(u)}$, in the sense discussed at the start of this section; see Theorem 9.1.VI of [18].

Now we argue as in [1], page 251. By the Skorohod representation theorem, we can choose coupled point processes $\tilde{\mathcal{X}}_n$ and $\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)}$, all on the same probability space, such that $\tilde{\mathcal{X}}_n$ has the same distribution as $\mathcal{X}_n \cap U_i$, and $\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)}$ has the same distribution as $\mathcal{H}_{\tilde{\kappa}_i(u)}$, and such that $n^{1/m}(-u + \tilde{\mathcal{X}}_n)$ converges almost surely to $\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)}$. That is [see (5.1) and (5.2)], we can list the points of $\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)}$ as $x_1, x_2, \dots$ and the points of $\tilde{\mathcal{X}}_n$ as $x_{n,1}, x_{n,2}, \dots, x_{n,N_n}$, in such a way that for each j we have almost surely

$$n^{1/m}(x_{n,j} - u) \to x_j, \tag{5.8}$$

and for any $L > 0$ with no point of $\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)}$ in $B_L(\mathbf{0})$, we have almost surely

$$\mathrm{card}\big(n^{1/m}(-u + \tilde{\mathcal{X}}_n) \cap B_L(\mathbf{0})\big) \to \mathrm{card}\big(\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)} \cap B_L(\mathbf{0})\big). \tag{5.9}$$

By (5.8), $x_{n,j} \to u$ as $n \to \infty$, so by the differentiability of g_i, we can write

$$g_i(x_{n,j}) - g_i(u) = g_i'(u)(x_{n,j} - u) + w_{n,j}, \tag{5.10}$$

with $\|w_{n,j}\| = o(\|x_{n,j} - u\|)$ so $n^{1/m} w_{n,j} \to \mathbf{0}$ as $n \to \infty$. By (5.8) and (5.10),

$$n^{1/m}\big(g_i(x_{n,j}) - g_i(u)\big) \to g_i'(u)(x_j) \qquad \text{as } n \to \infty. \tag{5.11}$$

We claim that the point process $n^{1/m}(-g_i(u) + g_i(\tilde{\mathcal{X}}_n))$ converges a.s. to $g_i'(u)(\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)})$. The condition corresponding to (5.1) follows from (5.11). To demonstrate the condition corresponding to (5.2), set $y := g_i(u)$, and let L be such that no point $g_i'(u)(x_j)$ lies on the boundary of $B_L(\mathbf{0})$. We need to show that

$$\lim_{n\to\infty} \mathrm{card}\big[n^{1/m}(-y + g_i(\tilde{\mathcal{X}}_n)) \cap B_L(\mathbf{0})\big] = \mathrm{card}\big[g_i'(u)(\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)}) \cap B_L(\mathbf{0})\big]. \tag{5.12}$$

Choose $\delta > 0$ such that $B_{2\delta}(y) \subset g_i(U_i)$. By Lemma 4.1 we may define finite K by

$$K := \sup_{z,z' \in B_\delta(y), z \neq z'} \|g_i^{-1}(z') - g_i^{-1}(z)\| / \|z' - z\|.$$

Let $K' > K$, and suppose $x \in U_i$ with $\|x - u\| > n^{-1/m} K' L$. By definition of K, if $g(x) \in B_\delta(y)$, then $n^{1/m}\|g(x) - y\| > L$, and this also holds if $g(x) \notin B_\delta(y)$, provided $\delta n^{1/m} > L$.

Hence, for n large the contribution to the left-hand side of (5.12) comes only from $x \in \tilde{\mathcal{X}}_n \cap B_{n^{-1/m} K' L}(u)$. For large enough n, the set of such x consists precisely of those $x_{n,j}$ such that $x_j \in B_{K'L}(\mathbf{0})$, provided K' is chosen so that no point of $\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)}$ lies on the boundary of $B_{K'L}(\mathbf{0})$.

By (5.11), for large enough n the set of j such that $x_{n,j}$ contributes to the left-hand side of (5.7) is precisely those j such that $g'(u)(x_j) \in B_L(\mathbf{0})$. Thus we have (5.12), and therefore $n^{1/m}(-y + g_i(\tilde{\mathcal{X}}_n))$ converges almost surely to $g_i'(u)(\tilde{\mathcal{H}}_{\tilde{\kappa}_i(u)})$ as claimed. Hence $n^{1/m}(-y + g_i(\mathcal{X}_n \cap U_i))$ converges in distribution to $g_i'(u)(\mathcal{H}_{\tilde{\kappa}_i(u)})$. Together with (5.6) this yields

$$n^{1/m}\big(-y + \big(\mathcal{Y}_{\ell(n)} \cap g_i(U_i)\big)\big) \xrightarrow{\mathcal{D}} g_i'(u)\big(\mathcal{H}'_{\tilde{\kappa}_i(u)}\big). \tag{5.13}$$

By again using Lemma 9.1.VI of [18] (equivalence of weak convergence and convergence of fidi distributions) and the fact that $g_i(U_i)$ is a neighborhood of y in $\mathcal{M}$, we can deduce that (5.13) still holds with $\mathcal{Y}_{\ell(n)} \cap g_i(U_i)$, replaced by $\mathcal{Y}_{\ell(n)}$ on the left-hand side, so that (5.5) holds as asserted. $\square$

The next lemma is a two-dimensional version of Lemma 5.1.

LEMMA 5.2. *Suppose $1 \in \mathcal{I}$ and $2 \in \mathcal{I}$, and $g_1(U_1) \cap g_2(U_2) = \varnothing$ and also $U_1 \cap U_2 = \varnothing$ and $U_1 \cup U_2$ is bounded. Suppose that for $i = 1, 2$, $x_i \in U_i$ is a Lebesgue point of $\tilde{\kappa}_i$, and set $y_i = g_i(x_i)$. Suppose $(\ell(n), n \in \mathbb{N})$ is a sequence of positive integers such that $\ell(n) \sim n$ as $n \to \infty$. Then as $n \to \infty$,*

$$\begin{aligned} &\big[n^{1/m}(-y_1 + \mathcal{Y}_{\ell(n)}), n^{1/m}(-y_2 + \mathcal{Y}_{\ell(n)})\big] \\ &\qquad \xrightarrow{\mathcal{D}} \big[g_1'(x_1)(\mathcal{H}_{\tilde{\kappa}_1(x_1)}), g_2'(x_2)(\tilde{\mathcal{H}}_{\tilde{\kappa}_2(x_2)})\big], \end{aligned} \tag{5.14}$$

where $\tilde{\mathcal{H}}_a$ here is an independent copy of $\mathcal{H}_a$.

PROOF. By (5.4) we have $\int_{U_1} \tilde{\kappa}_1(x)\,dx + \int_{U_2} \tilde{\kappa}_2(x)\,dx \le 1$. Since we assume $U_1 \cap U_2 = \varnothing$ and $U_1 \cup U_2$ is bounded, we can therefore find a probability density function $\tilde{\kappa}$ on $\mathbb{R}^m$ which is an extension of both $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$, that is, with $\tilde{\kappa}(x) = \tilde{\kappa}_i(x)$ for $x \in U_i$, $i \in \{1, 2\}$.

Let $\mathcal{X}_n$ be a point process in $\mathbb{R}^m$ consisting of n independent identically distributed random m-vectors $X_1, \dots, X_n$ with density κ. Then

$$(5.15)\qquad \left(g_1^{-1}(\mathcal{Y}_{\ell(n)} \cap g_1(U_1)), g_2^{-1}(\mathcal{Y}_n \cap g_2(U_2))\right) \stackrel{\mathcal{D}}{=} (\mathcal{X}_{\ell(n)} \cap U_1, \mathcal{X}_{\ell(n)} \cap U_2)$$

because for $i = 1, 2$ and Borel $B \subseteq U_i$, (5.4) shows that $P[Y_1 \in g_i(B)] = P[X_1 \in B]$.

By Lemma 3.2 of [36] (restated as Lemma 3.2 of [35]), the joint distribution of the point processes $n^{1/m}(-x_1 + \mathcal{X}_{\ell(n)})$, $n^{1/m}(-x_2 + \mathcal{X}_{\ell(n)})$, converges weakly to that of $\mathcal{H}_{\tilde{\kappa}_1(x_1)}$, $\tilde{\mathcal{H}}_{\tilde{\kappa}_2(x_2)}$, in the metric of [36]. We can then follow the proof of Lemma 5.1 with straightforward modifications to deduce (5.2). □

Before proceeding we shall re-express the Poisson processes appearing in the limits (5.5) and (5.14) in a manner that is intrinsic to $\mathcal{M}$, that is, not dependent on the choice of atlas. Recalling that $\mathrm{Gr}_m(d)$ is the Grassmannian, given $\mathcal{M} \in \mathbb{M}$ and $y \in \mathcal{M}$, let $T_y\mathcal{M} \in \mathrm{Gr}_m(d)$ be the hyperplane tangent to $-y + \mathcal{M}$ at $\mathbf{0}$, that is, the image of $\mathbb{R}^m$ under the linear map $g_i'(x)$ when $x \in \mathbb{R}^m$ and (U_i, g_i) is any chart such that $x \in U_i$ and $y = g_i(x)$. We normalize the Lebesgue measure on $T_y\mathcal{M}$ (with volume element denoted du) in such a way that for any orthonormal basis $(f_i)_{i=1}^m$ of the subspace, the set $\{\sum_{i=1}^m a_i f_i : 0 \le a_i \le 1\}$ has unit Lebesgue measure. For $y, z \in \mathcal{M}$, let $\mathcal{H}'_{y,\kappa(y)}$ denote a homogeneous Poisson point process on $T_y\mathcal{M}$ with intensity $\kappa(y)$, and let $\tilde{\mathcal{H}}'_{z,\kappa(z)}$ denote a homogeneous Poisson point process on $T_z\mathcal{M}$ with intensity $\kappa(z)$, independent of $\mathcal{H}'_{y,\kappa(y)}$.

LEMMA 5.3. *Suppose $z_1 \in \mathcal{M}$ and $z_2 \in \mathcal{M}$ are distinct Lebesgue points for κ. Suppose $(\ell(n), n \ge 1)$ is a sequence of positive integers with $\ell(n) \sim n$ as $n \to \infty$. Then*

$$(5.16)\qquad [n^{1/m}(-z_1 + \mathcal{Y}_{\ell(n)}), n^{1/m}(-z_2 + \mathcal{Y}_{\ell(n)})] \xrightarrow{\mathcal{D}} (\mathcal{H}'_{z_1,\kappa(z_1)}, \tilde{\mathcal{H}}'_{z_2,\kappa(z_2)}).$$

PROOF. It is easy to see that we can choose our atlas $(U_i, g_i)_{i \in \mathcal{I}}$ such that $z_i \in g_i(U_i)$ for $i = 1, 2$, and such that moreover $g_1(U_1) \cap g_2(U_2) = \varnothing$, and $U_1 \cap U_2 = \varnothing$, and $U_1 \cup U_2$ is bounded. Let $x_i := g_i^{-1}(z_i)$ for $i = 1, 2$. Then by Lemma 5.2, to prove (5.16), it suffices to demonstrate for $i = 1, 2$ the distributional equality

$$(5.17)\qquad g_i'(x_i)(\mathcal{H}_{\tilde{\kappa}_i(x_i)}) \stackrel{\mathcal{D}}{=} \mathcal{H}'_{z_i,\kappa(z_i)}.$$

Let $i = 1$ or $i = 2$. By the mapping theorem on page 18 of [27], $g_i'(x_i)(\mathcal{H}_{\tilde{\kappa}_i(x_i)})$ is a Poisson process on the linear space $g_i'(x_i)(\mathbb{R}^m)$ with intensity measure μ

where $\mu(B)$ is $\tilde{\kappa}_i(x_i)$ times $|(g_i'(x_i))^{-1}(B)|$, where $|\cdot|$ denotes the m-dimensional Lebesgue measure.

Recall from Section 2.1 the definition of $D_{g_i}(x)$. For bounded measurable $A \subset \mathbb{R}^m$, it is a fact from linear algebra that

$$|g_i'(x_i)(A)| = D_{g_i}(x_i)|A|. \tag{5.18}$$

Indeed, the columns of the Jacobian matrix $J_{g_i}(x_i)$ are the images under $g'(x_i)$ of the standard basis vectors of $\mathbb{R}^m$, so (5.18) clearly holds when the standard basis vectors map to an orthonormal system, but then it can be deduced in the general case using standard properties of determinants. Equation (5.18) is the basis of the formula (2.2) given earlier.

By (5.18), if $A = (g_i'(x_i))^{-1}(B)$, then $|B| = D_{g_i}(x_i)|A|$ so that $\mu(B) = \tilde{\kappa}_i(x_i)|B|/D_{g_i}(x_i)$ so by (5.3), $\mu(B) = \kappa(z_i)|B|$ and (5.17) follows. $\square$

Next we give weak convergence results for ξ. Recall that $\mathbf{0}$ is the origin of $\mathbb{R}^m$. The next lemma is an analog of Lemma 3.6 of [35] and Lemma 3.6 of [36].

LEMMA 5.4. *Suppose ξ is continuous in the sense of Definition 5.1, and rotation invariant. Let $y \in \mathcal{M}$ and $z \in \mathcal{M}$ be a pair of distinct Lebesgue points for κ with $\kappa(y) > 0$ and $\kappa(z) > 0$. Suppose $(\ell(n), n \geq 1)$ is a sequence of positive integers such that $\ell(n) \sim n$ as $n \to \infty$. Let $k \in \mathbb{Z}^+$ and $\rho \in (0, \infty]$. Then as $n \to \infty$ we have*

$$[\xi_{n,k,\rho}(y, \mathcal{Y}_{\ell(n)}), \xi_{n,k,\rho}(z, \mathcal{Y}_{\ell(n)})] \xrightarrow{\mathcal{D}} [\xi(\mathbf{0}, \mathcal{H}_{\kappa(y)}), \xi(\mathbf{0}, \tilde{\mathcal{H}}_{\kappa(z)})]. \tag{5.19}$$

Also, if we choose a chart (U, g) and $u \in U$ such that $y = g(u)$, then for almost all (fixed) $x \in \mathbb{R}^m$, setting $v_n := u + n^{-1/m}x$, we have as $n \to \infty$ that

$$\begin{aligned} &\xi_{n,k,\rho}(y, \mathcal{Y}_{\ell(n)}^{g(v_n)})\xi_{n,k,\rho}(g(v_n), \mathcal{Y}_{\ell(n)}^{y}) \\ &\quad \xrightarrow{\mathcal{D}} \xi(\mathbf{0}, \mathcal{H}'^{g'(u)(x)}_{y,\kappa(y)})\xi(g'(u)(x), \mathcal{H}'^{\mathbf{0}}_{y,\kappa(y)}). \end{aligned} \tag{5.20}$$

PROOF. First suppose $\rho = \infty$. Then the left-hand side of (5.19) is equal to

$$[\xi(\mathbf{0}, n^{1/m}(-y + \mathcal{Y}_{\ell(n)})), \xi(\mathbf{0}, n^{1/m}(-z + \mathcal{Y}_{\ell(n)}))].$$

Also, by rotation invariance, the right-hand side of (5.19) has the same distribution as $[\xi(\mathbf{0}, \mathcal{H}'_{y,\kappa(y)}), \xi(\mathbf{0}, \tilde{\mathcal{H}}'_{z,\kappa(z)})]$. Therefore, under the assumptions given, (5.19) is immediate from Lemma 5.3 and the continuous mapping theorem ([11], Chapter 1, Theorem 5.1).

Next we prove (5.20) in the case $\rho = \infty$. By (3.1) and translation invariance of ξ,

$$\begin{aligned} &\xi_n(y, \mathcal{Y}_{\ell(n)}^{g(v_n)})\xi_n(g(v_n), \mathcal{Y}_{\ell(n)}^{y}) \\ &\quad = \xi(\mathbf{0}, n^{1/m}(-y + \mathcal{Y}_{\ell(n)}) \cup \{n^{1/m}(-y + g(v_n))\}) \end{aligned} \tag{5.21}$$

$$\times \xi\big(n^{1/m}(-y+g(v_n)), n^{1/m}(-y+\mathcal{Y}_{\ell(n)}) \cup \{\mathbf{0}\}\big)$$
$$= F\big(n^{1/m}(-y+g(v_n)), n^{1/m}(-y+\mathcal{Y}_{\ell(n)})\big),$$

where for any $y \in \mathbb{R}^d$ and any locally finite $\mathbf{y} \subset \mathbb{R}^d$, we set

$$F(y,\mathbf{y}) := \xi\big(\mathbf{0}, \mathbf{y} \cup \{y\}\big)\xi\big(y, \mathbf{y} \cup \{\mathbf{0}\}\big) = \xi\big(\mathbf{0}, \mathbf{y} \cup \{y\}\big)\xi\big(\mathbf{0}, -y + \big(\mathbf{y} \cup \{\mathbf{0}\}\big)\big).$$

By definition, $n^{1/m}(v_n - u) = x$, and by the same argument as for (5.11) earlier on, $n^{1/m}(g(v_n) - y)$ converges to $g'(u)(x)$ as $n \to \infty$. Combining this with Lemma 5.1, we have by (5.17) the convergence in distribution

$$\big(n^{1/m}(g(v_n)-y), n^{1/m}[-y+\mathcal{Y}_{\ell(n)}]\big) \xrightarrow{\mathcal{D}} \big(g'(u)(x), \mathcal{H}'_{y,\kappa(y)}\big).$$

By the continuity assumption, for almost every x the point set $\mathcal{H}'_{y,\kappa(y)} \cup \{g'(u)(x)\}$ is a.s. a continuity point of $\xi(\mathbf{0},\cdot)$, and so is the point set $-g'(u)(x) + (\mathcal{H}'_{y,\kappa(y)} \cup \{\mathbf{0}\})$. Thus $(g'(u)(x), \mathcal{H}'_{y,\kappa(y)})$ is a.s. at a continuity point of F, for almost all x. Hence we have the desired convergence in distribution (5.20) (when $\rho = \infty$) by (5.21) and the continuous mapping theorem.

Finally, we consider the case with $0 < \rho < \infty$. It is easy to see that

$$\lim_{n\to\infty} P\big[\xi_{n,k,\rho}(y,\mathcal{Y}_{\ell(n)})\xi_{n,k,\rho}(z,\mathcal{Y}_{\ell(n)}) \neq \xi_n(y,\mathcal{Y}_{\ell(n)})\xi_n(z,\mathcal{Y}_{\ell(n)})\big] = 0,$$

so the general case of (5.19) follows from the special case with $\rho = \infty$ (already proved) along with Slutsky's theorem. The proof of the general case of (5.20) is similar. □

6. Proofs of Theorems 3.1–3.3.

We first give some definitions. Assume $\mathcal{M} \in \mathbb{M}$ and $\kappa \in \mathbb{P}(\mathcal{M})$ are given, and set $\mathcal{K} := \mathcal{K}(\kappa)$. We adapt to the manifold setting the definition of exponentially stabilizing functionals [5, 35]. Suppose $k \in \mathbb{Z}^+$, $r \in [0,\infty)$ are given, along with the density κ. For $y \in \mathcal{K}$ and locally finite $\mathcal{Y} \subset \mathcal{K}$, define

$$R_\lambda(y,\mathcal{Y}) := \begin{cases} \max[r, N_k(\lambda^{1/m}y, \lambda^{1/m}\mathcal{Y})], & \text{if } \operatorname{card}(\mathcal{Y} \setminus \{y\}) \geq k, \\ \lambda^{1/m}\operatorname{diam}(\mathcal{K}), & \text{otherwise.} \end{cases}$$

Thus if $k = 0$, then $R_\lambda(y,\mathcal{Y}) = r$.

It is easy to see that $R := R_\lambda(y,\mathcal{Y})$ serves as a *radius of stabilization* for any $\xi \in \Xi(k,r)$, in the following sense: for all finite $\mathcal{A} \subset (\mathcal{K} \setminus B_{\lambda^{-1/m}R}(y))$, we have

$$\xi_\lambda\big(y, (\mathcal{Y} \cap B_{\lambda^{-1/m}R}(y)) \cup \mathcal{A}\big) = \xi_\lambda\big(y, \mathcal{Y} \cap B_{\lambda^{-1/m}R}(y)\big). \tag{6.1}$$

For all $k \in \mathbb{Z}^+$, $\rho \in (0,\infty)$, note that R also serves as a radius of stabilization for $\xi_{\lambda,k,\rho}$ in the sense that (6.1) holds if ξ_λ is replaced by $\xi_{\lambda,k,\rho}$. Recall the definition of point processes $\mathcal{Y}_n$ and $\mathcal{P}_\lambda$ in Section 2.1, and recall that $\mathcal{S}_2$ is the collection of all subsets of $\mathcal{K}(\kappa)$ of cardinality at most 2, including the empty set. Given

$\varepsilon > 0$ and $t > 0$, we define the tail probabilities for R_λ denoted $\tau(t)$ and $\tau_\varepsilon(t)$, for Poisson and binomial input, respectively, as follows:

$$\tau(t) := \sup_{\lambda \geq 1} \operatorname*{ess\,sup}_{y \in \mathcal{K}} P[R_\lambda(y, \mathcal{P}_\lambda) > t];$$

$$\tau_\varepsilon(t) := \sup_{\lambda \geq 1, n \in \mathbb{N} \cap ((1-\varepsilon)\lambda, (1+\varepsilon)\lambda), \mathcal{A} \in \mathcal{S}_2} \operatorname*{ess\,sup}_{y \in \mathcal{K}} P[R_\lambda(y, \mathcal{Y}_n \cup \mathcal{A}) > t],$$

where the ess sup denotes essential supremum with respect to the measure $\kappa(y)\,dy$.

DEFINITION 6.1. Given k and r, we say that every $\xi \in \Xi(k,r)$ is *exponentially stabilizing* for κ if $\limsup_{t\to\infty} t^{-1} \log \tau(t) < 0$. We say that every $\xi \in \Xi(k,r)$ is *binomially exponentially stabilizing* for κ if there exists $\varepsilon > 0$ such that $\limsup_{t\to\infty} t^{-1} \log \tau_\varepsilon(t) < 0$.

We next show that functionals in $\Xi(k,r)$ have the continuity property of Definition 5.1 as well as the binomial and exponential stabilization properties.

LEMMA 6.1. *Let $k \in \mathbb{Z}^+$ and $r \geq 0$. Then every $\xi \in \Xi(k,r)$ is continuous. If either $k = 0$ or $\kappa \in \mathbb{P}_c(\mathcal{M})$, then every $\xi \in \Xi(k,r)$ is exponentially stabilizing and binomially exponentially stabilizing for κ.*

PROOF. To prove continuity, let $\xi \in \Xi(k,r)$, let $z \in \mathbb{R}^m$, and let $F : \mathbb{R}^m \to \mathbb{R}^d$ be linear and of full rank. Assume that the points of $F(\mathcal{H}^z)$ have distinct Euclidean norms, and that for all $n \in \mathbb{N}$ there are no points of $F(\mathcal{H})$ on the boundary of the ball $B_n(\mathbf{0})$. List the elements of $F(\mathcal{H})$ in order of increasing Euclidean norm as as $x_1, x_2, \ldots$. Suppose $(\mathbf{y}_n)_{n\in\mathbb{N}}$ is a sequence of locally finite point sets in $\mathbb{R}^d$ converging in $\mathcal{T}$ to $F(\mathcal{H})$, and list the elements of $\mathbf{y}_n$ in order of increasing Euclidean norm as $y_{n,1}, y_{n,2}, y_{n,3}, \ldots$ (possibly a terminating sequence).

Given the realization of $\mathcal{H}$, we pick the smallest $K \in \mathbb{N}$ such that

$$K > \max(r, N_k(\mathbf{0}, F(\mathcal{H})), \|F(z)\|).$$

Let N denote the number of points of $F(\mathcal{H})$ in B_K, and assume $(y_1, \ldots, y_N)$ lies at a continuity point of the mapping $(x_1, \ldots, x_N) \mapsto \xi(\mathbf{0}, \{x_1, \ldots, x_N\})$. By the convergence of $\mathbf{y}_n$ to $\mathcal{H}_a$, for all large enough n we have $y_{n,N} \in B_K$ and $y_{n,N+1} \notin B_K$, and moreover $y_{n,j} \to y_j$ as $n \to \infty$ for all $j \leq N$. Therefore by the continuity assumption we have

$$\xi(\mathbf{0}, F(\mathcal{H}_a)) = \xi(\mathbf{0}, \{y_1, \ldots, y_N\}) = \lim_{n\to\infty} \xi(\mathbf{0}, \{y_{n,1}, \ldots, y_{n,N}\})$$
$$= \lim_{n\to\infty} \xi(\mathbf{0}, F(\mathbf{y}_n)).$$

Similarly, if $(F(z), y_1, \ldots, y_N)$ lies at a continuity point of the mapping $(x_0, x_1, \ldots, x_N) \mapsto \xi(\mathbf{0}, \{x_0, x_1, \ldots, x_N\})$ and if $(\mathbf{v}_n)_{n\in\mathbb{N}}$ is any sequence of locally finite point sets in $\mathbb{R}^d$ converging in $\mathcal{T}$ to $F(\mathcal{H}^z)$, then $\xi(\mathbf{0}, F(\mathcal{H}_a^z)) = \lim_{n\to\infty} \xi(\mathbf{0}, \mathbf{v}_n)$. Thus, ξ is continuous.

We prove the exponential stabilization of $\xi \in \Xi(k, r)$, that is, the uniform exponential tail bound for $R_\lambda(y, \mathcal{P}_\lambda)$, as follows. Suppose $y \in \mathcal{K}$, $\lambda \in [1, \infty)$ and $r < t \le \lambda^{1/d} \operatorname{diam}(\mathcal{K})$. Then

$$P[R_\lambda(y, \mathcal{P}_\lambda) > t] = P[N_k(\lambda^{1/m} y, \lambda^{1/m}\mathcal{P}_\lambda) > t],$$

and the last event occurs if and only if the number of points from $\mathcal{P}_\lambda$ in $B_{t\lambda^{-1/m}}(y) \setminus \{y\}$ is less than k. The number of such points is Poisson distributed with parameter $\alpha(t) := \alpha(t, y, \lambda)$ equal to the $\lambda\kappa$ measure of $B_{t\lambda^{-1/m}}(y) \cap \mathcal{M}$. By Lemma 4.3 there is a constant $C_2 > 0$ such that we have uniformly in $\lambda \in [1, \infty)$, $y \in \mathcal{K}$ and $t \in (0, \lambda^{1/m} \operatorname{diam}(\mathcal{K}))$ that $\alpha(t) \ge C_2^{-1} t^m$. Thus by a Chernoff bound for the Poisson distribution (see, e.g., Lemma 1.2 of [34]), there is a constant C_3 such that for $\max(r, (2kC_2)^{1/m}) < t < \lambda^{1/m} \operatorname{diam}(\mathcal{K})$ we have

$$P[R_\lambda(y, \mathcal{P}_\lambda) > t] \le k \exp(-C_3^{-1} t^m),$$

and moreover this also holds for $t \ge \lambda^{1/m} \operatorname{diam}(\mathcal{K})$ since $P[R_\lambda(y, \mathcal{P}_\lambda) > t] = 0$ in this case. This gives the desired exponential stabilization of ξ for Poisson input. Modifications of this argument yields the exponential stabilization of ξ with respect to binomial input. $\square$

For finite $\mathcal{Y} \subset \mathbb{R}^d$ and $y \in \mathcal{Y}$, and $k \in \mathbb{Z}^+$, $n \in \mathbb{N}$, $\rho \in (0, \infty]$, define

$$\begin{aligned} \xi^*_{n,k,\rho}(y, \mathcal{Y}) &:= \xi_{n,k,\rho}(y, \mathcal{Y}) \mathbf{1}\{|\xi_{n,k,\rho}(y, \mathcal{Y})| \le n^{5/12}\}; \\ H^*_{n,k,\rho}(\mathcal{Y}) &:= \sum_{y\in\mathcal{Y}} \xi^*_{n,k,\rho}(y, \mathcal{Y}). \end{aligned} \tag{6.2}$$

Recall the (similar) definition of $H^\xi_{n,k,\rho}(\mathcal{Y})$ at (3.3). Given $n, i, \nu \in \mathbb{N}$ with $i \le \nu$, define

$$G_{i,\nu,n} := H^\xi_{n,k,\rho}(\mathcal{Y}_\nu) - H^\xi_{n,k,\rho}(\mathcal{Y}_\nu \setminus \{Y_i\}), \tag{6.3}$$

$$G^*_{i,\nu,n} := H^*_{n,k,\rho}(\mathcal{Y}_\nu) - H^*_{n,k,\rho}(\mathcal{Y}_\nu \setminus \{Y_i\}). \tag{6.4}$$

LEMMA 6.2. *Suppose ξ is binomially exponentially stabilizing, and $\kappa \in \mathbb{P}_b(\mathcal{M})$. Suppose $h(n)/n \to 0$ as $n \to \infty$ and suppose for some $p \in \mathbb{N}$ that (3.5) holds. Then*

$$\limsup_{n\to\infty} \sup_{n-h(n)\le\nu\le n+h(n)} \mathbb{E}|G_{\nu,\nu,n}|^p < \infty \tag{6.5}$$

and

$$\limsup_{n\to\infty} \sup_{n-h(n)\le\nu\le n+h(n)} \mathbb{E}|G^*_{\nu,\nu,n}|^p < \infty. \tag{6.6}$$

PROOF. We prove only (6.5); the proof of (6.6) is virtually the same. Write ξ_n for $\xi_{n,k,\rho}$, and define $\Delta^x \xi_n(y, \mathcal{Y}) := \xi_n(y, \mathcal{Y} \cup \{x\}) - \xi_n(y, \mathcal{Y})$. Putting $Y = Y_\nu$, observe that

$$|G_{\nu,\nu,n}| \le |\xi_n(Y, \mathcal{Y}_{\nu-1})| + \sum_{i=1}^{\nu-1} |\Delta^Y \xi_n(Y_i, \mathcal{Y}_{\nu-1})|. \tag{6.7}$$

The pth moment of the first term in the right-hand side of (6.7) is uniformly bounded by (3.5). The pth moment of the sum in the right-hand side of (6.7) is given by

$$\begin{aligned} &(\nu-1)\mathbb{E}|\Delta^Y \xi_n(Y_1, \mathcal{Y}_\nu)|^p \\ &\quad + (\nu-1)(\nu-2)\mathbb{E}|\Delta^Y \xi_n(Y_1, \mathcal{Y}_\nu)|^{p-1}|\Delta^Y \xi_n(Y_2, \mathcal{Y}_\nu)| + \cdots \\ &\quad + \left(\frac{(\nu-1)!}{(\nu-1-p)!}\right)\mathbb{E}\prod_{i=1}^{p}|\Delta^Y \xi_n(Y_i, \mathcal{Y}_\nu)|. \end{aligned} \tag{6.8}$$

Let $\mathcal{I}$, y_i, δ_i, U_i be as in Section 2.1. Using compactness, let $\mathcal{I}_0 \subset \mathcal{I}$ be a finite set such that $\mathcal{K} \subset \bigcup_{i \in \mathcal{I}_0} B_{\delta_i}(y_i)$, and set $\delta := \min_{i \in \mathcal{I}_0}(\delta_i)$. Then

$$\begin{aligned} &\nu\mathbb{E}|\Delta^Y \xi_n(Y_1, \mathcal{Y}_{\nu-1})|^p \\ &\quad = \int_{\mathcal{M}} \int_{g_\ell(U_\ell)} \mathbb{E}|\Delta^y \xi_n(z, \mathcal{Y}_{\nu-2})|^p \nu\kappa(z)\, dz\, \kappa(y)\, dy. \end{aligned} \tag{6.9}$$

For $z, y \in \mathcal{K}$ with $\|z - y\| \ge \delta$, by binomial exponential stabilization we have $P[\Delta^y \xi_n(z, \mathcal{Y}^z_{\nu-2}) \ne 0]$ decaying exponentially in $n^{1/m}$, uniformly over such (z, y) and over $\nu \in [n - h(n), n + h(n)]$. By this, bound (3.5) and Hölder's inequality, the contribution to (6.9) from such (z, y) tends to zero as $n \to \infty$ and is uniformly bounded.

Now take $y \in \mathcal{M}$, and choose $i \in \mathcal{I}_0$ such that $y \in B_{\delta_i}(y_i)$. Then $B_{2\delta}(y) \subset g_i(U_i)$. Assume without loss of generality that $g_\ell(\mathbf{0}) = y$, and let $U_i' := g_i^{-1}(B_\delta(y))$. Then the contribution to the inner integral in the right-hand side of (6.9) from $z \in B_\delta(y)$ can be rewritten as the expression

$$\begin{aligned} &\nu \int_{U_i'} \mathbb{E}|\Delta^y \xi_n(g_\ell(u), \mathcal{Y}_{\nu-2})|^p \tilde{\kappa}_\ell(u)\, du \\ &\quad = \int_{\nu^{1/m} U_i'} \mathbb{E}|\Delta^y \xi_n(g_\ell(\nu^{-1/m} v), \mathcal{Y}_{\nu-2})|^p \tilde{\kappa}_i(\nu^{-1/m} v)\, dv, \end{aligned}$$

and by binomial exponential stabilization, and (3.5), and Hölder's inequality and comparability of norms in U_i and in $g_i(U_i)$ (see Lemma 4.1), there is a constant C, independent of y, such that the integrand is bounded by $C\exp(-C^{-1}\|v\|^{1/m})$, which is integrable in v. This shows that (6.9) is uniformly bounded.

Turning to the second term in (6.8), we condition on $Y = y$ with $i \in \mathcal{I}_0$ again chosen so that $y \in B_{\delta_i}(y_i)$. By Hölder's inequality we have

$$\begin{aligned}
&\nu^2 \mathbb{E}\big|\Delta^y \xi_n(Y_1, \mathcal{Y}_\nu)\big|^{p-1}\big|\Delta^y \xi_n(Y_2, \mathcal{Y}_\nu)\big| \\
&\quad = \nu^2 \int_{\mathcal{K}} \int_{\mathcal{K}} \mathbb{E}\big|\Delta^y \xi_n\big(w, \mathcal{Y}^z_{\nu-2}\big)\big|^{p-1}\big|\Delta^y \xi_n\big(w, \mathcal{Y}^z_{\nu-2}\big)\big|\kappa(w)\kappa(z)\,dz\,dw \\
&\quad \le \nu^2 \int_{\mathcal{K}} \int_{\mathcal{K}} \big(\mathbb{E}\big|\Delta^y \xi_n\big(w, \mathcal{Y}^z_{\nu-2}\big)\big|^p\big)^{(p-1)/p} \\
&\qquad\qquad \times \big(\mathbb{E}\big|\Delta^y \xi_n\big(z, \mathcal{Y}^w_{\nu-2}\big)\big|^p\big)^{1/p}\kappa(w)\kappa(z)\,dz\,dw.
\end{aligned}$$

The contribution to the last expression from $(w, z) \notin B_\delta(y) \times B_\delta(y)$ is uniformly bounded (and in fact tending to zero as $n \to \infty$) by a similar argument to the one given above for the contribution to (6.9) from $w \notin B_\delta(y)$. Assuming $g_\ell(\mathbf{0}) = y$, the contribution to the last integral from $(w, z) \in B_\delta(y) \times B_\delta(y)$ is given by

$$\begin{aligned}
&\int_{\nu^{1/m}U'_i} \int_{\nu^{1/m}U'_i} \big(\mathbb{E}\big|\Delta^y \xi_n\big(g_\ell\big(\nu^{-1/m}v\big), \mathcal{Y}_{\nu-2} \cup \big\{g_\ell\big(\nu^{-1/m}v\big)\big\}\big)\big|^p\big)^{(p-1)/p} \\
&\qquad \times \big(\mathbb{E}\big|\Delta^y \xi_n\big(g_\ell\big(\nu^{-1/m}v\big), \mathcal{Y}_{\nu-2} \cup \big\{g_\ell\big(\nu^{-1/m}u\big)\big\}\big)\big|^p\big)^{1/p} \\
&\qquad \times \tilde{\kappa}_\ell\big(\nu^{-1/m}u\big)\,du\,\tilde{\kappa}_\ell\big(\nu^{-1/m}v\big)\,dv.
\end{aligned}$$

By binomial exponential stabilization, Hölder's inequality and comparability of norms in U_ℓ and in $g_\ell(U_\ell)$ (see Lemma 4.1), there is a constant C such that the integrand is bounded by $C\exp(-C^{-1}(\|u\|^{1/m} + \|v\|^{1/m}))$, which is integrable in (u, v). This shows that the integral is uniformly bounded, and hence the second term in (6.8) is bounded. The remaining terms in (6.8) are handled similarly, showing that all terms in (6.8) are uniformly bounded, and hence by (6.7), we have (6.5). □

PROOF OF THEOREM 3.1. We first sketch the proof for $q = 2$. Using (5.19) we obtain, as in the proof of equations (4.3) and (4.4) of [36], the distributional convergence

$$\xi_{n,k,\rho}(Y_1, \mathcal{Y}_n) \xrightarrow{\mathcal{D}} \xi(\mathbf{0}, \mathcal{H}_{\kappa(Y_1)}),$$

where $\mathcal{H}_{\kappa(Y_1)}$ is a Cox process in $\mathbb{R}^d$ whose distribution, conditional on the value y of Y_1, is that of $\mathcal{H}_{\kappa(y)}$, and also,

$$\xi_{n,k,\rho}(Y_1, \mathcal{Y}_n)\xi_{n,k,\rho}(Y_2, \mathcal{Y}_n) \xrightarrow{\mathcal{D}} \xi(\mathbf{0}, \mathcal{H}_{\kappa(Y_1)})\xi(\mathbf{0}, \tilde{\mathcal{H}}_{\kappa(Y_2)}),$$

where $\tilde{\mathcal{H}}_{\kappa(Y_2)}$ is an independent copy of the Cox process $\mathcal{H}_{\kappa(Y_1)}$.

Put $\mu := \mathbb{E}\xi(\mathbf{0}, \mathcal{H}_{\kappa(Y_1)})$. Under the assumed moment condition it follows that

$$\lim_{n\to\infty} \mathbb{E}\xi_{n,k,\rho}(Y_1, \mathcal{Y}_n)\xi_{n,k,\rho}(Y_2, \mathcal{Y}_n) = \mathbb{E}\xi(\mathbf{0}, \mathcal{H}_{\kappa(Y_1)})\xi(\mathbf{0}, \tilde{\mathcal{H}}_{\kappa(Y_2)}) = \mu^2, \tag{6.10}$$

where the last equality follows by independence. Recalling definition (3.3) of $H^{\xi}_{n,k\rho}(\cdot)$, we have that

$$n^{-2}\mathbb{E}H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)^2 = n^{-1}\mathbb{E}\xi_{n,k,\rho}(Y_1,\mathcal{Y}_n)^2 + (1-n^{-1})\mathbb{E}\xi_{n,k,\rho}(Y_1,\mathcal{Y}_n)\xi_n(Y_2,\mathcal{Y}_n),$$

so we obtain from (6.10) that $n^{-2}\mathbb{E}H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)^2 \to \mu^2$ as $n\to\infty$. Since $n^{-1}\mathbb{E}H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)\to\mu$ as $n\to\infty$ it follows that $n^{-1}H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)$ converges in L^2 to μ.

Since μ equals the right-hand side of (3.7), we have (3.7) with L^2 convergence when $q=2$. To obtain L^1 convergence when $q=1$, we use a truncation argument and follow the proof of Proposition 3.2 in [39]. We leave the details to the reader.

It remains to prove that if we assume (3.5) holds for some $p>5$ and $\kappa\in\mathbb{P}_b(\mathcal{M})$, and that either $\kappa\in\mathbb{P}_c(\mathcal{M})$ or $k=0$, then (3.7) holds with a.s. convergence. Under these extra assumptions, Lemmas 6.1 and 6.2 show that $\mathbb{E}|H^*_{n,k,\rho}(\mathcal{Y}_n)-H^*_{n,k,\rho}(\mathcal{Y}_{n-1})|^5$ is bounded by a constant that is independent of n. Then (2.11) of [36] holds with $\beta=4/3$ and $p'=5$ (and $f\equiv 1$ in the notation of [36]). By following the proof of Theorem 2.2 of [36] we obtain for all $\varepsilon>0$ that

$$\sum_{n=1}^{\infty} P\bigl[\bigl|H^*_{n,k,\rho}(\mathcal{Y}_n)-\mathbb{E}H^*_{n,k,\rho}(\mathcal{Y}_n)\bigr|>\varepsilon n\bigr]<\infty. \tag{6.11}$$

Also, by (3.4) [i.e., by taking $\mathcal{A}=\varnothing$ in (3.5)] and (6.2) and Markov's inequality,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\bigl[H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)\neq H^*_{n,k,\rho}(\mathcal{Y}_n)\bigr]\\
&\quad\le \sum_{n=1}^{\infty} nP\bigl[\xi_{n,k,\rho}(Y_1,\mathcal{Y}_n)\neq \xi^*_{n,k,\rho}(Y_1,\mathcal{Y}_n)\bigr] \qquad (6.12)\\
&\quad\le \sum_{n=1}^{\infty} n^{1-25/12}\mathbb{E}\bigl[\bigl|\xi_{n,k,\rho}(Y_1,\mathcal{Y}_n)\bigr|^5\bigr]<\infty.
\end{aligned}$$

By (6.11), (6.12) and the Borel–Cantelli lemma,

$$\lim_{n\to\infty} n^{-1}\bigl(H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)-\mathbb{E}H^*_{n,k,\rho}(\mathcal{Y}_n)\bigr)=0 \qquad \text{a.s.} \tag{6.13}$$

Also, $\{\xi_{n,k,\rho}(Y_1,\mathcal{Y}_n), n\ge 1\}$ are uniformly integrable by (3.4), so

$$\begin{aligned}
&n^{-1}\bigl|\mathbb{E}H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)-\mathbb{E}H^*_{n,k,\rho}(\mathcal{Y}_n)\bigr|\\
&\quad\le \mathbb{E}\bigl|\xi_{n,k,\rho}(Y_1,\mathcal{Y}_n)-\xi^*_{n,k,\rho}(Y_1,\mathcal{Y}_n)\bigr| \qquad (6.14)\\
&\quad= \mathbb{E}\bigl[\bigl|\xi_{n,k,\rho}(Y_1,\mathcal{Y}_n)\bigr|\mathbf{1}\bigl\{\bigl|\xi_{n,k,\rho}(Y_1,\mathcal{Y}_n)\bigr|>n^{5/12}\bigr\}\bigr]\to 0 \qquad \text{as } n\to\infty.
\end{aligned}$$

Finally, by (3.7) we have that

$$\mathbb{E}[n^{-1} H^{\xi}_{n,k,\rho}(\mathcal{Y}_n)] \to \int_{\mathcal{M}} \mathbb{E}[\xi(\mathbf{0}, \mathcal{H}_{\kappa(y)})]\kappa(y)\,dy,$$

and by (6.13) and (6.14) we have (3.7) with a.s. convergence. □

The following lemma will be used in the proof of Theorem 3.2. It can be proved by following verbatim the proof of Lemma 4.2 of [35], so we omit details here. Again we write $\mathcal{Y}^y$ for $\mathcal{Y} \cup \{y\}$.

LEMMA 6.3. *Suppose that $\mathcal{M} \in \mathbb{M}(m,d)$ and $\kappa \in \mathbb{P}_b(\mathcal{M})$, and ξ is exponentially stabilizing. Let $k \in \mathbb{Z}^+$, $\rho \in (0,\infty]$ and suppose for some $p > 2$ that ξ satisfies (3.6). Then there is a constant $C > 0$ such that for all $\lambda \geq 1$ and all $y, z \in \mathcal{K}$,*

$$\begin{aligned}&\bigl|\mathbb{E}\xi_{\lambda,k,\rho}(y, \mathcal{P}^z_\lambda)\xi_{\lambda,k,\rho}(z, \mathcal{P}^y_\lambda) - \mathbb{E}\xi_{\lambda,k,\rho}(y, \mathcal{P}_\lambda)\mathbb{E}\xi_{\lambda,k,\rho}(z, \mathcal{P}_\lambda)\bigr| \\ &\quad \leq C \exp(-C^{-1}\lambda^{1/m}\|z-y\|).\end{aligned}$$

PROOF OF THEOREM 3.3. We show first the asymptotic variance convergence (3.17). Recalling from (2.5) that $\mathcal{P}_\lambda$ is the Poisson point process on $\mathcal{M}$ having intensity measure $\lambda\kappa(y)\,dy$, we have (cf. the proof of Lemma 4.1 of [35])

$$\begin{aligned}&\lambda^{-1}\operatorname{Var}[H^{\xi}_{\lambda,k,\rho}] \\ &\quad = \int_{\mathcal{M}} \mathbb{E}[\xi_{\lambda,k,\rho}(y, \mathcal{P}_\lambda)^2]\kappa(y)\,dy \\ &\qquad + \int_{\mathcal{M}}\int_{\mathcal{M}} \{\mathbb{E}\xi_{\lambda,k,\rho}(z, \mathcal{P}^y_\lambda)\xi_\lambda(y, \mathcal{P}^z_\lambda) \\ &\qquad\qquad - \mathbb{E}\xi_{\lambda,k,\rho}(z, \mathcal{P}_\lambda)\mathbb{E}\xi_{\lambda,k,\rho}(y, \mathcal{P}_\lambda)\}\lambda\kappa(z)\kappa(y)\,dz\,dy.\end{aligned} \tag{6.15}$$

Since we assume $\kappa \in \mathbb{P}_b(\mathcal{M})$, we can choose an index set $\mathcal{I} \subset \mathbb{N}$, and a set of quadruples $(y_i, \delta_i, U_i, g_i)_{i \in \mathcal{I}}$ as in Section 2.1, such that the support κ of $\mathcal{K}$ is contained in finitely many $B_{\delta_i}(y_i)$, $i \in \mathcal{I}_0 := \{1, \dots, m\} \subset \mathcal{I}$. Also, we define our partition of unity here by $\psi_i = \mathbf{1}\{B_{\delta_i}(y_i) \setminus \bigcup_{j<i} B_{\delta_j}(y_j)\}$. Let $\delta := \min_{i \in \mathcal{I}_0} \delta_i$.

Suppose $y, z \in \mathcal{K}$ with $\|z - y\| > \delta$. Then by Lemma 6.3, the integrand inside the braces in the double integral in (6.15) is bounded by $C\lambda\exp(-C^{-1}(\delta\lambda)^{1/m})$, where the constant C does not depend on λ, y or z. Hence the contribution to the double integral from such y, z tends to zero.

To estimate the remaining contribution to the double integral, given y take $i = i(y)$ such that $\psi_i(y) = 1$ [so in particular $B_\delta(y) \subset g_i(U_i)$], and let $U_i' := g_i^{-1}(B_\delta(y))$ and $u := g_i^{-1}(y)$. Then the contribution to inner integral in the double integral from $z \in B_\delta(y)$ is given by

$$\int_{U_i'} \{\mathbb{E}\xi_{\lambda,k,\rho}(y, \mathcal{P}_\lambda^{g_i(x)})\xi_{\lambda,k,\rho}(g_i(x), \mathcal{P}_\lambda^y) - \mathbb{E}\xi_{\lambda,k,\rho}(g_i(x), \mathcal{P}_\lambda)\mathbb{E}\xi_{\lambda,k,\rho}(y, \mathcal{P}_\lambda)\}\lambda\tilde{\kappa}_i(x)\,dx = \int_{\lambda^{1/m}(-u+U_i')} F_\lambda(v, y)\tilde{\kappa}_i(u + \lambda^{-1/m}v)\,dv,$$

where we set

$$F_\lambda(v, y) := \mathbb{E}\xi_{\lambda,k,\rho}(y, \mathcal{P}_\lambda^{g_i(u+\lambda^{-1/m}v)})\xi_{\lambda,k,\rho}(g_i(u + \lambda^{-1/m}v), \mathcal{P}_\lambda^y) - \mathbb{E}\xi_{\lambda,k,\rho}(y, \mathcal{P}_\lambda)\mathbb{E}\xi_\lambda(g_i(u + \lambda^{-1/m}v), \mathcal{P}_\lambda).$$

By the Poisson analog of Lemma 5.4, together with the moment condition (3.6), provided $y \in g_i(U_i)$ is a Lebesgue point of κ_i, for almost all $v \in \mathbb{R}^m$ we have as $\lambda \to \infty$ that $F_\lambda(v, y) \to F(g_i'(u)(v), y)$, where for $x \in T_y\mathcal{M}$ we set

$$F(x, y) := \mathbb{E}\xi(\mathbf{0}, \mathcal{H}_{\kappa(y)}^x)\xi(x, \mathcal{H}_{\kappa(y)}^{\mathbf{0}}) - (\mathbb{E}\xi(\mathbf{0}, \mathcal{H}_{\kappa(y)}))^2.$$

Also, by (5.3) and the assumed a.e. continuity of κ, almost every $y \in g_i(U_i)$ is a Lebesgue point of $\tilde{\kappa}_i$. Moreover, by Lemmas 6.1, 6.3 and 4.1, we have for some constant C, independent of (λ, y), that $|F_\lambda(v, y)| \le C\exp(-C^{-1}\|v\|)$. Hence, using the representation (2.2) with our chosen partition of unity, by dominated convergence the double integral in (6.15) converges as $\lambda \to \infty$ to

$$\sum_{i\in\mathcal{I}} \int_{g(U_i)} \kappa(y)\psi_i(y)\,dy \int_{\mathbb{R}^m} F(g_i'(u)(v), y)\tilde{\kappa}_i(u)\,dv. \tag{6.16}$$

We can simplify this limit by the change of variable $w = g_i'(u)(v)$. Then $dw = D_g(u)\,dv$ and by (5.3) and then (2.2) the expression (6.16) equals

$$\begin{aligned}\sum_{i\in\mathcal{I}} \int_{g(U_i)} \kappa(y)^2\psi_i(y)\,dy \int_{\mathcal{T}_{y_0}\mathcal{M}} F(w, y)\,dw \\ = \int_{\mathcal{M}} \int_{T_y M} F(w, y)\,dw\kappa(y)^2\,dy,\end{aligned} \tag{6.17}$$

so that the double integral at (6.15) tends to this.

On the other hand, by the Poisson analog of Lemma 5.4, the moment bound (3.6), and dominated convergence, the single integral at (6.15) tends to

$$\lim_{\lambda\to\infty} \int_{\mathcal{M}} \mathbb{E}[\xi_\lambda(y, \mathcal{P}_\lambda)^2]\kappa(y)\,dy = \int_{\mathcal{M}} \mathbb{E}[\xi(\mathbf{0}, \mathcal{H}_{\kappa(y)})^2]\kappa(y)\,dy. \tag{6.18}$$

Combining the right-hand sides of (6.17) and (6.18), recalling the definition of V^ξ at (3.8), we get from (6.15) that $\lambda^{-1}\operatorname{Var}[H_{\lambda,k,\rho}^\xi]$ tends to $\int_{\mathcal{M}} V^\xi(y, \kappa(y))\kappa(y)\,dy$, which is (3.17) as desired.

Now assume (3.6) holds for some $p > 3$. To prove the normal approximation result (3.19), we adapt the proof of Corollary 2.4 of [40] (Corollary 2.1 in the arXiv version), putting $f \equiv 1$. Recall that the partition of unity has been chosen in such a way that $\psi_i(y) \in \{0, 1\}$ for all $y \in \mathcal{M}$ and all $i \in \mathcal{I}$. For $i \in \mathcal{I}$ let

$$\mathcal{K}_i := \{y \in \mathcal{K} : \psi_i(y) = 1\}.$$

As in Sections 4.2 and 4.3 of [40], let $\rho_\lambda := \alpha \log \lambda$ for $\lambda > 0$, where α is a suitably chosen large constant; see right after (4.9) of [40]. For each $\ell \in \mathcal{I}_0$, cover the bounded set $g_\ell^{-1}(\mathcal{K}_\ell)$ by a (minimal) collection of cubes of side $\lambda^{-1/m}\rho_\lambda$, denoted $Q_{i,\ell}$, $1 \le i \le V_\ell(\lambda)$, where $V_\ell = O(\lambda \rho_\lambda^{-m})$.

Fix λ for now. For $\ell \in \mathcal{I}_0$ and $1 \le i \le V_\ell(\lambda)$, let $N_{i,\ell}$ be the number of points of $\mathcal{P}_\lambda$ in $\mathcal{K}_\ell \cap g_\ell(Q_{i,\ell})$, a Poisson variable with parameter $\nu_{i,\ell}$ given by

$$\nu_{i,\ell} := \lambda \int_{\mathcal{K}_\ell \cap g_\ell(Q_{i,\ell})} \kappa(y)\, dy = \lambda \int_{g_\ell^{-1}(\mathcal{K}_\ell)\cap Q_{i,\ell}} \tilde{\kappa}_\ell(x)\, dx.$$

Note that the densities $\tilde{\kappa}_\ell$ are uniformly bounded because $D_{g_\ell}(\cdot)$ is uniformly bounded on $g_\ell^{-1}(\mathcal{K}_\ell)$ by compactness.

Let $X_{i,\ell,j}$ denote the jth point of $\mathcal{P}_\lambda \cap \mathcal{K}_\ell \cap g_\ell(Q_{i,\ell})$, when these points are listed in a randomized order; cf. Section 4.2 of [40]. Then with obvious modifications, Lemmas 4.2 and 4.3 of [40] still hold in the present setting.

Now follow Section 4.3 of [40], but now defining the graph $G_\lambda := (\mathcal{V}_\lambda, \mathcal{E}_\lambda)$ as follows. The set $\mathcal{V}_\lambda$ consists of pairs (i, ℓ), $1 \le i \le V_\ell(\lambda)$, $\ell \in \mathcal{I}_0$, and the adjacency $\mathcal{E}_\lambda$ is given by $\{(i, \ell), (j, \ell')\} \in \mathcal{E}_\lambda$ if and only if the distance between $g_\ell(Q_{i,\ell}) \cap \mathcal{K}$ and $g_{\ell'}(Q_{j,\ell'}) \cap \mathcal{K}$ is at most $2\alpha\lambda^{-1/m}\rho_\lambda$. With $S_{i,\ell}$ defined similarly to S_i in [40], the variables $(S_{i,\ell}, (i, \ell) \in \mathcal{V}_\lambda)$ have G_λ as a dependency graph.

Next we show that the degrees of the graphs G_λ to be bounded by a constant, uniformly in λ. By Lemma 4.1, there exists a constant K such that for all (large enough) λ,

$$\sup_{\ell \in \mathcal{I}_0} \sup_{y,z \in B_{\delta_\ell}(y_\ell)} \frac{\|g_\ell^{-1}(y) - g_\ell^{-1}(z)\|}{\|y - z\|} \le K \tag{6.19}$$

and also

$$\sup_{\ell \in \mathcal{I}_0} \sup_{1 \le i \le V_\ell(\lambda)} \frac{\operatorname{diam}(g_\ell(Q_{i,\ell}))}{\lambda^{-1/m}\rho_\lambda} \le K. \tag{6.20}$$

If $\{(i, \ell), (j, \ell)\} \in \mathcal{E}$, then $\operatorname{dist}(Q_{i,\ell}, Q_{j,\ell}) \le 2\alpha K \lambda^{-1/m}\rho_\lambda$, and for any i the number of such j is bounded by a constant. Here, for subsets E and F of $\mathbb{R}^d$ we put $\operatorname{dist}(E, F) := \inf\{\|x - y\| : x \in E, y \in F\}$.

Now suppose $\ell \neq \ell'$, and fix $i \le V_\ell(\lambda)$. Set $B' := B_{\delta_{\ell'}}(y'_\ell)$. Suppose $\operatorname{dist}(B', g_\ell(Q_{i,\ell}) \cap \mathcal{K}) > (3\alpha + K)\lambda^{-1/m}\rho_\lambda$. Then for each $j \le V_{\ell'}(\lambda)$, since $g_{\ell'}(Q_{j,\ell'}) \cap B' \neq \varnothing$, using (6.20) we have $\operatorname{dist}(g_\ell(Q_{i,\ell}), g_{\ell'}(Q_{j,\ell'}) \ge 3\alpha\lambda^{-1/m}\rho_\lambda$, so there are no (j, ℓ') adjacent to (i, ℓ).

Suppose instead that $\mathrm{dist}(B', g_\ell(Q_{i,\ell}) \cap \mathcal{K}) \le (3\alpha + K)\lambda^{-1/m}\rho_\lambda$. Choose $w \in B'$ such that $\mathrm{dist}(w, g_\ell(Q_{i,\ell}) \cap \mathcal{K}) \le (3\alpha + K)\lambda^{-1/m}\rho_\lambda$. Suppose j is such that $\{(i,\ell),(j,\ell')\} \in \mathcal{E}$. Then by the triangle inequality and (6.20),

$$\begin{aligned}\mathrm{dist}\big(g_{\ell'}(Q_{j,\ell'}), w\big) &\le \mathrm{dist}\big(g_{\ell'}(Q_{j,\ell'}), g_\ell(Q_{i,\ell})\big)\\ &\qquad + \mathrm{diam}\big(g_\ell(Q_{i,\ell})\big) + (3\alpha + K)\lambda^{-1/m}\rho_\lambda\\ &\le (5\alpha + 2K)\lambda^{-1/m}\rho_\lambda\end{aligned}$$

so by (6.19), $\mathrm{dist}(Q_{j,\ell'}, g_\ell^{-1}(w)) \le K(5\alpha + 2K)\lambda^{-1/m}\rho_\lambda$. Hence the number of such j is bounded by a constant.

Thus the graphs G_λ have degrees bounded uniformly by a constant independent of λ, and we can follow the argument in [40] to complete the proof of (3.19).

Finally, if the condition that (3.6) holds for some $p > 3$ is weakened to (3.6) holding for some $p > 2$, then we may similarly adapt the proof of Theorem 2.3 of [40] and show that when $\sigma^2(\xi,\kappa) > 0$, the left-hand side of (3.19) goes to zero, albeit at a slower rate. That is, in this case (3.18) holds. □

PROOF OF THEOREM 3.2. We now prove the variance asymptotics (3.12). Recall the definition of $V^\xi(y,a)$ at (3.8), and note that the definition (3.10) gives

$$\sigma^2(\xi,\kappa) := \int_{\mathcal{M}} V^\xi\big(y,\kappa(y)\big)\kappa(y)\,dy - \left(\int_{\mathcal{M}} \delta^\xi\big(y,\kappa(y)\big)\kappa(y)\,dy\right)^2. \tag{6.21}$$

The idea here is to follow the de-Poissonization argument in Section 5 of [35] (with $f \equiv 1$). To ease notation we write ξ_λ for $\xi_{\lambda,k,\rho}$ and also $\Delta^x\xi_\lambda(y,\mathcal{Y})$ for $\xi_\lambda(y,\mathcal{Y}^x) - \xi_\lambda(y,\mathcal{Y})$. First we seek an analog of Lemma 5.1 of [35]. Set

$$\begin{aligned}\gamma_1 &:= \int_{\mathcal{M}} \mathbb{E}\xi(\mathbf{0},\mathcal{H}_{\kappa(y)})\kappa(y)\,dy;\\ \gamma_2 &:= \int_{\mathcal{M}} \kappa(y)^2\,dy \int_{T_y\mathcal{M}} dv\,\mathbb{E}\big[\Delta^v\xi(\mathbf{0},\mathcal{H}'_{y,\kappa(y)})\big].\end{aligned}$$

We can show by a similar argument to that used already in the proof of Theorem 3.1 that the analog of equation (5.6) of [35] holds; namely if $\ell \sim \lambda$ and $\tilde{m} \sim \lambda$ with $\ell < \tilde{m}$, then

$$\mathbb{E}\xi_\lambda(Y_{\ell+1},\mathcal{Y}_{\ell+1})\xi_\lambda(Y_{\tilde{m}+1},\mathcal{Y}_{\tilde{m}+1}) \to \gamma_1^2.$$

Taking the same partition of unity $\{\psi_i\}$ as in the proof of (3.17) above, analogously to (5.7) of [35] we have

$$\begin{aligned}&\ell\mathbb{E}\big[\xi_\lambda(Y_{\tilde{m}+1},\mathcal{Y}_{\tilde{m}})\Delta^{Y_{\ell+1}}\xi_\lambda(Y_1,\mathcal{Y}_\ell)\big]\\ &\quad = \ell\sum_{i\in\mathcal{I}}\int_{\mathcal{M}}\kappa(y)\,dy\int_{g_i(U_i)}\psi_i(x)\kappa(x)\,dx\\ &\qquad \times\int_{\mathcal{M}}\kappa(z)\,dz\,\mathbb{E}\big[\xi_\lambda\big(y,\mathcal{Y}_{\tilde{m}-2}\cup\{x,z\}\big)\Delta^z\xi_\lambda(x,\mathcal{Y}_{\ell-1})\big].\end{aligned} \tag{6.22}$$

Using the binomial exponential stabilization and moment conditions, the contribution to (6.22) from $z \notin g_i(U_i)$ can be shown to vanish as $\lambda \to \infty$. By (5.3), the remaining contribution to (6.22) can be written, using the change of variable $u = g_i^{-1}(x)$ and $v = g_i^{-1}(z)$, as

$$\ell \sum_{i \in \mathcal{I}} \int_{\mathcal{M}} \kappa(y)\, dy \int_{U_i} \psi_i(g_i(u))\tilde{\kappa}_i(u)\, du \int_{U_i} \tilde{\kappa}_i(v)\, dv \times \mathbb{E}\xi_\lambda(y, \mathcal{Y}_{\tilde{m}-2} \cup \{g_i(u), g_i(v)\})\Delta^{g_i(v)}\xi_\lambda(g_i(u), \mathcal{Y}_{\ell-1}).$$

By the change of variables $w = \lambda^{1/m}(v-u)$, this equals

$$\begin{aligned}&\frac{\ell}{\lambda} \sum_{i \in \mathcal{I}} \int_{\mathcal{M}} \kappa(y)\, dy \int_{U_i} \psi_i(g_i(u))\tilde{\kappa}_i(u)\, du \int_{\lambda^{1/m}(U_i-u)} \tilde{\kappa}_i(u + \lambda^{-1/m}w)\, dw \\ &\quad\times \mathbb{E}[\xi_\lambda(y, \mathcal{Y}_{m-2} \cup \{g_i(u), g_i(u+\lambda^{-1/m}w)\})\Delta^{g_i(u+\lambda^{-1/m}w)} \\ &\qquad\times \xi_\lambda(g_i(u), \mathcal{Y}_{\ell-1})],\end{aligned} \tag{6.23}$$

and by the analog of Lemma 3.7 of [35] [see also (5.19) and (5.20) of the present paper], along with the moments conditions and the binomial exponential stabilization to provide a dominating function, as $\lambda \to \infty$ with $\ell \sim \lambda$ and $\tilde{m} \sim \lambda$ and $\ell < \tilde{m}$, expression (6.23), and hence expression (6.22), tend to the expression

$$\begin{aligned}&\sum_{i \in \mathcal{I}} \int_{\mathcal{M}} \kappa(y)\, dy \int_{U_i} \psi_i(g_i(u))\tilde{\kappa}_i(u)^2\, du\, \mathbb{E}\xi(\mathbf{0}, \mathcal{H}_{\kappa(y)}) \\ &\quad\times \int_{\mathbb{R}^m} dw\, \mathbb{E}\Delta^{g_i'(u)(w)}\xi(\mathbf{0}, \mathcal{H}'_{g_i(u),\kappa(g_i(u))}) \\ &\quad= \gamma_1 \sum_{i \in \mathcal{I}} \int_{U_i} \psi_i(g_i(u))\tilde{\kappa}_i(u)^2\, du \\ &\qquad\times \int_{\mathcal{T}_{g_i(u)}\mathcal{M}} \mathbb{E}\Delta^v\xi(\mathbf{0}, \mathcal{H}'_{g_i(u),\kappa(g_i(u))})(D_{g_i}(u))^{-1}\, dv \\ &\quad= \gamma_1\gamma_2,\end{aligned} \tag{6.24}$$

which is analogous to (5.13) of [35]. Similarly, $\ell\mathbb{E}\xi_\lambda(Y_{\ell+1}, \mathcal{Y}_\ell)\Delta^{Y_{\tilde{m}+1}}\xi_\lambda(Y_1, \mathcal{Y}_{\tilde{m}})$ converges to $\gamma_1\gamma_2$, analogously to equation (5.16) in [35]. Moreover, as in (5.17) of [35], $\mathbb{E}\Delta^{Y_{\ell+1}}\xi(Y_1, \mathcal{Y}_\ell)\Delta^{Y_{\tilde{m}+1}}\xi(Y_2, \mathcal{Y}_{\tilde{m}})$ is here equal to

$$\begin{aligned}&\lambda^{-2} \sum_{i \in \mathcal{I}_0} \sum_{j \in \mathcal{I}_0} \int_{U_i} \int_{U_j} \psi_i(g_i(x))\tilde{\kappa}_i(x)\, dx\, \psi_j(g_j(w))\tilde{\kappa}_j(w)\, dw \\ &\quad\times \int_{\lambda^{-1/d}(-x+U_i)} \tilde{\kappa}_i(x + \lambda^{-1/d}u)\lambda\, du \\ &\quad\times \int_{\lambda^{-1/d}(-w+U_i)} \tilde{\kappa}_j(w + \lambda^{-1/d}v)\lambda\, dv\end{aligned}$$

$$\times \mathbb{E}\Delta^{g_i(x+\lambda^{-1/d}u)}\xi_\lambda\big(g_i(x), \mathcal{Y}_{\ell-2}^{g_j(w)}\big)\Delta^{g_j(w+\lambda^{-1/d}v)}\xi_\lambda\big(g_j(w), \mathcal{Y}_{\tilde{m}-3}^{g_i(x)} \cup \{g_i(x+\lambda^{-1/d}u)\}\big) + o(1)$$

and by the analog of Lemma 3.7 of [35] [see also (5.19) of the present paper], along with the moments conditions and the binomial exponential stabilization to provide a dominating function, as $\lambda \to \infty$ with $\ell \sim \lambda$ and $\tilde{m} \sim \lambda$, this tends to the expression

$$\sum_{i\in\mathcal{I}_0}\sum_{j\in\mathcal{I}_0}\int_{U_i}\int_{U_j}\psi_i(g_i(x))\tilde{\kappa}_i(x)^2\,dx\,\psi_j(g_j(w))\tilde{\kappa}_j(w)^2\,dw$$

$$\times \int_{\mathbb{R}^m}\mathbb{E}\Delta^{g_i'(x)(u)}\xi(\mathbf{0}, \mathcal{H}_{\kappa(g_i(x))})\,du \times \int_{\mathbb{R}^m}\mathbb{E}\Delta^{g_j'(w)(v)}\xi\big(\mathbf{0}, \mathcal{H}'_{g_i(w),\kappa(g_i(w))}\big)\,dv$$

$$= \left(\sum_{i\in\mathcal{I}_0}\int_{g_i(U_i)}\psi_i(y)\kappa(y)^2\,dy\int_{T_y\mathcal{M}}\mathbb{E}\Delta^z\xi\big(\mathbf{0}, \mathcal{H}'_{y,\kappa(y)}\big)\,dz\right)^2$$

$$= \left(\int_{\mathcal{M}}\kappa(y)^2\,dy\int_{T_y\mathcal{M}}\mathbb{E}\Delta^z\xi\big(\mathbf{0}, \mathcal{H}'_{y,\kappa(y)}\big)\,dz\right)^2 = \gamma_2^2.$$

Then by arguments similar to those in the proof of Lemma 5.1 of [35], we have a similar result here with the squared integral in equation (5.2) of [35] replaced here by

$$(\gamma_1+\gamma_2)^2 = \left(\int_{\mathcal{M}}\delta^\xi\big(y, \kappa(y)\big)\,dy\right)^2,$$

and so we obtain $\lim_{n\to\infty} n^{-1}\operatorname{Var}[H_n^\xi] = \sigma^2(\xi,\kappa)$. Given this, by using Theorem 3.3, following verbatim the proof of Theorem 2.3 of [35], using the case $p = 2$ of (6.5) (in place of Lemma 5.2 of [35]), we can obtain the desired (3.11) and (3.12). □

REMARK. The method of proof given in this section shows that our general results can be extended to a broader class of ξ, potentially providing the limit theory for some of the statistics mentioned in the penultimate paragraph of Section 1. This goes as follows. Consider the class $\mathcal{F}_c$ of functionals ξ which are continuous in the sense of Definition 5.1 and which stabilize over homogeneous Poisson point processes in the sense that (6.1) holds when $\mathcal{Y}$ is a homogeneous Poisson point process and $R_\lambda(y, \mathcal{Y}) < \infty$ a.s. Then the above proof of Theorem 3.1 shows that the conclusion of Theorem 3.1 holds when $\Xi(k,r)$ is replaced by $\mathcal{F}_c$. Likewise, if $\mathcal{F}_c(\kappa) \subset \mathcal{F}_c$ consists of $\xi \in \mathcal{F}_c$ which are exponentially stabilizing and binomially exponentially stabilizing for κ, then Theorems 3.2 and 3.3 hold when $\Xi(k,r)$ is replaced by $\mathcal{F}_c(\kappa)$.

7. Proofs of Theorems 2.1–2.5.

7.1. *Proof of Theorem* 2.1. We require some additional lemmas. Observe that the functional $\zeta_k(y, \mathcal{Y})$, defined at (2.9), is scale invariant, namely satisfies $\zeta_k(y, \mathcal{Y}) = \zeta_k(ay, a\mathcal{Y})$ for all $a > 0$; cf. remark (ii) in Section 3. Recall that for all $a \in (0, \infty)$, $\mathcal{H}_a$ is a homogeneous Poisson point process of intensity a in $\mathbb{R}^m$.

LEMMA 7.1. *Let $k > 3$. For all $a > 0$ we have $\mathbb{E}\zeta_k(\mathbf{0}, \mathcal{H}_a) = m$ and* $\operatorname{Var}[\zeta_k(\mathbf{0}, \mathcal{H}_a)] = m^2/(k-3)$.

PROOF. If $N_j := N_j(\mathbf{0}, \mathcal{H}_a)$, then conditionally on N_k, the random variables $(N_j/N_k)^m$, $1 \le j \le k-1$, are distributed as the order statistics of a sample of size $k-1$ from the uniform distribution on $[0, 1]$, and therefore $-m \log(N_j/N_k)$, $1 \le j \le k-1$, are distributed as the order statistics of a sample of size $k-1$ from a standard exponential distribution. Thus the sum $U := m \sum_{j=1}^{k-1} \log(N_k/N_j)$ has a Gamma$(k-1, 1)$ distribution and $\zeta_k(\mathbf{0}, \mathcal{H}_a) = (k-2)mU^{-1}$. Since $\mathbb{E}[U^{-1}] = (k-2)^{-1}$ and since $\mathbb{E}[U^{-2}] = ((k-2)(k-3))^{-1}$, we have $\mathbb{E}\zeta_k(\mathbf{0}, \mathcal{H}_a) = m$ and $\mathbb{E}\zeta_k^2(\mathbf{0}, \mathcal{H}_a) = m^2(k-2)/(k-3)$, which gives the result. $\square$

For all $k \in \mathbb{Z}^+, \lambda > 0, \rho > 0$ we put $\zeta_{\lambda,k,\rho}(y, \mathcal{Y}) := \zeta_k(\lambda^{1/m} y, \lambda^{1/m}\mathcal{Y}) \times \mathbf{1}\{N_k(y, \mathcal{Y}) \le \rho\}$; by scale invariance $\zeta_{\lambda,k,\rho}(y, \mathcal{Y}) = \zeta_{\lambda,k,\rho}(y, \mathcal{Y})$. The next lemma shows that $\zeta_{\lambda,k,\rho}$ satisfies the moment bounds in the hypotheses of Theorems 3.1–3.2. Recall that for $i \in \mathbb{Z}^+$, $\mathcal{S}_i$ denotes all subsets of $\mathcal{K}(\kappa)$ of cardinality at most i.

LEMMA 7.2. *Let $\mathcal{M} \in \mathbb{M}$ and $\kappa \in \mathbb{P}_c(\mathcal{M})$. With ρ_1 as in Lemma 4.3, we have for $\rho \in (0, \rho_1)$, $p > 1$, $k \in \mathbb{Z}^+$ and $i \in \mathbb{Z}^+$ that*

$$\sup_{y \in \mathcal{K}, n \in \mathbb{N}, \mathcal{A} \in \mathcal{S}_i} \mathbb{E}\big[\zeta_{k,\rho}(y, \mathcal{Y}_n \cup \mathcal{A})^p\big] < \infty \qquad \text{if } k > p+1+i; \tag{7.1}$$

$$\sup_{y \in \mathcal{K}, \lambda \ge 1, \mathcal{A} \in \mathcal{S}_i} \mathbb{E}\big[\zeta_{k,\rho}(y, \mathcal{P}_\lambda \cup \mathcal{A})^p\big] < \infty \qquad \text{if } k > p+1+i. \tag{7.2}$$

PROOF. We first show (7.1). Fix $p > 1$. Let $t_0 \in (2^p, \infty)$ be such that $\exp(t^{-1/p}) < 1 + 2t^{-1/p}$ for all $t \in (t_0, \infty)$. Let $x \in \mathcal{K}, n \in \mathbb{N}, \mathcal{A} \in \mathcal{S}_i$. For $j \in \{1, \dots, k-1\}$, let $N_j := N_j(y; \mathcal{Y}_n \cup \mathcal{A})$. Then $\zeta_{k,\rho}(y, \mathcal{Y}_n \cup \mathcal{A}) = (k-2)(\sum_{j=1}^{k-1} \log(N_k/N_j))^{-1}\mathbf{1}\{N_k < \rho\}$, and

$$\begin{aligned}
(k-2)^{-p}\mathbb{E}\xi_{k,\rho}(y, \mathcal{Y}_n \cup \mathcal{A})^p &= \mathbb{E}\left[\left(\sum_{j=1}^{k-1} \log \frac{N_k}{N_j}\right)^{-p} \mathbf{1}\{N_k < \rho\}\right] \\
&\le t_0 + \int_{t_0}^{\infty} P\left[\left(\sum_{j=1}^{k-1} \log \frac{N_k}{N_j}\right)^{-p} \mathbf{1}\{N_k < \rho\} > t\right] dt.
\end{aligned}$$

Let P_{N_k} denote the probability distribution of N_k. Conditioning on N_k we obtain

$$(k-2)^{-p}\mathbb{E}\xi_{k,\rho}(y,\mathcal{Y}_n\cup\mathcal{A})^p$$

$$\le t_0+\int_{t_0}^{\infty}\int_0^{\rho}P\left[\sum_{j=1}^{k-1}\log\frac{N_k}{N_j}<t^{-1/p}|N_k=u\right]dP_{N_k}(u)\,dt \tag{7.3}$$

$$\le t_0+\int_{t_0}^{\infty}\int_0^{\rho}P\left[\log\frac{N_k}{N_1}<t^{-1/p}|N_k=u\right]dP_{N_k}(u)\,dt,$$

and by choice of t_0, for all $t\in(t_0,\infty)$, we have

$$\begin{aligned}P\big[\log(N_k/N_1)<t^{-1/p}|N_k=u\big]&\le P\big[(N_k/N_1)<1+2t^{-1/p}|N_k=u\big]\\ &=P\big[N_1>N_k/\big(1+2t^{-1/p}\big)|N_k=u\big]\\ &\le P\big[N_1>N_k\big(1-2t^{-1/p}\big)|N_k=u\big].\end{aligned}\tag{7.4}$$

Given $N_k=u$, there are $k-1-\text{card}(\mathcal{A}\cap B_u)$ points of $\mathcal{Y}_n$ in the interior of $B_u(y)$, and to have $N_1>u(1-2t^{-1/p})$ it is necessary for these points to all lie in $A_{t,u}(y)$, where we set $A_{t,u}(y):=B_u(y)\setminus B_{u(1-2t^{-1/p})}(y)$. For $u\in(0,\rho]$ and $\rho<\rho_1$, Lemma 4.3 gives

$$\int_{A_{t,u}(y)\cap\mathcal{M}}dy\le C_1\big(u^m-\big(u\big(1-2t^{-1/p}\big)\big)^m\big)\le 2^mC_1mt^{-1/p}u^m,$$

and $\int_{B_u^{\mathcal{K}}(y_1)}dy\ge C_0^{-1}u^m$. Thus by the boundedness assumptions on κ, for all $u\in(0,\rho]$,

$$\begin{aligned}P\big[N_1>N_k\big(1-2t^{-1/p}\big)|N_k=u\big]&\le\left(\frac{\int_{A_{t,u}(y)\cap\mathcal{M}}\kappa(z)\,dz}{\int_{B_u(y)\cap\mathcal{M}}\kappa(z)\,dz}\right)^{k-1-i}\\ &\le C\big(t^{-1/p}\big)^{k-1-i}.\end{aligned}\tag{7.5}$$

Combining this with (7.3) and (7.4) yields

$$(k-2)^{-p}\mathbb{E}\zeta_{k,\rho}(y,\mathcal{Y}_n\cup\mathcal{A})^p\le t_0+\int_{t_0}^{\infty}C\big(t^{-1/p}\big)^{k-1-i}\,dt,$$

which is finite and independent of n if $k>p+1+i$. This gives us (7.1), and the proof of (7.2) is just the same. □

PROOF OF THEOREM 2.1. First we prove (2.12). Given $\rho>0$, we have

$$\begin{aligned}P\big[\hat{m}_{k,\rho}(\mathcal{Y}_n)\ne\hat{m}_k(\mathcal{Y}_n)\big]&\le P\left[\bigcup_{i\le n}\{N_k(Y_i,\mathcal{Y}_n)\ge\rho\}\right]\\ &\le nP\big[N_k(Y_n,\mathcal{Y}_n)\ge\rho\big].\end{aligned}\tag{7.6}$$

Now $N_k(Y_n, \mathcal{Y}_n) \geq \rho$ if and only if there are at most $k-1$ points of $\mathcal{Y}_{n-1}$ in $B_\rho(Y_n)$. By Lemma 4.3 there exists $p > 0$ such that $P[Y_1 \in B_\rho(y)] \geq p$ for all $y \in \mathcal{K}$. Hence by a Chernoff-type large deviations estimate for the binomial distribution (see, e.g., Lemma 1.1 of [34]), provided $(n-1)p \geq 2(k-1)$ we have for some constant C independent of n that

$$nP[N_k(Y_n, \mathcal{Y}_n) \geq \rho] \leq n \exp(-C^{-1}(n-1)p),$$

which is summable in n, so that assertion in (2.12) follows by (7.6) and the Borel–Cantelli lemma.

We now prove the remaining assertions of Theorem 2.1 with ρ_1 as in Lemma 4.3. It suffices to show that $\zeta_{k,\rho}$, $k \in \mathbb{N}$, as defined at (2.10), satisfy the conditions of Theorems 3.1–3.2. Since ζ_k is a continuous function of the k nearest neighbor distances, it belongs to the class $\Xi(k, r)$. Lemma 7.2 establishes that $\zeta_{k,\rho}$ satisfies the moment condition (3.4) when $k > p + 1$. It follows by Theorem 3.1 with $q = 2$ there, and Lemma 7.1 that $\lim_{n\to\infty} \hat{m}_{k,\rho}(\mathcal{Y}_n) = m$ in L^2. Combined with (2.12), this implies the stated convergence in probability (2.13) for $\hat{m}_k(\mathcal{Y}_n)$.

If also $k \geq 10$, then by taking $i = 3$ in (7.1), $\xi_{k,\rho}$ satisfies (3.5) for $p = 5.5$. Hence by Theorem 3.1 we have $\lim_{n\to\infty} \hat{m}_{k,\rho}(\mathcal{Y}_n) = m$ almost surely. Combined with (2.12) this gives (2.13) with a.s. convergence.

To obtain the limits (2.14) and (2.15), observe that for $k \geq 7$, the moment bound (7.1) (with $i = 3$) is satisfied for $p = 2.5$, so taking $\xi \equiv \zeta_k$ we have conditions (3.5) and (3.6) with $p = 2.5$. Therefore we can apply Theorem 3.2 for this ξ, and since it is scale invariant, by (3.14) and Lemma 7.1 the limiting variance $\sigma^2(\zeta_k, \kappa)$ appearing in Theorem 3.2 equals the right-hand side of (2.14), so (2.14) and (2.15) follow. Finally, Lemma 7.3 completes the proof of Theorem 2.1. □

LEMMA 7.3. *With σ_k^2 given by (2.14), it is the case that $\sigma_k^2 > 0$.*

PROOF. Since they are given by (2.6) and (2.7), the expressions V^{ζ_k} and δ^{ζ_k} in (2.14) depend only on m and not on d or κ. Hence by using the part of Theorem 2.1 that we have already proved, in the special case where $d = m$ and κ is a uniform distribution on the unit cube in $\mathbb{R}^m$, we have $\sigma_k^2 = \lim_{n\to\infty} n \operatorname{Var}[\hat{m}_{k,\rho}(\mathcal{X}_n)]$, where $\mathcal{X}_n$ is a point process consisting of n independent uniformly random vectors in a unit cube in $\mathbb{R}^m$. By the definition (2.8) and scale invariance of ζ_k, we have

$$\sigma_k^2 = \lim_{n\to\infty} n^{-1} \operatorname{Var} \sum_{X \in \mathcal{X}_n} \zeta_k(n^{1/m} X, n^{1/m} \mathcal{X}_n),$$

and so it is enough to show that

$$\liminf_{n\to\infty} n^{-1} \operatorname{Var} \sum_{X \in \mathcal{X}_n} \zeta_k(n^{1/m} X, n^{1/m} \mathcal{X}_n) > 0.$$

This can be done, either by the method of Penrose and Yukich (Section 5 of [38]), or by the method of Avram and Bertsimas (Proposition 5 of [2], which can be adapted to binomial input). The particular functional under consideration here is not considered in those references, but the general approaches are well known so we omit further details. □

7.2. *Proof of Theorems 2.2–2.5*. Recall that Y_i are i.i.d. with density κ and that $N_1(y, \mathcal{Y})$ is the Euclidean distance between y and its nearest neighbor in $\mathcal{Y}$, or $+\infty$ if $\mathcal{Y} \setminus y$ is empty. To help deal with the possibility that a Poisson process $\mathcal{P}_\lambda$ has no elements, define

$$\tilde{N}_1(y, \mathcal{Y}) := \begin{cases} N_1(y, \mathcal{Y}), & \mathcal{Y} \setminus y \neq \varnothing, \\ 0, & \mathcal{Y} \setminus y = \varnothing. \end{cases}$$

The proofs of Theorems 2.2–2.4 depend in part on the following lemmas.

LEMMA 7.4. *Suppose $\kappa \in \mathbb{P}_c(\mathcal{M})$. There is a constant C such that for all $n \geq 3$, $\lambda \geq 1$, $n \geq 4$ and $\ell \in [n/2, 3n/2]$, $y \in \mathcal{K}$, $\mathcal{A} \in \mathcal{S}_3$ and $t \in (0, \infty)$ we have*

$$P\big[N_1\big(n^{1/m} y, n^{1/m}(\mathcal{Y}_\ell \cup \mathcal{A})\big) > t\big] \leq \exp\big(-C^{-1} t^m\big); \tag{7.7}$$

$$P\big[\tilde{N}_1\big(\lambda^{1/m} y, \lambda^{1/m} \mathcal{P}_\lambda\big) > t\big] \leq \exp\big(-C^{-1} t^m\big). \tag{7.8}$$

PROOF. These bounds can be deduced from the proof of Lemma 6.1, but we prefer to argue directly, as follows. Letting $\alpha(t, y, n)$ be the $n\kappa$ measure of $B^{\mathcal{M}}_{tn^{-1/m}}(y)$, we have

$$\begin{aligned} P\big[N_1\big(n^{1/m} y, n^{1/m}(\mathcal{Y}_\ell \cup \mathcal{A})\big) > t\big] &= \big(1 - \alpha(t, y, n)\big)^\ell \\ &\leq \exp\big(-\ell \alpha(t, y, n)/n\big). \end{aligned} \tag{7.9}$$

By Lemma 4.3, there is a constant C such that uniformly in $n \geq 3$, $y \in \mathcal{K}$, and $t \in (0, \Delta)$, we have $\alpha(t, y, n) \geq C^{-1} t^m$, which gives (7.7) for $t < \Delta$, and clearly (7.7) holds for $t \geq \Delta$. The second bound (7.8) is proved similarly. □

LEMMA 7.5. *If κ is bounded and $\delta \in (0, m)$, then* $\sup_n \mathbb{E}[N_1(n^{1/m} Y_1, n^{1/m} \mathcal{Y}_n)^{-\delta}] < \infty$.

PROOF. Set $F_{n,y}(t) := P[N_1(n^{1/m} y, n^{1/m} \mathcal{Y}_{n-1}) \leq t]$. Then

$$\mathbb{E}\big[N_1\big(n^{1/m} Y_1, n^{1/m} \mathcal{Y}_n\big)^{-\delta}\big] = \int_{\mathcal{M}} \int_0^\infty t^{-\delta}\, dF_{n,y}(t) \kappa(y)\, dy.$$

As in (7.9), we have for n large and all $y \in \mathcal{K}$, $t \in (0, 1)$ that

$$P\big[N_1\big(n^{1/m} y, n^{1/m} \mathcal{Y}_{n-1}\big) > t\big] = \big(1 - \alpha(t, y, n)\big)^{n-1} \geq \exp\big(-2(n-1)\alpha(t, y, n)\big),$$

where $\alpha(t, y, n) \leq Ct^m/n$ by Lemma 4.3. Thus for $t \in (0, 1)$ we obtain that

$$F_{n,y}(t) = 1 - P[N_1(n^{1/m} y, n^{1/m} \mathcal{Y}_{n-1}) > t] \leq 1 - \exp(-3Ct^m) \leq 3Ct^m. \tag{7.10}$$

Hence by Fubini's theorem we have for all $y \in \mathcal{K}$ that

$$\int_0^\infty t^{-\delta} \, dF_{n,y}(t) = \delta \int_0^\infty t^{-\delta-1} F_{n,y}(t) \, dt \leq \delta C \int_0^1 t^{m-\delta-1} \, dt + \delta \int_1^\infty t^{-\delta-1} \, dt,$$

which is finite if $\delta \in (0, m)$. Integrating over $y \in \mathcal{K}$ gives the result. □

PROOF OF THEOREM 2.2. Let $q = 1$ or $q = 2$. If $\alpha > 0$ and $\kappa \in \mathbb{P}_c(\mathcal{M})$, then Lemma 7.4 shows that $\sup_n \mathbb{E}[N_1(n^{1/m} Y_1, n^{1/m} \mathcal{Y}_n)^{\alpha p}] < \infty$ for all $p > 0$. If $\alpha \in (-m/q, 0)$ and κ is bounded, and if then $p > q$ is chosen so that $-m < \alpha p < 0$, then Lemma 7.5 gives $\mathbb{E}[N_1(n^{1/m} Y_1, n^{1/m} \mathcal{Y}_n)^{\alpha p}] < \infty$.

Thus, in all these cases the moment condition (3.4) holds for $\xi = N_1^\alpha$ and some $p > q$. Since N_1^α belongs to the class $\Xi(k, r)$, limit (2.18) (with L^q convergence) thus follows from Theorem 3.1, the fact that N_1^α is homogeneous of order α [see (3.13)], and the identity $\mathbb{E}[N_1^\alpha(\mathbf{0}, \mathcal{H})] = m\omega_m \int_0^\infty u^{\alpha+m-1} \exp(-\omega_m u^m) \, du$ which yields

$$\mathbb{E}[N_1^\alpha(\mathbf{0}, \mathcal{H})] = \pi^{-\alpha/2} \left(\Gamma\left(1 + \frac{m}{2}\right) \right)^{\alpha/m} \Gamma\left(1 + \frac{\alpha}{m}\right); \tag{7.11}$$

see also (15) of Wade [44]. Moreover, in the first case [$\alpha > 0$ and $\kappa \in \mathbb{P}_c(\mathcal{M})$], Lemma 7.4 shows that the moment condition (3.5) holds for $p > 6$, and so by Theorem 3.1 we obtain the a.s. convergence at (2.18). This completes the proof of Theorem 2.2. □

PROOF OF THEOREM 2.3. First assume $\alpha > 0$. To prove variance asymptotics and (2.20) it suffices to show that the functional $\tilde{N}_1^\alpha$ satisfies the conditions of Theorem 3.2. Since we assume $\kappa \in \mathbb{P}_c(\mathcal{M})$, Lemma 7.4 is applicable, showing that the moment conditions (3.5) and (3.6) hold when $\xi \equiv \tilde{N}_1^\alpha$. The conditions of Theorem 3.2 are all satisfied, so that result gives variance asymptotics (3.11) and the central limit theorem (3.12) for $\xi \equiv \tilde{N}_1^\alpha$. Also, this choice of ξ is homogeneous of order α, so that (3.14) is applicable with $\beta = \alpha$, and applying this identity to (3.11) and (3.12) gives the results (2.19) and (2.20) for $\alpha > 0$, subject to showing that $\sigma^2 > 0$ in (2.19).

Now assume $\alpha \in (-m/2, 0)$. We cannot directly apply Theorem 3.2 because the moment bound (3.6) fails since if $\mathcal{A} = \{z\}$, the distance between y and z can be made arbitrarily small, and thus $\tilde{N}_{1,\lambda}^\alpha(y, \mathcal{P}_\lambda \cup \{z\})$ can be made arbitrarily large. Instead we use a truncation argument and follow the approach of [3]. Put $\phi(x) = x^\alpha$ for $x > 0$ with $\phi(0) = 0$. Given $\varepsilon > 0$ define the functions

$$\phi^{(\varepsilon)}(x) := \begin{cases} \phi(x), & \text{if } x \geq \varepsilon, \\ 0, & \text{otherwise} \end{cases}$$

and $\phi_{(\varepsilon)}(x) := \phi(x) - \phi^{(\varepsilon)}(x)$. Let $\tilde{N}_1^{\alpha,\varepsilon}(y, \mathcal{Y}) := \phi^{(\varepsilon)}(\tilde{N}_1(y, \mathcal{Y}))$.

Let $\varepsilon > 0$. Then the moment bounds (3.6) and (3.5) (with $\rho = \infty$) hold for $\xi \equiv \tilde{N}_1^{\alpha,\varepsilon}$ for $p = 3$ (say), because $\phi^{(\varepsilon)}(y, \mathcal{Y}) \le \varepsilon^\alpha$ for all $y, \mathcal{Y}$. Thus we may apply Theorem 3.2 to deduce that as $n \to \infty$, $n^{-1}\operatorname{Var}[H_n^{\tilde{N}_1^{\alpha,\varepsilon}}(\mathcal{Y}_n)]$ converges to $\sigma^2(\tilde{N}_1^{\alpha,\varepsilon}, \kappa)$ and

$$n^{-1/2}\big(H_n^{\tilde{N}_1^{\alpha,\varepsilon}}(\mathcal{Y}_n) - \mathbb{E}H_n^{\tilde{N}_1^{\alpha,\varepsilon}}(\mathcal{Y}_n)\big) \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, \sigma^2\big(\tilde{N}_1^{\alpha,\varepsilon}, \kappa\big)\big). \tag{7.12}$$

To complete the proof, we adapt arguments in [4], given in detail in [3], as follows. To make the link with [3], for $1 \le i \le m$ we set $N_{i,m} := N_1(Y_i, \mathcal{Y}_m)$, giving the identification $\tilde{N}_{1,n}^\alpha(Y_i, \mathcal{Y}_n) = \phi(nN_{i,n}^m)$.

The equivalent of Lemma 5.1 of [3] is given by Lemma 7.6 below. Let us assume this for now. Lemma 5.2 of [3] remains valid here modulo some small notational modifications; the $\mathcal{H}$ featuring in that result should be considered now as a homogeneous Poisson process in $\mathbb{R}^m$. Likewise, Lemma 5.3 of [3] carries over with straightforward modifications. By the proof of Lemma 5.5 of [3], we can show that

$$\lim_{\varepsilon \to 0} \sigma^2\big(\tilde{N}_1^{\alpha,\varepsilon}, \kappa\big) = \sigma^2\big(\tilde{N}_1^\alpha, \kappa\big). \tag{7.13}$$

With (7.13) established, the variance asymptotics (2.19) and central limit theorem (2.20) follow along the lines of the proof of Theorem 2.1 of [3].

Now assume either $\alpha > 0$ or $\alpha \in (-m/2, 0)$. To show positivity of the limiting variance in (2.19), we would like to follow the approach used to show positivity of σ_k^2 in Lemma 7.3. In this case we do not have scale invariance. However, since κ is a probability density function, for any $\xi \in \Xi(k, r)$ we have from (3.10) and Jensen's inequality that

$$\sigma^2(\xi, \kappa) \ge \int_{\mathcal{M}} \{V^\xi(\kappa(y)) - \delta^\xi(\kappa(y))^2\}\kappa(y)\,dy, \tag{7.14}$$

so it suffices to show that when we take $\xi \equiv N_1^\alpha$, the integrand in the braces in the right-hand side of (7.14) is strictly positive. By what we have already proved,

$$V^{N_1^\alpha}(a) - \big(\delta^{N_1^\alpha}(a)\big)^2 = \lim_{n\to\infty} n^{-1}\operatorname{Var} R^\alpha\big(n^{1/m}\mathcal{X}_n\big),$$

where now $\mathcal{X}_n$ is a point process consisting of n independent, uniformly distributed random vectors in a cube of volume a^{-1} in $\mathbb{R}^m$. This limit can be shown to be strictly positive by using the methods of [2] or [38].

Finally we prove (2.21). By (2.7) and (7.11) we have

$$\delta^{N_1^\alpha} = \pi^{-\alpha/2}\left(\Gamma\left(1 + \frac{m}{2}\right)\right)^{1/m}\Gamma\left(1 + \frac{\alpha}{m}\right) + \int_{\mathbb{R}^m} f(x)\,dx, \tag{7.15}$$

where for $x \in \mathbb{R}^m$ we here set $f(x) := \mathbb{E}N_1^\alpha(\mathbf{0}, \mathcal{H}^x) - \mathbb{E}N_1^\alpha(\mathbf{0}, \mathcal{H})$, so that

$$f(x) = \int_{\mathbb{R}^m \setminus B_{\|x\|}(\mathbf{0})} \big(\|x\|^\alpha - \|y\|^\alpha\big)\exp\big(-\omega_m\|y\|^m\big)\,dy$$

and hence by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^m} f(x)\,dx &= \int_{\mathbb{R}^m} dy \exp(-\omega_m \|y\|^m) \int_{B_{\|y\|}(\mathbf{0})} (\|x\|^\alpha - \|y\|^\alpha)\,dx \\ &= \left(\frac{-\alpha\omega_m^2 m}{\alpha+m}\right) \int_0^\infty r^{\alpha+2m-1} \exp(-\omega_m r^m)\,dr \\ &= -\left(\frac{\alpha\omega_m^{-\alpha/m}}{\alpha+m}\right)\Gamma(2+\alpha/m) = -(\alpha/m)\omega_m^{-\alpha/m}\Gamma(1+\alpha/m).\end{aligned}$$

Substituting back into (7.15) gives us (2.21), completing the proof of Theorem 2.3. □

We state the equivalent of Lemma 5.1 of [3]. Recall $N_{i,m} := N_1(Y_i, \mathcal{Y}_m)$.

LEMMA 7.6. *Suppose $\mathcal{K} \in \mathbb{P}_c(\mathcal{M})$. Let $\phi_{(\varepsilon)}(\cdot)$ be as in the proof of Theorem 2.3. Given $\delta > 0$ there exists $\varepsilon_0 > 0$ and $n_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $n \geq n_0$ we have* $\operatorname{Var} \sum_{i=1}^n \phi_{(\varepsilon)}(nN_{i,n}^m) \leq \delta$.

PROOF. We may proceed as in [3] with only minor modifications as far as (5.4) of [3], but more effort is required to adapt the proof of (5.19) of [3]. To do this, set $Y = Y_{n+1}$ (corresponding to X in [3]), and for $1 \leq j \leq J$ define the open cones W_j with vertex Y, just as in [3] (these cones cover $\mathbb{R}^d$, not just the tangent hyperplane at Y).

Let $I_{j,n} := \mathbf{1}\{\mathcal{Y}_n \cap W_j(Y) \neq \varnothing\}$. If $I_{j,n} = 1$, then set $Z_{j,n}$ to be the nearest neighbor of Y in $\mathcal{Y}_n \cap W_j(Y)$; otherwise set $Z_{j,n} := Y$. Set $R_{j,n} := \|Z_{j,n} - Y\|$. Noting that $|\phi_{(\varepsilon)}(\cdot)|$ is nonincreasing on $(0, \varepsilon)$, we have as in (5.5) of [3] that

$$\left|\sum_{i=1}^n \left(\phi_{(\varepsilon)}(nN_{i,n+1}^m) - \phi_{(\varepsilon)}(nN_{i,n}^m)\right)\right| \leq 2\sum_{j=1}^J I_{j,n}|\phi_{(\varepsilon)}(nR_{j,n}^m)|. \tag{7.16}$$

Using the last inequality in (4.12) of the current paper, we have, similarly to (5.6) of [3], that there is a constant K_6 such that $P[R_{j,n}^m > r] \geq (1 - K_6 r)^n$, and therefore there is a constant K_7 such that for $0 < t \leq 1$ and large enough n we have

$$P[nR_{j,n}^m > t] \geq (1 - K_6 t/n)^n \geq \exp(-K_7 t).$$

We may follow the argument given after (5.7) of [3] to obtain the analog to (5.9) of [3] (with $N_{i,n}^m$ instead of the $D_{i,n,k}^g$ of [3]). We can then continue as in the proof of the case $\mathcal{K} = \mathbb{R}^d$ of Lemma 5.1 of [3], to complete the proof. □

Before proving Theorem 2.4 we need some preliminary results. Recall that $\psi(y, \mathcal{Y}) := \log(e^\gamma \omega_m N_1^m(y, \mathcal{Y}))$. If $\mathcal{Y} \setminus \{y\} = \varnothing$, let us set $\psi(y, \mathcal{Y}) := 0$. For all

$a \in (0, \infty)$ we claim that

$$\mathbb{E}\psi(\mathbf{0}, \mathcal{H}_a) = \gamma + a \int_0^\infty (\log u) e^{-au}\, du = -\log a. \tag{7.17}$$

The first equality of (7.17) follows since the probability that the volume of the nearest neighbor ball around $\mathbf{0}$ exceeds u is e^{-au}, and the second equality follows since $\int_0^\infty \log(w) e^{-w}\, dw = -\gamma$; see, for example, [22], page 107. Also, since $\int_0^\infty (\log w)^2 e^{-w}\, dw = \gamma^2 + \pi^2/6$, we have

$$\begin{aligned} \mathbb{E}\psi(\mathbf{0}, \mathcal{H}_a)^2 &= \int_0^\infty (\gamma + \log u)^2 a e^{-au}\, du \\ &= \gamma^2 + 2(-\gamma - \log a)\gamma + a \int_0^\infty (\log u)^2 e^{-au}\, du \qquad (7.18) \\ &= \pi^2/6 + (\log a)^2. \end{aligned}$$

Let V^ψ and $V^\psi(a)$ be given by setting $\xi \equiv \psi$ in (2.6) and (3.8), respectively.

LEMMA 7.7. *It is the case that* $\sigma^2(\psi, \kappa) = V^\psi - m^{-2} + \mathrm{Var}[\log(\kappa(Y_1))]$.

PROOF. Using (3.8), (7.17) and (7.18), and setting $v = a^{1/m} u$, we have that

$$\begin{aligned} &V^\psi(a) - (\log a)^2 - \pi^2/6 \\ &\quad = \int_{\mathbb{R}^m} \{\mathbb{E}\psi(\mathbf{0}, \mathcal{H}_a^{a^{-1/m} v}) \psi(a^{-1/m} v, \mathcal{H}_a^{\mathbf{0}}) - (\mathbb{E}\psi(\mathbf{0}, \mathcal{H}_a))^2\}\, dv \end{aligned}$$

and since $\mathcal{H}_a = a^{-1/m}\mathcal{H}$ in distribution, the last displayed expression is

$$= \int_{\mathbb{R}^m} \{\mathbb{E}\psi(\mathbf{0}, a^{-1/m}\mathcal{H}^v) \psi(a^{-1/m} v, a^{-1/m}\mathcal{H}^{\mathbf{0}}) - (\mathbb{E}\psi(\mathbf{0}, a^{-1/m}\mathcal{H}))^2\}\, dv.$$

By definition we always have $\psi(ty, t\mathcal{Y}) = \psi(y, \mathcal{Y}) + m \log t$, so the preceding display is

$$\begin{aligned} &= \int_{\mathbb{R}^m} \{\mathbb{E}[(\psi(\mathbf{0}, \mathcal{H}^v) - \log a)(\psi(v, \mathcal{H}^{\mathbf{0}}) - \log a)] - (\mathbb{E}\psi(\mathbf{0}, \mathcal{H}) - \log a)^2\}\, dv \\ &= V^\psi - \pi^2/6 - 2(\log a)\delta^\psi, \end{aligned}$$

so

$$V^\psi(a) = (\log a)^2 + V^\psi - 2\delta^\psi \log a. \tag{7.19}$$

Using (3.9) and (7.17), and setting $v = a^{1/m} u$, we have that

$$\begin{aligned} \delta^\psi(a) + \log a &= \int_{\mathbb{R}^m} \mathbb{E}[\psi(\mathbf{0}, \mathcal{H}_a^{a^{-1/m} v}) - \psi(\mathbf{0}, \mathcal{H}_a)]\, dv \\ &= \int_{\mathbb{R}^m} \mathbb{E}[\psi(\mathbf{0}, a^{-1/m}\mathcal{H}^v) - \psi(\mathbf{0}, a^{-1/m}\mathcal{H})]\, dv = \delta^\psi. \end{aligned}$$

Setting $I_{1,j}(\kappa) := \int_{\mathcal{M}} (\log \kappa(y))^j \kappa(y)\, dy$ for $j = 1, 2$, we may use (3.10) and (7.19) to deduce that

$$\begin{aligned} \sigma^2(\psi, \kappa) &= V^\psi + I_{1,2}(\kappa) - 2\delta^\psi I_{1,1}(\kappa) - (\delta^\psi - I_{1,1}(\kappa))^2 \\ &= V^\psi - (\delta^\psi)^2 + I_{1,2}(\kappa) - I_{1,1}^2(\kappa). \end{aligned} \tag{7.20}$$

Moreover, by (2.7) and (7.17) we have $\delta^\psi = \int_{\mathbb{R}^d} f(x)\, dx$, where here we set

$$\begin{aligned} f(x) &:= -m\mathbb{E}[\log(N_1(\mathbf{0}, \mathcal{H})/N_1(\mathbf{0}, \mathcal{H}^x))] \\ &= -\int_0^\infty P[N_1(0, \mathcal{H}) > \|x\| e^t]\, dt = -\int_0^\infty \exp(-\omega_m \|x\|^m e^{tm})\, dt \end{aligned}$$

so that setting $v = \omega_m \|x\|^m$ we have

$$-\delta^\psi = \int_0^\infty \int_0^\infty \exp(-ve^{tm})\, dt\, dv = \int_0^\infty e^{-tm}\, dt = m^{-1}.$$

Substituting this into (7.20) gives the result claimed. □

LEMMA 7.8. *Suppose either* (i) *that* $\kappa \in \mathbb{P}_b(\mathcal{M})$, *or* (ii) *that* $m = d$ *and* $\mathcal{M} = \mathbb{R}^d$ *and* $r_c(\kappa) > 0$. *Then there exists* $\delta > 0$ *such that* $\sup_n \mathbb{E}[N_1(n^{1/m} Y_1, n^{1/m} \mathcal{Y}_n)^\delta] < \infty$.

PROOF. First consider case (ii). Choose δ small enough so that $r_c(\kappa) > \delta d/(d - \delta)$. Then by the proof of Theorem 13.3 of [41] (Theorem 2.3 in the arXiv version) (with δ corresponding to αp in that proof), $\mathbb{E}[N_1(n^{1/m} Y_1, n^{1/m} \mathcal{Y}_n)^\delta]$ is bounded as asserted.

Now assume case (i) instead. We adapt the proof of Theorem 13.3 of [41] to manifolds. Let $((y_i, \delta_i, U_i, g_i), i \in \mathcal{I}_0)$ be as in Section 2.1, and take a finite $\mathcal{I}_0 \subset \mathcal{I}$ such that $\mathcal{K} \subset \bigcup_{i \in \mathcal{I}_0} B_{\delta_i}(y_i)$. Assume $\mathcal{I}_0 = \{1, \dots, i_0\}$ for some i_0, and write A_i for $B_{\delta_i}(y_i) \setminus \bigcup_{j<i} B_{\delta_j}(y_j)$. For any finite $\mathcal{Y}$ write $L^\delta(\mathcal{Y})$ for $\sum_{y \in \mathcal{Y}} N_1(y, \mathcal{Y})^\delta$ with $L^\delta(\mathcal{Y}) = 0$ if $\mathrm{card}(\mathcal{Y}) \le 1$. Similarly to (3.4) of [41], we have

$$\mathbb{E}[N_1(n^{1/m} Y_1, n^{1/m} \mathcal{Y}_n)^\delta] = n^{\delta/m - 1} \mathbb{E}[L^\delta(\mathcal{Y}_n)], \tag{7.21}$$

and using the boundedness of $\mathcal{K}$ we have, similarly to (3.6) of [41],

$$L^\delta(\mathcal{Y}_n) \le \sum_{i=1}^{i_0} L^\delta(\mathcal{Y}_n \cap A_i) + C. \tag{7.22}$$

Similarly to (3.7) of [41], by combining (7.21) and (7.22) we have

$$\mathbb{E}[N_1(n^{1/m} Y_1, n^{1/m} \mathcal{Y}_n)^\delta] = n^{\delta/m - 1} \mathbb{E}\left[\sum_{i=1}^{i_0} L^\delta(\mathcal{Y}_n \cap A_i) + C\right],$$

and by Jensen's inequality this remains bounded, provided we can establish the deterministic bound, for all finite $\mathcal{Y} \subset A_i$,

$$L^\delta(\mathcal{Y}) \le C(\mathrm{card}(\mathcal{Y}))^{1-\delta/m}. \tag{7.23}$$

By Lemma 4.1, there is a constant C such that for all finite $\mathcal{Y} \subset A_i$ and $x \in \mathcal{Y}$, taking y to be a nearest neighbor of x in $\mathcal{Y}$, and also taking $g_i^{-1}(z)$ to be a nearest neighbor of $g_i^{-1}(x)$ in $g_i^{-1}(\mathcal{Y})$, we have that

$$N_1(x, \mathcal{Y}) = \|y - x\| \le \|z - x\| \le C\|g_i^{-1}(z) - g_i^{-1}(x)\| = CN_1(g_i^{-1}(x), g_i^{-1}(\mathcal{Y})).$$

Hence, $L^\delta(\mathcal{Y}) \le CL^\delta(g_i^{-1}\mathcal{Y})$, and using Lemma 3.3 of [45], we have (7.23) as asserted. □

PROOF OF THEOREM 2.4. To prove the L^2 convergence at (2.22) we shall apply Theorem 3.1. Note that ψ belongs to $\Xi(k, r)$, and thus it suffices to verify under either of the hypotheses of Theorem 2.4 that $\xi \equiv \psi$ satisfies the moment condition (3.4) for $p = 3$. Set $N_1 := N_1(n^{1/m}Y_1, n^{1/m}\mathcal{Y}_n)$. It will suffice to show that $\sup_n \mathbb{E}|\log N_1^m|^p < \infty$. Given $\delta \in (0, 1)$, we can choose a constant C such that $|\log t|^p \le Ct^{-\delta}$ for $t \in [0, 1]$ and $|\log t|^p \le Ct^\delta$ for $t \in [1, \infty)$. Then

$$\mathbb{E}|\log N_1^m|^p \le C\mathbb{E}[N_1^{-\delta} + N_1^\delta]. \tag{7.24}$$

By Lemmas 7.5 and 7.8, the right-hand side of (7.24) is bounded, provided δ is chosen small enough. Hence we can apply Theorem 3.1, yielding

$$\lim_{n\to\infty} n^{-1}S(n^{1/m}\mathcal{Y}_n) = \int_{\mathcal{M}} \mathbb{E}[\psi(\mathbf{0}, \mathcal{H}_{\kappa(y)})]\kappa(y)\,dy = -\int_{\mathcal{M}} \log(\kappa(y))\kappa(y)\,dy \quad \text{in } L^2,$$

by (7.17). Thus (2.22) holds.

Now we suppose $\kappa \in \mathbb{P}_c(\mathcal{M})$ and prove the variance asymptotics (2.23) and central limit theorem (2.24). We cannot directly apply Theorem 3.2 because the moment bound (3.6) fails since if $\mathcal{A} = \{z\}$, the distance between y and z can be made arbitrarily small. As in the proof of the case $\alpha < 0$ of Theorem 2.3 we use a truncation argument, defining for all $\varepsilon > 0$ the function

$$\log^{(\varepsilon)}(x) := \begin{cases} \log(x), & \text{if } x \ge \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Let ψ^ε be defined as ψ, with log replaced by $\log^{(\varepsilon)}$.

As in the proof of Theorem 2.3 we may show that the moment bounds (3.6) and (3.5) hold (with $\rho = \infty$) for ψ^ε for some $p > 2$. The case $\mathcal{A} = \varnothing$ of (3.6) follows from the bound

$$\sup_{\lambda \ge 1, y \in \mathcal{K}} \mathbb{E}|\psi(\lambda^{1/d}y, \lambda^{1/d}\mathcal{P}_\lambda)|^p < \infty, \tag{7.25}$$

which is proved similar to (3.4) which we have already established. For the case $\mathcal{A} = \{z\}$, observe that

$$\begin{aligned} &\psi^\varepsilon(\lambda^{1/m} y, \lambda^{1/m}(\mathcal{P}_\lambda \cup \{z\})) \\ &\quad = \psi^\varepsilon(\lambda^{1/m} y, \lambda^{1/m}\mathcal{P}_\lambda)\mathbf{1}\{N_1(y, \mathcal{P}_\lambda) \le \|y - z\|\} \\ &\qquad + \log^{(\varepsilon)}(e^\gamma \omega_m \lambda \|y - z\|^m)\mathbf{1}\{N_1(y, \mathcal{P}_\lambda) > \|y - z\|\}. \end{aligned} \tag{7.26}$$

The first term on the right-hand side of (7.26) has bounded pth moment, which may be proved similarly to the proof of (3.4). Since the function $|\log(\cdot)|$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$, provided $\varepsilon < 1$ the absolute value of the last term in (7.26) is bounded by $|\log(\varepsilon)|$ if $e^\gamma \omega_m \lambda \|y - z\|^m < 1$, and otherwise is bounded by $|\psi^\varepsilon(\lambda^{1/m} y, \lambda^{1/m}\mathcal{P}_\lambda)|$, which has bounded pth moment, as noted already. This gives us (3.6), and the argument for (3.5) is similar.

By Theorem 3.2, as $n \to \infty$, we deduce that $n^{-1}\operatorname{Var}[H_n^{\psi^\varepsilon}(\mathcal{Y}_n)]$ converges to $\sigma^2(\psi^\varepsilon, \kappa)$ and

$$n^{-1/2}(H_n^{\psi^\varepsilon}(\mathcal{Y}_n) - \mathbb{E}H_n^{\psi^\varepsilon}(\mathcal{Y}_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(\psi^\varepsilon, \kappa)). \tag{7.27}$$

To complete the proof, we adapt arguments in [4], given in detail in [3], as follows. To make the link with [3], put $\phi(x) := \log(e^\gamma \omega_m x)$ in Section 5 of [3], and also for $1 \le i \le m$, set $N_{i,m} := N_1(Y_i, \mathcal{Y}_m)$, giving the identification $\psi_n(Y_i, \mathcal{Y}_n) = \phi(nN_{i,n}^m)$. Set $\log_{(\varepsilon)}(x) = \log(x) - \log^{(\varepsilon)}(x)$ and $\phi_\varepsilon(x) := \log_{(\varepsilon)}(e^\gamma x)$. It is easily seen with this choice of ϕ that Lemma 7.6 holds [follow that proof verbatim, choosing ε small enough so that $|\phi_\varepsilon(\cdot)|$ is nonincreasing on $(0, \varepsilon)$]. It now suffices to follow the proof of Theorem 2.3 for the case $\alpha \in (-m/2, 0)$.

It remains only to show $\sigma^2(\psi, \kappa) > 0$. We again follow the approach used in the proof of positivity in Theorem 2.3. By (2.23), it is enough to show that the expression $V^\psi - (\delta^\psi)^2$, is strictly positive. This can be shown to be nonnegative as in the proof of positivity in Theorem 2.3, and we leave the details to the reader. □

PROOF OF THEOREM 2.5. Let $\beta \in (0, \infty)$. Let $\varphi_k(y, \mathcal{Y}) := \varphi_k^{(\beta)}(y, \mathcal{Y})$ be $(k+1)^{-1}$ times the number of k-simplices containing y in $\mathcal{R}^\beta(\mathcal{Y})$, that is, $(k+1)^{-1}$ times the number of unordered $(k+1)$-tuples of points in $\mathcal{Y}$, all pairwise within β of each other, and including y. Then

$$\mathcal{C}_k^{(\beta)}(\mathcal{Y}) = \sum_{y \in \mathcal{Y}} \varphi_k(y, \mathcal{Y}). \tag{7.28}$$

We want to show that φ_k satisfies the conditions of Theorems 3.1 and 3.2. Note that $\varphi_k \in \Xi(0, r)$ if $r > \beta$. For any $y \in \mathcal{K}$, $\mathcal{A} \in \mathcal{S}_3$ and $k, \ell \in \mathbb{N}$, $\varphi_k(n^{1/m} y, n^{1/m}(\mathcal{Y}_\ell \cup \mathcal{A}))$ is bounded by $(3 + \mathcal{C}_\beta(n))^k$, where we set $\mathcal{C}_\beta(n) := \sum_{i=1}^{\ell} \mathbf{1}\{\|Y_i - Y_1\| < \beta n^{-1/m}\}$, which is binomially distributed with parameters ℓ and the κ measure of $B_{\beta n^{-1/m}}(y)$. Assuming κ is bounded, there is a constant C such that the latter parameter is at most $C\ell^{-1}$, uniformly in y. Hence $\mathcal{C}_\beta(n)$ is stochastically bounded by a binomial random variable with parameters ℓ and $C\ell^{-1}$, and thus for any $p \ge 1$, φ_k satisfies the moment condition (3.5), and hence also (3.4). Therefore by using Theorem 3.1 and (7.28), we have

$$\begin{aligned}\lim_{n\to\infty} n^{-1}\mathcal{C}_k^\beta(n^{1/m}\mathcal{Y}_n) &= \lim_{n\to\infty} n^{-1}H_n^{\varphi_k}(\mathcal{Y}_n)\\ &= \int_{\mathcal{M}} \mathbb{E}\varphi_k^{(\beta)}(\mathbf{0},\mathcal{H}_{\kappa(y)})\kappa(y)\,dy,\end{aligned} \tag{7.29}$$

with both L^2 and a.s. convergence.

Define $h_k^{(\beta)}:(\mathbb{R}^m)^{k+1}\to\mathbb{R}$ by $h_k^{(\beta)}(x_1,\dots,x_{k+1}) := \prod_{1\le i<j\le k+1}\mathbf{1}\{\|x_i - x_j\|\le\beta\}$, that is, the indicator of the event that $x_1,\dots,x_{k+1}$ are all within distance β of each other. By the Palm theory of Poisson processes (e.g., Theorem 1.6 of [34]) we have

$$\begin{aligned}\mathbb{E}\varphi_k^{(\beta)}(\mathbf{0},\mathcal{H}'_\lambda) &= \frac{\lambda^k}{(k+1)!}\int_{\mathbb{R}^m}\cdots\int_{\mathbb{R}^m} h_k^{(\beta)}(\mathbf{0},x_1,\dots,x_k)\,dx_1\cdots dx_k\\ &= \frac{(\lambda\beta^m)^k}{(k+1)!}\int_{\mathbb{R}^m}\cdots\int_{\mathbb{R}^m} h_k^{(1)}(\mathbf{0},x_1,\dots,x_k)\,dx_1\cdots dx_k\\ &= \frac{(\lambda\beta^m)^k}{(k+1)!}J_{k,k+1},\end{aligned}$$

where the last equality comes from (2.25). Combined with (7.29), this gives us (2.27).

A slight modification of the above argument shows that φ_k satisfies the Poisson moment condition (3.6) for any $p\ge 1$. By Theorem 3.2, the variance asymptotics (2.28) and central limit theorem (2.29) follow, with $\sigma_k^2(\beta,\kappa) := \sigma^2(\varphi_k^{(\beta)},\kappa)$ given by (3.10). We need to show that this is consistent with (2.26).

The expression $\delta^{\varphi_k}(a)$, given by (3.9), simplifies further as

$$\delta^{\varphi_k}(a) = (a\beta^m)^k J_{k,k+1} + a\int_{\mathbb{R}^m}\mathbb{E}[\Delta^u\varphi_k(\mathbf{0},\mathcal{H}_a)]\,du. \tag{7.30}$$

Using the Palm theory of the Poisson process again, we have for all $u\in\mathbb{R}^m$ that

$$\begin{aligned}&\mathbb{E}[\Delta^u\varphi_k(\mathbf{0},\mathcal{H}'_a)]\\ &\quad= \frac{a^{k-1}}{(k+1)(k-1)!}\int_{\mathbb{R}^m}\cdots\int_{\mathbb{R}^m} h_k^{(\beta)}(\mathbf{0},u,x_1,\dots,x_k)\,dx_1\cdots dx_{k-1}.\end{aligned}$$

Together with (7.30), this gives

$$\begin{aligned}\delta^{\varphi_k}(a) &= (a\beta^m)^k J_{k,k+1} + \frac{a^k k}{(k+1)!}\int\cdots\int h^{(\beta)}(\mathbf{0},x_1,\dots,x_k)\,dx_1\cdots dx_k\\ &= (k+1)(a\beta^m)^k J_{k,k+1}.\end{aligned} \tag{7.31}$$

For the first term in (3.10) we simplify the expression $V^{\varphi_k}(a)$ as follows. Consider the special case where $m = d$ and $\mathcal{M}$ is a smoothly bounded region of volume a^{-1}, and let $\mathcal{P}'_\lambda$ be a homogeneous Poisson point process in this region with expected total number of points equal to λ. By applying (3.17) in this case, recalling notation $\mathcal{C}_k^{(\beta)}$ from Section 2.4 we get that

$$V^{\varphi_k}(a) = \lim_{\lambda\to\infty} \lambda^{-1} \operatorname{Var} \sum_{y\in\mathcal{P}'_\lambda} \varphi_k(\lambda^{1/m} y, \lambda^{1/m}\mathcal{P}'_\lambda) = \lim_{\lambda\to\infty} \lambda^{-1} \operatorname{Var}[\mathcal{C}_k^{(\beta)}(\lambda^{1/m}\mathcal{P}'_\lambda)]$$

$$= \lim_{\lambda\to\infty} \lambda^{-1} \operatorname{Var}[\mathcal{C}_k^{(\beta(a/\lambda)^{1/m})}(a^{1/m}\mathcal{P}'_\lambda)].$$

Hence by Proposition 3.7 of [34], setting $k' = k + 1$ we have

$$V^{\varphi_k}(a) = \sum_{j=1}^{k'} J_{k,j} (a\beta^m)^{2k'-j-1}.$$

Using this identity in the first term of (3.10), and using (7.31) for the second term of (3.10) enables us to establish the identity (2.26).

It remains to show that $\sigma^2(\varphi_k^{(\beta)}, \kappa) > 0$. This can be done as in the proof of Lemma 2.3, that is, using (7.14) to reduce the problem to showing positivity in the case where $d = m$ and κ is a uniform distribution on a cube, and using the methods of [2] or [38] to demonstrate positivity in this case. This completes the proof of Theorem 2.5. □

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