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ROBUST MAXIMIZATION OF ASYMPTOTIC GROWTH UNDER COVARIANCE UNCERTAINTY

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This paper resolves a question proposed in Kardaras and Robertson [*Ann. Appl. Probab.* **22** (2012) 1576–1610]: how to invest in a robust growth-optimal way in a market where precise knowledge of the covariance structure of the underlying assets is unavailable. Among an appropriate class of admissible covariance structures, we characterize the optimal trading strategy in terms of a generalized version of the principal eigenvalue of a fully nonlinear elliptic operator and its associated eigenfunction, by slightly restricting the collection of nondominated probability measures.

1. Introduction. In this paper, we consider the problem of how to trade optimally in a market when the investing horizon is long and the dynamics of the underlying assets are uncertain. For the case where the uncertainty lies only in the instantaneous expected return of the underlying assets, this problem has been studied by Kardaras and Robertson [16]. They identify the optimal trading strategy using a generalized version of the principle eigenfunction for a linear elliptic operator which depends on the given covariance structure of the underlying assets. We intend to generalize their results to the case where *even* the covariance structure of the underlying assets is not known precisely, which is suggested in [16], Discussion. More precisely, we would like to determine a *robust* trading strategy under which the asymptotic growth rate of one's wealth, defined below, can be maximized no matter which admissible covariance structure materializes.

Uncertainty in variance (or, equivalently, in covariance) has been drawing increasing attention. The main difficulty lies in the absence of one single dominating probability measure among Π, the collection of all probability measures induced by variance uncertainty. In their pioneering works, Avellaneda, Levy and Paras [2] and Lyons [19] introduced the uncertain volatility model (UVM), where the volatility process is only known to lie in a fixed interval $[\underline{\sigma}, \bar{\sigma}]$. Under the Markovian framework, they obtained a duality formula for the superhedging price of (nonpath-dependent) European contingent claims. Under a generalized version of

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the UVM, Denis and Martini [11] extended the above duality formula, by using the capacity theory, to incorporate path-dependent European contingent claims. For the capacity theory to work, they required some continuity of the random variables being hedged. Taking a different approach based on the underlying partial differential equations, Peng [25] derived results very similar to [11]. The connection between [11] and [25] was then elaborated and extended in Denis, Hu and Peng [10]. On the other hand, instead of imposing some continuity assumptions on the random variables being hedged, Soner, Touzi and Zhang [31] chose to restrict slightly the collection of nondominated probability measures, and derived under this setting a duality formulation for the superhedging problem. With all these developments, superhedging under volatility uncertainty has then been further studied in Nutz and Soner [24] and Nutz [23], among others. Also notice that Fernholz and Karatzas [12] characterized the highest return relative to the market portfolio under covariance uncertainty. Moreover, a controller-and-stopper game with controlled drift and volatility was considered in [3], which can be viewed as an optimal stopping problem under volatility uncertainty.

While we also take covariance uncertainty into account, we focus on robust growth-optimal trading, which is different by nature from the superhedging problem. Here, an investor intends to find a trading strategy such that her wealth process can achieve maximal growth rate, in certain sense, uniformly over all possible probability measures in Π, or at least in a large enough subset Π^* of Π. Previous research on this problem can be found in [16] and the references therein. It is worth noting that this problem falls under the umbrella of ergodic control, for which the dynamic programming heuristic cannot be directly applied; see, for example, Arapostathis, Borkar and Ghosh [1] and Borkar [6], where they consider ergodic control problems with controlled drift.

Following the framework in [16], we first observe that the associated differential operator under covariance uncertainty is a variant of Pucci's extremal operator. We define the "principal eigenvalue" for this fully nonlinear operator, denoted by λ^*, in some appropriate sense, and then investigate the connection between λ^* and the generalized principal eigenvalue in [16] where the covariance structure is a priori given. This connection is first established on smooth bounded domains, thanks to the theory of continuous selection in Michael [22] and Brown [7]. Next, observing that a Harnack inequality holds under current context, we extend the result to unbounded domains. Finally, as a consequence of this connection, we generalize [16], Theorem 2.1, to the case with covariance uncertainty: we characterize the largest possible asymptotic growth rate as λ^* (which is robust among probabilities in a large enough subset Π^* of Π) and identify the optimal trading strategy in terms of λ^* and the corresponding eigenfunction; see Theorem 3.3.

The structure of this paper is as follows. In Section 2, we introduce the framework of our study and formulate the problem of robust maximization of asymptotic growth under covariance uncertainty. In Section 3, we first introduce several different notions of the generalized principal eigenvalue and then investigate the relation between them. The main technical result we obtain is Theorem 3.2, using which we resolve the problem of robust maximization of asymptotic growth in Theorem 3.3.

1.1. *Notation.* We collect some notation and definitions here for readers' convenience:

- $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, and Leb denotes Lebesgue measure in $\mathbb{R}^n$.
- $B_\delta(x)$ denotes the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $\delta > 0$.
- $\bar{D}$ denotes the closure of D, and ∂D denotes the boundary of D.
- Given $x \in \mathbb{R}^n$ and $D_1, D_2 \subset \mathbb{R}^n$, $d(x, D_1) := \inf\{|x - y| \mid y \in D_1\}$ and $d(D_1, D_2) := \inf\{|x - y| \mid x \in D_1, y \in D_2\}$.
- Given $D \subset \mathbb{R}^n$, $C(D) = C^0(D)$ denotes the set of continuous functions on D. If D is open, $C^k(D)$ denotes the set of functions having derivatives of order $\leq k$ continuous in D, and $C^k(\bar{D})$ denotes the set of functions in $C^k(D)$ whose derivatives of order $\leq k$ have continuous extension on $\bar{D}$.
- Given $D \subset \mathbb{R}^n$, $C^{k,\beta}(D)$ denotes the set of functions in $C^k(D)$ whose derivatives of order $\leq k$ are Hölder continuous on D with exponent $\beta \in (0, 1]$. Moreover, $C^{k,\beta}_{\text{loc}}(D)$ denotes the set of functions belonging to $C^{k,\beta}(K)$ for every compact subset K of D.
- We say $D \subset \mathbb{R}^n$ is a domain if it is an open connected set. We say D is a smooth domain if it is a domain whose boundary is of $C^{2,\beta}$ for some $\beta \in (0, 1]$.
- Given $D \subset \mathbb{R}^n$ and $u : D \mapsto \mathbb{R}$, $\text{osc}_D := \sup\{|u(x) - u(y)| \mid x, y \in D\}$.

2. The set-up. Fix $d \in \mathbb{N}$. Consider an open connected set $E \subseteq \mathbb{R}^d$, and two functions $\theta, \Theta : E \mapsto (0, \infty)$. The following assumption will be in force throughout this paper.

ASSUMPTION 2.1. (i) θ and Θ are of $C^{0,\alpha}_{\text{loc}}(E)$ for some $\alpha \in (0, 1]$, and $\theta < \Theta$ in E.

(ii) There exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of bounded open convex subsets of E such that ∂E_n is of $C^{2,\alpha'}$ for some $\alpha' \in (0, 1]$, $\bar{E}_n \subset E_{n+1}$ for all $n \in \mathbb{N}$ and $E = \bigcup_{n=1}^{\infty} E_n$.

Let $\mathbb{S}^d$ denote the space of $d \times d$ symmetric matrices, equipped with the norm

$$\|M\| := \max_{i=1,\dots,d} |e_i(M)|, \qquad M \in \mathbb{S}^d, \tag{2.1}$$

where $e_i(M)$'s are the eigenvalues of M. In some cases, we will also consider the norm $\|M\|_{\max} := \max |m_{ij}|$, for $M = \{m_{ij}\}_{i,j} \in \mathbb{S}^d$. These two norms are equivalent with $\|\cdot\|_{\max} \leq \|\cdot\| \leq d\|\cdot\|_{\max}$.

DEFINITION 2.1. Let $\mathcal{C}$ be the collection of functions $c : E \mapsto \mathbb{S}^d$ such that:

(i) for any $x \in E$, $\theta(x)|\xi|^2 \le \xi' c(x)\xi \le \Theta(x)|\xi|^2$, $\forall \xi \in \mathbb{R}^d \setminus \{0\}$;
(ii) $c_{ij}(x)$ is of $C^{1,\alpha}_{\mathrm{loc}}(E)$, $1 \le i, j \le d$.

Let $\widehat{E} := E \cup \triangle$ be the one-point compactification of E, where $\triangle$ is identified with ∂E if E is bounded with ∂E plus the point at infinity if E is unbounded. Following the set-up in [16], Section 1, or [26], page 40, we consider the space $C([0,\infty), \widehat{E})$ of continuous functions $\omega : [0,\infty) \mapsto \widehat{E}$, and define for each $\omega \in C([0,\infty), \widehat{E})$ the exit times

$$\zeta_n(\omega) := \inf\{t \ge 0 \mid \omega_t \notin E_n\}, \qquad \zeta(\omega) := \lim_{n\to\infty} \zeta_n(\omega).$$

Then, we introduce $\Omega := \{\omega \in C([0,\infty), \widehat{E}) \mid \omega_{\zeta+t} = \triangle \text{ for all } t \ge 0, \text{ if } \zeta(\omega) < \infty\}$. Let $X = \{X_t\}_{t\ge 0}$ be the coordinate mapping process for $\omega \in \Omega$. Set $\{\mathcal{B}_t\}_{t\ge 0}$ to be the natural filtration generated by X, and denote by $\mathcal{B}$ the smallest σ-algebra generated by $\bigcup_{t\ge 0} \mathcal{B}_t$. Similarly, set $(\mathcal{F}_t)_{t\ge 0}$ to be the right-continuous enlargement of $(\mathcal{B}_t)_{t\ge 0}$, and denote by $\mathcal{F}$ the smallest σ-algebra generated by $\bigcup_{t\ge 0} \mathcal{F}_t$.

REMARK 2.1. For financial applications, $X = \{X_t\}_{t\ge 0}$ represents the (relative) price process of certain underlying assets, and each $c \in \mathcal{C}$ represents a possible covariance structure that might eventually materialize. In view of Definition 2.1(i), the extent of the uncertainty in covariance is captured by the functions θ and Θ: they act as the pointwise lower and upper bounds uniformly over all possible covariance structures $c \in \mathcal{C}$.

2.1. *The generalized martingale problem.* For any $M = \{m_{ij}\}_{i,j} \in \mathbb{S}^d$, define the operator L^M which acts on $f \in C^2(E)$ by

$$\bigl(L^M f\bigr)(x) := \frac{1}{2} \sum_{i,j=1}^d m_{ij} \frac{\partial^2 f}{\partial x_i\, \partial x_j}(x) = \frac{1}{2} \operatorname{Tr}\bigl[M D^2 f(x)\bigr], \qquad x \in E.$$

For each $c \in \mathcal{C}$, we define similarly the operator $L^{c(\cdot)}$ as

$$\bigl(L^{c(\cdot)} f\bigr)(x) := \frac{1}{2} \sum_{i,j=1}^d c_{ij}(x) \frac{\partial^2 f}{\partial x_i\, \partial x_j}(x) = \frac{1}{2} \operatorname{Tr}\bigl[c(x) D^2 f(x)\bigr], \qquad x \in E.$$

Given $c \in \mathcal{C}$, a solution to the generalized martingale problem on E for the operator $L^{c(\cdot)}$ is a family of probability measures $(\mathbb{Q}^c_x)_{x\in\widehat{E}}$ on $(\Omega, \mathcal{B})$ such that $\mathbb{Q}^c_x[X_0 = x] = 1$ and

$$f(X_{s\wedge\zeta_n}) - \int_0^{s\wedge\zeta_n} \bigl(L^{c(\cdot)} f\bigr)(X_u)\, du$$

is a $(\Omega, (\mathcal{B}_t)_{t\ge 0}, \mathbb{Q}^c_x)$-martingale for all $n \in \mathbb{N}$ and $f \in C^2(E)$.

The following result, taken from [26], Theorem 1.13.1, states that Assumption 2.1 guarantees the existence and uniqueness of the solutions to the generalized martingale problem on E for the operator $L^{c(\cdot)}$, for each fixed $c \in \mathcal{C}$.

PROPOSITION 2.1. *Under Assumption 2.1, for each $c \in \mathcal{C}$, there is a unique solution $(\mathbb{Q}_x^c)_{x \in \widehat{E}}$ to the generalized martingale problem on E for the operator $L^{c(\cdot)}$.*

REMARK 2.2. For each $c \in \mathcal{C}$, as mentioned in [16], Section 1,

$$f(X_{s\wedge\zeta_n}) - \int_0^{s\wedge\zeta_n} \big(L^{c(\cdot)} f\big)(X_u)\,du$$

is also a $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}_x^c)$-martingale for all $n \in \mathbb{N}$ and $f \in C^2(E)$, as f and $L^{c(\cdot)} f$ are bounded in each E_n. Now, by taking $f(x) = x^i$, $i = 1, \dots, d$ and $f(x) = x^i x^j$ with $i, j, = 1 \cdots d$, we get $X_{t\wedge\zeta_n}$ is a $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}_x^c)$-martingale with quadratic covariation process $\int_0^\cdot 1_{\{t\leq\zeta_n\}} c(X_t)\,dt$, for each $n \in \mathbb{N}$ and $x \in \widehat{E}$.

2.2. *Asymptotic growth rate.* For any fixed $x_0 \in E$, we will simply write $\mathbb{Q}^c = \mathbb{Q}_{x_0}^c$ for all $c \in \mathcal{C}$, when there is no confusion on the initial value x_0 of X. Let us denote by Π the collection of probability measures on $(\Omega, \mathcal{F})$ which are locally absolutely continuous with respect to $\mathbb{Q}^c$ (written $\mathbb{P} \ll_{\text{loc}} \mathbb{Q}^c$) for some $c \in \mathcal{C}$, and for which the process X does not explode. That is,

$$\Pi := \big\{\mathbb{P} \in P(\Omega, \mathcal{F}) \mid \exists c \in \mathcal{C} \text{ s.t. } \mathbb{P}|_{\mathcal{F}_t} \ll \mathbb{Q}^c|_{\mathcal{F}_t} \text{ for all } t \geq 0, \text{ and } \mathbb{P}[\zeta < \infty] = 0\big\},$$

where $P(\Omega, \mathcal{F})$ denotes the collection of all probability measures on $(\Omega, \mathcal{F})$. As observed in [16], Section 1, for each $\mathbb{P} \in \Pi$, X is a $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$-semimartingale such that $\mathbb{P}[X \in C([0, \infty), E)] = 1$. Moreover, if we take $c \in \mathcal{C}$ such that $\mathbb{P} \ll_{\text{loc}} \mathbb{Q}^c$, then X admits the representation

$$X_\cdot = x_0 + \int_0^\cdot b_t^{\mathbb{P}}\,dt + \int_0^\cdot \sigma(X_t)\,dW_t^{\mathbb{P}},$$

where $W^{\mathbb{P}}$ is a standard d-dimensional Brownian motion on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, σ is the unique symmetric strictly positive definite square root of c, and $b^{\mathbb{P}}$ is a d-dimensional $\{F_t\}_{t\geq 0}$-progressively measurable process.

Let $(Z_t)_{t\geq 0}$ be an adapted process. For $\mathbb{P} \in \Pi$, define

$$\mathbb{P}\text{-}\liminf_{t\to\infty} Z_t := \operatorname{ess\,sup}^{\mathbb{P}}\Big\{\chi \text{ is } \mathcal{F}\text{-measurable} \;\big|\; \lim_{t\to\infty} \mathbb{P}[Z_t \geq \chi] = 1\Big\}.$$

For any d-dimensional predictable process π which is X-integrable under $\mathbb{Q}^c$ for all $c \in \mathcal{C}$, we can define the process $V_\cdot^\pi := 1 + \int_0^\cdot \pi_t'\,dX_t$ under $\mathbb{Q}^c$ for all $c \in \mathcal{C}$. Let $\mathcal{V}$ denote the collection of all such processes π which in addition satisfy the following: for each $c \in \mathcal{C}$, $\mathbb{Q}^c[V_t^\pi > 0] = 1, \forall t \geq 0$. Here, $\pi \in \mathcal{V}$ represents an admissible trading strategy and V^π represents the corresponding wealth process. Now, for any $\pi \in \mathcal{V}$, we define the asymptotic growth rate of V^π under $\mathbb{P} \in \Pi$ as

$$g(\pi; \mathbb{P}) := \sup\Big\{\gamma \in \mathbb{R} \;\big|\; \mathbb{P}\text{-}\liminf_{t\to\infty}\big(t^{-1} \log V_t^\pi\big) \geq \gamma, \mathbb{P}\text{-a.s.}\Big\}.$$

2.3. *The problem.* The problem we consider in this paper is how to choose a trading strategy $\pi^* \in \mathcal{V}$ such that the wealth process V^{π^*} attains the robust maximal asymptotic growth rate under all possible probabilities in Π, or at least, in a large enough subset of Π which readily contains all "nonpathological" cases. More precisely, in Theorem 3.3 below, we will construct a large enough suitable subset Π^* of Π, and determine

$$\sup_{\pi \in \mathcal{V}} \inf_{\mathbb{P} \in \Pi^*} g(\pi; \mathbb{P}),$$

the robust maximal asymptotic growth rate (robust in Π^*). Moreover, we will find $\pi^* \in \mathcal{V}$ such that V^{π^*} attains (or surpasses) the maximal growth rate no matter which $\mathbb{P} \in \Pi^*$ materializes. This generalizes [16], Theorem 2.1, to the case with covariance uncertainty.

3. The min–max result. In this section, we will first introduce generalized versions of the principal eigenvalue for the linear operator $L^{c(\cdot)}$ and a fully nonlinear operator F defined below. Then, we will investigate the relation between them on smooth bounded domains, and eventually extend the result to the entire domain E. The main technical result we obtain is Theorem 3.2. Finally, by using Theorem 3.2, we are able to resolve in Theorem 3.3 the problem proposed in Section 2.3.

Let us first recall the definition of Pucci's extremal operators. Given $0 < \lambda \leq \Lambda$, we define for any $M \in \mathbb{S}^d$ the following matrix operators:

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(M) &:= \Lambda \sum_{e_i(M)>0} e_i(M) + \lambda \sum_{e_i(M)<0} e_i(M), \\ \mathcal{M}^-_{\lambda,\Lambda}(M) &:= \lambda \sum_{e_i(M)>0} e_i(M) + \Lambda \sum_{e_i(M)<0} e_i(M). \end{aligned} \tag{3.1}$$

From [9], page 15, we see that these operators can be expressed as

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A \in \mathcal{A}(\lambda,\Lambda)} \operatorname{Tr}(AM), \qquad \mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A \in \mathcal{A}(\lambda,\Lambda)} \operatorname{Tr}(AM),$$

where $\mathcal{A}(a,b)$ denotes the set of matrices in $\mathbb{S}^d$ with eigenvalues lying in $[a,b]$ for some real numbers $a \leq b$. For general properties of Pucci's extremal operators, see, for example, [28] and [9], Section 2.2. Now, let us define the operator $F : E \times \mathbb{S}^d \mapsto \mathbb{R}$ by

$$F(x, M) := \frac{1}{2}\mathcal{M}^+_{\theta(x),\Theta(x)}(M) = \frac{1}{2} \sup_{A \in \mathcal{A}(\theta(x),\Theta(x))} \operatorname{Tr}(AM). \tag{3.2}$$

Let D be an open connected subset of E. Fixing $c \in \mathcal{C}$, we consider, for any given $\lambda \in \mathbb{R}$, the cone of positive harmonic functions with respect to $L^{c(\cdot)} + \lambda$ as

$$H^c_\lambda(D) := \{\eta \in C^2(D) \mid L^{c(\cdot)}\eta + \lambda\eta = 0 \text{ and } \eta > 0 \text{ in } D\} \tag{3.3}$$

and set

$$\lambda^{*,c}(D) := \sup\{\lambda \in \mathbb{R} \mid H^c_\lambda(D) \neq \varnothing\}. \tag{3.4}$$

Note that if D is a smooth bounded domain, $\lambda^{*,c}(D)$ coincides with the principal eigenvalue for $L^{c(\cdot)}$ on D; see, for example, [26], Theorem 4.3.2. In our case, since we do not require the boundedness of D, $\lambda^{*,c}(D)$ is a generalized version of the principal eigenvalue for $L^{c(\cdot)}$ on D, which is also used in [16]. On the other hand, for any $\lambda \in \mathbb{R}$, we define

$$H_\lambda(D) := \{\eta \in C^2(D) \mid F(x, D^2\eta) + \lambda\eta \leq 0 \text{ and } \eta > 0 \text{ in } D\} \tag{3.5}$$

and set

$$\lambda^*(D) := \sup\{\lambda \in \mathbb{R} \mid H_\lambda(D) \neq \varnothing\}, \tag{3.6}$$

which is a generalized version of the principal eigenvalue for the fully nonlinear operator F on D. For auxiliary purposes, we also consider, for any $\lambda \in \mathbb{R}$, the set

$$H^+_\lambda(D) := \{\eta \in C(\bar{D}) \mid F(x, D^2\eta) + \lambda\eta \leq 0 \text{ and } \eta > 0 \text{ in } D\}, \tag{3.7}$$

where the inequality holds in the viscosity sense. From this, we define

$$\lambda^+(D) := \sup\{\lambda \in \mathbb{R} \mid H^+_\lambda(D) \neq \varnothing\}. \tag{3.8}$$

For the special case where D is a smooth bounded domain, $\lambda^+(D)$ is the principal half-eigenvalue of the operator F on D that corresponds to positive eigenfunctions; see, for example, [29].

LEMMA 3.1. *Given a smooth bounded domain $D \subset E$, there exists $\eta_D \in C(\bar{D})$ such that $\eta_D > 0$ in D and satisfies in the viscosity sense the equation*

$$\begin{cases} F(x, D^2\eta_D) + \lambda^+(D)\eta_D = 0, & \text{in } D, \\ \eta_D = 0, & \text{on } \partial D. \end{cases} \tag{3.9}$$

Moreover, for any pair $(\lambda, \eta) \in \mathbb{R} \times C(\bar{D})$ with $\eta > 0$ in D which solves

$$\begin{cases} F(x, D^2\eta) + \lambda\eta = 0, & \text{in } D, \\ \eta = 0, & \text{on } \partial D, \end{cases} \tag{3.10}$$

(λ, η) must be of the form $(\lambda^+(D), \mu\eta_D)$ for some $\mu > 0$.

PROOF. Let us introduce some properties of F. By definition, we see that

$$F(x, \mu M) = \mu F(x, M) \qquad \text{for any } x \in E \text{ and } \mu \geq 0; \tag{3.11}$$

$$F \text{ is convex in } M. \tag{3.12}$$

Also, by [9], Lemma 2.10(5), for any $x \in E$ and $M, N \in \mathbb{S}^d$, we have

$$\tfrac{1}{2}\mathcal{M}^-_{\theta(x),\Theta(x)}(M - N) \leq F(x, M) - F(x, N) \leq \tfrac{1}{2}\mathcal{M}^+_{\theta(x),\Theta(x)}(M - N). \tag{3.13}$$

Finally, we observe from (3.1) that F can be expressed as

$$F(x,M)=\frac{1}{2}\mathcal{M}^{+}_{\theta(x),\Theta(x)}(M)=\frac{1}{2}\biggl\{\Theta(x)\sum_{e_i(M)>0}e_i(M)+\theta(x)\sum_{e_i(M)<0}e_i(M)\biggr\}.$$

From the continuity of θ and Θ in x, and the continuity of $e_i(M)$ in M for each i (see, e.g., [21], page 497), we conclude that

$$F \text{ is continuous in } E\times\mathbb{S}^d. \tag{3.14}$$

Now, thanks to (3.11)–(3.14) and [29], Lemma 1.1, this lemma follows from [29], Theorems 1.1, 1.2. □

3.1. *Regularity of η_D.* In this subsection, we will show that, for any smooth bounded domain $D\subset E$, the continuous viscosity solution η_D given in Lemma 3.1 is actually smooth up to the boundary ∂D.

Let us consider the operator $J:\bar{D}\times\mathbb{S}^d\mapsto\mathbb{R}$ defined by

$$J(x,M):=F(x,M)+\lambda^{+}(D)\eta_D(x).$$

LEMMA 3.2. *η_D belongs to $C^{0,\beta}(\bar{D})$, for any $\beta\in(0,1)$.*

PROOF. For any $x\in\bar{D}$ and $M,N\in\mathbb{S}^d$ with $M\geq N$, we deduce from (3.13) and (3.1) that

$$\begin{aligned}\frac{\theta_D}{2}\operatorname{Tr}(M-N)&\leq\frac{\theta(x)}{2}\operatorname{Tr}(M-N)=\frac{1}{2}\mathcal{M}^{-}_{\theta(x),\Theta(x)}(M-N)\\&\leq F(x,M)-F(x,N)\leq\frac{1}{2}\mathcal{M}^{+}_{\theta(x),\Theta(x)}(M-N)\\&=\frac{\Theta(x)}{2}\operatorname{Tr}(M-N)\leq\frac{\Theta_D}{2}\operatorname{Tr}(M-N),\end{aligned}\tag{3.15}$$

where $\theta_D:=\min_{x\in\bar{D}}\theta(x)$ and $\Theta_D:=\max_{x\in\bar{D}}\Theta(x)$. On the other hand, recall that under Assumption 2.1, $\theta,\Theta\in C^{0,\alpha}(\bar{D})$. Let K be a Hölder constant for both θ and Θ on $\bar{D}$. By (3.2) and (3.1), for any $x,y\in\bar{D}$ and $M\in\mathbb{S}^d$,

$$\begin{aligned}&\bigl|F(x,M)-F(y,M)\bigr|\\&\quad\leq\frac{1}{2}\biggl\{\bigl|\Theta(x)-\Theta(y)\bigr|\sum_{e_i(M)>0}e_i(M)+\bigl|\theta(x)-\theta(y)\bigr|\sum_{e_i(M)<0}\bigl|e_i(M)\bigr|\biggr\}\\&\quad\leq Kd\|M\||x-y|^{\alpha}.\end{aligned}\tag{3.16}$$

Under (3.11), (3.15) and (3.16), [4], Proposition 6, states that every bounded non-negative viscosity solution to

$$J\bigl(x,D^2\eta\bigr)=0 \text{ in } D,\qquad \eta=0 \text{ on } \partial D \tag{3.17}$$

is of the class $C^{0,\beta}(\bar{D})$ for all $\beta \in (0,1)$. Thanks to Lemma 3.1, η_D is indeed a bounded nonnegative viscosity solution to the above equation, and thus the lemma follows. □

LEMMA 3.3. *η_D is the unique continuous viscosity solution to* (3.17).

PROOF. By Lemma 3.1, we immediately have the viscosity solution property. To prove the uniqueness, it suffices to show that a comparison principle holds for $J(x, D^2\eta) = 0$. For any $x \in \bar{D}$ and $M, N \in \mathbb{S}^d$ with $M \geq N$, we see from the definition of J and (3.15) that

$$\frac{\theta_D}{2}\operatorname{Tr}(M-N) \leq J(x,M) - J(x,N) \leq \frac{\Theta_D}{2}\operatorname{Tr}(M-N). \tag{3.18}$$

Thanks to this inequality, we conclude from [17], Theorem 2.6, that a comparison principle holds for $J(x, D^2\eta) = 0$. □

The following regularity result is taken from [30], Theorem 1.2.

LEMMA 3.4. *Suppose $H : D \times \mathbb{S}^d \mapsto \mathbb{R}$ satisfies the following conditions*:

(a) *H is lower convex in $M \in \mathbb{S}^d$*;
(b) *there is a $\nu \in (0,1]$ s.t. $\nu|\xi|^2 \leq H(x, M + \xi\xi') - H(x, M) \leq \nu^{-1}|\xi|^2$ for all $\xi \in \mathbb{R}^d$*;
(c) *there is a $K_1 > 0$ s.t. $|H(x, 0)| \leq K_1$ for all $x \in D$*;
(d) *there are $K_2, K_3 > 0$ and $\beta \in (0,1)$ s.t. $\langle H(\cdot, M)\rangle_D^{(\beta)} \leq K_2 \sum_{i,j} |m_{ij}| + K_3$ for all $M = \{m_{ij}\}_{i,j} \in \mathbb{S}^d$, where $\langle u\rangle_D^{(\beta)} := \sup_{x\in D,\rho>0} \rho^{-\beta} \operatorname{osc}_{D\cap B_\rho(x)} u$, for any $u : D \mapsto \mathbb{R}$.*

Then

$$H(x, D^2\eta) = 0 \text{ in } D, \qquad \eta = 0 \text{ on } \partial D$$

has a unique solution in the class $C^{2,\beta}(\bar{D})$ if $\beta \in (0, \bar{\alpha})$, where the constant $\bar{\alpha} \in (0,1)$ depends only on d and ν.

PROPOSITION 3.1. *η_D belongs to $C^{2,\beta}(\bar{D})$ for any $\beta \in (0, \alpha \wedge \bar{\alpha})$, where $\bar{\alpha}$ is given in Lemma 3.4. This in particular implies $\lambda^+(D) = \lambda^*(D)$, and thus we have*

$$\begin{cases} F(x, D^2\eta_D) + \lambda^*(D)\eta_D = 0, & \text{in } D, \\ \eta_D = 0, & \text{on } \partial D. \end{cases} \tag{3.19}$$

PROOF. Let us show that the operator J satisfies conditions (a)–(d) in Lemma 3.4. It is obvious from (3.12) that J satisfies (a). Since $\xi\xi' \geq 0$ and $\operatorname{Tr}(\xi\xi') = |\xi|^2$ for all $\xi \in \mathbb{R}^d$, we see from (3.18) that J satisfies (b). By the continuity of η_D on $\bar{D}$, (c) is also satisfied as $|J(x,0)| = 0 + \lambda^+(D)\eta_D(x) \leq K_1 :=$

$\lambda^+(D)\max_{\bar{D}}\eta_D$. To prove (d), let us first observe that: for any $\beta\in(0,1)$ and $u\in C^{0,\beta}(D)$ with a Hölder constant K, we have $\operatorname{osc}_{D\cap B_\rho(x)} u\le K\rho^\beta$, which yields $\langle u\rangle_D^{(\beta)}\le K$. Recall that $\theta,\Theta\in C^{0,\alpha}(D)$ (Assumption 2.1) and $\eta_D\in C^{0,\beta}(\bar{D})$ for all $\beta\in(0,1)$ (Lemma 3.2). Now, for any $\beta\in(0,\alpha\wedge\bar{\alpha})$, we have $\theta,\Theta,\eta_D\in C^{0,\beta}(D)$. Let K' be a Hölder constant for all the three functions. Then, from the definition of J, the calculation (3.16) and the fact that $\|M\|\le d\|M\|_{\max}\le d\sum_{i,j}|m_{ij}|$ for any $M=\{m_{ij}\}_{i,j}\in\mathbb{S}^d$, we conclude that $J(\cdot,M)\in C^{0,\beta}(D)$ with a Hölder constant $d^2(\sum_{i,j}|m_{ij}|)K'+\lambda^+(D)K'$. It follows that $\langle J(\cdot,M)\rangle_D^{(\beta)}\le d^2(\sum_{i,j}|m_{ij}|)K'+\lambda^+(D)K'$. Thus, (d) is satisfied for all $\beta\in(0,\alpha\wedge\bar{\alpha})$, with $K_2:=d^2K'$ and $K_3:=\lambda^+(D)K'$. Now, we conclude from Lemma 3.4 that there is a unique solution in $C^{2,\beta}(\bar{D})$ to (3.17) for all $\beta\in(0,\alpha\wedge\bar{\alpha})$. However, in view of Lemma 3.3, this unique $C^{2,\beta}(\bar{D})$ solution can only be η_D.

The fact that η_D is of the class $C^{2,\beta}(\bar{D})$ and solves (3.9) implies that $\lambda^+(D)\le\lambda^*(D)$. Since we have the opposite inequality just from the definitions of $\lambda^+(D)$ and $\lambda^*(D)$, we conclude that $\lambda^+(D)=\lambda^*(D)$. Then (3.9) becomes (3.19). $\square$

3.2. *Relation between $\lambda^*(D)$ and $\lambda^{*,c}(D)$.* In this subsection, we will show that $\lambda^*(D)=\inf_{c\in\mathcal{C}}\lambda^{*,c}(D)$ for any smooth bounded domain D.

Let us first state a maximum principle on small domains for the operator $G_\delta: E\times\mathbb{R}\times\mathbb{S}^d\mapsto\mathbb{R}$ defined by

$$G_\delta(x,u,M):=-F(x,-M)-\delta|u|=\tfrac{1}{2}\mathcal{M}^-_{\theta(x),\Theta(x)}(M)-\delta|u|,$$

where δ can be any nonnegative real number.

LEMMA 3.5. *For any smooth bounded domain $D\subset E$, there exists $\varepsilon_0>0$, depending on D, such that if a smooth bounded domain $U\subset D$ satisfies* $\mathrm{Leb}(U)<\varepsilon_0$, *then if $\eta\in C(\bar{U})$ is a viscosity solution to*

$$\begin{cases} G_\delta(x,\eta,D^2\eta)\le 0, & \text{in } U,\\ \eta\ge 0, & \text{on } \partial U,\end{cases}$$

then $\eta\ge 0$ in U.

PROOF. Consider the operator $\bar{F}: E\times\mathbb{R}\times\mathbb{S}^d\mapsto\mathbb{R}$ defined by $\bar{F}(x,u,M):=F(x,M)+\delta|u|$. For any $x\in E$, $u,v\in\mathbb{R}$ and $M,N\in\mathbb{S}^d$, we see from (3.13) that

$$\begin{aligned}\tfrac{1}{2}\mathcal{M}^-_{\theta(x),\Theta(x)}(M-N)-\delta|u-v|&\le\bar{F}(x,u,M)-\bar{F}(x,v,N)\\ &\le\tfrac{1}{2}\mathcal{M}^+_{\theta(x),\Theta(x)}(M-N)+\delta|u-v|.\end{aligned}\tag{3.20}$$

Moreover, by (3.14), we immediately have

$$\bar{F}(x,0,M)=F(x,M)\text{ is continuous in }E\times\mathbb{S}^d.\tag{3.21}$$

Noting that $G_\delta(x,u,M) = -\bar{F}(x,-u,-M)$, we have $G_\delta(x,u,M) - G_\delta(x,v,N) = \bar{F}(x,-v,-N) - \bar{F}(x,-u,-M)$. Then, by using (3.20), we get

$$
\begin{aligned}
G_\delta(x,u-v,M-N) &= \tfrac{1}{2}\mathcal{M}^-_{\theta(x),\Theta(x)}(M-N) - \delta|u-v| \\
&\le G_\delta(x,u,M) - G_\delta(x,v,N) \\
&\le \tfrac{1}{2}\mathcal{M}^+_{\theta(x),\Theta(x)}(M-N) + \delta|u-v| \\
&= \bar{F}(x,u-v,M-N),
\end{aligned}
\tag{3.22}
$$

which implies that the operator G_δ satisfies the (D_F) condition in [29], page 107 (with F replaced by $\bar{F}$). Now, thanks to (3.20)–(3.22), this lemma follows from [29], Theorem 3.5. □

PROPOSITION 3.2. *For any smooth bounded domain $D \subset E$, $\lambda^*(D) \le \inf_{c\in\mathcal{C}} \lambda^{*,c}(D)$.*

PROOF. Assume the contrary that $\lambda^*(D) > \inf_{c\in\mathcal{C}} \lambda^{*,c}(D)$. Then there exists $\bar{c} \in \mathcal{C}$ such that $\lambda^*(D) > \lambda^{*,\bar{c}}(D)$. Take $\bar{\eta} \in C^2(D)$ with $\bar{\eta} > 0$ in D such that

$$
\begin{cases}
L^{\bar{c}(\cdot)}\bar{\eta} + \lambda^{*,\bar{c}}(D)\bar{\eta} = 0, & \text{in } D,\\
\bar{\eta} = 0, & \text{on } \partial D.
\end{cases}
$$

From the definition of F, we see that $\bar{\eta}$ is a viscosity subsolution to

$$
F(x, D^2\eta) + \lambda^{*,\bar{c}}(D)\eta = 0 \qquad \text{in } D. \tag{3.23}
$$

On the other hand, the function η_D, given in Lemma 3.1, is a viscosity supersolution to (3.23) as it solves (3.19) and $\lambda^*(D) > \lambda^{*,\bar{c}}(D)$. We claim that there exists $\ell > 0$ such that $\bar{\eta} \le \ell\eta_D$ in D. We will show this by following an argument used in the proof of Theorem 4.1 in [29]. Take a compact subset K of D such that $\mathrm{Leb}(D \setminus K) < \varepsilon_0$, where ε_0 is given in Lemma 3.5. By the continuity of $\bar{\eta}$ and η_D, there exists $\ell > 0$ such that $\ell\eta_D - \bar{\eta} > 0$ on K. Consider the function $f_\ell := \ell\eta_D - \bar{\eta}$. By (3.13) and (3.11),

$$
\begin{aligned}
G_{\lambda^{*,\bar{c}}(D)}(x, f_\ell, D^2 f_\ell) &= -F(x, -D^2 f_\ell) - \lambda^{*,\bar{c}}(D)|f_\ell| \\
&\le -F(x, -D^2 f_\ell) + \lambda^{*,\bar{c}}(D) f_\ell \\
&\le \ell F(x, D^2\eta_D) - F(x, D^2\bar{\eta}) + \lambda^{*,\bar{c}}(D)(\ell\eta_D - \bar{\eta}) \\
&\le 0 \qquad \text{in } D,
\end{aligned}
$$

where the last inequality follows from the supersolution property of η_D and the subsolution property of $\bar{\eta}$ to (3.23). Since $f_\ell \ge 0$ on $\partial(D \setminus K)$, we obtain from Lemma 3.5 that $f_\ell \ge 0$ on $D \setminus K$. Thus, we conclude that $\bar{\eta} \le \ell\eta_D$ in D. Now, by Perron's method we can construct a continuous viscosity solution v to (3.23) on D such that $\bar{\eta} \le v \le \ell\eta_D$. This in particular implies $v > 0$ in D and the pair

$(\lambda^{*,\bar{c}}(D), v)$ solves (3.10). Recalling that $\lambda^+(D) = \lambda^*(D)$ from Proposition 3.1, we see that this is a contradiction to Lemma 3.1 as $\lambda^{*,\bar{c}}(D) < \lambda^*(D) = \lambda^+(D)$. □

To prove the opposite inequality $\lambda^*(D) \geq \inf_{c\in\mathcal{C}} \lambda^{*,c}(D)$ for any smooth bounded domain $D \subset E$, we will make use of the theory of continuous selection pioneered by [22], and follow particularly the formulation in [7]. For a brief introduction to this theory and its adaptation to the current context, see Appendix A.

PROPOSITION 3.3. *Let $D \subset E$ be a smooth bounded domain. If D is convex, then $\lambda^*(D) \geq \inf_{c\in\mathcal{C}} \lambda^{*,c}(D)$.*

PROOF. We will construct a sequence $\{\bar{c}'_m\}_{m\in\mathbb{N}} \subset \mathcal{C}$ such that

$$\limsup_{m\to\infty} \lambda^{*,\bar{c}'_m}(D) \leq \lambda^*(D),$$

which gives the desired result.

Step 1: *Constructing* $\{\bar{c}'_m\}_{m\in\mathbb{N}}$. Recall that $\eta_D \in C^2(\bar{D})$ by Proposition 3.1. Then, we deduce from (3.1) that there exists $\kappa > 0$ such that

$$\begin{aligned} &\max\{|\lambda - \lambda'|, |\Lambda - \Lambda'|\} < \kappa \\ &\quad \Rightarrow \quad \bigl|\mathcal{M}^+_{\lambda,\Lambda}\bigl(D^2\eta_D(x)\bigr) - \mathcal{M}^+_{\lambda',\Lambda'}\bigl(D^2\eta_D(x)\bigr)\bigr| < 2/m \qquad \text{for all } x \in \bar{D}. \end{aligned} \tag{3.24}$$

Also, since $\|\cdot\|_{\max} \leq \|\cdot\|$, the map $(M, x) \mapsto L^M\eta_D(x)$ is continuous in M, uniformly in $x \in \bar{D}$. It follows that there exists $\beta > 0$ such that

$$\|N - M\| < \beta \quad \Rightarrow \quad \bigl|L^N\eta_D(x) - L^M\eta_D(x)\bigr| < 1/m \qquad \text{for all } x \in \bar{D}. \tag{3.25}$$

Set $\xi := \min_{x\in\bar{D}}(\Theta - \theta)(x) > 0$ (recall that $\Theta > \theta$ in E under Assumption 2.1). Now, by taking $\gamma := \theta + \frac{\kappa\wedge\xi}{4}$ and $\Gamma := \Theta - \frac{\kappa\wedge\xi}{4}$ in Proposition A.3, we obtain that there is a continuous function $c_m : \bar{D} \mapsto \mathbb{S}^d$ such that

$$\begin{aligned} &c_m(x) \in \mathcal{A}\bigl(\gamma(x), \Gamma(x)\bigr) \quad \text{and} \\ &F_{\gamma,\Gamma}\bigl(x, D^2\eta_D\bigr) \leq L^{c_m(\cdot)}\eta_D(x) + 1/m \qquad \text{for all } x \in \bar{D}, \end{aligned} \tag{3.26}$$

where $F_{\gamma,\Gamma}(x, M)$ is defined in (A.2). By mollifying the function c_m, we can construct a function $\bar{c}_m : \bar{D} \mapsto \mathbb{S}^d$ such that $\bar{c}_m \in C^\infty(\bar{D})$ and $\|\bar{c}_m(x) - c_m(x)\|_{\max} < (\beta \wedge \frac{\kappa\wedge\xi}{4})/d$ for all $x \in \bar{D}$ (more precisely, $c_m \in C(\bar{D})$ implies that for any open set D' containing $\bar{D}$, there is a function $\tilde{c}_m \in C(D')$ such that $\tilde{c}_m = c_m$ on $\bar{D}$; see, e.g., [15], Lemma 6.37. Then by mollifying $\tilde{c}_m$, we get a sequence of smooth functions converging uniformly to $\tilde{c}_m$ on $\bar{D}$). It follows that

$$\begin{aligned} \bigl\|\bar{c}_m(x) - c_m(x)\bigr\| &\leq d\bigl\|\bar{c}_m(x) - c_m(x)\bigr\|_{\max} \\ &< \beta \wedge \frac{\kappa \wedge \xi}{4} \qquad \text{for all } x \in \bar{D}. \end{aligned} \tag{3.27}$$

Combining (3.24)–(3.27), for each $x \in \bar{D}$, we see that $\bar{c}_m(x) \in \mathcal{A}(\theta(x), \Theta(x))$ and

$$\begin{aligned} F(x, D^2\eta_D) &= \frac{1}{2}\mathcal{M}^+_{\theta(x),\Theta(x)}(D^2\eta_D(x)) < \frac{1}{2}\mathcal{M}^+_{\gamma(x),\Gamma(x)}(D^2\eta_D(x)) + \frac{1}{m} \\ &= F_{\gamma,\Gamma}(x, D^2\eta_D) + \frac{1}{m} \le L^{c_m(\cdot)}\eta_D(x) + \frac{2}{m} \\ &\le L^{\bar{c}_m(\cdot)}\eta_D(x) + \frac{3}{m}. \end{aligned} \tag{3.28}$$

Now, take some $\bar{c}'_m \in \mathcal{C}$ such that $\bar{c}'_m$ and $\bar{c}_m$ coincide on $\bar{D}$. Then (3.28) and the fact that $F(x, D^2\eta_D) + \lambda^*(D)\eta_D = 0$ in D (Proposition 3.1) imply

$$|h_m| < 3/m \text{ in } D \qquad \text{where } h_m := L^{\bar{c}'_m(\cdot)}\eta_D + \lambda^*(D)\eta_D. \tag{3.29}$$

Step 2: *Showing* $\limsup_{m\to\infty} \lambda^{*,\bar{c}'_m}(D) \le \lambda^*(D)$. In the following, we will use the argument in [14], Section 3, starting from (3.3). Let η_m be the eigenfunction associated with the eigenvalue problem

$$\begin{cases} L^{\bar{c}'_m(\cdot)}\eta + \lambda^{*,\bar{c}'_m}(D)\eta = 0, & \text{in } D, \\ \eta = 0, & \text{on } \partial D. \end{cases}$$

Pick $x_0 \in D$. We define the normalized eigenfunction $\tilde{\eta}_m := \frac{\eta_D(x_0)}{\eta_m(x_0)}\eta_m$. By [27], lemma on page 789, there exist $k_1, k_2 > 0$, independent of m, such that

$$k_1 d(x, \partial D) \le \tilde{\eta}_m(x) \le k_2 d(x, \partial D) \qquad \text{for all } x \in D. \tag{3.30}$$

Also, thanks to (3.11) and (3.15), we may apply [4], Proposition 1, and obtain some $\delta > 0$ and $C > 0$ such that $\eta_D(x) \le C d(x, \partial D)$ if $d(x, \partial D) < \delta$. Thus, we conclude that

$$1 \le t_m := \sup_{x\in D} \frac{\eta_D(x)}{\tilde{\eta}_m(x)} < \infty. \tag{3.31}$$

By setting $s_m := t_m\lambda^*(D)/\lambda^{*,\bar{c}'_m}(D)$, we deduce from the definitions of t_m and s_m that

$$L^{\bar{c}'_m(\cdot)}(s_m\tilde{\eta}_m - \eta_D) + h_m = -t_m\lambda^*(D)\tilde{\eta}_m + \lambda^*(D)\eta_D \le 0 \qquad \text{in } D. \tag{3.32}$$

Let w_m be the unique solution of the class $C^{2,\alpha}(D) \cap C(\bar{D})$ to the equation

$$L^{\bar{c}'_m(\cdot)}w_m = h_m \text{ in } D, \qquad w_m = 0 \text{ on } \partial D. \tag{3.33}$$

Note that by [14], Remark 3.1, the convexity of D and (3.29) guarantee the existence of a constant $M > 0$, independent of m, such that

$$|w_m(x)| \le \frac{M d(x, \partial D)}{m} \qquad \text{for all } x \in D. \tag{3.34}$$

Combining (3.32) and (3.33), we get

$$\begin{cases} L^{\bar{c}'_m(\cdot)}(s_m\tilde{\eta}_m - \eta_D + w_m) \le 0, & \text{in } D, \\ s_m\tilde{\eta}_m - \eta_D + w_m = 0, & \text{on } \partial D. \end{cases}$$

We then conclude from the maximum principle that $s_m\tilde{\eta}_m - \eta_D + w_m \ge 0$ in D. From the definition of s_m, this inequality gives

$$\frac{\lambda^*(D)}{\lambda^{*,\bar{c}'_m}(D)} \ge \frac{\eta_D(x)}{t_m\tilde{\eta}_m(x)} - \frac{w_m(x)}{t_m\tilde{\eta}_m(x)} \ge \frac{\eta_D(x)}{t_m\tilde{\eta}_m(x)} - \frac{M}{k_1 m} \qquad \text{for all } x \in D,$$

where the last inequality follows from (3.34), (3.31) and (3.30). Now, take a sequence $\{x_k\}_{k\in\mathbb{N}}$ in D such that $\frac{\eta_D(x_k)}{\eta_m(x_k)} \to t_m$. By plugging x_k into the above inequality and taking limit in k, we get

$$\frac{\lambda^*(D)}{\lambda^{*,\bar{c}'_m}(D)} \ge 1 - \frac{M}{k_1 m},$$

which implies $\lambda^*(D) \ge \limsup_{m\to\infty} \lambda^{*,\bar{c}'_m}(D)$. $\square$

Combining Propositions 3.2 and 3.3, we have the following result:

THEOREM 3.1. *Let $D \subset E$ be a smooth bounded domain. If D is convex, $\lambda^*(D) = \inf_{c\in\mathcal{C}} \lambda^{*,c}(D)$.*

3.3. *Relation between $\lambda^*(E)$ and $\lambda^{*,c}(E)$.* In this subsection, we will first characterize $\lambda^*(E)$ in terms of $\lambda^*(E_n)$, and then generalize Theorem 3.1 from bounded domains to the entire space E.

Let us first consider some Harnack-type inequalities. Note that for any $D \subset \mathbb{R}^d$ and $p \in [1,\infty)$, we will denote by $\mathcal{L}^p(D)$ the space of measurable functions f satisfying $(\int_D |f(x)|^p\,dx)^{1/p} < \infty$.

LEMMA 3.6. *Let $D \subset E$ be a smooth bounded domain. Let $H : E \times \mathbb{S}^d \mapsto \mathbb{R}$ be such that*

$$\exists 0 < \lambda \le \Lambda \quad s.t. \quad \mathcal{M}^-_{\lambda,\Lambda}(M) \le H(x,M) \le \mathcal{M}^+_{\lambda,\Lambda}(M) \quad \text{for all } (x,M) \in D \times \mathbb{S}^d. \tag{3.35}$$

If $\{u_n\}_{n\in\mathbb{N}}$ is sequence of continuous nonnegative viscosity solutions to

$$H(x, D^2u_n) + \delta_n u_n = f_n \qquad \text{in } D, \tag{3.36}$$

where $\{\delta_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $[0,\infty)$ and $f_n \in \mathcal{L}^d(D)$, then we have:

(i) *for any compact set $K \subset D$, there is a constant $C > 0$, depending only on D, K, d, λ, Λ, $\sup_n \delta_n$, such that*

$$\sup_K u_n \le C\Big\{\inf_K u_n + \|f_n\|_{\mathcal{L}^d(D)}\Big\}. \tag{3.37}$$

(ii) *Suppose H satisfies* (3.11). *Given $x_0 \in D$ and $R_0 > 0$ such that $B_{R_0}(x_0) \subset D$, there exists a constant $C > 0$, depending only on $R_0, d, \lambda, \Lambda, \sup_n \delta_n$, such that for any $0 < R < R_0$,*

$$\sup_{\bar{B}_R(x_0)} u_n \le C\Big\{\inf_{\bar{B}_R(x_0)} u_n + R^2\|f_n\|_{\mathcal{L}^d(B_{R_0}(x_0))}\Big\}. \tag{3.38}$$

As a consequence, if we assume further that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded, and $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}^d(D)$, then for any compact connected set $K \subset D$ and $\beta \in (0,1)$, $u_n \in C^{0,\beta}(K)$ for all $n \in \mathbb{N}$, with one fixed Hölder constant.

PROOF. (i) Set $\delta^* := \sup_n \delta_n < \infty$. By (3.35), we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + \delta^* u_n \ge H(x, D^2u_n) + \delta_n u_n \ge \mathcal{M}^-_{\lambda,\Lambda}(D^2u_n) - \delta^* u_n \qquad \text{in } D.$$

In view of (3.36), we obtain $\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + \delta^* u_n \ge f_n \ge \mathcal{M}^-_{\lambda,\Lambda}(D^2u_n) - \delta^* u_n$ in D. Thanks to this inequality, estimate (3.37) follows from [29], Theorem 3.6.

(ii) Thanks to estimate (3.37) and [15], Lemma 8.23, we can prove part (ii) by following the argument in the proof of Corollary 3.2 in [5]. For a detailed proof, see Appendix B. $\square$

PROPOSITION 3.4. *$\lambda^*(E) =\downarrow \lim_{n\to\infty} \lambda^*(E_n)$ and there exists some $\eta^* \in H_{\lambda^*(E)}(E)$ such that*

$$F(x, D^2\eta^*) + \lambda^*(E)\eta^* = 0 \qquad \text{in } E. \tag{3.39}$$

PROOF. It is obvious from the definition that $\lambda^*(E_n)$ is decreasing in n and $\lambda^*(E) \le \lambda^*(E_n)$ for all $n \in \mathbb{N}$. It follows that $\lambda^*(E) \le \lambda_0 :=\downarrow \lim_{n\to\infty} \lambda^*(E_n)$. To prove the opposite inequality, it suffices to show that $H_{\lambda_0}(E) \ne \varnothing$. To this end, we take η_n as the eigenfunction given in Lemma 3.1 with $D = E_n$. Pick an arbitrary $x_0 \in E_1$, and define $\tilde{\eta}_n(x) := \frac{\eta_n(x)}{\eta_n(x_0)}$ such that $\tilde{\eta}_n(x_0) = 1$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. In view of Proposition 3.1, $\{\tilde{\eta}_m\}_{m>n}$ is a sequence of positive smooth solutions to

$$F(x, D^2\tilde{\eta}_m) + \lambda^*(E_m)\tilde{\eta}_m = 0 \qquad \text{in } E_{n+1}. \tag{3.40}$$

From the definition of F, we see that F satisfies (3.35) in E_n with $\lambda = \min_{x\in\bar{E}_n} \theta(x)$ and $\Lambda = \max_{x\in\bar{E}_n} \Theta(x)$. Thus, by Lemma 3.6(i), there is a constant $C > 0$, independent of m, such that

$$\sup_{\bar{E}_n} \tilde{\eta}_m \le C \inf_{\bar{E}_n} \tilde{\eta}_m \le C,$$

which implies $\{\tilde{\eta}_m\}_{m>n}$ is uniformly bounded in $\bar{E}_n$. On the other hand, given $\beta \in (0,1)$, Lemma 3.6(ii) guarantees that $\tilde{\eta}_m \in C^{0,\beta}(\bar{E}_n)$ for all $m > n$, with a fixed Hölder constant. Therefore, by using the Arzela–Ascoli theorem, we conclude that

$\tilde{\eta}_m$ converges uniformly, up to some subsequence, to some function η^* on $\bar{E}_n$. Thanks to the stability result of viscosity solutions (see, e.g., [13], Lemma II.6.2), we obtain from (3.40) that η^* is a nonnegative continuous viscosity solution in E_n to

$$F\big(x, D^2\eta^*\big) + \lambda_0\eta^* = 0. \tag{3.41}$$

Furthermore, since $\eta^*(x_0) = \lim_{m\to\infty} \eta_m(x_0) = 1$, we conclude from [4], Theorem 2, a strict maximum principle for eigenvalue problems of fully nonlinear operators, that $\eta^* > 0$ in E_n. Finally, noting that for any $\beta \in (0, 1)$, $\eta^* \in C^{0,\beta}(\bar{E}_n)$ with its Hölder constant same as $\tilde{\eta}_m$'s, we may use Lemma 3.4, as in the proof of Proposition 3.1, to show that $\eta^* \in C^2(\bar{E}_n)$.

Since the results above hold for each $n \in \mathbb{N}$, we conclude that η^* belongs to $C^2(E)$, takes positive values in E and satisfies (3.41) in E. It follows that $\eta^* \in H_{\lambda_0}(E)$, which yields $\lambda_0 \le \lambda^*(E)$. Therefore, we get $\lambda^*(E) = \lambda_0$, and then (3.41) becomes (3.39). $\square$

Now, we are ready to present the main technical result of this paper.

THEOREM 3.2. $\lambda^*(E) = \inf_{c\in\mathcal{C}} \lambda^{*,c}(E)$.

PROOF. Thanks to Theorem 4.4.1(i) in [26], Theorem 3.1 and Proposition 3.4, we have

$$\inf_{c\in\mathcal{C}} \lambda^{*,c}(E) = \inf_{c\in\mathcal{C}} \inf_{n\in\mathbb{N}} \lambda^{*,c}(E_n) = \inf_{n\in\mathbb{N}} \inf_{c\in\mathcal{C}} \lambda^{*,c}(E_n) = \inf_{n\in\mathbb{N}} \lambda^*(E_n) = \lambda^*(E). \qquad \square$$

REMARK 3.1. For the special case where θ and Θ are merely two positive constants, the derivation of Theorem 3.2 can be much simpler. Since the operator $F(x, M) = \frac{1}{2}\mathcal{M}^+_{\theta,\Theta}(M)$ is now Pucci's operator with elliptic constants θ and Θ, we may apply [5], Theorem 3.5, and obtain a positive Hölder continuous viscosity solution η^* to

$$F\big(x, D^2\eta^*\big) + \bar{\lambda}(E)\eta^* = 0 \qquad \text{in } E,$$

where $\bar{\lambda}(E) := \inf\{\lambda^+(D) \mid D \subset E \text{ is a smooth bounded domain}\}$. Then, Lemma 3.4 implies η^* is actually smooth, and thus $\bar{\lambda}(E) \le \lambda^*(E)$. Since $\bar{\lambda}(E) \ge \lambda^*(E)$ by definition, we conclude that $\bar{\lambda}(E) = \lambda^*(E)$. Now, thanks to [26], Theorem 4.4.1(i), and the standard result $\lambda^+(E_n) = \inf_{c\in\mathcal{C}} \lambda^{*,c}(E_n)$ for Pucci's operator (see, e.g., [8], Proposition 1.1(ii), and [27], Theorem I), we get

$$\begin{aligned}\inf_{c\in\mathcal{C}} \lambda^{*,c}(E) &= \inf_{c\in\mathcal{C}} \inf_{n\in\mathbb{N}} \lambda^{*,c}(E_n) = \inf_{n\in\mathbb{N}} \inf_{c\in\mathcal{C}} \lambda^{*,c}(E_n)\\ &= \inf_{n\in\mathbb{N}} \lambda^+(E_n) = \bar{\lambda}(E) = \lambda^*(E).\end{aligned}$$

However, as pointed out in [16], Discussion, it is not reasonable for financial applications to assume that each $c \in \mathcal{C}$ is both continuous and uniformly elliptic in E. Therefore, we consider in this paper the more general setting where θ and Θ are functions defined on E, which includes the case without uniform ellipticity.

3.4. *Application*. By Theorem 3.2 and mimicking the proof of Theorem 2.1 in [16], we have the following result. Note that, for simplicity, we will write $\lambda^* = \lambda^*(E)$.

THEOREM 3.3. *Take $\eta^* \in H_{\lambda^*}(E)$ and normalize it so that $\eta^*(x_0) = 1$. Define $\pi^*_t := e^{\lambda^* t}\nabla\eta^*(X_t)$ for all $t \geq 0$, and set*

$$\Pi^* := \Big\{\mathbb{P} \in \Pi \mid \mathbb{P}\text{-}\liminf_{t\to\infty}\big(t^{-1}\log\eta^*(X_t)\big) \geq 0, \mathbb{P}\text{-}a.s.\Big\}.$$

Then, we have $\pi^* \in \mathcal{V}$ and $g(\pi^*; \mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*$. Moreover,

$$\lambda^* = \sup_{\pi\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi^*} g(\pi; \mathbb{P}) = \inf_{\mathbb{P}\in\Pi^*} \sup_{\pi\in\mathcal{V}} g(\pi; \mathbb{P}). \tag{3.42}$$

PROOF. Set $V^*_t := V^{\pi^*}_t = 1 + \int_0^t e^{\lambda^* s}\nabla\eta^*(X_s)'\,dX_s$, $t \geq 0$. By applying Itô's rule to the process $e^{\lambda^* t}\eta^*(X_t)$ we see that $V^*_t \geq e^{\lambda^* t}\eta^*(X_t) > 0$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \Pi$. This already implies $\pi^* \in \mathcal{V}$. Also, by the construction of Π^*, we have $\mathbb{P}$-$\liminf_{t\to\infty}(t^{-1}\log(V^*_t)) \geq \lambda^*$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \Pi^*$. It follows that $g(\pi^*; \mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*$, which in turn implies $\lambda^* \leq \sup_{\pi\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi^*} g(\pi; \mathbb{P})$.

Now, for any $c \in \mathcal{C}$ and $n \in \mathbb{N}$, set $\lambda^{*,c}_n = \lambda^{*,c}(E_n)$, take $\eta^{*,c}_n \in H^c_{\lambda^{*,c}_n}(E_n)$ with $\eta^{*,c}_n(x_0) = 1$ and define the process $\tilde{V}^c_n(t) := e^{\lambda^{*,c}_n t}\eta^{*,c}_n(X_t)$. Note that under any $\mathbb{P} \in \Pi$ such that $\mathbb{P} \ll_{\text{loc}} \mathbb{Q}^c$, we have $\tilde{V}^c_n(t) = 1 + \int_0^t (\pi^{*,c}_n)'_s\,dX_s$ with $(\pi^{*,c}_n)_t := e^{\lambda^{*,c}_n t}\nabla\eta^{*,c}_n(X_t)$. This, however, may not be true for general $\mathbb{P} \in \Pi$. As shown in the proof of Theorem 2.1 in [16], for any fixed $c \in \mathcal{C}$ and $n \in \mathbb{N}$, we have the following: (1) there exists a solution $(\mathbb{P}^{*,c}_{x,n})_{x\in E_n}$ to the generalized martingale problem for the operator $L^{c(\cdot),\eta^{*,c}_n} := L^{c(\cdot)} + c\nabla\log\eta^{*,c}_n \cdot \nabla$; (2) the coordinate process X under $(\mathbb{P}^{*,c}_{x,n})_{x\in E_n}$ is recurrent in E_n; (3) $\mathbb{P}^{*,c}_{x,n} \ll_{\text{loc}} \mathbb{Q}^c$ (note that we conclude from the previous two conditions that $\mathbb{P}^{*,c}_{x,n} \in \Pi^*$); (4) the process $V^\pi/\tilde{V}^c_n$ is a nonnegative $\mathbb{P}^{*,c}_{x,n}$-supermartingale for all $\pi \in \mathcal{V}$. We therefore have the analogous result $g(\pi; \mathbb{P}^{*,c}_n) \leq g(\pi^{*,c}_n; \mathbb{P}^{*,c}_n) \leq \lambda^{*,c}_n$ for all $\pi \in \mathcal{V}$, which yields $\inf_{\mathbb{P}\in\Pi^*}\sup_{\pi\in\mathcal{V}} g(\pi; \mathbb{P}) \leq \lambda^{*,c}_n$. Now, thanks to Theorem 4.4.1(i) in [26] and Theorem 3.2, we have

$$\inf_{\mathbb{P}\in\Pi^*}\sup_{\pi\in\mathcal{V}} g(\pi; \mathbb{P}) \leq \inf_{c\in\mathcal{C}}\lim_{n\to\infty}\lambda^{*,c}_n = \lambda^*. \qquad \square$$

REMARK 3.2. Note that the normalized eigenfunction η^* in the statement of Theorem 3.3 may not be unique. It follows that the set of measures Π^* and the min–max problem in (3.42) may differ with our choice of η^*. In spite of this, we would like to emphasize the following:

(i) No matter which η^* we choose, the robust maximal asymptotic growth rate λ^* stays the same.

(ii) At the first glance, it may seem restrictive to work with Π^*. However, by the same calculation in [16], Remark 2.2, we see that: no matter which η^* we choose, Π^* is large enough to contain all the probabilities in Π under which X is tight in E, and thus corresponds to those $\mathbb{P} \in \Pi$ such that X is stable.

APPENDIX A: CONTINUOUS SELECTION RESULTS NEEDED FOR PROPOSITION 3.3

The goal of this Appendix is to state and prove Proposition A.3, which is used in the proof of Proposition 3.3. Before we do that, we need some preparations concerning the theory of continuous selection in [22] and [7].

DEFINITION A.1. Let X be a topological space.

(i) We say X is a T_1 space if for any distinct points $x, y \in X$, there exist open sets U_x and U_y such that U_x contains x but not y, and U_y contains y but not x.
(ii) We say X is a T_2(Hausdorff) space if for any distinct points $x, y \in X$, there exist open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \varnothing$.
(iii) We say X is a paracompact space if for any collection $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of open sets in X such that $\bigcup_{\alpha \in \mathcal{A}} X_\alpha = X$, there exists a collection $\{X_\beta\}_{\beta \in \mathcal{B}}$ of open sets in X satisfying:
    (1) each X_β is a subset of some X_α;
    (2) $\bigcup_{\beta \in \mathcal{B}} X_\beta = X$;
    (3) given $x \in X$, there exists an open neighborhood of x which intersects only finitely many elements in $\{X_\beta\}_{\beta \in \mathcal{B}}$.

DEFINITION A.2. Let X, Y be topological spaces. A set-valued map $\phi : X \mapsto 2^Y$ is lower semicontinuous if, whenever $V \subset Y$ is open in Y, the set $\{x \in X \mid \phi(x) \cap V \neq \varnothing\}$ is open in X.

The main theorem in [22], Theorem 3.2″, gives the following result for continuous selection.

PROPOSITION A.1. *Let X be a T_1 paracompact space, Y be a Banach space and $\phi : X \mapsto 2^Y$ be a set-valued map such that $\phi(x)$ is a closed convex subset of Y for each $x \in X$. Then, if ϕ is lower semicontinuous, there exists a continuous function $f : X \mapsto Y$ such that $f(x) \in \phi(x)$ for all $x \in X$.*

Since the lower semicontinuity of ϕ can be difficult to prove in general, one may wonder whether there is a weaker condition sufficient for continuous selection. Brown [7] worked toward this direction and characterized the weakest possible condition (it is therefore sufficient and necessary). For the special case where X is a Hausdorff paracompact space and Y is a real linear space with finite dimension n^*, given a set-valued map $\phi : X \mapsto 2^Y$, a sequence $\{\phi^{(n)}\}_{n\in\mathbb{N}}$ of set-valued maps was introduced in [7] via the following iteration:

$$\begin{aligned}
&\phi^{(1)}(x) := \{y \in \phi(x) \mid \text{Given } V \text{ open in } Y \text{ s.t. } y \in V,\\
&\qquad\qquad \text{there is a neighborhood } U \text{ of } x \text{ s.t. } \forall x' \in U, \exists y' \in \phi(x') \cap V\};\\
&\phi^{(n)}(x) := (\phi^{(n-1)})^{(1)}(x) \qquad \text{for } n \geq 2.
\end{aligned}\tag{A.1}$$

The following result, taken from [7], Theorem 4.3, characterizes the possibility of continuous selection using $\phi^{(n^*)}$.

PROPOSITION A.2. *Let X be a Hausdorff paracompact space, Y be a real linear space with finite dimension n^* and $\phi : X \mapsto 2^Y$ be a set-valued map such that $\phi(x)$ is a closed convex subset of Y for each $x \in X$. Then, there exists a continuous function $f : X \mapsto Y$ such that $f(x) \in \phi(x)$ for all $x \in X$ if and only if $\phi^{(n^*)}(x) \neq \varnothing$ for all $x \in X$.*

In this paper, we would like to take $X = \bar{D}$ and $Y = \mathbb{S}^d$, where $D \subset E$ is a smooth bounded domain. Note that $\bar{D}$ is Hausdorff and paracompact as it is a metric space in $\mathbb{R}^d$ (see, e.g., [18], Corollary 5.35), and $\mathbb{S}^d$ is a real linear space with dimension $n^* := d(d+1)/2$. Fix two continuous functions $\gamma, \Gamma : E \mapsto (0,\infty)$ with $\gamma \leq \Gamma$, we consider the operator $F_{\gamma,\Gamma} : E \times \mathbb{S}^d \mapsto \mathbb{R}$ defined by

$$F_{\gamma,\Gamma}(x, M) := \frac{1}{2}\mathcal{M}^+_{\gamma(x),\Gamma(x)}(M) = \frac{1}{2}\sup_{A\in\mathcal{A}(\gamma(x),\Gamma(x))} \operatorname{Tr}(AM).\tag{A.2}$$

Observe that $F_{\gamma,\Gamma}$ also satisfies (3.11)–(3.14), and in particular $F_{\theta,\Theta} = F$. Given $m \in \mathbb{N}$, we intend to show that there exists a continuous function $c_m : \bar{D} \mapsto \mathbb{S}^d$ such that for all $x \in \bar{D}$, $c_m(x) \in \mathcal{A}(\gamma(x), \Gamma(x))$ and $F_{\gamma,\Gamma}(x, D^2\eta_D) \leq L^{c_m(\cdot)}\eta_D(x) + 1/m$, with η_D given in Lemma 3.1. Note that since $\eta_D \in C^2(\bar{D})$ by Proposition 3.1, $D^2\eta_D$ is well defined on ∂D. Also, see Proposition 3.3 for the purpose of finding such a function c_m. We then define the set-valued map $\varphi : D \mapsto \mathbb{S}^d$ by

$$\begin{aligned}
\varphi(x) := \{M \in \mathbb{S}^d \mid\ & M \in \mathcal{A}(\gamma(x), \Gamma(x)) \text{ and}\\
& F_{\gamma,\Gamma}(x, D^2\eta_D) \leq L^M\eta_D(x) + 1/m\}.
\end{aligned}\tag{A.3}$$

For any $x \in \bar{D}$, we see from the definition of $F_{\gamma,\Gamma}$ that $\varphi(x) \neq \varnothing$. Moreover, $\varphi(x)$ is by definition a closed convex subset of $\mathbb{S}^d$. Then, we define $\varphi^{(n)}$ inductively as in (A.1) for all $n \in \mathbb{N}$. In view of Proposition A.2, such a function c_m exists if $\varphi^{(n^*)}(x) \neq \varnothing$ for all $x \in \bar{D}$. We claim that this is true. Actually, we will prove a stronger result in the next lemma: given $x \in \bar{D}$, $\varphi^{(n)}(x) \neq \varnothing$ for all $n \in \mathbb{N}$.

Recall that $B_\delta(x)$ denotes the open ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with radius $\delta > 0$. In the following, we will denote by $B^{\bar{D}}_\delta(x)$ the corresponding open ball in $\bar{D}$ under the relative topology, that is, $B^{\bar{D}}_\delta(x) := B_\delta(x) \cap \bar{D}$. Similarly, we will denote by $B_\delta^{\mathbb{S}^d}(M)$ the corresponding open ball in $\mathbb{S}^d$ under the topology induced by $\|\cdot\|$ in (2.1).

LEMMA A.1. *Fix a smooth bounded domain $D \subset E$, two continuous functions $\gamma, \Gamma : E \mapsto (0,\infty)$ with $\gamma \le \Gamma$, and $m \in \mathbb{N}$. Let η_D be given as in Lemma 3.1. Then, given $x \in \bar{D}$, if $M \in \varphi(x)$ satisfies*

$$F_{\gamma,\Gamma}(x, D^2\eta_D) < L^M \eta_D(x) + 1/m, \tag{A.4}$$

then $M \in \varphi^{(n)}(x)$ for all $n \in \mathbb{N}$.

PROOF. Fix $M \in \varphi(x)$ such that (A.4) holds. We will first show that $M \in \varphi^{(1)}(x)$, and then complete the proof by an induction argument. Take $0 \le \zeta < 1/m$ such that $F_{\gamma,\Gamma}(x, D^2\eta_D) = L^M\eta_D(x) + \zeta$. Set $\nu := 1/m - \zeta > 0$. Recall that $\eta_D \in C^2(\bar{D})$ from Proposition 3.1. By the continuity of the maps $x \mapsto F_{\gamma,\Gamma}(x, D^2\eta_D(x))$ [thanks to (3.14)] and $x \mapsto L^M\eta_D(x)$, we can take $\delta_1 > 0$ small enough such that the following holds for any $x' \in B_{\delta_1}^{\bar{D}}(x)$:

$$\begin{aligned} F_{\gamma,\Gamma}(x', D^2\eta_D) &< F_{\gamma,\Gamma}(x, D^2\eta_D) + \frac{\nu}{3} = L^M\eta_D(x) + \zeta + \frac{\nu}{3} \\ &< L^M\eta_D(x') + \zeta + \frac{2\nu}{3}. \end{aligned} \tag{A.5}$$

Since $\|\cdot\|_{\max} \le \|\cdot\|$, the map $(M, y) \mapsto L^M\eta_D(y)$ is continuous in M, uniformly in $y \in \bar{D}$. It follows that there exists $\beta > 0$ such that

$$\|N - M\| < \beta \quad \Rightarrow \quad \left|L^N\eta_D(y) - L^M\eta_D(y)\right| < \frac{\nu}{3} \qquad \text{for all } y \in \bar{D}. \tag{A.6}$$

Now, by the continuity of γ and Γ on $\bar{D}$, we can take $\delta_2 > 0$ such that $\max\{|\gamma(x') - \gamma(x)|, |\Gamma(x') - \Gamma(x)|\} < \beta$ for all $x' \in B_{\delta_2}^{\bar{D}}(x)$. For each $x' \in B_{\delta_2}^{\bar{D}}(x)$, we pick $M' \in \mathbb{S}^d$ satisfying

$$e_i(M') = \begin{cases} \gamma(x'), & \text{if } e_i(M) < \gamma(x'), \\ e_i(M), & \text{if } e_i(M) \in [\gamma(x'), \Gamma(x')], \\ \Gamma(x'), & \text{if } e_i(M) > \Gamma(x'). \end{cases}$$

By construction, $M' \in \mathcal{A}(\gamma(x'), \Gamma(x'))$ and $\|M' - M\| \le \max\{|\gamma(x') - \gamma(x)|, |\Gamma(x') - \Gamma(x)|\} < \beta$, which implies

$$\left|L^{M'}\eta_D(y) - L^M\eta_D(y)\right| < \frac{\nu}{3} \qquad \text{for all } y \in \bar{D}. \tag{A.7}$$

Finally, set $U := B_\delta^{\bar{D}}(x)$ with $\delta := \delta_1 \wedge \delta_2$. Then by (A.5) and (A.7), for any $x' \in U$ there exists $M' \in B_\beta^{\mathbb{S}^d}(M)$ such that $M' \in \mathcal{A}(\gamma(x'), \Gamma(x'))$ and

$$F_{\gamma,\Gamma}(x', D^2\eta_D) < L^{M'}\eta_D(x') + 1/m, \tag{A.8}$$

which shows that $M' \in \varphi(x')$. Given any open set V in $\mathbb{S}^d$ such that $M \in V$, since we may take $\beta > 0$ in (A.6) small enough such that $B_\beta^{\mathbb{S}^d}(M) \subset V$, we conclude that $M' \in V$ also. It follows that $M \in \varphi^{(1)}(x)$.

Notice that what we have proved is the following result: for any $x \in \bar{D}$, if $M \in \varphi(x)$ satisfies (A.4), then $M \in \varphi^{(1)}(x)$. Since $M' \in \varphi(x')$ satisfies (A.8), the above result immediately gives $M' \in \varphi^{(1)}(x')$. We then obtain a stronger result: for any $x \in \bar{D}$, if $M \in \varphi(x)$ satisfies (A.4), then $M \in \varphi^{(2)}(x)$. But this stronger result, when applied again to $M' \in \varphi(x')$ satisfying (A.8), gives $M' \in \varphi^{(2)}(x')$. We, therefore, obtain that: for any $x \in \bar{D}$, if $M \in \varphi(x)$ satisfies (A.4), then $M \in \varphi^{(3)}(x)$. We can then argue inductively to conclude that $M \in \varphi^{(n)}(x)$ for all $n \in \mathbb{N}$. □

PROPOSITION A.3. *Fix a smooth bounded domain $D \subset E$ and two continuous functions $\gamma, \Gamma : E \mapsto (0, \infty)$ with $\gamma \leq \Gamma$. Let η_D be given as in Lemma 3.1. For any $m \in \mathbb{N}$, there exists a continuous function $c_m : \bar{D} \mapsto \mathbb{S}^d$ such that*

$$c_m(x) \in \mathcal{A}(\gamma(x), \Gamma(x)) \quad \text{and} \quad F_{\gamma,\Gamma}(x, D^2\eta_D) \leq L^{c_m(\cdot)}\eta_D(x) + 1/m \quad \text{for all } x \in \bar{D}.$$

PROOF. Fix $m \in \mathbb{N}$. As explained before Lemma A.1, $\bar{D}$ is a Hausdorff paracompact space, $\mathbb{S}^d$ is a real linear space with dimension $n^* := d(d+1)/2$ and $\varphi(x)$ is a closed convex subset of $\mathbb{S}^d$ for all $x \in \bar{D}$. For each $x \in \bar{D}$, by the definition of $F_{\gamma,\Gamma}$ in (A.2), we can always find some $M \in \varphi(x)$ satisfying (A.4). By Lemma A.1, this implies $\varphi^{(n)}(x) \neq \varnothing$ for all $n \in \mathbb{N}$. In particular, we have $\varphi^{(n^*)}(x) \neq \varnothing$ for all $x \in \bar{D}$. Then the desired result follows from Proposition A.2. □

APPENDIX B: PROOF OF LEMMA 3.6(ii)

PROOF OF (3.38). Pick $x_0 \in D$ and $R_0 > 0$ such that $B_{R_0}(x_0) \subset D$. For any $0 < R < R_0$, define

$$v_n(x) := u_n(x_0 + Rx) \quad \text{and} \quad \bar{H}(x, M) := H(x_0 + Rx, M).$$

Then we deduce from (3.11) and (3.36) that

$$\begin{aligned}\bar{H}(x, D^2v_n(x)) + R^2\delta_n v_n(x) &= H(x_0 + Rx, D^2v_n(x)) + R^2\delta_n v_n(x)\\ &= R^2 f_n(x_0 + Rx) \qquad \text{in } B_{R_0/R}(0).\end{aligned}$$

Since $\bar{H}(x, M)$ satisfies (3.35) in $B_{R_0/R}(0)$, we can apply the estimate (3.37) to v_n and get

$$\begin{aligned}\sup_{\bar{B}_R(x_0)} u_n = \sup_{\bar{B}_1(0)} v_n &\leq C\Big\{\inf_{\bar{B}_1(0)} v_n + R^2\|f_n\|_{\mathcal{L}^d(B_{R_0}(x_0))}\Big\}\\ &= C\Big\{\inf_{\bar{B}_R(x_0)} u_n + R^2\|f_n\|_{\mathcal{L}^d(B_{R_0}(x_0))}\Big\},\end{aligned}$$

where $C > 0$ depends only on $R_0, d, \lambda, \Lambda, \sup_n \delta_n$. $\square$

PROOF OF THE HÖLDER CONTINUITY. Now, fix a compact connected set $K \subset D$. Set $R_0 := \frac{1}{2}d(\partial K, \partial D) > 0$. By [20], Lemma 2, there exists some $k^* \in \mathbb{N}$ such that the set $K' := \{x \in \mathbb{R}^d \mid d(x, K) \le R_0\} \subset D$ has the following property: any two points in K' can be joined by a polygonal line of at most k^* segments which lie entirely in K'. Fix $x_0 \in K'$. By the definition of R_0, we have $B_{R_0}(x_0) \subset D$. For each $n \in \mathbb{N}$, we consider the nondecreasing function $w^n : (0, R_0] \mapsto \mathbb{R}$ defined by

$$w^n(R) := M_R^n - m_R^n \qquad \text{where } M_R^n := \max_{\bar{B}_R(x_0)} u_n, m_R^n := \min_{\bar{B}_R(x_0)} u_n.$$

For each $R \in (0, R_0]$, we obtain from (3.36) that $\{u_n - m_R^n\}_{n\in\mathbb{N}}$ is sequence of nonnegative continuous viscosity solution to

$$H(x, D^2(u_n - m_R^n)) + \delta_n(u_n - m_R^n) = f_n - \delta_n m_R^n \qquad \text{in } B_R(x_0).$$

By the estimate (3.38), there is a constant $C > 0$, independent of n and x_0, such that

$$\begin{aligned} M_{R/4}^n - m_R^n &= \sup_{\bar{B}_{R/4}(x_0)} (u_n(x) - m_R^n) \le C \inf_{\bar{B}_{R/4}(x_0)} (u_n(x) - m_R^n) + AR^2 \\ &= C(m_{R/4}^n - m_R^n) + AR^2, \end{aligned} \tag{B.1}$$

where $A > 0$ is a constant that depends on C and R_0, but not n [thanks to the uniform boundedness of $\{u_n\}_{n\in\mathbb{N}}$ and the boundedness of $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}^d(D)$]. Define $\bar{H}(x, M) := -H(x, -M)$. Then we deduce again from (3.36) that $\{M_R^n - u_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative continuous viscosity solutions to

$$\begin{aligned} \bar{H}(x, D^2(M_R^n - u_n)) + \delta_n(M_R^n - u_n) &= -H(x, D^2 u_n) + \delta_n(M_R^n - u_n) \\ &= -f_n + \delta_n M_R^n \qquad \text{in } B_R(x_0). \end{aligned}$$

Observe that $\bar{H}$ also satisfies (3.11) and (3.35). Thus, we can apply estimate (3.38) and get

$$\begin{aligned} M_R^n - m_{R/4}^n &= \sup_{\bar{B}_{R/4}(x_0)} (M_R^n - u_n(x)) \le C \inf_{\bar{B}_{R/4}(x_0)} (M_R^n - u_n(x)) + AR^2 \\ &= C(M_R^n - M_{R/4}^n) + AR^2, \end{aligned} \tag{B.2}$$

where C and A are as above. Summing (B.1) and (B.2), we get

$$w^n(R/4) = M_{R/4}^n - m_{R/4}^n \le \frac{C-1}{C+1}(M_R^n - m_R^n) + A'R^2 = \frac{C-1}{C+1} w^n(R) + A'R^2,$$

where $A' > 0$ depends on C and R_0, and is independent of R and n. By applying [15], Lemma 8.23, to the above inequality, for any $\beta \in (0, 1)$, we can find some

$\tilde{C} > 0$ (depending on C, R_0 and A', but not n) such that $w^n(R) \le \tilde{C} R^\beta$, for all $R \le R_0$. This implies the following result: for any $x, y \in K'$ with $|x - y| \le R_0$, we can take $x_0 = x$ in the above analysis and obtain $|u_n(x) - u_n(y)| \le w^n(|x - y|) \le \tilde{C}|x - y|^\beta$ for all $n \in \mathbb{N}$. For the case where $|x - y| > R_0$, recall that x and y can be joined by a polygonal line of k segments which lie entirely in K', for some $k \le k^*$. On the jth segment, pick points $x_1^j, x_2^j, \ldots, x_{\ell_j}^j$ along the segment such that $x_1^j, x_{\ell_j}^j$ are the two endpoints, $|x_i^j - x_{i+1}^j| = R_0$ for $i = 1, \ldots, \ell_j - 2$ and $|x_{\ell_j - 1}^j - x_{\ell_j}^j| \le R_0$. Since K' is bounded, there must be a uniform bound $\ell^* > 0$ such that $\ell_j \le \ell^*$ for all j. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
|u_n(x) - u_n(y)| &\le \sum_{j=1}^{k} \sum_{i=1}^{\ell_j - 1} |u_n(x_i^j) - u_n(x_{i+1}^j)| \le \sum_{j=1}^{k} \sum_{i=1}^{\ell_j - 1} \tilde{C}|x_i^j - x_{i+1}^j|^\beta \\
&\le k^* \ell^* \tilde{C} |x - y|^\beta. \qquad \square
\end{aligned}$$

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