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KALIKOW-TYPE DECOMPOSITION FOR MULTICOLOR INFINITE RANGE PARTICLE SYSTEMS

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We consider a particle system on $\mathbb{Z}^d$ with real state space and interactions of infinite range. Assuming that the rate of change is continuous we obtain a Kalikow-type decomposition of the infinite range change rates as a mixture of finite range change rates. Furthermore, if a high noise condition holds, as an application of this decomposition, we design a feasible perfect simulation algorithm to sample from the stationary process. Finally, the perfect simulation scheme allows us to forge an algorithm to obtain an explicit construction of a coupling attaining Ornstein's $\bar{d}$-distance for two ordered Ising probability measures.

1. Introduction. In this paper we present a Kalikow-type decomposition for interacting multicolor systems on $\mathbb{Z}^d$ having real state space and interactions of infinite range. By a Kalikow-type decomposition we mean a representation of the infinite range rates as a countable mixture of local change rates of increasing range. This decomposition extends the notion of random Markov chains to interacting particle systems and has many potential theoretical consequences and applications. As a first example we present a perfect simulation algorithm which is based on the decomposition. As a corollary we obtain a result about the existence and uniqueness of the invariant measure of the system as well as a rate of convergence to stationarity. As a second application we construct a coupling attaining the $\bar{d}$-distance for two ordered Ising probability measures.

By a perfect simulation algorithm we mean a simulation which samples in a finite window precisely from the stationary law of the infinite process. More precisely, for any finite set of sites F we want to sample the projection of the stationary law on F. Our approach is feasible in the sense that it stops almost surely after a finite number of steps. It does not require any duality or monotonicity properties. We do not assume that the system has a dual, or is attractive, or monotone in any

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sense. Our system is not spatially homogeneous. The basic assumptions are the continuity of the infinite range change rates together with a high-noise condition [Condition (5.2): fast decay of the range influence on the change rate and a certain subcriticality-criterion].

Concerning possible applications, perfect simulation of infinite range continuous (or discrete) systems has shown to be an important tool, for instance, in statistical inference for Gibbs distributions, for Bayesian statistics, for maximum likelihood estimation, rates of convergence of estimators. This field of research has enormous relevance due to its applications in image processing, spatial statistics, gene expression, to cite just a few of the possible applications. Some of the applications use discrete state space whereas others deal with continuous state space, some deal with nearest neighbor interaction (image processing) whereas others need infinite range interactions (tumor growth, dynamics of populations of neurons or gene expression). We refer the reader to the books of Møller and Waagepetersen (2004), Gaetan and Guyon (2010) for some applications.

Let us stress that from an applied point of view, it is important to be able to deal with infinite range systems. In neuroscience, for example, for all practical purposes, the interactions between neurons have infinite range [see, e.g., Cessac et al. (2009) and Cessac, Nasser and Vasquez (2010)]. Also, very simple error structures superposed on finite range systems will produce infinite range systems. This is the case, for example, of blurred images [see, e.g., Nishimori and Wong (1999) and Tanaka (2002)]. Up to now, denoising of images has been studied mostly with very simple models (finite lattice, nearest neighbor interaction) due to the difficulty to sample from a Gibbs distribution with infinite range interactions and/or continuous state space.

So it is important to build up a unifying approach that enables us to deal with infinite range interactions and/or continuous state space at the same time. Although perfect simulation of Gibbs random fields are known for discrete state spaces and finite range interaction, very little is known for simulation of continuous state spaces and/or infinite range interaction. To the best of our knowledge ours is the first concrete result concerning perfect simulation for interacting particle systems with continuous state space and infinite range interactions.

This paper is organized as follows. The model and the Kalikow-type decomposition (Theorem 1) are presented in Section 2. The aim of Section 4 is to present some examples where the decomposition can be explicitly done. In particular, we apply Theorem 1 to the important case of Gibbs measures with infinite range interactions and continuous spin values. In Section 5 we present the perfect simulation algorithm as a main application of the Kalikow-type decomposition. In particular, Theorem 3 shows that the proposed algorithm is feasible under a high noise condition. The proofs are given in Sections 3 and 6. In Section 7 we give as a main application of the perfect simulation algorithm an explicit construction of a coupling attaining Ornstein's $\bar{d}$-distance for two ordered Ising probability measures. We present a section discussing the user impatient bias. We conclude the article with final comments and a bibliographical discussion.

2. Definitions, notation and convex decomposition. We consider interacting particle systems on $\mathbb{Z}^d$ having state space A and interactions of infinite range. The elements of the state space A are called *colors*. To each site in $\mathbb{Z}^d$ we assign a color. The coloring of the sites changes as time goes by. The rate at which the color of a fixed site i changes from a color a to a new color b is a function of the entire configuration and depends on b.

In what follows, we suppose that A is a Borel subset of $\mathbb{R}$, equipped with its Borel sigma-field $\mathcal{A}$. ϱ will be a finite nonnegative reference measure on $(A, \mathcal{A})$. The initial lowercase letters a, b, $c, \ldots$ will denote elements of A. We denote by $S = A^{\mathbb{Z}^d}$ the configuration space with its product sigma algebra, $\mathcal{S}$. We call an element of S a configuration. Configurations will be denoted by Greek letters $\eta, \zeta, \xi, \ldots$. A point $i \in \mathbb{Z}^d$ will be called a site. We define on $\mathbb{Z}^d$ the L^1 norm, $\|i\| = \sum_{k=1}^{d} |i_k|$. For $k \geq 0$, let the ball of radius k be

$$V_i(k) = \{j \in \mathbb{Z}^d; \|j - i\| \leq k\}.$$

As usual, for any $i \in \mathbb{Z}^d$, $\eta(i)$ will denote the value of the configuration η at site i. By extension, for any subset $V \subset \mathbb{Z}^d$, $\eta(V) \in A^V$ will denote the restriction of the configuration η to the set of positions in V. For any η, i and a, we shall denote $\eta^{i,a}$ the modified configuration

$$\eta^{i,a}(j) = \eta(j) \qquad \text{for all } j \neq i, \text{ and } \eta^{i,a}(i) = a.$$

For any $i \in \mathbb{Z}^d$, let $c_i(a, \eta)$ be a positive $\mathcal{A} \otimes \mathcal{S} - \mathcal{B}(\mathbb{R}_+)$-measurable function such that the following two properties hold. First, for ϱ-almost all $a \in A$, $\eta \mapsto c_i(a, \eta)$ is continuous. Second, we have

$$\sup_{i \in \mathbb{Z}^d} \sup_{\eta} \int_A c_i(a, \eta) \varrho(da) < \infty. \tag{2.1}$$

A multicolor system with interactions of infinite range is a Markov process on S whose generator is defined on cylinder functions by

$$\mathcal{G} f(\eta) = \sum_{i \in \mathbb{Z}^d} \int_A \varrho(da) c_i(a, \eta) \big[f\big(\eta^{i,a}\big) - f(\eta) \big], \tag{2.2}$$

where $f \in D(\mathcal{G}) = \{f : |||f||| = \sum_{i \in \mathbb{Z}^d} \Delta_f(i) < \infty\}$ with $\Delta_f(i) = \sup\{|f(\eta) - f(\zeta)| : \eta(j) = \zeta(j) \text{ for all } j \neq i\}$.

By Theorem 3.9 of Chapter 1 of Liggett (1985) the following condition, together with (2.1), implies that $\mathcal{G}$ is the generator of a Feller process (σ_t) on S:

$$\sup_{i \in \mathbb{Z}^d} \sum_{j \neq i} \sup_{\eta} \sup_{b \in A} \left\{ \int_A \rho(da) |c_i(a, \eta) - c_i\big(a, \eta^{j,b}\big)| \right\} < \infty. \tag{2.3}$$

In the following we shall work under conditions stronger than (2.3) ensuring not only that $\mathcal{G}$ is the generator of a unique Feller process, but also the possibility of perfectly simulating the stationary distribution corresponding to this infinitesimal

generator. As a byproduct this implies that the system admits the existence of a unique invariant measure μ.

The main result of this article is a Kalikow-type convex decomposition of the change rates. We will prove that the change rate can be decomposed as

$$c_i(a,\eta) = M_i\Big[\lambda_i(-1)\,p_i^{[-1]}(a) + \sum_{k\geq 0}\lambda_i(k)\,p_i^{[k]}\big(a|\eta(V_i(k))\big)\Big], \tag{2.4}$$

where:

- $M_i, i \in \mathbb{Z}^d$ are positive constants,
- for each $i \in \mathbb{Z}^d$, $\{\lambda_i(k), k \geq -1\}$ is a probability distribution on $\{-1, 0, 1, 2, \ldots\}$,
- for each $i \in \mathbb{Z}^d$, $p_i^{[-1]}(\cdot)$ is a probability density on A with respect to the reference measure ϱ, which does not depend on the configuration,
- for each $k \geq 0$ and for each $\eta \in S$, $p_i^{[k]}(\cdot|\eta(V_i(k)))$ is a probability density with respect to the reference measure ϱ, depending only on the local configuration $\eta(V_i(k))$.

For convenience of the presentation we will add additional invisible jumps in (2.2). This is obtained by adding a cemetery Δ to A and defining $A^* := A \cup \{\Delta\}$. Define also

$$\varrho^* := \varrho + \delta_\Delta.$$

Denote

$$M_i := \sup_{\eta\in A^{\mathbb{Z}^d}} \int c_i(a,\eta)\varrho(da). \tag{2.5}$$

Notice that M_i is finite under condition (2.1), and define

$$c_i(\Delta,\eta) := M_i - \int_A c_i(a,\eta)\varrho(da). \tag{2.6}$$

Observe that

$$\inf_\eta c_i(\Delta,\eta) = 0. \tag{2.7}$$

Therefore we can rewrite the generator given by (2.2) as

$$\mathcal{G}f(\eta) = \sum_{i\in\mathbb{Z}^d}\int_{A^*}\varrho^*(da)c_i(a,\eta)\big[f\big(\eta^{i,a}\big) - f(\eta)\big], \tag{2.8}$$

where, by convention, for any $i \in \mathbb{Z}^d$ and any $\eta \in S = A^{\mathbb{Z}^d}$, we define

$$\eta^{i,\Delta} = \eta.$$

It follows that (2.8) is a representation of the same generator as (2.2).

In order to obtain the decomposition we need the following continuity condition.

Continuity condition.

$$\sup_{i\in\mathbb{Z}^d}\int_A \sup_{\eta,\zeta:\eta(V_i(k))=\zeta(V_i(k))} |c_i(a,\eta)-c_i(a,\zeta)|\varrho(da)\to 0 \tag{2.9}$$

as $k\to\infty$.

To describe the convex decomposition of the rate function c_i, we have to introduce the following quantities. Define

$$\alpha_i(-1)=\int_{A^*}\inf_{\zeta\in A^{\mathbb{Z}^d}} c_i(a,\zeta)\varrho^*(da), \tag{2.10}$$

and for any $k\geq 0$,

$$\alpha_i(k)=\inf_{w\in A^{V_i(k)}}\left(\int_{A^*}\inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho^*(da)\right). \tag{2.11}$$

The continuity of $c_i(a,\eta)$ in η and the separability of S imply the measurability of $\inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)$ and $\inf_{\zeta\in A^{\mathbb{Z}^d}} c_i(a,\zeta)$ with respect to a.

Note that by (2.7)

$$\int_{A^*}\inf_{\zeta\in A^{\mathbb{Z}^d}} c_i(a,\zeta)\varrho^*(da)=\int_A \inf_{\zeta\in A^{\mathbb{Z}^d}} c_i(a,\zeta)\varrho(da).$$

Further, by construction, we have that $\alpha_i(k)\leq\alpha_i(k+1)$, for each $k\geq -1$. We claim that

$$M_i=\lim_{k\to\infty}\alpha_i(k). \tag{2.12}$$

To obtain equality (2.12), fix some $w\in A^{V_i(k)}$; from (2.6), we have that

$$\begin{aligned}
&\int_{A^*}\inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho^*(da)\\
&\quad=\int_A \inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho^*(da)+\inf_{\zeta:\zeta(V_i(k))=w} c_i(\Delta,\zeta)\\
&\quad=\int_A \inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho(da)+M_i-\sup_{\zeta:\zeta(V_i(k))=w}\int_A c_i(a,\zeta)\varrho(da).
\end{aligned}$$

But

$$\int_A \inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho(da)-\sup_{\zeta:\zeta(V_i(k))=w}\int_A c_i(a,\zeta)\varrho(da)\to 0$$

as $k\to\infty$ thanks to condition (2.9).

Hence, to each site i we can associate a probability distribution λ_i by

$$\lambda_i(-1)=\frac{\alpha_i(-1)}{M_i}, \tag{2.13}$$

and for $k \geq 0$

$$\lambda_i(k) = \frac{\alpha_i(k) - \alpha_i(k-1)}{M_i}. \tag{2.14}$$

Now we are ready to state the decomposition theorem.

THEOREM 1. *Let $(c_i)_{i\in\mathbb{Z}^d}$ be a family of measurable rate functions satisfying conditions (2.1), (2.6) and (2.9). Then, for each site i, for M_i defined by (2.5) and $\lambda_i(\cdot)$ defined by (2.13) and (2.14), there exist*:

- *$p_i^{[-1]}$ a probability density with respect to ϱ with support A,*
- *a family of conditional probability densities $p_i^{[k]}$ [given by (3.4)], $k \geq 0$ on A^*, with respect to ϱ^*, depending on the local configurations $\eta(V_i(k)) \in A^{V_i(k)}$ such that*

$$c_i(a,\eta) = M_i\, p_i(a|\eta) \qquad \text{for } \varrho^*\text{-almost all } a \in A^*, \tag{2.15}$$

where

$$p_i(a|\eta) = \lambda_i(-1)\, p_i^{[-1]}(a) + \sum_{k\geq 0} \lambda_i(k)\, p_i^{[k]}\big(a|\eta\big(V_i(k)\big)\big). \tag{2.16}$$

As a consequence, the infinitesimal generator $\mathcal{G}$ given by (2.8) can be rewritten as

$$\begin{aligned}\mathcal{G}f(\eta) = \sum_{i\in\mathbb{Z}^d} M_i \Bigg[&\lambda_i(-1) \int_A p_i^{[-1]}(a)\big[f\big(\eta^{i,a}\big) - f(\eta)\big]\varrho(da) \\ &+ \sum_{k\geq 0} \lambda_i(k) \int_{A^*} p_i^{[k]}\big(a|\eta\big(V_i(k)\big)\big)\big[f\big(\eta^{i,a}\big) - f(\eta)\big]\varrho^*(da)\Bigg].\end{aligned} \tag{2.17}$$

Note that for $k = -1$, $p_i^{[-1]}(a)$ does not depend on the configuration and $\lambda_i(-1)$ represents the spontaneous self-coloring rate of site i in the process. We will see in the proof that $p_i^{[-1]}$ is defined in such way that $p_i^{[-1]}(\Delta) = 0$ and therefore, the choice $k = -1$ always implies a choice of a real color $a \in A$, not of $a = \Delta$.

The decomposition given in Theorem 1 was designed in such way that the probability of self-coloring is maximized. This is important to speed up the perfect simulation algorithm. Obviously, slight modifications can be employed for different purposes as we will see in Example 2 (Section 4).

The representation given by (2.17) provides a random finite range description of the time evolution of the process. We start with an initial configuration η at time zero. For each site $i \in \mathbb{Z}^d$, we consider a rate M_i Poisson point process N^i. The Poisson processes corresponding to distinct sites are all independent. If at time t, the Poisson clock associated to site i rings, we choose a range k with probability

$\lambda_i(k)$ independently of everything else. Then, we update the value of the configuration at this site by choosing a symbol a with probability $p_i^{[k]}(a|\sigma_t(V_i(k)))\varrho^*(da)$. Choosing the symbol Δ means that we actually keep the current value of the spin.

In Section 4 we give examples of infinite range interacting systems where Theorem 1 can be applied.

3. Proof of Theorem 1. Put for any $a \in A^*$,

$$c_i^{[-1]}(a) = \inf_{\zeta} c_i(a,\zeta),$$

$$\Delta_i^{[-1]}(a) = c_i^{[-1]}(a),$$

$$c_i^{[0]}\big(a|\eta(i)\big) = \inf_{\zeta:\zeta(i)=\eta(i)} c_i(a,\zeta),$$

$$\Delta_i^{[0]}\big(a|\eta(i)\big) = c_i^{[0]}\big(a|\eta(i)\big) - c_i^{[-1]}(a).$$

For any $k \geq 1$, define

$$c_i^{[k]}\big(a|\eta(V_i(k))\big) = \inf_{\zeta:\zeta(V_i(k))=\eta(V_i(k))} c_i(a,\zeta),$$

$$\Delta_i^{[k]}\big(a|\eta(V_i(k))\big) = c_i^{[k]}\big(a|\eta(V_i(k))\big) - c_i^{[k-1]}\big(a|\eta(V_i(k-1))\big).$$

Then we have that for any $a \in A$,

$$c_i(a,\eta) = \sum_{j=-1}^{k} \Delta_i^{[j]}\big(a|\eta(V_i(j))\big) + \big[c_i(a,\eta) - c_i^{[k]}\big(a|\eta(V_i(k))\big)\big]. \tag{3.1}$$

Note that

$$c_i^{[-1]}(\Delta) = \inf_{\eta} c_i(\Delta,\eta) = M_i - \sup_{\eta} \int_A c_i(a,\eta)\varrho(da) = 0.$$

Therefore, for $a = \Delta$ decomposition (3.1) starts with $j = 0$,

$$c_i(\Delta,\eta) = \sum_{j=0}^{k} \Delta_i^{[j]}\big(\Delta|\eta(V_i(j))\big) + \big[c_i(\Delta,\eta) - c_i^{[k]}\big(\Delta|\eta(V_i(k))\big)\big].$$

By monotonicity, we have for ϱ^*-almost all $a \in A^*$ that

$$c_i^{[k]}\big(a|\eta(V_i(k))\big) \to \lim_k c_i^{[k]}\big(a|\eta(V_i(k))\big) \leq c_i(a,\eta) \qquad \text{as } k \to \infty.$$

Hence, by monotone convergence,

$$\int_A c_i^{[k]}\big(a|\eta(V_i(k))\big)\varrho(da) \to \int_A \Big[\lim_k c_i^{[k]}\big(a|\eta(V_i(k))\big)\Big]\varrho(da) \leq \int_A c_i(a,\eta)\varrho(da).$$

On the other hand, by (2.9),

$$\int_A c_i^{[k]}\big(a|\eta(V_i(k))\big)\varrho(da) \to \int_A c_i(a,\eta)\varrho(da).$$

Hence, for ϱ^*-almost all a and all η,

$$\lim_k c_i^{[k]}(a|\eta(V_i(k))) = \sum_{j=-1}^{\infty} \Delta_i^{[j]}(a|\eta(V_i(j))) = c_i(a,\eta).$$

Taking into account (2.10) and (2.13),

$$M_i\lambda_i(-1) = \int_A \Delta_i^{[-1]}(a)\varrho(da).$$

Hence, we can define

$$p_i^{[-1]}(a) = \frac{\Delta_i^{[-1]}(a)}{M_i\lambda_i(-1)}$$

and

$$p_i^{[-1]}(\Delta) = 0.$$

Hence, $p_i^{[-1]}(a)$ is a probability density with respect to ϱ^*. Now, for $k \geq 0$, put

$$\tilde{\lambda}_i(k,\eta(V_i(k))) = \frac{1}{M_i}\int_{A^*} \Delta_i^{[k]}(a|\eta(V_i(k)))\varrho^*(da), \tag{3.2}$$

and for any i,k such that $\tilde{\lambda}_i(k,\eta(V_i(k))) > 0$, we define

$$\tilde{p}_i^{[k]}(a|\eta(V_i(k))) = \frac{\Delta_i^{[k]}(a|\eta(V_i(k)))}{M_i\tilde{\lambda}_i(k,\eta(V_i(k)))}.$$

For i,k such that $\tilde{\lambda}_i(k,\eta(V_i(k))) = 0$, define $\tilde{p}_i^{[k]}(a|\eta(V_i(k)))$ in an arbitrary fixed way.

Hence, for ϱ^*-almost all $a \in A^*$,

$$c_i(a,\eta) = M_i\left[\lambda_i(-1)p_i^{[-1]}(a) + \sum_{k=0}^{\infty}\tilde{\lambda}_i(k,\eta(V_i(k)))\tilde{p}_i^{[k]}(a|\eta(V_i(k)))\right]. \tag{3.3}$$

In (3.3) the factors $\tilde{\lambda}_i(k,\eta(V_i(k)))$, $k \geq 0$, still depend on $\eta(V_i(k))$. To obtain the decomposition as in the theorem, we must rewrite it as follows.

For any i, take M_i as in (2.12) and the sequences $\alpha_i(k), \lambda_i(k), k \geq -1$, as defined in (2.11) and (2.14), respectively. Define the new quantities

$$\alpha_i(k,\eta(V_i(k))) = M_i\sum_{l\leq k}\tilde{\lambda}_i(l,\eta(V_i(l)))$$

and notice that

$$\alpha_i(k,\eta(V_i(k))) = \int_{A^*} c_i^{[k]}(a,\eta(V_i(k)))\varrho^*(da)$$

is the total mass associated to $c_i^{[k]}(\cdot,\eta(V_i(k)))$.

By definition of $\alpha_i(k)$ in (2.11), $\alpha_i(k)$ is the smallest total mass associated to $c_i^{[k]}$, uniformly with respect to all possible neighborhoods $\eta(V_i(k))$. Hence, in order to get a decomposition with weights $\lambda_i(k)$ not depending on the configuration, we have to define a partition of the interval $[0, \alpha_i(k, \eta(V_i(k)))]$ according to the values of $\alpha_i(k)$.

This yields, for any $k \geq 0$, the following definition of the conditional finite range probability densities.

$$
\begin{aligned}
&p_i^{[k]}\big(a|\eta\big(V_i(k)\big)\big) \\
&\quad= \sum_{-1=l'\leq l}^{k-1} 1_{\{\alpha_i(l'-1,\eta(V_i(l'-1)))<\alpha_i(k-1)\leq\alpha_i(l',\eta(V_i(l')))\}} \\
&\qquad\times 1_{\{\alpha_i(l,\eta(V_i(l)))<\alpha_i(k)\leq\alpha_i(l+1\eta(V_i(l+1)))\}} \\
&\qquad\times\Bigg[\frac{\alpha_i(l',\eta(V_i(l'))) - \alpha_i(k-1)}{M_i\lambda_i(k)}\tilde{p}_i^{[l']}\big(a|\eta\big(V_i(l')\big)\big) \\
&\qquad\quad+\sum_{m=l'+1}^{l}\frac{\tilde{\lambda}_i(m,\eta(V_i(m)))}{M_i\lambda_i(k)}\tilde{p}_i^{[m]}\big(a|\eta\big(V_i(m)\big)\big) \\
&\qquad\quad+\frac{\alpha_i(k)-\alpha_i(l,\eta(V_i(l)))}{M_i\lambda_i(k)}\tilde{p}_i^{[l+1]}\big(a|\eta\big(V_i(l+1)\big)\big)\Bigg].
\end{aligned}
\tag{3.4}
$$

The desired decomposition now follows from this.

4. Examples. In this section we give examples where the decomposition of Theorem 1 can be applied. We start with an example from Bayesian statistics and image reconstruction.

EXAMPLE 1 (Autonormal distribution). This model can be seen as the spatial analogue of the autoregressive model. The usual way to describe its dynamics is through the simultaneous schemes: Each pixel updates its value using a normal distribution with mean depending on the values of its neighbors.

In this work, we are going to generalize this definition to incorporate long-range interactions and arbitrary neighborhoods, but we are going to limit the values of the process to be on a compact interval $[l, u]$. The existence of a unique invariant measure for this process was studied by McBryan and Spencer (1977) and revisited by Ferrari and Grynberg (2008). Gibbs (2004) proposes a finite version of this model as the posterior distribution for Bayesian restoration of grayscale images. Huber (2007) studies perfect simulation of these distributions in a finite box.

Let $\{\sigma(i), i \in \mathbb{Z}^d\}$ be a collection of positive real numbers. Consider that each pixel in $\mathbb{Z}^d$ has an independent exponential clock and when the clock rings at the

pixel i it updates its value depending on the values of its neighbors using a normal distribution with mean $h(i,\eta)$ and variance $\sigma^2(i)$ conditioned to lie in a given compact interval. Without loss of generality we can consider the interval $[0,1]$. The term $h(i,\eta)$ depends on η only through the values on the neighborhood of the site i. Typically,

$$h(i,\eta)=\sum_{j\neq i} J(i-j)\eta(j),$$

where $J:\mathbb{Z}^d\to\mathbb{R}^+$ is summable, nonnegative and symmetric: $J(i)\geq 0$ for all $i\in\mathbb{Z}^d$, $J(0)=0$ and $0<J:=\sum_{i\in\mathbb{Z}^d} J(i)=1$.

In this case, $\varrho(da)=\mathbf{1}_{[0,1]}(a)\,da$ and

$$c_i(a,\eta)=\frac{1}{\sigma(i)}\frac{\phi((a-h(i,\eta))/\sigma(i))}{\Phi((1-h(i,\eta))/\sigma(i))-\Phi(-h(i,\eta)/\sigma(i))},\qquad 0\leq a\leq 1, \tag{4.1}$$

where ϕ and Φ are the density and cumulative function of the standard normal distribution, respectively.

Applying the bounds given by Proposition 1 of Fernández, Ferrari and Grynberg (2007) we can show that (4.1) satisfies the assumptions (2.1) and (2.9) needed in order to apply the decomposition of Theorem 1. In our case, $M_i=1$ and

$$\begin{aligned}\alpha_i(-1)=&\frac{1}{A^+}\biggl(\Phi\biggl(\frac{x_i(\sigma(i))-\mu^+}{\sigma(i)}\biggr)-\Phi\biggl(\frac{-\mu^+}{\sigma(i)}\biggr)\biggr)\\&+\frac{1}{A^-}\biggl(\Phi\biggl(\frac{1-\mu^-}{\sigma(i)}\biggr)-\Phi\biggl(\frac{x_i(\sigma(i))-\mu^-}{\sigma(i)}\biggr)\biggr),\end{aligned} \tag{4.2}$$

where $\mu^-=\inf_\eta h(i,\eta)$, $\mu^+=\sup_\eta h(i,\eta)$,

$$A^-=\Phi\biggl(\frac{1-\mu^-}{\sigma(i)}\biggr)-\Phi\biggl(\frac{-\mu^-}{\sigma(i)}\biggr),\qquad A^+=\Phi\biggl(\frac{1-\mu^+}{\sigma(i)}\biggr)-\Phi\biggl(\frac{-\mu^+}{\sqrt{\sigma(i)}}\biggr)$$

and

$$x_i\bigl(\sigma(i)\bigr)=\frac{\mu^-+\mu^+}{2}-\frac{\sigma(i)^2}{\mu^+-\mu^-}\log\frac{A^-}{A^+}.$$

Also,

$$\begin{aligned}\alpha_i(k)=\inf_{w\in A^{V_i(k)}}&\frac{1}{A_i^+(k)}\biggl(\Phi\biggl(\frac{x_i(k,\sigma(i))-\mu_i^+(k)}{\sigma(i)}\biggr)-\Phi\biggl(\frac{-\mu_i^+(k)}{\sigma(i)}\biggr)\biggr)\\&+\frac{1}{A_i^-(k)}\biggl(\Phi\biggl(\frac{1-\mu_i^-(k)}{\sigma(i)}\biggr)-\Phi\biggl(\frac{x_i(k,\sigma(i))-\mu_i^-(k)}{\sigma(i)}\biggr)\biggr),\end{aligned} \tag{4.3}$$

where $\mu_i^-(k) = \inf_{\eta:\eta(V_i(k))=w} h(i,\eta)$, $\mu_i^+(k) = \sup_{\eta:\eta(V_i(k))=w} h(i,\eta)$,

$$A_i^-(k) = \Phi\left(\frac{1-\mu_i^-(k)}{\sigma(i)}\right) - \Phi\left(\frac{-\mu_i^-(k)}{\sigma(i)}\right),$$

$$A_i^+(k) = \Phi\left(\frac{1-\mu_i^+(k)}{\sigma(i)}\right) - \Phi\left(\frac{0-\mu_i^+(k)}{\sqrt{\sigma(i)}}\right)$$

and

$$x_i\big(k,\sigma(i)\big) = \frac{\mu^- + \mu^+}{2} - \frac{\sigma(i)^2}{\mu^+ - \mu^-}\log\frac{A_i^-(k)}{A_i^+(k)}.$$

As a second example we show that the decomposition presented in Theorem 1 can be effectively implemented in Gibbsian systems with compact-valued spins. We take $A = [-1, 1]$ and introduce the following definitions.

DEFINITION 1. A pairwise potential is a collection $\{J(i,j), (i,j) \in \mathbb{Z}^d \times \mathbb{Z}^d\}$ of real numbers which satisfies

$$J(i,i) = 0, \ \sup_{i\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \big|J(i,j)\big| < \infty. \tag{4.4}$$

In what follows we use the notation

$$\Sigma_i = \sum_{j\in\mathbb{Z}^d} \big|J(i,j)\big|.$$

For any $i \in \mathbb{Z}^d$, let $\eta(i)$ be the value of the spin at site i in the configuration $\eta \in S$.

DEFINITION 2. A probability measure μ on $(S, \mathcal{S})$ is said to be a Gibbs state relative to the potential $\{J(i,j)\}$ if for all $i \in \mathbb{Z}^d$, a version of the conditional probability density of $\eta(i)$, given $\eta(j)$, $j \neq i$, is given by

$$\mu\big(\eta(i) = a | \eta(j) \text{ for all } j \neq i\big) = \frac{\exp(a\sum_{j\neq i} J(i,j)\eta(j))}{Z^\eta},$$

where

$$Z^\eta = \int_A \exp\left(a\sum_{j\neq i} J(i,j)\eta(j)\right)\varrho(da).$$

In the following we consider the interaction $J_\beta = \beta J$, where β is a positive parameter. The associated Gibbs measure will be denoted μ without indicating explicitly the dependence on β. Now, put

$$c_i(a,\eta) = e^{\beta a \sum_{j\in\mathbb{Z}^d} J(i,j)\eta(j)}. \tag{4.5}$$

Then, by construction, the process (σ_t) with generator (2.2) and this choice of change rates is reversible with respect to the Gibbs state μ corresponding to the potential $J_\beta(i, j) = \beta J(i, j)$. It is immediate to see that condition (4.4) implies the continuity condition (2.9).

We now give the explicit decomposition in one specific case.

EXAMPLE 2. The following example is a Gibbsian time evolution with infinite range interaction. The decomposition we present here is inspired by the one presented in Galves, Löcherbach and Orlandi (2010) in the case of two color systems. In Galves, Löcherbach and Orlandi (2010), for coupling reasons, it was convenient to give a slightly different decomposition. The goal there was to be able to couple together the infinite range Gibbsian system with the finite range Gibbsian system obtained by truncating the potential interaction. We suppose that the spin distribution ϱ is symmetric.

Define for any $i \in \mathbb{Z}^d$ and any $k \geq -1$,

$$S_i^{>k} := \sum_{j:\|i-j\|>k} |J(i, j)|, \qquad S_i^{\leq k} := \sum_{j:\|i-j\|\leq k} |J(i, j)|.$$

Note that $\Sigma_i = S_i^{>-1}$.

Then the decomposition (2.16) holds with

$$M_i = \int_0^1 (e^{a\beta\Sigma_i} + e^{-a\beta\Sigma_i})\varrho(da). \tag{4.6}$$

Moreover,

$$\alpha_i(-1) = 2\int_0^1 e^{-a\beta\Sigma_i}\varrho(da) \tag{4.7}$$

and

$$\alpha_i(k) = M_i + \int_0^1 e^{a\beta S_i^{\leq k}} e^{-a\beta S_i^{>k}}\varrho(da) - \int_0^1 e^{a\beta\Sigma_i}\varrho(da). \tag{4.8}$$

Finally,

$$\lambda_i(-1) = 2\frac{\int_0^1 e^{-a\beta\Sigma_i}\varrho(da)}{\int_0^1 (e^{a\beta\Sigma_i} + e^{-a\beta\Sigma_i})\varrho(da)} \tag{4.9}$$

and

$$\begin{aligned}\lambda_i(k) = {} & \int_0^1 e^{a\beta S_i^{\leq k-1}} e^{-a\beta S_i^{>k}} \\ & \times (e^{a\beta\sum_{j:\|j-i\|=k}|J(i,j)|} - e^{-a\beta\sum_{j:\|j-i\|=k}|J(i,j)|})\varrho(da) \\ & \Big/ \int_0^1 (e^{a\beta\Sigma_i} + e^{-a\beta\Sigma_i})\varrho(da).\end{aligned} \tag{4.10}$$

5. Perfect simulation. The goal of this section is to give an application of the Kalikow-type decomposition given by Theorem 1. This application is a perfect simulation algorithm for the invariant measure of an interacting multicolor system. We assume that the interaction rates are continuous in the sense of (2.9) and satisfy a high noise condition. The basis of the algorithm is the convex decomposition given in Theorem 1. First of all, Proposition 1 gives a sufficient condition for exponential ergodicity which is based on the construction of a dominating branching process.

From now on we will denote by (σ_t^η) [and (σ_t^μ)] the multicolor system having generator $\mathcal{G}$ given by (2.8) with a fixed initial configuration η (a random configuration chosen with probability distribution μ).

PROPOSITION 1. *Let $(c_i)_{i\in\mathbb{Z}^d}$ be a family of rate functions satisfying the conditions of Theorem 1. Furthermore, assume that*

$$\underline{M} = \inf_{i\in\mathbb{Z}^d} M_i > 0 \tag{5.1}$$

and

$$\sup_{i\in\mathbb{Z}^d} \sum_{k\geq 0} |V_i(k)|\lambda_i(k) = \gamma < 1. \tag{5.2}$$

Then the following two statements hold.

(1) *The process (σ_t) admits a unique invariant probability measure μ.*

(2) *For any finite set of sites $F \subset \mathbb{Z}^d$, for any $T > 0$ and any initial configuration η, there exists a coupling between the process (σ_t^η) and the stationary process (σ_t^μ) such that*

$$P\big(\sigma_T^\eta(F) \neq \sigma_T^\mu(F)\big) \leq |F| e^{-\underline{M}(1-\gamma)T}.$$

Let us compare the above proposition to known results in the literature on particle systems.

(1) Condition (5.2) is stronger than Liggett's existence condition (2.3) which does not imply the uniqueness of the invariant measure.

(2) Let us compare our result to the $M < \varepsilon$-criterion of Theorem 4.1 of Liggett (2000), page 31. Recall that Liggett's quantity M (translated into our context) is given by

$$M = \sup_{i\in\mathbb{Z}^d} \sum_{j\neq i} \sup\big\{\big\|c_i(a,\eta)\varrho(da) - c_i(a,\zeta)\varrho(da)\big\|_{\mathrm{TV}} : \eta(k) = \zeta(k)\ \forall k \neq j\big\}.$$

By our decomposition of $c_i(a,\eta)$, this expression can be upper bounded by

$$M \leq \sup_{i\in\mathbb{Z}^d} M_i \sum_{k\geq 0} \lambda_i(k)|V_i(k)|.$$

Since $\sup_i M_i < \infty$, condition $\sup_{i\in\mathbb{Z}^d} \sum_{k\geq 0} |V_i(k)|\lambda_i(k) < \infty$ implies condition (3.8) of Liggett (2000).

Concerning the quantity ε defined on page 24 of Liggett (2000), note that in our case, it can be written as follows:

$$\varepsilon = \inf_{i\in\mathbb{Z}^d} \inf_{\eta, a\neq b} \big[c_i(a, \eta^{i,b})\varrho(\{a\}) + c_i(b, \eta^{i,a})\varrho(\{b\})\big],$$

which will be equal to zero in general. Hence, with our techniques we are able to treat cases where the $M < \varepsilon$-condition of Liggett (2000), Theorem 4.1, is not satisfied.

(3) Condition (5.2) is a high-noise condition reminiscent of Dobrushin–Shlosman condition [see Maes and Shlosman (1991)]. It is a sufficient condition ensuring that there is no phase transition.

We are now in position to present the perfect simulation scheme. Suppose we want to sample the configuration at site i under the invariant measure μ. In a first step, we determine the set of sites whose spins influence the spin at site i under equilibrium. We call this set of sites *ancestors* of i and this stage *backward sketch procedure*. First, we climb up from time 0 using a reverse time Poisson point process with rate M_i. We stop when the last Poisson clock before time 0 rings. At that time, we choose a range k with probability $\lambda_i(k)$. If $k = -1$, we decide the value of the spin using the law $p_i^{[-1]}\,d\varrho$, independently of everything else. If k is different from -1, we restart the above procedure from every site $j \in V_i(k)$. The procedure stops once each site involved has chosen range -1. When this occurs, we can start the second stage, in which we go back to the future assigning spins to all sites visited during the first stage. We call this procedure forward spin assignment procedure. This is done from the past to the future by using the update probability densities $p_i^{[k]}$ starting at the sites which ended the first procedure by choosing range -1. For each one of these sites a spin is chosen according to $p^{[-1]}\,d\varrho$. The values obtained in this way enter successively in the choice of the values of the spins depending on a neighborhood of range greater or equal to 0.

We now give the precise form of the algorithm. Fix a finite set $F \subset \mathbb{Z}^d$. The following variables will be used:

- N is an auxiliary variable taking values in the set of nonnegative integers $\{0, 1, 2, \ldots\}$.
- $N_{\mathrm{STOP}}^{(F)}$ is a counter taking values in the set of nonnegative integers $\{0, 1, 2, \ldots\}$.
- I is a variable taking values in $\mathbb{Z}^d$.
- K is a variable taking values in $\{-1, 0, 1, \ldots\}$.
- B is an array of elements of $\mathbb{Z}^d \times \{-1, 0, 1, \ldots\}$.
- C is a variable taking values in the set of finite subsets of $\mathbb{Z}^d$.
- W is an auxiliary variable taking values in A^*.
- σ is a function from $\mathbb{Z}^d$ to A^*.

ALGORITHM 1 (Backward sketch procedure).

1. *Input:* F; *Output:* $N_{\mathrm{STOP}}^{(F)}$, B
2. $N \leftarrow 0$, $N_{\mathrm{STOP}}^{(F)} \leftarrow 0$, $B \leftarrow \varnothing$, $C \leftarrow F$
3. WHILE $C \neq \varnothing$
4. $N \leftarrow N + 1$
5. Choose randomly a position $I \in C$ and an integer $K \geq -1$ according to the probability distribution

$$P(I = i, K = k) = \frac{M_i \lambda_i(k)}{\sum_{j \in C} \sum_{l \geq -1} M_j \lambda_j(l)}$$

6. IF $K = -1$, $C \leftarrow C \setminus \{I\}$
7. ELSE $C \leftarrow C \cup B_I(K)$
8. ENDIF
9. $B(N) \leftarrow (I, K)$
10. ENDWHILE
11. $N_{\mathrm{STOP}}^{(F)} \leftarrow N$
12. RETURN $N_{\mathrm{STOP}}^{(F)}$, B

Now we use the following forward spin assignment procedure to sample from the invariant measure μ. Recall that the choice of Δ in (2.8) implies that the system does not change its colors. This explains Step 9 in Algorithm 2.

ALGORITHM 2 (Forward spin assignment procedure).

1. *Input:* $N_{\mathrm{STOP}}^{(F)}$, B; *Output:* $\{(i, \sigma(i)) : i \in F\}$
2. $N \leftarrow N_{\mathrm{STOP}}^{(F)}$
3. $\sigma(j) \leftarrow \Delta$ for all $j \in \mathbb{Z}^d$
4. WHILE $N \geq 1$
5. $(I, K) \leftarrow B(N)$
6. IF $K = -1$ choose W randomly in A according to the probability distribution

$$p_I^{[-1]}\, d\varrho$$

7. ELSE choose W randomly in A^* according to the probability distribution

$$p_I^{[K]}(\cdot|\sigma)\, d\varrho^*$$

8. ENDIF
9. IF $W \neq \Delta$ put $\sigma(I) \leftarrow W$
10. ENDIF
11. $N \leftarrow N - 1$
12. ENDWHILE
13. RETURN $\{(i, \sigma(i)) : i \in F\}$

The next theorems summarize the properties of Algorithms 1 and 2. In order to distinguish clearly to which part of the two algorithms we refer, we shall write P_{sketch} for the probability associated to the backward sketch procedure.

THEOREM 2. *Suppose that*

$$\text{for all } F \subset \mathbb{Z}^d \text{ finite} \qquad P_{\text{sketch}}\big(N_{\text{STOP}}^{(F)} < \infty\big) = 1. \tag{5.3}$$

Then the following two statements hold.

(1) *Algorithms 1 and 2 are successful.*

(2) *The process (σ_t) admits a unique invariant measure μ. The law of the set $\{(i, \sigma(i)) : i \in F\}$ printed at the end of Algorithms 1 and 2 is the projection on A^F of μ.*

The next theorem states sufficient conditions ensuring (5.3) and gives also a control on the rate of convergence.

THEOREM 3. (1) *The sub-criticality condition* (5.2) *implies* (5.3). *More precisely, we have*

$$P_{\text{sketch}}\big(N_{\text{STOP}}^{(F)} > N\big) \le |F|\gamma^N, \tag{5.4}$$

where γ is given in (5.2).

(2) *Suppose, in addition to* (5.2), *that* (5.1) *holds. Fix a time $t > 0$, some finite set of sites $F \subset \mathbb{Z}^d$ and two initial configurations η and $\zeta \in A^{\mathbb{Z}^d}$. Then there exists a coupling of the two processes $(\sigma_s^\eta)_s$ and $(\sigma_s^\zeta)_s$ such that*

$$P\big(\sigma_t^\eta(F) \ne \sigma_t^\zeta(F)\big) \le |F| e^{-\underline{M}(1-\gamma)t}.$$

The proofs of Proposition 1, Theorems 2 and 3 will be given in the next section.

6. Proofs of Proposition 1, Theorems 2 and 3. The proofs rely on the notion of *black and white time-reverse sketch process* that we will introduce now. The black and white time-reverse sketch process gives the mathematically precise description of the backward black and white Algorithm 1 given in Section 5.

For each $i \in \mathbb{Z}^d$, denote by $\cdots T_{-2}^i < T_{-1}^i < T_0^i < 0 < T_1^i < T_2^i < \cdots$ the occurrence times of the rate M_i Poisson point process N^i on the real line. The Poisson point processes associated to different sites are independent. To each point T_n^i associate an independent mark K_n^i according to the probability distribution $(\lambda_i(k))_{k \ge -1}$. As usual, we identify the Poisson point processes and the associated counting measures.

For each $i \in \mathbb{Z}^d$ and $t \in \mathbb{R}$ we define the time-reverse point process starting at time t, associated to site i,

$$\begin{aligned} \tilde{T}_n^{(i,t)} &= t - T^i_{N^i(0,t]-n+1}, \qquad t \ge 0, \\ \tilde{T}_n^{(i,t)} &= t - T^i_{-N^i(t,0]-n+1}, \qquad t < 0. \end{aligned} \tag{6.1}$$

To these time-reverse point processes we can associate in an obvious way the corresponding marks $\tilde{K}_n^{(i,t)}$, $n \in \mathbb{Z}$. Finally, for each site $i \in \mathbb{Z}^d$, $k \geq -1$, the reversed k-marked Poisson point process returning from time t is defined as

$$\tilde{N}^{(i,t,k)}[s,u] = \sum_n \mathbf{1}_{\{s \leq \tilde{T}_n^{(i,t)} \leq u\}} \mathbf{1}_{\{\tilde{K}_n^{(i,t)} = k\}}. \tag{6.2}$$

To define the black and white time-reverse sketch process we need to introduce a family of transformations $\{\pi^{(i,k)}, i \in \mathbb{Z}^d, k \geq -1\}$ on the set of finite subsets of $\mathbb{Z}^d$, $\mathcal{F}(\mathbb{Z}^d)$, defined as follows. For any unitary set $\{j\}$,

$$\pi^{(i,k)}(\{j\}) = \left\{ \begin{matrix} V_i(k), & \text{if } j = i \\ \{j\}, & \text{otherwise} \end{matrix} \right\}. \tag{6.3}$$

Notice that for $k = -1$, $\pi^{(i,k)}(\{i\}) = \varnothing$. For any finite set $F \subset \mathbb{Z}^d$, we similarly define

$$\pi^{(i,k)}(F) = \bigcup_{j \in F} \pi^{(i,k)}(\{j\}). \tag{6.4}$$

The black and white time-reverse sketch process starting at site i at time t will be denoted by $(C_s^{(i,t)})_{s \geq 0}$. $C_s^{(i,t)}$ is the set of sites at time s whose colors affect the color of site i at time t. We call this set $C_s^{(i,t)}$ *set of ancestors of* i *at time* s *before time* t. The evolution of this process is defined through the following equation: $C_0^{(i,t)} := \{i\}$, and

$$\begin{aligned} f\big(C_s^{(i,t)}\big) &= f\big(C_0^{(i,t)}\big) \\ &\quad + \sum_{k \geq -1} \sum_{j \in \mathbb{Z}^d} \int_0^s \big[f\big(\pi^{(j,k)}\big(C_{u-}^{(i,t)}\big)\big) - f\big(C_{u-}^{(i,t)}\big) \big] \tilde{N}^{(j,t,k)}(du), \end{aligned} \tag{6.5}$$

where $f : \mathcal{F}(\mathbb{Z}^d) \to \mathbb{R}$ is any bounded cylindrical function. This family of equations characterizes completely the time evolution $\{C_s^{(i,t)}, s \geq 0\}$. For any finite set $F \subset \mathbb{Z}^d$ define

$$C_s^{(F,t)} = \bigcup_{i \in F} C_s^{(i,t)}.$$

The following proposition summarizes the properties of the family of processes defined above.

PROPOSITION 2. *For any finite set* $F \subset \mathbb{Z}^d$, $\{C_s^{(F,t)}, s \geq 0\}$ *is a Markov jump process having as infinitesimal generator*

$$\begin{aligned} Lf(C) &= M_i \sum_{i \in C} \sum_{k \geq 0} \lambda_i(k) \big[f\big(C \cup V_i(k)\big) - f(C) \big] \\ &\quad + \lambda_i(-1) \big[f\big(C \setminus \{i\}\big) - f(C) \big], \end{aligned} \tag{6.6}$$

where f *is any bounded cylindrical function.*

PROOF. The proof follows in a standard way from the construction (6.5). □

If we are interested in simulating from the invariant measure of the process, then we will start the black and white time-reverse sketch process at time $t = 0$; if, however, we wish to construct the process at time t, we shall start the black and white time-reverse sketch process at that time t precisely.

6.1. *Backward oriented percolation and sub-criticality.* For the algorithm to be successful it is crucial to show that $\bigcup_{s\geq 0} C_s^{(i,t)}$, the set of ancestors of any site i, is finite with probability one. Formally, let

$$T_{\mathrm{STOP}}^{(i)} = \inf\{s : C_s^{(i,0)} = \varnothing\}$$

be the relaxation time. We introduce the sequence of successive jump times $\tilde{T}_n^{(i)}, n \geq 1$, of processes $N^{(j,k)}$ whose jumps occur in (6.5), for $t = 0$. Let $\tilde{T}_1^{(i)} = T_1^{(i,0)}$ and define successively for $n \geq 2$

$$\tilde{T}_n^{(i)} = \inf\{t > \tilde{T}_{n-1}^{(i)} : \exists j \in C_{\tilde{T}_{n-1}^{(i)}}^{(i,0)}, \exists k : N^{(j,k)}(]\tilde{T}_{n-1}^{(i)}, t]) = 1\}. \tag{6.7}$$

We write $\tilde{K}_n^{(i)}$ for the associated marks. Now we put

$$\mathbf{C}_n^{(i)} = C_{\tilde{T}_n^{(i)}}^{(i,0)} \tag{6.8}$$

and

$$N_{\mathrm{STOP}}^{(i)} = \inf\{n : \mathbf{C}_n^{(i)} = \varnothing\}.$$

This is the number of steps of the backward sketch process—and it is exactly the number of steps of Algorithm 1. For the perfect simulation algorithm to be successful, it is crucial to show that the number of steps $N_{\mathrm{STOP}}^{(i)}$ is finite. Since at every step of the algorithm a finite interaction range k is chosen, this implies automatically that also $T_{\mathrm{STOP}}^{(i)}$ is finite almost surely. However, in order to control the speed of convergence, we need a precise control on the tail probabilities of $T_{\mathrm{STOP}}^{(i)}$. To this aim we estimate the volume of the set $C_s^{(F,t)} = \bigcup_{i\in F} C_s^{(i,t)}$ where F is a bounded set of $\mathbb{Z}^d$.

LEMMA 1.

$$E(|C_s^{(F,t)}|) \leq |F| e^{-\underline{M}(1-\gamma)s}, \tag{6.9}$$

where $\underline{M}$ *is defined in* (5.1) *and* γ *in* (5.2).

PROOF. Fix some $N \in \mathbb{N}$. Let $L_s^i = |C_s^{(i,t)}|$ and

$$T_N = \inf\{t : L_t^i \geq N\}.$$

Then by (6.5),

$$L^i_{s\wedge T_N} \le 1 + \sum_{k\ge 1}\sum_{j\in\mathbb{Z}^d}\int_0^{s\wedge T_N}\big[|V_j(k)|-1\big]1_{\{j\in C^{(i,t)}_{u-}\}}\tilde N^{(j,t,k)}(du)$$
$$- \sum_{j\in\mathbb{Z}^d}\int_0^{s\wedge T_N} 1_{\{j\in C^{(i,t)}_{u-}\}}\tilde N^{(j,t,-1)}(du). \tag{6.10}$$

Recall that $\underline{M} = \inf_{i\in\mathbb{Z}^d} M_i > 0$. Passing to expectation and using that, by condition (5.2),

$$M_j\Big(\Big(\sum_{k\ge 1}\lambda_j(k)\big[|V_j(k)|-1\big]\Big) - \lambda_j(-1)\Big) \le -\underline{M}(1-\gamma) < 0,$$

which yields

$$E\big(L^i_{s\wedge T_N}\big) \le 1 + \sum_{j\in\mathbb{Z}^d} M_j\Big(\Big(\sum_{k\ge 1}\lambda_j(k)\big[|V_j(k)|-1\big]\Big) - \lambda_j(-1)\Big)$$
$$\times E\int_0^{s\wedge T_N} 1_{\{j\in C^{(i,t)}_{u-}\}}\,du \tag{6.11}$$
$$\le 1 - \underline{M}(1-\gamma)E\int_0^{s\wedge T_N} L^i_u\,du.$$

Letting $N\to\infty$, we thus get by Fatou's lemma that

$$E\big(L^i_s\big) \le 1 - \underline{M}(1-\gamma)\int_0^s E\big(L^i_u\big)\,du.$$

This implies that

$$E\big(L^i_s\big) \le 1 \qquad \text{for all } s\ge 0.$$

Hence, we may apply Gronwall's lemma which yields

$$E\big(L^i_s\big) \le e^{-\underline{M}(1-\gamma)s}. \tag{6.12}$$

Hence, since $|C^{(F,t)}_s| \le \sum_{i\in F}|C^{(i,t)}_s| = \sum_{i\in F} L^i_s$,

$$E\big(|C^{(F,t)}_s|\big) \le |F|e^{-\underline{M}(1-\gamma)s}. \tag{6.13}$$

□

6.2. *Proof of Proposition 1 and Theorem 2.* Proposition 1 is an immediate consequence of Theorem 2, item 2, and Theorem 3, item 2.

PROOF OF THEOREM 2. Item 1 of Theorem 2 is evident. We give the proof of item 2 of Theorem 2.

Write μ_F for the law of the output $\{(i,\sigma(i)) : i\in F\}$ of Algorithms 1 and 2; μ_F is a probability measure on $(A^F, \mathcal{A}^F)$. By construction, the family of probability

laws $\{\mu_F, F \subset \mathbb{Z}^d \text{ finite}\}$ is a consistent family of finite dimensional distributions. Hence, there exists a unique probability measure μ on $(S, \mathcal{S})$ such that μ_F is the projection onto A^F of μ, for any fixed finite set of sites $F \subset \mathbb{Z}^d$.

We show that μ is the unique invariant measure of the process (σ_t). In order to do so, we use a slight modification of Algorithms 1 and 2 in order to construct σ_t^η, for some fixed initial configuration $\eta \in S$. The modification is defined as follows. Let T_{STOP} and T be variables taking values in $(0, \infty)$. Replace Steps 1–3 of Algorithm 1 by:

1. Input: F; Output: $N_{\text{STOP}}^{(F)}, B, C$
2. $N \leftarrow 0$, $N_{\text{STOP}}^F \leftarrow 0$, $B \leftarrow \varnothing$, $C \leftarrow F$, $T_{\text{STOP}} \leftarrow 0$
3. WHILE $T_{\text{STOP}} < t$ and $C \neq \varnothing$

3′. Choose a time $T \in (0, +\infty)$ randomly according to the exponential distribution with parameter $\sum_{j \in C} M_j$. Update

$$T_{\text{STOP}} \leftarrow T_{\text{STOP}} + T.$$

Finally replace Step 12 of Algorithm 1 by:

12. RETURN $N_{\text{STOP}}^{(F)}, B, C$.

In this modified version, we stop the algorithm after time t, hence, the output set C might not be empty. The output C is exactly the set $C_t^{(F,t)}$, the set of sites at time 0 whose colors influence the colors of sites in F at time t. Finally, notice that if $C = \varnothing$, then $T_{\text{STOP}} < t$, and in this case, $T_{\text{STOP}} = T_{\text{STOP}}^F$ is the relaxation time introduced in the previous subsection.

Concerning Algorithm 2, replace Step 1 of Algorithm 2 by:

1. Input: $N_{\text{STOP}}^{(F)}, B, C$; Output: $\{(i, \sigma(i)) : i \in F\}$

and Step 3 by:

3. $\sigma(j) \leftarrow \eta(j)$ for all $j \in C$; $\sigma(j) \leftarrow \Delta$ for all $j \in \mathbb{Z}^d \setminus C$.

Then the law of the set $\{(i, \sigma(i)) : i \in F\}$ printed at the end of the modified Algorithm 2 is the law of $\sigma_t^\eta(F)$. Notice that the output of the modified Algorithm 2 equals the output of the unmodified Algorithm 2 if $T_{\text{STOP}} < t$.

We first give an intuitive argument showing that μ must be invariant for (σ_t). Write P_t for the transition semigroup of (σ_t). Fix $t > 0$ and a finite set of sites $F \subset \mathbb{Z}^d$. Suppose we want to determine the projection on A^F of μP_t. This means that we have first to run the above modified Algorithm 1 up to time t. It gives as output the set of ancestor sites $C = C_t^{(F,t)}$. We then have to run the modified Algorithm 2 with initial configuration $\{\sigma(i), i \in C = C_t^{(F,t)}\}$ chosen according to μ; cf. Step 3. But this means that we have to concatenate the modified Algorithm 1 with the original Algorithm 1, where the Algorithm 1 is now starting from $C_t^{(F,t)}$ instead of F. In other words, we concatenate two backward sketch processes and consider $C^{(C^{(F,t)},0)}$ up to the time of its extinction. By the Markov property and the stationarity of the Poisson processes, this means that we directly consider $C^{(F,t)}$ up to the time of its extinction—which is finite by our assumptions. Hence, $\mu P_t = \mu$ in restriction to finite cylinder sets.

We now give a more formal argument. Recall that the output of the modified Algorithm 2 equals the output of the unmodified Algorithm 2 if $T_{\mathrm{STOP}} < t$. Let $f : A^F \to \mathbb{R}_+$ be a bounded measurable function. Then

$$\begin{aligned} E\big[f\big(\sigma_t^\eta(i), i \in F\big)\big] &= E\big[f\big(\sigma_t^\eta(i), i \in F\big), T_{\mathrm{STOP}} < t\big] \\ &\quad + E\big[f\big(\sigma_t^\eta(i), i \in F\big), T_{\mathrm{STOP}} \geq t\big] \\ &= E\big[f\big(\sigma(i), i \in F\big), T_{\mathrm{STOP}}^F < t\big] \\ &\quad + E\big[f\big(\sigma_t^\eta(i), i \in F\big), T_{\mathrm{STOP}} \geq t\big], \end{aligned} \tag{6.14}$$

where $(\sigma(i), i \in F)$ is the output of the unmodified Algorithms 1 and 2.

But

$$E\big[f\big(\sigma_t^\eta(i), i \in F\big), T_{\mathrm{STOP}} \geq t\big] \leq \|f\|_\infty P_{\mathrm{sketch}}\big(T_{\mathrm{STOP}}^F \geq t\big) \to 0 \qquad \text{as } t \to \infty,$$

since finiteness of N_{STOP}^F implies the finiteness of T_{STOP}^F. Hence, we obtain that

$$\lim_{t\to\infty} E\big[f\big(\sigma_t^\eta(i), i \in F\big)\big] = E\big[f\big(\sigma(i), i \in F\big)\big],$$

since $1_{\{T_{\mathrm{STOP}}^F < t\}} \to 1$ almost surely.

This implies that μ is an invariant measure of the process. Replacing the initial condition η by any stationary initial condition, we finally also get uniqueness of the invariant measure. Thus Theorem 2 is proved. □

6.3. *Proof of Theorem 3.* We start by proving (5.4). Let

$$L_n^{(i)} = |\mathbf{C}_n^{(i)}|$$

be the cardinal of the set $\mathbf{C}_n^{(i)}$ after n steps of the algorithm [recall (6.8)]. Then due to our assumptions, $L_n^{(i)}$ can be compared to a multi-type branching process Z_n having offspring mean which is bounded by γ at each step, such that $L_n^{(i)} \leq Z_n$ for all n. The details are given in Galves, Löcherbach and Orlandi (2010), proof of Theorem 1, and are omitted here. Thus,

$$P\big(N_{\mathrm{STOP}}^{(i)} > n\big) = P\big(L_n^{(i)} > 0\big) = P\big(L_n^{(i)} \geq 1\big) \leq P(Z_n \geq 1) \leq E(Z_n) = \gamma^n.$$

When starting with the initial set F instead of the singleton $\{i\}$, then the above estimates remain true by multiplying with $|F|$, due to the independence properties of the branching process.

Concerning item 2 of Theorem 3, we use once more the modified Algorithms 1 and 2 introduced in the proof of Theorem 2 above. In order to realize the coupling, we use the same realizations of T, I and K for the construction of σ_t^η and σ_t^ζ. Write L_s for the cardinal of $C_s^{(F,t)}$. Clearly, both realizations of σ_t^η and σ_t^ζ do

not depend on the initial configuration η, ζ, respectively, if the output C of the modified Algorithm 1 is void. Thus, by Lemma 1,

$$\begin{aligned} P\big(\sigma_t^{\eta}(F) \neq \sigma_t^{\zeta}(F)\big) &\leq P(T_{\mathrm{STOP}} \geq t) \\ &= P(L_t \geq 1) \\ &\leq E(L_t) \leq |F| e^{-\underline{M}(1-\gamma)t}. \end{aligned}$$

This implies that the convergence toward the unique invariant measure takes place exponentially fast. The proof of Theorem 3 is complete.

7. Applications for perfect simulation.

7.1. *Maximum likelihood estimation in Gibbs distributions.* Parameter estimation for Gibbs distributions in the infinite lattice is usually based on the maximum likelihood approach [see, e.g., Gidas (1988, 1991)]. The maximum likelihood estimation is theoretically well understood in this framework. Comets (1992) proved the consistency of the MLE for exponential families of Markov random fields on the lattice. Also, in the case of no phase transition, Janžura (1997) proved asymptotic normality and efficiency of the MLE inside the uniqueness region of the Gibbs distributions considered. Comets and Gidas (1992) considered maximum likelihood estimators for Markov random fields over $\mathbf{Z}^d$ from incomplete data. They prove the strong consistency of maximum likelihood estimators in this case. Their results hold irrespective of the presence of long-range correlations or nonanalytic behavior of the underlying quantities. The parameter space is thereby allowed to be noncompact.

However, the numerical feasibility of the ML method is strongly limited, due to the computation of the normalizing constant for each relevant parameter, in particular, for each temperature. Geyer and Thompson (1992) devised a rather ingenious method for this computation based on an MCMC computation of the equilibrium distribution for a fixed value of the parameter.

Consider the family of probability densities with respect to a reference measure μ given by

$$f(x,\theta) = \frac{1}{Z(\theta)} \exp\langle T(x), \theta\rangle, \tag{7.1}$$

where $\langle T(x), \theta\rangle$ denotes the inner product between the canonical parameter θ and the sufficient statistics $T(x)$ and

$$Z(\theta) = \int \exp\langle T(x), \theta\rangle \, d\mu(x). \tag{7.2}$$

Denote by P_ψ the measure having density $f(\cdot, \psi)$ with respect to μ. Then,

$$Z(\theta) = Z(\psi) \int \exp\langle T(x), \theta - \psi\rangle \, dP_\psi(x). \tag{7.3}$$

Therefore, if we have $X_1, X_2, \ldots$ i.i.d. random objects with distribution P_ψ, we have that

$$d_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} \exp\langle T(X_i), \theta - \psi\rangle \to d(\theta) = \frac{Z(\theta)}{Z(\psi)} \qquad \text{almost surely.} \tag{7.4}$$

The maximum likelihood of θ can be taken as

$$\hat{\theta} = \operatorname{argmax} \log f(x,\theta) + \log Z(\psi) = \operatorname{argmax}\langle T(x), \theta\rangle - \log d(\theta), \tag{7.5}$$

and its Monte Carlo approximant

$$\hat{\theta}_n = \operatorname{argmax}\langle T(x), \theta\rangle - \log d_n(\theta). \tag{7.6}$$

Notice that if we can perfectly simulate from P_ψ, we have trivially that $\hat{\theta}_n \to \hat{\theta}$ as $n \to \infty$ along with the rate of convergence of such convergence.

7.2. *Attaining Ornstein's $\bar{d}$-distance for ordered pairs of Ising probability distributions*. In this section we show how to use the decomposition of Theorem 1 and the above perfect simulation algorithm in order to construct an explicit coupling attaining Ornstein's $\bar{d}$-distance for two ordered Ising probability measures. Let $A := \{-1, 1\}$ and $S = A^{\mathbb{Z}^d}$.

We consider a ferromagnetic pairwise interaction J, that is, a collection $\{J(i,j), i \neq j, i, j \in \mathbb{Z}^d\}$ of positive real numbers satisfying $J(i,j) = J(j,i)$ for all $i, j \in \mathbb{Z}^d$ and for all $i \in \mathbb{Z}^d$,

$$J(i,i) = 0, \qquad \sup_{i\in\mathbb{Z}^d} \sum_j J(i,j) < \infty. \tag{7.7}$$

Let $\{\tilde{J}(i,j), i \neq j, i, j \in \mathbb{Z}^d\}$ be another pairwise interaction satisfying an analogous summability assumption such that

$$J(i,j) \le \tilde{J}(i,j) \qquad \text{for all } i, j \in \mathbb{Z}^d.$$

Moreover, let $\{h_i, i \in \mathbb{Z}^d\}$ and $\{\tilde{h}_i, i \in \mathbb{Z}^d\}$ be two collections of positive real numbers representing an external field such that

$$h_i \le \tilde{h}_i \qquad \text{for all } i \in \mathbb{Z}^d, \sup_i \tilde{h}_i < \infty.$$

Finally we suppose that,

$$\text{for all } i \in \mathbb{Z}^d, \qquad \sum_j \big[\tilde{J}(i,j) - J(i,j)\big] \le \tilde{h}_i - h_i. \tag{7.8}$$

Recall that a probability measure μ on S is said to be a Gibbs measure relative to the interaction J and the external field $\{h_i\}$ if for all $i \in \mathbb{Z}^d$ and for any fixed $\zeta \in S$,

a version of the conditional probability $\mu(\{\sigma : \sigma(i) = \zeta(i) | \sigma(j) = \zeta(j) \text{ for all } j \neq i\})$ is given by

$$\mu\big(\{\sigma : \sigma(i) = \zeta(i) | \sigma(j) = \zeta(j) \text{ for all } j \neq i\}\big) = \frac{1}{1 + \exp(-2\beta[\sum_j J(i,j)\zeta(i)\zeta(j) + h_i\zeta(i)])}. \tag{7.9}$$

The Gibbs measure $\tilde{\mu}$ associated to the interaction $\tilde{J}$ and the external field $\{\tilde{h}_i\}$ is introduced analogously.

We consider a Glauber dynamics $(\sigma_t(i), i \in \mathbb{Z}^d, t \in \mathbb{R})$ taking values in $S = A^{\mathbb{Z}^d}$ and having μ as reversible measure. The process is defined by the rates $c_i(\sigma)$, $i \in \mathbb{Z}^d$, where $c_i(\sigma)$ is the rate at which the spin i flips (i.e., changes its sign) when the system is in the configuration σ. We take

$$c_i(\sigma) = \exp\left(-\beta\left[\sum_j J(i,j)\sigma(i)\sigma(j) + h_i\sigma(i)\right]\right). \tag{7.10}$$

By construction, the process $(\sigma_t)_t$ is reversible with respect to the Gibbs measure μ. In the same way, we can define a Glauber dynamics $(\tilde{\sigma}_t)_t$ reversible with respect to the Gibbs measure $\tilde{\mu}$, associated to the interaction $\tilde{J}$ and the external field $\{\tilde{h}_i\}$.

The main idea of our approach is a coupled construction of the processes (σ_t) and $(\tilde{\sigma}_t)$ which is order preserving. More precisely, let us write

$$\sigma \leq \tilde{\sigma} \quad \text{if and only if} \quad \sigma(i) \leq \tilde{\sigma}(i) \qquad \text{for all } i \in \mathbb{Z}^d.$$

Write

$$\mathcal{S} = \{(\sigma, \tilde{\sigma}) \in A^{\mathbb{Z}^d} \times A^{\mathbb{Z}^d} : \sigma \leq \tilde{\sigma}\} = \{(-1,-1), (-1,+1), (+1,+1)\}^{\mathbb{Z}^d}.$$

We now describe the coupled time evolution of σ_t and $\tilde{\sigma}_t$. Start with an ordered couple of initial configurations $\eta \leq \tilde{\eta}$ at time 0. Let

$$M_i = 2e^{\beta[\sum_{j \in \mathbb{Z}^d} \tilde{J}(i,j) + \tilde{h}_i]}.$$

For each site $i \in \mathbb{Z}^d$, consider a rate M_i Poisson process N^i. The Poisson processes corresponding to distinct sites are independent. If at time t, the Poisson clock ate site i rings, then both processes try simultaneously to update their spin at site i. Process σ replaces spin $\sigma(i)$ by $-\sigma(i)$ with probability

$$\frac{c_i(\sigma_t)}{M_i},$$

and the process $\tilde{\sigma}$ replaces spin $\tilde{\sigma}(i)$ by $-\tilde{\sigma}(i)$ with probability

$$\frac{\tilde{c}_i(\tilde{\sigma}_t)}{M_i}.$$

Hence, we can introduce the following probability measures. For any configuration σ with $\sigma(i) = -1$, we put

$$p_i(+1|\sigma) = \frac{c_i(\sigma)}{M_i}, \qquad p_i(-1|\sigma) = 1 - p_i(+1|\sigma).$$

In the same way, for any configuration σ with $\sigma(i) = +1$, we put

$$p_i(-1|\sigma) = \frac{c_i(\sigma)}{M_i}, \qquad p_i(+1|\sigma) = 1 - p_i(-1|\sigma).$$

The same definitions hold for $\tilde{c}_i$ with obvious modifications. Then we have by construction and thanks to condition (7.8) that

$$p_i(+1|\sigma) \le \tilde{p}_i(+1|\tilde{\sigma}) \qquad \text{whenever } \sigma \le \tilde{\sigma}.$$

This stochastic order makes it possible to construct a coupled Glauber dynamics $(\sigma_t, \tilde{\sigma}_t)_t$ taking values in the space of ordered configurations $\mathcal{S}$.

At each jump time t of one of the Poisson processes N^i, the ordered configuration $(\sigma_t, \tilde{\sigma}_t)$ is replaced at site i by the ordered pair

$$\begin{aligned}
&(+1,+1) \qquad \text{with probability } P_i\big((+1,+1)|(\sigma_t,\tilde{\sigma}_t)\big) = p_i(+1|\sigma_t),\\
&(-1,-1) \qquad \text{with probability } P_i\big((-1,-1)|(\sigma_t,\tilde{\sigma}_t)\big) = \tilde{p}_i(-1|\tilde{\sigma}_t),\\
&(-1,+1)\\
&\qquad \text{with probability } P_i\big((-1,+1)|(\sigma_t,\tilde{\sigma}_t)\big) = p_i(-1|\sigma_t) - \tilde{p}_i(-1|\tilde{\sigma}_t).
\end{aligned} \tag{7.11}$$

Now, it is straightforward to show that under our summability condition (7.7) on the interaction J and $\tilde{J}$ and due to the boundedness of the force of the external field, the transition probability P_i satisfies the continuity assumption (2.9). Hence, the decomposition of Theorem 1 can be applied and yields the following corollary [compare also to Theorem 3.3 of Galves, Garcia and Prieur (2010)].

COROLLARY 1. *There exists a sequence of transition probabilities* $P_k, k \ge -1$, *such that for any pair of symbols* $(a,b) \in \{(-1,-1),(-1,1),(1,1)\}$ *and any ordered pair of configurations* $(\sigma,\tilde{\sigma}) \in \mathcal{S}$,

$$P_i\big((a,b)|(\sigma,\tilde{\sigma})\big) = \sum_{k=-1}^{\infty} \lambda_i(k) P_i^{[k]}\big((a,b)|(\sigma,\tilde{\sigma})\big(V_i(k)\big)\big).$$

As in Galves, Löcherbach and Orlandi (2010), it can be shown that for this decomposition, a sufficient condition for (5.2) is

$$\sup_{i\in\mathbb{Z}^d} \sum_k |V_i(k)| \Bigg(\sum_{j:\|j-i\|=k} \tilde{J}(i,j) \Bigg) < \infty$$

and $\beta < \beta_c$, where β_c is solution of

$$2\beta \sum_{k\geq 1}\Bigg(|V_i(k)| \sum_{j:\|j-i\|=k} \tilde{J}(i,j)\Bigg) = 1.$$

In this case, we can extend the ideas of Galves, Garcia and Prieur (2010) from the case of chains of infinite order to infinite range Gibbs measures.

Our perfect simulation algorithm simulates two ordered configurations belonging to $\mathcal{S}$ according to the invariant distribution of the coupled Glauber dynamics. This yields an explicit coupling of the two Gibbs measures μ and $\tilde{\mu}$. Since this coupling is ordered, the very nice argument of the proof of Theorem 3.6 in Galves, Garcia and Prieur (2010) tells us that this coupling necessarily attains the $\bar{d}$-distance

$$\bar{d}(\mu,\tilde{\mu}) = \inf \sup_i \{\mathbb{P}(\sigma(i) \neq \tilde{\sigma}(i)) : (\sigma,\tilde{\sigma}) \text{ is a coupling of } \mu \text{ and } \tilde{\mu}\}.$$

Hence, our perfect simulation algorithm enables us to construct explicit couplings achieving this distance, and as far as we know the problem of finding explicit solutions was addressed only for finite volume Gibbs measures up to now.

8. Impatient user bias. Perfect simulation procedures, very often cannot be run until the algorithm stops, either by limitations of time or limitations of buffer. In this section we give upper bounds for the probability of these two types of errors.

According to our construction, the perfect simulation algorithm of μ presented in this article is a function $F : [0,1]^{\mathbb{N}\times\mathbb{Z}^d}$ to S such that, if $(U_n)_n = (U_n(i), i \in \mathbb{Z}^d)_n$ is a sequence of i.i.d. families, indexed by $\mathbb{Z}^d$, of uniform in $[0,1]$ random variables, then for any site $i \in \mathbb{Z}^d$, there exists a stopping time $N^{(i)}_{\mathrm{STOP}}$, such that F depends only on the first $N^{(i)}_{\mathrm{STOP}}$ families of $(U_n)_n$, that is, for any measurable $B \in \mathcal{A}$,

$$P\big[F\big((U_1(j))_j, \ldots, (U_{N^{(i)}_{\mathrm{STOP}}}(j))_j\big)(i) \in B\big] = \mu\big(\sigma(i) \in B\big).$$

Note that $N^{(i)}_{\mathrm{STOP}}$ is *not* the number of uniform random variables that have to be simulated in order to sample from μ; this number will, in general, be considerably larger. $N^{(i)}_{\mathrm{STOP}}$ is the number of steps of the backward sketch procedure.

A first kind of "impatient user bias" occurs whenever the user, for reasons independent of the algorithm, has to stop the algorithm after, say N steps maximal. In this case, we do not sample from μ, but instead sample from

$$P\big[F\big((U_1(j))_j, \ldots, (U_{N^{(i)}_{\mathrm{STOP}}}(j))_j\big)(i) \in B | N^{(i)}_{\mathrm{STOP}} \leq N\big].$$

By Proposition 6.2 of Fill (1998) [compare also to Section 6 of Ferrari, Fernández and Garcia (2002)] the error made above can be bounded by

$$\frac{P(N^{(i)}_{\mathrm{STOP}} > N)}{1 - P(N^{(i)}_{\mathrm{STOP}} > N)} \leq \frac{\gamma^N}{1-\gamma^N}$$

(see Theorem 3 above).

At each step of the backward sketch procedure, a range of order k is chosen, where k is, in general, not bounded from above. In practical situations, however, a user will be limited in the choice of the interaction range and will restrict the simulation to the choice of ranges bounded by a certain upper bound L that he decided to fix in advance. More precisely, writing

$$T_L^{(i)} := \inf\{\tilde{T}_n^{(i)} : \tilde{K}_n^{(i)} > L\},$$

the user will therefore sample from the measure

$$P\big[F\big(\big(U_1(j)\big)_j, \ldots, \big(U_{N_{\mathrm{STOP}}^{(i)}}(j)\big)_j\big)(i) \in B | \{N_{\mathrm{STOP}}^{(i)} \le N\} \cap \{T_L^{(i)} > T_{\mathrm{STOP}}^{i)}\}\big].$$

In order to control the error made induced by this "space–time impatient user bias," we have to control

$$P\big(T_L^{(i)} \le T_{\mathrm{STOP}}^{i)}\big).$$

Using arguments similar to Lemma 2 of Galves, Löcherbach and Orlandi (2010), this can be bounded by

$$P\big(T_L^{(i)} \le T_{\mathrm{STOP}}^{i)}\big) \le \sup_{i \in \mathbb{Z}^d} \left(\frac{M_i - \alpha_i(L)}{M_i}\right) \frac{1}{1-\gamma}.$$

9. Final comments and bibliographical discussion. In this work we study the equilibrium measure of systems with infinite range interactions satisfying fast decay of the long range influence on the change rate and a certain subcriticality-criterion. For Gibbs random fields, this regime has traditionally been studied via cluster-expansion methods which either rely on sophisticated combinatorial estimations [Malyshev (1980), Seiler (1982), Brydges (1986)] or inductive hypothesis and complex analysis [Kotecký and Preiss (1986), Dobrushin (1996a, 1996b)].

This is not the approach we follow here. Our approach is probabilistic, based on an explicit construction of the dynamics and gives probabilistic insight into the structure of the stationary law of the process, without combinatorial or complex-analysis techniques. Let us stress that our approach is not an alternative to cluster expansions. It has a different regime of validity and different aims.

Our construction is reminiscent of Harris's graphical representation for particle systems and it is similar in spirit to procedures adopted in Bertein and Galves (1977/78), Ferrari (1990), van den Berg and Steif (1999), Ferrari, Fernández and Garcia (2002) and Garcia and Marić (2006) among others. However, all these papers only consider particular models, satisfying restrictive assumptions which are not assumed in the present paper. Our approach works for any infinite range continuous interaction under the only assumption of high-noise.

There are several techniques for perfect simulation of Markov processes. Among the most popular ones figure *coupling from the past* (CFTP) originally

proposed by Propp and Wilson (1996) and applied to several special cases in a vast literature. A good review can be found in Kendall (2005). This kind of technique applies to invariant measures of Markov processes with finite coalescence time. One main point of the CFTP technique is that one has to be able to control the coalescence times uniformly with respect to all possible starting points. This is an issue that becomes particularly difficult in the case of "big" state spaces. The problem of large state spaces can be overcome for processes with certain monotonicity properties or for some specific cases. For example, for spatial point processes there is a vast literature on the subject; we point out the works of Kendall (1998), Kendall and Thönnes (1999), Kendall and Møller (2000) among others.

In our case, we sample directly from a time stationary realization of the process. There is no coalescence criterion, either between coupled realizations or between sandwiching processes. The scheme neither requires nor takes advantage of monotonicity properties. The scheme directly samples a finite window of the equilibrium measure in *infinite-volume*. In contrast, other CFTP algorithms [e.g., Kendall (1997, 1998)] focus on finite windows with fixed boundary conditions, and the infinite-volume limit requires an additional process of "perfect simulation in space." We point out Ferrari (1990), van den Berg (1993) and van den Berg and Maes (1994) have also proposed construction schemes for (infinite-volume) Gibbs measures of spin systems that can be easily transcribed into perfect-simulation algorithms.

For continuous state spaces in systems with finite number of components, Cai (2005) proposes a nonmonotone CFTP but as he points out "the detailed construction of the nonmonotone CFTP algorithm is problem specific." Fernández, Ferrari and Grynberg (2007) construct perfect simulation for random distributions supported on a d dimensional box, in particular, multivariate normal distributions restricted to a compact set. Connor and Kendall (2007) show that for a large class for positive recurrent Markov processes it is always possible to perform CFTP, although not always feasible. In general, for interacting particle systems with continuous state spaces, it seems to be out of reach to apply CFTP successfully. A recent paper by Huber (2007) succeeded in using CFTP for a very specific case of a continuous autonormal system restricted to a finite box.

The notion of random Markov chains was introduced explicitly in Kalikow (1990) and Bramson and Kalikow (1993) and appeared implicitly in Ferrari et al. (2000) and Comets, Fernández and Ferrari (2002).

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