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The complementary exponential power series distribution

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Abstract. In this paper, we introduce the complementary exponential power series distributions, with failure rate increasing, which is complementary to the exponential power series model proposed by Chahkandi and Ganjali [*Comput. Statist. Data Anal.* **53** (2009) 4433–4440]. The new class of distribution arises on latent complementary risks scenarios, where the lifetime associated with a particular risk is not observable, rather we observe only the maximum lifetime value among all risks. This new class contains several distributions as a particular case. The properties of the proposed distribution class are discussed, such as quantiles, moments and order statistics. Estimation is carried out via maximum likelihood. Simulation results on maximum likelihood estimation are presented. A real application illustrates the usefulness of the new distribution class.

1 Introduction

The exponential distribution is widely used for modeling many problems in lifetime testing and reliability studies. However, the exponential distribution does not provide a reasonable parametric fit for some practical applications where the underlying failure rates are nonconstant, presenting monotone shapes. Recently, new distributions to model the failure rate have appeared in the literature, such as those obtained by compounding the exponential distribution with several discrete distributions. For example, a distribution with a decreasing failure rate has been obtained by assuming the minimum of a random sample of the exponential distribution and a random sample size. In order to do this, the geometric, the Poisson and the logarithmic distributions were considered by Adamidis and Loukas (1998), Kus (2007) and Tahmashi and Rezaei (2008), respectively. These works were generalized by Chahkandi and Ganjali (2009) by showing that the composition of the exponential distribution with the power series distribution yields a distribution with a decreasing failure rate, the exponential power series (EPS) distribution. Later Morais and Barreto-Souza (2011) replaced the exponential distribution by the Weibull generating distributions with decreasing failure rates when the form

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parameter is lower or equal than 1 and different types of failure rates when the form parameter is greater than 1. The distribution proposed by Kus (2007) was generalized by including a power parameter in his distribution, by Barreto-Souza and Cribari-Neto (2009). A different approach, that considers the maximum, instead of the minimum, has been also considered. In this case the distributions obtained have increasing failure rates. The geometric distribution was considered by Adamidis et al. (2005), later generalized in Silva, Barreto-Souza and Cordeiro (2010) by including a power parameter in his distribution. The Poisson distribution was assumed to be obtained by the distribution of Cancho, Louzada-Neto and Barriga (2011), which later was generalized in Cordeiro, Rodrigues and de Castro (2012) by assuming a COMPoisson distribution, by including a power parameter in his distribution. The power series distribution, however, has not been considered yet when the maximum number of competing causes is considered, leading to a complementary risk scenario. In this paper, assuming a power series distribution, we propose a new family distribution based on a complementary risk problem (Basu and Klein, 1982) in the presence of latent risks. We assume that there is no information about which factor was responsible for the component failure, but only the maximum lifetime value among all risks is observed instead of the minimum lifetime value among all risks as in Chahkandi and Ganjali (2009) and Morais and Barreto-Souza (2011). The new distribution is a counterpart of the EPS distribution and then, hereafter, it shall be called the complementary exponential power series (CEPS) distribution.

The paper is organized as follows. In Section 2 we define the CEPS distribution. In Section 3 the survival and failure rate function, the quantiles, moments, order statistics and moments of the order statistics are provided. In Section 4 some special cases are studied in detail and are compared to the graphics of failure rate functions of these cases with the ones analogues EPS distributions for some particular parameters. Estimation of the parameters by maximum likelihood is given in Section 5. Section 6 presents the results of a simulation study. In Section 7 an application to one real data set is provided. Final remarks in Section 8 conclude the paper.

2 The CEPS distribution

In the classical complementary risks scenarios (Basu and Klein, 1982) the lifetime associated with a particular risk is not observable, rather we observe only the maximum lifetime value among all risks. Simplistically, in reliability, we observe only the maximum component lifetime of a parallel system, that is, the observable quantities for each component are maximum lifetime value to failure among all risks, and the cause of failure. Complementary risk problems arise in several areas, such as medical, industrial and financial ones [interested readers can refer to Lawless (2003), Crowder et al. (1991) and Cox and Oakes (1984)].

A difficulty arises if the risks are latent in the sense that there is no information about which factor was responsible for the component failure, which can be often observed in field data. We call these latent complementary risks data. On many occasions this information is not available or it is impossible that the true cause of failure is specified by an expert. In reliability, the components can be totally destroyed in the experiment. Further, the true cause of failure can be masked from our view. In modular systems, the need to keep a system running means that a module that contains many components can be replaced without the identification of the exact failing component. Goetghebeur and Ryan (1995) addressed the problem of assessing covariate effects based on a semi-parametric proportional hazards structure for each failure type when the failure type is unknown for some individuals. Reiser et al. (1995) considered statistical procedures for analyzing masked data, but their procedure can not be applied when all observations have an unknown cause of failure. Lu and Tsiatis (2001) present a multiple imputation method for estimating regression coefficients for risk modeling with missing cause of failure. A comparison of two partial likelihood approaches for risk modeling with missing cause of failure is presented in Lu and Tsiatis (2005).

The proposed distribution can be derived as follows. Let Z be a random variable denoting the number of failure causes, $z = 1, 2, \dots,$ and considering Z following a power series distribution (truncated at zero) with probability function given by

$$P[Z = z; \theta] = \frac{a_z \theta^z}{A(\theta)}, \qquad z = 1, 2, \dots, \theta \in (0, s), \tag{2.1}$$

where $a_1, a_2, \dots$ is a sequence of non-negative real numbers, where at least one of them is strictly positive, s is a positive number no greater than the ratio of convergence of the power series $\sum_{z=1}^{\infty} a_z \theta^z$, and $A(\theta) = \sum_{z=1}^{\infty} a_z \theta^z$, $\forall \theta \in (0, s)$. Notice, in particular, that A is positive and infinitely many differentiable. For more details on the power series class of distributions, see Johnson, Kemp and Kotz (2005). Table 1, reported in Morais and Barreto-Souza (2011), shows useful quantities of some power series distributions (truncated at zero) such as Poisson, logarithmic, geometric and binomial (with m being the number of replicas) distributions. The quantities $A'(\theta)$ and $A''(\theta)$ are the derivation of $A(\theta)$ and $A'(\theta)$ with respect to θ, respectively. The quantile $A^{-1}(\theta)$ is the inverse function of $A(\theta)$.

Table 1 *Useful quantities of some power series distributions*

Distribution	a_z	$A(\theta)$	$A'(\theta)$	$A''(\theta)$	$A^{-1}(\theta)$	s
Poisson	$z!^{-1}$	$e^{\theta} - 1$	e^{θ}	e^{θ}	$\log(1 + \theta)$	∞
Logarithmic	z^{-1}	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	1
Geometric	1	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(1 + \theta)^{-1}$	1
Binomial	$\binom{m}{z}$	$(1 + \theta)^m - 1$	$m(1 + \theta)^{m-1}$	$\frac{m(m-1)}{(1+\theta)^{2-m}}$	$(\theta - 1)^{1/m} - 1$	∞

Let's also consider $Y_1, Y_2, \ldots,$ be a sequence of independent, identically distributed, continuous random variables, independent of Z, with exponential distribution with parameter $\beta > 0$, that is, the probability density function (p.d.f.) is given by

$$f(y; \beta) = \beta \exp\{-\beta y\}, \qquad y > 0. \tag{2.2}$$

These random variables represent the lifetimes associated with the failure causes. In the latent complementary risks scenario, the number of causes Z and the lifetime Y_i associated with a particular cause are not observable (latent variables), but only the maximum lifetime X among all independent causes is usually observed. So, we only observe the random variable given by

$$X = \max\{Y_1, \ldots, Y_Z\}. \tag{2.3}$$

Then, $f(x|z; \beta) = z\beta e^{-\beta x}(1 - e^{-\beta x})^{z-1}$ and the marginal p.d.f. X is

$$\begin{aligned} f(x; \theta, \beta) &= \sum_{z=1}^{\infty} f(x|z, \beta) P(Z = z; \theta) \\ &= \sum_{z=1}^{\infty} z\beta e^{-\beta x}(1 - e^{-\beta x})^{z-1} \frac{a_z \theta^z}{A(\theta)} \\ &= \sum_{z=1}^{\infty} a_z z \theta \beta e^{-\beta x} \frac{[\theta(1 - e^{-\beta x})]^{z-1}}{A(\theta)} \end{aligned} \tag{2.4}$$

but

$$\sum_{z=1}^{\infty} a_z z \theta \beta e^{-\beta x}[\theta(1 - e^{-\beta x})]^{z-1} = \frac{\partial}{\partial x} A\big(\theta(1 - e^{-\beta x})\big).$$

So

$$\begin{aligned} f(x; \theta, \beta) &= \frac{\partial}{\partial x} A\big(\theta(1 - e^{-\beta x})\big) / A(\theta) \\ &= \frac{\theta \beta e^{-\beta x} A'(\theta(1 - e^{-\beta x}))}{A(\theta)}, \qquad x > 0, \theta, \beta > 0, \end{aligned} \tag{2.5}$$

where $A'(\theta(1 - e^{-\beta x}))$ is the derivative of $A(\cdot)$ evaluated at $\theta(1 - e^{-\beta x})$. We denote a random variable X following CEPS distribution with parameters θ, β by $X \sim \text{CEPS}(\theta, \beta)$.

Notice by equation (2.4) that p.d.f. of the CEPS distribution can be written as a mixture of densities as follows:

$$f(x) = \sum_{z=1}^{\infty} \frac{a_z \theta^z}{A(\theta)} f_z(x), \tag{2.6}$$

where f_z is the p.d.f. of the maximum of a sample of size z of a exponential distribution with parameter β, that is,

$$f_z(x) = z\beta e^{-\beta x}(1 - e^{-\beta x})^{z-1}, \qquad x > 0. \tag{2.7}$$

3 Some properties of the CEPS distribution

3.1 The distribution, survivor, failure rate functions

Let X be a non-negative random variable denoting the lifetime of a component in some population with CEPS distribution with parameters θ and β, that is, $X \sim \mathrm{CEPS}(\theta, \beta)$. The distribution function is given by

$$F(x; \theta, \beta) = \frac{A(\theta(1 - e^{-\beta x}))}{A(\theta)}, \qquad x > 0, \tag{3.1}$$

and the survival function is

$$S(x; \theta, \beta) = 1 - \frac{A(\theta(1 - e^{-\beta x}))}{A(\theta)}, \qquad x > 0. \tag{3.2}$$

The following proposition shows that our distribution has exponential distribution as limiting distribution, when $a_1 > 0$.

Proposition 3.1. *If $a_1 > 0$, the exponential distribution with parameter β is a limiting special case of the CEPS distribution when $\theta \to 0^+$. In general, $\lim_{\theta\to 0^+} F(x; \theta, \beta) = (1 - e^{-\beta x})^k$, with $k = \min\{n \in \mathbb{N}^+ : a_n > 0\}$.*

Proof. Considering equation (3.1) and using the L'Hospital's rule k times, it follows that

$$\begin{aligned}\lim_{\theta\to 0^+} F(x; \theta, \beta) &= \lim_{\theta\to 0^+} \frac{(1 - e^{-\beta x})^k A^{(k)}(\theta(1 - e^{-\beta x}))}{A^{(k)}(\theta)} \\ &= \frac{(1 - e^{-\beta x})^k a_k}{a_k} = (1 - e^{-\beta x})^k.\end{aligned}$$

□

The failure rate function is given by

$$h(x; \theta, \beta) = \frac{f(x; \theta, \beta)}{S(x; \theta, \beta)} = \frac{\theta\beta e^{-\beta x} A'(\theta(1 - e^{-\beta x}))}{A(\theta) - A(\theta(1 - e^{-\beta x}))}, \qquad x > 0. \tag{3.3}$$

Proposition 3.2. *The failure rate function is increasing for x sufficiently large.*

Proof. The derivative of the failure rate function is given by

$$h'(x) = \frac{\theta\beta^2 e^{-\beta x} w(x)}{[A(\theta) - A(\theta(1 - e^{-\beta x}))]^2},$$

where

$$\begin{aligned} w(x) = {} & \big[-A'\big(\theta(1-e^{-\beta x})\big) + \theta e^{-\beta x} A''\big(\theta(1-e^{-\beta x})\big)\big] \\ & \times \big[A(\theta) - A\big(\theta(1-e^{-\beta x})\big)\big] \\ & + \theta e^{-\beta x}\big[A'\big(\theta(1-e^{-\beta x})\big)\big]^2. \end{aligned}$$

Notice that $\lim_{x\to\infty} w(x) = 0$. Therefore, the claim follows by showing that $w'(x) < 0$, for x sufficiently large. But $w'(x) = \theta\beta e^{-\beta x} w_1(x)$, with

$$\begin{aligned} w_1(x) = {} & \big[-2A''\big(\theta(1-e^{-\beta x})\big) + \theta e^{-\beta x} A'''\big(\theta(1-e^{-\beta x})\big)\big] \\ & \times \big[A(\theta) - A\big(\theta(1-e^{-\beta x})\big)\big] \\ & + \theta e^{-\beta x} A''\big(\theta(1-e^{-\beta x})\big) A'\big(\theta(1-e^{-\beta x})\big) \end{aligned}$$

and $\lim_{x\to\infty} w_1(x) = 0$. Then it's sufficient to show that $w_1'(x)$ is positive for x sufficiently large. Notice that $w_1'(x) = \theta\beta e^{-\beta x} w_2(x)$, with

$$\begin{aligned} w_2(x) = {} & \big[-3A'''\big(\theta(1-e^{-\beta x})\big) + \theta e^{-\beta x} A^{(\mathrm{iv})}\big(\theta(1-e^{-\beta x})\big)\big] \\ & \times \big[A(\theta) - A\big(\theta(1-e^{-\beta x})\big)\big] \\ & + A''\big(\theta(1-e^{-\beta x})\big) A'\big(\theta(1-e^{-\beta x})\big) + \theta e^{-\beta x}\big[A''\big(\theta(1-e^{-\beta x})\big)\big]^2 \end{aligned}$$

and $\lim_{x\to\infty} w_2(x) = A''(\theta)A'(\theta) > 0$. Then $w_2(x)$ and $w_1'(x)$ are positive for x sufficiently large. □

The following proposition gives the initial and long-term values for the failure rate function. This follows by (3.3).

Proposition 3.3. *The failure rate function has the following limits*:

$$\lim_{x\to 0^+} h(x) = \frac{a_1\theta}{A(\theta)}\beta \quad \textit{and} \quad \lim_{x\to\infty} h(x) = \beta.$$

Notice that $\lim_{x\to 0^+} h(x) \le \lim_{x\to\infty} h(x)$.

3.2 Quantiles, moments, mean residual and order statistics

From (3.1) the quantile γ of the CEPS distribution, $x_\gamma = F^{-1}(\gamma; \theta, \beta)$ is given by

$$x_\gamma = -\beta^{-1}\log\{1 - \theta^{-1}A^{-1}(\gamma A(\theta))\}, \tag{3.4}$$

where $A^{-1}(\cdot)$ is the inverse function of $A(\cdot)$.

An expression for the moments of a CEPS distribution can be derived as the following.

Proposition 3.4. *The rth moment of* CEPS(θ, β) *distribution is finite and is given by*

$$E(X^r) = \frac{\Gamma(r+1)}{\beta^r A(\theta)} \sum_{z=1}^{\infty} \sum_{j=0}^{z-1} a_z \theta^z z \binom{z-1}{j} (-1)^j \frac{1}{(j+1)^{r+1}}. \tag{3.5}$$

Proof. Since A' is a nondecreasing function, $A'(\theta(1 - e^{-\beta x})) \leq A'(\theta)$. Hence, by (2.5) it follows that $f(x) \leq \frac{A'(\theta)}{A(\theta)} \theta \beta e^{-\beta x}$, which implies that $E(X^r)$ is finite.

By equation (2.6), that describes the density CEPS as a mixture, it follows

$$E(X^r) = \sum_{z=1}^{\infty} \frac{a_z \theta^z}{A(\theta)} E(Y_z^r),$$

where Y_z has f_z as its density function, defined in (2.7). Equation (3.5) follows from this equality and that $E(Y_z^r) = \frac{\Gamma(r+1)}{\beta^r} \sum_{j=0}^{z-1} z \binom{z-1}{j} (-1)^j \frac{1}{(j+1)^{r+1}}$. □

An expression for the mean residual of a CEPS distribution can be derived as the following.

Proposition 3.5. *The mean residual, given the survival to time x, until the time of failure, of the CEPS distribution can be obtained as follows*:

$$\begin{aligned} m(x) &= E(X - x | X \geq x) \\ &= \frac{1}{\beta A(\theta) S(x)} \sum_{z=1}^{\infty} \sum_{j=0}^{z-1} a_z \theta^z z \binom{z-1}{j} (-1)^{j-1} \frac{e^{-\beta(j+1)x}}{(j+1)^2}. \end{aligned} \tag{3.6}$$

Proof. Since the conditional density of $X - x_0$ given $X \geq x_0$ is $f(x + x_0)/S(x_0)$, then $E(X - x | X \geq x) = \frac{1}{S(x)} \int_0^{\infty} y f(y + x)\, dy$ and the claim follows by writing f as a mixture of the density functions f_z as in (2.7). □

An explicit expression for the density of the ith order statistic $X_{i:n}$, say, $f_{i:n}(x)$, in a random sample of size n from the CEPS distribution is derived in the sequel. It is well known that

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) F^{i-1}(x) \big(1 - F(x)\big)^{n-i}$$

for $i = 1, \ldots, n$, where $B(\cdot, \cdot)$ is the beta function. Using the binomial expansion in the last equation, $f_{i:n}(x)$ becomes

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \frac{(-1)^k}{B(i, n-i+1)} \binom{n-i}{k} f(x) F^{i+k-1}(x), \tag{3.7}$$

where $f(\cdot)$ and $F(\cdot)$ are p.d.f. and c.d.f. given by (2.5) and (3.1), respectively. By inserting these equations in (3.7), we obtain for $x > 0$

$$\begin{aligned} f_{i:n}(x;\theta,\beta) = {} & \frac{\theta\beta e^{-\beta x} A'(\theta(1-e^{-\beta x}))}{A(\theta)B(i,n-i+1)} \\ & \times \sum_{k=0}^{n-i}(-1)^k \binom{n-i}{k} \left[\frac{A(\theta(1-e^{-\beta x}))}{A(\theta)}\right]^{i+k-1}. \end{aligned} \tag{3.8}$$

The moments of the CEPS distribution order statistics are obtained by using a result due to Barakat and Abdelkader (2004) applied to the independent and identically distributed (i.i.d.) case, leading to

$$\begin{aligned} E[X_{i:n}^r;\theta,\beta] &= r \sum_{k=n-i+1}^{n} (-1)^{k-n+i-1} \binom{k-1}{n-i} \binom{n}{k} \int_0^\infty x^{r-1} S^k(x;\beta,p)\,dx \\ &= r \sum_{k=n-i+1}^{n} (-1)^{k-n+i-1} \binom{k-1}{n-i} \binom{n}{k} \\ &\quad \times \int_0^\infty x^{r-1} \left[1 - \frac{A(\theta(1-e^{-\beta x}))}{A(\theta)}\right]^k dx. \end{aligned} \tag{3.9}$$

4 Special cases

In this section we present some special cases of the CEPS distribution. Expressions for mean, variance and mean residual are presented.

4.1 Complementary exponential binomial distribution

The complementary exponential binomial (CEB) distribution is defined from the c.d.f. (2.5) with $A(\theta) = (1+\theta)^m - 1$, which is given by

$$F(x;\theta,\beta) = \frac{(1+\theta(1-e^{-\beta x}))^m - 1}{(1+\theta)^m - 1}, \qquad x > 0, \tag{4.1}$$

where m is integer positive.

The associated p.d.f. and failure rate function are given, respectively, by

$$f(x;\theta,\beta) = \frac{\theta\beta e^{-\beta x} m(1+\theta(1-e^{-\beta x}))^{m-1}}{(1+\theta)^m - 1}$$

and

$$h(x;\theta,\beta) = \frac{\theta\beta e^{-\beta x} m(1+\theta(1-e^{-\beta x}))^{m-1}}{(1+\theta)^m - (1+\theta(1-e^{-\beta x}))^m}$$

for $x > 0$.

The mean, variance and mean residual of the CEB distribution are given, respectively, by

$$E[X] = \frac{(1+\theta)^m}{\beta((1+\theta)^m - 1)} \sum_{z=1}^{m} \frac{(-1)^{z+1}}{z} \binom{m}{z} \left(\frac{\theta}{1+\theta}\right)^z,$$

$$\mathrm{Var}[X] = \frac{(1+\theta)^m}{\beta^2((1+\theta)^m - 1)} \Bigg[2 \sum_{z=1}^{m} \frac{(-1)^{z+1}}{z^2} \binom{m}{z} \left(\frac{\theta}{1+\theta}\right)^z - \frac{(1+\theta)^m}{(1+\theta)^m - 1} \left(\sum_{z=1}^{m} \frac{(-1)^{z+1}}{z} \binom{m}{z} \left(\frac{\theta}{1+\theta}\right)^z \right)^2 \Bigg]$$

and

$$m(x) = \frac{(1+\theta)^m}{\beta[(1+\theta)^m - (1+\theta(1-e^{-\beta x}))^m]} \sum_{z=1}^{m} \frac{(-1)^z}{z} \binom{m}{z} \left(\frac{\theta e^{-\beta x}}{1+\theta}\right)^x.$$

4.2 Complementary exponential Poisson distribution

The complementary exponential Poisson (CEP) distribution was introduced by Cancho, Louzada-Neto and Barriga (2011). The p.d.f. and survival function are given by

$$f(x;\theta,\beta) = \frac{\theta\beta e^{-\beta x - \theta e^{-\beta x}}}{1 - e^{-\theta}} \quad \text{and} \quad S(x;\theta,\beta) = \frac{1 - e^{-\theta e^{-\beta x}}}{1 - e^{-\theta}}$$

for $x > 0$, respectively. The next proposition gives us another characterization of the CEP distribution.

Proposition 4.1. *The CEP distribution can be obtained as limiting of CEB distribution with c.d.f. given by* (4.1), *if* $m\theta \to \lambda > 0$, *when* $m \to \infty$ *and* $\theta \to 0^+$.

Proof. We shall show that the survival function of the CEB distribution converges to the survival function of the CEP distribution under conditions of the proposition, that is,

$$\lim_{\substack{m\to\infty \\ \theta\to 0^+}} 1 - \frac{(1+\theta(1-e^{-\beta x}))^m - 1}{(1+\theta)^m - 1} = \frac{1 - e^{-\lambda e^{-\beta x}}}{1 - e^{-\lambda}}.$$

The claim follows by the following limits:

$$\lim_{\substack{m\to\infty \\ \theta\to 0^+}} (1+\theta(1-e^{-\beta x}))^m = \lim_{\substack{m\to\infty \\ \theta\to 0^+}} \left(1 + \frac{m\theta(1-e^{-\beta x})}{m}\right) = e^{\lambda(1-e^{-\beta x})}$$

and

$$\lim_{\substack{m\to\infty\\ \theta\to 0^+}} (1+\theta)^m = \lim_{\substack{m\to\infty\\ \theta\to 0^+}} \left(1+\frac{m\theta}{m}\right)^m = e^{\lambda}. \qquad \square$$

The failure rate function of the CEP distribution is given by

$$h(x;\theta,\beta) = \frac{\theta\beta e^{-\beta x}}{e^{\theta e^{-\beta x}} - 1}, \qquad x > 0.$$

The mean, variance and mean residual of the CEP distribution are given, respectively, by

$$E[X] = \frac{\theta}{\beta(1-e^{-\theta})} F_{2,2}([1,1],[2,2],-\theta),$$

$$\operatorname{Var}[X] = \frac{2\theta}{\beta^2(1-e^{-\theta})}\Big[2F_{3,3}([1,1,1],[2,2,2],-\theta) - \frac{\theta}{1-e^{-\theta}} F^2_{2,2}([1,1],[2,2],-\theta)\Big]$$

and

$$m(x) = \frac{\theta e^{-\beta x}}{\lambda(1-e^{-\theta e^{-\beta x}})} F_{2,2}([1,1],[2,2],-\theta e^{-\beta x}),$$

where $F_{p,q}(\mathbf{n},\mathbf{d},\lambda)$ is the generalized hypergeometric function. This function is also known as Barnes's extended hypergeometric function. The definition of $F_{p,q}(\mathbf{n},\mathbf{d},\lambda)$ is

$$F_{p,q}(\mathbf{n},\mathbf{d},\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k \prod_{i=1}^{p} \Gamma(n_i+k)\Gamma^{-1}(n_i)}{\Gamma(k+1)\prod_{i=1}^{q}\Gamma(d_i+k)\Gamma^{-1}(d_i)},$$

where $\mathbf{n} = [n_1, n_2, \ldots, n_p]$, p is the number of operands of $\mathbf{n}$, $\mathbf{d} = [d_1, d_2, \ldots, d_q]$ and q is the number of operands of $\mathbf{d}$. The generalized hypergeometric function is quickly evaluated and readily available in standard software such as Maple or R (R Development Core Team, 2008).

4.3 Complementary exponential geometric distribution

The complementary exponential geometric (CEG) distribution, introduced by Adamidis et al. (2005), is defined by the c.d.f. (2.5) with $A(\theta) = \theta(1-\theta)^{-1}$, which is given by

$$F(x;\theta,\beta) = \frac{(1-\theta)(1-e^{-\beta x})}{(1-\theta(1-e^{-\beta x}))}, \qquad x > 0, \theta \in (0,1).$$

The associated p.d.f. and failure rate function are given, for $x > 0$, respectively, by

$$f(x;\theta,\beta) = \frac{(1-\theta)\beta e^{-\beta x}}{(1-\theta(1-e^{-\beta x}))^2} \quad \text{and} \quad h(x;\theta,\beta) = \frac{(1-\theta)\beta}{1-\theta(1-e^{-\beta x})}.$$

The mean, variance and mean residual are given, respectively, by

$$E[X] = -\frac{1}{\theta\beta}\log(1-\theta), \qquad \mathrm{Var}[X] = -\frac{2}{\theta\beta^2}\left[2\,\mathrm{Li}_2\left(\frac{\theta}{\theta-1}\right) + \frac{1}{\theta}\log^2(1-\theta)\right]$$

and

$$m(x) = -\frac{1-\theta(1-e^{-\beta x})}{\beta\theta e^{-\beta x}}\left[\log(1-\theta) - \log\bigl(1-\theta(1-e^{-\beta x})\bigr)\right],$$

where $\mathrm{Li}_s(z)$ is the polylogarithm function defined by

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)}\int_0^\infty \frac{u^{s-1}e^{-u}}{1-ze^{-u}}\,du, \qquad z < 1, s > 0.$$

The polylogarithm function is quickly evaluated in standard software such as R.

4.4 Complementary exponential logarithmic distribution

The c.d.f. of the complementary exponential logarithmic (CEL) distribution is defined by (2.5) with $A(\theta) = -\log(1-\theta)$, $0 < \theta < 1$. The associated p.d.f. and failure rate function are

$$f(x;\theta,\beta) = -\frac{\theta\beta e^{-\beta x}}{\log(1-\theta)(1-\theta(1-e^{-\beta x}))}$$

and

$$h(x;\theta,\beta) = -\frac{\theta\beta e^{-\beta x}}{\log((1-\theta)/(1-\theta(1-e^{-\beta x})))(1-\theta(1-e^{-\beta x}))}$$

for $x > 0$, respectively.

The mean, variance and mean residual of the CEL distribution are given, respectively, by

$$E[X] = \frac{1}{\beta\log(1-\theta)}\mathrm{Li}_2\left(\frac{\theta}{\theta-1}\right),$$

$$\mathrm{Var}[X] = \frac{2}{\beta^2\log(1-\theta)}\mathrm{Li}_3\left(\frac{\theta}{\theta-1}\right) - \frac{1}{\beta^2\log^2(1-\theta)}\mathrm{Li}_2^2\left(\frac{\theta}{\theta-1}\right)$$

and

$$m(x) = \frac{1}{\beta[\log(1-\theta) - \log(1-\theta(1-e^{-\beta x}))]}\mathrm{Li}_2\left(\frac{\theta}{\theta-1}e^{-\beta x}\right).$$

Figure 1 shows the behavior of failure rate functions of the EPS and CEPS distributions for some values of the parameters. The CEPS failure rate function increases while the EPS failure rate function decreases with x, but both converge to β when $x \to \infty$, corroborating Proposition 3.3.

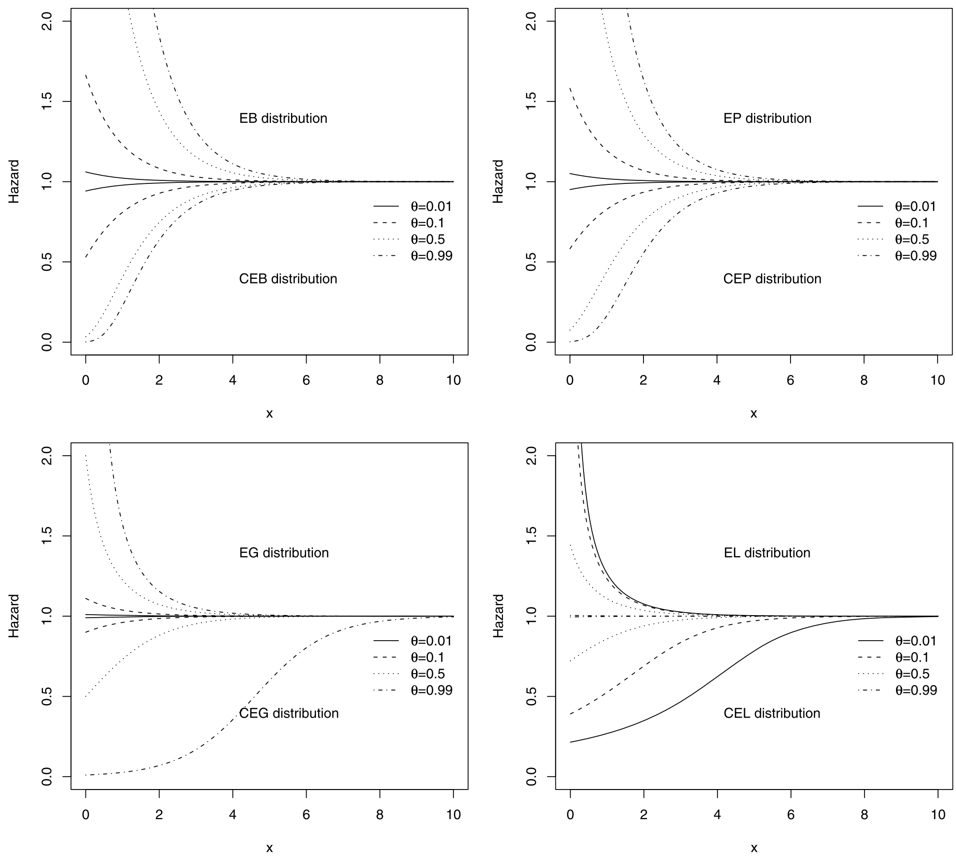

Figure 1 *Comparing the failure rate function of the EPS and CEPS distributions for the same fixed θ values. We fixed $\beta = 1.0$.*

5 Inference

5.1 Maximum likelihood estimation

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a random sample of the CEPS distribution with unknown parameter vector $\xi = (\theta, \beta)$. The log-likelihood $l = l(\xi; \mathbf{x})$ is given by

$$l = n \log(\theta\beta) - \beta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\bigl(A'\bigl(\theta(1 - e^{-\beta x_i})\bigr)\bigr) - n \log(A(\theta)). \tag{5.1}$$

The maximum likelihood estimations (MLEs) of θ and β can be derived directly either from the log-likelihood (5.1) or by solving the following nonlinear system:

$$\frac{\partial l(\xi; \mathbf{x})}{\partial \theta} = \frac{n}{\theta} - \frac{nA'(\theta)}{A(\theta)} + \sum_{i=1}^{n} \frac{(1 - e^{-\beta x_i})A''(\theta(1 - e^{-\beta x_i}))}{A'(\theta(1 - e^{-\beta x_i}))} = 0, \tag{5.2}$$

$$\frac{\partial l(\xi;\mathbf{x})}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}x_i+\theta\sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}A''(\theta(1-e^{-\beta x_i}))}{A'(\theta(1-e^{-\beta x_i}))}=0. \tag{5.3}$$

Large sample inference for the parameters can be based, in principle, on the MLEs and their estimated standard errors. According to Cox and Hinkley (1974), it can be shown, under suitable regularity conditions, the following convergence in distribution:

$$\sqrt{n}(\widehat{\theta}-\theta,\widehat{\beta}-\beta)\xrightarrow{D}N_2(\mathbf{0},\mathbf{I}^{-1}(\theta,\beta)),$$

where $\mathbf{I}(\theta,\beta)$ is the Fisher information matrix, that is,

$$\mathbf{I}(\theta,\beta)=-\begin{bmatrix}E\left(\frac{\partial^2 L(\theta,\beta;X)}{\partial\theta^2}\right) & E\left(\frac{\partial^2 L(\theta,\beta;X)}{\partial\theta\,\partial\beta}\right)\\ E\left(\frac{\partial^2 L(\theta,\beta;X)}{\partial\theta\,\partial\beta}\right) & E\left(\frac{\partial^2 L(\theta,\beta;X)}{\partial\beta^2}\right)\end{bmatrix},$$

where $L(\theta,\beta;x)=\log(f(x;\theta,\beta))$ is the logarithm of density CEPS(θ,β) distribution (given by equation (2.5)), then the second derivatives are given by

$$\begin{aligned}\frac{\partial^2 L(\theta,\beta;x)}{\partial\theta^2}&=-\frac{1}{\theta^2}-\frac{A''(\theta)A(\theta)-A'^2(\theta)}{A^2(\theta)}\\&\quad+\frac{b^2(x)[A'''(\theta b(x))A'(\theta b(x))-A''^2(\theta b(x))]}{A'^2(\theta b(x))},\\ \frac{\partial^2 L(\theta,\beta;x)}{\partial\theta\,\partial\beta}&=\frac{xe^{-\beta x}}{A'^2(\theta b(x))}\{A'(\theta b(x))[A''(\theta b(x))+\theta b(x)A'''(\theta b(x))]\\&\quad-\theta b(x)A''^2(\theta b(x))\},\\ \frac{\partial^2 L(\theta,\beta;x)}{\partial\beta^2}&=-\frac{1}{\beta^2}-\frac{\theta x^2e^{-\beta x}}{A'^2(\theta b(x))}\{A'(\theta b(x))[A''(\theta b(x))-\theta e^{-\beta x}A'''(\theta b(x))]\\&\quad+\theta e^{-\beta x}A''^2(\theta b(x))\},\end{aligned}$$

where $A'''(\theta b(x))$ is the third derivative of $A(\cdot)$ evaluated at $b(x)$ and $b(x)=\theta(1-e^{-\beta x})$.

6 Simulation study

This section presents the results of a simulation study carried out to assess the accuracy of the approximation of the variance and covariance of the MLEs determined from the Fisher information matrix. Five thousand samples of sizes $n=20, 30, 50, 100$ and 200 were generated from a Binomial–Exponential distribution (with $m=13$) for each combination of the parameter values $(\theta;\beta)=$

(0, 1; 2), (0, 3; 2), (0, 5; 2), (0, 7; 2), (0, 9; 2), (0, 1; 6), (0, 3; 6), (0, 5; 6), (0, 7; 6) and (0, 9; 6). Overall, 50 different setups were considered.

The simulated values of $\mathrm{Var}(\hat{\theta})$, $\mathrm{Var}(\hat{\beta})$ and $\mathrm{Cov}(\hat{\theta}, \hat{\beta})$, as well as their approximate values obtained by averaging the corresponding values obtained from the expected and observed information matrices, are presented in Table 2. It is observed that the approximate values determined from the expected and observed information matrices are close to the simulated values for large values of n. Furthermore, it is noted that the approximation becomes quite accurate as n increases. Additionally, variances and covariances of MLEs obtained from the observed information matrix are quite close from the variances and covariances obtained from the expected information matrix for large value of n. Also, the simulation study include the mean square error (mse) of the MLEs, as well as the empirical coverage probabilities of the 95% confidence intervals for the parameters θ and β, which are closer to the nominal coverage as the sample size increases.

7 Application

In this section we reanalyze the data set considered by Cancho, Louzada-Neto and Barriga (2011). The lifetimes are the number of million revolutions before failure for each one of the 23 ball bearings on an endurance test of deep groove ball bearings. From the practical point of view, even though the risks for deep groove ball bearing failure are unobserved, one may speculate on some possible competing risks. For instance, we can consider the risk of contamination from dirt from the casting of the casing, the wear particles from hardened steel gear wheels and the harsh working environments, amongst others, which can lead to the deep groove ball bearing failure.

The shape of the failure rate function for the data set can be determined from a TTT plot (Aarset, 1985). This plot is built from the points $(\frac{r}{n}, G(\frac{r}{n}))$, where $G(\frac{r}{n}) = [\sum_{i=1}^{r} Y_{(i)} + (n-r)Y_{(r)}]/(\sum_{i=1}^{r} Y_{(i)})$, $r = 1, \ldots, n$, and $Y_{(i)}$ is the ith order statistic of the sample. As pointed out in the literature, it is shown that the rate of failure is increasing (decreasing) if the TTT plot is concave (convex). However, since the TTT plot is a sufficient condition, but not necessary, to indicate the rate of failure, it is taken here only as a reference to determine the shape of the rate of failure. Figure 2 shows the TTT plot for the considered data, which is concave, indicating an increasing failure rate function, which can be properly accommodated by the CEPS distribution, but not for the EPS ones proposed by Chahkandi and Ganjali (2009), since they can only accommodate decreasing failure rate functions; see Figure 1.

Therefore, the data set may be fitted by the CEPS distribution. Thus, the Poisson, geometric, logarithmic and binomial distributions can be used for this purpose. The maximum likelihood estimation is obtained by direct maximization of (5.1) via the optim function of the R program (R Development Core Team,

Table 2 *Mean of the variances and covariances of the MLEs, mean square error (mse) of the MLEs and coverage probabilities of the 95% confidence intervals for the parameters*

				Simulated					Expected information			Observed information			Coverage	
n	$(\theta;\beta)$	$\hat{\theta}$	$\hat{\beta}$	Var($\hat{\theta}$)	Var($\hat{\beta}$)	mse($\hat{\theta}$)	mse($\hat{\beta}$)	Cov($\hat{\theta}$, $\hat{\beta}$)	Var($\hat{\theta}$)	Var($\hat{\beta}$)	Cov($\hat{\theta}$, $\hat{\beta}$)	Var($\hat{\theta}$)	Var($\hat{\beta}$)	Cov($\hat{\theta}$, $\hat{\beta}$)	θ	β
20	(0, 1; 2)	0.151	2.228	0.020	0.420	0.022	0.472	0.063	0.015	0.401	0.061	0.022	0.447	0.071	0.999	0.953
	(0, 3; 2)	0.390	2.128	0.051	0.221	0.059	0.237	0.073	0.026	0.198	0.053	0.065	0.220	0.077	0.985	0.955
	(0, 5; 2)	0.609	2.078	0.069	0.142	0.081	0.148	0.064	0.063	0.148	0.070	0.143	0.160	0.100	0.964	0.968
	(0, 7; 2)	0.763	2.032	0.057	0.099	0.061	0.100	0.043	0.145	0.131	0.103	0.209	0.133	0.115	0.956	0.978
	(0, 9; 2)	0.857	1.979	0.039	0.071	0.041	0.072	0.026	0.305	0.125	0.149	0.248	0.115	0.118	0.953	0.982
30	(0, 1; 2)	0.129	2.140	0.011	0.261	0.012	0.280	0.038	0.010	0.268	0.041	0.012	0.290	0.045	0.998	0.958
	(0, 3; 2)	0.356	2.079	0.031	0.151	0.034	0.157	0.049	0.018	0.132	0.035	0.032	0.142	0.045	0.978	0.954
	(0, 5; 2)	0.587	2.052	0.053	0.099	0.061	0.102	0.049	0.042	0.099	0.047	0.085	0.104	0.064	0.964	0.960
	(0, 7; 2)	0.760	2.024	0.049	0.069	0.053	0.070	0.036	0.097	0.088	0.068	0.140	0.088	0.078	0.954	0.976
	(0, 9; 2)	0.861	1.978	0.033	0.049	0.034	0.050	0.022	0.203	0.083	0.099	0.172	0.077	0.082	0.953	0.979
50	(0, 1; 2)	0.118	2.084	0.006	0.162	0.006	0.169	0.023	0.006	0.161	0.025	0.007	0.167	0.026	0.986	0.955
	(0, 3; 2)	0.333	2.050	0.015	0.087	0.017	0.090	0.027	0.011	0.079	0.021	0.015	0.083	0.024	0.975	0.949
	(0, 5; 2)	0.558	2.037	0.035	0.062	0.038	0.063	0.033	0.025	0.059	0.028	0.043	0.062	0.035	0.966	0.962
	(0, 7; 2)	0.756	2.027	0.037	0.043	0.040	0.044	0.026	0.058	0.053	0.041	0.083	0.053	0.047	0.963	0.976
	(0, 9; 2)	0.874	1.980	0.025	0.030	0.026	0.030	0.015	0.122	0.050	0.060	0.110	0.047	0.052	0.957	0.979
100	(0, 1; 2)	0.107	2.035	0.003	0.078	0.003	0.079	0.012	0.003	0.080	0.012	0.003	0.082	0.013	0.980	0.962
	(0, 3; 2)	0.315	2.023	0.006	0.041	0.006	0.041	0.011	0.005	0.040	0.011	0.006	0.040	0.011	0.966	0.956
	(0, 5; 2)	0.529	2.022	0.017	0.031	0.017	0.032	0.016	0.013	0.030	0.014	0.017	0.030	0.016	0.966	0.954
	(0, 7; 2)	0.741	2.018	0.026	0.024	0.028	0.024	0.017	0.029	0.026	0.021	0.039	0.027	0.023	0.961	0.969
	(0, 9; 2)	0.882	1.985	0.018	0.016	0.018	0.016	0.010	0.061	0.025	0.030	0.057	0.024	0.027	0.960	0.976
200	(0.1;2)	0.102	2.016	0.002	0.042	0.002	0.042	0.006	0.001	0.040	0.006	0.002	0.041	0.006	0.961	0.956
	(0, 3; 2)	0.307	2.010	0.003	0.020	0.003	0.020	0.006	0.003	0.020	0.005	0.003	0.020	0.005	0.962	0.956
	(0, 5; 2)	0.518	2.012	0.008	0.015	0.008	0.015	0.008	0.006	0.015	0.007	0.007	0.015	0.008	0.962	0.956
	(0, 7; 2)	0.728	2.015	0.015	0.013	0.016	0.013	0.010	0.015	0.013	0.010	0.018	0.013	0.011	0.965	0.965
	(0, 9; 2)	0.891	1.993	0.012	0.009	0.012	0.009	0.007	0.031	0.012	0.015	0.030	0.012	0.014	0.963	0.979

Table 2 (*Continued*)

				Simulated					Expected information			Observed information			Coverage	
n	$(\theta; \beta)$	$\hat{\theta}$	$\hat{\beta}$	Var($\hat{\theta}$)	Var($\hat{\beta}$)	mse($\hat{\theta}$)	mse($\hat{\beta}$)	Cov($\hat{\theta}$, $\hat{\beta}$)	Var($\hat{\theta}$)	Var($\hat{\beta}$)	Cov($\hat{\theta}$, $\hat{\beta}$)	Var($\hat{\theta}$)	Var($\hat{\beta}$)	Cov($\hat{\theta}$, $\hat{\beta}$)	θ	β
20	(0, 1; 6)	0.149	6.666	0.018	3.677	0.020	4.120	0.174	0.015	3.613	0.184	0.021	3.985	0.21	0.999	0.961
	(0, 3; 6)	0.385	6.353	0.051	2.010	0.058	2.134	0.217	0.026	1.781	0.158	0.065	1.987	0.231	0.981	0.958
	(0, 5; 6)	0.615	6.248	0.069	1.256	0.082	1.317	0.188	0.063	1.334	0.210	0.147	1.448	0.307	0.969	0.970
	(0, 7; 6)	0.771	6.111	0.056	0.883	0.061	0.895	0.130	0.145	1.182	0.308	0.212	1.191	0.346	0.958	0.980
	(0, 9; 6)	0.854	5.943	0.040	0.661	0.042	0.664	0.085	0.305	1.123	0.447	0.246	1.035	0.352	0.946	0.980
30	(0, 1; 6)	0.128	6.411	0.011	2.348	0.012	2.516	0.114	0.010	2.408	0.123	0.012	2.595	0.134	0.999	0.965
	(0, 3; 6)	0.358	6.243	0.032	1.317	0.035	1.376	0.144	0.018	1.187	0.105	0.033	1.274	0.136	0.980	0.951
	(0, 5; 6)	0.594	6.185	0.054	0.882	0.063	0.916	0.147	0.042	0.889	0.140	0.088	0.946	0.195	0.967	0.966
	(0, 7; 6)	0.761	6.081	0.048	0.598	0.052	0.605	0.106	0.097	0.788	0.205	0.140	0.797	0.234	0.961	0.978
	(0, 9; 6)	0.862	5.924	0.033	0.429	0.034	0.435	0.063	0.203	0.749	0.298	0.172	0.691	0.245	0.954	0.982
50	(0, 1; 6)	0.115	6.248	0.006	1.420	0.006	1.481	0.069	0.006	1.445	0.074	0.007	1.532	0.078	0.986	0.960
	(0, 3; 6)	0.332	6.161	0.015	0.779	0.016	0.805	0.077	0.011	0.712	0.063	0.015	0.748	0.073	0.976	0.953
	(0, 5; 6)	0.559	6.123	0.035	0.531	0.038	0.546	0.094	0.025	0.533	0.084	0.043	0.559	0.106	0.968	0.961
	(0, 7; 6)	0.755	6.070	0.038	0.388	0.041	0.392	0.079	0.058	0.473	0.123	0.083	0.479	0.142	0.958	0.974
	(0, 9; 6)	0.869	5.938	0.026	0.269	0.027	0.273	0.047	0.122	0.449	0.179	0.108	0.423	0.155	0.951	0.980
100	(0, 1; 6)	0.106	6.099	0.003	0.750	0.003	0.760	0.037	0.003	0.723	0.037	0.003	0.745	0.038	0.981	0.958
	(0, 3; 6)	0.314	6.074	0.006	0.374	0.006	0.379	0.034	0.005	0.356	0.032	0.006	0.365	0.034	0.967	0.953
	(0, 5; 6)	0.531	6.072	0.016	0.269	0.017	0.274	0.047	0.013	0.267	0.042	0.017	0.274	0.048	0.972	0.956
	(0, 7; 6)	0.738	6.043	0.025	0.212	0.026	0.214	0.050	0.029	0.236	0.062	0.038	0.240	0.069	0.963	0.969
	(0, 9; 6)	0.880	5.959	0.018	0.152	0.018	0.153	0.032	0.061	0.225	0.089	0.057	0.216	0.082	0.961	0.979
200	(0, 1; 6)	0.104	6.058	0.002	0.366	0.002	0.369	0.019	0.001	0.361	0.018	0.002	0.367	0.019	0.957	0.955
	(0, 3; 6)	0.307	6.039	0.003	0.183	0.003	0.184	0.016	0.003	0.178	0.016	0.003	0.181	0.016	0.962	0.954
	(0, 5; 6)	0.515	6.040	0.007	0.133	0.008	0.134	0.023	0.006	0.133	0.021	0.007	0.135	0.022	0.965	0.958
	(0, 7; 6)	0.726	6.035	0.016	0.120	0.016	0.121	0.032	0.015	0.118	0.031	0.018	0.120	0.034	0.963	0.959
	(0, 9; 6)	0.893	5.981	0.012	0.076	0.012	0.077	0.020	0.031	0.112	0.045	0.030	0.110	0.043	0.962	0.960

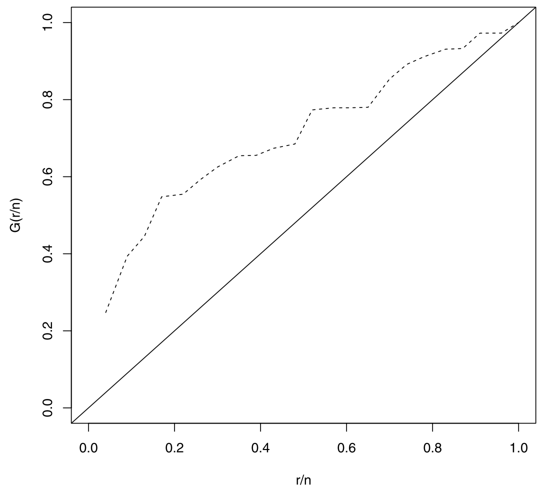

Figure 2 *Empirical scaled TTT-transform for the data.*

Table 3 *MLEs and their standard errors (in brackets) for the parameters of the fitted distributions*

Distribution	θ	β
CEB	600 (280.3460)	0.0315 (0.0036)
CEP	7.3259 (2.594)	0.0358 (0.0061)
CEG	0.9447 (0.0415)	0.0436 (0.0094)
CEL	0.9982 (0.0077)	0.0516 (0.0215)
Weibull	2.1026 (0.3437)	0.0122 (0.0014)

2008). For the complementary exponential-binomial model it is taken $m = 5$, a value that determines the CEB model with the greatest likelihood. Also, for sake of comparison, we fit more one usual lifetime distribution generally used for fitting a data set with increasing failure rate function: the Weibull distribution. As well known, the Weibull distribution is indexing by two parameters as it is the case for the CEPS distribution. The density functions of the Weibull distributions, with parameters θ and β, is given by $\theta\beta^\theta x^{\theta-1}\exp(-(\beta x)^\theta)$, respectively. Table 3 shows the MLEs and their standard errors for the CEPS distribution parameters as well as for the Weibull distribution parameters.

We compare the fitting of the CEPS particular distributions and the Weibull one by considering the AIC (Akaike's information criterion,$-2l(\hat{\theta}, \hat{\beta}) + 2p$, where p is the number of parameters in the model) and BIC [Schawartz's Bayesian information criterion, $-2l(\hat{\theta}, \hat{\beta}) + 2\log(n)$, where n is the size sample]. Both criterion penalize overfitting and the preferred model is the one with the smaller value on each criterion. Also, Table 4 presents the maximum values of the log-likelihood

Table 4 *$l(\cdot)$ value, and AIC and BIC values for the fitted distributions*

Distribution	$l(\cdot)$	AIC	BIC	χ^2 (p-value)	K–S (p-value)
CEB	−112.9874	229.9748	232.2459	0.7117 (0.8704)	0.1061 (0.9339)
CEP	−113.1521	230.3042	232.5752	0.8630 (0.8344)	0.1150 (0.8875)
CEG	−114.3502	232.7004	234.9714	2.4302 (0.4880)	0.1387 (0.7173)
CEL	−116.7022	237.4044	239.6754	6.5446 (0.0879)	0.2066 (0.2441)
Weibull	−113.6887	231.3774	233.6484	2.4634 (0.4819)	0.1512 (0.6157)

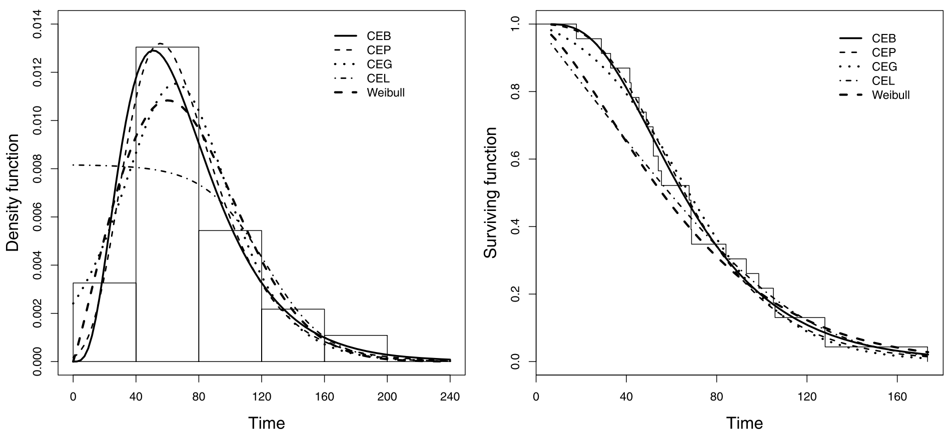

Figure 3 *Left panel*: *The plots of the fitted CEB*, *CEP*, *CEG*, *CEL and Weibull densities*. *Right panel*: *Kaplan–Meir curve together with the fitted survival functions*.

function ($l(\cdot)$), the estimated AIC and BIC criteria, the Pearson χ^2 statistic (obtained with the partition used in the histogram presented in the Figure 3) and the Kolmogorov–Smirnov distance (K–S) considering the all fitted distributions.

The CEB distribution outperforms its concurrent distributions in all considered criteria. The parameter estimates of the CEB distribution of θ and β (and their standard errors) are 600 (280.3460) and 0.0315 (0.0036), respectively. Figure 3 shows the fitted density functions of the all fitted distributions superimposed to the histogram and the fitted survival superimposed to the empirical survival function.

8 Concluding remarks

In this paper we propose the CEPS distribution, which is complementary to the EPS distribution proposed by Chahkandi and Ganjali (2009), and accommodate increasing failure rate functions. It arises on latent complementary risks scenarios, where the lifetime associated with a particular risk is not observable but only the maximum lifetime value among all risks. We provide a mathematical treatment of the new distribution including expansions for its density and cumulative distributions, survival and failure rate functions. We derive expansions for the moments and quantile function, obtain the density of the order statistics and provide expansions for moments of the order statistics. Maximum likelihood inference is implemented straightforwardly. Finally, we fit the CEPS distribution to a real data set in order to show its flexibility and potentially as a lifetime distribution compared with an usual two-parameters lifetime distributions.

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References

Aarset, M. V. (1985). The null distribution for a test of constant versus “bathtub” failure rate. *Scandinavian Journal of Statistics* **12**(1), 55–68. MR0804225

Adamidis, K. and Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics & Probability Letters* **39**(1), 35–42. MR1649319

Adamidis, K., Dimitrakopoulou, T. and Loukas, S. (2005). On an extension of the exponential-geometric distribution. *Statistics & Probability Letters* **73**(3), 259–269. MR2179285

Barakat, H. M. and Abdelkader, Y. H. (2004). Computing the moments of order statistics from nonidentical random variables. *Statistical Methods and Applications* **13**, 15–26. MR2081962

Barreto-Souza, W. and Cribari-Neto, F. (2009). A generalization of the exponential–Poisson distribution. *Statistics & Probability Letters* **79**, 2493–2500. MR2556316

Basu, A. and Klein, J. (1982). Some recent development in competing risks theory. In *Survival Analysis* (Crowley, J. and Johnson, R. A, eds.) **1**, 216–229. Hayward: IMS. MR0734204

Cancho, V. G., Louzada-Neto, F. and Barriga, G. D. (2011). The Poisson–exponential lifetime distribution. *Computational Statistics & Data Analysis* **55**, 677–686. MR2736587

Chahkandi, M. and Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. *Computational Statistics & Data Analysis* **53**, 4433–4440. MR2744336

Cordeiro, G., Rodrigues, J. and de Castro, M. (2012). The exponential COM–Poisson distribution. *Statistical Papers* **53**, 653–664. MR2958745

Cox, D. R. and Hinkley, D. V. (1974). *Theoretical Statistics*. London: Chapman & Hall. MR0370837

Cox, D. R. and Oakes, D. (1984). *Analysis of Survival Data*. London: Chapman & Hall. MR0751780

Crowder, M., Kimber, A., Smith, R. and Sweeting, T. (1991). *Statistical Analysis of Reliability Data*. London: Chapman & Hall. MR1122148

Goetghebeur, E. and Ryan, L. (1995). A modified log rank test for competing risks with missing failure type. *Biometrics* **77**, 207–211. MR1049423

Johnson, N. L., Kemp, A. W. and Kotz, S. (2005). *Univariate Discrete Distribution*. Hoboken, NJ: Wiley. MR2163227

Kus, C. (2007). A new lifetime distribution. *Computational Statistics & Data Analysis* **51**, 4497–4509. MR2364461

Lawless, J. F. (2003). *Statistical Models and Methods for Lifetime Data*, 2nd ed. Hoboken, NJ: Wiley. MR1940115

Lu, K. and Tsiatis, A. A. (2001). Multiple imputation methods for estimating regression coefficients in the competing risks model with missing cause of failure. *Biometrics* **54**, 1191–1197. MR1950427

Lu, K. and Tsiatis, A. A. (2005). Comparision between two partial likelihood approaches for the competing risks model with missing cause of failure. *Lifetime Data Analysis* **11**, 29–40. MR2139480

Morais, A. L. and Barreto-Souza, W. (2011). A compound class of weibull and power series distributions. *Computational Statistics & Data Analysis* **55**(3), 1410–1425.

Reiser, B., Guttman, I., Lin, D., Guess, M. and Usher, J. (1995). Bayesian inference for masked system lifetime data. *Applied Statistics* **44**, 79–90. MR1323353

R Development Core Team (2008). *R: A Language and Environment for Statistical Computing*. Vienna, Austria: R Foundation for Statistical Computing.

Silva, R. B., Barreto-Souza, W. and Cordeiro, G. M. (2010). A new distribution with decreasing, increasing and upside-down bathtub failure rate. *Computational Statistics & Data Analysis* **54**(4), 935–944. MR2580928

Tahmashi, R. and Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate. *Computational Statistics & Data Analysis* **52**, 3889–3901. MR2432214

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