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SOME ASYMPTOTIC RESULTS OF GAUSSIAN RANDOM FIELDS WITH VARYING MEAN FUNCTIONS AND THE ASSOCIATED PROCESSES

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In this paper, we derive tail approximations of integrals of exponential functions of Gaussian random fields with varying mean functions and approximations of the associated point processes. This study is motivated naturally by multiple applications such as hypothesis testing for spatial models and financial applications.

1. Introduction. Gaussian random fields and multivariate Gaussian random vectors constitute a cornerstone of statistics models in many disciplines, such as physical oceanography and hydrology [4, 48], atmosphere study [26], geostatistics [21, 22], astronomy [33, 52] and brain imaging [49, 59–61]. The difficulty very often lies in assessing the significance of the test statistics due to the dependence structure induced by the random field. In recent studies, closed form approximations of the tail probabilities of supremum of random fields (the p-values) have been studied intensively such as in [45, 59, 60]. In this paper, we develop asymptotic results of the integrals of exponential functions of smooth Gaussian random fields with varying mean functions and the associated point processes.

For concreteness, let $\{f(t): t \in T\}$ be a centered Gaussian random field with unit variance and living on a d-dimensional domain $T \subset R^d$. For every finite subset of $\{t_1, \dots, t_n\} \subset T$, $(f(t_1), \dots, f(t_n))$ is a mean-zero multivariate Gaussian random vector. In addition, let $\mu(t)$ be a (deterministic) function. The main quantity of interest is the probability

$$P\left(\int_T e^{\sigma f(t)+\mu(t)}\, dt > b\right), \tag{1.1}$$

where $\sigma \in (0, \infty)$ is the scale factor. In particular, we consider the asymptotic regime where b tends to infinity and develop closed form approximations of the above tail probabilities. We further consider a doubly stochastic Poisson process $\{N(A): A \subset T\}$, with intensity $\{\lambda(t): t \in T\}$. More specifically, let $\log \lambda(t) = \sigma f(t) + \mu(t)$ be a continuous Gaussian process. Conditional on $\{\lambda(t): t \in T\}$,

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$\{N(A): A \subset T\}$ is an inhomogeneous Poisson process with intensity $\lambda(t)$. Note that, conditional on the process $f(t)$, $N(A)$ is a Poisson random variable with expectation $\int_A e^{\sigma f(t)+\mu(t)}\,dt$. Then, we are interested in approximating the tail probability

$$P\big(N(T) > b\big). \tag{1.2}$$

The approximations of tail probabilities in (1.1) and (1.2) are motivated by multiple applications such as hypothesis testing for spatial models and financial applications; see detailed discussions in Section 2. In fact, (1.1) and (1.2) are asymptotically the same. Therefore, the main result of this paper lies in developing approximations for (1.1).

In the statistics literature, closed form approximations of the tail probabilities of Gaussian random fields have been widely employed for the computation of p-values such as significance levels of the scanning statistics [7, 45, 49, 51]. The works of [19, 59, 60] used expected Euler characteristics of the excursion set as an approximation and applied it to neuroimaging. Rabinowitz and Siegmund [47] used saddlepoint approximation for the tail of a smoothed Poisson point process. Using a change of measure idea, [45] derived the approximations for non-Gaussian fields in the context of a likelihood-based hypothesis test. In the probability literature, the extreme behavior of Gaussian random fields is also intensively studied. The results range from general bounds to sharp asymptotic approximations. An incomplete list of works includes [13, 16, 17, 23, 37, 39, 40, 43, 55, 57]. A few lines of investigations on the supremum norm are given as follows. Assuming locally stationary structure, the double-sum method [46] provides the exact asymptotic approximation of $\sup_T f(t)$ over a compact set T, which is allowed to grow as the threshold tends to infinity. For almost surely at least twice differentiable fields, [1, 6, 58] derive the analytic form of the expected Euler–Poicaré Characteristics of the excursion set $[\chi(A_b)]$ which serves as a good approximation of the tail probability of the supremum. The tube method [56] takes advantage of the Karhune–Loève expansion and Weyl's formula. A recent related work along this line is given by [45]. The Rice method [9–11] provides an implicit description of $\sup_T f(t)$. The discussions also go beyond the Gaussian fields. For instance, [38] discusses the situations of Gaussian process with random variances. See also [5] for other discussions.

The analysis of integrals of nonlinear functions of Gaussian random fields is less developed compared with that of the supremum. In the case that $f(t)$ is the Brownian motion, the distribution of $\int_0^\infty e^{f(t)}\,dt$ is discussed by [31, 63]. For smooth and homogeneous Gaussian random fields, the tail approximation of $\int_T e^{f(t)}\,dt$ is given by [41] using a technique similar to the double-sum method. The result and technique in [41] are restricted to the homogeneous fields (with a constant mean). In statistical analysis, however, allowing spatially varying mean functions is usually very important especially in presence of spatially varying covariates.

Meanwhile, developing sharp asymptotic approximations for random fields with spatially varying means is a much more complicated and more difficult problem. The current work substantially generalizes the result of [41] and is applicable to more practical settings such as the presence of spatially varying covariates.

The contribution of this paper is to develop asymptotic approximations of the probabilities as in (1.1) and (1.2) by introducing a change-of-measure technique. This change of measure was first proposed by [45] to derive tail asymptotics of supremum of non-Gaussian random fields. This technique substantially simplifies the analysis (though the derivations are still complicated) and may potentially lead to efficient importance sampling algorithms to numerically compute (1.1) and (1.2); see [2, 3, 15, 50] for a connection between the change of measure and efficient computation of tail probabilities. In addition, without too many modifications, one can foresee that the proposed change of measure can be adapted to certain non-Gaussian random fields such as those in [45], which uses a change-of-measure technique to develop the approximations of suprema of non-Gaussian random fields with functional expansions.

The organization of the rest of this paper is as follows. In Section 2, we present several applications of the current study. The main results are given in Section 3 with proofs provided in Section 4. Some useful lemmas are stated in Section 5. A simulation study and technical proofs of several lemmas are provided as supplemental article [42].

2. Applications. The integrals of exponential functions of random fields play an important role in many applications. We present a few of them in this section.

2.1. *Hypothesis testing.*

Hypothesis testing for point processes. Consider the doubly stochastic Poisson process $N(\cdot)$ with intensity $\lambda(t)$ as defined in the Introduction. The mean function of the log-intensity $\mu(t)$ is typically modeled as a linear combination of the observed spatially varying covariates, that is,

$$\mu(t) = \mathbf{x}^\top(t)\beta, \tag{2.1}$$

where $\mathbf{x}(t) = (x_1(t), \dots, x_p(t))^\top$. The Gaussian process $f(t)$ is then employed to build in a spatial dependence structure by letting $\log \lambda(t) = f(t) + \mu(t)$. This modeling approach has been widely used in the literature. For instance, [20] considers the time series setting in which T is a one-dimensional interval, $\mu(t)$ in (2.1) is modeled as the observed covariate process and $f(t)$ is an autoregressive process. See [18, 24, 25, 27, 64] for more examples of such kind. For applications, this approach has been used in many disciplines, such as astronomy, epidemiology, geography, ecology and material science. Particularly, in the epidemiological study, this model is used to describe the spatial distribution of positive (e.g., cancer) cases over a region T and the latent intensity process is used to account for the unobserved factors that may affect the hazard.

Under the above assumptions, we consider the related hypothesis testing problems admitting a (simple) null hypothesis that the point process has log-intensity $f(t) + \mathbf{x}^\top(t)\beta$ with β and the covariance function known. The alternative hypothesis could be any probability model under which the distribution of $N(T)$ stochastically dominates the one under the null hypothesis. The one-sided p-value is then given by

$$P\big(N(T) > b\big), \tag{2.2}$$

where b is the observed count. This is equivalent to testing that a region T is of a higher hazard level than the typical (null) level.

For concreteness, we consider one situation that is frequently encountered in epidemiology. Let the process $N(\cdot)$ denote the spatial locations of positive asthma cases in a certain region T such as the New York City metropolitan area. We assume that $N(\cdot)$ admits the doubly stochastic structure described previously. To keep the example simple, we only include one covariate in the model, that is, $\log\lambda(t) = \beta_0 + \beta_1 x(t)$, where $x(t)$ is the pollution level at location t and has been standardized so that $\int_T x(t)\,dt = 0$. Suppose that β_1 is known or an accurate estimate of β_1 is available. One is interested in testing the simple hypothesis $H_0 : \beta_0 = \beta_0^*$ against $H_1 : \beta_0 > \beta_0^*$, where β_0^* is the national-wise log-intensity. A p-value is given by $P(N(T) > b)$. In this case, it is necessary to consider a spatially varying mean of the log-intensity to account for the inhomogeneity given that regression coefficient β_1 for pollution level is nonzero under the null hypothesis.

For other instances, the spatially varying covariates are sometimes chosen to functions to reflect certain periodicity. One such case study is discussed in [64] and further in [27] under the time series setting. In that example, positive cases of poliomyelitis in the United States for the years 1970–1983 were observed. The time-varying covariates $\mathbf{x}(t)$ are set to be a linear trend and harmonics at periods of 6 and 12 months and more precisely

$$\begin{aligned}
\mu(t) &= \mathbf{x}(t)^\top \beta,\\
\mathbf{x}(t) &= \big(1, t, \cos(2\pi t/12), \sin(2\pi t/12), \cos(2\pi t/6), \sin(2\pi t/6)\big),
\end{aligned}$$

where t is in the unit of one month. Similar hypothesis testing problems to the asthma case may be considered. The coefficients for most terms are significantly nonzero. Thus, spatially varying covariates are ubiquitous in practice, which results in a nonconstant mean under the null hypothesis.

In order to apply the results in this paper to a composite null hypothesis, such as the case in which the covariance function of $f(t)$ is unknown, we need the corresponding estimates for some characteristics of the covariance function of f (see Theorem 3.4). The uncertainty of these estimates will definitely introduce additional difficulty to the p-value calculation. On the other hand, with reasonably large sample size, the necessary parameters can be estimated accurately. In addition, the approximations stated in later theorems only consist of the derivatives

of the covariance function (equivalently spectral moments) and $\mu(t)$ at the global maximum. Then, one can design estimators simply for these distributional characteristics, which are much easier to estimate than the entire covariance functional form. There are extensive discussions on the estimations of spectral moments both parametrically and nonparametrically such as in the textbook [54]. This plug-in-estimate strategy is used by [59] to handle such a composite null hypothesis combined with a closed form p-value approximation by means of the expected Euler characteristics. In that paper, the authors estimate and plug in the estimate of the Lipschitz–Killing curvature to the expected Euler characteristic function to approximate the tail probability of the supremum of a t-field. Given that the main focus of this paper is on developing the approximations for the tail probabilities, we do not pursue parameter estimation aspects.

Hypothesis testing for aggregated data. The tail probability of $\int e^{\mu(t)+f(t)}\,dt$ itself can also serve as a p-value. In environmental science, the ozone concentration fluctuation is typically modeled to follow a log-normal distribution. For instance, it is found that the hourly averaged zone concentration typically admits a log-normal distribution; see [35]. We let $\log\lambda(t) = \mu(t) + f(t)$ be the log-concentration of ozone at location t. One is interested in testing whether a region T has an unusually high ozone level, that is, $H_0 : E(\log\lambda(t)) = \mu(t)$ and $H_A : E(\log\lambda(t)) > \lambda(t)$ for $t \in T$. Similarly to the previous motivating example, in a regression setting, one may model $\mu(t) = \beta_0 + \mathbf{x}^\top(t)\beta$ and consider $H_0 : \beta_0 = \beta_0^*$ and $H_1 : \beta_0 > \beta_0^*$. One may reject the null if the observed aggregated ozone level, b, in region T is too high and a p-value is given by

$$P\left(\int_T \lambda(t)\,dt = \int_T e^{\mu(t)+f(t)}\,dt > b\right).$$

A similar argument as that for the point process application applies for the necessity of incorporating a nonconstant mean function $\mu(t)$ to reflect spatial inhomogeneity such as spatially varying covariates and periodicity.

2.2. *Financial applications.* The integrals of exponential functions also play an important role in financial applications. This is more related to the applied probability literature. In asset pricing, the asset price indexed by time t is typically modeled as an exponential function of a Gaussian process, that is, $S(t) = e^{f(t)}$. For instance, the Black–Scholes–Merton formula [14, 44] assumes that the price follows a geometric Brownian motion. Then, the payoff of an Asian option is the function of the averaged price $\int_0^T e^{f(t)}\,dt$ and (1.1) is the probability of exercising an Asian call option.

In the portfolio risk analysis, consider a portfolio consisting of n assets $(S_1, \dots, S_n)$ each of which is associated with a weight (e.g., number of shares) $(w_1, \dots, w_n)$. One popular model assumes that $(\log S_1, \dots, \log S_n)$ is a multivariate Gaussian random vector. The value of the portfolio, $S = \sum_{i=1}^n w_i S_i$, is then the

sum of correlated log-normal random variables (see [8, 12, 29, 30, 32]). Without loss of generality, we let $\sum w_i = n$.

One typical situation is that the portfolio size is large and the asset prices are usually highly correlated. One may employ a latent space approach used in the literature of social network. More specifically, we construct a Gaussian process $\{f(t): t \in T\}$ and associate each asset i with a latent variable $t_i \in T$ so that $\log S_i = f(t_i)$. Then, the log asset prices fall into a subset of the continuous Gaussian process. Further, there exists a (deterministic) process $w(t)$ so that $w(t_i) = w_i$. Then, the total asset one unit share value of the portfolio is $\frac{1}{n}\sum w_i S_i = \frac{1}{n}\sum w(t_i)e^{f(t_i)}$.

In this representation, the dependence among two assets is determined by $|t_i - t_j|$, which indicates the economical distance between two firms. For instance, if firm i is the supplier of firm j, then $t_i - t_j$ tends to be small. The spatial index t may also include other social–economical indices. This latent space approach has become popular in recent social network studies. For instance, [36] considers a graph of n nodes. The authors associate each node i with a spatial latent variable t_i and model the probability of generating an edge between two nodes (i and j) in a graph as a logistic function of $|t_i - t_j|$. Similar latent space models that project nodes onto a latent space can be found in [34]. Other approaches to represent interactions among variables via latent structures have been used; see, for instance, [53, 62] and references therein.

In the asymptotic regime that $n \to \infty$ and the correlations among the asset prices become close to 1, the subset $\{t_i\}$ becomes denser in T. Ultimately, we obtain the limit

$$\frac{1}{n}\sum_{i=1}^{n} w_i S_i \to \int w(t)e^{f(t)}h(t)\,dt,$$

where $h(t)$ indicates the limiting spatial distribution of $\{t_i\}$ in T. Let $\mu(t) = \log w(t) + \log h(t)$. Then the tail probability of the (limiting) unit share price is

$$P\left(\int e^{f(t)+\mu(t)}\,dt > b\right).$$

It is necessary to include a varying mean in the above representation to incorporate the variation of the weights assigned to different assets and the inhomogeneity of the limiting distribution of t_i's.

3. Main results.

3.1. *Problem setting.* Consider a homogeneous Gaussian random field $\{f(t): t \in T\}$ living on a domain $T \subset R^d$. Let the covariance function be

$$C(t-s) = \mathrm{Cov}(f(t), f(s)).$$

We impose the following assumptions:

(C1) f is homogeneous with $Ef(t) = 0$ and $Ef^2(t) = 1$.

(C2) f is almost surely at least three times differentiable with respect to t and $\mu(t) \in C^3(T)$.

(C3) T is a d-dimensional Borel measurable compact set of R^d with piecewise smooth boundary.

(C4) The Hessian matrix of $C(t)$ at the origin is $-I$, where I is the $d \times d$ identity matrix.

(C5) For each $t \in R^d$, the function $C(\lambda t)$ is a nonincreasing function of $\lambda \in R^+$.

(C6) If $\mu(t)$ is not a constant, the maximum of $\mu(t)$ is not attained at the boundary of T.

Let $N(\cdot)$ be a point process such that conditional on $\{f(t) : t \in T\}$, $N(\cdot)$ is distributed as a Poisson process with intensity $\lambda(t) = e^{\mu(t)+\sigma f(t)}$. For each Borel measurable set $A \subset T$, let

$$I(A) = \int_A e^{\mu(t)+\sigma f(t)}\, dt. \tag{3.1}$$

Throughout this paper, we are interested in developing closed form approximations to

$$P\left(\int_T e^{\mu(t)+\sigma f(t)}\, dt > b\right) \quad \text{and} \quad P\big(N(T) > b\big) \tag{3.2}$$

as $b \to \infty$.

REMARK 3.1. Condition (C1) assumes unit variance. We treat the standard deviation σ as an additional parameter and consider $\int e^{\mu(t)+\sigma f(t)}\, dt$. Condition (C2) is rather a strong assumption. It implies that $C(t)$ is at least six times differentiable and the first, third and fifth derivatives at the origin are all zero. Condition (C3) restricts the results to finite horizon. Condition (C4) is introduced to simplify notation. For any Gaussian process $g(t)$ with covariance function $C_g(t)$ and $\Delta C_g(0) = -\Sigma$ and $\det(\Sigma) > 0$, (C4) can be obtained by an affine transformation by letting $g(t) = f(\Sigma^{1/2} t)$ and

$$\int_T e^{\mu(t)+\sigma g(t)}\, dt = \det(\Sigma^{-1/2}) \int_{\{s\,:\,\Sigma^{-1/2} s \in T\}} e^{\mu(\Sigma^{-1/2} s)+\sigma f(s)}\, ds,$$

where for each positive semidefinite matrix Σ we let $\Sigma^{1/2}$ be a symmetric matrix such that $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$. Conditions (C5) and (C6) are imposed for technical reasons.

REMARK 3.2. The setting in (3.2) also incorporates the case in which the integral is with respect to other measures with smooth densities with respect to the Lebesgue measure. Then, if $\nu(dt) = \kappa(t)\, dt$, we will have that

$$\int_A e^{\mu(t)+\sigma f(t)} \nu(dt) = \int_A e^{\mu(t)+\log\kappa(t)+\sigma f(t)}\, dt,$$

which shows that the density can be absorbed by the mean function as long as $\kappa(t)$ is bounded away from zero and infinity on T.

REMARK 3.3. The results presented in the current paper are directly applicable to some of the applications in Section 2 such as the approximation of p-value for simple null hypothesis. Some conditions required by the theorems may need to be relaxed to reflect practical circumstances for other applications. Nonetheless, the current analysis forms a standpoint of further study of more general cases.

3.2. *Notation*. To simplify the discussion, we define a set of notation constantly used in the later development and provide some basic calculations of Gaussian random field. Let "∂" denote the gradient and "Δ" denote the Hessian matrix with respect to t. The notation "∂^2" is used to denote the vector of second derivatives. The difference between $\partial^2 f(t)$ and $\Delta f(t)$ is that $\Delta f(t)$ is a $d \times d$ symmetric matrix whose diagonal and upper triangle consists of elements of $\partial^2 f(t)$. Similarly, we will later use $\mathbf{z}$ (and $\tilde{\mathbf{z}}$) to denote the matrix version of the vector z (and $\tilde{z}$), that is, $\mathbf{z}$ (and $\tilde{\mathbf{z}}$) is a symmetric matrix whose upper triangle consists of elements in the vector z (and $\tilde{z}$). Further, let $\partial_j f(t)$ be the partial derivative with respect to the jth element of t. We define u as the larger solution to

$$\left(\frac{2\pi}{\sigma}\right)^{d/2} u^{-d/2} e^{\sigma u} = b.$$

Note that when b is large, the above equation generally has two solutions. One is on the order of $\log b$; the other one is close to zero. We choose the larger solution as our u. Lastly, we define the following notation:

$$\begin{aligned}
&\mu_1(t) = -(\partial_1 C(t), \dots, \partial_d C(t)),\\
&\mu_2(t) = \left(\partial_{ii}^2 C(t), i = 1, \dots, d; \partial_{ij}^2 C(t), i = 1, \dots, d-1, j = i+1, \dots, d\right),\\
&\mu_{02}^\top = \mu_{20} = \mu_2(0).
\end{aligned}$$

It is well known that (cf. Chapter 5.5 of [6]) $(f(0), \partial^2 f(0), \partial f(0), f(t))$ is a multivariate Gaussian random vector with mean zero and covariance matrix

$$\begin{pmatrix} 1 & \mu_{20} & 0 & C(t) \\ \mu_{02} & \mu_{22} & 0 & \mu_2^\top(t) \\ 0 & 0 & I & \mu_1^\top(t) \\ C(t) & \mu_2(t) & \mu_1(t) & 1 \end{pmatrix},$$

where the matrix μ_{22} is a $d(d+1)/2$-dimensional positive definite matrix and contains the fourth-order spectral moments arranged in an appropriate order according to the order of elements in $\partial^2 f(0)$. Define

$$\Gamma = \begin{pmatrix} 1 & \mu_{20} \\ \mu_{02} & \mu_{22} \end{pmatrix}. \tag{3.3}$$

For notational convenience, we write $a_u = O(b_u)$ if there exists a constant $c > 0$ independent of everything such that $a_u \le c b_u$ for all $u > 1$, and $a_u = o(b_u)$ if $a_u/b_u \to 0$ as $u \to \infty$ and the convergence is uniform in other quantities. We write $a_u = \Theta(b_u)$ if $a_u = O(b_u)$ and $b_u = O(a_u)$. In addition, we write $X_u = o_p(a_u)$ if $X_u/a_u \xrightarrow{p} 0$ as $u \to \infty$ and $E^{\theta X_u/a_u} \to 1$ uniformly for θ over a compact interval around the origin. Similarly, we define that $X_u = O_p(a_u)$ if $E^{\theta X_u/a_u}$ is bounded away from zero and infinity for all $u \in R^+$ and θ in a compact interval around zero. We write $a_u \sim b_u$ if $a_u/b_u \to 1$ as $u \to \infty$.

3.3. *The main theorems.* The main theorems of this paper are presented as follows. The following theorem is the central result of this paper whose proof is provided in Section 4.

THEOREM 3.4. *Consider a Gaussian random field $\{f(t) : t \in T\}$ living on a domain T satisfying conditions* (C1)–(C6). *Let $I(T)$ be as defined in* (3.1). *Then,*

$$P\big(I(T) > b\big) \sim u^{d-1} \int_T \exp\left\{-\frac{(u - \mu_\sigma(t))^2}{2}\right\} \cdot H(\mu, \sigma, t)\, dt$$

as $b \to \infty$, where

$$\mu_\sigma(t) = \mu(t)/\sigma, \tag{3.4}$$

u is the larger solution to

$$\left(\frac{2\pi}{\sigma}\right)^{d/2} u^{-d/2} e^{\sigma u} = b,$$

$H(\mu, \sigma, t)$ is defined as

$$\begin{aligned}
&\frac{|\Gamma|^{-1/2}}{(2\pi)^{(d+1)(d+2)/4}} \exp\left\{\frac{\mathbf{1}^T \mu_{22} \mathbf{1} + \sum_i \partial^4_{iiii} C(0)}{8\sigma^2}\right. \\
&\qquad\qquad \left. + \frac{d \cdot \mu_\sigma(t) + \operatorname{Tr}(\Delta \mu_\sigma(t))}{2\sigma} + |\partial \mu_\sigma(t)|^2\right\} \\
&\quad \times \int_{z \in R^{d(d+1)/2}} \exp\left\{-\frac{1}{2}\left[\frac{|\mu_{20}\mu_{22}^{-1} z|^2}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} + \left|\mu_{22}^{-1/2} z - \frac{\mu_{22}^{1/2}\mathbf{1}}{2\sigma}\right|^2\right]\right\} dz
\end{aligned}$$

and

$$\mathbf{1} = (\underbrace{1, \dots, 1}_{d}, \underbrace{0, \dots, 0}_{d(d-1)/2})^\top.$$

COROLLARY 3.5. *Under the conditions of Theorem* 3.4, *if $\mu(t)$ has one unique maximum in T denoted by t_*, then*

$$\begin{aligned}
P\big(I(T) > b\big) &\sim (2\pi)^{d/2} \det(\Delta \mu_\sigma(t_*))^{-1/2} H(\mu, \sigma, t_*) u^{d/2-1} \\
&\quad \times \exp\left\{-\frac{(u - \mu_\sigma(t_*))^2}{2}\right\}.
\end{aligned}$$

PROOF. The result is immediate by expanding $\mu_\sigma(t)$ around t_* up to the second order. $\square$

THEOREM 3.6. *Assume that the Gaussian process $f(t)$ satisfies the conditions in Theorem 3.4. Consider a point process $\{N(A): A \in \mathcal{B}(T)\}$, where $\mathcal{B}(T)$ denotes the Borel subsets of T. Suppose that there exists a process $\log\lambda(t) = \mu(t) + \sigma f(t)$ such that given $\{\lambda(t): t \in T\}$, $N(\cdot)$ is a Poisson process with intensity $\lambda(t)$. Then,*

$$P\big(N(T) > b\big) \sim P\big(I(T) > b\big)$$

as $b \to \infty$.

PROOF. We prove this approximation from both sides. For $\varepsilon > 0$ small enough, we have that

$$\begin{aligned} P\big(N(T) > b\big) &\geq P\big(N(T) > b; I(T) \geq b + b^{1/2+\varepsilon}\big) \\ &= \big(1 + o(1)\big) P\big(I(T) \geq b + b^{1/2+\varepsilon}\big) \\ &= \big(1 + o(1)\big) P\big(I(T) \geq b\big). \end{aligned} \tag{3.5}$$

The second step is due to the fact that conditional on $I(T)$

$$\frac{N(T) - I(T)}{\sqrt{I(T)}} \to N(0, 1)$$

in distribution as $I(T) \to \infty$. Therefore, we obtain that

$$P\big(N(T) > b | I(T) \geq b + b^{1/2+\varepsilon}\big) \to 1.$$

Together with the fact that

$$\begin{aligned} &P\big(N(T) > b; I(T) \geq b + b^{1/2+\varepsilon}\big) \\ &\quad = P\big(N(T) > b | I(T) \geq b + b^{1/2+\varepsilon}\big) P\big(I(T) \geq b + b^{1/2+\varepsilon}\big), \end{aligned}$$

we obtain the second step of (3.5). The last step that $P(I(T) \geq b + b^{1/2+\varepsilon}) = (1+o(1)) P(I(T) \geq b)$ is a direct application of Theorem 3.4. For the upper bound, we have that

$$\begin{aligned} P\big(N(T) > b\big) &= P\big(N(T) > b; I(T) \geq b - b^{1-\varepsilon}\big) \\ &\quad + P\big(N(T) > b; I(T) \leq b - b^{1-\varepsilon}\big) \\ &\leq \big(1 + o(1)\big) P\big(I(T) > b\big) + P\big(N(T) > b | I(T) = b - b^{1-\varepsilon}\big) \\ &= \big(1 + o(1)\big) P\big(I(T) > b\big). \end{aligned}$$

The last step uses the fact that

$$P\big(N(T) > b | I(T) = b - b^{1-\varepsilon}\big) \leq \exp\big\{-\big(1/2 + o(1)\big) b^{1-2\varepsilon}\big\} = o(1) P\big(I(T) > b\big).$$

The bound of the tail of a Poisson distribution can be derived by the standard technique of large deviations theory [28] and therefore is omitted. $\square$

REMARK 3.7. The result in Theorem 3.6 suggests that an observation of a large number of points in a region T is mainly caused by a high level of its underlying intensity. Technically, this is because the distribution of $N(T)$ can be roughly considered as a convolution of the distribution of $\int e^{f(t)}\,dt$ and a Poisson distribution. Note that $\int e^{f(t)}\,dt$ is approximately a log-normal random variable, which has a much heavier tail than that of a Poisson random variable. Therefore, the tail behavior of $N(T)$ is mostly dominated by the tail of its underlying intensity.

3.4. *The change of measure.* In this subsection, we propose a change of measure Q which is central to the proof of Theorem 3.4. Let P be the original measure. The measure Q is defined such that P and Q are mutually absolutely continuous with the Radon–Nikodym derivative being

$$\frac{dQ}{dP} = \int_T \frac{1}{\operatorname{mes}(T)} \cdot \frac{\exp\{-(1/2)(f(t) - u + \mu_\sigma(t))^2\}}{\exp\{-(1/2)f(t)^2\}}\,dt, \tag{3.6}$$

where $\operatorname{mes}(\cdot)$ denotes Lebesgue measure. This change of measure was first proposed by [45] to derive the high excursion probabilities of approximately Gaussian processes. It is more intuitive to describe the measure Q from a simulation point of view. In order to simulate $f(t)$ under the measure Q, one can do the following two steps:

(1) Simulate a random variable τ uniformly over T with respect to the Lebesgue measure.

(2) Given the realized τ, simulate the Gaussian process $f(t)$ with mean $(u - \mu_\sigma(\tau))C(t-\tau)$ and covariance function $C(t)$.

It is not hard to verify that the above two-step procedure is consistent with the Radon–Nikodym derivative in (3.6). The measure Q is designed such that the distribution of f under the measure Q is approximately the conditional distribution of f given $\int_T e^{f(t)}\,dt > b$ under the measure P. Under Q, a random variable τ is first sampled uniformly over T, then $f(\tau)$ is simulated with a large mean at level $u - \mu_\sigma(\tau)$. This implies that the high level of the integral $\int_T e^{\mu(t)+\sigma f(t)}\,dt$ is mostly caused by the fact that the field reaches a high level at one location t^* and such a location t^* is very close to τ. Therefore, the random index τ localizes the maximum of the field. In particular, one can write the tail probability as

$$P\left(\int_T e^{\mu(t)+\sigma f(t)}\,dt > b\right) = E^Q\left[\frac{dP}{dQ}; \int_T e^{\mu(t)+\sigma f(t)}\,dt > b\right],$$

where we use E^Q to denote the expectation under Q and E to denote that under P.

In what follows, we explain the main result in Theorem 3.4 and how the change of measure helps in deriving the asymptotics. To simplify the discussion, we proceed by assuming that $\mu(t) \equiv 0$ and $\sigma = 1$. Upon considering zero (constant) mean, we obtain from the result of Theorem 3.4 that $P(\int_T e^{f(t)}\,dt > b) =$

$\Theta(1)P(\sup_T f(t) > u)$ (cf. [6]). This suggests that the large value of the exponential integral at the level b is largely caused by the high excursion of $\sup_T f(t)$ at a level u. The conditional distribution of $f(t)$ given a high excursion at level u (the Slepian model) is well known [7]. We proceed with a rough mean calculation. Suppose that $f(t)$ attains a large value at the origin of level u. Then the conditional field will have expectation $E[f(t)|f(0) = u] = uC(t)$. We expand the covariance function as

$$uC(t) \approx u - \frac{u}{2}|t|^2.$$

Therefore, one may expect to choose u such that conditional on $f(0) = u$

$$\int_T e^{f(t)}\,dt \approx \int_{R^d} e^{u-(u/2)|t|^2}\,dt = (2\pi)^{d/2}u^{-d/2}e^u = b. \tag{3.7}$$

This is precisely how u is selected in Theorem 3.4. The above calculation ignores the higher-order expansions of $C(t)$ and the deviation of the conditional field from its expectation. It turns out that these variations do not affect the asymptotic decaying rate of the tail probability. They only contribute to the constant term.

4. Proof of Theorem 3.4. The proof of Theorem 3.4 requires several lemmas. To facilitate the reading, we arrange their statements in Section 5.

Note that

$$\begin{aligned} P\left(\int_T e^{\mu(t)+\sigma f(t)}\,dt > b\right) &= E^Q\left[\frac{dP}{dQ}; \int_T e^{\mu(t)+\sigma f(t)}\,dt > b\right] \\ &= \int_T \frac{1}{\operatorname{mes}(T)} E^Q\left[\frac{dP}{dQ}; \int_T e^{\mu(t)+\sigma f(t)}\,dt > b \middle| \tau\right] d\tau. \end{aligned}$$

Furthermore, we use the notation that $E_\tau^Q[\cdot] = E^Q[\cdot|\tau]$. For each τ, we plug in (3.6) and further write the expectation inside the above integral as

$$\begin{aligned} &E_\tau^Q\left[\frac{dP}{dQ}; \int_T e^{\mu(t)+\sigma f(t)}\,dt > b\right] \\ &\quad = \operatorname{mes}(T) E_\tau^Q\left[\frac{1}{\int_T e^{-(1/2)(f(t)-u+\mu_\sigma(t))^2+(1/2)f^2(t)}\,dt}; \right. \\ &\qquad\qquad \left. \int_T e^{\mu(t)+\sigma f(t)}\,dt > b\right] \\ &\quad = \operatorname{mes}(T) e^{u^2/2} E_\tau^Q\left[\frac{1}{\int_T e^{(u-\mu_\sigma(t))(f(t)+\mu_\sigma(t))+(1/2)\mu_\sigma^2(t)}\,dt}; \right. \\ &\qquad\qquad \left. \int_T e^{\mu(t)+\sigma f(t)}\,dt > b\right]. \end{aligned} \tag{4.1}$$

We write

$$\Lambda(\tau) = e^{u^2/2} E_\tau^Q \Bigg[\frac{1}{\int_T e^{(u-\mu_\sigma(t))(f(t)+\mu_\sigma(t))+(1/2)\mu_\sigma^2(t)}\,dt}; \int_T e^{\mu(t)+\sigma f(t)}\,dt > b \Bigg]. \tag{4.2}$$

Note that conditional on τ, for every set A,

$$Q\big(f(\cdot) \in A|\tau\big) = P\big(f(\cdot) + \big(u - \mu_\sigma(\tau)\big)C(\cdot - \tau) \in A\big), \tag{4.3}$$

that is, the conditional distribution of $f(t)$ (given τ) under Q equals the distribution of $f(t) + (u - \mu_\sigma(\tau))C(t - \tau)$ under P. This equivalence can be derived from the two-step simulation procedure in Section 3.4. Therefore, we can simply replace $f(t)$ by $f(t) + (u - \mu_\sigma(\tau))C(t - \tau)$, replace Q by P, and write

$$\Lambda(\tau) = e^{u^2/2} E \Bigg[\frac{1}{\int_T e^{(u-\mu_\sigma(t))[f(t)+(u-\mu_\sigma(\tau))C(t-\tau)+\mu_\sigma(t)]+(1/2)\mu_\sigma^2(t)}\,dt}; \int_T e^{\sigma\{f(t)+(u-\mu_\sigma(\tau))C(t-\tau)+\mu_\sigma(t)\}}\,dt > b \Bigg]. \tag{4.4}$$

Let

$$\mathcal{E}_b = \Bigg\{ \int_T e^{\sigma\{f(t)+(u-\mu_\sigma(t))C(t-\tau)+\mu_\sigma(t)\}}\,dt > b \Bigg\}, \tag{4.5}$$

$$K = \int_T e^{(u-\mu_\sigma(t))[f(t)+(u-\mu_\sigma(\tau))C(t-\tau)+\mu_\sigma(t)]+(1/2)\mu_\sigma^2(t)}\,dt. \tag{4.6}$$

Then, (4.4) can be written as

$$\Lambda(\tau) = e^{u^2/2} \int E[K^{-1}; \mathcal{E}_b | f(\tau) = w, \partial f(\tau) = \tilde{y}, \partial^2 f(\tau) = \tilde{z}] \times h(w, \tilde{y}, \tilde{z})\,dw\,d\tilde{y}\,d\tilde{z}, \tag{4.7}$$

where $h(w, \tilde{y}, \tilde{z})$ is the density function of $(f(\tau), \partial f(\tau), \partial^2 f(\tau))$ evaluated at $(w, \tilde{y}, \tilde{z})$.

For a given $\delta' > 0$ small enough, we consider two cases for τ: first, $\{t : |t - \tau| \le u^{-1/2+\delta'}\} \subset T$ and otherwise. For the first situation, τ is "far away" from the boundary of T, which is the important case in our analysis. For the second situation, τ is close to the boundary. We will show that the second situation is of less importance given that the maximum of $\mu(t)$ is attained at the interior of T.

For the first situation, the analysis consists of three main parts.

Part 1. Conditional on $(\tau, f(\tau), \partial f(\tau), \partial^2 f(\tau))$, we study the event

$$\mathcal{E}_b = \Bigg\{ \int_T e^{\sigma\{f(t)+(u-\mu_\sigma(\tau))C(t-\tau)+\mu_\sigma(t)\}}\,dt > b \Bigg\} \tag{4.8}$$

and write the occurrence of this event almost as a deterministic function of $f(\tau)$, $\partial f(\tau)$ and $\partial^2 f(\tau)$.

Part 2. Conditional on $(\tau, f(\tau), \partial f(\tau), \partial^2 f(\tau))$, we write K defined in (4.6) as a function of $f(\tau)$, $\partial f(\tau)$, $\partial^2 f(\tau)$ and a small correction term.

Part 3. We combine the results from the first two parts and obtain an approximation of (4.1) through the right-hand side of (4.7).

All the subsequent derivations are conditional on a specific value of τ.

Preliminary calculations. For $0 < \varepsilon < \delta'$ sufficiently small, let

$$\begin{aligned} \mathcal{L}_Q = \{ & |f(\tau) - u + \mu_\sigma(\tau)| \le u^{1/2+\varepsilon}, \\ & |\partial f(\tau)| < u^{1/2+\varepsilon}, \\ & \left|\partial^2 f(\tau) - \big(u - \mu_\sigma(\tau)\big)\mu_{02}\right| < u^{1/2+\varepsilon}\}. \end{aligned} \tag{4.9}$$

According to Lemma 5.2, we only need to consider the integral on the set $\mathcal{L}_Q$, that is,

$$E_\tau^Q\Bigg[\frac{1}{\int \exp\{(u-\mu_\sigma(t))(f(t)+\mu_\sigma(t)) + (1/2)\mu_\sigma^2(t)\}\,dt}; \\ \int_T e^{\mu(t)+\sigma f(t)}\,dt > b, \mathcal{L}_Q\Bigg].$$

The above display equals

$$E[K^{-1}; \mathcal{E}_b, \mathcal{L}],$$

where

$$\mathcal{L} = \{|f(\tau)| \le u^{1/2+\varepsilon}, |\partial f(\tau)| < u^{1/2+\varepsilon}, |\partial^2 f(\tau)| < u^{1/2+\varepsilon}\} \tag{4.10}$$

corresponds to $\mathcal{L}_Q$ under the transform (4.3). Therefore, throughout the rest of the proof, all the derivations are on the set $\mathcal{L}$. Note that the sets $\mathcal{L}_Q$ and $\mathcal{L}$ depend on τ and u. Since all the subsequent derivations are for specific τ and u, we omit the indices of τ and u in the notation $\mathcal{L}$ and $\mathcal{L}_Q$.

We first provide the Taylor expansions for $f(t)$, $C(t)$ and $\mu(t)$.

- Expansion of $f(t)$ given $(f(\tau), \partial f(\tau), \partial^2 f(\tau))$. Let $t - \tau = ((t-\tau)_1, \dots, (t-\tau)_d)$. Conditional on $(f(\tau), \partial f(\tau), \partial^2 f(\tau))$, we first expand the random function

$$\begin{aligned} f(t) &= E[f(t)|f(\tau), \partial f(\tau), \partial^2 f(\tau)] + g(t-\tau) \\ &= f(\tau) + \partial f(\tau)^\top (t-\tau) + \tfrac{1}{2}(t-\tau)^\top \Delta f(\tau)(t-\tau) \\ &\quad + g_3(t-\tau) + R_f(t-\tau) + g(t-\tau), \end{aligned} \tag{4.11}$$

where

$$g_3(t-\tau) = \frac{1}{6}\sum_{i,j,k} E[\partial^3_{ijk} f(\tau) | f(\tau), \partial f(\tau), \partial^2 f(\tau)](t-\tau)_i (t-\tau)_j (t-\tau)_k.$$

Note that $\partial^3_{ijk} f(\tau)$ is independent of $(f(\tau), \Delta f(\tau))$ and

$$E[\partial^3_{ijk} f(\tau) | f(\tau), \partial f(\tau), \partial^2 f(\tau)] = -\sum_l \partial^4_{ijkl} C(0) \partial_l f(\tau).$$

$g(t)$ is a mean-zero Gaussian random field such that $Eg^2(t) = O(|t|^6)$ as $t \to 0$. In addition, the distribution of $g(t)$ is independent of τ, $f(\tau)$, $\partial f(\tau)$ and $\partial^2 f(\tau)$. $R_f(t-\tau) = O(|t-\tau|^4)$ is the remainder term of the Taylor expansion of $E[f(t) | f(\tau), \partial f(\tau), \partial^2 f(\tau)]$.

- Expansion of $C(t)$:

$$C(t) = 1 - \tfrac{1}{2} t^\top t + C_4(t) + R_C(t), \tag{4.12}$$

where $R_C(t) = O(|t|^6)$ and

$$C_4(t) = \frac{1}{24}\sum_{ijkl} \partial^4_{ijkl} C(0) t_i t_j t_k t_l.$$

- Expansion of $\mu(t)$:

$$\begin{aligned} \mu_\sigma(t) &= \mu_\sigma(\tau) + \partial \mu_\sigma(\tau)^\top (t-\tau) \\ &\quad + \tfrac{1}{2}(t-\tau)^\top \Delta \mu_\sigma(\tau)(t-\tau) + R_\mu(t-\tau), \end{aligned} \tag{4.13}$$

where $R_\mu(t-\tau) = O(|t-\tau|^3)$.

Let I be the $d \times d$ identity matrix. We define the following notation that will be constantly used later:

$$\begin{aligned} \tilde{u} &= u - \mu_\sigma(\tau), \qquad \tilde{y} = \partial f(\tau), \\ \tilde{\mathbf{z}} &= \Delta f(\tau), \qquad y = \partial f(\tau) + \partial \mu_\sigma(\tau), \\ \mathbf{z} &= \Delta f(\tau) + \mu_\sigma(\tau) I + \Delta \mu_\sigma(\tau), \\ R(t) &= R_f(t) + \big(u - \mu_\sigma(\tau)\big) R_C(t) + R_\mu(t). \end{aligned}$$

As mentioned earlier, we let z and $\tilde{z}$ be the vector version of the matrices $\mathbf{z}$ and $\tilde{\mathbf{z}}$.

Now, we start to carry out our three-step program.

Part 1. All the derivations in this part are conditional on specific values of τ, $f(\tau)$, $\partial f(\tau)$ and $\partial^2 f(\tau)$. Define

$$I_1 \triangleq \int_T e^{\sigma\{f(t) + (u - \mu_\sigma(\tau)) C(t-\tau) + \mu_\sigma(t)\}}\, dt.$$

We insert the expansions in (4.11), (4.12) and (4.13) into the expression of I_1 and obtain that

$$
\begin{aligned}
I_1 = \int_{t\in T} \exp\Bigg\{\sigma\Bigg[& f(\tau) + \partial f(\tau)^\top (t-\tau) + \frac{1}{2}(t-\tau)^\top \Delta f(\tau)(t-\tau) \\
& + g_3(t-\tau) + R_f(t-\tau) + g(t-\tau) \\
& + \big(u - \mu_\sigma(\tau)\big)\Bigg(1 - \frac{1}{2}(t-\tau)^\top (t-\tau) \\
& \qquad + C_4(t-\tau) + R_C(t-\tau)\Bigg) \\
& + \mu_\sigma(\tau) + \partial \mu_\sigma(\tau)^\top (t-\tau) \\
& \qquad + \frac{1}{2}(t-\tau)^\top \Delta \mu_\sigma(\tau)(t-\tau) + R_\mu(t-\tau)\Bigg]\Bigg\}\, dt.
\end{aligned} \tag{4.14}
$$

We write the exponent inside the integral in a quadratic form of $(t-\tau)$ and obtain that

$$
\begin{aligned}
I_1 = \exp\Big\{\sigma u + \sigma f(\tau) + \frac{\sigma}{2} y^\top (uI - \mathbf{z})^{-1} y\Big\} & \\
\times \int_{t\in T} \exp\Big\{-\frac{\sigma}{2}\big(t - (uI-\mathbf{z})^{-1}y\big)^\top (uI-\mathbf{z})\big(s - (uI-\mathbf{z})^{-1}y\big)\Big\} & \\
\times \exp\{\sigma g_3(t) + \sigma\big(u - \mu_\sigma(\tau)\big)C_4(t) + \sigma R(t)\} & \\
\times \exp\{\sigma g(t)\}\, dt. &
\end{aligned} \tag{4.15}
$$

Further, consider the change of variable that $s = (uI - \mathbf{z})^{1/2}(t-\tau)$, write the big integral in above display as a product of expectations and a normalizing constant, and obtain that

$$
\begin{aligned}
I_1 = {} & \det(uI-\mathbf{z})^{-1/2} \exp\Big\{\sigma u + \sigma f(\tau) + \frac{\sigma}{2} y^\top (uI-\mathbf{z})^{-1} y\Big\} \\
& \times \int_{(uI-\mathbf{z})^{-1/2}s+\tau\in T} \exp\Big\{-\frac{\sigma}{2}\big(s - (uI-\mathbf{z})^{-1/2}y\big)^\top \big(s - (uI-\mathbf{z})^{-1/2}y\big)\Big\}\, ds \\
& \times E\Big[\exp\Big\{\sigma g_3\big((uI-\mathbf{z})^{-1/2}S\big) + \sigma\big(u - \mu_\sigma(\tau)\big)C_4\big((uI-\mathbf{z})^{-1/2}S\big) \\
& \qquad\qquad + \sigma R\big((uI-\mathbf{z})^{-1/2}S\big)\Big\}\Big] \\
& \times E\big[\exp\{\sigma g\big((uI-\mathbf{z})^{-1/2}\tilde{S}\big)\}\big].
\end{aligned}
$$

The two expectations in the above display are taken with respect to S and $\tilde{S}$ given the process $g(t)$. S is a random variable taking values in the set $\{s : (uI-\mathbf{z})^{-1/2}s +$

$\tau \in T\}$ with density proportional to

$$\exp\left\{-\frac{\sigma}{2}\big(s-(uI-\mathbf{z})^{-1/2}y\big)^{\top}\big(s-(uI-\mathbf{z})^{-1/2}y\big)\right\}$$

and $\tilde{S}$ is a random variable taking values in the set $\{s:(uI-\mathbf{z})^{-1/2}s+\tau\in T\}$ with density proportional to

$$\begin{aligned}&\exp\left\{-\frac{\sigma}{2}\big(s-(uI-\mathbf{z})^{-1/2}y\big)^{\top}\big(s-(uI-\mathbf{z})^{-1/2}y\big)\right\}\\ &\quad\times\exp\big\{\sigma g_3\big((uI-\mathbf{z})^{-1/2}s\big)+\sigma\big(u-\mu_\sigma(\tau)\big)C_4\big((uI-\mathbf{z})^{-1/2}s\big)\\ &\qquad\qquad+\sigma R\big((uI-\mathbf{z})^{-1/2}s\big)\big\}.\end{aligned}\tag{4.16}$$

Together with the definition of u that

$$\left(\frac{2\pi}{\sigma}\right)^{d/2}u^{-d/2}e^{\sigma u}=b,$$

we obtain that

$$I_1=\int_T e^{\sigma\{f(t)+(u-\mu_\sigma(t))C(t-\tau)+\mu_\sigma(t)\}}\,dt>b$$

if and only if

$$\begin{aligned}I_1&=\det(uI-\mathbf{z})^{-1/2}\exp\left\{\sigma u+\sigma f(\tau)+\frac{\sigma}{2}y^{\top}(uI-\mathbf{z})^{-1}y\right\}\\ &\quad\times\int_{(uI-\mathbf{z})^{-1/2}s+\tau\in T}\exp\left\{-\frac{\sigma}{2}\big(s-(uI-\mathbf{z})^{-1/2}y\big)^{\top}\right.\\ &\qquad\qquad\left.\times\big(s-(uI-\mathbf{z})^{-1/2}y\big)\right\}ds\\ &\quad\times E\exp\big\{\sigma g_3\big((uI-\mathbf{z})^{-1/2}S\big)\\ &\qquad\qquad+\sigma\big(u-\mu_\sigma(\tau)\big)C_4\big((uI-\mathbf{z})^{-1/2}S\big)+\sigma R\big((uI-\mathbf{z})^{-1/2}S\big)\big\}\\ &\quad\times\exp\{-u^{-1}\xi_u\}\\ &>\left(\frac{2\pi}{\sigma}\right)^{d/2}u^{-d/2}e^{\sigma u},\end{aligned}\tag{4.17}$$

where

$$\xi_u=-u\log\big\{E\exp\big[\sigma g\big((uI-\mathbf{z})^{-1/2}\tilde{S}\big)\big]\big\}.\tag{4.18}$$

We take log on both sides and plug in the result of Lemma 5.4 that handles the big expectation term in (4.17). Then, inequality (4.17) is equivalent to

$$\begin{aligned} A &\triangleq \sigma f(\tau) + \frac{\sigma}{2} y^\top (uI - \mathbf{z})^{-1} y \\ &\quad - \frac{1}{2}\log\det(I - u^{-1}\mathbf{z}) + \sigma B + o(u^{-1}) \\ &> u^{-1}\xi_u, \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} B &\triangleq -\frac{1}{8u}(u^{-1}Y + \mathbf{1}/\sigma)^\top \mu_{22}(u^{-1}Y + \mathbf{1}/\sigma) \\ &\quad + \frac{\mathbf{1}^\top \mu_{22}\mathbf{1}}{8\sigma^2 u} + \frac{1}{8\sigma^2 u}\sum_i \partial^4_{iiii} C(0) \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} Y &= \{y_i^2, i = 1, \dots, d; 2y_i y_j, 1 \le i < j \le d\}, \\ \mathbf{1} &= (\underbrace{1, \dots, 1}_{d}, \underbrace{0, \dots, 0}_{d(d-1)/2})^\top. \end{aligned}$$

Roughly speaking, according to Lemma 5.3, the event $\mathcal{E}_b$ is the same as the event $\{A > O_p(u^{-3/2+3\delta})\}$.

Part 2. Similarly to part 1, all the derivations in this part are conditional on $(\tau, f(\tau), \partial f(\tau), \partial^2 f(\tau))$. We now proceed to the second part of the proof. More precisely, we simplify the term K defined as in (4.6) and write it as a deterministic function of $(f(\tau), \partial f(\tau), \partial^2 f(\tau))$ with a small correction term. For $\varepsilon < \delta < \delta'$ with all of them sufficiently small, we let $\lambda_u = u^{-1/2+\delta}$. We first split the integral into two parts, that is,

$$\begin{aligned} K &= \int_T e^{(u-\mu_\sigma(t))[f(t)+(u-\mu_\sigma(\tau))C(t-\tau)+\mu_\sigma(t)]+(1/2)\mu_\sigma^2(t)}\,dt \\ &= \int_{|t-\tau|<\lambda_u} + \cdots + \int_{|t-\tau|>\lambda_u} + \cdots \\ &= I_2 + I_3. \end{aligned}$$

For the leading term, note that $|t - \tau| \le \lambda_n = u^{-1/2+\delta}$. We insert the Taylor expansion of $\mu_\sigma(t)$:

$$\begin{aligned} I_2 &= \int_{|t-\tau|<\lambda_u} e^{(u-\mu_\sigma(t))[f(t)+(u-\mu_\sigma(\tau))C(t-\tau)+\mu_\sigma(t)]+(1/2)\mu_\sigma^2(t)}\,dt \\ &= \big(1 + o(1)\big)e^{u^2 - u\mu_\sigma(\tau)+(1/2)\mu_\sigma^2(\tau)} \\ &\quad \times \int_{|t-\tau|<\lambda_u} e^{(u-\mu_\sigma(t))[f(t)+(u-\mu_\sigma(\tau))C(t-\tau)-u+\mu_\sigma(\tau)]}\,dt. \end{aligned}$$

Let $\zeta_u = O(u^{-1/2+\delta})$. In what follows, we insert the expansions in (4.11), (4.12) and (4.13), write the exponent as a quadratic function of $t-\tau$, and obtain that on the set $\mathcal{L}$:

$$\begin{aligned}
&\int_{|t-\tau|<\lambda_u} e^{(u-\mu_\sigma(t))[f(t)+(u-\mu_\sigma(\tau))C(t-\tau)-u+\mu_\sigma(\tau)]}\,dt \\
&\quad = \int_{|t-\tau|<\lambda_u} \exp\Big\{(\tilde{u}+\zeta_u)\Big[f(\tau)+(t-\tau)^\top\tilde{y} \\
&\qquad\qquad - \frac{1}{2}(t-\tau)^\top(\tilde{u}I-\tilde{\mathbf{z}})(t-\tau)+g_3(t-\tau) \\
&\qquad\qquad + \tilde{u}C_4(t-\tau)+g(t-\tau)+O(u^{-3/2+3\delta})\Big]\Big\}\,dt \\
&\quad = \big(1+o(1)\big)\exp\Big\{(\tilde{u}+\zeta_u)\Big(f(\tau)+\frac{1}{2}\tilde{y}^\top(\tilde{u}I-\tilde{\mathbf{z}})^{-1}\tilde{y}\Big)\Big\} \\
&\qquad \times \int_{|t-\tau|<\lambda_u} \exp\Big\{(\tilde{u}+\zeta_u)\Big[-\frac{1}{2}\big(t-\tau-(\tilde{u}I-\tilde{\mathbf{z}})^{-1}\tilde{y}\big)^\top \\
&\qquad\qquad \times (\tilde{u}I-\tilde{\mathbf{z}})\big(t-\tau-(\tilde{u}I-\tilde{\mathbf{z}})^{-1}\tilde{y}\big) \\
&\qquad\qquad + g_3(t-\tau)+\tilde{u}C_4(t-\tau) \\
&\qquad\qquad + g(t-\tau)+O(u^{-3/2+3\delta})\Big]\Big\}\,dt.
\end{aligned} \tag{4.21}$$

We consider the change of variable that $s=(\tilde{u}+\zeta_u)^{1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{1/2}(t-\tau)$ and obtain that (4.21) equals

$$\begin{aligned}
&\big(1+o(1)\big) \\
&\quad \times \det(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}\tilde{u}^{-d/2} \\
&\quad \times \exp\Big\{(\tilde{u}+\zeta_u)\Big(f(\tau)+\frac{1}{2}\tilde{y}^\top(\tilde{u}I-\tilde{\mathbf{z}})^{-1}\tilde{y}\Big)\Big\} \\
&\quad \times \int_{|\tilde{u}^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s|<\lambda_u} \exp\Big\{-\frac{1}{2}|s-(\tilde{u}+\zeta_u)^{1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}\tilde{y}|^2\Big\} \\
&\qquad \times \exp\big\{(\tilde{u}+\zeta_u)g_3\big((\tilde{u}+\zeta_u)^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s\big) \\
&\qquad\quad + \tilde{u}(\tilde{u}+\zeta_u)C_4\big((\tilde{u}+\zeta_u)^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s\big) \\
&\qquad\quad + (\tilde{u}+\zeta_u)g\big((\tilde{u}+\zeta_u)^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s\big)\big\}\,ds.
\end{aligned} \tag{4.22}$$

Note that the variation of the last term in (4.22), the $g(t)$ term, is tiny. Then, we first focus on the leading term. Similar to the proof of Lemma 5.4, we can write

the integral [without the $g(t)$ term] as

$$\begin{aligned}
&\int_{|\tilde{u}^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s|<\lambda_u} \exp\Big\{-\frac{1}{2}|s-(\tilde{u}+\zeta_u)^{1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}\tilde{y}|^2\Big\} \\
&\qquad\qquad \times \exp\big\{(\tilde{u}+\zeta_u)g_3\big((\tilde{u}+\zeta_u)^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s\big) \\
&\qquad\qquad\qquad + \tilde{u}(\tilde{u}+\zeta_u)C_4\big((\tilde{u}+\zeta_u)^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s\big)\big\}\,ds \\
&\quad = \big(1+o(1)\big)e^{-(\tilde{u}^{-2}/8)\tilde{Y}^\top\mu_{22}\tilde{Y}} \\
&\qquad \times \int_{|\tilde{u}^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s|<\lambda_u} e^{-(1/2)|s-(\tilde{u}+\zeta_u)^{1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}\tilde{y}|^2}\,ds,
\end{aligned}$$

where

$$\tilde{Y} = \{\tilde{y}_i^2, i=1,\dots,d; 2\tilde{y}_i\tilde{y}_j, 1\le i<j\le d\}$$

is arranged in the same order as that of the elements in Y. Therefore, (4.21) equals

$$\begin{aligned}
&\big(1+o(1)\big)\det(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}\tilde{u}^{-d/2} \\
&\quad \times \exp\Big\{(\tilde{u}+\zeta_u)\Big(f(\tau)+\frac{1}{2}\tilde{y}^\top(\tilde{u}I-\tilde{\mathbf{z}})^{-1}\tilde{y}\Big)-\frac{\tilde{u}^{-2}}{8}\tilde{Y}^\top\mu_{22}\tilde{Y}\Big\} \\
&\quad \times \int_{|\tilde{u}^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s|<\lambda_u} \exp\Big\{-\frac{1}{2}|s-(\tilde{u}+\zeta_u)^{1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}\tilde{y}|^2\Big\}\,ds \\
&\quad \times E\big[\exp\big\{(\tilde{u}+\zeta_u)g\big((\tilde{u}+\zeta_u)^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}S'\big)\big\}\big],
\end{aligned} \tag{4.23}$$

where S' is a random variable taking values on the set $\{s : |\tilde{u}^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s| < \lambda_u\}$ with density proportional to

$$\begin{aligned}
&\exp\big\{-\tfrac{1}{2}|s-(\tilde{u}+\zeta_u)^{1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}\tilde{y}|^2\big\} \\
&\qquad \times \exp\big\{(\tilde{u}+\zeta_u)g_3\big((\tilde{u}+\zeta_u)^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s\big) \\
&\qquad\qquad + \tilde{u}(\tilde{u}+\zeta_u)C_4\big((\tilde{u}+\zeta_u)^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s\big)\big\}.
\end{aligned}$$

We use κ to denote the last two terms of (4.23):

$$\begin{aligned}
\kappa = &\int_{|\tilde{u}^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}s|<\lambda_u} \exp\Big\{-\frac{1}{2}|s-(\tilde{u}+\zeta_u)^{1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}\tilde{y}|^2\Big\}\,ds \\
&\times E\big[\exp\big\{(\tilde{u}+\zeta_u)g\big((\tilde{u}+\zeta_u)^{-1/2}(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}S'\big)\big\}\big].
\end{aligned} \tag{4.24}$$

It is helpful to keep in mind that $\kappa = (2\pi)^{d/2}+o(1)$. Now, we continue the calculations in (4.23) and write (4.21) in the form of A defined as in (4.19) to facilitate the change of variable later. Then, on the set $\mathcal{L}$, we plug in the form of A and B

defined in part 1 of the proof and obtain that (4.21) equals

$$
\begin{aligned}
(4.21) &= \big(1+o(1)\big)\kappa \det(\tilde{u}I-\tilde{\mathbf{z}})^{-1/2}u^{-d/2} \\
&\qquad \times \exp\Big\{(\tilde{u}+\zeta_u)\Big(f(\tau)+\frac{1}{2}\tilde{y}^\top(\tilde{u}I-\tilde{z})^{-1}\tilde{y}-\frac{\tilde{u}^{-2}}{8}\tilde{Y}^\top\mu_{22}\tilde{Y}\Big)\Big\} \\
&= \big(1+o(1)\big)\kappa u^{-d}\exp\Big\{\frac{\tilde{u}}{\sigma}A-\tilde{u}B+\frac{\tilde{u}}{2\sigma}\log\det(I-u^{-1}\mathbf{z}) \\
&\qquad\qquad + \frac{\tilde{u}}{2}\big(\tilde{y}^\top(\tilde{u}I-\tilde{\mathbf{z}})^{-1}\tilde{y}-y^\top(uI-\mathbf{z})^{-1}y\big) \\
&\qquad\qquad\qquad + u^{-1/2+\delta}O\big(|y|^2+f(\tau)\big)\Big\} \\
&= \big(1+o(1)\big)\kappa u^{-d}\exp\Big\{\frac{\tilde{u}}{\sigma}A-\tilde{u}B-\frac{1}{2\sigma}\operatorname{Tr}\big(\tilde{\mathbf{z}}+\mu_\sigma(\tau)I+\Delta\mu_\sigma(\tau)\big) \\
&\qquad\qquad - \tilde{y}^\top\partial\mu_\sigma(\tau)-\frac{1}{2}|\partial\mu_\sigma(\tau)|^2 \\
&\qquad\qquad\qquad + u^{-1/2+\delta}O\big(|y|^2+|z|^2+f(\tau)\big)\Big\},
\end{aligned}
\tag{4.25}
$$

where $\operatorname{Tr}(\mathbf{z})$ is the trace of matrix $\mathbf{z}$. The last step in the above display is thanks to Lemma 5.6 and the fact that $y=\tilde{y}+\partial\mu_\sigma(\tau)$. We insert the result of (4.25) into the definition of I_2 and obtain that

$$
\begin{aligned}
I_2 &= \big(1+o(1)\big)\kappa u^{-d}e^{u^2-u\mu_\sigma(\tau)+(1/2)\mu_\sigma^2(\tau)} \\
&\quad \times \exp\Big\{\frac{\tilde{u}}{\sigma}A-\tilde{u}B-\frac{1}{2\sigma}\operatorname{Tr}\big(\tilde{\mathbf{z}}+\mu_\sigma(\tau)I+\Delta\mu_\sigma(\tau)\big) \\
&\qquad - \tilde{y}^\top\partial\mu_\sigma(\tau)-\frac{1}{2}|\partial\mu_\sigma(\tau)|^2+u^{-1/2+\delta}O\big(|y|^2+|z|^2+f(\tau)\big)\Big\}.
\end{aligned}
$$

Thanks to Lemma 5.5, I_3 is of a much smaller order than I_2 and we obtain that

$$
\begin{aligned}
K &= I_2+I_3 \\
&= \big(1+o(1)\big)\big(\kappa+O\big(e^{-\delta^* u^{1+2\delta}}e^{u\sup|g(t)|}\big)\big)u^{-d}e^{u^2-u\mu_\sigma(\tau)+(1/2)\mu_\sigma^2(\tau)} \\
&\quad \times \exp\Big\{\frac{\tilde{u}}{\sigma}A-\tilde{u}B-\frac{1}{2\sigma}\operatorname{Tr}\big(\tilde{\mathbf{z}}+\mu_\sigma(\tau)I+\Delta\mu_\sigma(\tau)\big) \\
&\qquad - \tilde{y}^\top\partial\mu_\sigma(\tau)-\frac{1}{2}|\partial\mu_\sigma(\tau)|^2+u^{-1/2+\delta}O\big(|y|^2+|z|^2+f(\tau)\big)\Big\}.
\end{aligned}
\tag{4.26}
$$

Part 3. Use the notation $f(\tau)=w$, $\partial f(\tau)=\tilde{y}$ and $\partial^2 f(\tau)=\tilde{z}$. We now put together the results from parts 1 and 2 and obtain an approximation of $\Lambda(\tau)$ defined

as in (4.2). Note that

$$\begin{aligned}\Lambda^*(\tau) &\triangleq E_\tau^Q\Bigg[\frac{1}{\int_T e^{-(1/2)(f(t)-u+\mu_\sigma(t))^2+(1/2)f^2(t)}\,dt};\\ &\qquad\qquad \int_T e^{\mu(t)+\sigma f(t)}\,dt > b, \mathcal{L}_Q\Bigg]\\ &= e^{u^2/2}E\Bigg[\frac{1}{I_2+I_3}; u\cdot A > \xi_u, \mathcal{L}\Bigg]\\ &= e^{u^2/2}\int_{\mathcal{L}} E\Bigg[\frac{1}{I_2+I_3}; u\cdot A > \xi_u \Big| f(\tau)=w, \partial f(\tau)=\tilde{y}, \partial^2 f(\tau)=\tilde{z}\Bigg]\\ &\qquad\qquad \times h(w,\tilde{y},\tilde{z})\,dw\,d\tilde{y}\,d\tilde{z}.\end{aligned}$$

Plugging in (4.26), we have that

$$\begin{aligned}\Lambda^*(\tau) &= \big(1+o(1)\big)u^d \exp\Big\{-u^2/2 + u\mu_\sigma(\tau) - \frac{1}{2}\mu_\sigma^2(\tau)\Big\}\\ &\quad\times \int_{\mathcal{L}} \gamma_u(u\cdot A)\\ &\qquad\times \exp\Big\{-\frac{\tilde{u}}{\sigma}A + \tilde{u}B + \frac{1}{2\sigma}\operatorname{Tr}\big(\tilde{\mathbf{z}} + \mu_\sigma(\tau)I + \Delta\mu_\sigma(\tau)\big)\\ &\qquad\qquad + \tilde{y}^\top \partial\mu_\sigma(\tau) + \frac{1}{2}|\partial\mu_\sigma(\tau)|^2 + u^{-1/2+\delta}O(|\tilde{y}|^2+|\tilde{z}|^2+w)\Big\}\\ &\qquad\times h(w,\tilde{y},\tilde{z})\,dw\,d\tilde{y}\,d\tilde{z},\end{aligned}$$

where h is the density function of $(f(\tau), \partial f(\tau), \partial^2 f(\tau))$ under the measure P. The above display also uses the fact that both A and B are functions of $(w,\tilde{y},\tilde{z})$ and therefore can be pulled outside of the conditional expectation. The notation

$$\gamma_u(x) = E\Bigg[\frac{1}{\kappa + O(e^{-\delta^* u^{1+2\delta}} e^{u\sup|g(t)|})}; x > \xi_u \Big| w, \tilde{y}, \tilde{z}\Bigg], \tag{4.27}$$

where the expectation is taken with respect to the process $g(t)$. Note that $\mathbf{1}^\top\tilde{z} = \operatorname{Tr}(\tilde{\mathbf{z}})$. Plugging in the analytic forms of h (Lemma 5.7) and B as in (4.20) and moving all the constants out of the integral, we obtain that

$$\begin{aligned}\Lambda^*(\tau) &= \big(1+o(1)\big)\\ &\qquad\times u^d \exp\Big\{-u^2/2 + u\mu_\sigma(\tau) - \frac{1}{2}\mu_\sigma^2(\tau)\\ &\qquad\qquad + \frac{1}{2\sigma}\operatorname{Tr}\big(\mu_\sigma(\tau)I + \Delta\mu_\sigma(\tau)\big)\\ &\qquad\qquad + \frac{1}{2}|\partial\mu_\sigma(\tau)|^2 + \frac{\mathbf{1}^\top\mu_{22}\mathbf{1}}{8\sigma^2} + \frac{1}{8\sigma^2}\sum_i \partial^4_{iiii}C(0)\Big\}\end{aligned}$$

$$
\begin{aligned}
&\times \frac{|\Gamma|^{-1/2}}{(2\pi)^{(d+1)(d+2)/4}} \\
&\times \int_{\mathcal{L}} \gamma_u(u \cdot A) \exp\Big\{-\frac{\tilde{u}}{\sigma} A + \frac{1}{2\sigma} \mathbf{1}^\top \tilde{z} + \tilde{y}^\top \partial \mu_\sigma(\tau)\Big\} \\
&\qquad \times \exp\Big\{-\frac{1}{8}(u^{-1}Y + \mathbf{1}/\sigma)^\top \mu_{22}(u^{-1}Y + \mathbf{1}/\sigma) \\
&\qquad\qquad - \frac{1}{2}\Big[\tilde{y}^\top \tilde{y} + \frac{(w - \mu_{20}\mu_{22}^{-1}\tilde{z})^2}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} + \tilde{z}^\top \mu_{22}^{-1}\tilde{z}\Big]\Big\} \\
&\qquad \times \exp\{u^{-1/2+\delta} O(|\tilde{y}|^2 + |\tilde{z}|^2 + w)\}\, dw\, d\tilde{y}\, d\tilde{z}.
\end{aligned}
$$

We insert

$$
\begin{aligned}
&-\frac{1}{8}(u^{-1}Y + \mathbf{1}/\sigma)^\top \mu_{22}(u^{-1}Y + \mathbf{1}/\sigma) \\
&\qquad = -\frac{1}{8\sigma^2}\mathbf{1}^\top \mu_{22}\mathbf{1} + u^{-1/2+\delta} O(|\tilde{y}|^2 + 1)
\end{aligned}
$$

into the above display. With some elementary calculation, we obtain that

$$
\begin{aligned}
\Lambda^*(\tau) &= \big(1 + o(1)\big) u^d \\
&\quad \times \exp\Big\{-u^2/2 + u\mu_\sigma(\tau) - \frac{1}{2}\mu_\sigma^2(\tau) \\
&\qquad + \frac{1}{2\sigma}\operatorname{Tr}\big(\mu_\sigma(\tau) I + \Delta\mu_\sigma(\tau)\big) + |\partial\mu_\sigma(\tau)|^2 \\
&\qquad\qquad + \frac{\mathbf{1}^\top \mu_{22}\mathbf{1}}{8\sigma^2} + \frac{1}{8\sigma^2}\sum_i \partial^4_{iiii} C(0)\Big\} \\
&\quad \times \frac{|\Gamma|^{-1/2}}{(2\pi)^{(d+1)(d+2)/4}} \\
&\quad \times \int_{\mathcal{L}} \gamma_u(u \cdot A) \exp\Big\{-\frac{\tilde{u}}{\sigma} A + u^{-1/2+\delta} O(|\tilde{y}|^2 + |\tilde{z}|^2 + w + 1)\Big\} \\
&\qquad \times \exp\Big\{-\frac{1}{2}\Big[|\tilde{y} - \partial\mu_\sigma(\tau)|^2 + \frac{(w - \mu_{20}\mu_{22}^{-1}\tilde{z})^2}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} \\
&\qquad\qquad + \Big|\mu_{22}^{-1/2}\tilde{z} - \mu_{22}^{1/2}\frac{\mathbf{1}}{2\sigma}\Big|^2\Big]\Big\}\, dw\, d\tilde{y}\, d\tilde{z}.
\end{aligned}
$$

Furthermore, on the set $\mathcal{L}$, according to the definition of A in (4.19) and

$$
w = A/\sigma + O(u^{-1/2+\varepsilon}|\tilde{y}|) + o(1),
$$

we obtain that

$$
\begin{aligned}
\Lambda^*(\tau) &= \big(1+o(1)\big)u^d \\
&\quad \times \exp\Big\{-u^2/2 + u\mu_\sigma(\tau) - \frac{1}{2}\mu_\sigma^2(\tau) + \frac{1}{2\sigma}\operatorname{Tr}\big(\mu_\sigma(\tau)I + \Delta\mu_\sigma(\tau)\big) \\
&\qquad\qquad + |\partial\mu_\sigma(\tau)|^2 + \frac{\mathbf{1}^\top\mu_{22}\mathbf{1}}{8\sigma^2} + \frac{1}{8\sigma^2}\sum_i \partial^4_{iiii}C(0)\Big\} \\
&\quad \times \frac{|\Gamma|^{-1/2}}{(2\pi)^{(d+1)(d+2)/4}} \\
&\quad \times \int_{\mathcal{L}} \gamma_u(u\cdot A)\exp\Big\{-\frac{\tilde{u}}{\sigma}A + u^{-1/2+\delta}O(|\tilde{y}|^2 + |\tilde{z}|^2 + A)\Big\} \\
&\quad \times \exp\Big\{-\frac{1}{2}\Big[|\tilde{y} - \partial\mu_\sigma(\tau)|^2 \\
&\qquad\qquad + \frac{(A/\sigma + O(u^{-1/2+\varepsilon}|\tilde{y}|) + o(1) - \mu_{20}\mu_{22}^{-1}\tilde{z})^2}{1-\mu_{20}\mu_{22}^{-1}\mu_{02}} \\
&\qquad\qquad\qquad + \Big|\mu_{22}^{-1/2}\tilde{z} - \mu_{22}^{1/2}\frac{\mathbf{1}}{2\sigma}\Big|^2\Big]\Big\}\,dw\,d\tilde{y}\,d\tilde{z} \\
&= \big(1+o(1)\big)u^{d-1} \\
&\quad \times \exp\Big\{-\frac{1}{2}\big(u-\mu_\sigma(\tau)\big)^2 + \frac{1}{2\sigma}\operatorname{Tr}\big(\mu_\sigma(\tau)I + \Delta\mu_\sigma(\tau)\big) \\
&\qquad\qquad + |\partial\mu_\sigma(\tau)|^2 + \frac{\mathbf{1}^\top\mu_{22}\mathbf{1}}{8\sigma^2} + \frac{1}{8\sigma^2}\sum_i \partial^4_{iiii}C(0)\Big\} \\
&\quad \times \frac{|\Gamma|^{-1/2}}{(2\pi)^{(d+1)(d+2)/4}} \\
&\quad \times \int_{\mathcal{L}} \gamma_u\big((\sigma+o(1))\tilde{A}\big)\exp\{-\tilde{A} + u^{-1/2+\delta}O(|\tilde{y}|^2 + |\tilde{z}|^2 + A)\} \\
&\quad \times \exp\Big\{-\frac{1}{2}\Big[|\tilde{y} - \partial\mu_\sigma(\tau)|^2 \\
&\qquad\qquad + \frac{(\tilde{A}/u + O(u^{-1/2+\varepsilon}|\tilde{y}|) + o(1) - \mu_{20}\mu_{22}^{-1}\tilde{z})^2}{1-\mu_{20}\mu_{22}^{-1}\mu_{02}} \\
&\qquad\qquad\qquad + \Big|\mu_{22}^{-1/2}\tilde{z} - \mu_{22}^{1/2}\frac{\mathbf{1}}{2\sigma}\Big|^2\Big]\Big\}\,d\tilde{A}\,d\tilde{y}\,d\tilde{z}.
\end{aligned}
$$

The last step changes the integral from "$dw\,d\tilde{y}\,d\tilde{z}$" to "$d\tilde{A}\,d\tilde{y}\,d\tilde{z}$," where $\tilde{A} = \tilde{u}A/\sigma$. Thanks to the Borel–TIS inequality (Lemma 5.1), Lemma 5.3 and the def-

inition of κ in (4.24), for $x > 0$, $\gamma_u(x)$ is bounded and as $b \to \infty$,

$$\gamma_u(x) = E\left[\frac{1}{\kappa + O(e^{-\delta^* u^{1+2\delta}} e^{u \sup |g(t)|})}; x > \xi_u\right] \to (2\pi)^{-d/2}.$$

Note that on the set $\mathcal{L}$, $\tilde{A} > -u^{3/2+\varepsilon}$. By Lemma 5.8, for $-u^{3/2+\varepsilon} < x < 0$, we have that

$$\gamma_u(x) \le e^{u^{\delta^*} x}.$$

Therefore,

$$\begin{aligned} &\gamma_u(x) I(x \neq 0, x > -u^{3/2+\varepsilon}) \\ &\quad = I(x > 0)\big((2\pi)^{-d/2} + o(1)\big) \\ &\qquad + I(-u^{3/2+\varepsilon} < x < 0) O(e^{u^{\delta^*} x}). \end{aligned} \tag{4.28}$$

The above approximation of $\gamma_u(x)$ and the dominated convergence theorem (applied to the region where $A > 0$) imply that

$$\int_{-u^{3/2+\varepsilon}}^{\infty} \gamma_u(A) e^{-A}\, dA = \frac{1 + o(1)}{(2\pi)^{d/2}} \int_0^{\infty} e^{-A}\, dA = \frac{1 + o(1)}{(2\pi)^{d/2}}.$$

Then, we continue the calculation of $\Lambda^*(\tau)$ and obtain, by the dominated convergence theorem and the above results, that

$$\begin{aligned} \Lambda(\tau) &= \big(1 + o(1)\big)\Lambda^*(\tau) \\ &= \big(1 + o(1)\big) u^{d-1} \\ &\quad \times \exp\Big\{-\frac{1}{2}\big(u - \mu_\sigma(\tau)\big)^2 \\ &\qquad + \frac{1}{2\sigma} \operatorname{Tr}\big(\mu_\sigma(\tau) I + \Delta\mu_\sigma(\tau)\big) \\ &\qquad + |\partial\mu_\sigma(\tau)|^2 + \frac{\mathbf{1}^\top \mu_{22} \mathbf{1}}{8\sigma^2} + \frac{1}{8\sigma^2} \sum_i \partial^4_{iiii} C(0)\Big\} \frac{|\Gamma|^{-1/2}}{(2\pi)^{(d+1)(d+2)/4}} \\ &\quad \times \int \frac{I(\tilde{A} > 0)}{(2\pi)^{d/2}} e^{-\tilde{A}} \\ &\qquad \times \exp\Big\{-\frac{1}{2}\Big[|\tilde{y} - \partial\mu_\sigma(\tau)|^2 + \frac{|\mu_{20}\mu_{22}^{-1}\tilde{z}|^2}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} \\ &\qquad\qquad + \Big|\mu_{22}^{-1/2}\tilde{z} - \mu_{22}^{1/2}\frac{\mathbf{1}}{2\sigma}\Big|^2\Big]\Big\}\, d\tilde{A}\, d\tilde{y}\, d\tilde{z}. \end{aligned} \tag{4.29}$$

The integrand factorizes. Then, we integrate out $d\tilde{A}$ and $d\tilde{y}$ and obtain that

$$\begin{aligned}\Lambda(\tau) &= \big(1+o(1)\big)u^{d-1} \\ &\quad\times \exp\Big\{-\frac{1}{2}\big(u-\mu_\sigma(\tau)\big)^2 + \frac{1}{2\sigma}\operatorname{Tr}\big(\mu_\sigma(\tau)I + \Delta\mu_\sigma(\tau)\big) \\ &\qquad + |\partial\mu_\sigma(\tau)|^2 + \frac{\mathbf{1}^\top\mu_{22}\mathbf{1}}{8\sigma^2} + \frac{1}{8\sigma^2}\sum_i \partial^4_{iiii}C(0)\Big\}\frac{|\Gamma|^{-1/2}}{(2\pi)^{(d+1)(d+2)/4}} \\ &\quad\times \int_{\tilde{z}\in R^{d(d+1)/2}} \exp\Big\{-\frac{1}{2}\Big[\frac{|\mu_{20}\mu_{22}^{-1}\tilde{z}|^2}{1-\mu_{20}\mu_{22}^{-1}\mu_{02}} + \Big|\mu_{22}^{-1/2}\tilde{z} - \mu_{22}^{1/2}\frac{\mathbf{1}}{2\sigma}\Big|^2\Big]\Big\}\,d\tilde{z}.\end{aligned}$$

Thus, we conclude the situation when τ is at least $u^{-1/2+\delta'}$ away from the boundary.

The case in which τ is close to the boundary of T. For the case in which τ is within $u^{-1/2+\delta'}$ distance from the boundary of T, Lemma 5.9 establishes that the contribution of the boundary case is ignorable. An intuitive interpretation of Lemma 5.9 is that the important region of the integral $\int e^{f(t)}\,dt$ might be cut off by the boundary of T. Therefore, in cases that τ is too close to the boundary, the tail $\int e^{f(t)}\,dt$ is not heavier than that of the interior case.

Summary. Note that $\mu_\sigma(\tau)$ does not achieve its maximum at the boundary of T. Together with (4.29) and Lemma 5.9, we obtain that

$$\begin{aligned}&P\Big(\int_T e^{\mu(t)+\sigma f(t)}\,dt > b\Big) \\ &\quad= E^Q\Big[\frac{dP}{dQ}; \int_T e^{\mu(t)+\sigma f(t)}\,dt > b\Big] \\ &\quad= \int_T \Lambda(\tau)\,d\tau = \big(1+o(1)\big)\int_T \Lambda^*(\tau)\,d\tau \\ &\quad= \big(1+o(1)\big)u^{d-1}\int_T H(\mu,\sigma,\tau)\exp\Big\{-\frac{1}{2}\big(u-\mu_\sigma(\tau)\big)^2\Big\}\,d\tau,\end{aligned}$$

where

$$\begin{aligned}&H(\mu,\sigma,\tau) \\ &\quad= \frac{|\Gamma|^{-1/2}}{(2\pi)^{(d+1)(d+2)/4}}\exp\Big\{\frac{1}{2\sigma}\operatorname{Tr}\big(\mu_\sigma(\tau)I + \Delta\mu_\sigma(\tau)\big) \\ &\qquad\qquad + |\partial\mu_\sigma(\tau)|^2 + \frac{\mathbf{1}^\top\mu_{22}\mathbf{1}}{8\sigma^2} + \frac{1}{8\sigma^2}\sum_i \partial^4_{iiii}C(0)\Big\} \\ &\quad\times \int_{\tilde{z}\in R^{d(d+1)/2}} \exp\Big\{-\frac{1}{2}\Big[\frac{|\mu_{20}\mu_{22}^{-1}\tilde{z}|^2}{1-\mu_{20}\mu_{22}^{-1}\mu_{02}} + \Big|\mu_{22}^{-1/2}\tilde{z} - \mu_{22}^{1/2}\frac{\mathbf{1}}{2\sigma}\Big|^2\Big]\Big\}\,d\tilde{z}.\end{aligned}$$

5. Lemmas. In this section, we state all the lemmas used in the previous section. To facilitate reading, we move several lengthy proofs to the supplemental article [42].

The first lemma is known as the Borel–TIS lemma, which was proved independently by [16, 23].

LEMMA 5.1 (Borel–TIS). *Let $f(t)$, $t \in \mathcal{U}$, $\mathcal{U}$ is a parameter set, be a mean-zero Gaussian random field. f is almost surely bounded on $\mathcal{U}$. Then,*

$$E\Big(\sup_{\mathcal{U}} f(t)\Big) < \infty$$

and

$$P\Big(\max_{t\in\mathcal{U}} f(t) - E\Big[\max_{t\in\mathcal{U}} f(t)\Big] \geq b\Big) \leq e^{-b^2/(2\sigma_{\mathcal{U}}^2)},$$

where

$$\sigma_{\mathcal{U}}^2 = \max_{t\in\mathcal{U}} \mathrm{Var}[f(t)].$$

LEMMA 5.2. *For each $\varepsilon > 0$, let $\mathcal{L}_Q$ be as defined in (4.9). There exists some $\lambda > 0$, such that*

$$\begin{aligned} E_\tau^Q\Bigg[\frac{1}{\int_T \exp\{(u-\mu_\sigma(t))(f(t)+\mu_\sigma(t)) + (1/2)\mu_\sigma^2(t)\}\,dt}; \\ \int_T e^{\mu(t)+\sigma f(t)}\,dt > b, \mathcal{L}_Q^c\Bigg] \\ = o(1)e^{-u^2-u^{1+\lambda}}, \end{aligned}$$

where $\mathcal{L}_Q^c$ is the complement of set $\mathcal{L}_Q$.

LEMMA 5.3. *Let ξ_u be as defined in (4.18); then there exists a $\lambda > 0$ such that for all $x > 0$*

$$P(u^{1/2-3\delta}|\xi_u| > x) \leq e^{-\lambda x^2} + e^{-\lambda u^2}$$

for u sufficiently large.

PROOF. We split the expectation into two parts $\{|\tilde{S}| \leq u^\delta\}$ and $\{|\tilde{S}| > u^\delta, \tau + (uI - \mathbf{z})^{-1/2}\tilde{S} \in T\}$. Note that $|S| \leq \kappa u^\delta$ and $g(t)$ is a mean-zero Gaussian random field with $\mathrm{Var}(g(t)) = O(|t|^6)$. A direct application of the Borel–TIS inequality (Lemma 5.1) yields the result of this lemma. $\square$

LEMMA 5.4. *Let S be a random variable taking values in $\{s : (uI - \mathbf{z})^{-1/2}s + \tau \in T\}$ with density proportional to*

$$\exp\left\{-\frac{\sigma}{2}\big(s - (uI - \mathbf{z})^{-1/2}y\big)^\top\big(s - (uI - \mathbf{z})^{-1/2}y\big)\right\}.$$

Then, on the set $\mathcal{L}$

$$\begin{aligned}
&\log\{E\exp[\sigma g_3((uI - \mathbf{z})^{-1/2}S) \\
&\qquad + \sigma(u - \mu_\sigma(\tau))C_4((uI - \mathbf{z})^{-1/2}S) + \sigma R((uI - \mathbf{z})^{-1/2}S)]\} \\
&\quad = -\frac{\sigma}{8u}(u^{-1}Y + \mathbf{1}/\sigma)^\top\mu_{22}(u^{-1}Y + \mathbf{1}/\sigma) + \frac{\mathbf{1}^\top\mu_{22}\mathbf{1}}{8\sigma u} \\
&\qquad + \frac{1}{8\sigma u}\sum_i \partial^4_{iiii}C(0) + o(u^{-1}),
\end{aligned}$$

where the expectation is taken with respect to S as in (4.15).

LEMMA 5.5. *Let I_3 be as defined in part* 2 *of the proof of Theorem* 3.4. *On the set $\mathcal{L}$, there exists some $\delta^* > 0$ such that*

$$\begin{aligned}
I_3 &= O(e^{-\delta^* u^{1+2\delta}})e^{u\sup|g(t)|}u^{-d}e^{u^2 - u\mu_\sigma(\tau) + (1/2)\mu_\sigma^2(\tau)} \\
&\quad \times \exp\left\{\frac{\tilde{u}}{\sigma}A - \tilde{u}B - \frac{1}{2\sigma}\operatorname{Tr}\big(\Delta f(\tau) + \mu_\sigma(\tau)I + \Delta\mu_\sigma(\tau)\big)\right. \\
&\qquad \left. - \tilde{y}^\top\partial\mu_\sigma(\tau) - \frac{1}{2}|\partial\mu_\sigma(\tau)|^2 + u^{-1/2+\delta}O\big(|y|^2 + |z|^2 + f(\tau)\big)\right\}.
\end{aligned}$$

LEMMA 5.6.

$$\log\big(\det(I - u^{-1}\mathbf{z})\big) = -u^{-1}\operatorname{Tr}(\mathbf{z}) + \tfrac{1}{2}u^{-2}\mathcal{I}_2(\mathbf{z}) + o(u^{-2}),$$

where Tr *is the trace of a matrix, $\mathcal{I}_2(\mathbf{z}) = \sum_{i=1}^d \lambda_i^2$, and λ_i's are the eigenvalues of $\mathbf{z}$.*

PROOF. The result is immediate by noting that

$$\det(I - u^{-1}\mathbf{z}) = \prod_{i=1}^d (1 - \lambda_i/u)$$

and $\operatorname{Tr}(\mathbf{z}) = \sum_{i=1}^d \lambda_i$. $\square$

LEMMA 5.7. *For the homogeneous Gaussian random field $f(t)$ in Theorem* 3.4, *let $h(w, \tilde{y}, \tilde{z})$ be the density of $(f(\tau), \partial f(\tau), \partial^2 f(\tau))$ evaluated at*

$(w, \tilde{y}, \tilde{z})$. *Then*,

$$h(w, \tilde{y}, \tilde{z}) = \frac{|\Gamma|^{-1/2}}{(2\pi)^{(d+1)(d+2)/4}} \times \exp\left\{-\frac{1}{2}\left[\tilde{y}^\top \tilde{y} + \frac{(w - \mu_{20}\mu_{22}^{-1}\tilde{z})^2}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} + \tilde{z}^\top \mu_{22}^{-1}\tilde{z}\right]\right\}, \tag{5.1}$$

where Γ *is the covariance matrix of* $(f(\tau), \partial^2 f(\tau))$ *whose inverse is*

$$\Gamma^{-1} = \begin{pmatrix} \dfrac{1}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} & \dfrac{-\mu_{20}\mu_{22}^{-1}}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} \\ \dfrac{-\mu_{20}\mu_{22}^{-1}}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} & \mu_{22}^{-1} + \dfrac{\mu_{22}^{-1}\mu_{02}\mu_{20}\mu_{22}^{-1}}{1 - \mu_{20}\mu_{22}^{-1}\mu_{02}} \end{pmatrix}. \tag{5.2}$$

PROOF. The form of (5.2) results from direct application of the block matric inverse of linear algebra. Note that

$$h(w, \tilde{y}, \tilde{z}) = \frac{1}{(2\pi)^{(d+1)(d+2)/4}} |\Gamma|^{-1/2} \times \exp\left\{-\frac{1}{2}(w, \tilde{z}^\top, \tilde{y}^\top) \begin{pmatrix} \Gamma^{-1} & 0 \\ 0 & I \end{pmatrix} (w, \tilde{z}^\top, \tilde{y}^\top)^\top\right\}.$$

By plugging in the form of Γ^{-1}, we get the conclusion. $\square$

LEMMA 5.8. *Consider that* $\{t : |t - \tau| \le u^{-1/2+\delta'}\} \subset T$. *Let* $\gamma_u(x)$ *be as defined in* (4.27). *There exists some* $\delta^* > 0$ *such that for all* $0 < x < u^{3/2+\varepsilon}$,

$$\gamma_u(-x) \le e^{-u^{\delta^*} x}.$$

LEMMA 5.9. *For each* τ *within* $u^{-1/2+\delta'}$ *distance from the boundary of* T, *that is, there exists an* s *such that* $|s - \tau| \le u^{-1/2+\delta'}$ *and* $s \notin T$, *we have that*

$$E_\tau^Q\left[\frac{dP}{dQ}; \int_T e^{\mu(t)+\sigma f(t)}\, dt > b, \mathcal{L}_Q\right] = O(1) u^{d-1} e^{-(1/2)(u-\mu_\sigma(\tau))^2}.$$

SUPPLEMENTARY MATERIAL

Proofs of several lemmas in Section 5 and the numerical results (DOI: 10.1214/11-AOS960SUPP; .pdf). This supplement contains proofs of Lemmas 5.2, 5.4, 5.5, 5.8 and 5.9 as well as numerical results.

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