The Annals of Probability
2013, Vol. 41, No. 1, 206–228
DOI: 10.1214/11-AOP732

ERGODIC PROPERTIES OF SUM- AND MAX-STABLE STATIONARY RANDOM FIELDS VIA NULL AND POSITIVE GROUP ACTIONS

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We establish characterization results for the ergodicity of stationary symmetric α-stable (SαS) and α-Fréchet random fields. We show that the result of Samorodnitsky [*Ann. Probab.* **33** (2005) 1782–1803] remains valid in the multiparameter setting, that is, a stationary SαS ($0 < \alpha < 2$) random field is ergodic (or, equivalently, weakly mixing) if and only if it is generated by a null group action. Similar results are also established for max-stable random fields. The key ingredient is the adaption of a characterization of positive/null recurrence of group actions by Takahashi [*Kōdai Math. Sem. Rep.* **23** (1971) 131–143], which is dimension-free and different from the one used by Samorodnitsky.

1. Introduction. A process is called sum-stable (max-stable, resp.) if so are its finite-dimensional distributions and it arises as a limit, under suitable affine transformations, of sums (maxima, resp.) of independent processes. Convenient stochastic integral representations have been developed and actively used to study the structure and properties of sum-stable processes and random fields (see, e.g., Samorodnitsky and Taqqu [35], Rosiński [20, 21], Rosiński and Samorodnitsky [23], Pipiras and Taqqu [18], Samorodnitsky [32–34], Roy and Samorodnitsky [30] and Roy [28, 29]). On the other hand, the seminal works of de Haan [4] and de Haan and Pickands [5] as well as the recent developments by Stoev and Taqqu [37], Wang and Stoev [41, 42] and Kabluchko [9] have developed similar tools to represent and handle general classes of max-stable processes.

The ergodic properties of stationary stochastic processes and random fields are of fundamental importance and hence well-studied. See, for example, Maruyama [14], Rosiński and Żak [24, 25] and Roy [26, 27] for results on infinite divisible processes and Cambanis et al. [2], Podgórski [19], Gross and Robertson [8] and Gross [7] for results on stable processes. These culminated in the characterization of Samorodnitsky [34], which shows that the ergodicity of a stationary symmetric

Received October 2009; revised December 2010.

[1]Supported in part by NSF Grants DMS-08-06094 and DMS-11-06695 at the University of Michigan.

MSC2010 subject classifications. Primary 60G10, 60G52, 60G60; secondary 37A40, 37A50.

Key words and phrases. Stable, max-stable, random field, ergodic theory, nonsingular group action, null action, positive action, ergodicity.

stable process is equivalent to the null-recurrence of the underlying nonsingular flow. On the other hand, the ergodic properties of max-stable processes have been recently studied by Stoev [36], Kabluchko [9] and Kabluchko and Schlather [10]. In particular, Kabluchko [9] has shown that as in the sum-stable case, one can associate a nonsingular flow to the stationary max-stable process and that the characterization of Samorodnitsky [34] remains valid. The case of random fields, however, remained open in both sum- and max-stable settings.

Our goal in this paper is to establish a Samorodnitsky-type characterization for sum-stable and max-stable *random fields*. The main obstacle is the unavailability of a higher-dimensional analogue of the work of Krengel [12], which plays a crucial role in Samorodnitsky's approach for processes. We resolve this problem by providing an alternative dimension-free characterization of ergodicity for both classes of sum- and max-stable stationary random fields. For simplicity of exposition as well as mathematical tractability, we work with symmetric α-stable (SαS), $(0 < \alpha < 2)$ sum-stable random fields and α-Fréchet max-stable random fields $(\alpha > 0)$.

The key ingredient of our results is the adaptation of the work of Takahashi [39]. Thanks to Takahashi's result, we are able to develop tractable and dimension-free criteria for verifying whether a given spectral representation corresponds to an SαS random field generated by a null (or positive) action. We also extend a well-known result of Gross [7] and give necessary and sufficient condition for a stationary SαS random field to be weakly mixing and in the process fill a gap in the proof of [7] (see Remark A.6 below). Similar results for α-Fréchet random fields are obtained. Furthermore, these results offer alternative characterizations of ergodicity in the one-dimensional case.

The paper is organized as follows. In Section 2 we start with some auxiliary results from ergodic theory. In Section 3 we establish the positive-null decomposition for measurable stationary SαS random fields. Section 4 characterizes the ergodicity of SαS random fields. The max-stable setting is discussed in Section 5. We conclude with a couple of examples in Section 6. Some technical proofs and auxiliary results are given in the Appendix.

2. Preliminaries on ergodic theory. Throughout this paper, we let $(S, \mathcal{B}, \mu)$ denote a standard Lebesgue space (see Appendix A in [18]). Let ϕ denote a bimeasurable and invertible transformation on S. We say that ϕ is *nonsingular*, if the measure $\mu \circ \phi^{-1}$ and μ are equivalent, written $\mu \circ \phi^{-1} \sim \mu$. In this case, one can define the *dual operator* $\widehat{\phi}$ as a mapping from $L^1(S, \mu)$ to $L^1(S, \mu)$:

$$\widehat{\phi} f(s) \equiv [\widehat{\phi} f](s) := \left(\frac{\mathrm{d}(\mu \circ \phi^{-1})}{\mathrm{d}\mu}\right)(s) f \circ \phi^{-1}(s). \tag{2.1}$$

Note that $\widehat{\phi}$ is a positive linear isometry (hence a contraction) on $L^1(S, \mu)$. The characterization results in the next section are in terms of dual operators.

2.1. *Group actions*. Let $G \equiv (G, +)$ be a locally compact, topological Abelian group with identity element 0. Equip G with the Borel σ-algebra $\mathcal{A}$.

DEFINITION 2.1. A collection of measurable transformations $\phi_t : S \to S$, $t \in G$ is called a *group action* of G on S (or a G-*action*), if:

(i) $\phi_0(s) = s$ for all $s \in S$,
(ii) $\phi_{v+u}(s) = \phi_u \circ \phi_v(s)$ for all $s \in S, u, v \in G$,
(iii) $(s, u) \mapsto \phi_u(s)$ is measurable w.r.t. the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$.

A G-action $\mathcal{G} = \{\phi_t\}_{t \in G}$ on (S, μ) is *nonsingular* if ϕ_t is nonsingular for all $t \in G$. In this paper, all the group actions are assumed to be nonsingular.

The existence of a $\mathcal{G}$-invariant finite measure ν, $\nu \sim \mu$ (equivalently, the existence of a fixed point of the dual operator $\widehat{\phi}$, see, e.g., Proposition 1.4.1 in [1]), is an important problem in ergodic theory. The investigation of this problem was initiated by Neveu [15] and further explored by Krengel [12] and Takahashi [39], among others. In the rest of this section we present results due essentially to Takahashi [39]. We will see that the invariant finite measures induce a modulo μ unique decomposition of S. This decomposition will play an important role in the characterization of ergodicity for sum- and max-stable random fields. The proofs of the results mentioned in this section are given in the Appendix.

Consider the class of finite (positive) $\mathcal{G}$-invariant measures on S absolutely continuous with respect to μ:

$$\Lambda(\mathcal{G}) := \{\nu \ll \mu : \nu \text{ finite measure on } S, \nu \circ \phi^{-1} = \nu \text{ for all } \phi \in \mathcal{G}\}.$$

For all $\nu \in \Lambda(\mathcal{G})$, let $S_\nu \equiv \text{supp}(\nu) := \{d\nu/d\mu > 0\}$ denote the support of ν (mod μ) and set $I(\mathcal{G}) := \{S_\nu : \nu \in \Lambda(\mathcal{G})\}$.

LEMMA 2.2. *There exists a modulo μ unique maximal element $P_{\mathcal{G}} \in I(\mathcal{G})$, that is*:

(i) *For all $S_\nu \in I(\mathcal{G})$, $S_\nu \subset P_{\mathcal{G}}$, that is, $\mu(S_\nu \setminus P_{\mathcal{G}}) = 0$.*
(ii) *If* (i) *holds for $Q_{\mathcal{G}} \in I(\mathcal{G})$, then $P_{\mathcal{G}} = Q_{\mathcal{G}}$ modulo μ.*

This result suggests the decomposition

$$S = P_{\mathcal{G}} \cup N_{\mathcal{G}}, \tag{2.2}$$

where $N_{\mathcal{G}} := S \setminus P_{\mathcal{G}}$. The set $P_{\mathcal{G}} \equiv S_{\nu_0}$, $\nu_0 \in \Lambda(\mathcal{G})$ is the largest (mod μ) set where one can have a finite $\mathcal{G}$-invariant measure ν_0, equivalent to $\mu|_{P_{\mathcal{G}}}$. Consequently, there are no finite measures supported on $N_{\mathcal{G}}$, invariant w.r.t. $\mathcal{G}$ and absolutely continuous w.r.t. μ. The next theorem provides a convenient characterization of the decomposition (2.2).

THEOREM 2.3. *Consider any* $f \in L^1(S, \mu)$, $f > 0$. *Let* $P_{\mathcal{G}}$ *denote the unique maximal element of* $I(\mathcal{G})$ *and set* $N_{\mathcal{G}} := S \setminus P_{\mathcal{G}}$. *We have the following*:

(i) *The sets* $P_{\mathcal{G}}$ *and* $N_{\mathcal{G}}$ *are invariant w.r.t.* $\mathcal{G}$, *that is, for all* $\phi \in \mathcal{G}$, *we have*

$$\mu(\phi^{-1}(P_{\mathcal{G}}) \triangle P_{\mathcal{G}}) = 0 \quad \textit{and} \quad \mu(\phi^{-1}(N_{\mathcal{G}}) \triangle N_{\mathcal{G}}) = 0.$$

(ii) *Restricted to* $P_{\mathcal{G}}$,

$$\sum_{n=1}^{\infty} \widehat{\phi}_{u_n} f(s) = \infty, \qquad \mu\text{-a.e. for all } \{\phi_{u_n}\}_{n \in \mathbb{N}} \subset \mathcal{G}. \tag{2.3}$$

(iii) *Restricted to* $N_{\mathcal{G}}$,

$$\sum_{n=1}^{\infty} \widehat{\phi}_{u_n} f(s) < \infty, \qquad \mu\text{-a.e. for some } \{\phi_{u_n}\}_{n \in \mathbb{N}} \subset \mathcal{G}. \tag{2.4}$$

The decomposition in (2.2) is unique (mod μ). It is referred to as the *positive-null* decomposition w.r.t. $\mathcal{G}$. The sets $P_{\mathcal{G}}$ and $N_{\mathcal{G}}$ are referred to as the *positive* and *null* parts of S w.r.t. $\mathcal{G}$, respectively. If $\mu(N_{\mathcal{G}}) = 0$ [$\mu(P_{\mathcal{G}}) = 0$, resp.], then $\mathcal{G}$ is said to be a *positive* (*null*, resp.) G-action.

The next result provides an equivalent characterization of (2.2) based on the notion of a weakly wandering set. Recall that a measurable set $W \subset S$ is *weakly wandering*, w.r.t. $\mathcal{G}$, if there exists $\{\phi_{t_n}\}_{n \in \mathbb{N}} \subset \mathcal{G}$ such that $\mu(\phi_{t_n}^{-1}(W) \cap \phi_{t_m}^{-1}(W)) = 0$ for all $n \neq m$.

THEOREM 2.4. *Under the assumptions of Theorem* 2.3, *we have the following*:

(i) *The positive part* $P_{\mathcal{G}}$ *has no weakly wandering set of positive measure.*
(ii) *The null part* $N_{\mathcal{G}}$ *is a union of weakly wandering sets w.r.t.* $\mathcal{G}$.

REMARK 2.5. In the one-dimensional case, Krengel [12] (for $G = \mathbb{Z}$) and Samorodnitsky [34] (for $G = \mathbb{R}$) establish alternative characterizations of the decomposition (2.2). These results involve certain integral tests, which we were unable to extend to multiple dimensions. Takahashi's characterizations, employed in Theorem 2.3, are valid for all dimensions.

2.2. *Multiparameter ergodic theorems.* In the rest of the paper we focus on $\mathbb{T}^d$-actions, where $\mathbb{T}$ stands for either $\mathbb{Z}$ or $\mathbb{R}$. We equip $\mathbb{T}^d$ with the measure $\lambda \equiv \lambda_{\mathbb{T}^d}$, which is either the counting (if $\mathbb{T} = \mathbb{Z}$) or the Lebesgue (if $\mathbb{T} = \mathbb{R}$) measure. In the sequel we establish multiparameter versions of the *stochastic ergodic theorem* and *Birkhoff theorem* for the case of $\mathbb{T}^d$-actions. They are extensions of the well-known results in the one-dimensional case. The proofs follow from the works of Krengel and Tempel'man (see, e.g., [13]).

Introduce the *average functional* A_T, defined for all locally integrable $h:\mathbb{T}^d \to \mathbb{R}$:

$$A_T h \equiv A_{\mathbb{T}^d,T} h := \frac{1}{C(T)} \int_{B(T)} h(t)\lambda(\mathrm{d}t)$$

with $B(T) \equiv B_{\mathbb{T}^d}(T) := (-T, T]^d \cap \mathbb{T}^d$ and $C(T) \equiv C_{\mathbb{T}^d}(T) := (2T)^d$.

Consider now a collection of functions $\{f_t\}_{t\in\mathbb{T}^d} \subset L^1(S,\mu)$ such that $(t,s) \mapsto f(t,s) \equiv f_t(s)$ is jointly measurable when $\mathbb{T} \equiv \mathbb{R}$. Then, one can define the *average operator*:

$$(A_T f)(s) := \frac{1}{C(T)} \int_{B(T)} f_t(s)\lambda(\mathrm{d}t). \tag{2.5}$$

Let $\|\cdot\|$ denote the L^1 norm. If $t \mapsto \|f_t\|$ is locally integrable (i.e., integrable on finite intervals), then Fubini's theorem implies that $A_T f \in L^1(S,\mu)$, for all $T > 0$. Recall also that a sequence of measurable functions $\{f_n\}_{n\in\mathbb{N}} \subset L^\alpha(S,\mu)$ converges *stochastically* (or locally in measure) to $g \in L^\alpha(S,\mu)$, in short, $f_n \overset{\mu}{\to} g$, as $n \to \infty$, if

$$\lim_{n\to\infty} \mu\big(\{s : |f_n(s) - g(s)| > \varepsilon\} \cap B\big) = 0 \qquad \text{for all } \varepsilon > 0,\ B \in \mathcal{B} \text{ with } \mu(B) < \infty. \tag{2.6}$$

REMARK 2.6. By Theorem A.1 in [11], there exists a strictly positive measurable function $(t,s) \mapsto w(t,s)$, such that for all $t \in \mathbb{T}^d$, $w(t,s) = \mathrm{d}(\mu\circ\phi_t)/\mathrm{d}\mu(s)$ for μ-almost all s, and for all $t, h \in \mathbb{T}^d$ and for all $s \in S$,

$$w(t+h,s) = w(h,s)w(t,\phi_h(s)). \tag{2.7}$$

From now on, we shall use $w(t,s)$ as the version of the Radon–Nikodym derivative $\mathrm{d}(\mu\circ\phi_t)/\mathrm{d}\mu(s)$.

THEOREM 2.7 (Multiparameter stochastic ergodic theorem for nonsingular actions). *Let $\{\phi_t\}_{t\in\mathbb{T}^d}$ be a nonsingular $\mathbb{T}^d$-action on the measure space (S,μ). Let $f_0 \in L^1(S,\mu)$ and define $f(t,s) \equiv (\widehat{\phi}_{-t} f_0)(s) := w(t,s) f_0\circ\phi_t(s)$. Then, there exists $\widetilde{f} \in L^1(S,\mu)$, such that*

$$A_T f \equiv \frac{1}{C(T)} \int_{B(T)} f(t,\cdot)\lambda(\mathrm{d}t) \overset{\mu}{\to} \widetilde{f} \qquad \text{as } T \to \infty. \tag{2.8}$$

Moreover, $\widetilde{f}$ is invariant w.r.t. $\widehat{\mathcal{G}}$, that is, $\widehat{\phi}_t \widetilde{f} = \widetilde{f}$ for all $t \in \mathbb{T}^d$.

PROOF. *Suppose first that* $\mathbb{T} = \mathbb{Z}$. The existence of $\widetilde{f}$ follows from Krengel's stochastic ergodic theorem (Theorem 6.3.10 in [13]). To see that $\widetilde{f}$ is L^1-integrable, pick a subsequence T_n such that $A_{T_n} f \to \widetilde{f}$, μ-a.e., as $n \to \infty$. By Fatou's lemma, $\|\widetilde{f}\| = \|\lim_{n\to\infty} A_{T_n} f\| \le \liminf_{n\to\infty} \|A_{T_n} f\| \le \|f_0\| <$

∞, which implies $\widetilde{f} \in L^1(S,\mu)$. Here we used the fact that $\int_S |A_T f|\,\mathrm{d}\mu \leq A_T \int_S |\widehat{\phi}_{-t} f_0|\,\mathrm{d}\mu = A_T\|f_0\| = \|f_0\|$.

We now prove that $\widetilde{f}$ is invariant w.r.t. $\widehat{\mathcal{G}}$. Fix $\tau \in \mathbb{T}^d$ and let $T_n \to \infty$ be such that $g_n := A_{T_n} f \to \widetilde{f}$, μ-a.e., as $n \to \infty$. Then, since ϕ_τ is nonsingular,

$$(\widehat{\phi}_{-\tau} g_n)(s) \equiv \frac{\mathrm{d}(\mu \circ \phi_\tau)}{\mathrm{d}\mu}(s) g_n \circ \phi_\tau(s) \tag{2.9}$$
$$\longrightarrow \quad \frac{\mathrm{d}(\mu \circ \phi_\tau)}{\mathrm{d}\mu}(s)\widetilde{f} \circ \phi_\tau(s) \equiv (\widehat{\phi}_{-\tau}\widetilde{f})(s), \qquad \mu\text{-a.e.}$$

as $n \to \infty$. On the other hand, since $f(t,\phi_\tau(s)) = w(t,\phi_\tau(s)) f_0 \circ \phi_{t+\tau}(s)$, we obtain by (2.7) and Fubini's theorem that

$$\begin{aligned}(\widehat{\phi}_{-\tau} g_n)(s) &= \frac{1}{C(T_n)} \int_{B(T_n)} w(\tau+t,s) f_0(\phi_{\tau+t}(s))\lambda(\mathrm{d}t) \\ &= \frac{1}{C(T_n)} \int_{B(T_n)+\tau} f(t,s)\lambda(\mathrm{d}t), \qquad \mu\text{-a.e.}\end{aligned}$$

Therefore, by performing cancelations and applying Fubini's theorem, we get

$$\|\widehat{\phi}_{-\tau} g_n - g_n\| \leq \frac{\lambda((B(T_n)+\tau)\Delta B(T_n))}{C(T_n)}\|f_0\|,$$

where $D\Delta E = (D \setminus E) \cup (E \setminus D)$ is the symmetric difference of sets. The last term vanishes, as $n \to \infty$, since $\tau \in \mathbb{Z}^d$ is fixed. This implies that $\widehat{\phi}_{-\tau} g_n \stackrel{\mu}{\to} \widetilde{f}$, as $n \to \infty$, which, in view of (2.9), yields $\widehat{\phi}_{-\tau}\widetilde{f} = \widetilde{f}$, μ-a.e. This, since $\tau \in \mathbb{Z}^d$ was arbitrary, establishes the desired invariance of the limit $\widetilde{f}$.

Suppose now that $\mathbb{T} = \mathbb{R}$. Since we will use the result proved for $\mathbb{T} = \mathbb{Z}$, we explicitly write $A_{\mathbb{Z}^d,T}$ and $A_{\mathbb{R}^d,T}$ to distinguish between the discrete and integral average operators, respectively. In view of part (i), for all $\delta > 0$, we have

$$\begin{aligned} A_{\mathbb{R}^d,n\delta} f_0 &\equiv \frac{1}{(2n\delta)^d} \int_{(-n\delta,n\delta]^d} \widehat{\phi}_{-\tau} f\,\mathrm{d}\tau \\ &= \frac{1}{(2n)^d} \sum_{t \in (-n,n]^d \cap \mathbb{Z}^d} \widehat{\phi}_{-\delta t} g^{(\delta)} \equiv A_{\mathbb{Z}^d,n} g^{(\delta)}, \end{aligned} \tag{2.10}$$

where

$$g^{(\delta)}(s) := \frac{1}{\delta^d} \int_{(-\delta,0]^d} (\widehat{\phi}_{-\tau} f_0)(s)\,\mathrm{d}\tau \in L^1(S,\mu).$$

As already shown for the case $\mathbb{T} = \mathbb{Z}$, the right-hand side of (2.10) converges stochastically, as $n \to \infty$, to $\widetilde{g}^{(\delta)} \in L^1(S,\mu)$, where $\widetilde{g}^{(\delta)}$ is $\widehat{\phi}_{-\delta t}$-invariant, for all $t \in \mathbb{Z}^d$. Write $T_\delta = \lfloor T/\delta \rfloor \delta$. Since for all $\delta > 0$, the volume of $(-T,T]^d \setminus$

$(-T_\delta, T_\delta]^d$ is $o(C(T))$ as $T \to \infty$, it follows that $\|A_{\mathbb{R}^d,T} f - A_{\mathbb{R}^d,T_\delta} f\| \to 0$ as $T \to \infty$. Therefore, we have that

$$A_{\mathbb{R}^d,T} f \xrightarrow{\mu} \widetilde{g}^{(\delta)} \qquad \text{as } T \to \infty,$$

which shows, in particular, that $\widetilde{g}^{(\delta)} = \widetilde{g} \in L^1(S, \mu)$ must be independent of $\delta > 0$. Since $\widetilde{g}$ is invariant w.r.t. $\widehat{\phi}_{\delta t}$ for all $\delta > 0$ and $t \in \mathbb{Z}^d$, it follows that $\widetilde{g}$ is $\widehat{\mathcal{G}}$-invariant. □

THEOREM 2.8 (Multiparameter Birkhoff theorem). *Assume the conditions of Theorem 2.7 hold. Suppose, moreover, that the action $\{\phi_t\}_{t\in\mathbb{T}^d}$ is measure preserving on (S, μ), and that μ is a probability measure. Then,*

$$A_T f \to \widetilde{f} := \mathbb{E}_\mu(f|\mathcal{I}) \qquad \textit{almost surely and in } L^1,$$

where $\mathcal{I}$ is the σ-algebra of all $\mathcal{G}$-invariant measurable sets.

PROOF. Suppose first that $\mathbb{T} = \mathbb{Z}$. The almost sure convergence and the structure of the limit $\widetilde{f}$ follow from Tempel'man's theorem (Theorem 6.2.8 in [13], page 205). The L^1-convergence is clear when f_0 is bounded. Suppose now that $f_0 \in L^1(S, \mu)$. Consider the sequence $A_T f, T \in \mathbb{N}$. For all $\varepsilon > 0$ there exists a bounded $f_0^{(\varepsilon)} \in L^\infty(S, \mu)$ such that $\|f_0 - f_0^{(\varepsilon)}\| < \varepsilon/3$. Then, by the triangle inequality and the fact that A_T is a linear contraction, we get

$$\begin{aligned}\|A_{T_1} f - A_{T_2} f\| &\le \|A_{T_1} f^{(\varepsilon)} - A_{T_2} f^{(\varepsilon)}\| + 2\|f_0 - f_0^{(\varepsilon)}\| \\ &\le \|A_{T_1} f^{(\varepsilon)} - A_{T_2} f^{(\varepsilon)}\| + 2\varepsilon/3 < \varepsilon\end{aligned}$$

for all sufficiently large T_1 and T_2. This is because $A_T f^{(\varepsilon)}$ converges in L^1. We have thus shown that $A_T f, T \in \mathbb{N}$, is a Cauchy sequence in the Banach space $L^1(S, \mu)$ and, hence, it has a limit, which is necessarily $\widetilde{f}$.

Let now $\mathbb{T} = \mathbb{R}$. First, by a discretization argument as in the proof of Theorem 2.7, we can show $A_T f \to \widetilde{f}$ almost surely, for all $f_0 \in L^1(S, \mu)$. The L^1-convergence can be established as in the proof in the discrete case. □

3. Stationary sum-stable random fields. We focus on SαS ($0 < \alpha < 2$) random fields $\mathbf{X} = \{X_t\}_{t\in\mathbb{T}^d}$, with a *spectral representation*:

$$\{X_t\}_{t\in\mathbb{T}^d} \stackrel{\mathrm{d}}{=} \left\{\int_S f_t(s) M_\alpha(\mathrm{d}s)\right\}_{t\in\mathbb{T}^d}. \tag{3.1}$$

Here $\{f_t\}_{t\in\mathbb{T}^d} \subset L^\alpha(S, \mu)$, and the integral is with respect to an independently scattered *SαS random measure* M_α on S with control measure μ (see Chapters 3 and 13 in [35] for more details). Without loss of generality, we shall also assume that $\{f_t\}_{t\in\mathbb{T}^d}$ has *full support* in $L^\alpha(S, \mu)$. Namely, there is no $B \in \mathcal{B}$ with $\mu(B) > 0$, such that $\int_B |f_t(s)|^\alpha \mu(\mathrm{d}s) = 0$, for all $t \in \mathbb{T}^d$.

All measurable SαS random fields $\mathbf{X}$ have a spectral representation (3.1), where (S,μ) can be chosen to be a standard Lebesgue space and the functions $(t,s)\mapsto f_t(s)$ to be jointly measurable (see, e.g., Proposition 11.1.1 and Theorem 13.2.1 in [35]).

It is known from Rosiński [20, 21] that when $\mathbf{X}$ is stationary, there exists a *minimal* spectral representation (3.1) with

$$f_t(s)=c_t(s)\left(\frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}(s)\right)^{1/\alpha}f_0\circ\phi_t(s),\qquad t\in\mathbb{T}^d, \tag{3.2}$$

where $f_0\in L^\alpha(S,\mu)$, $\{\phi_t\}_{t\in\mathbb{T}^d}$ is a nonsingular $\mathbb{T}^d$-action on $(S,\mathcal{B},\mu)$, and $\{c_t\}_{t\in\mathbb{T}^d}$ is a cocycle for $\{\phi_t\}_{t\in\mathbb{T}^d}$ taking values in $\{-1,1\}$. Namely, $(t,s)\mapsto c_t(s)\in\{-1,1\}$ is a measurable map, such that for all $u,v\in\mathbb{T}^d$, $c_{u+v}(s)=c_v(s)c_u(\phi_v(s))$, μ-a.e. $s\in S$. The representation (3.1) is minimal, if the ratio σ-algebra $\sigma(f_t/f_\tau:t,\tau\in\mathbb{T}^d)$ is equivalent to $\mathcal{B}$ (see Definition 2.1 in [20]). The minimality is an indispensable tool to study the spectral representations, although it is hard to check in practice. For more equivalent conditions and insights, see Rosiński [22] and Pipiras [17].

We say that a random field $\{X_t\}_{t\in\mathbb{T}^d}$ with the minimal representation (3.1) and (3.2) is *generated by the* $\mathbb{T}^d$*-action* $\{\phi_t\}_{t\in\mathbb{T}^d}$ *and the cocycle* $\{c_t\}_{t\in\mathbb{T}^d}$. In this case, we also say $\{X_t\}_{t\in\mathbb{T}^d}$ has an action representation $(f_0,\mathcal{G}\equiv\{\phi_t\}_{t\in\mathbb{T}^d},\{c_t\}_{t\in\mathbb{T}^d})$.

It turns out, moreover, the action $\{\phi_t\}_{t\in\mathbb{T}^d}$ is determined by the distribution of $\{X_t\}_{t\in\mathbb{T}^d}$, up to the equivalence relationship of $\mathbb{T}^d$-actions (see Theorem 3.6 in [20]). Thus, structural results for the $\mathbb{T}^d$-actions imply important structural results for the corresponding SαS random fields. In particular, by using Theorem 2.3, we obtain the following result:

THEOREM 3.1. *Let $\{X_t\}_{t\in\mathbb{T}^d}$ be a measurable stationary SαS random field with spectral representation* (3.1). *We suppose that $(S,\mathcal{B},\mu)$ is a standard Lebesgue space and the spectral representation $\{f_t(s)\}_{t\in\mathbb{T}^d}$ is measurable. Assume, in addition, that*

$$g(s):=\int_{T_0}a_\tau|f_\tau(s)|^\alpha\lambda(\mathrm{d}\tau)\ \textit{is } L^1\textit{-integrable}\quad\textit{and}\quad \operatorname{supp}(g)=S \tag{3.3}$$

for some $T_0\in\mathcal{B}_{\mathbb{T}^d}$ and $a_\tau>0,\forall\tau\in T_0$. Then:

(i) *$\{X_t\}_{t\in\mathbb{T}^d}$ is generated by a positive $\mathbb{T}^d$-action if and only if*

$$\sum_{n=1}^\infty\int_{T_0}a_\tau|f_{\tau+t_n}(s)|^\alpha\lambda(\mathrm{d}\tau)=\infty,\qquad \mu\textit{-a.e. for all } \{t_n\}_{n\in\mathbb{N}}\subset\mathbb{T}^d. \tag{3.4}$$

(ii) *$\{X_t\}_{t\in\mathbb{Z}^d}$ is generated by a null $\mathbb{T}^d$-action if and only if*

$$\sum_{n=1}^\infty\int_{T_0}a_\tau|f_{\tau+t_n}(s)|^\alpha\lambda(\mathrm{d}\tau)<\infty,\qquad \mu\textit{-a.e. for some } \{t_n\}_{n\in\mathbb{N}}\subset\mathbb{T}^d. \tag{3.5}$$

In particular, the classes of stationary SαS random fields generated by positive and null $\mathbb{T}^d$-actions are disjoint.

REMARK 3.2. One can always choose $\{a_\tau\}_{\tau\in T_0}$ such that (3.3) holds, if the spectral functions $\{f_t\}_{t\in\mathbb{T}^d}$ have full support in $L^\alpha(S,\mu)$.

PROOF OF THEOREM 3.1. Suppose first that $\{f_t\}_{t\in\mathbb{T}^d}$ is minimal and, hence, it has the form (3.2). Observe that, for all $t,\tau\in\mathbb{T}^d$, we have

$$|f_{\tau+t}(s)|^\alpha = \frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}(s)\,\frac{\mathrm{d}(\mu\circ\phi_\tau)}{\mathrm{d}\mu}\circ\phi_t(s)|f_0\circ\phi_\tau\circ\phi_t(s)|^\alpha, \qquad \mu\text{-a.e.}$$

Since both the left-hand side and the right-hand side are measurable in (τ,s), by Fubini's theorem,

$$\begin{aligned}\int_{T_0} a_\tau|f_{\tau+t}(s)|^\alpha\lambda(\mathrm{d}\tau) &= \frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}(s)\int_{T_0} a_\tau|f_\tau\circ\phi_t(s)|^\alpha\lambda(\mathrm{d}\tau)\\ &= (\widehat{\phi}_{-t}g)(s), \qquad \mu\text{-a.e.},\end{aligned}$$

where the last relation follows from (2.1). Therefore,

$$\sum_{n=1}^\infty\int_{T_0} a_\tau|f_{\tau+t_n}(s)|^\alpha\lambda(\mathrm{d}\tau) = \sum_{n=1}^\infty \widehat{\phi}_{-t_n}g, \qquad \mu\text{-a.e. } \forall\{t_n\}_{n\in\mathbb{N}}\subset\mathbb{T}^d.$$

Hence, Theorem 2.3(ii) and (iii), applied to the strictly positive function $g\in L^1(S,\mu)$, implies the statements of parts (i) and (ii), respectively.

Using Remark 2.5 in [20] and a standard Fubini argument, it can be shown that a test function (3.3) in the general case corresponds to one in the situation when the integral representation $\{f_t\}_{t\in\mathbb{T}^d}$ of the field is of the form (3.2). Therefore, an argument parallel to the proof of Corollary 4.2 in [20] shows that the tests described in this theorem can be applied to any full support integral representation, not necessarily minimal or of the form (3.2). This completes the proof. □

The above characterization motivates the following decomposition of an arbitrary measurable stationary SαS random field $\mathbf{X}=\{X_t\}_{t\in\mathbb{T}^d}$. Without loss of generality, let $\mathbf{X}$ have a representation $(f_0, \mathcal{G}\equiv\{\phi_t\}_{t\in\mathbb{T}^d}, \{c_t\}_{t\in\mathbb{T}^d})$ as in (3.1) and (3.2). Then, by Lemma 2.2, $S=P_\mathcal{G}\cup N_\mathcal{G}$ and one can write

$$\{X_t\}_{t\in\mathbb{T}^d} \stackrel{\mathrm{d}}{=} \{X_t^P + X_t^N\}_{t\in\mathbb{T}^d} \tag{3.6}$$

with

$$X_t^P = \int_{P_\mathcal{G}} f_t(s)M_\alpha(\mathrm{d}s) \quad\text{and}\quad X_t^N = \int_{N_\mathcal{G}} f_t(s)M_\alpha(\mathrm{d}s) \qquad \text{for all } t\in\mathbb{T}^d.$$

COROLLARY 3.3. (i) *The decomposition* (3.6) *is unique in law. That is, if there is another representation* $(f_0^{(2)}, \mathcal{G}^{(2)} \equiv \{\phi_t^{(2)}\}_{t\in\mathbb{T}^d}, \{c_t^{(2)}\}_{t\in\mathbb{T}^d})$ *satisfying* (3.1) *and* (3.2), *then*

$$\{X_t^P\} \stackrel{d}{=} \left\{\int_{P_{\mathcal{G}^{(2)}}} f_t^{(2)}\,\mathrm{d}M_\alpha\right\} \quad \textit{and} \quad \{X_t^N\} \stackrel{d}{=} \left\{\int_{N_{\mathcal{G}^{(2)}}} f_t^{(2)}\,\mathrm{d}M_\alpha\right\}.$$

(ii) *The components* $\mathbf{X}^P = \{X_t^P\}_{t\in\mathbb{T}^d}$ *and* $\mathbf{X}^N = \{X_t^N\}_{t\in\mathbb{T}^d}$ *are independent*, $\mathbf{X}^P$ *is generated by a positive* $\mathbb{T}^d$*-action and* $\mathbf{X}^N$ *is generated by a null* $\mathbb{T}^d$*-action.*

PROOF. Proof of (ii) is trivial. To prove (i), observe that by Remark 2.5 in [20], there exist measurable functions $\Phi : S_2 \to S$ and $h : S_2 \to \mathbb{R} \setminus \{0\}$ such that for all $t \in \mathbb{T}^d$,

$$f_t^{(2)}(s) = h(s) f_t \circ \Phi(s), \qquad \mu_2\text{-almost all } s \in S_2 \tag{3.7}$$

and $\mathrm{d}\mu = (|h|^\alpha\,\mathrm{d}\mu_2) \circ \Phi^{-1}$. Using (3.7) and an argument parallel to the proof of (2.18) in [34], it can be shown that $P_{\mathcal{G}^{(2)}} = \Phi^{-1}(P_{\mathcal{G}})$ and $N_{\mathcal{G}^{(2)}} = \Phi^{-1}(N_{\mathcal{G}})$ modulo μ_2, from which the distributional equality in (i) follows as in the proof of Theorem 4.3 in [20]. □

4. Ergodic properties of stationary SαS fields. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\{\theta_t\}_{t\in\mathbb{T}^d}$ a measure-preserving $\mathbb{T}^d$-action on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the random field $X_t(\omega) = X_0 \circ \theta_t(\omega)$, $t \in \mathbb{T}^d$. The random field $\{X_t\}_{t\in\mathbb{T}^d}$ defined in this way is stationary and, conversely, any stationary measurable random field can be expressed in this form.

We start by introducing some notation. For $t \in \mathbb{T}^d$, let $\|t\|$ denote its sup norm. We consider the class $\mathcal{T}$ of all sequences that converge to infinity:

$$\mathcal{T} := \left\{\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{T}^d : \lim_{n\to\infty} \|t_n\| = \infty\right\}.$$

Recall that a set $E \subset \mathbb{T}^d$ is said to have *density zero* in $\mathbb{T}^d$ if

$$\lim_{T\to\infty} \frac{1}{C(T)} \int_{B(T)} \mathbf{1}_E(t)\lambda(\mathrm{d}t) = 0. \tag{4.1}$$

A set $D \subset \mathbb{T}^d$ is said to have *density one* in $\mathbb{T}^d$ if $\mathbb{T}^d \setminus D$ has density zero in $\mathbb{T}^d$. The class of all sequences on D that converge to infinity will be denoted by

$$\mathcal{T}_D := \left\{\{t_n\}_{n\in\mathbb{N}} : t_n \in \mathbb{T}^d \cap D, \lim_{n\to\infty} \|t_n\| = \infty\right\}.$$

Now we recall some basic definitions. Write $\sigma_{\mathbf{X}} := \sigma(\{X_t : t \in \mathbb{T}^d\})$ for the σ-algebra generated by the field $\{X_t\}_{t\in\mathbb{T}^d}$. We say $\{X_t\}_{t\in\mathbb{T}^d}$ is:

(i) *ergodic*, if

$$\lim_{T\to\infty} \frac{1}{C(T)} \int_{B(T)} \mathbb{P}\big(A \cap \theta_t(B)\big)\lambda(\mathrm{d}t) = \mathbb{P}(A)\mathbb{P}(B) \qquad \text{for all } A, B \in \sigma_{\mathbf{X}}. \tag{4.2}$$

(ii) *weakly mixing*, if there exists a density one set D such that

$$\lim_{n\to\infty} \mathbb{P}\big(A \cap \theta_{t_n}(B)\big) = \mathbb{P}(A)\mathbb{P}(B) \qquad \text{for all } A, B \in \sigma_{\mathbf{X}}, \{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}_D. \tag{4.3}$$

(iii) *mixing*, if

$$\lim_{n\to\infty} \mathbb{P}\big(A \cap \theta_{t_n}(B)\big) = \mathbb{P}(A)\mathbb{P}(B) \qquad \text{for all } A, B \in \sigma_{\mathbf{X}}, \{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}. \tag{4.4}$$

In general, we always have that

$$\text{mixing} \Rightarrow \text{weakly mixing} \Rightarrow \text{ergodicity}.$$

For stationary SαS random fields, however, we have the following result.

THEOREM 4.1. *Let $\{X_t\}_{t\in\mathbb{T}^d}$ denote a measurable SαS random field with spectral representation* (3.1) *and $\alpha \in (0,2)$. The following are equivalent*:

(i) $\{X_t\}_{t\in\mathbb{T}^d}$ *is ergodic*.
(ii) $\{X_t\}_{t\in\mathbb{T}^d}$ *is weakly mixing*.
(iii) $\lim_{T\to\infty} C(T)^{-1}\int_{B(T)} \exp(2\|f_0\|_\alpha^\alpha - \|f_0 - f_t\|_\alpha^\alpha)\lambda(\mathrm{d}t) = 1$.
(iv) *The $\mathbb{T}^d$-action $\{\phi_t\}_{t\in\mathbb{T}^d}$ has no nontrivial positive component*.

PROOF. Using Theorem 2.8 and proceeding as in Theorems 2 and 3 in [19], one can show the equivalence of (i), (ii) and (iii).

To prove the equivalence of (ii) and (iv), we need the following result, which is an extension of Theorem 2.7 in [7]. The proof is given in the Appendix. We also fill a gap in the results of Gross [7] (see Remark A.6).

PROPOSITION 4.2. *Assume $\alpha \in (0,2)$ and $\{X_t\}_{t\in\mathbb{T}^d}$ is a stationary SαS random field with spectral representation $\{f_t\}_{t\in\mathbb{T}^d} \subset L^\alpha(S,\mathcal{B},\mu)$. Then, the process $\{X_t\}_{t\in\mathbb{T}^d}$ is weakly mixing if and only if there exists a density one set $D \subset \mathbb{T}^d$, such that*

$$\lim_{n\to\infty} \mu\{s : |f_0(s)|^\alpha \in K, |f_{t_n^*}(s)|^\alpha > \varepsilon\} = 0 \tag{4.5}$$
$$\textit{for all compact } K \subset \mathbb{R}\setminus\{0\}, \varepsilon > 0 \textit{ and } \{t_n^*\}_{n\in\mathbb{N}} \in \mathcal{T}_D.$$

Now we prove the equivalence of (ii) and (iv) by following closely the proof of Theorem 3.1 in [34]. The proof of (ii) implying (iv) remains the same. To show that (iv) implies (ii), however, we treat the discrete and the continuous parameter scenarios together by virtue of Theorem 2.7, which unifies the two cases (which were treated differently in [34]). More specifically, in view of (4.5) and a multivariate extension of Lemma 6.2 in [16], page 65, it is enough to show that for all $\varepsilon > 0$ and compact sets $K \subset \mathbb{R}\setminus\{0\}$,

$$\lim_{T\to\infty} A_T \mu\big\{s : |f_0(s)|^\alpha \in K, \big|f_{(\cdot)}(s)\big|^\alpha > \varepsilon\big\} = 0, \tag{4.6}$$

where A_T is the average operator defined by (2.5). Following verbatim the argument in the proof of (3.1) in [34], we obtain (4.6) for both discrete and continuous parameter cases with the help of Theorem 2.7. □

REMARK 4.3. From the structural results [29, 30] and Theorem 4.1 above, we obtain a unique in law decomposition of $\mathbf{X}$ into three independent stable processes in parallel to the one-dimensional case [34], that is,

$$\mathbf{X} = \mathbf{X}^{(1)} + \mathbf{X}^{(2)} + \mathbf{X}^{(3)},$$

where $\mathbf{X}^{(1)}$ is a mixed moving average in the sense of [38], $\mathbf{X}^{(2)}$ is weakly mixing with no mixed moving average component and $\mathbf{X}^{(3)}$ has no weakly mixing component.

5. Max-stable stationary random fields. In this section we discuss the structure and ergodic properties of stationary max-stable random fields, indexed by $\mathbb{T}^d$. For simplicity and without loss of generality, we will focus on α-Fréchet random fields. The random field $\mathbf{Y} = \{Y_t\}_{t\in\mathbb{T}^d}$ is said to be α-Fréchet, if for all $a_j > 0, \tau_j \in \mathbb{T}^d, 1 \le j \le n$, the max-linear combinations $\xi := \max_{1\le j\le n} a_j Y_{\tau_j} \equiv \bigvee_{1\le j\le n} a_j Y_{\tau_j}$ have α-Fréchet distributions. Namely, $\mathbb{P}(\xi \le x) = \exp\{-\sigma^\alpha x^{-\alpha}\}$ for all $x \in (0,\infty)$, where $\sigma > 0$ is referred to as the *scale coefficient* and $\alpha > 0$ is the tail index of ξ. The α-Fréchet random fields are max-stable. Conversely, all max-stable random fields with α-Fréchet marginals are α-Fréchet random fields.

The spectral representations for α-Fréchet random fields have been developed by de Haan [4] and developed by [37, 42]. Any measurable α-Fréchet random field $\mathbf{Y} = \{Y_t\}_{t\in\mathbb{T}^d}$ $(\alpha > 0)$ can be represented as

$$\{Y_t\}_{t\in\mathbb{T}^d} \stackrel{\mathrm{d}}{=} \left\{ {\int_S}^{e} f_t(s) M_{\alpha,\vee}(\mathrm{d}s) \right\}_{t\in\mathbb{T}^d}, \tag{5.1}$$

where $\{f_t\}_{t\in\mathbb{T}^d} \subset L_+^\alpha(S,\mu) := \{f \in L^\alpha(S,\mu) : f \ge 0\}$, "${}^e\!\!\int$" stands for the *extremal integral*, $M_{\alpha,\vee}$ is an *independently scattered α-Fréchet random sup-measure* with control measure μ and (S,μ) can be chosen to be a standard Lebesgue space (see [37, 42]). The functions $\{f_t\}_{t\in\mathbb{T}^d}$ in (5.1) are called *spectral functions* of the α-Fréchet random field. If the representation in (5.1) is *minimal*, as in the sum-stable case, it then follows that

$$f_t(s) = \left(\frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}\right)^{1/\alpha} f_0\circ\phi_t(s) \qquad \text{for all } t \in \mathbb{T}^d, \tag{5.2}$$

where $\phi = \{\phi_t\}_{t\in\mathbb{T}^d}$ is a nonsingular group action and $f_0 \in L_+^\alpha(S,\mu)$ (see, e.g., [42], Theorems 3.1 and 3.2). Thus, the α-Fréchet random field $\mathbf{Y}$ is said to be generated by the group action ϕ if (5.1) is a minimal representation such that (5.2) holds. This allows us to extend the available classification results in the sum-stable case to the max-stable setting. Note that compared to (3.2), the cocycle $\{c_t\}_{t\in\mathbb{T}^d}$ disappears, as $\{f_t\}_{t\in\mathbb{T}^d}$ are nonnegative. By a similar argument as in Theorem 3.1, we obtain the following result.

THEOREM 5.1. *Suppose $\{Y_t\}_{t\in\mathbb{T}^d}$ is a measurable stationary α-Fréchet random field with spectral representation $\{f_t\}_{t\in\mathbb{T}^d}$ as in (5.1). Let $T_0 \in \mathcal{B}_{\mathbb{T}^d}$ and $\{a_\tau\}_{\tau\in T_0}$, $a_\tau > 0$, be such that (3.3) holds. Then:*

(i) *$\{Y_t\}_{t\in\mathbb{T}^d}$ is generated by a positive $\mathbb{T}^d$-action, if and only if (3.4) holds.*
(ii) *$\{Y_t\}_{t\in\mathbb{T}^d}$ is generated by a null $\mathbb{T}^d$-action, if and only if (3.5) holds.*

In particular, the classes of stationary α-Fréchet random fields generated by positive and null $\mathbb{T}^d$-actions are disjoint.

An intimate connection between the α-Fréchet and SαS processes $(0 < \alpha < 2)$ was recently revealed through the notion of *association*, independently by Kabluchko [9] and Wang and Stoev [41]. By the association tool established in [41], the decomposition results for α-Fréchet random fields follow immediately from the corresponding ones for SαS random fields. Indeed, for an α-Fréchet random field $\{Y_t\}_{t\in\mathbb{T}^d}$ with spectral functions $\{f_t\}_{t\in\mathbb{T}^d}$, $\alpha \in (0, 2)$, consider the SαS random field $\{X_t\}_{t\in\mathbb{T}^d}$ with the same spectral functions. Naturally, the random fields $\{X_t\}_{t\in\mathbb{T}^d}$ and $\{Y_t\}_{t\in\mathbb{T}^d}$ are said to be *associated*, according to [41]. Then, applying Theorem 5.1 in [41] to Corollary 3.3, we obtain the following results on α-Fréchet random fields.

COROLLARY 5.2. *Let $\{Y_t\}_{t\in\mathbb{T}^d}$ be a measurable stationary α-Fréchet random field with representation in the form of (5.1) and (5.2). We have the unique-in-law decomposition $\{Y_t\}_{t\in\mathbb{T}^d} \stackrel{d}{=} \{Y_t^P \vee Y_t^N\}_{t\in\mathbb{T}^d}$, with*

$$Y_t^P = \int^e P_{\mathcal{G}} f_t(s) M_{\alpha,\vee}(\mathrm{d}s) \quad \textit{and} \quad Y_t^N = \int^e N_{\mathcal{G}} f_t(s) M_{\alpha,\vee}(\mathrm{d}s) \qquad \textit{for all } t \in \mathbb{T}^d$$

with $\mathcal{G} \equiv \{\phi_t\}_{t\in\mathbb{T}^d}$. The two components are independent, $\{Y_t^P\}_{t\in\mathbb{T}^d}$ is generated by positive $\mathbb{T}^d$-action and $\{Y_t^N\}_{t\in\mathbb{T}^d}$ is generated by null $\mathbb{T}^d$-action.

The ergodic properties of stationary α-Fréchet random fields can be characterized in terms of the recurrence properties of the nonsingular group actions, as in the sum-stable case. The following theorem extends the known results in the one-dimensional case (see [9, 10, 36]). These results, however, cannot be established by the association method.

THEOREM 5.3. *Let $\{Y_t\}_{t\in\mathbb{T}^d}$ denote a measurable α-Fréchet random field with spectral representation (5.1) and (5.2). The following are equivalent:*

(i) *$\{Y_t\}_{t\in\mathbb{T}^d}$ is ergodic.*

(ii) $\{Y_t\}_{t\in\mathbb{T}^d}$ *is weakly mixing.*
(iii) $\lim_{T\to\infty} C(T)^{-1}\int_{B(T)} \|f_t \wedge f_0\|_\alpha^\alpha \lambda(\mathrm{d}t) = 0$.
(iv) *The* $\mathbb{T}^d$*-action* $\{\phi_t\}_{t\in\mathbb{T}^d}$ *has no nontrivial positive component.*

PROOF. The equivalence of (i), (ii) and (iii) for $\mathbb{R}$-action is proved by Kabluchko and Schlather [10], Theorem 1.2. Their proof generalizes to $\mathbb{T}^d$-actions as well. The equivalence of (i) and (iv) can be proved by extending the proof of Theorem 8 in [9] to the multiparameter setting, using Theorems 2.7 and 2.8 accordingly. □

6. Examples. This section contains two examples of stable random fields and their ergodic properties via the positive-null decomposition of the underlying action. These examples show the usefulness of our results to check whether or not a stationary SαS (or max-stable) random field is ergodic (or, equivalently, weakly mixing).

The first example is based on a self-similar SαS processes with stationary increments introduced by [3] as a stochastic integral with respect to an SαS random measure, with the integrand being the local time process of a fractional Brownian motion. We extend these processes by replacing the fractional Brownian motion by a Brownian sheet. We can call it a *Brownian sheet local time fractional SαS random field* following the terminology of [3].

EXAMPLE 6.1. Suppose $(\Omega', \mathcal{F}', P')$ is a probability space supporting a Brownian sheet $\{B_u\}_{u\in\mathbb{R}_+^d}$. By [6], $\{B_u\}$ has a jointly continuous local time field $\{l(x,u): x\in\mathbb{R}, u\in\mathbb{R}_+^d\}$ defined on the same probability space. We will define an SαS random field based on this local time field, which inherits the stationary increments property from $\{B_u\}_{u\in\mathbb{R}_+^d}$. Let M_α be an SαS random measure on $\Omega'\times\mathbb{R}$ with control measure $P'\times\text{Leb}$ living on another probability space $(\Omega, \mathcal{F}, P)$. Following verbatim the calculations of [3], we have that

$$Z_u = \int_{\Omega'\times\mathbb{R}} l(x,u)(\omega') M_\alpha(\mathrm{d}\omega', \mathrm{d}x), \qquad u\in\mathbb{R}_+^d,$$

is a well-defined SαS random field, which has stationary increments over d-dimensional rectangles.

We now concentrate on the increments of $\{Z_u\}$ taken over d-dimensional rectangles. For any $t\in\mathbb{Z}_+^d$, define

$$X_t = \Delta Z_t := \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_d=0}^{1}(-1)^{i_1+i_2+\cdots+i_d+d} Z_{t+(i_1,i_2,\ldots,i_d)}. \tag{6.1}$$

Clearly, $\{X_t\}_{t\in\mathbb{Z}_+^d}$ is a stationary SαS random field, which can be extended (in law) to a stationary SαS random field $\mathbf{X} := \{X_t\}_{t\in\mathbb{Z}^d}$ by Kolmogorov's extension theorem. We claim that $\mathbf{X}$ is generated by a null $\mathbb{Z}^d$-action. To prove this, define, for all $n \geq 1$, $\tau^{(n)} := (n^{4/d}, n^{4/d}, \dots, n^{4/d})$, and for all $n \geq 1$ and $t \in \mathbb{Z}_+^d$,

$$T_{n,t} := \{s : t_i + n^{4/d} \leq s_i \leq 1 + t_i + n^{4/d} \text{ for all } i = 1, 2, \dots, d\}.$$

For each $t \in \mathbb{Z}_+^d$, take a positive real number a_t in such a way that $\sum_{t \in \mathbb{Z}_+^d} a_t = 1$. Defining $\Delta l(x,t)$ in parallel to (6.1) and following the proof of (4.7) in [3], we can establish that

$$\begin{aligned}\int_{\Omega'} \int_{\mathbb{R}} e^{-x^2/2} \sum_{t \in \mathbb{Z}_+^d} \sum_{n=1}^{\infty} a_t \Delta l\bigl(x, t + \tau^{(n)}\bigr) \, \mathrm{d}x \, \mathrm{d}P' \\ = \sum_{t \in \mathbb{Z}_+^d} a_t \sum_{n=1}^{\infty} \int_{T_{n,t}} \frac{\mathrm{d}s}{\sqrt{1 + \prod_{i=1}^d s_i}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{1 + n^4}} < \infty.\end{aligned}$$

This shows, in particular, that $\sum_{t \in \mathbb{Z}_+^d} \sum_{n=1}^{\infty} a_t \Delta l(x, t + \tau^{(n)})(\omega') < \infty$ for $P' \times$ Leb-almost all $(\omega', x) \in \Omega \times \mathbb{R}$. Besides, it can be easily shown that $\sum_{t \in \mathbb{Z}_+^d} a_t \Delta l(x, t)(\omega') > 0$ for $P' \times$ Leb-almost all $(\omega', x) \in \Omega \times \mathbb{R}$ (see, e.g., [40]). Hence, by Theorem 3.1, it follows that $\mathbf{X}$ is generated by a null action and hence is weakly mixing.

The next example is based on a class of mixing stationary SαS process considered in [23]. We look at a stationary SαS random field generated by d independent recurrent Markov chains, at least one of which is null-recurrent. This is a class of stationary SαS random fields which are weakly mixing as a field but not necessarily ergodic in every direction.

EXAMPLE 6.2. We start with d irreducible aperiodic recurrent Markov chains on $\mathbb{Z}$ with laws $P_i^{(1)}(\cdot), P_i^{(2)}(\cdot), \dots, P_i^{(d)}(\cdot)$, $i \in \mathbb{Z}$ and transition probabilities $(p_{jk}^{(1)}), (p_{jk}^{(2)}), \dots, (p_{jk}^{(d)})$, respectively. For all $l = 1, 2, \dots, d$, let $\pi^{(l)} = (\pi_i^{(l)})_{i \in \mathbb{Z}}$ be a σ-finite invariant measure corresponding to the family $(P_i^{(l)})$. Let $\widetilde{P}_i^{(l)}$ be the lateral extension of $P_i^{(l)}$ to $\mathbb{Z}^{\mathbb{Z}}$, that is, under $\widetilde{P}_i^{(l)}$, $x(0) = i$, $(x(0), x(1), \dots)$ is a Markov chain with transition probabilities $(p_{jk}^{(l)})$ and $(x(0), x(-1), \dots)$ is a Markov chain with transition probabilities $(\pi_k^{(l)} p_{kj}^{(l)} / \pi_j^{(l)})$. Assume at least one (say, the first one) of the Markov chains is null-recurrent and define a σ-finite measure μ on $S = (\mathbb{Z}^{\mathbb{Z}})^d$ by

$$\mu(A_1 \times A_2 \times \cdots \times A_d) = \prod_{l=1}^{d} \left(\sum_{i=-\infty}^{\infty} \pi_i^{(l)} \widetilde{P}_i^{(l)}(A_l) \right),$$

and observe that μ is invariant under the $\mathbb{Z}^d$-action $\{\phi_{(i_1, i_2, \dots, i_d)}\}_{(i_1, \dots, i_d) \in \mathbb{Z}^d}$ on S defined as the coordinatewise left shift, that is,

$$\phi_{(i_1,\dots,i_d)}\bigl(a^{(1)}, \dots, a^{(d)}\bigr)(u_1, \dots, u_d) = \bigl(a^{(1)}(u_1 + i_1), \dots, a^{(d)}(u_d + i_d)\bigr) \tag{6.2}$$

for all $(a^{(1)}, \dots, a^{(d)}) \in S$ and $u_1, \dots, u_d \in \mathbb{Z}$.

Let $\mathbf{X} = \{X_{(i_1,i_2,\dots,i_d)}\}_{(i_1,\dots,i_d)\in\mathbb{Z}^d}$ be a stationary SαS random field defined by the integral representation (3.1) with M_α being a SαS random measure on S with control measure μ and

$$f_{(i_1,i_2,\dots,i_d)} = f \circ \phi_{(i_1,i_2,\dots,i_d)}, \qquad i_1, i_2, \dots, i_d \in \mathbb{Z},$$

with

$$f\big(x^{(1)}, x^{(2)}, \dots, x^{(d)}\big) = \mathbf{1}_{\{x^{(1)}(0)=x^{(2)}(0)=\cdots=x^{(d)}(0)=0\}}, \qquad x^{(1)}, x^{(2)}, \dots, x^{(d)} \in \mathbb{Z}^{\mathbb{Z}}.$$

Clearly, the restriction of (6.2) to the first coordinate is a null flow because the first Markov chain is null-recurrent (see Example 4.1 in [34]) and, hence, (6.2) is a null $\mathbb{Z}^d$-action. This shows, in particular, that $\mathbf{X}$ is weakly mixing. However, if $d > 1$ and some of the Markov chains are positive-recurrent, then the restriction of μ in the corresponding coordinate directions are finite and, hence, by Theorem 4.1, $\mathbf{X}$ is not ergodic along those directions. In this case, the random field cannot be mixing because it is not mixing in every coordinate direction. This gives examples of stationary d-dimensional ($d > 1$) SαS random fields, which are weakly mixing but not mixing. See Example 4.2 in [8] for such an example in the $d = 1$ case.

REMARK 6.3. Correspondingly, we can define α-Fréchet random fields and apply Theorem 5.3. In particular, when $d > 1$, we can obtain an example of an α-Fréchet random field, which is weakly mixing but not mixing.

APPENDIX: PROOFS OF AUXILIARY RESULTS

A.1. Proof of Lemma 2.2. Set

$$u(I(\mathcal{G})) := \sup_{\nu\in\Lambda(\mathcal{G})} \mu(S_\nu). \tag{A.1}$$

Without loss of generality, we assume $\mu(S) < \infty$ (recall that μ is σ-finite), whence $u(I(\mathcal{G})) < \infty$. Then, there exists a sequence of measures $\{\nu_n\}_{n\in\mathbb{N}} \subset \Lambda(\mathcal{G})$, such that $u_n := \mu(S_{\nu_n}) \to u(I(\mathcal{G}))$ as $n \to \infty$. Set

$$P_{\mathcal{G}} := \bigcup_{n=1}^{\infty} S_{\nu_n}.$$

Clearly, $P_{\mathcal{G}}$ is measurable. We show that there exists $\nu_{\mathcal{G}} \in \Lambda(\mathcal{G})$ such that $S_{\nu_{\mathcal{G}}} = P_{\mathcal{G}}$ and $\mu(P_{\mathcal{G}}) = u(I(\mathcal{G}))$. Indeed, we can define on $(S, \mathcal{B})$ the measure

$$\nu_{\mathcal{G}}(A) := \sum_{n=1}^{\infty} \frac{1}{2^n u_n} \nu_n(A) \qquad \text{for all } A \in \mathcal{B}. \tag{A.2}$$

Clearly, $\nu_{\mathcal{G}} \in \Lambda(\mathcal{G})$, $S_{\nu_{\mathcal{G}}} = P_{\mathcal{G}} \bmod \mu$, and $\mu(P_{\mathcal{G}}) \le u(I(\mathcal{G}))$ by (A.1). It is also clear that for all $n \in \mathbb{N}$, $\nu_n \ll \nu_{\mathcal{G}}$ and, hence, $P_{\mathcal{G}} \supset S_{\nu_n} \bmod \mu$. This implies $\mu(P_{\mathcal{G}}) \ge u_n$ for all $n \in \mathbb{N}$. We have thus shown that $\mu(P_{\mathcal{G}}) = u(I(\mathcal{G}))$.

To complete the proof, we show $P_{\mathcal{G}}$ is unique modulo μ-null sets. Suppose there exist $P_{\mathcal{G}}^{(1)}$ and $P_{\mathcal{G}}^{(2)}$ such that $\mu(P_{\mathcal{G}}^{(1)}) = \mu(P_{\mathcal{G}}^{(2)}) = u(I(\mathcal{G}))$ and $\mu(P_{\mathcal{G}}^{(1)} \triangle P_{\mathcal{G}}^{(2)}) > 0$. Suppose $\nu^{(1)}, \nu^{(2)} \in \Lambda(\mathcal{G})$ are defined as in (A.2), so that $S_{\nu^{(i)}} = P_{\mathcal{G}}^{(i)}$ for $i = 1, 2$. Clearly, $\nu^{(1)} + \nu^{(2)} \in \Lambda(\mathcal{G})$. Then, we have $P_{\mathcal{G}}^{(1)} \cup P_{\mathcal{G}}^{(2)} \subset I(\mathcal{G})$ and $\mu(P_{\mathcal{G}}^{(1)} \cup P_{\mathcal{G}}^{(2)}) > u(I(\mathcal{G}))$, which contradicts (A.1). The proof is thus complete.

A.2. Proof of Theorem 2.3. First we introduce some notation. For all transformation ϕ on $(S, \mathcal{B}, \mu)$, write

$$\Lambda(\phi) := \{\nu \ll \mu : \nu \text{ finite positive measure on } S, \nu \circ \phi^{-1} = \nu\}.$$

We need the following lemma.

LEMMA A.4. *Suppose ϕ is an arbitrary invertible, bimeasurable and nonsingular transformation on $(S, \mathcal{B}, \mu)$. Then $\mu(\phi^{-1}(S_\nu) \triangle S_\nu) = 0$, for all $\nu \in \Lambda(\phi)$.*

PROOF. First, we show for all $\nu \in \Lambda(\phi)$, $\mu(\phi^{-1}(S_\nu) \triangle S_\nu) = 0$. If not, then set $E_0 := \phi^{-1}(S_\nu) \setminus S_\nu$, $F_0 = \phi(E_0)$ and suppose $\mu(E_0) > 0$. Since ϕ is nonsingular, $\mu(F_0) > 0$. Note that $F_0 \subset S_\nu$ and $\mu \sim \nu$ on S_ν, whence $\nu(F_0) > 0$. Note also that $\nu(S_\nu^c) = 0$ and $\nu \circ \phi^{-1} = \nu$ imply $\nu(F_0) = \nu \circ \phi^{-1}(F_0) = \nu(E_0) \le \nu(S_\nu^c) = 0$. This contradicts $\nu(F_0) > 0$. We have thus shown that $\mu(\phi^{-1}(S_\nu) \setminus S_\nu) = 0$.

Next, we show that $\mu(S_\nu \setminus \phi^{-1}(S_\nu)) = 0$. Indeed, setting $E_1 := S_\nu \setminus \phi^{-1}(S_\nu)$, we have $\nu(S_\nu) = \nu(E_1) + \nu(\phi^{-1}(S_\nu) \cap S_\nu)$. At the same time, $\nu(S_\nu) = \nu \circ \phi^{-1}(S_\nu) = \nu(\phi^{-1}(S_\nu) \cap S_\nu) + \nu(E_0)$, where $E_0 := \phi^{-1}(S_\nu) \setminus S_\nu$. Since $\nu(E_0) = 0$ as shown in the first part of the proof, the two equations above imply $\nu(E_1) = 0$, since ν is finite. Finally, by the fact that $\nu \sim \mu$ on S_ν, we have $\mu(S_\nu \setminus \phi^{-1}(S_\nu)) \equiv \mu(E_1) = 0$. $\square$

Now we prove Theorem 2.3.

(i) Fix $\phi \in \mathcal{G}$. Note that by Lemma 2.2, there exists $\nu_{\mathcal{G}} \in \Lambda(\phi) \subset I(\mathcal{G})$ such that $S_{\nu_{\mathcal{G}}} = P_{\mathcal{G}}$. Then, by Lemma A.4, $\mu(\phi^{-1}(P_{\mathcal{G}}) \triangle P_{\mathcal{G}}) = 0$. By the fact that all $\phi \in \mathcal{G}$ are invertible, we have that $\phi^{-1}(N_{\mathcal{G}})^c = \phi^{-1}(N_{\mathcal{G}}^c)$ and by the identity $A \triangle B = A^c \triangle B^c$, we have $\mu(\phi^{-1}(N_{\mathcal{G}}) \triangle N_{\mathcal{G}}) = 0$. The previous argument is valid for all $\phi \in \mathcal{G}$.

(ii) Consider $L^1(P_{\mathcal{G}}, \mathcal{B} \cap P_{\mathcal{G}}, \mu|_{P_{\mathcal{G}}})$, where $\mathcal{B} \cap P_{\mathcal{G}} := \{A \cap P_{\mathcal{G}} : A \in \mathcal{B}\}$ and $\mu|_{P_{\mathcal{G}}}$ is the restriction of μ tn $\mathcal{B} \cap P_{\mathcal{G}}$. Define

$$\widetilde{\phi} f(s) \equiv [\widetilde{\phi}(f)](s) := \frac{\mathrm{d}(\mu \circ \phi^{-1})}{\mathrm{d}\mu}(s) f \circ \phi^{-1}(s) \mathbf{1}_{P_{\mathcal{G}} \cap \phi(P_{\mathcal{G}})}(s) \quad \text{for all } f \in L^1(P_{\mathcal{G}}, \mu|_{P_{\mathcal{G}}}). \tag{A.3}$$

In this way, the mapping $\widetilde{\phi}$ is a restricted version of $\widehat{\phi}$ on $L^1(P_{\mathcal{G}}, \mu|_{P_{\mathcal{G}}})$ in the sense that

$$\widetilde{\phi} f = \widehat{\phi} f, \qquad \mu|_{P_{\mathcal{G}}}\text{-a.e. for all } f \in L^1(P_{\mathcal{G}}, \mu|_{P_{\mathcal{G}}}) \subset L^1(S, \mu). \tag{A.4}$$

Recall that by Lemma 2.2 there exists $\nu \in \Lambda(\mathcal{G})$ such that $\widehat{\phi}(\mathrm{d}\nu/\mathrm{d}\mu) = \mathrm{d}\nu/\mathrm{d}\mu$ for all $\phi \in \mathcal{G}$ and $\operatorname{supp}(\nu) = P_{\mathcal{G}}$. Whence, for $\widetilde{\nu} := \nu|_{P_{\mathcal{G}}}$, we have $\widetilde{\phi}(\mathrm{d}\widetilde{\nu}/\mathrm{d}\mu|_{P_{\mathcal{G}}}) = \mathrm{d}\widetilde{\nu}/\mathrm{d}\mu|_{P_{\mathcal{G}}}$ for all $\phi \in \mathcal{G}$ and $\widetilde{\nu} \sim \mu|_{P_{\mathcal{G}}}$. Note that all locally compact Abelian groups are amenable (see, e.g., Example 1.1.5(c) in [31]). Thus, Theorem 1 [parts (1) and (8)] in [39] applied to $\widetilde{G}$ and f implies that

$$\sum_{n=1}^{\infty} \widetilde{\phi}_{u_n} f(s) = \infty, \qquad \mu|_{P_{\mathcal{G}}}\text{-a.e. for all } \{\widetilde{\phi}_{u_n}\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{G}},$$

which, by (A.4), is equivalent to (2.3).

(iii) Similarly, as in (ii), restrict $\mathcal{G}$ to $L^1(N_{\mathcal{G}}, \mathcal{B} \cap N_{\mathcal{G}}, \mu|_{N_{\mathcal{G}}})$ and apply Theorem 2 [parts (1) and (8)] in [39].

A.3. Proof of Theorem 2.4. We only sketch the proof of this result.

(i) We apply Theorem 1 [parts (1) and (6)] in [39]. Recall that the adjoint operator of $\widehat{\phi}$, $\widehat{\phi}^* : (L^1)^* \to (L^1)^*$ $[(L^1)^* = L^\infty]$ is such that for all $f \in L^1(S, \mu)$ and $h \in L^\infty(S, \mu)$,

$$\int_S f(s)[\widehat{\phi}^*(h)](s)\mu(\mathrm{d}s) = \int_S [\widehat{\phi}(f)](s)h(s)\mu(\mathrm{d}s).$$

The last integral equals

$$\int_S \frac{\mathrm{d}(\mu\circ\phi^{-1})}{\mathrm{d}\mu}(s) f\circ\phi^{-1}(s) h\circ\phi\circ\phi^{-1}(s)\mu(\mathrm{d}s) = \int_S f(s)h\circ\phi(s)\mu(\mathrm{d}s),$$

whence $[\widehat{\phi}^*(h)](s) = h\circ\phi(s)$, μ-a.e. Thus, if W is a weakly wandering set w.r.t. $\mathcal{G}$, we have

$$\sum_{n=1}^{\infty} \widehat{\phi}_{t_n}^* \mathbf{1}_W(s) < 2 \qquad \text{for some } \{\phi_{t_n}\}_{n\in\mathbb{N}} \subset \mathcal{G}.$$

Now, part (6) of Theorem 1 in [39] is equivalent to the nonexistence of a weakly wandering set of positive measure.

(ii) The proof is similar to the proof of Proposition 1.4.7 in [1].

A.4. Proof of Proposition 4.2. We first need the following lemma.

LEMMA A.5. *Assume $\{X_t\}_{t\in\mathbb{T}^d}$ is a stationary $S\alpha S$ random field with spectral representation $\{f_t\}_{t\in\mathbb{T}^d} \subset L^\alpha(S, \mathcal{B}, \mu)$, $\alpha \in (0, 2)$. Then, $\{X_t\}_{t\in\mathbb{T}^d}$ is weakly* *mixing*, *if and only if*, *there exists a density one set* $D \subset \mathbb{T}^d$, *such that*

$$\lim_{n\to\infty} \mu\left\{s : \left|\sum_{j=1}^{p} \beta_j f_{\tau_j}(s)\right| \in K, \left|\sum_{k=1}^{q} \gamma_k f_{t_k+t_n^*}(s)\right| > \varepsilon\right\} = 0$$

$$\text{for all } p, q \in \mathbb{N}, \beta_j, \gamma_k \in \mathbb{R}, \tau_j, t_k \in \mathbb{T}^d, \text{compact } K \subset \mathbb{R} \setminus \{0\}, \varepsilon > 0 \text{ and } \{t_n^*\}_{n\in\mathbb{N}} \in \mathcal{T}_D. \tag{A.5}$$

PROOF. It transpires from the proofs in [14] that a stationary process $\{X_t\}_{t\in\mathbb{T}^d}$ is weakly mixing if and only if there exists a density one set $D \subset \mathbb{T}^d$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(i \sum_{j=1}^{p} \beta_j X_{\tau_j}\right) \exp\left(i \sum_{k=1}^{q} \gamma_k X_{t_k+t_n^*}\right)\right]$$
$$= \mathbb{E}\exp\left(i \sum_{j=1}^{p} \beta_j X_{\tau_j}\right) \mathbb{E}\exp\left(i \sum_{k=1}^{q} \gamma_k X_{t_k}\right) \tag{A.6}$$
$$\text{for all } p, q \in \mathbb{N}, \beta_j, \gamma_k \in \mathbb{R}, \tau_j, t_k \in \mathbb{T} \text{ and } \{t_n^*\}_{n\in\mathbb{N}} \in \mathcal{T}_D.$$

See the following remark on the equivalence of (A.5) and (A.6). □

REMARK A.6. In the one-dimensional case, to show that (4.5) is equivalent to the weak mixing of the process, Gross [7] proved that (4.5) is equivalent to the following weaker condition (A.6) (Theorem 2.7 in [7]):

$$\lim_{n\to\infty} \mathbb{E}[\exp(i\theta_1 X_0)\exp(i\theta_2 X_{t_n})] = \mathbb{E}\exp(i\theta_1 X_0)\mathbb{E}\exp(i\theta_2 X_0)$$
$$\text{for all } \theta_1, \theta_2 \in \mathbb{R}, \{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}_D. \tag{A.7}$$

The equivalence of (A.6) and (A.7), however, seems nontrivial and yet not mentioned in [7]. Nevertheless, parallel to the proof of Theorem 2.7 in [7], we can prove Lemma A.5.

To show Proposition 4.2, it suffices to prove the following lemma.

LEMMA A.7. *Assume* $\alpha \in (0, 2)$ *and* $\{X_t\}_{t\in\mathbb{T}^d}$ *is a stationary* $S\alpha S$ *process with spectral representation* $\{f_t\}_{t\in\mathbb{T}^d} \subset L^\alpha(S, \mathcal{B}, \mu)$. *Then* (A.5) *is true if and only if* (4.5) *is true*.

PROOF. Clearly, (A.5) implies (4.5). Now suppose that (4.5) is true. We will show (A.5). For any $p, q \in \mathbb{N}$ and $\tau_j, t_k \in \mathbb{T}^d$, write

$$g_p(s) := \sum_{j=1}^{p} \beta_j f_{\tau_j}(s) \quad \text{and} \quad h_q(s) := \sum_{k=1}^{q} \gamma_k f_{t_k}(s). \tag{A.8}$$

We will prove (A.5) by induction on (p, q). By (4.5), we have that (A.5) holds for $(p, q) = (1, 1)$.

(i) Suppose for fixed (p, q) (A.5) holds, then we will show that (A.5) holds for $(p + 1, q)$. If not, then there exists $\{t_n^*\}_{n\in\mathbb{N}} \in \mathcal{T}_D$ such that for some compact $K \subset \mathbb{R} \setminus \{0\}$ and $\delta > 0$, we have $\mu(E_n) \geq \delta$ with

$$E_n := \{s : |g_p(s) + \beta_{p+1} f_{\tau_{p+1}}(s)| \in K, |U_{t_n^*} h_q(s)| > \varepsilon\}.$$

Here for all $t \in \mathbb{T}^d$, $U_t(\sum_{k=1}^q \gamma_k f_{t_k})(s) := \sum_{k=1}^q \gamma_k f_{t_k+t}(s)$.

Without loss of generality, we can assume $K \subset (0, \infty)$. Then, since K is compact, there exists $0 < d_K < M$ such that $K \subset [d_K, M]$. Since $f_{\tau_1}, \dots, f_{\tau_{p+1}} \in L^\alpha(S, \mu)$, we can also choose M to be large enough so that $\mu(E_M^0) \leq \delta/2$, where

$$E_M^0 := \{s : |g_p(s)| > M \text{ or } |\beta_{p+1} f_{\tau_{p+1}}(s)| > M\}.$$

Then, we claim that for each n, either of the two sets

$$E_n^p := \left\{s : |g_p(s)| \in \left[\frac{d_K}{2}, M\right], |U_{t_n^*} h_q(s)| > \varepsilon\right\}$$

and

$$E_n^{p+1} := \left\{s : |\beta_{p+1} f_{\tau_{p+1}}(s)| \in \left[\frac{d_K}{2}, M\right], |U_{t_n^*} h_g(s)| > \varepsilon\right\}$$

has measure larger than $\delta/4$. Otherwise, observe that $E_n \subset E_n^p \cup E_n^{p+1} \cup E_M^0$, which implies that $\mu(E_n) < \delta$, a contradiction.

It then follows that either $\{E_n^p\}_{n\in\mathbb{N}}$ or $\{E_n^{p+1}\}_{n\in\mathbb{N}}$ will have a subsequence with measures larger than $\delta/4$. Namely, there exists $\{t_{n_k}^*\}_{k\in\mathbb{N}} \in \mathcal{T}_D$ such that

$$\mu(E_{n_k}^p) \geq \frac{\delta}{4} \qquad \text{for all } k \in \mathbb{N} \quad \text{or} \quad \mu(E_{n_k}^{p+1}) \geq \frac{\delta}{4} \qquad \text{for all } k \in \mathbb{N}.$$

But the first case contradicts the assumption that (A.5) holds for (p, q) and the second case contradicts (4.5). We have thus shown that (A.5) holds for $(p + 1, q)$.

(ii) Next, suppose (A.5) holds for (p, q) and we show that it holds for $(p, q + 1)$. If not, then there exists a compact $K \subset \mathbb{R} \setminus \{0\}$ such that

$$\mu\{s : |g_p(s)| \in K, |U_{t_n^*}(h_q + \gamma_{q+1} f_{t_{q+1}})(s)| > \varepsilon\} \nrightarrow 0 \qquad \text{as } n \to \infty.$$

Then, by a similar argument as in part (i), one can show that for all $\varepsilon > 0$, there exists $\{t_n^*\}_{n\in\mathbb{N}} \in \mathcal{T}_D$ and $\delta > 0$ such that we have either

$$\mu\left\{s : |g_p(s)| \in K, |U_{t_n^*} h_q(s)| > \frac{\varepsilon}{2}\right\} \geq \delta > 0$$

or

$$\mu\left\{s : |g_p(s)| \in K, |\gamma_{q+1} f_{t_{q+1}+t_n^*}(s)| > \frac{\varepsilon}{2}\right\} \geq \delta > 0.$$

Both cases lead to contradictions. We have thus shown that (A.5) holds for $(p, q + 1)$. The proof is thus complete. □

Acknowledgments. The authors are thankful to Jan Rosiński for suggesting the problem of equivalence of ergodicity and weak mixing for the max-stable case, and to Yimin Xiao for a number of useful discussions on the properties of the local times of a Brownian sheet. The authors are also thankful to the anonymous referees for their detailed comments.

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