VARIATIONAL CHARACTERIZATION OF THE CRITICAL CURVE FOR PINNING OF RANDOM POLYMERS

By Dimitris Cheliotis[1] and Frank den Hollander

University of Athens and Leiden University

In this paper we look at the pinning of a directed polymer by a one-dimensional linear interface carrying random charges. There are two phases, localized and delocalized, depending on the inverse temperature and on the disorder bias. Using quenched and annealed large deviation principles for the empirical process of words drawn from a random letter sequence according to a random renewal process [Birkner, Greven and den Hollander, *Probab. Theory Related Fields* **148** (2010) 403–456], we derive variational formulas for the quenched, respectively, annealed critical curve separating the two phases. These variational formulas are used to obtain a necessary and sufficient criterion, stated in terms of relative entropies, for the two critical curves to be different at a given inverse temperature, a property referred to as relevance of the disorder. This criterion in turn is used to show that the regimes of relevant and irrelevant disorder are separated by a unique inverse critical temperature. Subsequently, upper and lower bounds are derived for the inverse critical temperature, from which sufficient conditions under which it is strictly positive, respectively, finite are obtained. The former condition is believed to be necessary as well, a problem that we will address in a forthcoming paper.

Random pinning has been studied extensively in the literature. The present paper opens up a window with a variational view. Our variational formulas for the quenched and the annealed critical curve are new and provide valuable insight into the nature of the phase transition. Our results on the inverse critical temperature drawn from these variational formulas are not new, but they offer an alternative approach, that is, flexible enough to be extended to other models of random polymers with disorder.

1. Introduction and main results.

1.1. *Introduction.*

I. Model. Let $S = (S_n)_{n \in \mathbb{N}_0}$ be a Markov chain on a countable state space $\mathcal{S}$ in which a given point is marked 0 ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). Write **P** to denote the law of S given $S_0 = 0$ and **E** the corresponding expectation. Let K denote the distribution

of the first return time of S to 0, that is,

$$(1.1) \qquad K(n) := \mathbf{P}(S_n = 0, S_m \neq 0 \; \forall 0 < m < n), \qquad n \in \mathbb{N}.$$

We will assume that $\sum_{n \in \mathbb{N}} K(n) = 1$ (i.e., 0 is a recurrent state) and

$$(1.2) \qquad \lim_{n \to \infty} \frac{\log K(n)}{\log n} = -(1 + \alpha) \qquad \text{for some } \alpha \in [0, \infty).$$

Let $\omega = (\omega_k)_{k \in \mathbb{N}_0}$ be i.i.d. $\mathbb{R}$-valued random variables with marginal distribution μ_0. Write $\mathbb{P} = \mu_0^{\otimes \mathbb{N}_0}$ to denote the law of ω, and $\mathbb{E}$ to denote the corresponding expectation. We will assume that

$$(1.3) \qquad M(\lambda) := \mathbb{E}(e^{\lambda \omega_0}) < \infty \qquad \forall \lambda \in \mathbb{R},$$

and that μ_0 has mean 0 and variance 1.

Let $\beta \in [0, \infty)$ and $h \in \mathbb{R}$, and for fixed ω define the law $\mathbf{P}_n^{\beta, h, \omega}$ on $\{0\} \times \mathcal{S}^n$, the set of n-steps paths in $\mathcal{S}$ starting from 0, by putting

$$(1.4) \qquad \frac{d\mathbf{P}_n^{\beta, h, \omega}}{d\mathbf{P}_n}((S_k)_{k=0}^n) := \frac{1}{Z_n^{\beta, h, \omega}} \exp\left[\sum_{k=0}^{n-1}(\beta \omega_k - h) 1_{\{S_k = 0\}}\right] 1_{\{S_n = 0\}},$$

where $\mathbf{P}_n$ is the projection of $\mathbf{P}$ onto $\{0\} \times \mathcal{S}^n$. Here, β plays the role of the inverse temperature, h the role of the disorder bias, while $Z_n^{\beta, h, \omega}$ is the normalizing partition sum. Note that $k = 0$ contributes to the sum, while $k = n$ does not. Also note that the path is tied to 0 at both ends. This is done for later convenience.

REMARK 1.1. Note that (1.2) implies $p := \gcd[\mathrm{supp}(K)] = 1$. If $p \geq 2$, then the model can be trivially restricted to $p\mathbb{N}$, so there is no loss of generality. Moreover, if $\sum_{n \in \mathbb{N}} K(n) < 1$, then the model can be reduced to the recurrent case by a shift of h. Similarly, the restriction to μ_0 with mean 0 and variance 1 can be removed by a scaling of β and a shift of h.

REMARK 1.2. The key example of the above setting is the simple random walk on $\mathbb{Z}$, for which $p = 2$ and $\alpha = \frac{1}{2}$ (Spitzer [19], Section 1). In that case the process $(n, S_n)_{n \in \mathbb{N}_0}$ can be thought of as describing a directed polymer in $\mathbb{N}_0 \times \mathbb{Z}$, that is, pinned to the interface $\mathbb{N}_0 \times \{0\}$ by random charges $\beta \omega - h$; see Figure 1. When the polymer hits the interface at time k, it picks up a reward $\exp[\beta \omega_k - h]$, which can be either >1 or <1, depending on the value of ω_k. For $h \leq 0$ the polymer tends to intersect the interface with a positive frequency ("localization"), whereas for $h > 0$ large enough it tends to wander away from the interface ("delocalization"). Simple random walk on $\mathbb{Z}^2$ corresponds to $p = 2$ and $\alpha = 0$, while simple random walk on $\mathbb{Z}^d$, $d \geq 3$, conditioned on returning to 0 corresponds to $p = 2$ and $\alpha = \frac{d}{2} - 1$ (Spitzer [19], Section 1).
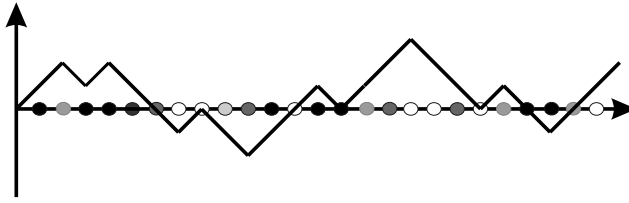
FIG. 1. *A directed polymer sampling random charges at an interface.*

II. Free energy and phase transition. The *quenched free energy* is defined as

$$(1.5) \qquad f^{\mathrm{que}}(\beta, h) := \lim_{n \to \infty} \frac{1}{n} \log Z_n^{\beta, h, \omega}.$$

Standard subadditivity arguments show that the limit exists ω-a.s. and in $\mathbb{P}$-mean, and is nonrandom; see, for example, Giacomin [11], Chapter 5, and den Hollander [8], Chapter 11. Moreover, $f^{\mathrm{que}}(\beta, h) \geq 0$ because $Z_n^{\beta, h, \omega} \geq e^{\beta \omega_0 - h} K(n)$, $n \in \mathbb{N}$, and $\lim_{n \to \infty} \frac{1}{n} \log K(n) = 0$ by (1.2). The lower bound $f^{\mathrm{que}}(\beta, h) = 0$ is attained when S visits the state 0 only rarely. This motivates the definition of two quenched phases,

$$(1.6) \qquad \begin{aligned} \mathcal{L} &:= \{(\beta, h) : f^{\mathrm{que}}(\beta, h) > 0\}, \\ \mathcal{D} &:= \{(\beta, h) : f^{\mathrm{que}}(\beta, h) = 0\}, \end{aligned}$$

referred to as the *localized* phase, respectively, the *delocalized* phase.

Since $h \mapsto f^{\mathrm{que}}(\beta, h)$ is nonincreasing for every $\beta \in [0, \infty)$, the two phases are separated by a *quenched critical curve*

$$(1.7) \qquad h_c^{\mathrm{que}}(\beta) := \inf\{h : f^{\mathrm{que}}(\beta, h) = 0\}, \qquad \beta \in [0, \infty),$$

with $\mathcal{L}$ the region below the curve and $\mathcal{D}$ the region on and above. Since $(\beta, h) \mapsto f^{\mathrm{que}}(\beta, h)$ is convex and $\mathcal{D} = \{(\beta, h) : f^{\mathrm{que}}(\beta, h) \leq 0\}$ is a level set of f^{que}, it follows that $\mathcal{D}$ is a convex set and h_c^{que} is a convex function. Since $\beta = 0$ corresponds to a homopolymer, we have $h_c^{\mathrm{que}}(0) = 0$; see Appendix A. It was shown in Alexander and Sidoravicius [2] that $h_c^{\mathrm{que}}(\beta) > 0$ for $\beta \in (0, \infty)$. Therefore we have the qualitative picture drawn in Figure 2. We further remark that $\lim_{\beta \to \infty} h_c^{\mathrm{que}}(\beta)/\beta$ is finite if and only if $\mathrm{supp}(\mu_0)$ is bounded from above.

The mean value of the disorder is $\mathbb{E}(\beta \omega_0 - h) = -h$. Thus, we see from Figure 2 that for the random pinning model localization may even occur for *moderately negative* mean values of the disorder, contrary to what happens for the homogeneous pinning model, where localization occurs only for a strictly positive parameter; see Appendix A. In other words, even a globally repulsive random interface can pin the polymer: all that the polymer needs to do is to hit some positive values of the disorder and avoid the negative values of the disorder.
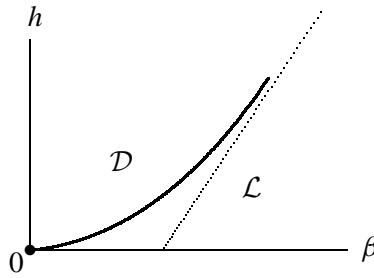
FIG. 2.   *Qualitative plot of $\beta \mapsto h_c^{\text{que}}(\beta)$. The fine details of this curve are not known.*

The *annealed free energy* is defined by

$$(1.8) \qquad f^{\text{ann}}(\beta, h) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(Z_n^{\beta,h,\omega}).$$

Since

$$(1.9) \qquad \mathbb{E}(Z_n^{\beta,h,\omega}) = \mathbf{E}\left(\exp\left[\sum_{k=0}^{n-1}[\log M(\beta) - h]1_{\{S_k=0\}}\right]1_{\{S_n=0\}}\right),$$

we have that $f^{\text{ann}}(\beta, h)$ is the free energy of the homopolymer with parameter $\log M(\beta) - h$. The associated *annealed critical curve*

$$(1.10) \qquad h_c^{\text{ann}}(\beta) := \inf\{h : f^{\text{ann}}(\beta, h) = 0\}, \qquad \beta \in [0, \infty),$$

therefore equals

$$(1.11) \qquad h_c^{\text{ann}}(\beta) = \log M(\beta).$$

Since $f^{\text{que}} \leq f^{\text{ann}}$, we have $h_c^{\text{que}} \leq h_c^{\text{ann}}$.

DEFINITION 1.3.   The disorder is said to be *relevant* for a given choice of K, μ_0 and β when $h_c^{\text{que}}(\beta) < h_c^{\text{ann}}(\beta)$, otherwise it is said to be *irrelevant*.

Note: In the physics literature, the term relevant disorder is reserved for the situation where the disorder not only changes the critical value but also changes the behavior of the free energy near the critical value. In the present paper we adopt the more narrow definition above.

Our main focus in the present paper will be on deriving *variational formulas* for h_c^{que} and h_c^{ann}, and on investigating under what conditions on K, μ_0 and β the disorder is relevant, respectively, irrelevant.

1.2. *Main results.* This section contains three theorems and four corollaries, all valid subject to (1.2) and (1.3). To state these we need some further notation.

I. Notation. Abbreviate

$$(1.12) \qquad E := \text{supp}[\mu_0] \subset \mathbb{R}.$$

Let $\widetilde{E} := \bigcup_{k \in \mathbb{N}} E^k$ be the set of *finite words* consisting of *letters* drawn from E. Let $\mathcal{P}(\widetilde{E}^{\mathbb{N}})$ denote the set of probability measures on *infinite sentences*, equipped with the topology of weak convergence. Write $\widetilde{\theta}$ for the left-shift acting on $\widetilde{E}^{\mathbb{N}}$, and $\mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$ for the set of probability measures that are invariant under $\widetilde{\theta}$.

For $Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$, let $\pi_{1,1} Q \in \mathcal{P}(E)$ denote the projection of Q onto the first letter of the first word. Define the set

$$(1.13) \qquad \mathcal{C} := \left\{ Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}) : \int_E |x| \, d(\pi_{1,1} Q)(x) < \infty \right\},$$

and on this set the function

$$(1.14) \qquad \Phi(Q) := \int_E x \, d(\pi_{1,1} Q)(x), \qquad Q \in \mathcal{C}.$$

We also need two rate functions on $\mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$, denoted by I^{ann} and I^{que}, which will be defined in Section 2. These are the rate functions of the annealed and the quenched large deviation principles that play a central role in the present paper, and they satisfy $I^{\text{que}} \geq I^{\text{ann}}$.

II. Theorems. With the above ingredients, we obtain the following *characterization of the critical curves.*

THEOREM 1.4.   *Fix μ_0 and K. For all $\beta \in [0, \infty)$,*

$$(1.15) \qquad h_c^{\text{que}}(\beta) = \sup_{Q \in \mathcal{C}} [\beta \Phi(Q) - I^{\text{que}}(Q)],$$

$$(1.16) \qquad h_c^{\text{ann}}(\beta) = \sup_{Q \in \mathcal{C}} [\beta \Phi(Q) - I^{\text{ann}}(Q)].$$

We know that $h_c^{\text{ann}}(\beta) = \log M(\beta)$. However, the variational formula for $h_c^{\text{ann}}(\beta)$ will be important for the comparison with $h_c^{\text{que}}(\beta)$.

Next, for $\beta \in [0, \infty)$ define the probability measures

$$(1.17) \qquad d\mu_\beta(x) := \frac{1}{M(\beta)} e^{\beta x} \, d\mu_0(x), \qquad x \in E,$$

and

$$(1.18) \qquad d q_\beta(x_1, x_2, \ldots, x_n) := K(n) \, d\mu_\beta(x_1) \, d\mu_0(x_2) \times \cdots \times d\mu_0(x_n),$$

$$n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in E.$$

Further, let $Q_\beta := q_\beta^{\otimes \mathbb{N}} \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$. Then Q_0 is the probability measure under which the words are i.i.d., with length drawn from K and i.i.d. letters drawn
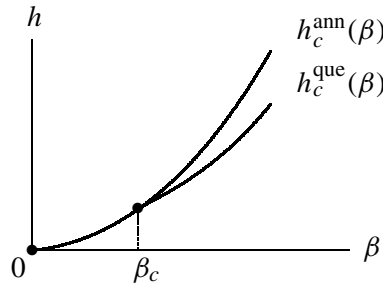
FIG. 3.   *Uniqueness of the critical inverse temperature β_c.*

from μ_0, while Q_β differs from Q_0 in that the first letter of each word is drawn from the tilted probability distribution μ_β. We will see that Q_β is the unique maximizer of the supremum in (1.16) [note that $Q_\beta \in \mathcal{C}$ because of (1.3)]. This leads to the following *necessary and sufficient criterion for disorder relevance*.

THEOREM 1.5.   *Fix μ_0 and K. For all $\beta \in [0, \infty)$,*

$$(1.19) \qquad h_c^{\mathrm{que}}(\beta) < h_c^{\mathrm{ann}}(\beta) \quad \Longleftrightarrow \quad I^{\mathrm{que}}(Q_\beta) > I^{\mathrm{ann}}(Q_\beta).$$

What is appealing about (1.19) is that the gap between I^{que} and I^{ann} needs to be established *only* for the measure Q_β, which has a simple and explicit form. We will see that the supremum in (1.15) is attained, which is to be interpreted as saying that there is a localization strategy *at* the quenched critical line.

Disorder relevance is *monotone* in β; see Figure 3.

THEOREM 1.6.   *For all μ_0 and K there exists a $\beta_c = \beta_c(\mu_0, K) \in [0, \infty]$ such that*

$$(1.20) \qquad h_c^{\mathrm{que}}(\beta) \begin{cases} = h_c^{\mathrm{ann}}(\beta), & \text{if } \beta \in [0, \beta_c], \\ < h_c^{\mathrm{ann}}(\beta), & \text{if } \beta \in (\beta_c, \infty). \end{cases}$$

III. Corollaries. From Theorems 1.4–1.6 we draw four corollaries. Abbreviate

$$(1.21) \qquad \chi := \sum_{n \in \mathbb{N}} [\mathbf{P}(S_n = 0)]^2, \qquad w := \sup[\mathrm{supp}(\mu_0)].$$

COROLLARY 1.7.   *If $\alpha = 0$, then $\beta_c = \infty$ for all μ_0.*

COROLLARY 1.8.   *If $\alpha \in (0, \infty)$, then the following bounds hold*:

(i) $\beta_c \geq \beta_c^*$ *with* $\beta_c^* = \beta_c^*(\mu_0, K) \in [0, \infty]$ *given by*

$$(1.22) \qquad \beta_c^* := 0 \vee \sup\{\beta : M(2\beta)/M(\beta)^2 < 1 + \chi^{-1}\}.$$

(ii) $\beta_c \leq \beta_c^{**}$ with $\beta_c^{**} = \beta_c^{**}(\mu_0, K) \in (0, \infty]$ given by

(1.23)
$$\beta_c^{**} := \inf\{\beta : h(\mu_\beta|\mu_0) > h(K)\},$$

where $h(\mu_\beta|\mu_0) = \int_E \log(d\mu_\beta/d\mu_0)\,d\mu_\beta$ is the relative entropy of μ_β w.r.t. μ_0, and $h(K) := -\sum_{n\in\mathbb{N}} K(n)\log K(n)$ is the entropy of K.

COROLLARY 1.9. *If $\alpha \in (0, \infty)$ and $\chi < \infty$, then $\beta_c > 0$ for all μ_0.*

COROLLARY 1.10. *If $\alpha \in (0, \infty)$, then $\beta_c < \infty$ for all μ_0 with $\mu_0(\{w\}) = 0$* (*which includes $w = \infty$*).

We close with a conjecture stating that the condition $\chi < \infty$ in Corollary 1.9 is not only sufficient for $\beta_c > 0$ but also necessary. This conjecture will be addressed in a forthcoming paper.

CONJECTURE 1.11. *If $\alpha \in (0, \infty)$ and $\chi = \infty$, then $\beta_c = 0$ for all μ_0.*

1.3. *Discussion.*
I. What is known from the literature? Before discussing the results in Section 1.2, we give a summary of what is known about the issue of relevant vs. irrelevant disorder from the literature. This summary is drawn from the papers by Alexander [1], Toninelli [20, 21], Giacomin and Toninelli [14], Derrida, Giacomin, Lacoin and Toninelli [9], Alexander and Zygouras [3, 4], Giacomin, Lacoin and Toninelli [12, 13] and Lacoin [18].

THEOREM 1.12. *Suppose that condition* (1.2) *is strengthened to*
$$K(n) = n^{-(1+\alpha)}L(n)$$
(1.24)
with $\alpha \in [0, \infty)$ and L strictly positive and slowy varying at infinity.

Then:

(1) $\beta_c = 0$ when $\alpha \in (\frac{1}{2}, \infty)$.
(2) $\beta_c = 0$ when $\alpha = \frac{1}{2}$ and $\lim_{n\to\infty}[\log n]^{\delta-1}L^2(n) = 0$ for some $\delta > 0$.
(3) $\beta_c > 0$ when $\alpha = \frac{1}{2}$ and $\sum_{n\in\mathbb{N}} n^{-1}[L(n)]^{-2} < \infty$.
(4) $\beta_c > 0$ when $\alpha \in (0, \frac{1}{2})$.
(5) $\beta_c = \infty$ when $\alpha = 0$.

The results in Theorem 1.12 hold irrespective of the choice of μ_0; see Remark 1.13 below. Toninelli [21] proves that if $\log M(\lambda) \sim C\lambda^\gamma$ as $\lambda \to \infty$ for some $C \in (0, \infty)$ and $\gamma \in (1, \infty)$, then $\beta_c < \infty$ irrespective of $\alpha \in (0, \infty)$ and L. Note that there is a small gap between cases (2) and (3) at the critical threshold $\alpha = \frac{1}{2}$.

For the cases of relevant disorder, bounds on the gap between $h_c^{\mathrm{ann}}(\beta)$ and $h_c^{\mathrm{que}}(\beta)$ have been derived in the above cited papers subject to (1.24). As $\beta \downarrow 0$, this gap decays like

$$(1.25) \qquad h_c^{\mathrm{ann}}(\beta) - h_c^{\mathrm{que}}(\beta) \asymp \begin{cases} \beta^2, & \text{if } \alpha \in (1, \infty), \\ \beta^2 \psi(1/\beta), & \text{if } \alpha = 1, \\ \beta^{2\alpha/(2\alpha-1)}, & \text{if } \alpha \in \left(\frac{1}{2}, 1\right) \end{cases}$$

for all choices of L, with ψ slowly varying and vanishing at infinity when $L(\infty) \in (0, \infty)$.

Partial results are known for $\alpha = \frac{1}{2}$. For instance, it is shown in Giacomin, Lacoin and Toninelli [13] that, under the condition in Theorem 1.12(2), the gap decays faster than any polynomial, namely, roughly like $\exp[-\beta^{-2/\delta}]$, $\beta \downarrow 0$, when $L^2(n) \asymp [\log n]^{1-\delta}$, $n \to \infty$. This implies that the disorder can at most be *marginally relevant*, a situation where standard perturbative arguments do not work.

REMARK 1.13. Some of the above mentioned results are proved for Gaussian disorder only, and are claimed to be true for arbitrary disorder subject to (1.3). Full proofs for arbitrary disorder are in [9, 13, 18, 21].

REMARK 1.14. The fact that $\alpha = \frac{1}{2}$ is critical for relevant vs. irrelevant disorder is in accordance with the so-called *Harris criterion* for disordered systems (see Harris [17]): "Arbitrary weak disorder modifies the nature of a phase transition when the order of the phase transition in the nondisordered system is < 2." The order of the phase transition for the homopolymer, which is briefly described in Appendix A, is < 2 precisely when $\alpha \in (\frac{1}{2}, \infty)$ (see Giacomin [11], Chapter 2). This link is emphasized in Toninelli [20].

II. What is new in the present paper? The main importance of our results in Section 1.2 is that they *open up a new window* on the random pinning problem. Whereas the results cited in Theorem 1.12 are derived with the help of a variety of *estimation techniques*, like fractional moment estimates and trial choices of localization strategies, Theorem 1.4 gives a *variational characterization* of the critical curves, that is, new. (It is very rare indeed that critical curves for disordered systems allow for a direct variational representation.) Theorem 1.5 gives a necessary and sufficient criterion for disorder relevance that, although not easy to handle, at least is explicit and offers a different handle. Theorem 1.6 shows that uniqueness of the inverse critical temperature is a direct consequence of this criterion, while Corollaries 1.7–1.10 show that the criterion can be used to obtain important information on the inverse critical temperature.

REMARK 1.15. Theorem 1.6 was proved in Giacomin, Lacoin and Toninelli [13] with the help of the FKG-inequality.

REMARK 1.16. Corollary 1.7 is the main result in Alexander and Zygouras [4].

REMARK 1.17. Since (see Section 8)

$$(1.26) \quad \lim_{\beta \downarrow 0} M(2\beta)/M(\beta)^2 = 1, \qquad \lim_{\beta \to \infty} h(\mu_\beta | \mu_0) = \log[1/\mu_0(\{w\})],$$

with the understanding that the second limit is ∞ when $\mu_0(\{w\}) = 0$, Corollary 1.8 implies Corollaries 1.9 and 1.10. Corollary 1.10 was noted also in Alexander and Zygouras [4].

REMARK 1.18. Note that $\chi = \mathbf{E}(|I_1 \cap I_2|)$ with I_1, I_2 two independent copies of the set of return times of S [recall (1.1)]. Thus, according to Corollary 1.9 and Conjecture 1.11, $\beta_c > 0$ is expected to be equivalent to the renewal process of *joint return times* to be recurrent. Note that $1/\mathbf{P}(I_1 \cap I_2 \neq \varnothing) = 1 + \chi^{-1}$ (see Spitzer [19], Section 1), the quantity appearing in Corollary 1.8(i).

REMARK 1.19. If μ_0 is Bernoulli(1/2) on $\{-1, 1\}$, (1.26) gives that $\lim_{\beta \to \infty} h(\mu_\beta | \mu_0) = \log 2$. For any $\alpha > 0$, we can find a distribution K that satisfies (1.2) and $H(K) < \log 2$, and thus (1.23) implies that $\beta_c = \beta_c(\mu_0, K) < \infty$. This shows that for $\alpha > 0$, the condition $\mu_0(\{w\}) = 0$ is not (!) necessary for $\beta_c < \infty$.

REMARK 1.20. As shown in Doney [10], subject to the condition of regular variation in (1.24),

$$\mathbf{P}(S_n = 0) \sim \frac{C_\alpha}{n^{1-\alpha} L(n)}$$

(1.27)

as $n \to \infty$ with $C_\alpha = (\alpha/\pi) \sin(\alpha\pi)$ when $\alpha \in (0, 1)$.

Hence the condition $\chi < \infty$ in Corollary 1.9 is satisfied exactly for $\alpha \in (0, \frac{1}{2})$ and L arbitrary, and for $\alpha = \frac{1}{2}$ and $\sum_{n \in \mathbb{N}} n^{-1}[L(n)]^{-2} < \infty$. This fits precisely with cases (3) and (4) in Theorem 1.12.

REMARK 1.21. Corollary 1.8(ii) is essentially Corollary 3.2 in Toninelli [21], where the condition for relevance, $h(\mu_\beta | \mu_0) > h(K)$, is given in an equivalent form (see equation (3.6) in [21]). Note that, by (1.2), $h(K) < \infty$ when $\alpha \in (0, \infty)$.

1.4. *Outline.* In Section 2 we formulate the annealed and the quenched large deviation principles (LDP) that are in Birkner, Greven and den Hollander [6], which are the key tools in the present paper. In Section 3 we use these LDP's to prove Theorem 1.4. In Section 4 we compare the variational formulas for the two critical curves and prove the criterion for disorder relevance stated in Theorem 1.5.
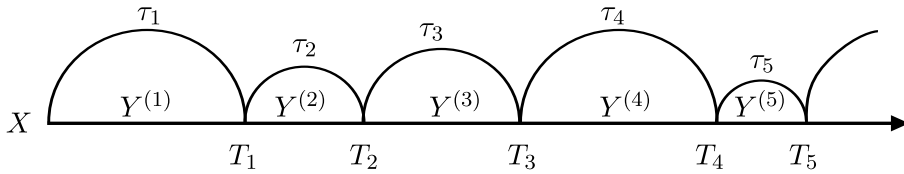
FIG. 4.   *Cutting words out from a sequence of letters according to renewal times.*

In Section 5 we reformulate this criterion to put it into a form, that is, more convenient for computations. In Section 6 we use the latter to prove Theorem 1.6. In Sections 7–8 we prove Corollaries 1.7–1.10. Appendix A collects a few standard facts about the homopolymer, while Appendix B provides the details of the proof of a key lemma in Section 3 based on an approximation argument in [6].

2. Annealed and quenched LDP.   In this section we recall the main results from Birkner, Greven and den Hollander [6] that are needed in the present paper. Section 2.1 introduces the relevant notation, while Sections 2.2 and 2.3 state the relevant annealed and quenched LDP's.

2.1. *Notation.*   Let E be a Polish space, playing the role of an alphabet, that is, a set of *letters*. Let $\widetilde{E} := \bigcup_{k \in \mathbb{N}} E^k$ be the set of *finite words* drawn from E, which can be metrized to become a Polish space.

Fix $\mu_0 \in \mathcal{P}(E)$, and $K \in \mathcal{P}(\mathbb{N})$ satisfying (1.2). Let $X = (X_k)_{k \in \mathbb{N}_0}$ be i.i.d. E-valued random variables with marginal law μ_0, and $\tau = (\tau_i)_{i \in \mathbb{N}}$ i.i.d. $\mathbb{N}$-valued random variables with marginal law K. Assume that X and τ are independent, and write $\mathbf{P}^*$ to denote their joint law. Cut words out of the letter sequence X according to τ (see Figure 4), that is, put

$$(2.1) \qquad T_0 := 0 \quad \text{and} \quad T_i := T_{i-1} + \tau_i, \qquad i \in \mathbb{N},$$

and let

$$(2.2) \qquad Y^{(i)} := (X_{T_{i-1}}, X_{T_{i-1}+1}, \ldots, X_{T_i - 1}), \qquad i \in \mathbb{N}.$$

Under the law $\mathbf{P}^*$, $Y = (Y^{(i)})_{i \in \mathbb{N}}$ is an i.i.d. sequence of words with marginal distribution q_0 on $\widetilde{E}$ given by

$$\begin{aligned}
& \mathrm{d}q_0(x_1, \ldots, x_n) \\
(2.3) \qquad & := \mathbf{P}^*\big(Y^{(1)} \in (\mathrm{d}x_1, \ldots, \mathrm{d}x_n)\big) \\
& = K(n)\,\mathrm{d}\mu_0(x_1) \times \cdots \times \mathrm{d}\mu_0(x_n), \qquad n \in \mathbb{N}, x_1, \ldots, x_n \in E.
\end{aligned}$$

The reverse operation of *cutting* words out of a sequence of letters is *glueing* words together into a sequence of letters. Formally, this is done by defining a *concatenation* map κ from $\widetilde{E}^{\mathbb{N}}$ to $E^{\mathbb{N}_0}$. This map induces in a natural way a map from $\mathcal{P}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}(E^{\mathbb{N}_0})$, the sets of probability measures on $\widetilde{E}^{\mathbb{N}}$ and $E^{\mathbb{N}_0}$ (endowed with the topology of weak convergence). The concatenation $q_0^{\otimes \mathbb{N}} \circ \kappa^{-1}$ of $q_0^{\otimes \mathbb{N}}$ equals $\mu_0^{\mathbb{N}_0}$, as is evident from (2.3)

2.2. *Annealed LDP.* Let $\mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$ be the set of probability measures on $\widetilde{E}^{\mathbb{N}}$ that are invariant under the left-shift $\widetilde{\theta}$ acting on $\widetilde{E}^{\mathbb{N}}$. For $N \in \mathbb{N}$, let $(Y^{(1)}, \ldots, Y^{(N)})^{\text{per}}$ be the periodic extension of the N-tuple $(Y^{(1)}, \ldots, Y^{(N)}) \in \widetilde{E}^N$ to an element of $\widetilde{E}^{\mathbb{N}}$, and define

$$(2.4) \qquad R_N := \frac{1}{N} \sum_{i=0}^{N-1} \delta_{\widetilde{\theta}^i (Y^{(1)}, \ldots, Y^{(N)})^{\text{per}}} \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}).$$

This is the *empirical process of N-tuples of words*. The following *annealed LDP* is standard; see, for example, Dembo and Zeitouni [7], Section 6.5. For $Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$, let $H(Q|q_0^{\otimes \mathbb{N}})$ be the *specific relative entropy of Q w.r.t. $q_0^{\otimes \mathbb{N}}$* defined by

$$(2.5) \qquad H(Q|q_0^{\otimes \mathbb{N}}) := \lim_{N \to \infty} \frac{1}{N} h(\pi_N Q | \pi_N q_0^{\otimes \mathbb{N}}),$$

where $\pi_N Q \in \mathcal{P}(\widetilde{E}^N)$ denotes the projection of Q onto the first N words, $h(\cdot|\cdot)$ denotes relative entropy, and the limit is nondecreasing.

THEOREM 2.1. *The family $\mathbf{P}^*(R_N \in \cdot)$, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate N and with rate function I^{ann} given by*

$$(2.6) \qquad I^{\text{ann}}(Q) := H(Q|q_0^{\otimes \mathbb{N}}), \qquad Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}).$$

This rate function is lower semi-continuous, has compact level sets, has a unique zero at $q_0^{\otimes \mathbb{N}}$, and is affine.

2.3. *Quenched LDP.* To formulate the quenched analog of Theorem 2.1, we need some more notation. Let $\mathcal{P}^{\text{inv}}(E^{\mathbb{N}_0})$ be the set of probability measures on $E^{\mathbb{N}_0}$ that are invariant under the left-shift θ acting on $E^{\mathbb{N}_0}$. For $Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$ such that $m_Q := \mathbf{E}_Q(\tau_1) < \infty$ (where $\mathbf{E}_Q$ denotes expectation under the law Q and τ_1 is the length of the first word), define

$$(2.7) \qquad \Psi_Q := \frac{1}{m_Q} \mathbf{E}_Q \left(\sum_{k=0}^{\tau_1 - 1} \delta_{\theta^k \kappa(Y)} \right) \in \mathcal{P}^{\text{inv}}(E^{\mathbb{N}_0}).$$

Think of Ψ_Q as the shift-invariant version of $Q \circ \kappa^{-1}$ obtained after *randomizing* the location of the origin. This randomization is necessary because a shift-invariant Q in general does not give rise to a shift-invariant $Q \circ \kappa^{-1}$.

For tr $\in \mathbb{N}$, let $[\cdot]_{\text{tr}} : \widetilde{E} \to [\widetilde{E}]_{\text{tr}} = \bigcup_{n=1}^{\text{tr}} E^n$ denote the *truncation map* on words defined by

$$(2.8) \quad y = (x_1, \ldots, x_n) \mapsto [y]_{\text{tr}} := (x_1, \ldots, x_{n \wedge \text{tr}}), \qquad n \in \mathbb{N}, x_1, \ldots, x_n \in E,$$

that is, $[y]_{\text{tr}}$ is the word of length $\le$ tr obtained from the word y by dropping all the letters with label $>$ tr. This map induces in a natural way a map from $\widetilde{E}^{\mathbb{N}}$ to

$[\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}}$, and from $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}^{\mathrm{inv}}([\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}})$. Note that if $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, then $[Q]_{\mathrm{tr}}$ is an element of the set

$$(2.9) \qquad \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}) = \{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) : m_Q < \infty\}.$$

THEOREM 2.2. (*Birkner, Greven and den Hollander* [6]) *Assume* (1.2). *Then, for* $\mu_0^{\otimes \mathbb{N}_0}$-*a.s. all* X, *the family of* (*regular*) *conditional probability distributions* $\mathbf{P}^*(R_N \in \cdot | X)$, $N \in \mathbb{N}$, *satisfies the LDP on* $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ *with rate* N *and with deterministic rate function* I^{que} *given by*

$$(2.10) \qquad I^{\mathrm{que}}(Q) := \begin{cases} I^{\mathrm{fin}}(Q), & \text{if } Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}), \\ \lim_{\mathrm{tr} \to \infty} I^{\mathrm{fin}}([Q]_{\mathrm{tr}}), & \text{otherwise}, \end{cases}$$

where

$$(2.11) \qquad I^{\mathrm{fin}}(Q) := H(Q | q_0^{\otimes \mathbb{N}}) + \alpha m_Q H(\Psi_Q | \mu_0^{\otimes \mathbb{N}_0}).$$

This rate function is lower semi-continuous, has compact level sets, has a unique zero at $q_0^{\otimes \mathbb{N}}$ *and is affine.*

There is no closed form expression for $I^{\mathrm{que}}(Q)$ when $m_Q = \infty$. For later reference we remark that, for all $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$,

$$(2.12) \qquad \begin{aligned} I^{\mathrm{ann}}(Q) &= \lim_{\mathrm{tr} \to \infty} I^{\mathrm{ann}}([Q]_{\mathrm{tr}}) = \sup_{\mathrm{tr} \in \mathbb{N}} I^{\mathrm{ann}}([Q]_{\mathrm{tr}}), \\ I^{\mathrm{que}}(Q) &= \lim_{\mathrm{tr} \to \infty} I^{\mathrm{que}}([Q]_{\mathrm{tr}}) = \sup_{\mathrm{tr} \in \mathbb{N}} I^{\mathrm{que}}([Q]_{\mathrm{tr}}) \end{aligned}$$

as shown in [6], Lemma A.1. A remarkable aspect of (2.11) in relation to (2.6) is that it *quantifies* the difference between I^{que} and I^{ann}. Note the explicit appearance of the tail exponent α. Also note that $I^{\mathrm{que}} = I^{\mathrm{ann}}$ when $\alpha = 0$.

3. Variational formulas: Proof of Theorem 1.4. In Section 3.1 we prove (1.16), the variational formula for the annealed critical curve. The proof of (1.15) in Sections 3.2–3.4, the variational formula for the quenched critical curve, is longer. In Section 3.2 we first give the proof for μ_0 with finite support. In Section 3.3 we extend the proof to μ_0 satisfying (1.3). In Section 3.4 we prove three technical lemmas that are needed in Section 3.3.

3.1. *Proof of* (1.16).

PROOF. Recall from (1.17) and (1.18) that $Q_\beta = q_\beta^{\otimes \mathbb{N}}$, and from (1.11) that $h_c^{\mathrm{ann}}(\beta) = \log M(\beta)$. Below we show that for every $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$,

$$(3.1) \qquad \beta \Phi(Q) - I^{\mathrm{ann}}(Q) = \log M(\beta) - H(Q | Q_\beta).$$

Taking the supremum over Q, we arrive at (1.16). Note that the unique probability measure that achieves the supremum in (3.1) is Q_β, which is an element of the set $\mathcal{C}$ defined in (1.13) because of (1.3).

To get (3.1), note that $H(Q|Q_\beta)$ is the limit as $N \to \infty$ of [recall (1.17) and (1.18)]

$$\frac{1}{N} \int_{\widetilde{E}^N} \log\left[\frac{d(\pi_N Q)}{d(\pi_N Q_\beta)}(y_1, \ldots, y_N)\right] d(\pi_N Q)(y_1, \ldots, y_N)$$

$$= \frac{1}{N} \int_{\widetilde{E}^N} \log\left[\frac{d(\pi_N Q)}{d(\pi_N Q_0)}(y_1, \ldots, y_N)\right.$$

(3.2)
$$\left. \times \frac{M(\beta)^N}{e^{\beta[c(y_1)+\cdots+c(y_N)]}}\right] d(\pi_N Q)(y_1, \ldots, y_N)$$

$$= \log M(\beta) + \frac{1}{N} h(\pi_N Q | \pi_N Q_0)$$

$$- \beta \frac{1}{N} \int_{\widetilde{E}^N} [c(y_1) + \cdots + c(y_N)] d(\pi_N Q)(y_1, \ldots, y_N),$$

where, $c(y)$ denotes the first letter of the word y. In the last line of (3.2), the limit as $N \to \infty$ of the second quantity is $H(Q|Q_0) = I^{\mathrm{ann}}(Q)$, while the integral equals $N\Phi(Q)$ by shift-invariance of Q. Thus, (3.1) follows. □

3.2. *Proof of* (1.15) *for* μ_0 *with finite support.*

PROOF.   The proof comes in three steps.

Step 1: An alternative way to compute the quenched free energy $f^{\mathrm{que}}(\beta, h)$ from (1.5) is through the radius of convergence $z^{\mathrm{que}}(\beta, h)$ of the power series

$$(3.3) \qquad \sum_{n \in \mathbb{N}} z^n Z_n^{\beta,h,\omega},$$

because

$$(3.4) \qquad z^{\mathrm{que}}(\beta, h) = e^{-f^{\mathrm{que}}(\beta,h)}.$$

Write

$$(3.5) \qquad Z_n^{\beta,h,\omega} = \sum_{N \in \mathbb{N}} \sum_{0=k_0<k_1<\cdots<k_N=n} \prod_{i=1}^N K(k_i - k_{i-1}) e^{\beta\omega_{k_{i-1}}-h},$$

so that, for $z \in (0, \infty)$,

$$(3.6) \qquad \sum_{n \in \mathbb{N}} z^n Z_n^{\beta,h,\omega} = \sum_{N \in \mathbb{N}} F_N^{\beta,h,\omega}(z),$$

where we abbreviate

$$(3.7) \qquad F_N^{\beta,h,\omega}(z) := \sum_{0=k_0<\cdots<k_N<\infty} \prod_{i=1}^N z^{k_i-k_{i-1}} K(k_i - k_{i-1}) e^{\beta\omega_{k_{i-1}}-h}.4$$

Step 2: We return to the setting of Section 2. The letter space is E, the word space is $\widetilde{E} = \bigcup_{k \in \mathbb{N}} E^k$, the sequence of letters is $\omega = (\omega_k)_{k \in \mathbb{N}_0}$, while the sequence of renewal times is $(T_i)_{i \in \mathbb{N}_0} = (k_i)_{i \in \mathbb{N}_0}$. Each interval $I_i := [k_{i-1}, k_i)$ of integers cuts out a word $\omega_{I_i} := (\omega_{k_{i-1}}, \ldots, \omega_{k_i-1})$. Let

$$(3.8) \qquad R_N^\omega = R_N^\omega((k_i)_{i=0}^N) := \frac{1}{N} \sum_{i=0}^{N-1} \delta_{\widetilde{\theta}^i(\omega_{I_1}, \ldots, \omega_{I_N})^{\mathrm{per}}}$$

denote the empirical process of N-tuples of words in ω cut out by the first N renewals. Then we can rewrite $F_N^{\beta, h, \omega}(z)$ as

$$
\begin{aligned}
(3.9) \quad F_N^{\beta, h, \omega}(z) &= \mathbf{E}\left(\exp\left[N \int_{\widetilde{E}} \{\tau(y) \log z + (\beta c(y) - h)\} \, d(\pi_1 R_N^\omega)(y) \right] \right) \\
&= e^{-Nh} \mathbf{E}(\exp[Nm_{R_N^\omega} \log z + N\beta \Phi(R_N^\omega)]),
\end{aligned}
$$

where $\tau(y)$ and $c(y)$ are the length, respectively, the first letter of the word y, $\pi_1 R_N^\omega$ is the projection of R_N^ω onto the first word, while $m_{R_N^\omega}$ and $\Phi(R_N^\omega)$ are the average word length, respectively, the average first letter of the first word under R_N^ω.

To identify the radius of convergence of the series in the left-hand side of (3.6), we apply the root test for the series in the right-hand side of (3.6) using the expression in (3.9). To that end, let

$$(3.10) \quad S^{\mathrm{que}}(\beta; z) := \limsup_{N \to \infty} \frac{1}{N} \log \mathbf{E}(\exp[Nm_{R_N^\omega} \log z + N\beta \Phi(R_N^\omega)]).$$

Then

$$(3.11) \qquad \limsup_{N \to \infty} \frac{1}{N} \log F_N^{\beta, h, \omega}(z) = -h + S^{\mathrm{que}}(\beta; z).$$

We know from (3.4) and the nonnegativity of $f^{\mathrm{que}}(\beta, h)$ that $z^{\mathrm{que}}(\beta, h) \leq 1$, and we are interested in knowing when it is < 1, respectively, $= 1$ [recall (1.6)]. Hence, the sign of the right-hand side of (3.11) for $z \uparrow 1$ will be important as the next lemma shows.

LEMMA 3.1. *For all* $\beta \in [0, \infty)$ *and* $h \in \mathbb{R}$,

$$
\begin{aligned}
(3.12) \qquad S^{\mathrm{que}}(\beta; 1-) < h &\implies f(\beta, h) = 0, \\
S^{\mathrm{que}}(\beta; 1-) > h &\implies f(\beta, h) > 0.
\end{aligned}
$$

PROOF. The first line holds because, by (3.11), $-h + S^{\mathrm{que}}(\beta; 1-) < 0$ implies that the sums in (3.6) converge for $|z| < 1$, so that $z^{\mathrm{que}}(\beta, h) \geq 1$, which gives $f^{\mathrm{que}}(\beta, h) \leq 0$. The second line holds because if $-h + S^{\mathrm{que}}(\beta; 1-) > 0$, then there exists a $z_0 < 1$ such that $-h + S^{\mathrm{que}}(\beta; z_0) > 0$, which implies that the sums in (3.6) diverge for $z = z_0$, so that $z^{\mathrm{que}}(\beta, h) \leq z_0 < 1$, which gives $f^{\mathrm{que}}(\beta, h) > 0$. □
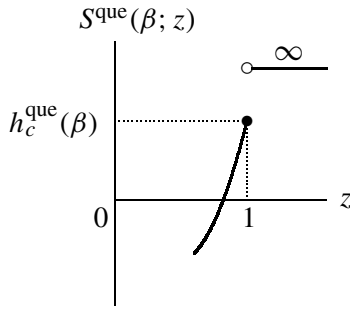
FIG. 5. *Qualitative plot of $z \mapsto S^{\mathrm{que}}(\beta; z)$.*

Lemma 3.1 implies that

$$(3.13) \qquad h_c^{\mathrm{que}}(\beta) = S^{\mathrm{que}}(\beta; 1-).$$

The rest of the proof is devoted to computing $S^{\mathrm{que}}(\beta; 1-)$.

Step 3: Since μ_0 has finite support, $Q \mapsto \Phi(Q)$ is continuous. Therefore we can apply Varadhan's lemma to the expression in (3.10) for $z = 1$ using the LDP of Theorem 2.2. This gives

$$(3.14) \qquad S^{\mathrm{que}}(\beta; 1) = \sup_{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})} [\beta \Phi(Q) - I^{\mathrm{que}}(Q)].$$

We would like to do the same for (3.10) with $z < 1$, and subsequently take the limit $z \uparrow 1$, to get (see Figure 5)

$$(3.15) \qquad S^{\mathrm{que}}(\beta; 1-) = \sup_{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})} [\beta \Phi(Q) - I^{\mathrm{que}}(Q)].$$

However, even though $Q \mapsto \Phi(Q)$ is continuous (because μ_0 has finite support), $Q \mapsto m_Q$ is only lower semicontinuous. Therefore we proceed by first showing that the term $N m_{R_N^\omega} \log z$ in (3.10) is harmless in the limit as $z \uparrow 1$.

LEMMA 3.2. $S^{\mathrm{que}}(\beta; 1-) = S^{\mathrm{que}}(\beta; 1)$ *for all* $\beta \in [0, \infty)$.

PROOF. Since $S^{\mathrm{que}}(\beta; 1-) \leq S^{\mathrm{que}}(\beta; 1)$, we need only prove the reverse inequality. The idea is to show that, for any $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ and in the limit as $N \to \infty$, R_N^ω can be arbitrarily close to Q with probability $\approx \exp[-N I^{\mathrm{que}}(Q)]$ while $m_{R_N^\omega}$ remains bounded by a large constant. Therefore, letting $N \to \infty$ followed by $z \uparrow 1$, we can remove the term $N m_{R_N^\omega} \log z$ in (3.10). The details are given in Appendix B. □

Combining Lemma 3.2 with (3.13) and (3.14), we obtain (1.15). □

3.3. *Proof of* (1.15) *for* μ_0 *satisfying* (1.3). The proof stays the same up to (3.13). Henceforth write $\mathcal{C} = \mathcal{C}(\mu_0)$ to exhibit the fact that the set $\mathcal{C}$ in (1.13) depends on μ_0 via its support E in (1.12), and define

$$(3.16) \qquad A(\beta) := \sup_{Q \in \mathcal{C}(\mu_0)} [\beta \Phi(Q) - I^{\text{que}}(Q)],$$

which replaces the right-hand side of (3.15). We will show the following.

LEMMA 3.3. $S^{\text{que}}(\beta; 1-) = A(\beta)$ *for all* $\beta \in (0, \infty)$.

PROOF. The proof of the lemma is accomplished in four steps. Along the way we use three technical lemmas, the proof of which is deferred to Section 3.4. Our starting point is the validity of the claim for μ_0 with finite support obtained in Lemma 3.2. (Note that $|E| < \infty$ implies $\mathcal{C} = \mathcal{C}(\mu_0) = \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$.)

Step 1: $S^{\text{que}}(\beta; 1-) \leq A(\beta)$ for all $\beta \in (0, \infty)$ when μ_0 satisfies (1.3).

PROOF. We have $S^{\text{que}}(\beta; 1-) \leq S^{\text{que}}(\beta; 1)$. We will show that $S^{\text{que}}(\beta; 1) \leq A(p\beta)/p$ for all $p > 1$. Taking $p \downarrow 1$ and using the continuity of A, proven in Lemma 3.4 below, we get the claim.

For $M > 0$, let

$$(3.17) \qquad \Phi^M(Q) := \int_E (x \wedge M) \, d(\pi_{1,1} Q)(x).$$

Then, for any $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, we have

$$\mathbf{E}(e^{N\beta\Phi(R_N^\omega)}) = \mathbf{E}\big(e^{\beta \sum_{i=1}^N c(y_i) 1_{\{c(y_i) \leq M\}}} e^{\beta \sum_{i=1}^N c(y_i) 1_{\{c(y_i) > M\}}}\big)$$

$$(3.18) \qquad \leq \big[\mathbf{E}\big(e^{p\beta \sum_{i=1}^N c(y_i) 1_{\{c(y_i) \leq M\}}}\big)\big]^{1/p} \big[\mathbf{E}\big(e^{q\beta \sum_{i=1}^N c(y_i) 1_{\{c(y_i) > M\}}}\big)\big]^{1/q}$$

$$\leq \big[\mathbf{E}\big(e^{Np\beta\Phi^M(R_N^\omega)}\big)\big]^{1/p} \big[\mathbf{E}\big(e^{q\beta \sum_{i=1}^N c(y_i) 1_{\{c(y_i) > M\}}}\big)\big]^{1/q},$$

where $y_1, \ldots, y_N$ are the N words determining R_N^ω and $c(y_i)$ is the first letter of the ith word. Hence

$$(3.19) \qquad \begin{aligned} \frac{1}{N} \log \mathbf{E}\big(e^{N\beta\Phi(R_N^\omega)}\big) &\leq \frac{1}{p} \frac{1}{N} \log \mathbf{E}\big(e^{Np\beta\Phi^M(R_N^\omega)}\big) \\ &\quad + \frac{1}{q} \frac{1}{N} \log \mathbf{E}\big(e^{q\beta \sum_{i=1}^N c(y_i) 1_{\{c(y_i) > M\}}}\big). \end{aligned}$$

Since $Q \mapsto \Phi^M(Q)$ is upper semicontinuous, Varadhan's lemma gives

$$(3.20) \quad \limsup_{N \to \infty} \frac{1}{N} \log \mathbf{E}\big(e^{Np\beta\Phi^M(R_N^\omega)}\big) \leq \sup_{Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})} [p\beta\Phi^M(Q) - I^{\text{que}}(Q)].$$

Clearly, Q's with $\int_E (x \wedge 0) \, d(\pi_{1,1} Q)(x) = -\infty$ do not contribute to the supremum. Also, Q's with $\int_E (x \vee 0) \, d(\pi_{1,1} Q)(x) = \infty$ do not contribute, because for

such Q we have $I^{\text{que}}(Q) = \infty$, by Lemma 3.5 below, and $\Phi^M(Q) < \infty$. Since $\Phi^M \le \Phi$, we therefore have

$$
\sup_{Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})} [p\beta \Phi^M(Q) - I^{\text{que}}(Q)] \le \sup_{Q \in \mathcal{C}(\mu_0)} [p\beta \Phi(Q) - I^{\text{que}}(Q)]
$$

(3.21)
$$
= A(p\beta).
$$

Next, we use the following observation. For any sequence $\Theta = (\Theta_N)_{N \in \mathbb{N}}$ of positive random variables on a space with probability measure $\mathbb{P}$, we have

$$
(3.22) \qquad \limsup_{N \to \infty} \frac{1}{N} \log \Theta_N \le \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}(\Theta_N) \qquad \mathbb{P}\text{-a.s.},
$$

by the first Borel–Cantelli lemma. Applying this to

$$
(3.23) \qquad\qquad \Theta_N := \mathbf{E}\big(e^{q\beta \sum_{i=1}^N c(y_i) 1_{\{c(y_i) > M\}}}\big)
$$

$$
\text{with } \mathbb{E}(\Theta_N) = \left(\int_E e^{q\beta x 1_{\{x > M\}}} \, d\mu_0(x) \right)^N =: (c_M)^N,
$$

we get, after letting $N \to \infty$ in (3.19),

$$
(3.24) \qquad\qquad S^{\text{que}}(\beta; 1) \le \frac{1}{p} A(p\beta) + \frac{1}{q} \log c_M.
$$

By (1.3), we have $c_M < \infty$ for all $M > 0$ and $\lim_{M \to \infty} c_M = 1$. Hence $S^{\text{que}}(\beta; 1) \le A(p\beta)/p$. $\square$

Step 2: $S^{\text{que}}(\beta; 1-) \ge A(\beta)$ for all $\beta \in (0, \infty)$ when μ_0 has bounded support.

PROOF.   In the estimates below, we abbreviate

$$
(3.25) \qquad\qquad\qquad L_N^\omega := N m_{R_N^\omega},
$$

the sum of the lengths of the first N words. The proof is based on a discretization argument similar to the one used in [6], Section 8. For $\delta > 0$ and $x \in E$, let $\langle x \rangle_\delta := \sup\{k\delta : k \in \mathbb{Z}, k\delta \le x\}$. The operation $\langle \cdot \rangle$ extends to measures on E, $\widetilde{E}$ and $\widetilde{E}^{\mathbb{N}}$ in the obvious way. Now, $\langle R_N^\omega \rangle_\delta$ satisfies the quenched LDP with rate function I_δ^{que}, the quenched rate function corresponding to the measure $\langle \mu_0 \rangle_\delta$. Clearly,

$$
(3.26) \qquad \mathbf{E}\big(e^{L_N^\omega \log z + N\beta \Phi(R_N^\omega)}\big) \ge \mathbf{E}\big(e^{L_N^\omega \log z + N\beta \Phi(\langle R_N^\omega \rangle_\delta)}\big),
$$

and so, by the results in Section 3.2, we have

$$
(3.27) \qquad S^{\text{que}}(\beta; 1-) \ge \sup_{Q \in \mathcal{C}(\langle \mu_0 \rangle_\delta)} [\beta \Phi(Q) - I_\delta^{\text{que}}(Q)].
$$

For every $Q \in \mathcal{C}(\mu_0)$, we have

$$
(3.28) \qquad \Phi(Q) = \lim_{\delta \downarrow 0} \Phi(\langle Q \rangle_\delta), \qquad I^{\text{que}}(Q) = \lim_{n \to \infty} I_{\delta_n}^{\text{que}}(\langle Q \rangle_{\delta_n}),
$$

where $\delta_n = 2^{-n}$. The first relation holds because $\Phi(\langle Q \rangle_\delta) \leq \Phi(Q) \leq \Phi(\langle Q \rangle_\delta) + \delta$, the second relation uses Lemma 3.6(i) below. Hence the claim follows by picking $\delta = \delta_n$ in (3.27) and letting $n \to \infty$. $\quad\square$

Step 3: $S^{\text{que}}(\beta; 1-) \geq A(\beta)$ for all $\beta \in (0, \infty)$ when μ_0 satisfies (1.3) with support bounded from below.

PROOF. For $M > 0$ and $x \in E$, let $x^M = x \wedge M$. This truncation operation acts on μ_0 by moving the mass in (M, ∞) to M, resulting in a measure μ_0^M with bounded support and with associated quenched rate function $I^{\text{que}, M}$. Let $R_N^{\omega, M}$ be the empirical process of N-tuples of words obtained from R_N^ω defined in (2.4) after replacing each letter $x \in E$ by x^M. We have

$$(3.29) \qquad \mathbf{E}\big(e^{L_N^\omega \log z + N\beta \Phi(R_N^\omega)}\big) \geq \mathbf{E}\big(e^{L_N^\omega \log z + N\beta \Phi(R_N^{\omega, M})}\big).$$

Combined with the result in Step 2, this bound implies that

$$(3.30) \qquad S(\beta; 1-) \geq \sup_{Q' \in \mathcal{C}(\mu_0^M)} [\beta \Phi(Q') - I^{\text{que}, M}(Q')].$$

For every $Q \in \mathcal{C}(\mu_0)$, we have

$$(3.31) \qquad \begin{aligned} \Phi(Q) &= \lim_{M \to \infty} \Phi(Q^M) = \lim_{M \to \infty} \int_E (x \wedge M) \, \mathrm{d}(\pi_{1,1} Q)(x), \\ I^{\text{que}}(Q) &= \lim_{M \to \infty} I^{\text{que}, M}(Q^M). \end{aligned}$$

The first relation holds by dominated convergence, and the second relation uses Lemma 3.6(ii) below. It follows from (3.31) that

$$(3.32) \qquad \begin{aligned} &\limsup_{M \to \infty} \sup_{Q' \in \mathcal{C}(\mu_0^M)} [\beta \Phi(Q') - I^{\text{que}, M}(Q')] \geq \beta \Phi(Q) - I^{\text{que}}(Q) \\ &\hspace{8cm} \forall Q \in \mathcal{C}(\mu_0), \end{aligned}$$

which combined with (3.30) yields

$$(3.33) \qquad S(\beta; 1-) \geq \beta \Phi(Q) - I^{\text{que}}(Q) \qquad \forall Q \in \mathcal{C}(\mu_0).$$

Take the supremum over $Q \in \mathcal{C}(\mu_0)$ to get the claim. $\quad\square$

Step 4: $S^{\text{que}}(\beta; 1-) \geq A(\beta)$ for all $\beta \in (0, \infty)$ when μ_0 satisfies (1.3).

PROOF. For $M > 0$ and $x \in E$, let $x^{-M} = x \vee (-M)$. This truncation operation acts on μ_0 by moving the mass in $(-\infty, -M)$ to $-M$, resulting in a measure μ_0^{-M} with support bounded from below and with associated quenched rate function $I^{\text{que}, -M}$. Let $R_N^{\omega, -M}$ be the empirical process of N-tuples of words obtained from R_N^ω defined in (2.4) after replacing each letter $x \in E$ by x^{-M}.

As in Step 1, for any $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, we have

$$\mathbf{E}\big(e^{L_N^\omega \log z + N\beta\Phi(R_N^{\omega,-M})}\big)$$

$$(3.34) \qquad \leq \mathbf{E}\big(e^{L_N^\omega \log z + N\beta\Phi(R_N^\omega)}e^{-\beta\sum_{i=1}^N c(y_i)1_{\{c(y_i)<-M\}}}\big)$$

$$\leq \big[\mathbf{E}\big(e^{pL_N^\omega \log z + Np\beta\Phi(R_N^\omega)}\big)\big]^{1/p}\big[\mathbf{E}\big(e^{-q\beta\sum_{i=1}^N c(y_i)1_{\{c(y_i)<-M\}}}\big)\big]^{1/q},$$

and hence

$$\frac{1}{N}\log\mathbf{E}\big(e^{L_N^\omega \log z + N\beta\Phi(R_N^{\omega,-M})}\big)$$

$$(3.35) \qquad\qquad \leq \frac{1}{p}\frac{1}{N}\log\mathbf{E}\big(e^{pL_N^\omega \log z + Np\beta\Phi(R_N^\omega)}\big)$$

$$+ \frac{1}{q}\frac{1}{N}\log\mathbf{E}\big(e^{-q\beta\sum_{i=1}^N c(y_i)1_{\{c(y_i)<-M\}}}\big).$$

Let $N \to \infty$ followed by $z \uparrow 1$. For the left-hand side, we have the lower bound in Step 3, while the second term in the right-hand side can be handled as in (3.22–3.24). Therefore, recalling (3.10) and writing $p \log z = \log z^p$, we get

$$\sup_{Q \in \mathcal{C}(\mu_0^{-M})}[\beta\Phi(Q) - I^{\text{que},-M}(Q)] \leq \frac{1}{p}S^{\text{que}}(p\beta; 1-) + \frac{1}{q}\log C_{-M}$$

$$(3.36)$$

$$\text{with } C_{-M} := \int_E e^{-q\beta x 1_{\{x<-M\}}}\,d\mu_0(x).$$

Letting $M \to \infty$ and using that $\lim_{M\to\infty} C_{-M} = 1$ by (1.3), we arrive at

$$(3.37) \quad \frac{1}{p}S^{\text{que}}(p\beta, 1-) \geq \limsup_{M\to\infty}\sup_{Q\in\mathcal{C}(\mu_0^{-M})}[\beta\Phi(Q) - I^{\text{que},-M}(Q)] \geq A(\beta),$$

where the last inequality is obtained via arguments similar to those following (3.30), which require the use of Lemma 3.6(iii) below. Finally, let $p \downarrow 1$, and use the continuity of $\beta \mapsto S(\beta; 1-)$, proven in Lemma 3.4 below. $\square$

This completes the proof of Lemma 3.3 and hence of Theorem 1.4. $\square$

3.4. *Technical lemmas.* In the proof of Lemma 3.3 we used three technical lemmas, which we prove in this section.

LEMMA 3.4. $\beta \mapsto A(\beta)$ and $\beta \mapsto S^{\text{que}}(\beta; 1-)$ *are finite and convex on* $[0, \infty)$ *and, consequently, are continuous on* $(0, \infty)$.

PROOF. For the first function, note that $A(\beta) \leq \sup_{Q\in\mathcal{C}(\mu_0)}[\beta\Phi(Q) - I^{\text{ann}}(Q)] \leq \log M(\beta) < \infty$ by (1.3) and (3.1), and convexity follows from the fact that A is a supremum of linear functions. For the second function, note that $S^{\text{que}}(\beta; 1-) \leq S^{\text{que}}(\beta; 1) = A(\beta)$, and convexity follows from Hölder's inequality. $\square$

LEMMA 3.5.   *If $\mu, \nu \in P(\mathbb{R})$ satisfy $h(\mu|\nu) < \infty$ and $\int_E e^{\lambda x} d\nu(x) < \infty$ for some $\lambda > 0$, then $\int_E (x \vee 0) d\mu(x) < \infty$.*

PROOF.   The claim follows from the inequality

$$(3.38) \qquad \int_E f \, d\mu \le h(\mu|\nu) + \log \int_E e^f \, d\nu,$$

which is valid for all bounded and measurable f (see Dembo and Zeitouni [7], Lemma 6.2.13) and, by monotone convergence, extends to measurable $f \ge 0$. Pick $f(x) = \lambda(x \vee 0)$, $x \in E$.   □

LEMMA 3.6.   *For every $Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$:*

(i) $\lim_{n \to \infty} I^{\text{que}}_{\delta_n}(\langle Q \rangle_{\delta_n}) = I^{\text{que}}(Q)$ *with* $\delta_n := 2^{-n}$;
(ii) $\lim_{M \to \infty} I^{\text{que}, M}(Q^M) = I^{\text{que}}(Q)$;
(iii) $\lim_{M \to \infty} I^{\text{que}, -M}(Q^{-M}) = I^{\text{que}}(Q)$.

PROOF.   (i) The proof proceeds by choosing an appropriate function I: $[0, 1] \to \mathbb{R}$ and proving that:

$$\text{(a)} \qquad I(0) = \lim_{\delta \downarrow 0} I(\delta);$$

$$(3.39) \qquad \text{(b)} \qquad I(0) \ge I(\delta_1) \ge I(\delta_2)$$

$$\text{whenever } \delta_2 = k\delta_1 \in (0, 1) \text{ for some } k \in \mathbb{N}.$$

Recalling (2.10) and (2.11), we see that we need the following choices for I:

$$(1) \quad I(\delta) = \begin{cases} N^{-1} h(\langle \pi_N Q \rangle_\delta | \langle \pi_N q_0^{\otimes \mathbb{N}} \rangle_\delta), & \delta > 0, \\ N^{-1} h(\pi_N Q | \pi_N q_0^{\otimes \mathbb{N}}), & \delta = 0, \end{cases}$$

$$(2) \quad I(\delta) = \begin{cases} H(\langle Q \rangle_\delta | \langle q_0^{\otimes \mathbb{N}} \rangle_\delta), & \delta > 0, \\ H(Q | q_0^{\otimes \mathbb{N}}), & \delta = 0, \end{cases}$$

$$(3.40)$$

$$(3) \quad I(\delta) = \begin{cases} N^{-1} h(\langle \pi_N \Psi_Q \rangle_\delta | \langle \pi_N \mu_0^{\otimes \mathbb{N}_0} \rangle_\delta), & \delta > 0, \\ N^{-1} h(\pi_N \Psi_Q | \pi_N \mu_0^{\otimes \mathbb{N}_0}), & \delta = 0, \end{cases}$$

$$(4) \quad I(\delta) = \begin{cases} H(\langle \Psi_Q \rangle_\delta | \langle \mu_0^{\otimes \mathbb{N}_0} \rangle_\delta), & \delta > 0, \\ H(\Psi_Q | \mu_0^{\otimes \mathbb{N}_0}), & \delta = 0, \end{cases}$$

with $N \in \mathbb{N}$. It is clear from the definition of specific relative entropy [recall 2.5)] that if (a) and (b) hold for the choices (1) and (3), then they also hold for the choices (2) and (4), respectively. We will not actually prove (a) and (b) for the choices (1) and (3), but for the simpler choice

$$(3.41) \qquad I(\delta) = \begin{cases} h(\langle \mu \rangle_\delta | \langle \mu_0 \rangle_\delta), & \delta > 0, \\ h(\mu | \mu_0), & \delta = 0. \end{cases}$$

The proof will make it evident how to properly deal with (1) and (3).

Let $B(\mathbb{R})$ be the set of real-valued, bounded and Borel measurable functions on $\mathbb{R}$ and, for $\phi \in B(\mathbb{R})$ and $\delta > 0$, let ϕ_δ be the function defined by $\phi_\delta(x) := \phi(\langle x \rangle_\delta)$. As shown in Dembo and Zeitouni [7], Lemma 6.2.13, we have

$$
(3.42) \quad
\begin{aligned}
h(\langle \mu \rangle_\delta | \langle \mu_0 \rangle_\delta) &= \sup_{\phi \in B(\mathbb{R})} \left\{ \int_\mathbb{R} \phi \, d\langle \mu \rangle_\delta - \log \int_\mathbb{R} e^\phi \, d\langle \mu_0 \rangle_\delta \right\} \\
&= \sup_{\phi \in B(\mathbb{R})} \left\{ \int_\mathbb{R} \phi_\delta \, d\mu - \log \int_\mathbb{R} e^{\phi_\delta} \, d\mu_0 \right\}.
\end{aligned}
$$

From this representation, property (b) follows for the choice in (3.41). Next, fix any $\varepsilon > 0$ and take a ϕ such that $\int_\mathbb{R} \phi \, d\mu - \log \int_\mathbb{R} e^\phi \, d\mu_0 \geq h(\mu|\mu_0) - \varepsilon$. Then, since ϕ_δ converges pointwise to ϕ as $\delta \downarrow 0$, the bounded convergence theorem together with (3.42) give

$$
(3.43) \quad \liminf_{\delta \downarrow 0} h(\langle \mu \rangle_\delta | \langle \mu_0 \rangle_\delta) \geq h(\mu|\mu_0) - \varepsilon.
$$

Hence $\liminf_{\delta \downarrow 0} I(\delta) \geq I(0) - \varepsilon$. Since $I(0) \geq I(\delta)$, property (a) follows after letting $\varepsilon \downarrow 0$.

Having thus convinced ourselves that (3.39) and (3.40) are true, we now know that for any $Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^\mathbb{N})$ the sequences

$$
(3.44) \quad H(\langle Q \rangle_{\delta_n} | \langle q_0^{\otimes \mathbb{N}} \rangle_{\delta_n}), \qquad H(\langle \Psi_Q \rangle_{\delta_n} | \langle \mu_0^{\otimes \mathbb{N}_0} \rangle_{\delta_n}), \qquad n \in \mathbb{N},
$$

are increasing and converge to $H(Q|q_0^{\otimes \mathbb{N}})$, respectively, $H(\Psi_Q|\mu_0^{\otimes \mathbb{N}_0})$. This implies the claim for Q with $m_Q < \infty$ [recall (2.11)]. For Q with $m_Q = \infty$ we use that $I^{\text{que}}(Q) = \sup_{\text{tr} \in \mathbb{N}} I([Q]_{\text{tr}})$ [recall (2.12)], to conclude that $I^{\text{que}}_{\delta_n}(\langle Q \rangle_{\delta_n})$ is increasing and converges to $I^{\text{que}}(Q)$.

(ii)–(iii) The proof is similar as for (i). $\quad \square$

4. Characterization of disorder relevance: Proof of Theorem 1.5.

PROOF.   We will need the following lemma, the proof of which is postponed.

LEMMA 4.1.   *The supremum* $\sup_{Q \in \mathcal{C}}[\beta \Phi(Q) - I^{\text{que}}(Q)]$ *is attained for all* $\beta \in (0, \infty)$.

Let Q^* be a measure achieving the supremum in Lemma 4.1. Suppose that $h^{\text{que}}_c(\beta) = h^{\text{ann}}_c(\beta)$. Then

$$
(4.1) \quad
\begin{aligned}
h^{\text{que}}_c(\beta) &= \beta \Phi(Q^*) - I^{\text{que}}(Q^*) \leq \beta \Phi(Q^*) - I^{\text{ann}}(Q^*) \\
&\leq \beta \Phi(Q_\beta) - I^{\text{ann}}(Q_\beta) = h^{\text{ann}}_c(\beta) = h^{\text{que}}_c(\beta),
\end{aligned}
$$

where the second equality uses that Q_β achieves the supremum in (1.16) [with $I^{\text{ann}}(Q_\beta) < \infty$], as shown by (3.1). It follows that both inequalities in (4.1) are

equalities. However, since Q_β uniquely achieves the supremum in (1.16), we must have $Q^* = Q_\beta$ and therefore $I^{\text{que}}(Q_\beta) = I^{\text{ann}}(Q_\beta)$.

Conversely, suppose that $I^{\text{que}}(Q_\beta) = I^{\text{ann}}(Q_\beta)$. Then

$$(4.2) \quad h_c^{\text{que}}(\beta) \geq [\beta\Phi(Q_\beta) - I^{\text{que}}(Q_\beta)] = [\beta\Phi(Q_\beta) - I^{\text{ann}}(Q_\beta)] = h_c^{\text{ann}}(\beta).$$

Since $h_c^{\text{que}}(\beta) \leq h_c^{\text{ann}}(\beta)$, this proves that $h_c^{\text{que}}(\beta) = h_c^{\text{ann}}(\beta)$. $\quad\square$

We now give the proof of Lemma 4.1.

PROOF. The proof is accomplished in three steps. The claims in Steps 1 and 2 are obvious when the support of μ_0 is bounded from above, because then Φ is bounded from above and upper semicontinuous. Thus, for these steps we may assume that the support of μ_0 is unbounded from above.

Step 1: The supremum can be restricted to the set $\mathcal{C} \cap \{Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}) : I^{\text{que}}(Q) \leq \gamma\}$ for some $\gamma < \infty$.

PROOF. We first prove that

$$(4.3) \qquad\qquad \lim_{a\to\infty} \sup_{\substack{Q\in\mathcal{C}\\ \Phi(Q)=a}} [\beta\Phi(Q) - I^{\text{que}}(Q)] = -\infty.$$

To that end we estimate, for $a \in (0, \infty)$,

$$(4.4) \quad \begin{aligned} \sup_{\substack{Q\in\mathcal{C}\\ \Phi(Q)=a}} [\beta\Phi(Q) - I^{\text{que}}(Q)] &\leq \sup_{\substack{Q\in\mathcal{C}\\ \Phi(Q)=a}} [\beta a - h(\pi_{1,1}Q|\mu_0)] \\ &= \sup_{\substack{\mu\in\mathcal{P}(E)\\ \int_E |x|\,d\mu(x)<\infty,\,\int_E x\,d\mu(x)=a}} [\beta a - h(\mu|\mu_0)], \end{aligned}$$

where we use that $I^{\text{que}}(Q) \geq I^{\text{ann}}(Q) = H(Q|Q_0) \geq h(\pi_{1,1}Q|\mu_0)$. The last supremum is achieved by a measure μ_λ of the form $d\mu_\lambda(x) = M(\lambda)^{-1}e^{\lambda x}\,d\mu_0(x)$, $x \in E$, with λ such that $\int_E x\,d\mu_\lambda(x) = a$ [recall (1.17)]. To see why, first note that such a $\lambda = \lambda(a)$ exists because $(\lambda \mapsto \int_E x\,d\mu_\lambda(x))$ is continuous with value 0 at $\lambda = 0$ and $\lim_{\lambda\to\infty} \int_E x\,d\mu_\lambda(x) = \sup[\text{supp}(\mu_0)] = w$, where $w = \infty$ by assumption. Next note that, for any other measure μ with $\int_E x\,d\mu(x) = a$, we have

$$(4.5) \quad h(\mu|\mu_\lambda) = h(\mu|\mu_0) - \lambda a + \log M(\lambda) = h(\mu|\mu_0) - h(\mu_\lambda|\mu_0),$$

which shows that $h(\mu|\mu_0) \geq h(\mu_\lambda|\mu_0)$ with equality if and only if $\mu = \mu_\lambda$. Consequently,

$$(4.6) \quad \begin{aligned} \sup_{\substack{\mu\in\mathcal{P}(E)\\ \int_E |x|\,d\mu(x)<\infty,\,\int_E x\,d\mu(x)=a}} [\beta a - h(\mu|\mu_0)] &= \beta\int_E x\,d\mu_\lambda(x) - h(\mu_\lambda|\mu_0) \\ &=: g(\lambda). \end{aligned}$$

Clearly, $a \to \infty$ implies $\lambda = \lambda(a) \to \infty$, and so to prove (4.3) we must show that $\lim_{\lambda \to \infty} g(\lambda) = -\infty$.

To achieve the latter, note that a lower bound on $h(\mu_\lambda | \mu_0)$ is obtained by applying (3.38) to $f(x) := \bar{\beta}(x \vee 0)$ for some $\bar{\beta} > \beta$. This yields

$$(4.7) \qquad g(\lambda) \leq -(\bar{\beta} - \beta) \int_E x \, d\mu_\lambda(x) + \log[M(\bar{\beta}) + 1].$$

The integral in the right-hand side tends to infinity as $\lambda \to \infty$, and so (4.3) indeed follows.

Finally, recall the definition of $A(\beta)$ in (3.16), which is finite because of Lemma 3.4. Then, by (4.3), there is an $a_0 < \infty$ such that

$$(4.8) \qquad \sup_{\substack{Q \in \mathcal{C} \\ \Phi(Q) = a}} [\beta \Phi(Q) - I^{\text{que}}(Q)] \leq A(\beta) - 1 \qquad \forall a \geq a_0,$$

and so all $Q \in \mathcal{C}$ with $\beta \Phi(Q) - I^{\text{que}}(Q) > A(\beta) - 1$ must satisfy $\Phi(Q) < a_0$ and $I^{\text{que}}(Q) < \beta \Phi(Q) + 1 - A(\beta) \leq \beta a_0 + 1 - A(\beta) =: \gamma$. Consequently, the supremum can be restricted to the set $\mathcal{C} \cap \{Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}) : I^{\text{que}}(Q) \leq \gamma\}$. $\quad\square$

Step 2: Φ is upper semicontinuous on $\{Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}) : I^{\text{que}}(Q) \leq \gamma\}$ for every $\gamma > 0$.

PROOF. From the definition of Φ and the inequality $h(\pi_{1,1} Q | \mu_0) \leq I^{\text{que}}(Q) \leq \gamma$, it follows that it is enough to show that the map $\mu \mapsto \Psi(\mu) := \int_E (5.16), (x \vee 0) \, d\mu(x)$ is upper semicontinuous on $K_\gamma := \{\mu \in \mathcal{P}(E) : h(\mu | \mu_0) \leq \gamma\}$. To do so, let $(\mu^M)_{M \in \mathbb{N}}$ be a sequence in K_γ converging to μ weakly as $M \to \infty$. Then

$$(4.9) \qquad \Psi(\mu^M) = \int_E [(x \vee 0) \wedge n] \, d\mu^M(x) + \int_E x \mathbf{1}_{\{x > n\}} \, d\mu^M(x),$$

and so

$$(4.10) \qquad \begin{aligned} \limsup_{M \to \infty} \Psi(\mu^M) &\leq \int_E [(x \vee 0) \wedge n] \, d\mu(x) \\ &\quad + \sup_{M \in \mathbb{N}} \int_E x \mathbf{1}_{\{x > n\}} \, d\mu^M(x) \qquad \forall n \in \mathbb{N}. \end{aligned}$$

By the inequality in (3.38), we have

$$(4.11) \qquad \lambda \int_E x \mathbf{1}_{\{x > n\}} \, d\mu^M(x) \leq h(\mu^M | \mu_0) + \log \int_E e^{\lambda x \mathbf{1}_{\{x > n\}}} \, d\mu_0(x) \qquad\qquad \forall M, n \in \mathbb{N}, \lambda > 0,$$

and so

$$(4.12) \qquad \sup_{M \in \mathbb{N}} \int_E x \mathbf{1}_{\{x > n\}} \, d\mu^M(x) \leq \frac{\gamma}{\lambda} + \frac{1}{\lambda} \log \int_E e^{\lambda x \mathbf{1}_{\{x > n\}}} \, d\mu_0(x).$$

By (1.3), the limit as $n \to \infty$ of the right-hand side is γ/λ. Since $\lambda > 0$ is arbitrary, we conclude that the limit as $n \to \infty$ of the left-hand side is zero. Letting $n \to \infty$ in (4.10) and using monotone convergence, we therefore get $\limsup_{M \to \infty} \Psi(\mu^M) \leq \Psi(\mu)$, as required. $\quad \square$

Step 3: Let $\Gamma(Q) := \beta \Phi(Q) - I^{\mathrm{que}}(Q)$. Then, by Step 1, we have that for some $\gamma > 0$,

$$(4.13) \qquad \sup_{Q \in \mathcal{C}} \Gamma(Q) = \sup_{\substack{Q \in \mathcal{C} \\ I^{\mathrm{que}}(Q) \leq \gamma}} \Gamma(Q) \leq \sup_{\substack{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \\ I^{\mathrm{que}}(Q) \leq \gamma}} \Gamma(Q).$$

By Theorem 2.2, I^{que} is lower semicontinuous. Hence, by Step 2, $\beta \Phi - I^{\mathrm{que}}$ is upper semicontinuous on the compact set $\{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) : I^{\mathrm{que}}(Q) \leq \gamma\}$, achieving its supremum at some Q^*. Let $\mu^* := \pi_{1,1} Q^*$. Then, by (1.3), the inequality in (3.38) gives

$$(4.14) \qquad \int_E (x \vee 0) \, \mathrm{d}\mu^*(x) \leq \gamma + \log \int_E \mathrm{e}^x \, \mathrm{d}\mu_0(x) < \infty,$$

and, since $\Phi(Q^*) > -\infty$, we also have $\int_E (x \wedge 0) \, \mathrm{d}\mu^*(x) > -\infty$, so that $Q^* \in \mathcal{C}$. Hence

$$(4.15) \qquad \sup_{Q \in \mathcal{C}} \Gamma(Q) = \sup_{\substack{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \\ I^{\mathrm{que}}(Q) \leq \gamma}} \Gamma(Q) = \Gamma(Q^*),$$

which completes the proof. $\quad \square$

5. Reformulation of the criterion for disorder relevance. Note that, by (2.10) and (2.12), for $\alpha > 0$, the necessary and sufficient condition for relevance, $I^{\mathrm{que}}(Q_\beta) > I^{\mathrm{ann}}(Q_\beta)$, in Theorem 1.5 translates into

$$(5.1) \qquad \lim_{\mathrm{tr} \to \infty} m_{[Q_\beta]_{\mathrm{tr}}} H\big(\Psi_{[Q_\beta]_{\mathrm{tr}}} | \mu_0^{\otimes \mathbb{N}_0}\big) > 0.$$

In Lemma 5.3 below, we give two alternative expressions for the specific relative entropy appearing in (5.1). These expressions will be needed in Sections 6 and 7.

I. Asymptotic mean stationarity. In what follows we will make use of the notion of *asymptotic mean stationarity* (see Gray [16], Section 1.7). Let A be a topological space and equip $A^{\mathbb{N}_0}$ with the product topology. A measure $\mathcal{P}$ on $A^{\mathbb{N}_0}$ is called *asymptotically mean stationary* if for every Borel measurable $G \subset A^{\mathbb{N}_0}$,

$$(5.2) \qquad \overline{\mathcal{P}}(G) := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{P}(\theta^{-k} G) \qquad \text{exists.}$$

As in Section 2, θ denotes the left-shift acting on $A^{\mathbb{N}_0}$. If $\mathcal{P}$ is asymptotically mean stationary, then $\overline{\mathcal{P}}$ is a stationary measure, called the *stationary mean* of $\mathcal{P}$.

For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, recall from Section 2.1 that $\kappa(Q) \in \mathcal{P}(E^{\mathbb{N}_0})$ is the probability measure induced by the concatenation map $\kappa : \widetilde{E}^{\mathbb{N}} \to E^{\mathbb{N}_0}$ that glues a sequence of words into a sequence of letters, that is, $\kappa(Q) = Q \circ \kappa^{-1}$. Our aim is to replace Ψ_Q in (5.1) by $\kappa(Q)$, which is not stationary but more convenient to work with. These two probability measures are related in the following way.

LEMMA 5.1. *If $m_Q < \infty$, then $\kappa(Q)$ is asymptotically mean stationary with stationary mean $\overline{\kappa(Q)} = \Psi_Q$.*

PROOF. Let $X := \kappa(Y) \in E^{\mathbb{N}_0}$, where Y is distributed according to Q. Let I denote the set of indices $i \in \mathbb{N}_0$ where a new word starts ($0 \in I$). For $i \in \mathbb{N}_0$, let $r_i := \inf\{j \in \mathbb{N} : i - j \in I\}$, that is, the distance from i to the beginning of the word it belongs to. For $j \in I$, let L^j denote the length of the word that starts at j. Then, for any $G \subset E^{\mathbb{N}_0}$ Borel measurable, we have

(5.3)
$$\sum_{i=0}^{n-1} \kappa(Q)(\theta^i X \in G) = \sum_{i=0}^{n-1} \sum_{k=0}^{i} Q(\theta^i X \in G, r_i = k)$$
$$= \sum_{k=0}^{n-1} \sum_{i=k}^{n-1} Q(\theta^i X \in G, r_i = k).$$

Next, note that

(5.4)
$$Q(\theta^i X \in G, r_i = k)$$
$$= Q(\theta^i X \in G, i - k \in I, L^{i-k} > k)$$
$$= Q(\theta^i X \in G, L^{i-k} > k | i - k \in I) Q(i - k \in I)$$
$$= Q(\theta^k X \in G, L^0 > k) Q(i - k \in I).$$

Hence, dividing the sum in (5.3) by n, we get

(5.5)
$$\frac{1}{n} \sum_{i=0}^{n-1} \kappa(Q)(\theta^i X \in G) = \sum_{k=0}^{n-1} Q(\theta^k X \in G, L^0 > k) f_{k,n},$$

where we abbreviate $f_{k,n} := n^{-1} \sum_{j=0}^{n-k-1} Q(j \in I)$. By the renewal theorem, $\lim_{n \to \infty} f_{k,n} = 1/m_Q$ for k fixed. Since

(5.6)
$$\sum_{k=0}^{\infty} Q(L^0 > k) = m_Q < \infty,$$

we can apply the bounded convergence theorem, and conclude that

$$\overline{\kappa(Q)}(G) = \frac{1}{m_Q} \sum_{k=0}^{\infty} Q(\theta^k X \in G, L^0 > k)$$

(5.7)
$$= \frac{1}{m_Q} \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} Q(\theta^k X \in G, L^0 = j)$$

$$= \frac{1}{m_Q} \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} Q(\theta^k X \in G, L^0 = j) = \Psi_Q(G).$$

The last equality is simply the definition of Ψ_Q in (2.7). $\quad\square$

To complement Lemma 5.1, we need the following fact stated in Birkner [5], Remark 5, where ergodicity refers to the left-shifts acting on $\widetilde{E}^{\mathbb{N}}$ and $E^{\mathbb{N}}$.

LEMMA 5.2. *If $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ is ergodic and $m_Q < \infty$, then $\Psi_Q \in \mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}})$ is ergodic.*

An asymptotic mean stationary measure can be interchanged with its stationary mean in several situations (see Gray [15], Chapter 6), for example, in relative entropy computations, as in Lemma 5.3 below. Before stating this lemma, we use an extension of the notion of specific relative entropy to measures that are not necessarily stationary. More precisely, for two measures $\mathcal{P}$ and $\mathcal{Q}$ on a product space $A^{\mathbb{N}}$, we define the specific relative entropy of $\mathcal{P}$ w.r.t. $\mathcal{Q}$ as

(5.8)
$$\overline{H}(\mathcal{P}|\mathcal{Q}) := \limsup_{n \to \infty} \frac{1}{n} h(\pi_n \mathcal{P}|\pi_n \mathcal{Q}),$$

where π_n is the projection onto the first n coordinates. For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, we introduce the following Radon–Nikodym derivative:

(5.9)
$$f_n(x) := \frac{\mathrm{d}\pi_n \kappa(Q)}{\mathrm{d}\mu_0^{\otimes n}}(x), \qquad x \in E^{\mathbb{N}_0}.$$

With this notation, the main result of this section is the following.

LEMMA 5.3. *For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ ergodic with $m_Q < \infty$,*

(5.10) $\quad H(\Psi_Q|\mu_0^{\otimes \mathbb{N}_0}) = \overline{H}(\kappa(Q)|\mu_0^{\otimes \mathbb{N}_0}),$

(5.11)
$$= \lim_{n \to \infty} \frac{1}{n} \log f_n(x) \qquad \textit{for } \kappa(Q)\text{-a.s. all } x \in E^{\mathbb{N}_0}.$$

The first equality holds also without the assumption of ergodicity.

PROOF. The first equality follows from Gray [16], Corollary 7.5.1, last equality in equation (7.32), which does not need the assumption of ergodicity. For the proof of the other equality, define

$$(5.12) \qquad \bar{f}_n(x) := \frac{d\pi_n \Psi_Q}{d\mu_0^{\otimes n}}(x).$$

Since Ψ_Q is stationary and ergodic (Lemma 5.2), Gray [16], Theorem 8.2.1, applied to the pair $\Psi_Q, \mu_0^{\otimes \mathbb{N}_0}$ gives that

$$(5.13) \qquad \lim_{n \to \infty} \frac{1}{n} \log \bar{f}_n(x) = H(\Psi_Q | \mu_0^{\otimes \mathbb{N}_0})$$

for Ψ_Q almost all x. But Ψ_Q is the stationary mean of $\kappa(Q)$ (Lemma 5.1), so that Gray [16], Theorem 8.4.1, combined with (5.13) gives

$$(5.14) \qquad \lim_{n \to \infty} \frac{1}{n} \log f_n(x) = H(\Psi_Q | \mu_0^{\otimes \mathbb{N}_0})$$

for $\kappa(Q)$ almost all x. $\quad\square$

II. Alternative formulation. We will apply Lemma 5.3 to the measure $[Q_\beta]_{\text{tr}}$, which is ergodic, being a product measure. The word length distribution of it is

$$(5.15) \qquad K^{\text{tr}}(n) := \begin{cases} K(n), & \text{if } 1 \le n \le \text{tr} - 1, \\ \displaystyle\sum_{m=\text{tr}}^{\infty} K(m), & \text{if } n = \text{tr}, \\ 0, & \text{if } n > \text{tr}. \end{cases}$$

For $[Q_\beta]_{\text{tr}}$, the function f_n in (5.9) becomes

$$(5.16) \quad f_n(x) = \mathbf{E}_{K^{\text{tr}}} \left(\prod_{k=0}^{n-1} \left(\frac{e^{\beta x_k}}{M(\beta)} \right)^{1_{\{S_k=0\}}} \right) = \mathbf{E}_{K^{\text{tr}}} \left(e^{\sum_{k=0}^{n-1} \{\beta x_k - \log M(\beta)\} 1_{\{S_k=0\}}} \right),$$

where $\mathbf{E}_{K^{\text{tr}}}$ denotes expectation with respect to law of the Markov chain S with renewal time distribution K^{tr} starting from 0. This follows from the definition of Q_β and (1.17). To emphasize the fact that in the last expression the sequence $x \in E^{\mathbb{N}_0}$ is picked from $\kappa([Q_\beta]_{\text{tr}})$, we take two independent sequences

$$(5.17) \qquad (x_k)_{k \in \mathbb{N}_0}, (\hat{x}_k)_{k \in \mathbb{N}_0} \text{ drawn from } \mu_0^{\otimes \mathbb{N}_0} \text{ and } \mu_\beta^{\otimes \mathbb{N}_0}, \text{ respectively,}$$

and an independent copy S' of S. Let $I := \{i \ge 0 : S_i = 0\}$, $I' := \{i \ge 0 : S_i' = 0\}$. Then

$$(5.18) \qquad \begin{aligned} & H\big(\Psi_{[Q_\beta]_{\text{tr}}} | \mu_0^{\otimes \mathbb{N}_0}\big) \\ & = \lim_{n \to \infty} \frac{1}{n} \log \mathbf{E}_{K^{\text{tr}}} \big[e^{\sum_{k=0}^{n-1} [\beta x_k 1_{\{k \notin I'\}} + \beta \hat{x}_k 1_{\{k \in I'\}} - \log M(\beta)] 1_{\{k \in I\}}} \big]. \end{aligned}$$

Note the appearance of two renewal sets I, I', which are the key to understanding the issue of relevant vs. irrelevant disorder; recall Remark 1.18.

6. Monotonicity of disorder relevance: Proof of Theorem 1.6.

PROOF.   In view of (5.10) in Lemma 5.3, the condition for relevance in (5.1) becomes

$$(6.1) \qquad \lim_{\mathrm{tr}\to\infty} m_{[Q_\beta]_{\mathrm{tr}}} \overline{H}(\kappa([Q_\beta]_{\mathrm{tr}})|\mu_0^{\otimes\mathbb{N}_0}) > 0.$$

We will show that $\beta \mapsto \overline{H}(\kappa([Q_\beta]_{\mathrm{tr}})|\mu_0^{\otimes\mathbb{N}_0})$ is nondecreasing for every $\mathrm{tr} \in \mathbb{N}$, which will imply the claim because $m_{[Q_\beta]_{\mathrm{tr}}} = m_{K^{\mathrm{tr}}}$ does not depend on β. It will be enough to show that $\beta \mapsto h(\pi_n\kappa([Q_\beta]_{\mathrm{tr}})|\mu_0^{\otimes n})$ is nondecreasing for all $\mathrm{tr}, n \in \mathbb{N}$.

Fix $\mathrm{tr}, n \in \mathbb{N}$. For $\beta \in [0, \infty)$ and $\bar{x} = (x_0, x_1, \dots, x_{n-1}) \in E^n$, let

$$(6.2) \qquad k(\beta, \bar{x}) := \frac{\mathrm{d}\pi_n\kappa([Q_\beta]_{\mathrm{tr}})}{\mathrm{d}\mu_0^n}(\bar{x}) = \mathbf{E}_{K^{\mathrm{tr}}}\left(\prod_{k\in J_n} \frac{e^{\beta x_k}}{M(\beta)}\right),$$

with $J_n := \{0 \le k < n : S_k = 0\}$ the set of renewal times prior to time n for the chain S that has renewal time distribution K^{tr}, to which we add 0 for convenience. Our goal is to prove that

$$(6.3) \qquad \beta \mapsto f(\beta) := \int_{\mathbb{R}^n} [k(\beta, \bar{x}) \log k(\beta, \bar{x})] \, \mathrm{d}\mu_0^{\otimes n}(\bar{x}) = h(\pi_n\kappa([Q_\beta]_{\mathrm{tr}})|\mu_0^{\otimes n})$$

is nondecreasing on $[0, \infty)$. We will do this by proving a stronger property. Namely, for $\bar{\beta} = (\beta_0, \beta_1, \dots, \beta_{n-1}) \in [0, \infty)^n$ and $\bar{x} \in E^n$, let

$$(6.4) \qquad k(\bar{\beta}, \bar{x}) := \mathbf{E}_{K^{\mathrm{tr}}}\left(\prod_{k\in J_n} \frac{e^{\beta_k x_k}}{M(\beta_k)}\right).$$

We will show that

$$(6.5) \qquad \bar{\beta} \mapsto f(\bar{\beta}) := \int_{\mathbb{R}^n} [k(\bar{\beta}, \bar{x}) \log k(\bar{\beta}, \bar{x})] \, \mathrm{d}\mu_0^{\otimes n}(\bar{x})$$

is nondecreasing on $[0, \infty)^n$ in each of its arguments.

We will prove monotonicity w.r.t. β_1 only. The argument is the same for the other variables, with one simplification for β_0; namely, we may drop the corresponding indicator $1_{\{0\in J_n\}}$ in the third line of (6.6) and in (6.8). First, using that $\int k(\bar{\beta}, \bar{x}) \, \mathrm{d}\mu_0^{\otimes n}(\bar{x}) = 1$ for all $\bar{\beta}$, we compute

$$
\begin{aligned}
(6.6) \quad & \partial_{\beta_1} f(\bar{\beta}) \\
&= \int_{\mathbb{R}^n} \partial_{\beta_1}[k(\bar{\beta}, \bar{x}) \log k(\bar{\beta}, \bar{x})] \, \mathrm{d}\mu_0^{\otimes n}(\bar{x}) \\
&= \int_{\mathbb{R}^n} \partial_{\beta_1}[k(\bar{\beta}, \bar{x})] \log k(\bar{\beta}, \bar{x}) \, \mathrm{d}\mu_0^{\otimes n}(\bar{x}) \\
&= \int_{\mathbb{R}^n} \partial_{\beta_1}\left(\frac{e^{\beta_1 x_1}}{M(\beta_1)}\right) \mathbf{E}_{K^{\mathrm{tr}}}\left(1_{\{1\in J_n\}} \prod_{k\in J_n\setminus\{1\}} \frac{e^{\beta_k x_k}}{M(\beta_k)}\right) \log k(\bar{\beta}, \bar{x}) \, \mathrm{d}\mu_0^{\otimes n}(\bar{x}).
\end{aligned}
$$

Next, we note that

$$\partial_{\beta_1}\left(\frac{e^{\beta_1 x_1}}{M(\beta_1)}\right)d\mu_0(x_1) = \frac{e^{\beta_1 x_1}x_1 M(\beta_1) - e^{\beta_1 x_1}M'(\beta_1)}{M(\beta_1)^2}d\mu_0(x_1)$$

$$(6.7) \qquad = \left(x_1 - \frac{M'(\beta_1)}{M(\beta_1)}\right)\frac{e^{\beta_1 x_1}}{M(\beta_1)}d\mu_0(x_1)$$

$$= (x_1 - E_{\beta_1})d\mu_{\beta_1}(x_1),$$

where $E_{\beta_1} := M'(\beta_1)/M(\beta_1) = \int x_1 \, d\mu_{\beta_1}(x_1)$. Now, let $\bar{x}^1$ be $\bar{x}$ without x_1, and abbreviate

$$(6.8) \qquad A(x_1; \bar{x}^1) := \mathbf{E}_{K^{tr}}\left(\prod_{k \in J_n \setminus \{1\}} \frac{e^{\beta_k x_k}}{M(\beta_k)}1_{\{1 \in J_n\}}\right)\log k(\bar{\beta}, \bar{x}).$$

Then, for fixed $\bar{x}^1$, the integral over x_1 in (6.6) equals

$$(6.9) \qquad \int_{\mathbb{R}^n}(x_1 - E_{\beta_1})A(x_1; \bar{x}^1)\,d\mu_{\beta_1}(x_1)$$

$$\geq \int_{\mathbb{R}^n}(x_1 - E_{\beta_1})\,d\mu_{\beta_1}(x_1)\int_{\mathbb{R}^n}A(x_1; \bar{x}^1)\,d\mu_{\beta_1}(x_1) = 0,$$

where the inequality holds because both $x_1 \mapsto x_1 - E_{\beta_1}$ and $x_1 \mapsto A(x_1; \bar{x}^1)$ are nondecreasing [for the latter we need that $\beta_1 \in [0, \infty)$]. It therefore follows from (6.6), after integrating over $\bar{x}^1$ as well, that $\partial_{\beta_1}f(\bar{\beta}) \geq 0$. $\quad\square$

7. Disorder irrelevance: Proof of Corollaries 1.7 and 1.8(i).

7.1. *Proof of Corollary* 1.7.

PROOF. This is immediate from Theorem 1.5 and the fact that $I^{que} = I^{ann}$ when $\alpha = 0$. The latter was already noted at the end of Section 2. $\quad\square$

7.2. *Proof of Corollary* 1.8(i).

PROOF. We will show disorder irrelevance for all β that satisfy $M(2\beta)/M(\beta)^2 < 1 + \chi^{-1}$. To show that for such β the limit in (5.1) is zero, we use an annealed bound on $H(\Psi_{[Q_\beta]_{tr}}|\mu_0^{\otimes \mathbb{N}_0})$ based on the expression (5.11) for it. We bound the limit in the right-hand side of that formula, using (3.22) with the role of Θ_n played by

$$(7.1) \qquad f_n(x) = \frac{d\pi_n \kappa([Q_\beta]_{tr})}{d\mu_0^{\otimes n}}(x), \qquad x \in E^{\mathbb{N}_0}.$$

This satisfies

$$(7.2) \qquad \mathbf{E}_{\kappa([Q_\beta]_{tr})}(f_n(x)) = \mathbf{E}_{\mu_0^{\otimes n}}(f_n(x)f_n(x)),$$

because $f_n(x)$ depends on the first n coordinates of x only, and the Radon–Nikodym derivative of $\pi_n \kappa([Q_\beta]_{\mathrm{tr}})$ with respect to $\mu_0^{\otimes n}$ is f_n. Using (5.16), we write the last expectation as

(7.3)
$$
\begin{aligned}
\mathbf{E}_{\mu_0^{\otimes n}}&(f_n(x) f_n(x)) \\
&= \mathbf{E}_{\mu_0^{\otimes n}}\left((\mathbf{E}_{K^{\mathrm{tr}}} \times \mathbf{E}_{K^{\mathrm{tr}}}) \left(\prod_{k=0}^{n-1} \left(\frac{e^{\beta x_k}}{M(\beta)} \right)^{1_{\{S_k=0\}}} \prod_{l=0}^{n-1} \left(\frac{e^{\beta x_l}}{M(\beta)} \right)^{1_{\{S'_l=0\}}} \right) \right) \\
&= (\mathbf{E}_{K^{\mathrm{tr}}} \times \mathbf{E}_{K^{\mathrm{tr}}}) \left(\mathbf{E}_{\mu_0^{\otimes n}} \left(\prod_{k=0}^{n-1} \left(\frac{e^{\beta x_k}}{M(\beta)} \right)^{1_{\{S_k=0\}}} \prod_{l=0}^{n-1} \left(\frac{e^{\beta x_l}}{M(\beta)} \right)^{1_{\{S'_l=0\}}} \right) \right) \\
&= (\mathbf{E}_{K^{\mathrm{tr}}} \times \mathbf{E}_{K^{\mathrm{tr}}}) \left(\Xi(\beta)^{\sum_{k=0}^{n-1} 1_{\{S_k=S'_k=0\}}} \right),
\end{aligned}
$$

where $\mathbf{E}_{K^{\mathrm{tr}}} \times \mathbf{E}_{K^{\mathrm{tr}}}$ is the expectation with respect to two independent copies S, S' of the Markov chain starting from 0 with renewal time distribution K^{tr}, and

(7.4)
$$
\Xi(\beta) := \frac{M(2\beta)}{M(\beta)^2}.
$$

If we now let

(7.5)
$$
f_2^{\mathrm{tr}}(\lambda) := \lim_{n\to\infty} \frac{1}{n} \log(\mathbf{E}_{K^{\mathrm{tr}}} \times \mathbf{E}_{K^{\mathrm{tr}}}) \left(e^{\lambda \sum_{k=0}^{n-1} 1_{\{S_k=S'_k=0\}}} \right),
$$

then (5.11), (3.22) and (7.1)–(7.5) imply that

(7.6)
$$
H\left(\Psi_{[Q_\beta]_{\mathrm{tr}}} | \mu_0^{\otimes \mathbb{N}_0} \right) \leq f_2^{\mathrm{tr}}(\log \Xi(\beta)), \qquad \beta \in [0,\infty), \mathrm{tr} \in \mathbb{N}.
$$

Combining this bound with the condition for relevance in (5.1), we see that to prove irrelevance it suffices to show that

(7.7)
$$
\lim_{\mathrm{tr}\to\infty} m_{[Q_\beta]_{\mathrm{tr}}} f_2^{\mathrm{tr}}(\log \Xi(\beta)) = 0.
$$

By (A.2) in Appendix A, we have

(7.8)
$$
f_2(\lambda) = 0 \quad \Longleftrightarrow \quad \lambda \leq \lambda_0 := -\log \mathbf{P}(I \cap I' \neq \varnothing),
$$

where I, I' are the sets of renewal times for S, S' without truncation, and $f_2(\lambda)$ as defined in Appendix A. By Lemma A.1, if $\lambda < \lambda_0$, then $\sup_{tr \in \mathbb{N}} \mathrm{tr}\, f_2^{\mathrm{tr}}(\lambda) < \infty$. Since $\lim_{\mathrm{tr}\to\infty} m_{[Q_\beta]_{\mathrm{tr}}} / \mathrm{tr} = 0$ always, (7.7) holds as soon as $\log \Xi(\beta) < \lambda_0$, that is, $\Xi(\beta) < 1/\mathbf{P}(I \cap I' \neq \varnothing)$. Now the claim of the corollary follows because $\mathbf{P}(I \cap I' \neq \varnothing) = \chi/(\chi+1)$ (see Spitzer [19], Section 1), with χ as defined in (1.21), and with the convention that the last ratio is 1 if $\chi = \infty$. $\quad\square$

8. Disorder relevance: Proof of Corollary 1.8(ii).

PROOF. We restrict the expectation in (5.18) to the set

$$(8.1) \qquad A_n := \{(S_k)_{k=0}^n : I \cap \{1, \ldots, n\} = I' \cap \{1, \ldots, n\}\},$$

that is, S follows I' and collects only the tilted charges $\hat{x}_k$ defined in (5.17). This gives for the expectation the lower bound

$$(8.2) \qquad \exp\left[\sum_{k=0}^{n-1} [\beta \hat{x}_k - \log M(\beta)] 1_{\{k \in I'\}}\right] \mathbf{P}(A_n).$$

Let $k_n := |I \cap \{1, \ldots, n\}|$, $\tau_0' = 0$ and $\tau_1' < \cdots < \tau_{k_n}'$ the elements of $I' \cap \{1, \ldots, n\}$. By the renewal theorem, we have $k_n/n \to 1/m_{\mathrm{tr}}$ as $n \to \infty$. Moreover,

$$(8.3) \qquad \mathbf{P}(A_n) = \mathbf{P}(\tau_1 > n - \tau_{k_n}') \prod_{i=1}^{k_n} K^{\mathrm{tr}}(\tau_i' - \tau_{i-1}'),$$

so that

$$(8.4) \qquad \begin{aligned} \frac{1}{n} \log \mathbf{P}(A_n) &= \frac{1}{n} \log \mathbf{P}(\tau_1 > n - \tau_{k_n}) + \frac{k_n}{n} \frac{1}{k_n} \sum_{i=1}^{k_n} \log K^{\mathrm{tr}}(\tau_i' - \tau_{i-1}') \\ &\to \frac{1}{m_{\mathrm{tr}}} \sum_{k=1}^{\mathrm{tr}} K^{\mathrm{tr}}(k) \log K^{\mathrm{tr}}(k), \end{aligned}$$

while

$$(8.5) \qquad \frac{1}{n} \sum_{k=0}^{n-1} \{\beta \hat{x}_k - \log M(\beta)\} 1_{\{k \in I'\}} \to \frac{1}{m_{\mathrm{tr}}} c(\beta)$$

with

$$(8.6) \quad c(\beta) := \beta E_{\mu_\beta}(\hat{x}_1) - \log M(\beta) = \beta [\log M(\beta)]' - \log M(\beta) = h(\mu_\beta | \mu_0).$$

Hence

$$(8.7) \qquad m_{\mathrm{tr}} H\big(\Psi_{[Q_\beta]_{\mathrm{tr}}} | \mu_0^{\otimes \mathbb{N}_0}\big) \ge h(\mu_\beta | \mu_0) + \sum_{k=0}^{\mathrm{tr}} K^{\mathrm{tr}}(k) \log K^{\mathrm{tr}}(k),$$

and

$$(8.8) \qquad \liminf_{\mathrm{tr} \to \infty} m_{[Q_\beta]_{\mathrm{tr}}} H(\kappa([Q_\beta]_{\mathrm{tr}}) | \mu_0^{\otimes \mathbb{N}_0}) \ge h(\mu_\beta | \mu_0) - H(K).$$

Consequently, $h(\mu_\beta | \mu_0) > H(K)$ is sufficient for disorder relevance. $\square$

We close by proving the second part of (1.26).

$$(8.9) \qquad \lim_{\beta \to \infty} h(\mu_\beta | \mu_0) = \log[1/\mu_0(\{w\})].$$

We distinguish three different cases:

(1) $w = \infty$. Apply (3.38) with $\mu = \mu_\beta$, $\nu = \mu_0$ and $f(x) = x \vee 0$, to get

$$(8.10) \qquad h(\mu_\beta | \mu_0) \geq \int_E (x \vee 0) \, d\mu_\beta(x) - \log[M(1) + 1].$$

The integral diverges as $\beta \to \infty$, and so (8.9) follows.

(2) $\mu_0(\{w\}) = 0$ with $w < \infty$. Now μ_β converges weakly as $\beta \to \infty$ to δ_w, the point measure at w. Hence (8.9) follows by using the lower semicontinuity of $\mu \mapsto h(\mu | \mu_0)$ and the fact that $h(\delta_w | \mu_0) = \infty$ because δ_w is not absolutely continuous w.r.t. μ_0.

(3) $\mu_0(\{w\}) > 0$ with $w < \infty$. Define

$$(8.11) \qquad f_\beta(x) := \frac{d\mu_\beta}{d\mu_0}(x) = \frac{e^{\beta x}}{M(\beta)}, \qquad x \in E.$$

This function satisfies

$$\lim_{\beta \to \infty} f_\beta(x) = 0 \qquad \text{for } x < w,$$

$$(8.12) \qquad \lim_{\beta \to \infty} f_\beta(w) = 1/\mu_0(\{w\}),$$

$$f_\beta(x) \leq 1/\mu_0(\{w\}) < \infty \qquad \text{for } x \leq w.$$

Since $t \mapsto t \log t$ is increasing on $[1, \infty)$ and on $(0, 1]$ takes values in $[-e^{-1}, 0]$, we can apply the bounded convergence theorem to the integral

$$(8.13) \qquad h(\mu_\beta | \mu_0) = \int_E f_\beta(x) \log f_\beta(x) \, d\mu_0(x),$$

to get (8.9).

APPENDIX A: STANDARD FACTS ABOUT THE HOMOPOLYMER

In this appendix we recall a few standard facts about the homopolymer. For proofs we refer to Giacomin [11], Chapter 2, and den Hollander [8], Chapter 7.

The homopolymer has a path measure as in (1.4), but with exponent $\lambda \sum_{k=0}^{n-1} 1_{\{S_k = 0\}}$, $\lambda \in [0, \infty)$. For a given renewal time distribution K, it is known that the free energy $f(\lambda)$ is the unique solution of the equation

$$(A.1) \qquad e^{-\lambda} = \sum_{n \in \mathbb{N}} K(n) e^{-nf(\lambda)}$$

whenever a solution exists, otherwise $f(\lambda) = 0$. Clearly

$$(A.2) \qquad f(\lambda) = 0 \quad \Longleftrightarrow \quad \lambda \leq -\log \mathbf{P}(I \neq \varnothing),$$

where $I = \{k \in \mathbb{N} : S_k = 0\}$ is the set of renewal times of S.

Let S, S' be two independent copies of the Markov chain starting form 0, with renewal time distribution K, and with sets of renewal times I, I'. Transience of the joint renewal process $I \cap I'$ is equivalent to $\mathbf{P}(I \cap I' \neq \varnothing) < 1$. In that case, let

$$(A.3) \qquad \lambda_0 := -\log \mathbf{P}(I \cap I' \neq \varnothing) > 0,$$

and denote by $f_2(\lambda)$ and $f_2^{\mathrm{tr}}(\lambda)$ the free energy of the homopolymer whose underlying Markov chain has renewal set $I \cap I'$ when the renewal times of S, S' are drawn from K, respectively, K^{tr} defined in (5.15). Then $\lim_{\mathrm{tr} \to \infty} f_2^{\mathrm{tr}}(\lambda) = f_2(\lambda)$. Note that $f_2(\lambda) = 0$ if and only if $\lambda \leq \lambda_0$. This property does not hold for $f_2^{\mathrm{tr}}(\lambda)$, but the following lemma shows that $f_2^{\mathrm{tr}}(\lambda)$ tends to zero fast as $\mathrm{tr} \to \infty$ when $\lambda < \lambda_0$.

LEMMA A.1. *Suppose that* $\mathbf{P}(I \cap I' \neq \varnothing) < 1$. *Then* $\sup_{\mathrm{tr} \in \mathbb{N}} \mathrm{tr}\, f_2^{\mathrm{tr}}(\lambda) < \infty$ *for all* $\lambda < \lambda_0$.

PROOF. As in the paragraph preceding the lemma, define I^{tr}, I'^{tr}, where now the Markov chains S, S' have renewal time distribution K^{tr}. Let K_2, K_2^{tr} be the renewal time distributions generating the sets $I \cap I'$, $I^{\mathrm{tr}} \cap I'^{\mathrm{tr}}$ respectively. Put $L_2(n) := \sum_{k=1}^n K_2(k)$ and $L_2^{\mathrm{tr}}(n) := \sum_{k=1}^n K_2^{\mathrm{tr}}(k)$. Then $L_2(\infty) = e^{-\lambda_0}$ and $L_2^{\mathrm{tr}}(\infty) = 1$ because the renewal process $I^{\mathrm{tr}} \cap I'^{\mathrm{tr}}$ is resurrent. Since $K_2^{\mathrm{tr}}(n) = K_2(n)$ for $1 \leq n < \mathrm{tr}$, it follows from (A.1) that

$$
\begin{aligned}
e^{-\lambda} &= \sum_{n=1}^{\mathrm{tr}-1} K_2(n) e^{-n f_2^{\mathrm{tr}}(\lambda)} + \sum_{n=\mathrm{tr}}^{\infty} K_2^{\mathrm{tr}}(n) e^{-n f_2^{\mathrm{tr}}(\lambda)} \\
&\leq L_2(\mathrm{tr}-1) + e^{-\mathrm{tr}\, f_2^{\mathrm{tr}}(\lambda)}[1 - L_2(\mathrm{tr}-1)],
\end{aligned}
$$

(A.4)

where the equality holds because $f_2^{\mathrm{tr}}(\lambda) > 0$ for $\lambda > 0$. Hence

$$(A.5) \qquad \mathrm{tr}\, f_2^{\mathrm{tr}}(\lambda) \leq \log\left[\frac{1 - L_2(\mathrm{tr}-1)}{e^{-\lambda} - L_2(\mathrm{tr}-1)} \right].$$

The term between brackets tends to $(1 - e^{-\lambda_0})/(e^{-\lambda} - e^{-\lambda_0})$ as $\mathrm{tr} \to \infty$, which is finite for $\lambda < \lambda_0$. □

The order of the phase transition for the homopolymer depends on the tail of K. If K satisfies (1.24), then (see [11], Theorem 2.1, [8], Theorem 7.4)

$$(A.6) \qquad f(\lambda) \sim \lambda^{1/(1 \wedge \alpha)} L^*(1/\lambda), \qquad \lambda \downarrow 0,$$

for some L^*, that is, strictly positive and slowly varying at infinity. Hence, the phase transition is order 1 when $\alpha \in [1, \infty)$ and order $m \in \mathbb{N} \setminus \{1\}$ when $\alpha \in [\frac{1}{m}, \frac{1}{m-1})$. This shows that the value $\alpha = \frac{1}{2}$ is critical in view of the Harris criterion mentioned in Remark 1.14.

APPENDIX B: PROOF OF LEMMA 3.2

We borrow ideas from the proof of the lower bound of the LDP in Theorem 2.2 given in Birkner, Greven and den Hollander [6], Proposition 4.1. What follows is a rewriting of the relevant parts of that proof, organized as Sections B.1–B.4. Our setting is the same as their setting because the assumption throughout Section 3.2 is that $E := \mathrm{supp}[\mu_0]$ is finite.

We will prove that $S^{\mathrm{que}}(\beta; 1) \leq S^{\mathrm{que}}(\beta; 1-)$. Fix $A < S^{\mathrm{que}}(\beta; 1)$. By (3.14) and (2.12), there is a $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with $m_Q < \infty$ such that $\beta \Phi(Q) - I^{\mathrm{que}}(Q) > A$. Because Φ and I^{que} are affine, we may assume without loss of generality that Q is ergodic.

B.1. Step 1: Good sentences.　For $\varepsilon > 0$, the set

$$\text{(B.1)} \qquad \mathcal{U}_\varepsilon(Q) := \{Q' \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) : \Phi(Q') > \Phi(Q) - \varepsilon\}$$

is open because Φ is continuous. Hence there is an $M_0 \in \mathbb{N}$ large enough, a $\delta_1 > 0$ and a finite set $\mathcal{A}_0 \subset \widetilde{E}^{M_0}$ such that

$$\text{(B.2)} \quad \mathcal{U}_{M,\delta_1} := \{Q' \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) : |(\pi_{M_0} Q')(s) - f_s| < 2\delta_1 \ \forall s \in \mathcal{A}_0\} \subset \mathcal{U}_\varepsilon(Q),$$

where we set $f_s := (\pi_{M_0} Q)(s)$ for $s \in \mathcal{A}_0$. Also, by (1.2), we can assume that

$$\text{(B.3)} \qquad K(n) \geq n^{-\alpha-1-\varepsilon} \qquad \forall n \geq M_0.$$

By the ergodicity of Q, for every $s \in \mathcal{A}_0$ we have

$$\text{(B.4)} \quad \lim_{M \to \infty} \frac{1}{M} |\{0 \leq j \leq M - M_0 : \pi_{M_0}(\tilde{\theta}^j Y) = s\}| = f_s \qquad \text{for } Q\text{-a.e. } Y.$$

Consequently, there is a large M and a finite set $\mathcal{A} \subset \widetilde{E}^M$ with

$$\text{(B.5)} \qquad (\pi_M Q)(\mathcal{A}) \geq 1 - \varepsilon$$

such that

$$\text{(B.6)} \quad \left| \frac{1}{M} |\{0 \leq j \leq M - M_0 : \pi_{M_0}(\tilde{\theta}^j z) = s\}| - f_s \right| < \delta_1 \qquad \forall s \in \mathcal{A}_0, z \in \mathcal{A}.$$

Moreover, we can assume for all $z \in \mathcal{A}$ the following relations, which are stated in [6], equation (3.6), and are consequences of ergodicity, too:

$$\text{(B.7)} \qquad\qquad |\kappa(z)| \in [M(m_Q - \varepsilon), M(m_Q + \varepsilon)],$$

$$\text{(B.8)} \quad \begin{aligned} &\log Q\big(\kappa\big(Y^{(1)}, Y^{(2)}, \ldots, Y^{(M)}\big) = \kappa(z)\big) \\ &\qquad \in \big[-M\big(m_Q H(\Psi_Q) + \varepsilon\big), -M\big(m_Q H(\Psi_Q) - \varepsilon\big)\big], \end{aligned}$$

$$\text{(B.9)} \quad \log Q\big((Y^{(1)}, Y^{(2)}, \ldots, Y^{(M)}) = z\big) \in \big[-M(H(Q) + \varepsilon), -M(H(Q) - \varepsilon)\big],$$

$$\text{(B.10)} \quad \sum_{i=1}^{|\kappa(z)|} \log \mu_0((\kappa(z))_i) - M m_Q \mathbf{E}_{\Psi_Q}[\log \mu_0(X_1)] \in [-M\varepsilon, M\varepsilon],$$

$$(B.11) \qquad \sum_{i=1}^{M} \log K\left(|z^{(i)}|\right) - M\mathbf{E}_Q[\log K(\tau_1)] \in [-M\varepsilon, M\varepsilon].$$

In the above relations, $|\kappa(z)|$ denotes the length of the string $\kappa(z)$, $(\kappa(z))_i$ is the ith letter of that string, $z^{(i)}$ is the ith word of the sentence z, $|z^{(i)}|$ is its length, while $H(Q)$, $H(\Psi_Q)$ are the specific entropies of the measures Q, Ψ_Q. In the last relation, τ_1 is distributed as the length of the first word of an element of $\widetilde{E}^{\mathbb{N}}$ drawn from Q. Finally, M can be chosen such that

$$(B.12) \qquad M > \frac{8M_0}{\delta_1}, \qquad \frac{1}{M}(\alpha + 1 + \varepsilon) \log[M(m_Q + \varepsilon) + M_0] < \varepsilon.$$

B.2. Step 2: Good trajectories. For given $\omega \in E^{\mathbb{N}_0}$, we define a set $\mathcal{T}^{\omega}_{\varepsilon,M}$ of trajectories for the renewal sequence $T = (T_i)_{i \in \mathbb{N}_0}$ on which $R^{\omega}_N \in \mathcal{U}_{M,\delta_1}$. In Step 3 we will control the probability that T follows a trajectory in $\mathcal{T}^{\omega}_{\varepsilon,M}$.

Let $\mathcal{B} := \{\kappa(z) : z \in \mathcal{A}\}$ be the set of concatenations of the sentences of $\mathcal{A}$. By (B.5) and (B.8),

$$(B.13) \qquad |\mathcal{B}| \geq (1 - \varepsilon)e^{M(m_Q H(\Psi_Q) - \varepsilon)}.$$

Divide ω into consecutive pieces of length $\Lambda := [M(m_Q + \varepsilon)] + M_0$, mark with 1 those pieces that start with an element of $\mathcal{B}$, and mark with 0 the remaining pieces, that is, for $j \geq 0$, let

$$(B.14) \qquad \sigma_j := 1_{\{\theta^{j\Lambda}\omega \text{ starts with an element of } \mathcal{B}\}}.$$

Let $\{j(r) : r \geq 1\}$ be the increasing sequence that picks out the $j \geq 1$ with $\sigma_j = 1$, and let $j(0) = 0$. The increments $\{j(r+1) - j(r) : r \geq 0\}$ are i.i.d. geometric random variables with probability of success $p_{\mathcal{B}} := \mathbb{P}(\omega \text{ starts with an element of } \mathcal{B})$. It follows from (B.10) and (B.13) that

$$(B.15) \qquad p_{\mathcal{B}} \geq (1 - \varepsilon)e^{M(m_Q H(\Psi_Q) + m_Q \mathbf{E}_{\Psi_Q}[\log \mu_0(X_1)] - 2\varepsilon)}$$

$$(B.16) \qquad = (1 - \varepsilon)e^{-Mm_Q H(\Psi_Q | \mu_0^{\otimes \mathbb{N}_0}) - 2\varepsilon M}.$$

The equality in the second line follows from [6], equation (1.26). In particular, for $\mathbb{P}$-a.e. ω we have $\sigma_j = 1$ for infinitely many j's, and so the sequence $\{j(r) : r \geq 1\}$ is well defined.

Pick any $N > 16M/\delta_1$. The set $\mathcal{T}^{\omega}_{\varepsilon,M}$ consists of all T that first jump to $j(1)\Lambda$ [i.e., $T_1 = j(1)\Lambda$], next make M jumps that cut out of $\theta^{j(1)\Lambda}\omega$ an element of $\mathcal{A}$ [which is possible by the definitions of $j(1)$ and $\mathcal{B}$], next jump to $j(2)\Lambda$ [i.e., $T_{M+2} = j(2)\Lambda$], next again cut out an element of $\mathcal{A}$, and continue likewise until they jump to $j(\lceil N/(M+1)\rceil + 1)\Lambda$ (no conditions are imposed afterwards). The words between two consecutive $j(r)\Lambda$'s we call a block. After the first jump to $j(1)\Lambda$ and up to the last jump to $j(\lceil N/(M+1)\rceil + 1)\Lambda$, at least N words are cut out, because T has created $\lceil N/(M+1)\rceil$ blocks each containing exactly $M+1$

words. We note that the first M words are important and of typical length, while the last word is of an untypically large length and its sole purpose is for T to move to a good position in ω. Call $Y^{(1)}, Y^{(2)}, \ldots, Y^{(N)}$ the first N words cut.

LEMMA B.1. $R_N^\omega \in \mathcal{U}_{M,\delta_1}$ for all T in $\mathcal{T}_{\varepsilon,M}^\omega$.

PROOF.   By the definition of R_N^ω, we need to show that every element $s \in \mathcal{A}_0$ occurs in the finite sequence

$$(B.17) \qquad \left(\pi_{M_0} \tilde{\theta}^j \left(Y^{(1)}, Y^{(2)}, \ldots, Y^{(N)}\right)^{\mathrm{per}}\right)_{0 \le j \le N-1}$$

the right number of times, that is, a number of times that falls in the interval $((f_s - 2\delta_1)N, (f_s + 2\delta_1)N)$.

For the lower bound, note that the sequence $(Y^{(1)}, Y^{(2)}, \ldots, Y^{(N)})$ contains at least the words of the first $[N/(M+1)] - 1$ blocks out of the $\lceil N/(M+1) \rceil$ blocks that T created, because the last word of these blocks has index at most $i^* = 1 + (N/(M+1) - 1)(M+1) = N - M < N$. Each such block offers at least $M(f_s - \delta_1)$ occurrences of the word s, because of (B.6) and $i^* < N - M_0$. Thus, we have at least

$$M(f_s - \delta_1)\left(\frac{N}{M+1} - 2\right) = N(f_s - \delta_1) - \frac{f_s - \delta_1}{M+1}N - 2M(f_s - \delta_1)$$

$$(B.18)$$

$$> N(f_s - 2\delta_1)$$

occurrences of s in the sequence in (B.17), where the last inequality holds because $N > 16M/\delta_1$ and $M > 8M_0/\delta_1$ by (B.12).

For the upper bound, note that, because of (B.6), the occurrences of s in the sequence in (B.17) are at most

$$1 + ((f_s + \delta_1)M + M_0)\left(\frac{N}{M+1} + 1\right) + M_0$$

$$(B.19)$$

$$\le N(f_s + \delta_1) + \frac{NM_0}{M+1} + M(f_s + \delta_1) + 2M_0 + 1 < N(f_s + 2\delta_1),$$

where the last inequality again uses $N > 16M/\delta_1$ and $M > 8M_0/\delta_1$. $\square$

B.3. Step 3: Probability of good trajectories.

For the quenched probability $\mathbf{P}(T \in \mathcal{T}_{\varepsilon,M}^\omega)$, we have the lower bound

$$\mathbf{P}(T \in \mathcal{T}_{\varepsilon,M}^\omega) \ge K(j(1)\Lambda)$$

$$(B.20) \qquad \times \left(e^{M(H(Q) - m_Q H(\Psi_Q) - 2\varepsilon)} e^{M(\mathbf{E}_Q[\log K(\tau_1)] - \varepsilon)}\right)^{\lceil N/(M+1) \rceil}$$

$$\times \prod_{r=1}^{\lceil N/(M+1) \rceil} \inf_{|\eta - Mm_Q| < M\varepsilon} K\left([j(r+1) - j(r)]\Lambda - \eta\right).$$

The last product is a lower bound for the probability of the large jumps that land at the points $j(r+1)\Lambda$, $1 \le r \le \lceil N/(M+N) \rceil$. The power preceding this product corresponds to the jumps inside each of the $\lceil N/(M+1) \rceil$ blocks, and uses that, by (B.8) and (B.9), for each element of $\mathcal{B}$ there are at least $e^{M(H(Q) - m_Q H(\Psi_Q) - 2\varepsilon)}$ different words of $\mathcal{A}$ having this element as concatenation, and that, by (B.11), the probability for M jumps to cut out a given word in $\mathcal{A}$ is at least $e^{M(\mathbb{E}_Q[\log K(\tau_1)] - \varepsilon)}$. It therefore follows that

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbf{P}(\tau \in \mathcal{T}_{\varepsilon,M}^{\omega})$$

(B.21)
$$\ge H(Q) - m_Q H(\Psi_Q) + \mathbf{E}_Q[\log K(\tau_1)] - 3\varepsilon$$

$$+ \frac{1}{M} \mathbb{E} \Big(\log \Big[\inf_{|\eta - M m_Q| < M\varepsilon_1} K\big([j(2) - j(1)]\Lambda - \eta \big) \Big] \Big).$$

To be more precise, (B.20) gives (B.21) with the right-hand side multiplied by $M/(M+1)$, but since the factors in (B.20) are probabilities, replacing $M/(M+1)$ by 1 still gives us a lower bound. Now, because of (B.3) and $\Lambda - \eta \ge M_0$, the last expectation is bounded from below by

$$\mathbb{E}\big[\log\big(([j(2) - j(1)]\Lambda)^{-\alpha - 1 - \varepsilon}\big)\big]$$

(B.22)
$$= -(\alpha + 1 + \varepsilon)\mathbb{E}\big[\log([j(2) - j(1)]\Lambda)\big]$$

$$\ge -(\alpha + 1 + \varepsilon)\big(\log \Lambda + \log \mathbb{E}[j(2) - j(1)]\big),$$

where we use the concavity of log. Since $\mathbb{E}[j(2) - j(1)] = 1/p_{\mathcal{B}}$, by combining (B.21) and (B.22) with the lower bound on $p_{\mathcal{B}}$ in (B.16), we get that

$$\liminf_{N \to \infty} N^{-1} \log \mathbf{P}(T \in \mathcal{T}_{\varepsilon,M}^{\omega})$$

$$\ge H(Q) - m_Q H(\Psi_Q) + \mathbf{E}_Q[\log K(\tau_1)] - 3\varepsilon$$

$$+ \frac{1}{M}\big[-(\alpha + 1 + \varepsilon)$$

(B.23)
$$\times \big(\log \Lambda - \log(1 - \varepsilon) + M m_Q H(\Psi_Q | \mu_0^{\otimes \mathbb{N}_0}) + 2\varepsilon M\big)\big]$$

$$= H(Q) - m_Q H(\Psi_Q) + \mathbf{E}_Q[\log K(\tau_1)] - m_Q H(\Psi_Q | \mu_0^{\otimes \mathbb{N}_0})$$

$$- \alpha m_Q H(\Psi_Q | \mu_0^{\otimes \mathbb{N}_0}) - 3\varepsilon - \varepsilon m_Q H(\Psi_Q | \mu_0^{\otimes \mathbb{N}_0})$$

$$- \frac{1}{M}(\alpha + 1 + \varepsilon)\big(\log \Lambda - \log(1 - \varepsilon)\big) - 2(\alpha + 1 + \varepsilon)\varepsilon.$$

The fourth line equals $-I^{\mathrm{que}}(Q)$ because of [6], equations (1.16), (1.30) and (1.32), where in using (1.16) we note that what we call in this paper α is called $\alpha - 1$ in [6]. The fifth line is at least $-\varepsilon C_Q$ for some positive constant C_Q that depends on Q, because of (B.12). Thus, we end up with

(B.24)
$$\liminf_{N \to \infty} N^{-1} \log \mathbf{P}(T \in \mathcal{T}_{\varepsilon,M}^{\omega}) \ge -I^{\mathrm{que}}(Q) - \varepsilon C_Q.$$

B.4. Step 4: Lower bound. For $T \in \mathcal{T}^\omega_{\varepsilon,M}$, we have

$$(B.25) \qquad Nm_{R^\omega_N} \leq j\left(\left\lceil \frac{N}{M+1} \right\rceil + 1\right)(M(m_Q + \varepsilon) + M_0),$$

and note that

$$(B.26) \qquad \lim_{N \to \infty} \frac{1}{N} j\left(\left\lceil \frac{N}{M+1} \right\rceil + 1\right) = \frac{1}{M+1}\frac{1}{p_\mathcal{B}}.$$

Hence

$$\mathbf{E}\left(e^{Nm_{R^\omega_N}\log z + N\beta\Phi(R^\omega_N)}\right)$$

$$(B.27) \qquad \geq \mathbf{E}\left(e^{Nm_{R^\omega_N}\log z + N\beta\Phi(R^\omega_N)}1_{\{T \in \mathcal{T}^\omega_{\varepsilon,M}\}}\right)$$

$$\geq e^{N\beta(\Phi(Q)-\varepsilon)}z^{j(\lceil N/(M+1)\rceil+1)(M(m_Q+\varepsilon)+M_0)}\mathbf{P}(\tau \in \mathcal{T}^\omega_{\varepsilon,M}).$$

Combining (3.10), (B.24), (B.26) and (B.27), we get

$$S^{\text{que}}(\beta;z) \geq \beta\Phi(Q) - \beta\varepsilon + \frac{M(m_Q+\varepsilon)+M_0}{(M+1)p_\mathcal{B}}\log z$$

$$(B.28) \qquad\qquad\qquad - I^{\text{que}}(Q) - C_Q\varepsilon.$$

Now let $z \uparrow 1$ and $\varepsilon \downarrow 0$, to get $S^{\text{que}}(\beta;1-) \geq \beta\Phi(Q) - I^{\text{que}}(Q) > A$. Since $A < S^{\text{que}}(\beta;1)$ was arbitrary, it follows that $S^{\text{que}}(\beta;1-) \geq S^{\text{que}}(\beta;1)$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ATHENS
PANEPISTIMIOPOLIS
15784 ATHENS
GREECE
E-MAIL: dcheliotis@math.uoa.gr

MATHEMATICAL INSTITUTE
LEIDEN UNIVERSITY
P.O. BOX 9512
2300 RA LEIDEN
THE NETHERLANDS
E-MAIL: denholla@math.leidenuniv.nl