The Annals of Probability
2013, Vol. 41, No. 3B, 2225–2260
DOI: 10.1214/11-AOP717

ON THE CHAOTIC CHARACTER OF THE STOCHASTIC HEAT EQUATION, BEFORE THE ONSET OF INTERMITTTENCY

BY DANIEL CONUS[1], MATHEW JOSEPH[2] AND DAVAR KHOSHNEVISAN[3]

Lehigh University, University of Utah and University of Utah

We consider a nonlinear stochastic heat equation $\partial_t u = \frac{1}{2}\partial_{xx} u + \sigma(u)\partial_{xt} W$, where $\partial_{xt} W$ denotes space–time white noise and $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous. We establish that, at every fixed time $t > 0$, the global behavior of the solution depends in a critical manner on the structure of the initial function u_0: under suitable conditions on u_0 and σ, $\sup_{x\in\mathbf{R}} u_t(x)$ is a.s. finite when u_0 has compact support, whereas with probability one, $\limsup_{|x|\to\infty} u_t(x)/(\log|x|)^{1/6} > 0$ when u_0 is bounded uniformly away from zero. This sensitivity to the initial data of the stochastic heat equation is a way to state that the solution to the stochastic heat equation is *chaotic* at fixed times, well before the onset of intermittency.

1. Introduction and main results. Let $\mathbf{W} := \{W(t, x)\}_{t\geq 0, x\in\mathbf{R}}$ denote a real-valued Brownian sheet indexed by two parameters $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$. That is, $\mathbf{W}$ is a centered Gaussian process with covariance

$$\mathrm{Cov}(W(t, x), W(s, y)) = \min(t, s) \times \min(|x|, |y|) \times \mathbf{1}_{(0,\infty)}(xy). \tag{1.1}$$

And let us consider the nonlinear stochastic heat equation

$$\frac{\partial}{\partial t} u_t(x) = \frac{\varkappa}{2} \frac{\partial^2}{\partial x^2} u_t(x) + \sigma(u_t(x)) \frac{\partial^2}{\partial t\, \partial x} W(t, x), \tag{1.2}$$

where $x \in \mathbf{R}$, $t > 0$, $\sigma : \mathbf{R} \to \mathbf{R}$ is a nonrandom and Lipschitz continuous function, $\varkappa > 0$ is a fixed viscosity parameter and the initial function $u_0 : \mathbf{R} \to \mathbf{R}$ is bounded, nonrandom and measurable. The mixed partial derivative $\partial^2 W(t, x)/(\partial t\, \partial x)$ is the so-called "space–time white noise," and is defined as a generalized Gaussian random field; see Chapter 2 of Gelfand and Vilenkin [17], Section 2.7, for example.

It is well known that the stochastic heat equation (1.2) has a (weak) solution $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$ that is jointly continuous; it is also unique up to evanescence; see, for example, Chapter 3 of Walsh [23], (3.5), page 312. And the solution can be

---

Received March 2011; revised November 2011.

[1]Supported in part by the Swiss National Science Foundation Fellowship PBELP2-122879.

[2]Supported in part by the NSF Grant DMS-07-47758.

[3]Supported in part by the NSF Grants DMS-07-06728 and DMS-10-06903.

MSC2010 subject classifications. Primary 60H15; secondary 35R60.

Key words and phrases. Stochastic heat equation, chaos, intermittency.

written in mild form as the (a.s.) solution to the following stochastic integral equation:

$$u_t(x) = (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))W(\mathrm{d}s\,\mathrm{d}y), \tag{1.3}$$

where

$$p_t(z) := \frac{1}{(2\pi\varkappa t)^{1/2}}\exp\left(-\frac{z^2}{2\varkappa t}\right) \qquad (t > 0, z \in \mathbf{R}) \tag{1.4}$$

denotes the free-space heat kernel, and the final integral in (1.3) is a stochastic integral in the sense of Walsh [23], Chapter 2. Chapter 1 of the minicourse by Dalang et al. [10] contains a quick introduction to the topic of stochastic PDEs of the type considered here.

We are interested solely in the physically interesting case that $u_0(x) \geq 0$ for all $x \in \mathbf{R}$. In that case, a minor variation of Mueller's comparison principle [21] implies that if in addition $\sigma(0) = 0$, then with probability one $u_t(x) \geq 0$ for all $t > 0$ and $x \in \mathbf{R}$; see also Theorem 5.1 of Dalang et al. [10], page 130, as well as Theorem 2.1 below.

We follow Foondun and Khoshnevisan [15], and say that the solution $u := \{u_t(x)\}_{t>0,x\in\mathbf{R}}$ to (1.2) is (weakly) *intermittent* if

$$0 < \limsup_{t\to\infty} \frac{1}{t}\log \mathrm{E}(|u_t(x)|^\nu) < \infty \qquad \text{for all } \nu \geq 2, \tag{1.5}$$

where "log" denotes the natural logarithm, to be concrete. Here, we refer to property (1.5), if and when it holds, as *mathematical intermittency* [to be distinguished from *physical intermittency*, which is a phenomenological property of an object that (1.2) is modeling].

If $\sigma(u) = \text{const}\cdot u$ and u_0 is bounded from above and below uniformly, then the work of Bertini and Cancrini [2] and Mueller's comparison principle [21] together imply (1.5). In the fully nonlinear case, Foondun and Khoshnevisan [15] discuss a connection to nonlinear renewal theory, and use that connection to establish (1.5) under various conditions; for instance, they have shown that (1.5) holds provided that $\liminf_{|x|\to\infty} |\sigma(x)/x| > 0$ and $\inf_{x\in\mathbf{R}} u_0(x)$ is sufficiently large.

If the lim sup in (1.5) is a bona fide limit, then we arrive at the usual description of intermittency in the literature of mathematics and theoretical physics; see, for instance, Molchanov [20] and Zeldovich et al. [24–26].

Mathematical intermittency is motivated strongly by a vast physics literature on (physical) intermittency and *localization*, and many of the references can be found in the combined bibliographies of [15, 20, 24, 25]. Let us say a few more words about "localization" in the present context.

It is generally accepted that if (1.5) holds, then $u := \{u_t(x)\}_{t>0,x\in\mathbf{R}}$ ought to undergo a separation of scales (or "pattern/period breaking"). In fact, one can argue that property (1.5) implies that, as $t \to \infty$, the random function $x \mapsto u_t(x)$ starts to develop very tall peaks, distributed over small x-intervals (see Section 2.4 of Bertini and Cancrini [2] and the Introduction of the monograph by Carmona and Molchanov [7]). This "peaking property" is called *localization*, and is experienced with very high probability, provided that: (i) the intermittency property (1.5) holds; and (ii) $t \gg 1$.

Physical intermittency is expected to hold both in space and time, and not only when $t \gg 1$. And it is also expected that there are physically-intermittent processes, not unlike those studied in the present paper, which, however, do not satisfy the (mathematical) intermittency condition (1.5) on Liapounov exponents; see, for example, the paper by Cherktov et al. [8].

Our wish is to better understand "physical intermittency" in the setting of the stochastic heat equation (1.2). We are motivated strongly by the literature on smooth finite-dimensional dynamical systems ([22], Section 1.3), which ascribes intermittency in part to "chaos," or slightly more precisely, sensitive dependence on the initial state of the system.

In order to describe the contributions of this paper, we first recall a consequence of a more general theorem of Foondun and Khoshnevisan [16]: if $\sigma(0) = 0$, and if u_0 is Hölder continuous of index $> \frac{1}{2}$ and has compact support, then for every $t > 0$ fixed,

$$\limsup_{z\to\infty} \frac{1}{\log z} \log \mathrm{P}\Big\{\sup_{x\in\mathbf{R}} u_t(x) > z\Big\} = -\infty. \tag{1.6}$$

It follows in particular that the global maximum of the solution (at a fixed time) is a finite (nonnegative) random variable.

By contrast, one expects that if

$$\inf_{x\in\mathbf{R}} u_0(x) > 0, \tag{1.7}$$

then the solution u_t is unbounded for all $t > 0$. Here we prove that fact and a good deal more; namely, we demonstrate here that there in fact exists a minimum rate of "blowup" that applies regardless of the parameters of the problem.

A careful statement requires a technical condition that turns out to be necessary as well as sufficient. In order to discover that condition, let us consider the case that $u_0(x) \equiv \rho > 0$ is a constant for all $x \in \mathbf{R}$. Then, (1.7) clearly holds; but there can be no blowup if $\sigma(\rho) = 0$. Indeed, in that case the unique solution to the stochastic heat equation is $u_t(x) \equiv \rho$, which is bounded. Thus, in order to have an unbounded solution, we need at the very least to consider the case that $\sigma(x) \neq 0$ for all $x > 0$. [Note that $\sigma(0) = 0$ is permitted.] Instead, we will assume the following seemingly stronger, but in fact more or less equivalent, condition from now on:

$$\sigma(x) > 0 \qquad \text{for all } x \in \mathbf{R} \setminus \{0\}. \tag{1.8}$$

We are ready to present the first theorem of this paper. Here and throughout we write "$f(R) \gtrsim g(R)$ as $R \to \infty$" in place of the more cumbersome "there exists a nonrandom $C > 0$ such that $\liminf_{R\to\infty} f(R)/g(R) \ge C$." The largest such C is called "the constant in $\gtrsim$." We might sometimes also write "$g(R) \lesssim f(R)$" in place of "$f(R) \gtrsim g(R)$." And there is a corresponding "constant in $\lesssim$."

THEOREM 1.1. *Let $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$ be a solution to*

$$\frac{\partial}{\partial t}u_t(x) = \frac{\varkappa}{2}\frac{\partial^2}{\partial x^2}u_t(x) + \sigma(u_t(x))\frac{\partial^2}{\partial t\,\partial x}W(t,x) \qquad (t > 0, x \in \mathbf{R}) \tag{1.9}$$

written in mild form (1.3), *where the initial function $u_0 : \mathbf{R} \to \mathbf{R}$ is bounded, nonrandom and satisfies*

$$\inf_{x\in\mathbf{R}} u_0(x) > 0. \tag{1.10}$$

Then, the following hold:

(1) *If $\inf_{x\in\mathbf{R}} \sigma(x) \ge \varepsilon_0 > 0$ and $t > 0$, then a.s.*,

$$\sup_{x\in[-R,R]} u_t(x) \gtrsim \frac{(\log R)^{1/6}}{\varkappa^{1/12}} \qquad \text{as } R \to \infty; \tag{1.11}$$

and the constant in $\gtrsim$ does not depend on $\varkappa$.

(2) *If $\sigma(x) > 0$ for all $x \in \mathbf{R}$ and there exists $\gamma \in (0, 1/6)$ such that*

$$\lim_{|x|\to\infty} \sigma(x)\log(|x|)^{(1/6)-\gamma} = \infty, \tag{1.12}$$

then for all $t > 0$ the following holds almost surely:

$$\sup_{x\in[-R,R]} |u_t(x)| \gtrsim \frac{(\log R)^{\gamma}}{\varkappa^{1/12}} \qquad \text{as } R \to \infty; \tag{1.13}$$

and the constant in $\gtrsim$ does not depend on $\varkappa$.

Note in particular that if σ is uniformly bounded below then a.s.,

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/6}} \ge \frac{\text{const}}{\varkappa^{1/12}}. \tag{1.14}$$

We believe that it is a somewhat significant fact that a rate $(\log R)^{1/6}$ of blowup exists that is valid for all u_0 and σ in the first part of Theorem 1.1. However, the actual numerical estimate—that is, the $(1/6)$th power of the logarithm—appears to be less significant, as the behavior in $\varkappa$ might suggest (see Remark 1.5). In fact, we believe that the actual blowup rate might depend critically on the fine properties of the function σ. Next, we highlight this assertion in one particularly interesting case. Here and throughout, we write "$f(R) \asymp g(R)$ as $R \to \infty$" as shorthand for "$f(R) \gtrsim g(R)$ and $g(R) \gtrsim f(R)$ as $R \to \infty$." The two constants in the preceding two $\gtrsim$'s are called the "constants in $\asymp$."

THEOREM 1.2. *If σ is uniformly bounded away from* 0 *and* ∞ *and* $t > 0$, *then*

$$\sup_{x\in[-R,R]} u_t(x) \asymp \frac{(\log R)^{1/2}}{\varkappa^{1/4}} \qquad \text{a.s. as } R \to \infty. \tag{1.15}$$

Moreover, for every fixed $\varkappa_0 > 0$, *the preceding constants in* $\asymp$ *do not depend on* $\varkappa \geq \varkappa_0$.

In particular, we find that if σ is bounded uniformly away from 0 and ∞, then there exist constants $c_*, c^* \in (0, \infty)$ such that

$$\frac{c_*}{\varkappa^{1/4}} \leq \limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \leq \frac{c^*}{\varkappa^{1/4}} \qquad \text{a.s.}, \tag{1.16}$$

uniformly for all $\varkappa \geq \varkappa_0$.

The preceding discusses the behavior in case σ is bounded uniformly away from 0; that is, a uniformly-noisy stochastic heat equation (1.2). In general, we can say little about the remaining case that $\sigma(0) = 0$. Nevertheless in the well-known parabolic Anderson model, namely (1.2) with $\sigma(x) = cx$ for some constant $c > 0$, we are able to obtain some results (Theorem 1.3) that parallel Theorems 1.1 and 1.2.

THEOREM 1.3. *If* $\sigma(x) = cx$ *for some* $c > 0$, *then a.s.*,

$$\log \sup_{x\in[-R,R]} u_t(x) \asymp \frac{(\log R)^{2/3}}{\varkappa^{1/3}} \qquad \text{as } R \to \infty, \tag{1.17}$$

and the constants in $\asymp$ *do not depend on* $\varkappa > 0$.

Hence, when $\sigma(x) = cx$ we can find constants $C_*, C^* \in (0, \infty)$ such that

$$0 < \limsup_{|x|\to\infty} \frac{u_t(x)}{\exp\{C_*(\log|x|)^{2/3}/\varkappa^{1/3}\}} \leq \limsup_{|x|\to\infty} \frac{u_t(x)}{\exp\{C^*(\log|x|)^{2/3}/\varkappa^{1/3}\}} < \infty,$$

almost surely.

REMARK 1.4. Thanks to (1.3), and since Walsh stochastic integrals have zero mean, it follows that $\mathrm{E}u_t(x) = (p_t * u_0)(x)$. In particular, $\mathrm{E}u_t(x) \leq \sup_{x\in\mathbf{R}} u_0(x)$ is uniformly bounded. Since $u_t(x)$ is nonnegative, it follows from Fatou's lemma that $\liminf_{|x|\to\infty} u_t(x) < \infty$ a.s. Thus, the behavior described by Theorem 1.1 is one about the highly-oscillatory nature of $x \mapsto u_t(x)$, valid for every fixed time $t > 0$. We will say a little more about this topic in Appendix B below.

REMARK 1.5. We pay some attention to the powers of the viscosity parameter $\varkappa$ in Theorems 1.2 and 1.3. Those powers suggest that at least two distinct universality classes can be associated to (1.2): (i) when σ is bounded uniformly away from zero and infinity, the solution behaves as random walk in weakly-interacting random environment; and (ii) when $\sigma(x) = cx$ for some $c > 0$, then the solution behaves as objects that arise in some random matrix models.

REMARK 1.6. In [18], (2), Kardar, Parisi and Zhang consider the solution u to (1.2) and apply formally the Hopf–Cole transformation $u_t(x) := \exp(\lambda h_t(x))$ to deduce that $\mathbf{h} := \{h_t(x)\}_{t\geq 0, x\in\mathbf{R}}$ satisfies the following "SPDE": for $t > 0$ and $x \in \mathbf{R}$,

$$\frac{\partial}{\partial t} h_t(x) = \frac{\varkappa}{2} \frac{\partial^2}{\partial x^2} h_t(x) + \frac{\varkappa\lambda}{2}\left(\frac{\partial}{\partial x} h_t(x)\right)^2 + \frac{\partial^2}{\partial t\,\partial x} W(t, x). \tag{1.18}$$

This is the celebrated "KPZ equation," named after the authors of [18], and the random field $\mathbf{h}$ is believed to be a universal object (e.g., it is expected to arise as a continuum limit of a large number of interacting particle systems). Theorem 1.3 implies that there exist positive and finite constants a_t and A_t—depending only on t—such that

$$\frac{a_t}{\varkappa^{1/3}} < \limsup_{|x|\to\infty} \frac{h_t(x)}{(\log|x|)^{2/3}} < \frac{A_t}{\varkappa^{1/3}} \qquad \text{a.s. for all } t > 0. \tag{1.19}$$

This is purely formal, but only because the construction of $\mathbf{h}$ via u is not rigorous. More significantly, our proofs suggest strongly a kind of asymptotic space–time scaling "$|\log x| \approx t^{\pm 1/2}$." If so, then the preceding verifies that fluctuation exponent $1/z$ of $\mathbf{h}$ is $2/3$ under quite general conditions on the h_0. The latter has been predicted by Kardar et al. [18], page 890, and proved by Balazs, Quastel and Seppäläinen [1] for a special choice of u_0 (hence h_0) and $t \to 0$.

The proofs of our three theorems involve a fairly long series of technical computations. Therefore, we conclude the Introduction with a few remarks on the methods of proofs for the preceding three theorems in order to highlight the "pictures behind the proofs."

Theorem 1.1 relies on two well-established techniques from interacting particle systems [14, 19]: namely, comparison and coupling. Comparison reduces our problem to the case that u_0 is a constant; at a technical level this uses Mueller's comparison principle [21]. And we use coupling on a few occasions: first, we describe a two-step coupling of $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$ to the solution $\{v_t(x)\}_{t>0,x\in\mathbf{R}}$ of (1.2)—using the same space–time white noise $\partial^2 W/(\partial t\,\partial x)$—in the case that σ is bounded below uniformly on $\mathbf{R}$. The latter quantity [i.e., $\{v_t(x)\}_{t>0,x\in\mathbf{R}}$] turns out to be more amenable to moment analysis than $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$, and in this way we obtain the following a priori estimate, valid for every $t > 0$ fixed:

$$\log \inf_{x\in\mathbf{R}} \mathrm{P}\{u_t(x) \geq \lambda\} \gtrsim -\sqrt{\varkappa}\lambda^6 \qquad \text{as } \lambda \to \infty. \tag{1.20}$$

Theorem 1.1 follows immediately from this and the Borel–Cantelli lemma, provided that we prove that if x and x' are "$O(1)$ distance apart," then $u_t(x)$ and $u_t(x')$

are “approximately independent.” A quantitative version of this statement follows from coupling $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$ to the solution $\{w_t(x)\}_{t>0,x\in\mathbf{R}}$ of a random evolution equation that can be thought of as the “localization” of the original stochastic heat equation (1.2). The localized approximation $\{w_t(x)\}_{t>0,x\in\mathbf{R}}$ has the property that $w_t(x)$ and $w_t(x')$ are (exactly) independent for “most” values of x and x' that are $O(1)$ distance apart. And this turns out to be adequate for our needs.

Theorem 1.2 requires establishing separately a lower and an upper bound on $\sup_{x\in[-R,R]} u_t(x)$. Both bounds rely heavily on the following quantitative improvement of (1.20): if σ is bounded, then

$$\log \inf_{x\in\mathbf{R}} \mathrm{P}\{u_t(x)\ge\lambda\} \gtrsim -\sqrt{\varkappa}\lambda^2 \qquad \text{as } \lambda\to\infty. \tag{1.21}$$

And, as it turns out, the preceding lower bound will perforce imply a corresponding upper estimate,

$$\log \inf_{x\in\mathbf{R}} \mathrm{P}\{u_t(x)\ge\lambda\} \lesssim -\sqrt{\varkappa}\lambda^2 \qquad \text{as } \lambda\to\infty. \tag{1.22}$$

The derivation of the lower bound on $\sup_{x\in[-R,R]} u_t(x)$ follows closely the proof of Theorem 1.1, after (1.21) and (1.22) are established. Therefore, the remaining details will be omitted.

The upper bound on $\sup_{x\in[-R,R]} u_t(x)$ requires only (1.22) and a well-known quantitative version of the Kolmogorov continuity theorem.

Our proof of Theorem 1.3 has a similar flavor to that of Theorem 1.1, for the lower bound, and Theorem 1.2, for the upper bound. We make strong use of the moments formulas of Bertini and Cancrini [2], Theorem 2.6. [This is why we are only able to study the linear equation in the case that $\sigma(0)=0$.]

Throughout this paper, we use the following abbreviation:

$$u_t^*(R) := \sup_{x\in[-R,R]} u_t(x) \qquad (R>0). \tag{1.23}$$

We will also need the following elementary facts about the heat kernel:

$$\|p_s\|_{L^2(\mathbf{R})}^2 = (4\pi\varkappa s)^{-1/2} \qquad \text{for every } s>0; \tag{1.24}$$

and therefore,

$$\int_0^t \|p_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s = \sqrt{t/(\pi\varkappa)} \qquad \text{for all } t\ge 0. \tag{1.25}$$

We will tacitly write Lip_σ for the optimal Lipschitz constant of σ; that is,

$$\mathrm{Lip}_\sigma := \sup_{-\infty<x\neq x'<\infty} \left|\frac{\sigma(x)-\sigma(x')}{x-x'}\right|. \tag{1.26}$$

Of course, Lip_σ is finite because σ is Lipschitz continuous. Finally, we use the following notation for the $L^\nu(\mathrm{P})$ norm of a random variable $Z\in L^\nu(\mathrm{P})$:

$$\|Z\|_\nu := \{\mathrm{E}(|Z|^\nu)\}^{1/\nu}. \tag{1.27}$$

2. Mueller's comparison principle and a reduction. Mueller's comparison principle [21] is one of the cornerstones of the theory of stochastic PDEs. In its original form, Mueller's comparison principle is stated for an equation that is similar to (1.2), but for two differences: (i) $\sigma(z) := \varkappa z$ for some $\varkappa > 0$; and (ii) the variable x takes values in a compact interval such as $[0, 1]$. In his Utah Minicourse ([10], Theorem 5.1, page 130), Mueller outlines how one can include also the more general functions σ of the type studied here. And in both cases, the proofs assume that the initial function u_0 has compact support. Below we state and prove a small variation of the preceding comparison principles that shows that Mueller's theory continues to work when: (i) the variable x takes values in $\mathbf{R}$; and (ii) the initial function u_0 is not necessarily compactly supported.

THEOREM 2.1 (Mueller's comparison principle). *Let $u_0^{(1)}$ and $u_0^{(2)}$ denote two nonnegative bounded continuous functions on $\mathbf{R}$ such that $u_0^{(1)}(x) \geq u_0^{(2)}(x)$ for all $x \in \mathbf{R}$. Let $u_t^{(1)}(x), u_t^{(2)}(x)$ be solutions to (1.2) with respective initial functions $u_0^{(1)}$ and $u_0^{(2)}$. Then,*

$$\mathrm{P}\{u_t^{(1)}(x) \geq u_t^{(2)}(x) \text{ for all } t > 0 \text{ and } x \in \mathbf{R}\} = 1. \tag{2.1}$$

PROOF. Because the solution to (1.2) is continuous in (t, x), it suffices to prove that

$$\mathrm{P}\{u_t^{(1)}(x) \geq u_t^{(2)}(x)\} = 1 \qquad \text{for all } t > 0 \text{ and } x \in \mathbf{R}. \tag{2.2}$$

In the case that $u_0^{(1)}$ and $u_0^{(2)}$ both have bounded support, the preceding is proved almost exactly as in Theorem 3.1 of Mueller [21]. For general $u_0^{(1)}$ and $u_0^{(2)}$, we proceed as follows.

Let $v_0 : \mathbf{R} \to \mathbf{R}_+$ be a bounded and measurable initial function, and define a new initial function $v_0^{[N]} : \mathbf{R} \to \mathbf{R}_+$ as

$$v_0^{[N]}(x) := \begin{cases} v_0(x), & \text{if } |x| \leq N, \\ v_0(N)(-x + N + 1), & \text{if } N < x < N + 1, \\ v_0(-N)(x + N + 1), & \text{if } -(N + 1) < x < -N, \\ 0, & \text{if } |x| \geq N + 1. \end{cases} \tag{2.3}$$

Then, let $v_t^{[N]}(x)$ be the solution to (1.2) with initial condition $v_0^{[N]}$. We claim that

$$\delta_t^{[N]}(x) := v_t(x) - v_t^{[N]}(x) \to 0 \qquad \text{in probability as } N \to \infty. \tag{2.4}$$

Let $u_t^{(1),[N]}$ and $u_t^{(2),[N]}$ denote the solutions to (1.2) with initial conditions $u_0^{(1),[N]}$ and $u_0^{(2),[N]}$, respectively, where the latter are defined similarly as $v_0^{[N]}$ above. Now, (2.4) has the desired result because it shows that $u_t^{(1),[N]}(x) \to u_t^{(1)}(x)$ and $u_t^{(2),[N]}(x) \to u_t^{(2)}(x)$ in probability as $N \to \infty$. Since $u_t^{(1),[N]}(x) \geq u_t^{(2),[N]}(x)$ a.s. for all $t > 0$ and $x \in \mathbf{R}$, (2.2) follows from taking limits.

In order to conclude we establish (2.4); in fact, we will prove that

$$\sup_{x\in\mathbf{R}}\sup_{t\in(0,T)} \mathrm{E}(|\delta_t^{[N]}(x)|^2) = O(1/N) \qquad \text{as } N\to\infty \tag{2.5}$$

for all $T > 0$ fixed. Recall that

$$\begin{aligned} \delta_t^{[N]}(x) &= (p_t * \delta_0^{[N]})(x) \\ &\quad + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\{\sigma(v_s(y)) - \sigma(v_s^{[N]}(y))\} W(\mathrm{d}s\,\mathrm{d}y). \end{aligned} \tag{2.6}$$

Because $(p_t * \delta_0^{[N]})(x) \le 2(\sup_{z\in\mathbf{R}} v_0(z)) \int_{|y|>N} p_t(y)\,\mathrm{d}y$, a direct estimate of the latter stochastic integral yields

$$\begin{aligned} \mathrm{E}(|\delta_t^{[N]}(x)|^2) &\le \mathrm{const}\cdot t^{-1/2}\mathrm{e}^{-N^2/(2t)} \\ &\quad + \mathrm{const}\cdot \mathrm{Lip}_\sigma^2 \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, p_{t-s}^2(y-x)\mathrm{E}(|\delta_s^{[N]}(y)|^2) \\ &\le \mathrm{const}\cdot t^{-1/2}\mathrm{e}^{-N^2/(2t)} \\ &\quad + \mathrm{const}\cdot \mathrm{Lip}_\sigma^2 \mathrm{e}^{\beta t} M_t^{[N]}(\beta)\int_0^\infty \mathrm{e}^{-\beta r}\|p_r\|_{L^2(\mathbf{R})}^2\,\mathrm{d}r, \end{aligned} \tag{2.7}$$

where $\beta > 0$ is, for the moment, arbitrary and

$$M_t^{[N]}(\beta) := \sup_{s\in(0,t),\,y\in\mathbf{R}} [\mathrm{e}^{-\beta s}\mathrm{E}(|v_s(y) - v_s^{[N]}(y)|^2)]. \tag{2.8}$$

We multiply both sides of (2.7) by $\exp(-\beta t)$ and take the supremum over all $t \in (0,T)$ where $T > 0$ is fixed. An application of (1.24) yields

$$M_T^{[N]}(\beta) \le \mathrm{const}\cdot\Big[\sup_{t\in(0,T)}\{t^{-1/2}\mathrm{e}^{-N^2/(2t)}\} + \beta^{-1/2}M_T^{[N]}(\beta)\Big]. \tag{2.9}$$

The quantity in $\sup_{t\in(0,T)}\{\cdots\}$ is proportional to $1/N$ (with the constant of proportionality depending on T), and the implied constant does not depend on β. Therefore, it follows that if β were selected sufficiently large, then $M_T^{[N]}(\beta) = O(1/N)$ as $N\to\infty$ for that choice of β. This implies (2.4). □

Next we apply Mueller's comparison principle to make a helpful simplification to our problem.

Because $\underline{B} := \inf_{x\in\mathbf{R}} u_0(x) > 0$ and $\overline{B} := \sup_{x\in\mathbf{R}} u_0(x) < \infty$, it follows from Theorem 2.1 that almost surely,

$$\underline{u}_t(x) \le u_t(x) \le \overline{u}_t(x) \qquad \text{for all } t>0 \text{ and } x\in\mathbf{R}, \tag{2.10}$$

where $\overline{u}$ solves the stochastic heat equation (1.2) starting from initial function $\overline{u}_0(x) := \overline{B}$, and $\underline{u}$ solves (1.2) starting from $\underline{u}_0(x) := \underline{B}$. This shows that it suffices to prove Theorems 1.1, 1.2 and 1.3 with $u_t(x)$ everywhere replaced by $\underline{u}_t(x)$

and $\overline{u}_t(x)$. In other words, we can assume without loss of generality that u_0 is identically a constant. In order to simplify the notation, we will assume from now on that the mentioned constant is one. A quick inspection of the ensuing proofs reveals that this assumption is harmless. Thus, from now on, we consider in place of (1.2), the following parabolic stochastic PDE:

$$\begin{cases} \frac{\partial}{\partial t}u_t(x) = \frac{\varkappa}{2}\frac{\partial^2}{\partial x^2}u_t(x) + \sigma(u_t(x))\frac{\partial^2}{\partial t\,\partial x}W(t,x), & (t>0, x\in\mathbf{R}), \\ u_0(x) = 1. \end{cases} \tag{2.11}$$

We can write its solution in mild form as follows:

$$u_t(x) = 1 + \int_{[0,t]\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))W(\mathrm{d}s\,\mathrm{d}y). \tag{2.12}$$

3. Tail probability estimates. In this section we derive the following corollary which estimates the tails of the distribution of $u_t(x)$, where $u_t(x)$ solves (2.11) and (2.12). In fact, Corollary 3.5, Propositions 3.7 and 3.8 below readily imply the following:

COROLLARY 3.1. *If* $\inf_{z\in\mathbf{R}}[\sigma(z)] > 0$, *then for all* $t > 0$,

$$-\sqrt{\varkappa}\lambda^6 \lesssim \log \mathrm{P}\{|u_t(x)| \geq \lambda\} \lesssim -\sqrt{\varkappa}(\log\lambda)^{3/2}, \tag{3.1}$$

uniformly for $x \in \mathbf{R}$ *and* $\lambda > \mathrm{e}$. *And the constants in* $\lesssim$ *do not depend on* $\varkappa$.

If (1.12) *holds for some* $\gamma \in (0, 1/6)$, *then for all* $t > 0$,

$$-\varkappa^{1/(12\gamma)}\lambda^{1/\gamma} \lesssim \log \mathrm{P}\{|u_t(x)| \geq \lambda\} \lesssim -\sqrt{\varkappa}(\log\lambda)^{3/2}, \tag{3.2}$$

uniformly for all $x \in \mathbf{R}$ *and* $\lambda > \mathrm{e}$. *And the constants in* $\lesssim$ *do not depend on* $\varkappa$.

3.1. *An upper-tail estimate.* We begin by working toward the upper bound in Corollary 3.1.

LEMMA 3.2. *Fix* $T > 0$, *and define* $a := T(\mathrm{Lip}_\sigma \vee 1)^4/(2\varkappa)$. *Then, for all real numbers* $k \geq 1$,

$$\sup_{x\in\mathbf{R}}\sup_{t\in[0,T]} \mathrm{E}(|u_t(x)|^k) \leq C^k \mathrm{e}^{ak^3} \qquad \textit{where } C := 8\left(1 + \frac{|\sigma(0)|}{2^{1/4}(\mathrm{Lip}_\sigma \vee 1)}\right).$$

PROOF. We follow closely the proof of Theorem 2.1 of [15], but matters simplify considerably in the present, more specialized, setting.

First of all, we note that because $u_0 \equiv 1$ is spatially homogeneous, the distribution of $u_t(x)$ does not depend on x; this property was observed earlier by Dalang [11], for example.

Therefore, an application of Burkholder's inequality, using the Carlen–Kree bound [6] on Davis's optimal constant [12] in the Burkholder–Davis–Gundy inequality [3–5] and Minkowski's inequality imply the following: for all $t \geq 0$, $\beta > 0$ and $x \in \mathbf{R}$,

$$
\begin{aligned}
\|u_t(x)\|_k &\leq 1 + \left\| \int_{[0,t]\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))W(\mathrm{d}s\,\mathrm{d}y) \right\|_k \\
&\leq 1 + \mathrm{e}^{\beta t} 2\sqrt{k}\Big(|\sigma(0)| + \mathrm{Lip}_\sigma \sup_{r\geq 0}[\mathrm{e}^{-2\beta r}\|u_r(x)\|_k]\Big) \\
&\qquad \times \left(\int_0^\infty \mathrm{e}^{-2\beta s}\|p_s\|_2^2\,\mathrm{d}s \right)^{1/2} \qquad (3.3)\\
&= 1 + \frac{\sqrt{k}\mathrm{e}^{\beta t}}{(8\varkappa\beta)^{1/4}}\Big(|\sigma(0)| + \mathrm{Lip}_\sigma \sup_{r\geq 0}[\mathrm{e}^{-2\beta r}\|u_r(x)\|_k]\Big).
\end{aligned}
$$

See Foondun and Khoshnevisan [15], Lemma 3.3, for the details of the derivation of such an estimate. (Although Lemma 3.3 of [15] is stated for even integers $k \geq 2$, a simple variation on the proof of that lemma implies the result for general $k \geq 1$; see Conus and Khoshnevisan [9].) It follows that

$$\psi(\beta, k) := \sup_{t\geq 0}[\mathrm{e}^{-\beta t}\|u_t(x)\|_k] \tag{3.4}$$

satisfies

$$\psi(\beta, k) \leq 1 + \frac{\sqrt{k}}{(4\varkappa\beta)^{1/4}}\big(|\sigma(0)| + \mathrm{Lip}_\sigma \psi(\beta, k)\big). \tag{3.5}$$

If $\mathrm{Lip}_\sigma = 0$, then clearly $\psi(\beta, k) < \infty$. If $\mathrm{Lip}_\sigma > 0$, then $\psi(\beta, k) < \infty$ for all $\beta > k^2 \mathrm{Lip}_\sigma^4 /(4\varkappa)$; therefore, the preceding proves that if $\beta > k^2 \mathrm{Lip}_\sigma^4 /(4\varkappa)$, then

$$\psi(\beta, k) \leq \frac{1}{1 - (\sqrt{k}\,\mathrm{Lip}_\sigma /(4\varkappa\beta)^{1/4})} \cdot \left(1 + \frac{\sqrt{k}|\sigma(0)|}{(4\varkappa\beta)^{1/4}}\right). \tag{3.6}$$

We apply this with $\beta := k^2(\mathrm{Lip}_\sigma \vee 1)^4/(2\varkappa)$ to obtain the lemma. □

REMARK 3.3. In the preceding results, the term $\mathrm{Lip}_\sigma \vee 1$ appears in place of the more natural quantity Lip_σ only because it can happen that $\mathrm{Lip}_\sigma = 0$. In the latter case, σ is a constant function, and the machinery of Lemma 3.2 is not needed since $u_t(x)$ is a centered Gaussian process with a variance that can be estimated readily. (We remind the reader that the case where σ is a constant is covered by Theorem 1.2; see Section 6.)

Next we describe a real-variable lemma that shows how to transform the moment estimate of Lemma 3.2 into subexponential moment estimates.

LEMMA 3.4. *Suppose X is a nonnegative random variable that satisfies the following*: *there exist finite numbers $a, C > 0$ and $b > 1$ such that*

$$\mathrm{E}(X^k) \le C^k \mathrm{e}^{ak^b} \qquad \text{for all real numbers } k \ge 1. \tag{3.7}$$

Then, $\mathrm{E}\exp\{\alpha(\log_+ X)^{b/(b-1)}\} < \infty$—*for* $\log_+ u := \log(u \vee \mathrm{e})$—*provided that*

$$0 < \alpha < \frac{1-b^{-1}}{(ab)^{1/(b-1)}}. \tag{3.8}$$

Lemmas 3.2, 3.4 and Chebyshev's inequality together imply the following result.

COROLLARY 3.5. *Choose and fix $T > 0$, and define $c_0 := \sqrt{2/3} \approx 0.8165$. Then for all $\alpha < c_0\sqrt{\varkappa}/(\sqrt{T}(\mathrm{Lip}_\sigma \vee 1))$,*

$$\sup_{x\in\mathbf{R}} \sup_{t\in[0,T]} \mathrm{E}\big(\mathrm{e}^{\alpha(\log_+ u_t(x))^{3/2}}\big) < \infty. \tag{3.9}$$

Consequently,

$$\limsup_{\lambda\uparrow\infty} \frac{1}{(\log\lambda)^{3/2}} \sup_{x\in\mathbf{R}} \sup_{t\in[0,T]} \log \mathrm{P}\{u_t(x) > \lambda\} \le \frac{-c_0\sqrt{\varkappa}}{\sqrt{T}(\mathrm{Lip}_\sigma \vee 1)}. \tag{3.10}$$

We skip the derivation of Corollary 3.5 from Lemma 3.4, as it is immediate. The result holds uniformly in $t \in [0, T]$ and $x \in \mathbf{R}$ as the constants a and C in Lemma 3.2 are independent of t and x. Instead we verify Lemma 3.4.

PROOF OF LEMMA 3.4. Because

$$\left[\log_+\left(\frac{X}{C}\right)\right]^{b/(b-1)} \le 2^{b/(b-1)} \cdot \big\{(\log_+ X)^{b/(b-1)} + (\log_+ C)^{b/(b-1)}\big\}, \tag{3.11}$$

we can assume without loss of generality that $C = 1$; for otherwise we may consider X/C in place of X from here on.

For all $z > \mathrm{e}$, Chebyshev's inequality implies that

$$\mathrm{P}\big\{\mathrm{e}^{\alpha(\log_+ X)^{b/(b-1)}} > z\big\} \le \mathrm{e}^{-\max_k g(k)}, \tag{3.12}$$

where

$$g(k) := k\left(\frac{\log z}{\alpha}\right)^{(b-1)/b} - ak^b. \tag{3.13}$$

One can check directly that $\max_k g(k) = c \log z$, where

$$c := \frac{1-b^{-1}}{\alpha \cdot (ab)^{1/(b-1)}}. \tag{3.14}$$

Thus, it follows that $\mathrm{P}\{\exp[\alpha(\log_+ X)^{b/(b-1)}] > z\} = O(z^{-c})$ for $z \to \infty$. Consequently, $\mathrm{E}\exp\{\alpha(\log_+ X)^{b/(b-1)}\} < \infty$ as long as $c > 1$; this is equivalent to the statement of the lemma. □

3.2. *Lower-tail estimates.* In this section we proceed to estimate the tail of the distribution of $u_t(x)$ from below. We first consider the simplest case in which σ is bounded uniformly from below, away from zero.

PROPOSITION 3.6. *If* $\varepsilon_0 := \inf_{z\in\mathbf{R}} \sigma(z) > 0$, *then for all* $t > 0$,

$$\inf_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^{2k}) \geq \big(\sqrt{2} + o(1)\big)(\mu_t k)^k \qquad (\text{as } k \to \infty) \tag{3.15}$$

where the "$o(1)$" term depends only on k, *and*

$$\mu_t := \frac{2}{\mathrm{e}} \cdot \varepsilon_0^2 \sqrt{\frac{t}{\pi\varkappa}}. \tag{3.16}$$

PROOF. Because the initial function in (2.11) is $u_0(x) \equiv 1$, it follows that the distribution of $u_t(x)$ does not depend on x; see Dalang [11]. Therefore, the "inf" in the statement of the proposition is superfluous.

Throughout, let us fix $x \in \mathbf{R}$ and $t > 0$. Now we may consider a mean-one martingale $\{M_\tau\}_{0\leq\tau\leq t}$ defined as follows:

$$M_\tau := 1 + \int_{(0,\tau)\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))W(\mathrm{d}s\,\mathrm{d}y) \qquad (0 \leq \tau \leq t). \tag{3.17}$$

The quadratic variation of this martingale is

$$\langle M\rangle_\tau = \int_0^\tau \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, p_{t-s}^2(y-x)\sigma^2(u_s(y)) \qquad (0 \leq \tau \leq t). \tag{3.18}$$

Therefore, by Itô's formula, for all positive integers k, and for every $\tau \in [0,t]$,

$$\begin{aligned} M_\tau^{2k} &= 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s + \binom{2k}{2}\int_0^\tau M_s^{2(k-1)}\,\mathrm{d}\langle M\rangle_s \\ &= 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s \\ &\quad + \binom{2k}{2}\int_0^\tau M_s^{2(k-1)}\,\mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, p_{t-s}^2(y-x)\sigma^2(u_s(y)). \end{aligned} \tag{3.19}$$

By the assumption of the lemma, $\sigma(u_s(y)) \geq \varepsilon_0$ a.s. Therefore,

$$\begin{aligned} M_\tau^{2k} &\geq 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s + \binom{2k}{2}\varepsilon_0^2\int_0^\tau M_s^{2(k-1)} \cdot \|p_{t-s}\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s \\ &= 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s + \binom{2k}{2}\varepsilon_0^2\int_0^\tau \frac{M_s^{2(k-1)}}{(4\pi\varkappa(t-s))^{1/2}}\,\mathrm{d}s. \end{aligned} \tag{3.20}$$

We set $\tau := t$ and then take expectations to find that

$$\begin{aligned} \mathrm{E}(M_t^{2k}) &\geq 1 + \binom{2k}{2}\varepsilon_0^2\int_0^t \mathrm{E}\big(M_s^{2(k-1)}\big)\frac{\mathrm{d}s}{(4\pi\varkappa(t-s))^{1/2}} \\ &= 1 + \binom{2k}{2}\varepsilon_0^2\int_0^t \mathrm{E}\big(M_s^{2(k-1)}\big)\nu(t,\mathrm{d}s), \end{aligned} \tag{3.21}$$

where the measures $\{\nu(t,\cdot)\}_{t>0}$ are defined as

$$\nu(t, \mathrm{d}s) := \frac{\mathbf{1}_{(0,t)}(s)}{(4\pi\varkappa(t-s))^{1/2}}\,\mathrm{d}s. \tag{3.22}$$

We may iterate the preceding in order to obtain

$$\begin{aligned} \mathrm{E}(M_t^{2k}) \geq 1 + \sum_{l=0}^{k-1} a_{l,k}\varepsilon_0^{2(l+1)} \cdot \int_0^t \nu(t, \mathrm{d}s_1) \int_0^{s_1} \nu(s_1, \mathrm{d}s_2) \cdots \\ \times \int_0^{s_l} \nu(s_l, \mathrm{d}s_{l+1}), \end{aligned} \tag{3.23}$$

where

$$a_{l,k} := \prod_{j=0}^{l} \binom{2k-2j}{2} \qquad \text{for } 0 \leq l < k \tag{3.24}$$

and $s_0 := t$. The right-hand side of (3.23) is exactly equal to $\mathrm{E}(M_t^{2k})$ in the case where $\sigma(z) \equiv \varepsilon_0$ for all $z \in \mathbf{R}$. Indeed, the same computation as above works with identities all the way through. In other words,

$$\mathrm{E}(|u_t(x)|^{2k}) = \mathrm{E}(M_t^{2k}) \geq \mathrm{E}(\eta_t(x)^{2k}), \tag{3.25}$$

where

$$\eta_t(x) := 1 + \varepsilon_0 \cdot \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\,W(\mathrm{d}s\,\mathrm{d}y). \tag{3.26}$$

We define

$$\zeta_t(x) := \varepsilon_0 \cdot \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\,W(\mathrm{d}s\,\mathrm{d}y), \tag{3.27}$$

so that $\eta_t(x) = 1 + \zeta_t(x)$. Clearly,

$$\mathrm{E}(\eta_t(x)^{2k}) \geq \mathrm{E}(\zeta_t(x)^{2k}). \tag{3.28}$$

Since ζ is a centered Gaussian process,

$$\mathrm{E}(\zeta_t(x)^{2k}) = [\mathrm{E}(\zeta_t(x)^2)]^k \cdot \frac{(2k)!}{k! \cdot 2^k} \tag{3.29}$$

and

$$\mathrm{E}(\zeta_t(x)^2) = \varepsilon_0^2 \cdot \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, p_{t-s}^2(y-x) = \varepsilon_0^2 \cdot \sqrt{\frac{t}{\pi\varkappa}}; \tag{3.30}$$

see (1.25). The proposition follows from these observations and Stirling's formula. □

We can now use Proposition 3.6 to obtain a lower estimate on the tail of the distribution of $u_t(x)$.

PROPOSITION 3.7. *If there exists* $\varepsilon_0 > 0$ *such that* $\sigma(x) \geq \varepsilon_0$ *for all* $x \in \mathbf{R}$, *then there exists a universal constant* $C \in (0, \infty)$ *such that for all* $t > 0$,

$$\liminf_{\lambda \to \infty} \frac{1}{\lambda^6} \inf_{x \in \mathbf{R}} \log \mathrm{P}\{|u_t(x)| \geq \lambda\} \geq -C \frac{(\mathrm{Lip}_\sigma \vee 1)^4 \sqrt{\varkappa}}{\varepsilon_0^6 \sqrt{t}}. \tag{3.31}$$

PROOF. Choose and fix $t > 0$ and $x \in \mathbf{R}$. We apply the celebrated Paley–Zygmund inequality in the following form: for every integer $k \geq 1$,

$$\begin{aligned}\mathrm{E}(|u_t(x)|^{2k}) &\leq \mathrm{E}\big(|u_t(x)|^{2k}; |u_t(x)| \geq \tfrac{1}{2}\|u_t(x)\|_{2k}\big) + \tfrac{1}{2}\mathrm{E}(|u_t(x)|^{2k}) \\ &\leq \sqrt{\mathrm{E}(|u_t(x)|^{4k})\mathrm{P}\{|u_t(x)| \geq \tfrac{1}{2}\|u_t(x)\|_{2k}\}} + \tfrac{1}{2}\mathrm{E}(|u_t(x)|^{2k}).\end{aligned} \tag{3.32}$$

This yields the following bound:

$$\begin{aligned}\mathrm{P}\left\{|u_t(x)| \geq \frac{1}{2}\|u_t(x)\|_{2k}\right\} &\geq \frac{[\mathrm{E}(|u_t(x)|^{2k})]^2}{4\mathrm{E}(|u_t(x)|^{4k})} \\ &\geq \exp\left(-\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\varkappa} k^3\big(1 + o(1)\big)\right)\end{aligned} \tag{3.33}$$

as $k \to \infty$; see Lemma 3.2 and Proposition 3.6. Another application of Proposition 3.6 shows that $\|u_t(x)\|_{2k} \geq (1 + o(1))(\mu_t k)^{1/2}$ as $k \to \infty$ where μ_t is defined in (3.16). This implies as $k \to \infty$,

$$\mathrm{P}\left\{|u_t(x)| \geq \frac{1}{2}(\mu_t k)^{1/2}\right\} \geq \exp\left[-\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\varkappa} k^3\big(1 + o(1)\big)\right]. \tag{3.34}$$

The proposition follows from this by setting k to be the smallest possible integer that satisfies $(\mu_t k)^{1/2} \geq \lambda$. □

Now, we study the tails of the distribution of $u_t(x)$ under the conditions of part (2) of Theorem 1.1.

PROPOSITION 3.8. *Suppose* $\sigma(x) > 0$ *for all* $x \in \mathbf{R}$ *and* (1.12) *holds for some* $\gamma \in (0, 1/6)$. *Then*

$$\liminf_{\lambda \to \infty} \inf_{x \in \mathbf{R}} \frac{\log \mathrm{P}\{|u_t(x)| > \lambda\}}{\lambda^{1/\gamma}} \geq -C\left(\frac{(\mathrm{Lip}_\sigma \vee 1)^{2/3} \varkappa^{1/12}}{t^{1/12}}\right)^{1/\gamma}, \tag{3.35}$$

where $C \in (0, \infty)$ *is a constant that depends only on* γ.

PROOF. For every integer $N \geq 1$, define

$$\sigma^{(N)}(x) := \begin{cases} \sigma(x), & \text{if } |x| \leq N, \\ \sigma(-N), & \text{if } x < -N, \\ \sigma(N), & \text{if } x > N. \end{cases} \tag{3.36}$$

It can be checked directly that $\sigma^{(N)}$ is a Lipschitz function, and that in fact: (i) $\mathrm{Lip}_{\sigma^{(N)}} \le \mathrm{Lip}_\sigma$; and (ii) and $\inf_{z\in\mathbf{R}} \sigma^{(N)}(z) > 0$.

Let $u_t^{(N)}(x)$ denote the solution to (2.11), when σ is replaced by $\sigma^{(N)}$. We first establish the bound

$$\mathrm{E}\bigl(\bigl|u_t^{(N)}(x) - u_t(x)\bigr|^2\bigr) = O(N^{-2}) \qquad \text{as } N \to \infty. \tag{3.37}$$

Let us observe, using the mild representation of the solution to (2.11), that

$$\mathrm{E}\bigl(\bigl|u_t^{(N)}(x) - u_t(x)\bigr|^2\bigr) \le 2(\mathcal{T}_1 + \mathcal{T}_2), \tag{3.38}$$

where

$$\begin{aligned}
\mathcal{T}_1 &:= \mathrm{E}\biggl(\biggl|\int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\bigl[\sigma^{(N)}\bigl(u_s^{(N)}(y)\bigr) - \sigma\bigl(u_s^{(N)}(y)\bigr)\bigr] W(\mathrm{d}s\,\mathrm{d}y)\biggr|^2\biggr) \\
&= \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, p_{t-s}^2(y-x) \mathrm{E}\bigl(\bigl|\sigma^{(N)}\bigl(u_s^{(N)}(y)\bigr) - \sigma\bigl(u_s^{(N)}(y)\bigr)\bigr|^2\bigr) \quad \text{and} \\
\mathcal{T}_2 &:= \mathrm{E}\biggl(\biggl|\int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\bigl[\sigma\bigl(u_s^{(N)}(y)\bigr) - \sigma(u_s(y))\bigr] W(\mathrm{d}s\,\mathrm{d}y)\biggr|^2\biggr) \\
&= \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, p_{t-s}^2(y-x) \mathrm{E}\bigl(\bigl|\sigma\bigl(u_s^{(N)}(y)\bigr) - \sigma(u_s(y))\bigr|^2\bigr).
\end{aligned} \tag{3.39}$$

We can estimate the integrand of $\mathcal{T}_1$ by the following:

$$\begin{aligned}
&\mathrm{E}\bigl(\bigl|\sigma^{(N)}\bigl(u_s^{(N)}(y)\bigr) - \sigma\bigl(u_s^{(N)}(y)\bigr)\bigr|^2\bigr) \\
&\quad \le \mathrm{Lip}_\sigma^2 \cdot \mathrm{E}\bigl(\bigl|N - u_s^{(N)}(y)\bigr|^2; u_s^{(N)}(y) > N\bigr) \\
&\qquad + \mathrm{Lip}_\sigma^2 \cdot \mathrm{E}\bigl(\bigl|-N - u_s^{(N)}(y)\bigr|^2; u_s^{(N)}(y) < -N\bigr) \\
&\quad \le 4\,\mathrm{Lip}_\sigma^2 \cdot \mathrm{E}\bigl(\bigl|u_s^{(N)}(y)\bigr|^2; \bigl|u_s^{(N)}(y)\bigr| > N\bigr).
\end{aligned} \tag{3.40}$$

We first apply the Cauchy–Schwarz inequality and then Chebyshev's inequality (in this order) to conclude that

$$\begin{aligned}
\mathrm{E}\bigl(\bigl|\sigma^{(N)}\bigl(u_s^{(N)}(y)\bigr) - \sigma\bigl(u_s^{(N)}(y)\bigr)\bigr|^2\bigr) &\le \frac{4\,\mathrm{Lip}_\sigma^2}{N^2} \cdot \mathrm{E}\bigl(\bigl|u_s^{(N)}(y)\bigr|^4\bigr) \\
&= O(N^{-2}) \qquad \text{as } N \to \infty,
\end{aligned} \tag{3.41}$$

uniformly for all $y \in \mathbf{R}$ and $s \in (0,t)$. Indeed, Lemma 3.2 ensures that $\mathrm{E}[|u_s^{(N)}(y)|^4]$ is bounded in N, because $\lim_{N\to\infty} \mathrm{Lip}_{\sigma^{(N)}} = \mathrm{Lip}_\sigma$. This implies readily that $\mathcal{T}_1 = O(N^{-2})$ as $N \to \infty$.

Next we turn to $\mathcal{T}_2$; thanks to (1.25), the quantity $\mathcal{T}_2$ can be estimated as follows:

$$\begin{aligned}
\mathcal{T}_2 &\le \mathrm{Lip}_\sigma^2 \cdot \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\, p_{t-s}^2(y-x) \mathrm{E}\bigl(\bigl|u_s^{(N)}(y) - u_s(y)\bigr|^2\bigr) \\
&\le \mathrm{const} \cdot \int_0^t \frac{\mathcal{M}(s)}{\sqrt{t-s}},
\end{aligned} \tag{3.42}$$

where

$$\mathcal{M}(s) := \sup_{y\in\mathbf{R}} \mathrm{E}\big(\big|u_s^{(N)}(y) - u_s(y)\big|^2\big) \qquad (0 \le s \le t). \tag{3.43}$$

Notice that the implied constant in (3.42) does not depend on t.

We now combine our estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$ to conclude that

$$\begin{aligned}\mathcal{M}(s) &\le \frac{\text{const}}{N^2} + \text{const}\cdot\int_0^s \frac{\mathcal{M}(r)}{\sqrt{s-r}}\,\mathrm{d}r \qquad (0\le s\le t)\\ &\le \text{const}\cdot\left\{\left(\int_0^s [\mathcal{M}(r)]^{3/2}\,\mathrm{d}r\right)^{2/3} + \frac{1}{N^2}\right\},\end{aligned}\tag{3.44}$$

thanks to Hölder's inequality. We emphasize that the implied constant depends only on the Lipschitz constant of σ, the variable t and the diffusion constant $\varkappa$. Therefore,

$$[\mathcal{M}(s)]^{3/2} \le \text{const}\cdot\left\{\int_0^s [\mathcal{M}(r)]^{3/2}\,\mathrm{d}r + \frac{1}{N^3}\right\}, \tag{3.45}$$

uniformly for $s \in (0,t)$. Gronwall's inequality then implies the bound $\mathcal{M}(t) = O(N^{-2})$, valid as $N\to\infty$.

Now we proceed with the proof of Proposition 3.8. For all $N \ge 1$, the function $\sigma^{(N)}$ is bounded below. Let $\varepsilon(N)$ be such that $\sigma^{(N)}(x) \ge \varepsilon(N)$ for all $x \in \mathbf{R}$. Let $D := D_t := (4t/(\mathrm{e}^2\pi\varkappa))^{1/4}$. According to the proof of Proposition 3.7, specifically (3.34) applied to $u^{(N)}$, we have

$$\begin{aligned}\mathrm{P}\left\{|u_t(x)| \ge \frac{D}{4}\varepsilon(N)k^{1/2}\right\} &\ge \exp\left[-\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\varkappa}k^3\big(1+o(1)\big)\right]\\ &\quad - \mathrm{P}\left\{\big|u_t(x) - u_t^{(N)}(x)\big| \ge \frac{D}{4}\varepsilon(N)k^{1/2}\right\}.\end{aligned}\tag{3.46}$$

Thanks to (1.12), we can write

$$\varepsilon(N) \gg (\log N)^{-(1/6-\gamma)} \qquad \text{as } N\to\infty, \tag{3.47}$$

using standard notation. Therefore, if we choose

$$N := \left\lfloor \exp\left\{\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4 k^3}{\varkappa}\right\}\right\rfloor, \tag{3.48}$$

then we are led to the bound

$$\varepsilon(N) \gg \left(\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\varkappa}\right)^{-(1/6-\gamma)} k^{3\gamma-(1/2)}. \tag{3.49}$$

We can use Chebyshev's inequality in order to estimate the second term on the right-hand side of (3.46). In this way we obtain the following:

$$\begin{aligned}\mathrm{P}\left\{|u_t(x)| \ge \frac{\tilde{D}}{4}k^{3\gamma}\right\} &\ge \exp\left\{-\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\varkappa}k^3\big(1+o(1)\big)\right\}\\ &\quad - \frac{1}{C_1 N^2 k^{6\gamma}},\end{aligned}\tag{3.50}$$

where

$$\tilde{D} := D\left\{\frac{\varkappa}{64t(\mathrm{Lip}_\sigma \vee 1)^4}\right\}^{(1/6)-\gamma}, \tag{3.51}$$

and C_1 is a constant that depends only on t, Lip_σ and $\varkappa$. For all sufficiently large integers N,

$$\frac{1}{C_1 N^2 k^{6\gamma}} \le \exp\left[-\frac{128t(\mathrm{Lip}_\sigma \vee 1)^4}{\varkappa} k^3\bigl(1+o(1)\bigr)\right], \tag{3.52}$$

and the proposition follows upon setting $\lambda := \tilde{D}k^{3\gamma}/4$. □

4. Localization. The next step in the proof of Theorem 1.1 requires us to show that if x and x' are $O(1)$ apart, then $u_t(x)$ and $u_t(x')$ are approximately independent. We show this by coupling $u_t(x)$ first to the solution of a localized version—see (4.1) below—of the stochastic heat equation (2.11). And then a second coupling to a suitably-chosen Picard-iteration approximation of the mentioned localized version.

Consider the following parametric family of random evolution equations (indexed by the parameter $\beta > 0$):

$$U_t^{(\beta)}(x) = 1 + \int_{(0,t)\times[x-\sqrt{\beta t}, x+\sqrt{\beta t}]} p_{t-s}(y-x)\sigma\bigl(U_s^{(\beta)}(y)\bigr)W(\mathrm{d}s\,\mathrm{d}y) \tag{4.1}$$

for all $x \in \mathbf{R}$ and $t \ge 0$.

LEMMA 4.1. *Choose and fix $\beta > 0$. Then, (4.1) has an almost surely unique solution $U^{(\beta)}$ such that for all $T > 0$ and $k \ge 1$,*

$$\sup_{\beta>0}\sup_{t\in[0,T]}\sup_{x\in\mathbf{R}} \mathrm{E}\bigl(\bigl|U_t^{(\beta)}(x)\bigr|^k\bigr) \le C^k \mathrm{e}^{ak^3}, \tag{4.2}$$

where a and C are defined in Lemma 3.2.

PROOF. A fixed-point argument shows that there exists a unique, up to modification, solution to (4.1) subject to the condition that for all $T > 0$,

$$\sup_{t\in[0,T]}\sup_{x\in\mathbf{R}} \mathrm{E}\bigl(\bigl|U_t^{(\beta)}(x)\bigr|^k\bigr) < \infty \qquad \text{for all } k \ge 1. \tag{4.3}$$

See Foondun and Khoshnevisan [15] for more details on the ideas of the proof; and the moment estimate follows as in the proof of Lemma 3.2. We omit the numerous remaining details. □

LEMMA 4.2. *For every $T > 0$ there exists a finite and positive constant $C := C(\varkappa)$ such that for sufficiently large $\beta > 0$ and $k \ge 1$,*

$$\sup_{t\in[0,T]}\sup_{x\in\mathbf{R}} \mathrm{E}\bigl(\bigl|u_t(x) - U_t^{(\beta)}(x)\bigr|^k\bigr) \le C^k k^{k/2} \mathrm{e}^{Fk(k^2-\beta)}, \tag{4.4}$$

where $F \in (0,\infty)$ depends on $(T, \varkappa)$ but not on (k, β).

PROOF. For all $x \in \mathbf{R}$ and $t > 0$, define

$$V_t(x) := 1 + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma\big(U_s^{(\beta)}(y)\big)W(\mathrm{d}s\,\mathrm{d}y). \tag{4.5}$$

Then,

$$\begin{aligned}
&\big\|V_t(x) - U_t^{(\beta)}(x)\big\|_k \\
&\quad = \bigg\|\int_{(0,t)\times\{y\in\mathbf{R}:\,|y-x|>\sqrt{\beta t}\}} p_{t-s}(y-x)\sigma\big(U_s^{(\beta)}(y)\big)W(\mathrm{d}s\,\mathrm{d}y)\bigg\|_k \\
&\quad \le 2\sqrt{k}\bigg\|\int_0^t \mathrm{d}s \int_{|y-x|\ge\sqrt{\beta t}} \mathrm{d}y\, p_{t-s}^2(y-x)\sigma^2\big(U_s^{(\beta)}(y)\big)\bigg\|_{k/2}^{1/2}.
\end{aligned} \tag{4.6}$$

The preceding hinges on an application of Burkholder's inequality, using the Carlen–Kree bound [6] on Davis's optimal constant [12] in the Burkholder–Davis–Gundy inequality [3–5]; see Foondun and Khoshnevisan [15] for the details of the derivation of such an estimate. Minkowski's inequality tells us then that the preceding quantity is at most

$$\begin{aligned}
&2\sqrt{k\int_0^t \mathrm{d}s \int_{|y-x|\ge\sqrt{\beta t}} \mathrm{d}y\, p_{t-s}^2(y-x)\big\|\sigma^2\big(U_s^{(\beta)}(y)\big)\big\|_{k/2}} \\
&\quad \le \mathrm{const}\cdot\sqrt{k\int_0^t \mathrm{d}s \int_{|y-x|\ge\sqrt{\beta t}} \mathrm{d}y\, p_{t-s}^2(y-x)\big(1 + \big\|U_s^{(\beta)}(y)\big\|_k^2\big)}.
\end{aligned} \tag{4.7}$$

Equation (4.7) holds because the Lipschitz continuity of the function σ ensures that it has at-most-linear growth: $|\sigma(x)| \le \mathrm{const}\cdot(1+|x|)$ for all $x \in \mathbf{R}$. The inequality in Lemma 4.1 implies that, uniformly over all $t \in [0, T]$ and $x \in \mathbf{R}$,

$$\begin{aligned}
\big\|V_t(x) - U_t^{(\beta)}(x)\big\|_k &\le \mathrm{const}\cdot\sqrt{kC^2\mathrm{e}^{2ak^2}\int_0^t \mathrm{d}r \int_{|z|\ge\sqrt{\beta t}} \mathrm{d}z\, p_r^2(z)} \\
&\le \mathrm{const}\cdot\frac{k^{1/2}\mathrm{e}^{ak^2}}{\sqrt{\varkappa}}\sqrt{\int_0^1 \frac{\mathrm{d}s}{\sqrt{s}}\int_{|w|\ge\sqrt{2\beta}} \mathrm{d}w\, p_s(w)},
\end{aligned} \tag{4.8}$$

where we have used (1.4). Now a standard Gaussian tail estimate yields

$$\int_{|w|\ge\sqrt{2\beta}} p_s(w)\,\mathrm{d}w \le 2\mathrm{e}^{-\beta/s\varkappa}, \tag{4.9}$$

and the latter quantity is at most $2\exp(-\beta/\varkappa)$ whenever $s \in (0, 1]$. Therefore, on one hand,

$$\sup_{x\in\mathbf{R}}\big\|V_t(x) - U_t^{(\beta)}(x)\big\|_k \le \mathrm{const}\cdot\frac{k^{1/2}\mathrm{e}^{ak^2}}{\sqrt{\varkappa}}\mathrm{e}^{-\beta/2\varkappa}. \tag{4.10}$$

On the other hand,

$$u_t(x) - V_t(x) = \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\big[\sigma(u_s(y)) - \sigma\big(U_s^{(\beta)}(y)\big)\big] W(\mathrm{d}s\,\mathrm{d}y),$$

whence

$$\begin{aligned}
&\|u_t(x) - V_t(x)\|_k \\
&\quad \le 2\sqrt{k}\left\| \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, p_{t-s}^2(y-x)\big[\sigma(u_s(y)) - \sigma\big(U_s^{(\beta)}(y)\big)\big]^2 \right\|_{k/2}^{1/2} \\
&\quad \le 2\sqrt{k}\,\mathrm{Lip}_\sigma \left\| \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, p_{t-s}^2(y-x)\big[u_s(y) - U_s^{(\beta)}(y)\big]^2 \right\|_{k/2}^{1/2} \\
&\quad \le 2\sqrt{k}\,\mathrm{Lip}_\sigma \cdot \sqrt{\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, p_{t-s}^2(y-x)\big\|u_s(y) - U_s^{(\beta)}(y)\big\|_k^2}.
\end{aligned} \tag{4.11}$$

Consequently, (4.10) implies that

$$\begin{aligned}
&\big\|u_t(x) - U_t^{(\beta)}(x)\big\|_k \\
&\quad \le 2\sqrt{k}\,\mathrm{Lip}_\sigma \cdot \sqrt{\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, p_{t-s}^2(y-x)\big\|u_s(y) - U_s^{(\beta)}(y)\big\|_k^2} \\
&\qquad + \mathrm{const} \cdot \frac{k^{1/2}\mathrm{e}^{ak^2}}{\sqrt{\varkappa}} \mathrm{e}^{-\beta/(2\varkappa)}.
\end{aligned} \tag{4.12}$$

Let us introduce a parameter $\delta > 0$ and define the seminorms

$$\mathcal{N}_{k,\delta}(Z) := \sup_{s\ge 0}\sup_{y\in\mathbf{R}}\big[\mathrm{e}^{-\delta s}\|Z_s(y)\|_k\big] \tag{4.13}$$

for every space–time random field $Z := \{Z_s(y)\}_{s>0, y\in\mathbf{R}}$. Then, we have

$$\begin{aligned}
&\mathcal{N}_{k,\delta}\big(u - U^{(\beta)}\big) \\
&\quad \le 2\sqrt{k}\,\mathrm{Lip}_\sigma\, \mathcal{N}_{k,\delta}\big(u - U^{(\beta)}\big) \cdot \sqrt{\int_0^\infty \mathrm{e}^{-2\delta r}\|p_r\|_{L^2(\mathbf{R})}^2\,\mathrm{d}r} \\
&\qquad + \mathrm{const} \cdot \frac{k^{1/2}\mathrm{e}^{ak^2-\beta/(2\varkappa)}}{\sqrt{\varkappa}}.
\end{aligned} \tag{4.14}$$

Thanks to (1.24), if $\delta := Dk^2$ for some sufficiently large constant D, then the square root is at most $[4\sqrt{k}(\mathrm{Lip}_\sigma \vee 1)]^{-1}$, whence it follows that (for that fixed choice of δ)

$$\mathcal{N}_{k,\delta}\big(u - U^{(\beta)}\big) \le \mathrm{const} \cdot \frac{k^{1/2}\mathrm{e}^{ak^2 - \mathrm{const}\cdot(\beta/\varkappa)}}{\sqrt{\varkappa}}. \tag{4.15}$$

The lemma follows from this. □

Now let us define $U_t^{(\beta,n)}(x)$ to be the nth Picard-iteration approximation to $U_t^{(\beta)}(x)$. That is, $U_t^{(\beta,0)}(x) := 1$, and for all $\ell \geq 0$,

$$U_t^{(\beta,\ell+1)}(x) := 1 + \int_{(0,t)\times[x-\sqrt{\beta t},x+\sqrt{\beta t}]} p_{t-s}(y-x)\sigma\big(U_s^{(\beta,\ell)}(y)\big) W(\mathrm{d}s\,\mathrm{d}y). \tag{4.16}$$

LEMMA 4.3. *There exist positive and finite constants C_* and G—depending on $(t, \varkappa)$—such that uniformly for all $k \in [2, \infty)$ and $\beta > \mathrm{e}$,*

$$\sup_{x\in\mathbf{R}} \mathrm{E}\big(\big|u_t(x) - U_t^{(\beta,[\log\beta]+1)}(x)\big|^k\big) \leq \frac{C_*^k k^{k/2} \mathrm{e}^{Gk^3}}{\beta^k}. \tag{4.17}$$

PROOF. The method of Foondun and Khoshnevisan [15] shows that if $\delta := D'k^2$ for a sufficiently-large D', then

$$\mathcal{N}_{k,\delta}\big(U^{(\beta)} - U^{(\beta,n)}\big) \leq \mathrm{const}\cdot \mathrm{e}^{-n} \qquad \text{for all } n \geq 0 \text{ and } k \in [2,\infty). \tag{4.18}$$

To elaborate, we follow the arguments in [15] of the proof of Theorem 2.1 leading up to equation (4.6) but with v_n there replaced by $U^{(\beta,n)}$ here. We then obtain

$$\big\|U^{(\beta,n+1)} - U^{(\beta,n)}\big\|_{k,\theta} \leq \mathrm{const}\cdot\sqrt{k\Upsilon\left(\frac{2\theta}{k}\right)}\big\|U^{(\beta,n)} - U^{(\beta,n-1)}\big\|_{k,\theta}, \tag{4.19}$$

where

$$\|f\|_{k,\theta} := \Big\{\sup_{t\geq 0}\sup_{x\in\mathbf{R}} \mathrm{e}^{-\theta t}\mathrm{E}(|f(t,x)|^k)\Big\}^{1/k} \tag{4.20}$$

and

$$\Upsilon(\theta) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{\theta + \xi^2}. \tag{4.21}$$

A quick computation reveals that by choosing $\theta := D''k^3$, for a large enough constant $D'' > 0$, we obtain

$$\big\|U^{(\beta,n+1)} - U^{(\beta,n)}\big\|_{k,\theta} \leq \mathrm{e}^{-1}\big\|U^{(\beta,n)} - U^{(\beta,n-1)}\big\|_{k,\theta}. \tag{4.22}$$

We get (4.18) from this.

Next we set $n := [\log\beta] + 1$ and apply the preceding together with Lemma 4.2 to finish the proof. □

LEMMA 4.4. *Choose and fix $\beta, t > 0$ and $n \geq 0$. Also fix $x_1, x_2, \ldots \in \mathbf{R}$ such that $|x_i - x_j| \geq 2n\sqrt{\beta t}$ whenever $i \neq j$. Then $\{U_t^{(\beta,n)}(x_j)\}_{j\in\mathbf{Z}}$ is a collection of i.i.d. random variables.*

PROOF. The proof uses induction on the variable n, and proceeds by establishing a little more. We will use the σ-algebras $\mathcal{P}(A)$ as defined in Appendix A, where $A \subset \mathbf{R}$ ranges over all Lebesgue-measurable sets of finite Lebesgue measure Proposition A.1.

Since $U_t^{(\beta,0)}(x) \equiv 1$, the statement of the lemma holds tautologically for $n = 0$. In order to understand the following argument better let us concentrate on the case $n = 1$. In that case,

$$U_t^{(\beta,1)}(x) = 1 + \sigma(1) \cdot \int_{(0,t)\times[x-\sqrt{\beta t},x+\sqrt{\beta t}]} p_{t-s}(y-x) W(\mathrm{d}s\,\mathrm{d}y). \tag{4.23}$$

In particular, if we define the process $\{\tilde{U}_s^{(\beta,1)}(t,x)\}_{0\le s\le t}$ by

$$\tilde{U}_s^{(\beta,1)}(t,x) = 1 + \sigma(1) \cdot \int_{(0,s)\times[x-\sqrt{\beta t},x+\sqrt{\beta t}]} p_{t-r}(y-x) W(\mathrm{d}r\,\mathrm{d}y), \tag{4.24}$$

it follows from Proposition A.1 that $U_\bullet^{(\beta,1)}(t,x) \in \mathcal{P}([x-\sqrt{\beta t}, x+\sqrt{\beta t}])$. Hence, Corollary A.2 shows that $\{\tilde{U}_\bullet^{(\beta,1)}(t,x_j)\}_{j\in\mathbf{Z}}$ are independent processes. Taking $s = t$ shows the independence part of the lemma for $n = 1$. It is not hard to see that the law of $U_t^{(\beta,1)}(x)$ is independent of x, namely a Gaussian distribution with mean and variance parameters that are independent of x. This concludes the proof for $n = 1$.

Now, let us define processes $\{\tilde{U}_s^{(\beta,n)}(t,x)\}_{0\le s\le t}$ by

$$\begin{aligned}\tilde{U}_s^{(\beta,n)}&(t,x)\\ &= 1 + \sigma(1) \cdot \int_{(0,s)\times[x-\sqrt{\beta t},x+\sqrt{\beta t}]} p_{t-r}(y-x)\sigma\big(\tilde{U}_r^{(\beta,n-1)}(r,y)\big)\\ &\qquad \times W(\mathrm{d}r\,\mathrm{d}y).\end{aligned} \tag{4.25}$$

If we proved that $\tilde{U}_\bullet^{(\beta,n)}(t,x) \in \mathcal{P}([x-n\sqrt{\beta t}, x+n\sqrt{\beta t}])$ for all $x \in \mathbf{R}$ and $t > 0$, it then would follow from (4.16) and Proposition A.1 that $\tilde{U}_\bullet^{(\beta,n+1)}(t,x) \in \mathcal{P}([x-(n+1)\sqrt{\beta t}, x+(n+1)\sqrt{\beta t}])$ for all $x \in \mathbf{R}$ and $t > 0$. Since this fact is true for $n = 1$, then we have proved that $\tilde{U}_\bullet^{(\beta,n)}(t,x) \in \mathcal{P}([x-n\sqrt{\beta t}, x+n\sqrt{\beta t}])$ for all $x \in \mathbf{R}$, $t > 0$ and $n \in \mathbf{N}$. We then use this result, together with Corollary A.2 to deduce that $\{\tilde{U}_\bullet^{(\beta,n)}(t,x_j)\}_{j\in\mathbf{Z}}$ are independent processes. The independence part of the lemma follows upon taking $s := t$ in this discussion.

Since the law of the noise W is invariant by translation, it is not hard to prove by induction that the law of $U_t^{(\beta,n)}(x)$ is indeed independent of x. (Notice that this is always true when the initial condition is constant [11], Lemma 18.) This concludes the (inductive) proof of the lemma. □

5. Proof of Theorem 1.1. We are ready to combine our efforts thus far in order to verify Theorem 1.1.

PROOF OF THEOREM 1.1. Parts (1) and (2) of Theorem 1.1 are proved similarly. Therefore, we present the details of the second part. For the proof of the first part, we can take $\gamma := 1/6$ in the following argument. We remind that the processes U^{β} and $U^{(\beta,n)}$ are defined, respectively, in (4.1) and (4.16).

For all $x_1, \ldots, x_N \in \mathbf{R}$,

$$\mathrm{P}\Big\{\max_{1\le j\le N}|u_t(x_j)| < \lambda\Big\} \le \mathrm{P}\Big\{\max_{1\le j\le N}\big|U_t^{(\beta,\log\beta)}(x_j)\big| < 2\lambda\Big\} + \mathrm{P}\Big\{\max_{1\le j\le N}\big|U_t^{(\beta,\log\beta)}(x_j) - u_t(x_j)\big| > \lambda\Big\}. \tag{5.1}$$

(To be very precise, we need to write $(\beta, [\log\beta]+1)$ in place of $(\beta, \log\beta)$.) The whole program of Section 3, that led to Proposition 3.8 can be carried out for $U^{(\beta)}$ instead of u. Only minor changes (typically in Proposition 3.6) are needed. Then, (4.18) shows that the same moment estimates are valid for $U^{(\beta,n)}$ as for $U^{(\beta)}$.

We can follow along the proof of Proposition 3.8 and prove similarly the existence of constants $c_1, c_2 > 0$—independent of β for all sufficiently large values of β—so that for all $x \in \mathbf{R}$ and $\lambda \ge 1$,

$$\mathrm{P}\big\{\big|U_t^{(\beta,\log\beta)}(x)\big| \ge \lambda\big\} \ge c_1 \mathrm{e}^{-c_2\lambda^{1/\gamma}}. \tag{5.2}$$

Suppose, in addition, that $|x_i - x_j| \ge 2\sqrt{\beta t}\log\beta$ whenever $i \ne j$. Then, Lemmas 4.3 and 4.4 together imply the following:

$$\mathrm{P}\Big\{\max_{1\le j\le N}|u_t(x_j)| < \lambda\Big\} \le \big(1 - c_1\mathrm{e}^{-c_2\cdot(2\lambda)^{1/\gamma}}\big)^N + NC_*^k k^{k/2}\mathrm{e}^{Gk^3}\beta^{-k}\lambda^{-k}. \tag{5.3}$$

The constants C_* and G may differ from the ones in Lemma 4.3. We now select the various parameters judiciously: choose $\lambda := k$, $N := \lceil k\exp(c_2\cdot(2k)^{1/\gamma})\rceil$ and $\beta := \exp(\rho k^{(1-\gamma)/\gamma})$ for a large-enough positive constant $\rho > 2\cdot 3^{1/\gamma}c_2$. In this way, (5.3) simplifies: for all sufficiently large integers k,

$$\begin{aligned}\mathrm{P}\Big\{\max_{1\le j\le N}|u_t(x_j)| < k\Big\} &\le \mathrm{e}^{-c_1k} + \exp\Big[c_2\cdot(2k)^{1/\gamma} + \log k + k\log C_* - \frac{k\log k}{2} + Gk^3 - \frac{\rho k^{1/\gamma}}{2}\Big] \\ &\le 2\mathrm{e}^{-c_1k}.\end{aligned} \tag{5.4}$$

Now we choose the x_i's as follows: set $x_0 := 0$, and define iteratively

$$\begin{aligned}x_{i+1} &:= x_i + 2\sqrt{\beta t}([\log\beta]+1) \\ &= 2(i+1)\sqrt{\beta t}([\log\beta]+1) \qquad \text{for all } i \ge 0.\end{aligned} \tag{5.5}$$

The preceding implies that for all sufficiently large k,

$$\mathrm{P}\Big\{\sup_{x\in[0,2(N+1)\sqrt{\beta t}([\log\beta]+1)]}|u_t(x)|<k\Big\}\le 2\mathrm{e}^{-c_1k}, \tag{5.6}$$

whence

$$\mathrm{P}\Big\{\sup_{|x|\le 2(N+1)\sqrt{\beta t}([\log\beta]+1)}|u_t(x)|<k\Big\}\le 2\mathrm{e}^{-c_1k} \tag{5.7}$$

by symmetry.

It is not hard to verify that, as $k\to\infty$,

$$2(N+1)\sqrt{\beta t}([\log\beta]+1)=O(\mathrm{e}^{\rho k^{1/\gamma}}). \tag{5.8}$$

Consequently, the Borel–Cantelli lemma, used in conjunction with a standard monotonicity argument, implies that $u_t^*(R)\ge \mathrm{const}\cdot(\log(R)/c_2)^{\gamma}$ a.s. for all sufficiently large values of R, where $u_t^*(R)$ is defined in (1.23). By Proposition 3.8, $c_2=\mathrm{const}\cdot\varkappa^{1/12\gamma}$. Therefore, the theorem follows. □

6. Proof of Theorem 1.2.

Next we prove Theorem 1.2. In order to obtain the upper bound, the proof requires an estimate of spatial continuity of $x\mapsto u_t(x)$. However, matters are somewhat complicated by the fact that we need a modulus of continuity estimate that holds simultaneously for every $x\in[-R,R]$, uniformly for all large values of R. This will be overcome in a few steps. The first is a standard moment bound for the increments of the solution; however, we need to pay close attention to the constants in the estimate.

LEMMA 6.1. *Choose and fix some $t>0$, and suppose σ is uniformly bounded. Then there exists a finite and positive constant A such that for all real numbers $k\ge 2$,*

$$\sup_{-\infty<x\neq x'<\infty}\frac{\mathrm{E}(|u_t(x)-u_t(x')|^{2k})}{|x-x'|^k}\le\left(\frac{Ak}{\varkappa}\right)^k. \tag{6.1}$$

PROOF. Throughout, let $S_0:=\sup_{x\in\mathbf{R}}|\sigma(x)|$.

If $x,x'\in[-R,R]$ and $t>0$ are held fixed, then we can write $u_t(x)-u_t(x')=N_t$, where $\{N_\tau\}_{\tau\in(0,t)}$ is the continuous mean-one martingale described by

$$N_\tau:=\int_{(0,\tau)\times\mathbf{R}}[p_{t-s}(y-x)-p_{t-s}(y-x')]\sigma(u_s(y))W(\mathrm{d}s\,\mathrm{d}y) \tag{6.2}$$

for $\tau\in(0,t)$. The quadratic variation of $\{N_\tau\}_{\tau\in(0,t)}$ is estimated as follows:

$$\begin{aligned}\langle N\rangle_\tau &\le S_0^2\cdot\int_0^\tau \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,[p_s(y-x)-p_s(y-x')]^2\\ &\le \mathrm{e}^{\tau}S_0^2\cdot\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,[p_s(y-x)-p_s(y-x')]^2.\end{aligned} \tag{6.3}$$

For every $s > 0$ fixed, we can compute the $\mathrm{d}y$-integral using Plancherel's theorem, and obtain $\pi^{-1}\int_{-\infty}^{\infty}(1-\cos(\xi|x-x'|))\exp(-\varkappa s\xi^2)\,\mathrm{d}\xi$. Therefore, there exists a finite and positive constant a such that

$$\langle N\rangle_\tau \le \frac{\mathrm{e}^\tau S_0^2}{\pi}\cdot\int_{-\infty}^{\infty}\frac{1-\cos(\xi|x-x'|)}{1+\varkappa\xi^2}\,\mathrm{d}\xi \le \frac{a}{\varkappa}|x-x'|, \tag{6.4}$$

uniformly for all $\tau \in (0, t)$; we emphasize that a depends only on S_0 and t. The Carlen–Kree estimate [6] for the Davis [12] optimal constant in the Burkholder–Davis–Gundy inequality [3–5] implies the lemma. □

The second estimate turns the preceding moment bounds into an maximal exponential estimate. We use a standard chaining argument to do this. However, once again we have to pay close attention to the parameter dependencies in the implied constants.

LEMMA 6.2. *Choose and fix $t > 0$, and suppose σ is uniformly bounded. Then there exist a constant $C \in (0,\infty)$ such that*

$$\mathrm{E}\Bigg[\sup_{\substack{x,x'\in I:\\ |x-x'|\le\delta}}\exp\bigg(\frac{\varkappa|u_t(x)-u_t(x')|^2}{C\delta}\bigg)\Bigg] \le \frac{2}{\delta}, \tag{6.5}$$

uniformly for every $\delta \in (0,1]$ and every interval $I \subset [0,\infty)$ of length at most one.

PROOF. Recall [10], (39), page 11, the Kolmogorov continuity in the following quantitative form: suppose there exist $\nu > \gamma > 1$ for which a stochastic process $\{\xi(x)\}_{x\in\mathbf{R}}$ satisfies the following:

$$\mathrm{E}\big(|\xi(x)-\xi(x')|^\nu\big) \le C|x-x'|^\gamma; \tag{6.6}$$

we assume that the preceding holds for all $x, x' \in \mathbf{R}$, and $C \in (0,\infty)$ is independent of x and x'. Then, for every integer $m \ge 0$,

$$\mathrm{E}\Big(\sup_{\substack{x,x'\in I:\\ |x-x'|\le 2^{-m}}}|\xi(x)-\xi(x')|^\nu\Big) \le \bigg(\frac{2^{(2-\gamma+\nu)/\nu}C^{1/\nu}}{1-2^{-(\gamma-1)/\nu}}\bigg)^\nu\cdot 2^{-m(\gamma-1)}. \tag{6.7}$$

(Reference [10], (39), page 11, claims this with $2^{-m(\gamma-1)}$ replaced with $2^{-m\gamma}$ on the right-hand side. But this is a typographical error; compare with [10], (38), page 11.)

If $\delta \in (0,1]$, then we can find an integer $m \ge 0$ such that $2^{-m-1} \le \delta \le 2^{-m}$, whence it follows that

$$\begin{aligned}\mathrm{E}\Big(\sup_{\substack{x,x'\in I:\\ |x-x'|\le\delta}}|\xi(x)-\xi(x')|^\nu\Big) &\le \bigg(\frac{2^{(2-\gamma+\nu)/\nu}C^{1/\nu}}{1-2^{-(\gamma-1)/\nu}}\bigg)^\nu\cdot 2^{-m(\gamma-1)}\\ &\le \bigg(\frac{2^{(2-\gamma+\nu)/\nu}C^{1/\nu}}{1-2^{-(\gamma-1)/\nu}}\bigg)^\nu\cdot(2\delta)^{\gamma-1}.\end{aligned} \tag{6.8}$$

We apply the preceding with $\xi(x) := u_t(x)$, $\gamma := \nu/2 := k$ and $C := (Ak/\varkappa)^k$, where A is the constant of Lemma 6.1. It follows that there exists a positive and finite constant A_* such that for all intervals I of length at most one, all integers $k \geq 2$, and every $\delta \in (0, 1]$,

$$\mathrm{E}\Big(\sup_{\substack{x,x'\in I:\\ |x-x'|\leq\delta}} |u_t(x) - u_t(x')|^{2k}\Big) \leq \left(\frac{A_* k}{\varkappa}\right)^k \delta^{k-1}. \tag{6.9}$$

Stirling's formula tells us that there exists a finite constant $B_* > 1$ such that $(A_*k)^k \leq B_*^k k!$ for all integers $k \geq 1$. Therefore, for all $\alpha, \delta > 0$,

$$\mathrm{E}\Big[\sup_{\substack{x,x'\in I:\\ |x-x'|\leq\delta}} \exp\Big(\frac{\alpha|u_t(x) - u_t(x')|^2}{\delta}\Big)\Big] \leq \frac{1}{\delta}\sum_{k=0}^{\infty}\left(\frac{\zeta B_*}{\varkappa}\right)^k. \tag{6.10}$$

And this is at most two if $\alpha := \varkappa/(2B_*)$. The result follows. $\square$

Next we obtain another moments bound, this time for the solution rather than its increments.

LEMMA 6.3. *Choose and fix $t > 0$, and suppose σ is uniformly bounded. Then for all integers $k \geq 1$,*

$$\sup_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^{2k}) \leq \big(2\sqrt{2} + o(1)\big)(\tilde{\mu}_t k)^k \qquad (\textit{as } k \to \infty) \tag{6.11}$$

where the "$o(1)$" term depends only on k, and

$$\tilde{\mu}_t := \frac{8}{\mathrm{e}} \cdot S_0^2 \sqrt{\frac{t}{\pi\varkappa}}. \tag{6.12}$$

PROOF. Let us choose and fix a $t > 0$. Define $S_0 := \sup_{x\in\mathbf{R}}|\sigma(x)|$, and recall the martingale $\{M_\tau\}_{\tau\in(0,t)}$ from (3.17). Itô's formula (3.19) tells us that a.s., for all $\tau \in (0, t)$,

$$M_\tau^{2k} \leq 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s + \binom{2k}{2} S_0^2 \int_0^\tau \frac{M_s^{2(k-1)}}{(4\pi\varkappa(t-s))^{1/2}}\,\mathrm{d}s. \tag{6.13}$$

[Compare with (3.20).] We can take expectations, iterate the preceding and argue as we did in the proof of Proposition 3.6. To summarize the end result, let us define

$$\eta_t(x) := 1 + S_0 \cdot \int_{(0,\tau)\times\mathbf{R}} p_{t-s}(y - x) W(\mathrm{d}s\,\mathrm{d}y) \qquad (0 \leq \tau \leq t) \tag{6.14}$$

and

$$\zeta_t(x) := S_0 \cdot \int_{(0,\tau)\times\mathbf{R}} p_{t-s}(y - x) W(\mathrm{d}s\,\mathrm{d}y) \qquad (0 \leq \tau \leq t). \tag{6.15}$$

Then we have $\mathrm{E}[M_t^{2k}] \le \mathrm{E}[\eta_t(x)^{2k}] \le 2^{2k}(1 + \mathrm{E}[\zeta_t(x)^{2k}])$, and similar computations as those in the proof of Proposition 3.6 yield the lemma. □

Next we turn the preceding moment bound into a sharp Gaussian tail-probability estimate.

LEMMA 6.4. *Choose and fix a* $t > 0$, *and suppose that* σ *is uniformly bounded. Then there exist finite constants* $C > c > 0$ *such that simultaneously for all* $\lambda > 1$ *and* $x \in \mathbf{R}$,

$$c\exp\bigl(-C\sqrt{\varkappa}\lambda^2\bigr) \le \mathrm{P}\{|u_t(x)| > \lambda\} \le C\exp\bigl(-c\sqrt{\varkappa}\lambda^2\bigr). \tag{6.16}$$

PROOF. The lower bound is proved by an appeal to the Paley–Zygmund inequality, in the very same manner that Proposition 3.7 was established. However, we apply the improved inequality in Lemma 6.3 (in place of the result of Lemma 3.2). As regards the upper bound, note that Lemma 6.3 implies that there exists a positive and finite constant $\tilde{A}$ such that for all integers $m \ge 0$, $\sup_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^{2m}) \le (\tilde{A}/\sqrt{\varkappa})^m m!$, thanks to the Stirling formula. Thus,

$$\sup_{x\in\mathbf{R}} \mathrm{E}\exp(\alpha|u_t(x)|^2) \le \sum_{m=0}^{\infty}\left(\frac{\alpha\tilde{A}}{\sqrt{\varkappa}}\right)^m = \frac{1}{1-\alpha\tilde{A}\varkappa^{-1/2}} < \infty, \tag{6.17}$$

provided that $\alpha \in (0, \sqrt{\varkappa}/\tilde{A})$. Notice that this has a different behavior than (6.10) in terms of κ. If we fix such an α, then we obtain from Chebyshev's inequality the bound $\mathrm{P}\{u_t(x) > \lambda\} \le (1-\alpha\tilde{A}\varkappa^{-1/2})^{-1}\cdot\exp(-\alpha\lambda^2)$, valid simultaneously for all $x \in \mathbf{R}$ and $\lambda > 0$. We write $\alpha := c\sqrt{\varkappa}$ to finish. □

We are finally ready to assemble the preceding estimates in order to establish Theorem 1.2.

PROOF OF THEOREM 1.2. Consider the proof of Theorem 1.1: if we replace the role of (3.2) by the bounds in Lemma 6.4 and choose $\lambda := k$, $N := \lceil k \times \exp(c\sqrt{\varkappa}k^2)\rceil$ and $\beta := \exp((\mathrm{Lip}_\sigma \vee 1)^4 k^2/\varkappa)$ in the equivalent of (5.3) with the appropriate estimates, then we obtain the almost-sure bound $\liminf_{R\to\infty} u_t^*(R)/(\log R)^{1/2} > \mathrm{const}\cdot\varkappa^{-1/4}$, where "const" is independent of $\varkappa$. It remains to derive a corresponding upper bound for the lim sup.

Suppose $R \ge 1$ is an integer. We partition the interval $[-R, R]$ using a length-1 mesh with endpoints $\{x_j\}_{j=0}^{2R}$ via

$$x_j := -R + j \qquad \text{for } 0 \le j \le 2R. \tag{6.18}$$

Then we write

$$\mathrm{P}\{u_t^*(R) > 2\alpha(\log R)^{1/2}\} \le \mathcal{T}_1 + \mathcal{T}_2, \tag{6.19}$$

where

$$\mathcal{T}_1 := \mathrm{P}\Big\{\max_{1\le j\le 2R} u_t(x_j) > \alpha(\log R)^{1/2}\Big\},$$
$$\mathcal{T}_2 := \mathrm{P}\Big\{\max_{1\le j\le 2R} \sup_{x\in(x_j,x_{j+1})} |u_t(x) - u_t(x_j)| > \alpha(\log R)^{1/2}\Big\}. \tag{6.20}$$

By Lemma 6.4,

$$\mathcal{T}_1 \le 2R \sup_{x\in\mathbf{R}} \mathrm{P}\{u_t(x) > \alpha(\log R)^{1/2}\} \le \frac{\mathrm{const}}{R^{-1+c\sqrt{\varkappa}\alpha^2}}. \tag{6.21}$$

Similarly,

$$\mathcal{T}_2 \le 2R \sup_I \mathrm{P}\Big\{\sup_{x,x'\in I} |u_t(x) - u_t(x')| > \alpha(\log R)^{1/2}\Big\}, \tag{6.22}$$

where "$\sup_I$" designates a supremum over all intervals I of length one. Chebyshev's inequality and Lemma 6.2 together imply that

$$\begin{aligned}\mathcal{T}_2 &\le 2R^{-(\varkappa/C)\alpha^2+1} \sup_I \mathrm{E}\Big[\sup_{x,x'\in I} \exp\Big(\frac{\varkappa}{C}|u_t(x) - u_t(x')|^2\Big)\Big]\\ &\le \frac{\mathrm{const}}{R^{-1+(\varkappa\alpha^2)/C}}.\end{aligned} \tag{6.23}$$

Let $q := \min(\varkappa/C, c\sqrt{\varkappa})$ to find that

$$\sum_{R=1}^{\infty} \mathrm{P}\{u_t^*(R) > 2\alpha(\log R)^{1/2}\} \le \mathrm{const}\cdot \sum_{R=1}^{\infty} R^{-q\alpha^2+1}, \tag{6.24}$$

and this is finite provided that $\alpha > (2/q)^{1/2}$. By the Borel–Cantelli lemma,

$$\limsup_{\substack{R\to\infty:\\ R\in\mathbf{Z}}} \frac{u_t^*(R)}{(\log R)^{1/2}} \le \Big(\frac{8}{q}\Big)^{1/2} < \infty \qquad \text{a.s.} \tag{6.25}$$

Clearly, $(8/q)^{1/2} \le \mathrm{const}\cdot \varkappa^{-1/4}$ for all $\varkappa \ge \varkappa_0$, for a constant depends only on $\varkappa_0$. And we can remove the restriction "$R \in \mathbf{Z}$" in the lim sup by a standard monotonicity argument; namely, we find—by considering in the following $R - 1 \le X \le R$—that

$$\limsup_{X\to\infty} \frac{u_t^*(X)}{(\log X)^{1/2}} \le \limsup_{\substack{R\to\infty:\\ R\in\mathbf{Z}}} \frac{u_t^*(R)}{(\log(R-1))^{1/2}} \le \Big(\frac{8}{q}\Big)^{1/2} \qquad \text{a.s.} \tag{6.26}$$

This proves the theorem. □

7. Proof of Theorem 1.3. This section is mainly concerned with the proof of Theorem 1.3. For that purpose, we start with tail-estimates.

LEMMA 7.1. *Consider* (2.11) *with* $\sigma(x) := cx$, *where* $c > 0$ *is fixed. Then*,

$$\log \mathrm{P}\{|u_t(x)| \geq \lambda\} \asymp -\sqrt{\varkappa}(\log \lambda)^{3/2} \qquad \text{as } \lambda \to \infty. \tag{7.1}$$

PROOF. Corollary 3.5 implies the upper bound (the boundedness of σ is not required in the results of Section 3.1).

As for the lower bound, we know from [2], Theorem 2.6,

$$\mathrm{e}^{k(k^2-1)t/24\varkappa} \leq \mathrm{E}(|u_t(x)|^k) \leq 2\mathrm{e}^{k(k^2-1)t/24\varkappa}, \tag{7.2}$$

uniformly for all integers $k \geq 2$ and $x \in \mathbf{R}$. Now we follow the same method as in the proof of Proposition 3.7, and use the Paley–Zygmund inequality to obtain

$$\begin{aligned} \mathrm{P}\left\{|u_t(x)| \geq \frac{1}{2}\|u_t(x)\|_{2k}\right\} &\geq \frac{[\mathrm{E}(|u_t(x)|^{2k})]^2}{4\mathrm{E}(|u_t(x)|^{4k})} \\ &\geq C_1 \mathrm{e}^{-D_1 k^3/\varkappa} \end{aligned} \tag{7.3}$$

for some nontrivial constants C_1 and D_1 that do not depend on x or k. We then obtain the following: uniformly for all $x \in \mathbf{R}$ and sufficiently-large integers k,

$$\mathrm{P}\left\{|u_t(x)| \geq \frac{C}{2}\mathrm{e}^{4Dk^2/\varkappa}\right\} \geq C_1 \mathrm{e}^{-D_1 k^3/\varkappa}. \tag{7.4}$$

Let $\lambda := (C/2)\exp\{4Dk^2/\varkappa\}$, and apply a direct computation to deduce the lower bound. □

We are now ready to prove Theorem 1.3. Our proof is based on roughly-similar ideas to those used in the course of the proof of Theorem 1.1. However, at a technical level, they are slightly different. Let us point out some of the essential differences: unlike what we did in the proof of Theorem 1.1, we now do not choose the values of N, β and λ as functions of k, but rather as functions of R; the order of the moments k will be fixed; and we will not sum on k, but rather sum on a discrete sequence of values of the parameter R. The details follow.

PROOF OF THEOREM 1.3. First we derive the lower bound by following the same method that was used in the proof of Theorem 1.1; see Section 5. But we now use Lemma 7.1 rather than Corollary 3.1.

The results of Section 4 can be modified to apply to the parabolic Anderson model, provided that we again apply Lemma 7.1 in place of Corollary 3.1. In this way we obtain the following, where the x_i's are defined by (5.5)[4]: consider the

[4]To be very precise, we once again need to write $(\beta, [\log \beta] + 1)$ in place of $(\beta, \log \beta)$.

event

$$\Lambda := \Big\{ \max_{1\le j\le N} |u_t(x_j)| < \Xi \Big\} \qquad \text{where } \Xi := \exp\bigg(C_1 \frac{(\log R)^{2/3}}{\varkappa^{1/3}} \bigg). \tag{7.5}$$

Then,

$$\begin{aligned} \mathrm{P}(\Lambda) &\le \mathrm{P}\Big\{ \max_{1\le j\le N} \big| U_t^{(\beta,\log\beta)}(x_j) \big| < 2\Xi \Big\} \\ &\quad + \mathrm{P}\big\{ \big| u_t(x_j) - U_t^{(\beta,\log\beta)} \big| > \Xi \text{ for some } 1\le j\le N \big\} \\ &\le \big(1 - \mathrm{P}\big\{ \big| U_t^{(\beta,\log\beta)}(x_j) \big| \ge 2\Xi \big\}\big)^N + \frac{N\beta^{-k} C_*^k k^{k/2} \mathrm{e}^{Gk^3}}{\Xi}. \end{aligned} \tag{7.6}$$

Note that we do not yet have a lower bound on $\mathrm{P}\{|U_t^{(\beta,\log\beta)}(x)| \ge \lambda\}$. However, we have

$$\begin{aligned} &\mathrm{P}\big\{ \big| U_t^{(\beta,\log\beta)}(x_j) \big| \ge 2\Xi \big\} \\ &\quad \ge \mathrm{P}\{ |u_t(x_j)| \ge 3\Xi \} - \mathrm{P}\big\{ \big| u_t(x_j) - U_t^{(\beta,\log\beta)}(x_j) \big| \ge \Xi \big\} \\ &\quad \ge \alpha_1 R^{-\alpha_2 C_1^{3/2}} - \frac{N\beta^{-k} C_*^k k^{k/2} \mathrm{e}^{Gk^3}}{\Xi}, \end{aligned} \tag{7.7}$$

valid for some positive constants α_1 and α_2. Now let us choose $N := \lceil R^a \rceil$ and $\beta := R^{1-a}$ for a fixed $a \in (0,1)$. With these values of N and β and the lower bound in (7.7), the upper bound in (7.6) becomes

$$\mathrm{P}(\Lambda) \le \bigg(1 - \alpha_1 R^{-\alpha_2 C_1^{3/2}} + \frac{C_*^k k^{k/2} \mathrm{e}^{Gk^3}}{R^{k(1-a)-a}\,\Xi} \bigg)^N + \frac{C_*^k k^{k/2} \mathrm{e}^{Gk^3}}{R^{k(1-a)-a}\,\Xi}. \tag{7.8}$$

Let us consider k large enough so that $k(1-a) - a > 2$. Notice that k will not depend on R; this is in contrast with what happened in the proof of Theorem 1.1.

We can choose the constant C_1 to be small enough to satisfy $\alpha_2 C_1^{3/2} < a/2$. Using these, we obtain

$$\mathrm{P}\Big\{ \sup_{x\in[0,R]} |u_t(x)| < \mathrm{e}^{C_1 (\log R)^{2/3}/\varkappa^{1/3}} \Big\} \le \exp(-\alpha_1 R^{a/2}) + \frac{\mathrm{const}}{R^2}. \tag{7.9}$$

The Borel–Cantelli lemma yields the lower bound of the theorem.

We can now prove the upper bound. Our derivation is modeled after the proof of Theorem 1.2.

First, we need a continuity estimate for the solution of (2.11) in the case that $\sigma(x) := cx$. In accord with (7.2),

$$\begin{aligned} &\mathrm{E}\big(|u_t(x) - u_t(y)|^{2k}\big) \\ &\quad \le (2\sqrt{2k})^{2k} \bigg[\int_0^t \mathrm{d}r\, \|u(r,0)\|_{2k}^2 \int_{\mathbf{R}} \mathrm{d}z\, |p_{t-r}(x-z) - p_{t-r}(y-z)|^2 \bigg]^k \\ &\quad \le (2\sqrt{2k})^{2k} \bigg[\int_0^t \mathrm{d}r\, 2^{1/k} \mathrm{e}^{8Dk^2/\varkappa} \int_{\mathbf{R}} \mathrm{d}z\, |p_{t-r}(x-z) - p_{t-r}(y-z)|^2 \bigg]^k \end{aligned} \tag{7.10}$$

for some constant D which depends on t. Consequently [see the derivation of (6.4)],

$$\mathrm{E}\left(|u_t(x) - u_t(y)|^{2k}\right) \le C^{k^2}\left(\frac{|y-x|}{\varkappa}\right)^k \exp\left(\frac{Bk^3}{\varkappa}\right) \tag{7.11}$$

for constants $B, C \in (0,\infty)$ that do not depend on k. We apply an argument, similar to one we used in the proof of Lemma 6.2, in order to deduce that for simultaneously all intervals I of length 1,

$$\mathrm{E}\left(\sup_{x,x'\in I\,:\,|x-x'|\le 1} |u_t(x) - u_t(x')|^{2k}\right) \le \frac{C_1^{k^2} \mathrm{e}^{C_2 k^3/\varkappa}}{\varkappa^k} \tag{7.12}$$

for constants $C_1, C_2 \in (0,\infty)$ that do not depend on k or $\varkappa$. Now, we follow the proof of Theorem 1.2 and partition $[-R, R]$ into intervals of length 1. Let $b > 0$ to deduce the following:

$$\mathrm{P}\left\{u_t^*(R) > 2\mathrm{e}^{b(\log R)^{2/3}/\varkappa^{1/3}}\right\} \le \mathcal{T}_1 + \mathcal{T}_2, \tag{7.13}$$

where

$$\mathcal{T}_1 := \mathrm{P}\left\{\max_{1\le j\le 2R} u_t(x_j) > \mathrm{e}^{b(\log R)^{2/3}/\varkappa^{1/3}}\right\} \tag{7.14}$$

and

$$\mathcal{T}_2 := \mathrm{P}\left\{\max_{1\le j\le 2R} \sup_{x\in(x_j,x_{j+1})} |u_t(x) - u_t(x_j)| > \mathrm{e}^{b(\log R)^{2/3}/\varkappa^{1/3}}\right\}. \tag{7.15}$$

[Compare with (6.19).]

On one hand, Lemma 7.1 implies that

$$\mathcal{T}_1 \le 2R\cdot \mathrm{P}\left\{u_t(x_j) > \mathrm{e}^{b(\log R)^{2/3}/\varkappa^{1/3}}\right\} \le \frac{2c_3 R}{R^{c_4 b^{3/2}}} \tag{7.16}$$

for some constants $c_3, c_4 > 0$. On the other hand (7.12) and Chebyshev's inequality imply that

$$\begin{aligned} \mathcal{T}_2 &\le 2R\mathrm{P}\left\{\sup_{x,x'\in I\,:\,|x-x'|\le 1} |u_t(x) - u_t(x')| \ge \mathrm{e}^{b(\log R)^{2/3}/\varkappa^{1/3}}\right\} \\ &\le \frac{2RC_1^{k^2}\mathrm{e}^{C_2k^3/\varkappa}}{\varkappa^k \mathrm{e}^{2kb(\log R)^{2/3}/\varkappa^{1/3}}}. \end{aligned} \tag{7.17}$$

Now we choose $k := \lceil \varkappa^{1/3}(\log R)^{1/3}\rceil$ in order to obtain $\mathcal{T}_2 \le \mathrm{const}\cdot R^{1+C_2-2b}$ where the constant depends on $\varkappa$. With these choices of parameters we deduce from (7.16) and (7.17) that if b were sufficiently large, then

$$\sum_{R=1}^{\infty} \mathrm{P}\left\{u_t^*(R) > 2\mathrm{e}^{b(\log R)^{2/3}/\varkappa^{1/3}}\right\} < \infty. \tag{7.18}$$

The Borel–Cantelli lemma and a monotonicity argument together complete the proof. □

APPENDIX A: WALSH STOCHASTIC INTEGRALS

Throughout this Appendix, $(\Omega, \mathcal{F}, \mathrm{P})$ denotes (as is usual) the underlying probability space. We state and prove some elementary properties of Walsh stochastic integrals [23].

Let $\mathcal{L}^d$ denote the collection of all Borel-measurable sets in $\mathbf{R}^d$ that have finite d-dimensional Lebesgue measure. (We could work with Lebesgue-measurable sets, also.)

Let us follow Walsh [23] and define for every $t > 0$ and $A \in \mathcal{L}^d$ the random field

$$W_t(A) := \int_{[0,t]\times A} W(\mathrm{d}s\,\mathrm{d}y). \tag{A.1}$$

The preceding stochastic integral is defined in the same sense as Wiener.

Let $\mathcal{F}_t(A)$ denote the sigma-algebra generated by all random variables of the form

$$\{W_s(B) : s \in (0,t], B \in \mathcal{L}^d, B \subseteq A\}. \tag{A.2}$$

We may assume without loss of generality that, for all $A \in \mathcal{L}^d$, $\{\mathcal{F}_t(A)\}_{t>0}$ is a right-continuous P-complete filtration (i.e., satisfies the "usual hypotheses" of Dellacherie and Meyer [13]). Otherwise, we augment $\{\mathcal{F}_t(A)\}_{t>0}$ in the usual way. Let

$$\mathcal{F}_t := \bigvee_{A\in\mathcal{L}^d} \mathcal{F}_t(A) \qquad (t > 0). \tag{A.3}$$

Let $\mathcal{P}$ denote the collection of all processes that are predictable with respect to $\{\mathcal{F}_t\}_{t>0}$. The elements of $\mathcal{P}$ are precisely the "predictable random fields" of Walsh [23].

For us, the elements of $\mathcal{P}$ are of interest because if $Z \in \mathcal{P}$ and

$$\|Z\|^2_{L^2(\mathbf{R}_+\times\mathbf{R}^d\times\Omega)} := \mathrm{E}\int_0^\infty \mathrm{d}t \int_{\mathbf{R}^d} \mathrm{d}x\,[Z_t(x)]^2 < \infty, \tag{A.4}$$

then the Walsh stochastic integral $I_t := \int_{[0,t]\times\mathbf{R}} Z_s(y)W(\mathrm{d}s\,\mathrm{d}y)$ is defined properly, and has good mathematical properties. Chief among those good properties are the following: $\{I_t\}_{t>0}$ is a continuous mean-zero L^2-martingale with quadratic variation $\langle I\rangle_t := \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\,[Z_s(y)]^2$.

Let us define $\mathcal{P}(A)$ to be the collection of all processes that are predictable with respect to $\{\mathcal{F}_t(A)\}_{t>0}$. Clearly, $\mathcal{P}(A) \subseteq \mathcal{P}$ for all $A \in \mathcal{L}^d$.

PROPOSITION A.1. *If $Z \in \mathcal{P}(A)$ for some $A \in \mathcal{L}^d$ and $\|Z\|_{L^2(\mathbf{R}_+\times\mathbf{R}^d\times\Omega)} < \infty$, then the martingale defined by $J_t := \int_{[0,t]\times A} Z_s(y)W(\mathrm{d}s\,\mathrm{d}y)$ is in $\mathcal{P}(A)$.*

PROOF. It suffices to prove this for a random field Z that has the form

$$Z_s(y)(\omega) = \mathbf{1}_{[a,b]}(s)X(\omega)\mathbf{1}_A(y) \qquad (s > 0, y \in \mathbf{R}, \omega \in \Omega) \tag{A.5}$$

where $0 \le a < b$, and X is a bounded $\mathcal{F}_a(A)$-measurable random variable. But in that case, $J_t(\omega) = X(\omega) \cdot \int_{[0,t]\cap[a,b]\times A} W(\mathrm{d}s\,\mathrm{d}y)$, whence the result follows easily from the easy-to-check fact that the stochastic process defined by $I_t := \int_{[0,t]\cap[a,b]\times A} W(\mathrm{d}s\,\mathrm{d}y)$ is continuous (up to a modification). The latter assertion follows from the Kolmogorov continuity theorem; namely, we check first that $\mathrm{E}(|I_t - I_r|^2) = |A| \cdot |t-r|$, where $|A|$ denotes the Lebesgue measure of A. Then use the fact, valid for all Gaussian random variables including $I_t - I_r$, that $\mathrm{E}(|I_t - I_r|^k) = \mathrm{const} \cdot \{\mathrm{E}(|I_t - I_r|^2)\}^{k/2}$ for all $k \ge 2$. □

Proposition A.1 is a small variation on Walsh's original construction of his stochastic integrals. We need this minor variation for the following reason:

COROLLARY A.2. *Let $A^{(1)}, \dots, A^{(N)}$ be fixed and nonrandom disjoint elements of $\mathcal{L}^d$. If $Z^{(1)}, \dots, Z^{(N)}$ are, respectively, in $\mathcal{P}(A^{(1)}), \dots, \mathcal{P}(A^{(N)})$ and $\|Z^{(j)}\|_{L^2(\mathbf{R}_+\times\mathbf{R}^d\times\Omega)} < \infty$ for all $j = 1, \dots, N$, then $J^{(1)}, \dots, J^{(N)}$ are independent processes, where*

$$J_t^{(j)} := \int_{[0,t]\times A_j} Z_s(y) W(\mathrm{d}s\,\mathrm{d}y) \qquad (j = 1, \dots, N, t > 0). \tag{A.6}$$

PROOF. Owing to Proposition A.1, it suffices to prove that if some sequence of random fields $X^{(1)}, \dots, X^{(N)}$ satisfies $X^{(j)} \in \mathcal{P}(A^{(j)})$ $(j = 1, \dots, N)$, then $X^{(1)}, \dots, X^{(N)}$ are independent. It suffices to prove this in the case that the $X^{(j)}$'s are simple predictable processes; that is, in the case that

$$X_t^{(j)}(\omega) = \mathbf{1}_{[a_j,b_j]}(s) Y_j(\omega) \mathbf{1}_{A^{(j)}}(y), \tag{A.7}$$

where $0 < a_j < b_j$ and Y_j is a bounded $\mathcal{F}_{a_j}(A^{(j)})$-measurable random variable. In turn, we may restrict attention to Y_j's that have the form

$$Y_j(\omega) := \varphi_j\left(\int_{[\alpha_j,\beta_j]\times A^{(j)}} W(\mathrm{d}s\,\mathrm{d}y)\right), \tag{A.8}$$

where $0 < \alpha_j < \beta_j \le a_j$ and $\varphi_j : \mathbf{R} \to \mathbf{R}$ is bounded and Borel measurable. But the assertion is now clear, since $Y_1, \dots, Y_N$ are manifestly independent. In order to see this we need only verify that the covariance between $\int_{[\alpha_j,\beta_j]\times A^{(j)}} W(\mathrm{d}s\,\mathrm{d}y)$ and $\int_{[\alpha_k,\beta_k]\times A^{(k)}} W(\mathrm{d}s\,\mathrm{d}y)$ is zero when $j \ne k$; and this is a ready consequence of the fact that $A^{(j)} \cap A^{(k)} = \varnothing$ when $j \ne k$. □

APPENDIX B: SOME FINAL REMARKS

Recall that $\|U\|_k$ denotes the usual $L^k(\mathrm{P})$-norm of a random variable U for all $k \in (0, \infty)$. According to Lemma 3.2,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^k) \le aCk^3 \qquad \text{if } k \ge 2. \tag{B.1}$$

This and Jensen's inequality together imply that

$$\overline{\gamma}(\nu) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^\nu) < \infty \qquad \text{for all } \nu > 0. \tag{B.2}$$

And Chebyshev's inequality implies that for all $\nu > 0$,

$$\begin{aligned} \sup_{x\in\mathbf{R}} \mathrm{P}\{u_t(x) \ge \mathrm{e}^{-qt}\} &\le \exp\left(\nu t\left[q + \frac{1}{\nu t} \log \sup_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^\nu)\right]\right) \\ &= \exp\left(\nu t\left[q + \frac{\overline{\gamma}(\nu)}{\nu} + o(1)\right]\right) \qquad (t \to \infty). \end{aligned} \tag{B.3}$$

Because $u_t(x) \ge 0$, it follows that $\nu \mapsto \overline{\gamma}(\nu)/\nu$ is nondecreasing on $(0, \infty)$, whence

$$\ell := \lim_{\nu\downarrow 0} \frac{\overline{\gamma}(\nu)}{\nu} = \inf_{\nu>0} \frac{\overline{\gamma}(\nu)}{\nu} \text{ exists and is finite.} \tag{B.4}$$

Therefore, in particular,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbf{R}} \mathrm{P}\{u_t(x) \ge \mathrm{e}^{-qt}\} < 0 \qquad \text{for all } q \in (-\infty, -\ell). \tag{B.5}$$

Now consider the case that $\sigma(0) = 0$, and recall that in that case $u_0(x) \ge 0$ for all $x \in \mathbf{R}$. Mueller's comparison principle tells us that $u_t(x) \ge 0$ a.s. for all $t \ge 0$ and $x \in \mathbf{R}$, whence it follows that $\|u_t(x)\|_1 = \mathrm{E}[u_t(x)] = (p_t * u_0)(x)$ is bounded in t. This shows that $\overline{\gamma}(1) = 0$, and hence $\ell \le 0$. We have proved the following:

PROPOSITION B.1. *If $\sigma(0) = 0$, then there exists $q \ge 0$ such that*

$$\frac{1}{t} \log u_t(x) \le -q + o_{\mathrm{P}}(1) \qquad \textit{as } t \to \infty, \textit{for every } x \in \mathbf{R}, \tag{B.6}$$

where $o_{\mathrm{P}}(1)$ is a term that converges to zero in probability as $t \to \infty$.

Bertini and Giacomin [5] have studied the case that $\sigma(x) = cx$ and have shown that, in that case, there exists a special choice of u_0 such that for all compactly supported probability densities $\psi \in C^\infty(\mathbf{R})$,

$$\frac{1}{t} \log \int_{-\infty}^{\infty} u_t(x)\psi(x)\,\mathrm{d}x = -\frac{c^2}{24\varkappa} + o_{\mathrm{P}}(1) \qquad \text{as } t \to \infty. \tag{B.7}$$

Equation (B.7) and more generally Proposition B.1 show that the typical behavior of the sample function of the solution to (1.2) is subexponential in time, as one might expect from the unforced linear heat equation. And yet, it frequently is the case that $u_t(x)$ grows in time exponentially rapidly in $L^k(\mathrm{P})$ for $k \geq 2$ [2, 7, 15]. This phenomenon is further evidence of physical intermittency in the sort of systems that are modeled by (1.2).

Standard predictions suggest that the typical behavior of $u_t(x)$ (in this and related models) is that it decays exponentially rapidly with time. [Equation (B.7) is proof of this fact in one special case.] In other words, one might expect that typically $q > 0$. We are not able to resolve this matter here, and therefore ask the following questions:

Open problem 1. Is $q > 0$ in Proposition B.1? Equivalently, is $\ell < 0$ in (B.4)?

Open problem 2. Can the $o_{\mathrm{P}}(1)$ in (B.6) be replaced by a term that converges almost surely to zero as $t \to \infty$?

REFERENCES

[1] BALÁZS, M., QUASTEL, J. and SEPPÄLÄINEN, T. (2011). Fluctuation exponent of the KPZ/stochastic Burgers equation. *J. Amer. Math. Soc.* **24** 683–708. MR2784327

[2] BERTINI, L. and CANCRINI, N. (1995). The stochastic heat equation: Feynman–Kac formula and intermittence. *J. Stat. Phys.* **78** 1377–1401. MR1316109

[3] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504. MR0208647

[4] BURKHOLDER, D. L., DAVIS, B. J. and GUNDY, R. F. (1972). Integral inequalities for convex functions of operators on martingales. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California*, *Berkeley*, *Calif.*, 1970/1971), *Vol. II*: *Probability Theory* 223–240. Univ. California Press, Berkeley, CA. MR0400380

[5] BURKHOLDER, D. L. and GUNDY, R. F. (1970). Extrapolation and interpolation of quasi-linear operators on martingales. *Acta Math.* **124** 249–304. MR0440695

[6] CARLEN, E. and KRÉE, P. (1991). L^p estimates on iterated stochastic integrals. *Ann. Probab.* **19** 354–368. MR1085341

[7] CARMONA, R. A. and MOLCHANOV, S. A. (1994). Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.* **108** viii+125. MR1185878

[8] CHERTKOV, M., FALKOVICH, G., KOLOKOLOV, I. and LEBEDEV, V. (1995). Statistics of a passive scalar advected by a large-scale two-dimensional velocity field: Analytic solution. *Phys. Rev. E* (3) **51** 5609–5627. MR1383089

[9] CONUS, D. and KHOSHNEVISAN, D. (2010). Weak nonmild solutions to some SPDEs. Preprint. Available at http://arxiv.org/abs/1004.2744.

[10] DALANG, R., KHOSHNEVISAN, D., MUELLER, C., NUALART, D. and XIAO, Y. (2009). *A Minicourse on Stochastic Partial Differential Equations*. *Lecture Notes in Math.* **1962**. Springer, Berlin. MR1500166

[11] DALANG, R. C. (1999). Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* **4** 29 pp. (electronic). MR1684157

[12] DAVIS, B. (1976). On the L^p norms of stochastic integrals and other martingales. *Duke Math. J.* **43** 697–704. MR0418219

[13] DELLACHERIE, C. and MEYER, P.-A. (1982). *Probabilities and Potential. B. Theory of Martingales. North-Holland Mathematics Studies* **72**. North-Holland, Amsterdam. Translated from the French by J. P. Wilson. MR0745449

[14] DURRETT, R. (1998). *Lecture Notes on Particle Systems and Percolation*. Wadsworth & Brooks Cole, Pacific Grove, CA.

[15] FOONDUN, M. and KHOSHNEVISAN, D. (2009). Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.* **14** 548–568. MR2480553

[16] FOONDUN, M. and KHOSHNEVISAN, D. (2010). On the global maximum of the solution to a stochastic heat equation with compact-support initial data. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 895–907. MR2744876

[17] GEL'FAND, I. M. and VILENKIN, N. Y. (1964). *Generalized Functions. Vol.* 4: *Applications of Harmonic Analysis*. Academic Press, New York. MR0173945

[18] KARDAR, M., PARISI, G. and ZHANG, Y. C. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** 889–892.

[19] LIGGETT, T. M. (1985). *Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231

[20] MOLCHANOV, S. A. (1991). Ideas in the theory of random media. *Acta Appl. Math.* **22** 139–282. MR1111743

[21] MUELLER, C. (1991). On the support of solutions to the heat equation with noise. *Stochastics Stochastics Rep.* **37** 225–245. MR1149348

[22] RUELLE, D. (1985). The onset of turbulence: A mathematical introduction. In *Turbulence, Geophysical Flows, Predictability and Climate Dynamics* (*"Turbolenza e Predicibilità nella Fluidodinamica Geofisica e la Dinamica del Clima," Rendiconti della Scuola Internazionale di Fisica "Enrico Fermi"*). North-Holland, Amsterdam.

[23] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour, XIV—*1984. *Lecture Notes in Math.* **1180** 265–439. Springer, Berlin. MR0876085

[24] ZEL'DOVICH, Y. B., MOLCHANOV, S. A., RUZMAĬKIN, A. A. and SOKOLOFF, D. D. (1988). Intermittency, diffusion, and generation in a nonstationary random medium. *Sov. Sci. Rev. C Math. Phys.* **7** 1–110.

[25] ZELDOVICH, Y. B., MOLCHANOV, S. A., RUZMAIKIN, A. A. and SOKOLOV, D. D. (1985). Intermittency of passive fields in random media. *J. Exp. Theor. Phys.* **89** 2061–2072. [Actual journal title: *Zhurnal eksperimentalnoi teoreticheskoi fiziki*.] (In Russian.)

[26] ZEL'DOVICH, Y. B., RUZMAĬKIN, A. A. and SOKOLOFF, D. D. (1990). *The Almighty Chance. World Scientific Lecture Notes in Physics* **20**. World Scientific, River Edge, NJ. MR1141627

D. CONUS
DEPARTMENT OF MATHEMATICS
LEHIGH UNIVERSITY
CHRISTMAS–SAUCON HALL
14 EAST PACKER AVENUE
BETHLEHEM, PENNSYLVANIA 18015
USA
E-MAIL: daniel.conus@lehigh.edu

M. JOSEPH
D. KHOSHNEVISAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
155 SOUTH 1400 EAST
SALT LAKE CITY, UTAH 84112-0090
USA
E-MAIL: joseph@math.utah.edu
davar@math.utah.edu