EXTREME GAPS BETWEEN EIGENVALUES OF RANDOM MATRICES

BY GÉRARD BEN AROUS[1] AND PAUL BOURGADE

New York University and Harvard University

This paper studies the extreme gaps between eigenvalues of random matrices. We give the joint limiting law of the smallest gaps for Haar-distributed unitary matrices and matrices from the Gaussian unitary ensemble. In particular, the kth smallest gap, normalized by a factor $n^{-4/3}$, has a limiting density proportional to $x^{3k-1}e^{-x^3}$. Concerning the largest gaps, normalized by $n/\sqrt{\log n}$, they converge in L^p to a constant for all $p > 0$. These results are compared with the extreme gaps between zeros of the Riemann zeta function.

1. Introduction. We address here the following question: what is the asymptotic size and the limit laws of the smallest and largest gaps or spacings in the spectra of random matrices? Typical spacings between eigenvalues of random matrices have been well understood for invariant ensembles for quite some time. More recently, the behavior of these typical spacings has even been proved to be universal for much larger classes of random matrices [2, 14, 33]. Much less is known for atypically large or small spacings. This question was first considered for the smallest spacings in the unpublished Ph.D. thesis of Vinson [34] and raised by Diaconis in [11] for the largest ones. It was also discussed in an interesting debate during a conference at the Courant Institute in 2006, in honor of Percy Deift. We solve here completely the question of the smallest spacings for the simplest invariant ensembles, that is, the CUE and the GUE. We give the scaling and the limit laws for the joint distribution of the smallest spacings. The answer is simple to state: it is given by the trivial Poissonian ansatz where the spacings would be treated as i.i.d. random variables. The answer we find for the largest spacings is less complete, since we can only obtain at this point a first-order approximation which gives the asymptotic size of the largest spacings and not their limit laws. We believe that the same Poissonian ansatz should work as well for the largest gaps, but this question is left open. The question of the universality of the behavior of the extremal spacings is also left open.

Dyson [12] showed that the repulsion between eigenvalues of the Gaussian unitary ensemble (the GUE) could be described asymptotically in terms of the *deter-*

minantal point process associated with the sine kernel

$$K(x, y) = \frac{1}{\pi} \frac{\sin(x - y)}{x - y}.$$

For this limiting determinantal point process, the probability of having no eigenvalue in an interval of size s is known to be the Fredholm determinant $\det(\mathrm{Id} - K_{(0,s)})$, where $K_{(0,s)}$ is the convolution operator acting on $\mathrm{L}^2(0, s)$ with kernel K. The density of the spacing between two successive points is then given by (see [25])

$$p_2(s) = \partial_{ss} \det(\mathrm{Id} - K_{(0,s)}).$$

This spacing distribution was shown to appear in a number-theoretic context: some statistics produced by Odlyzko [27] presented a correspondence between the histogram of normalized gaps between zeros of the Riemann zeta function and $p_2(s)$. This gave further evidence for the analogy discovered by Dyson and Montgomery: they realized that the local dependence of the zeros of ζ, previously calculated by Montgomery, involved the sine kernel; see [21, 22] for an historical account and other steps of this fruitful analogy. At the same mean or typical gap scale, a precise analysis of the joint distribution of the gaps between eigenvalues was performed by Katz and Sarnak [21] and Soshnikov [29] for the Circular Unitary Ensemble (the CUE).

Less attention was paid to eigenvalues statistics at smaller and larger scales. This paper concerns the extreme gaps. This study was initiated by Vinson [34]: he showed that the smallest gap between elements of the CUE, multiplied by $n^{4/3}$, converges in law to a random variable with distribution function e^{-x^3}, as the size n of the unitary matrix increases. In his thesis, similar results for the smallest gap between eigenvalues of a generalization of the GUE were obtained. Vinson also gives interesting heuristics suggesting that the largest gap between CUE eigenvalues should be of order $\sqrt{\log n}/n$, with Poissonian fluctuations around this limit. Using a different technique, Soshnikov [31] investigated the smallest gaps for determinantal point processes on the real line, with a translation invariant kernel: amongst points included in $[0, L]$, this extreme spacing multiplied by $L^{1/3}$ converges weakly to the distribution with distribution function e^{-x^3}, as $L \to \infty$.

Heuristically, the above extreme gaps asymptotics can be obtained using the known asymptotics [12, 25] of the spacing distribution

$$p_2(s) \underset{s \to 0}{\sim} \frac{\pi^2}{3} s^2, \qquad \log p_2(s) \underset{s \to \infty}{\sim} -\frac{s^2}{8},$$

and treating the gaps as independent random variables. The difficulty in obtaining rigorous results lies in showing that this Poissonian ansatz is asymptotically correct for the extreme gaps, and in making the above estimates uniform in the dimension n.

We first consider the *joint* law of the smallest gaps (Theorem 1.1, Corollary 1.2) between eigenvalues of unitary matrices. This relies both on Soshnikov's method and a convergence of the process of small gaps to a Poisson point process. The same reasoning equally applies to the small gaps between eigenvalues from the Gaussian unitary ensemble (Theorem 1.4, Corollary 1.5). The proofs of the small spacings asymptotics are in Section 2.

The first-order asymptotics of the largest gaps is then proved. Concerning unitary random matrices (Theorem 1.3), this makes use of two important tools. A key ingredient, by Deift et al. [7], is the uniform asymptotics about the probability for a given arc of the circle to be free of eigenvalues. The proof also requires the negative correlation property for the event that two disjoint arcs are free of eigenangles. On account of the GUE (Theorem 1.7), we also make a essential use of the negative correlation, and the large gap probability is evaluated by comparing the GUE Fredholm determinant with the unitary one. These large gaps asymptotics are proved in Section 3.

The extreme spacings between random eigenvalues are important quantities for statistical physics, computational mathematics and number theory. For this reason, Diaconis [11] mentions the open question of maximal spacings, answered in Theorem 1.3. After making our results explicit, successively for unitary matrices and the GUE ensemble, we give applications of our extremal spacings statistics at the end of this Introduction.

1.1. *The unitary group.* Let u_n, a Haar-distributed (measure $\mu_{U(n)}$) unitary matrix over $\mathbb{C}^n$. Suppose u_n has eigenvalues $e^{i\theta_k}$'s, with ordered eigenangles $0 < \theta_1 < \cdots < \theta_n < 2\pi$. Consider the point process on $\mathbb{R}^2$,

$$\chi^{(n)} = \sum_{i=1}^{n} \delta_{(n^{4/3}(\theta_{i+1}-\theta_i),\theta_i)}.$$

Our first result is about the convergence of $\chi^{(n)}$ to a Poisson point process, thanks to this normalization by $n^{-4/3}$.

THEOREM 1.1.   *Suppose $u_n \sim \mu_{U(n)}$. As $n \to \infty$, the process $\chi^{(n)}$ converges to a Poisson point process χ with intensity*

$$\mathbb{E}\chi(A \times I) = \left(\frac{1}{24\pi}\int_A u^2\,du\right)\left(\int_I \frac{du}{2\pi}\right)$$

for any bounded Borel sets $A \subset \mathbb{R}^+$ and $I \subset (0, 2\pi)$.

The intensity is proportional to $\int_I du$ because of the rotational invariance of the Haar measure. The corresponding factor will be less trivial in the case of the GUE ensemble. Our method to prove Theorem 1.1 relies on the *s-modified* random point field technique initiated by Soshnikov [29, 31]: one can calculate the

correlation functions of the process obtained by keeping only the θ_k's, for which $\theta_k + An^{-4/3}$ contains exactly one other eigenvalue. Contrary to [29, 31], we do not use the notion of cluster functions, because we characterize the convergence to Poisson random variables, thanks to the convergence of the factorial moments; this allows us also to consider easily nontranslation invariant kernels, like in the GUE case. Moreover, Theorem 1.1 gives information about the joint distribution of the number of gaps taking values in disjoint intervals (convergence in terms of point processes). In particular we can compute the limiting joint law of the smallest gaps.

Let $t_1^{(n)} < \cdots < t_k^{(n)}$ be the k smallest eigenangles gaps [i.e., of the form $|\theta_{i+1} - \theta_i|$, where the indexes are modulo n and $|\theta_{i+1} - \theta_i| \in (-\pi, \pi)$]. For the sake of brevity, write

$$\tau_k^{(n)} = (72\pi)^{-1/3} t_k^{(n)}.$$

The limiting joint law of the τ_k's is a corollary of Theorem 1.1.

COROLLARY 1.2.   *For any $0 \le x_1 < y_1 < \cdots < x_k < y_k$, under the Haar measure on $\mathrm{U}(n)$,*

$$(1.1) \qquad \mathbb{P}\big(x_\ell < n^{4/3}\tau_\ell^{(n)} < y_\ell, 1 \le \ell \le k\big) \xrightarrow[n\to\infty]{} (e^{-x_k^3} - e^{-y_k^3}) \prod_{\ell=1}^{k-1} (y_\ell^3 - x_\ell^3).$$

In particular, the kth smallest normalized space $n^{4/3}\tau_k^{(n)}$ converges in law to τ_k, with density

$$\mathbb{P}(\tau_k \in \mathrm{d}x) = \frac{3}{(k-1)!} x^{3k-1} e^{-x^3} \, \mathrm{d}x.$$

Note that this result, for $k = 1$, is proved in Vinson's thesis [34] by a different method: he characterizes the number of small gaps as a symmetric function of the eigenvalues, and computes its moments. It is not clear how his method can be extended to provide the joint law of the k smallest gaps.

We now turn to our next question about extreme gaps, that is, the asymptotic behavior of the largest gaps, which were guessed by Vinson, based on the supposed asymptotic independence of distant gaps. We obtain, thanks to the precise asymptotics of one gap probability, obtained by the steepest descent method for Riemann Hilbert problems in [7], and the negative association property of determinantal point processes; see, for example, [4]. Note that both results are posterior to Vinson's thesis.

Consider $\mathcal{T}_1^{(n)} > \mathcal{T}_2^{(n)} > \cdots$ the largest gaps between successive eigenangles of $u \sim \mu_{\mathrm{U}(n)}$, that is, of the form $|\theta_{i+1} - \theta_i|$, where the indexes are modulo n and $|\theta_{i+1} - \theta_i| \in (-\pi, \pi)$. Then, as n goes to infinity, the largest gap converges in L^p to a constant, for any $p > 0$,

$$\frac{n}{\sqrt{32 \log n}} \mathcal{T}_1^{(n)} \xrightarrow{\mathrm{L}^p} 1.$$

Actually the above limit holds for all the ℓ_n largest gaps if ℓ_n is subpolynomial.

THEOREM 1.3. *Let $\ell_n = n^{o(1)}$ be positive integers. Then for any $p > 0$,*

$$\frac{n}{\sqrt{32 \log n}} T_{\ell_n}^{(n)} \xrightarrow{L^p} 1$$

as $n \to \infty$.[2]

Note that, for independent uniform eigenangles on the unit circle, the largest gap is of order $(\log n)/n$, more than in the above theorem, as expected from the repulsion of the eigenvalues in the determinantal case.

1.2. *The Gaussian unitary ensemble.* Similar results hold for the GUE. For this ensemble, the distribution of the eigenvalues has density

$$(1.2) \qquad \frac{1}{Z_n} e^{-n \sum_{i=1}^{n} \lambda_i^2 / 2} \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^2$$

with respect to the Lebesgue product measure, on the simplex $\lambda_1 < \cdots < \lambda_n$. The empirical spectral distribution $\frac{1}{n} \sum \delta_{\lambda_i}$ converges in probability to the semicircle law (see, e.g., [1])

$$\rho_{\mathrm{sc}}(x) = \frac{1}{2\pi} \sqrt{(4 - x^2)_+}.$$

Like for the unitary group, we first consider the smallest gaps, studying the point process

$$\tilde{\chi}^{(n)} = \sum_{i=1}^{n-1} \delta_{(n^{4/3}(\lambda_{i+1} - \lambda_i), \lambda_i)} \mathbb{1}_{|\lambda_i| < 2 - \varepsilon_0}$$

for any arbitrarily small fixed $\varepsilon_0 > 0$ (this is a technical restriction allowing the use of the Plancherel–Rotach asymptotics of the Hermite polynomials).

THEOREM 1.4. *As $n \to \infty$, the process $\tilde{\chi}^{(n)}$ converges to a Poisson point $\tilde{\chi}$ process with intensity*

$$\mathbb{E}\tilde{\chi}(A \times I) = \left(\frac{1}{48\pi^2} \int_A u^2 \, du \right) \left(\int_I (4 - x^2)^2 \, dx \right)$$

for any bounded Borel sets $A \subset \mathbb{R}_+$ and $I \subset (-2 + \varepsilon_0, 2 - \varepsilon_0)$.

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[2] A detailed analysis of the proof gives a speed of convergence: for example, one can show that for any sequence $a_n = o(1)$,

$$(\log n)^{a_n} \left(\frac{n}{\sqrt{32 \log n}} T_1^{(n)} - 1 \right) \xrightarrow{L^p} 0.$$

The problem of the exact fluctuations will be addressed in a future work.

The following corollary about the smallest gaps is an easy consequence of the previous theorem. As for the unitary group, introduce $\tilde{t}_1^{(n)} < \cdots < \tilde{t}_k^{(n)}$ the k nearest spacings in I, that is, of the form $\lambda_{i+1} - \lambda_i$, $1 \le i \le n-1$, with $\lambda_i \in I$, $I = [a, b]$, $-2 < a < b < 2$. Let

$$\tilde{\tau}_k^{(n)} = \left(\int_I (4 - x^2)^2 \, dx \Big/ (144\pi^2) \right)^{1/3} \tilde{t}_k^{(n)}.$$

COROLLARY 1.5.   *For any $0 \le x_1 < y_1 < \cdots < x_k < y_k$, with the above notations and for the* GUE *ensemble measure* (1.2),

$$\mathbb{P}\big(x_\ell < n^{4/3}\tilde{\tau}_\ell^{(n)} < y_\ell, 1 \le \ell \le k\big) \xrightarrow[n \to \infty]{} (e^{-x_k^3} - e^{-y_k^3}) \prod_{\ell=1}^{k-1} (y_\ell^3 - x_\ell^3).$$

In particular, the kth smallest normalized space $n^{4/3}\tilde{\tau}_k^{(n)}$ converges in law to τ_k, with density

$$\mathbb{P}(\tau_k \in dx) = \frac{3}{(k-1)!} x^{3k-1} e^{-x^3} \, dx.$$

COROLLARY 1.6.   *Let* inf *be the index of the smallest gap between eigenvalues of the* GUE *in a compact subset $I \subset (-2, 2)$ with nonempty interior*

$$\lambda_{\inf+1}^{(n)} - \lambda_{\inf}^{(n)} = \inf\{\lambda_{i+1}^{(n)} - \lambda_i^{(n)} | \lambda_i^{(n)} \in I\}.$$

As $n \to \infty$, $\lambda_{\inf}^{(n)}$ converges weakly to the probability measure with density proportional to

$$(4 - x^2)^2 \mathbb{1}_{x \in I}.$$

We now turn to the largest gaps for the GUE ensemble. The result is completely different inside the bulk and on the edge. Indeed, for eigenvalues strictly inside the support of the limiting measure, the maximal spacings have order $\sqrt{\log n}/n$ (see the following Theorem 1.7), while the eigenvalues on the border have an average distance of higher order, $n^{-2/3}$: for any k,

$$n^{2/3}(\lambda_n - 2, \ldots, \lambda_{n-k} - 2)$$

converges weakly as $n \to \infty$ to a multivariate Tracy–Widom distribution; see, for example, [1]. Strictly inside the bulk, the result is analogous to the circular case, the only difference being the normalization, due to the average density of eigenvalues. Let $\tilde{T}_1^{(n)} > \tilde{T}_2^{(n)} > \cdots$ be the largest gaps of type $\lambda_{i+1} - \lambda_i$ with $\lambda_i \in I$, a compact subset of $(-2, 2)$ with nonempty interior.

THEOREM 1.7.   *Let $\ell_n = n^{o(1)}$ be positive integers. Then for any $p > 0$,*

$$\left(\inf_{x \in I} \sqrt{4 - x^2} \right) \frac{n}{\sqrt{32 \log n}} \tilde{T}_{\ell_n}^{(n)} \xrightarrow{L^p} 1.$$

1.3. *The ζ zeros.* When seen in a window of size proportional to the average gap, the spacings between the zeros of Dirichlet L-functions are distributed like particles of a determinantal point process with sine kernel; this is the Montgomery–Odlyzko law [26]. Here we want to discuss the accuracy of this analogy when looking at rare events, the extreme gaps between the zeta zeros, relying on Theorems 1.1 and 1.3.

Due to the availability of many numerical data, we focus on the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ ($\mathfrak{Re}(s) > 1$), which admits an analytic continuation to $\mathbb{C} - \{1\}$. Let $1/2 \pm it_k$ be the nontrivial zeta zeros, $\gamma_i = \mathfrak{Re}(t_i)$, with $0 < \gamma_1 \le \gamma_2 \le \cdots$. Then

$$\tilde{\gamma}_i = \frac{\gamma_{i+1} - \gamma_i}{2\pi} \log\left(\frac{\gamma_i}{2\pi}\right)$$

has an average value 1. The quantity

$$\lambda = \limsup_{i \to \infty} \tilde{\gamma}_i$$

has been widely studied. Conditionally to the generalized Riemann hypothesis, the best known result is $\lambda > 3.0155$ [5]. From the GUE hypothesis for the zeta zeros, it is expected that $\lambda = \infty$. However, to the best of our knowledge, more precise conjectures about the growth speed of large gaps between zeta zeros were not proposed. From Theorem 1.3, amongst n successive gaps with fermionic repulsion, the maximal gap has size about $\sqrt{32 \log n}/(2\pi)$ times the average gap, suggesting

$$\sup_{m \le k \le m+n} \tilde{\gamma}_i \underset{n \to \infty}{\sim} \frac{\sqrt{32 \log n}}{2\pi},$$

in particular

$$\limsup_{i \to \infty} \sqrt{\frac{\log \gamma_i}{32}} (\gamma_{i+1} - \gamma_i) = 1.$$

Odlyzko's numerical data [27] give 3.303 for the maximal value of $\tilde{\gamma}_i$, $1 \le i \le n = 10^6$, while $\frac{\sqrt{32 \log n}}{2\pi} = 3.346$, giving a difference of 1% with the observed gaps. Further tests can be performed at distinct heights along the critical axis, thanks to numerical data of Gourdon [17]: he computed $n = 2 \times 10^9$ successive zeta zeros at height 10^k along the critical axis for each $k \in [\![13, 24]\!]$. The extreme normalized gaps are given on the joint Figure 1, where the expectation from our random matrices result is the straight line $\frac{\sqrt{32 \log n}}{2\pi} = 4.166$. For example, amongst the 2×10^9 gaps following the height 10^{24}, $\sup \tilde{\gamma}_i = 4.158$, that is, a difference of 0.2% with the expected value.

Concerning the smallest gaps, does the Poisson intensity $\frac{1}{24\pi} u^2 \, du$ from Theorem 1.1 appear in the context of the zeta zeros? Note $\tilde{\theta}_i = n(\theta_{i+1} - \theta_i)/(2\pi)$ the normalized gaps. We know that, as $n \to \infty$, the set of gaps $\{2\pi n^{1/3} \tilde{\theta}_i, 1 \le i \le n\}$
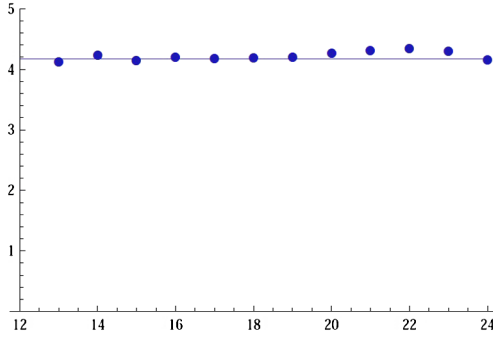
FIG. 1.   *Large gaps between the zeta zeros.*

converges weakly to a Poisson point process with intensity $\frac{1}{24\pi}u^2\,du$. We therefore expect that, as $n \to \infty$,

$$2\pi n^{1/3}\{\tilde{\gamma}_i, 1 \leq i \leq n\}$$

converges to a Poisson point process with the same intensity. The joint Figure 2 gives the histogram of the 3000 smallest gaps, normalized as previously, amongst the $n = 10^{13}$ first zeta zeros, based again on numerical data from [17]. More precisely, the histogram gives the 3000 smallest values of $2\pi 10^{13/3}\{\tilde{\gamma}_i, 1 \leq i \leq 10^{13}\}$. The straight line is the function $5 \times \frac{1}{24\pi}u^2$ (the step of the histogram is 5). This presents a good relevance of the GUE hypothesis for the Riemann zeta function, even at the scale of rare events, here the extreme spacings.

1.4. *Diagonalization speed with the Toda flow.*   The most classical method to diagonalize a matrix is the well-known QR algorithm. In the case of Hermitian matrices, an alternative approach was proposed by Deift et al. [9],[3] based on the isospectral property of the Toda flow. More precisely, given a $n \times n$ Hermitian
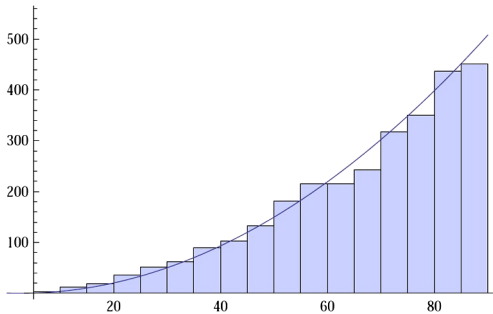

FIG. 2.   *Small gaps between the zeta zeros.*

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[3]Their approach is enounced for symmetric matrices, and naturally extends to the Hermitian case.

matrix M, the first step is to reduce it in a tridiagonal form T (which is a robust and fast operation), conjugating with successive Householder reflections,

$$
T = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{bmatrix},
$$

keeping the same spectrum as M. After such a tridiagonalization, the a_i and b_i's are real. The above matrix is important in the analysis of the Toda lattice: Flaschka, Hénon and Manakov proved independently in the 1970s that the following evolution of n particles on a line ($x_0 = -\infty$, $x_{n+1} = +\infty$, $1 \le k \le n$),

$$
\ddot{x}_k = e^{x_{k-1} - x_k} - e^{x_k - x_{k+1}}
$$

is an integrable system. More precisely, after the change of variables

$$
\begin{cases} a_i = -\dot{x}_i/2, \\ b_i = \frac{1}{2} e^{(x_i - x_{i+1})/2}, \end{cases}
$$

the differential equation takes the Lax pair form

$$
\frac{dT}{dt} = ST - TS,
$$

where

$$
S = \begin{bmatrix} 0 & b_1 & & \\ -b_1 & 0 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & -b_{n-1} & 0 \end{bmatrix}.
$$

In particular and importantly, the spectrum of $T(t)$ does not depend on time. Moser proved that $\dot{x}_k(t) = \lambda_k + \mathrm{o}(1)$, $x_k(t) = \lambda_k t + \mu_k + \mathrm{o}(1)$ as $t \to \infty$, with $\lambda_1 < \cdots < \lambda_n$. This implies that $b_k(t)$ converges to 0, hence T converges to a diagonal matrix, whose entries give the eigenvalues of $M = T(0)$. Deift [6] asked about the speed of convergence of the Toda flow till its equilibrium. More precisely, for a given $\varepsilon > 0$, what is the necessary time t such that $b_k(t) < \varepsilon$ for all $1 \le k \le n - 1$? As

$$
b_k(t) = e^{(1/2)(\lambda_k - \lambda_{k+1})t + (1/2)(\mu_k - \mu_{k+1}) + \mathrm{o}(1)},
$$

the speed of convergence to the spectrum is governed by the minimal gap between eigenvalues. A good choice for a *typical* Hermitian matrix is a matrix from the GUE, with independent (up to symmetry) complex Gaussian entries (of variance 2 on the diagonal, 1 elsewhere). From Corollary 1.5, the minimal gap for such matrices is of order

$$
\sqrt{n} n^{-4/3} = n^{-5/6}.
$$

For a given required precision ε, the Toda flow time necessary to evaluate the eigenvalues is expected to grow as $n^{5/6}$ with the dimension.[4]

2. Small gaps.

2.1. *Convergence to Poisson point processes.* We first review some results about convergence of processes, here with values in $\mathbb{R}^2$ and a finite number of atoms. A point process $\chi^{(n)} = \sum_{i=1}^{k_n} \delta_{X_{i,n}}$ is said to converge in distribution to $\chi = \sum_i \delta_{X_i}$ is, for any bounded continuous function f,

$$\chi^{(n)}(f) = \sum_{i=1}^{k_n} f(X_{i,n}) \xrightarrow{\text{law}} \sum_i f(X_i).$$

To show the convergence of $\chi^{(n)}$ to a Poisson point process χ, we only need to show the convergence in law of $\chi^{(n)}(A, I)$ to $\chi(A, I)$ for all bounded intervals A and I, the independence for disjoint $A \times I$'s being an automatic consequence: this is a very practical property of Poisson point processes, detailed in Proposition 2.1 below. Moreover, the convergence of $\chi^{(n)}(A, I)$ will be shown, thanks to the convergence of the factorial moments to those of a Poisson random variable. This is particularly adapted to our situation because, as we will see, the correlation functions of point processes are defined through factorial moments and are explicit in the case of determinantal point processes.

Note that this is the same technique employed in [3], where the following result is given, in the case of point processes with values in $\mathbb{R}$.

PROPOSITION 2.1.   *Let $\chi^{(n)} = \sum_{i=1}^{k_n} \delta_{X_{i,n}}$ be a sequence of point processes on $\mathbb{R}^2$, and χ a Poisson point process on $\mathbb{R}^2$ with intensity μ having no atoms (and σ-finite). Assume that for any bounded intervals A and I and all positive integers $k \geq 1$,*

$$(2.1) \qquad \lim_{n \to \infty} \mathbb{E}\left(\frac{\chi^{(n)}(A \times I)!}{(\chi^{(n)}(A \times I) - k)!} \right) = \mu(A \times I)^k.$$

Then the sequence of point processes $\chi^{(n)}$ converges in distribution to χ.

PROOF.   We need to check the three conditions of the following Theorem 2.2 by Kallenberg, which is written here in the specific case $\mathfrak{S} = \mathbb{R}^2$, $\mathcal{U}$ and $\mathcal{J}$ the set of compact rectangles $A \times I$ (A and I intervals), with the notation from [20]. Conditions (1), (2) and (3) will be verified if

$$(2.2) \qquad \chi^{(n)}(A \times I) \xrightarrow{\text{law}} \chi(A \times I).$$

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[4]Note that more precise estimates would need the asymptotics of $\mu_k - \mu_{k+1}$ as a function of n.

Indeed, if (2.2) holds, for any t,

$$\lim_{n\to\infty} \mathbb{P}\big(\chi^{(n)}(A \times I) > t\big) = \mathbb{P}\big(\chi(A \times I) > t\big),$$

and as $\chi(A \times I)$ is almost surely finite, this goes to 0 as $t \to \infty$ (is is a consequence of the dominated convergence theorem); this proves part (1). Equations (2) and (3) are also consequences of the above convergence in law.

To prove (2.2), as $\chi(A \times I)$ is a Poisson random variable, the convergence of all the moments is sufficient. A moment is a finite linear combination of factorial moments, which concludes the proof. □

THEOREM 2.2 (Kallenberg [20]).   *Let χ be a point process on $\mathbb{R}^2$, and assume χ is almost surely simple (i.e., the atoms of the measure χ all have weight* 1 *almost surely). Then $\chi^{(n)}$ converges weakly to χ if and only if the three following conditions are satisfied for any compact intervals A and I in $\mathbb{R}$:*

(1) $\lim_{t\to\infty} \lim_{n\to\infty} \mathbb{P}(\chi^{(n)}(A \times I) > t) = 0$;
(2) $\lim_{n\to\infty} \mathbb{P}(\chi^{(n)}(A \times I) = 0) = \mathbb{P}(\chi(A \times I) = 0)$;
(3) $\limsup_{n\to\infty} \mathbb{P}(\chi^{(n)}(A \times I) > 1) \le \mathbb{P}(\chi(A \times I) > 1)$.

2.2. *Correlation functions.*   References for the main properties of correlation functions of determinantal point processes are [19] and [30]. We follow this last survey to present the notions used in the following. If $\chi = \sum_i \delta_{X_i}$ is a simple point process on a complete separate metric space Λ, consider the point process

$$(2.3) \qquad \Xi^{(k)} = \sum_{X_{i_1},\ldots,X_{i_k} \text{ all distinct}} \delta_{(X_{i_1},\ldots,X_{i_k})}$$

on Λ^k. One can define this way a measure M_k on Λ^k by

$$M^{(k)}(\mathcal{A}) = \mathbb{E}\big(\Xi^{(k)}(\mathcal{A})\big)$$

for any Borel set $\mathcal{A}$ in Λ^k. Most of the time, there is a natural measure λ on Λ, in our cases $\Lambda = \mathbb{R}$ or $(0, 2\pi)$, and λ is the Lebesgue measure. If $M^{(k)}$ is absolutely continuous with respect to λ^k, there exists a function ρ_k on Λ^k such that for any Borel sets $B_1, \ldots, B_k$ in Λ

$$M^{(k)}(B_1 \times \cdots \times B_k) = \int_{B_1 \times \cdots \times B_k} \rho_k(x_1, \ldots, x_k)\, d\lambda(x_1) \cdots d\lambda(x_k).$$

Hence one can think about $\rho_k(x_1, \ldots, x_k)$ as the asymptotic (normalized) probability of having exactly one particle in neighborhoods of the x_k's. More precisely, under suitable smoothness assumptions, and for distinct points $x_1, \ldots, x_k$ in $\Lambda = \mathbb{R}$,

$$\rho_k(x_1, \ldots, x_k) = \lim_{\varepsilon \to 0} \frac{1}{\prod_{j=1}^k \lambda(x_j, x_j + \varepsilon)} \mathbb{P}\big(\chi(x_i, x_i + \varepsilon) = 1, 1 \le i \le k\big).$$

Note that ρ_k is not a probability density. Moreover, specifically, if $B_1, \ldots, B_\ell$ are disjoint in Λ and $n_1 + \cdots + n_\ell = k$,

$$M^{(k)}(B_1^{n_1} \times \cdots \times B_\ell^{n_\ell}) = \mathbb{E}\left(\prod_{i=1}^{\ell} \frac{(\chi(B_i))!}{(\chi(B_i) - n_i)!}\right);$$

see [19] for a proof. In particular,

$$(2.4) \quad M^{(k)}(B^k) = \mathbb{E}\left(\frac{(\chi(B))!}{(\chi(B) - k)!}\right) = \int_{B^k} \rho_k(x_1, \ldots, x_k)\, d\lambda(x_1) \cdots d\lambda(x_k).$$

Note the analogy with formula (2.1) we want to prove. For unitary matrices or the GUE ensemble, our method to prove convergence of small spacings counting measures is the same:

- For given compact intervals A and I, consider the modified process obtained from $\xi^{(n)} = \sum \delta_{X_{i,n}}$ by keeping only the points $X_{i,n}$ in I such that $\chi^{(n)}(X_{i,n} + An^{-4/3}) = 1$.
- Show that the correlation function $\tilde{\rho}_k^{(n)}(x_1, \ldots, x_n)$ of this new process uniformly converges to $\mu(A \times I)^k$. This is possible, thanks to the determinantal aspect of $\xi^{(n)}$ and the Hadamard–Fischer inequality, Lemma 2.3.
- Conclude that the factorial moments converge to those of the expected Poisson random variables, thanks to (2.1) and (2.4).

For the smallest gaps asymptotics, the following inequality will be repeatedly used. A concise proof can be found in [18].

LEMMA 2.3. *Let M be a positive-definite $n \times n$ (Hermitian) matrix. For any $\omega \subset [\![1, n]\!]$, let M_ω (resp., $M_{\overline{\omega}}$) be the submatrix of M using rows and columns numbered in ω (resp., $[\![1, n]\!]/\omega$). Then*

$$\det(M) \leq \det(M_\omega) \det(M_{\overline{\omega}}).$$

2.3. *The unitary group.*   We begin with the proof of Theorem 1.1. We know that for a unitary matrix $u_n \sim \mu_{\mathrm{U}(n)}$, the density of the eigenangles $0 \leq \theta_1 < \cdots < \theta_n < 2\pi$, with respect to the Lebesgue measure on the corresponding simplex is

$$\frac{1}{(2\pi)^n} \prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

Moreover, a remarkable fact about the point process $\sum \delta_{e^{i\theta_k}}$ is that it is determinantal: all its correlation functions $\rho_k^{\mathrm{U}(n)}$, $1 \leq k \leq n$, are determinants based on the same kernel,

$$\rho_k^{\mathrm{U}(n)}(\theta_1, \ldots, \theta_n) = \det_{k \times k}\left(K^{\mathrm{U}(n)}(\theta_i - \theta_j)\right), \qquad K^{\mathrm{U}(n)}(\theta) = \frac{1}{2\pi} \frac{\sin(n\theta/2)}{\sin(\theta/2)}.$$

This classical property relies on Gaudin's lemma; see [25]. In the following, for any bounded interval $A \subset \mathbb{R}^+$, we write $A_n = n^{-4/3}A$. We want to show that for an interval $I \subset (0, 2\pi)$,

$$\chi^{(n)}(A \times I) \xrightarrow{\text{law}} \text{Po}(\lambda)$$

with $\lambda = (\frac{1}{48\pi^2} \int_A u^2 \, du)(\int_I du)$. We consider the point process

$$\xi^{(n)} = \sum_{i=1}^n \delta_{\theta_i}$$

and its thinning $\tilde{\xi}^{(n)}$ obtained from $\xi^{(n)}$ by only keeping the eigenangles θ_k for which $\xi^{(n)}(\theta_k + A_n) = 1$. The following lemma means that $\chi^{(n)}(A \times I)$ is properly estimated by $\tilde{\xi}^{(n)}(I)$. It is analogous to Lemma 3 in [31].

LEMMA 2.4 (No successive small neighbors).   *For any interval $I \subset (0, 2\pi)$, as $n \to \infty$, $\chi^{(n)}(A \times I) - \tilde{\xi}^{(n)}(I) \xrightarrow{\text{law}} 0$.*

PROOF.   Let c be such that $A \subset (0, c)$, and $c_n = cn^{-4/3}$. If $\mathbb{1}_{\theta_{i+1} - \theta_i \in A_n} \neq \mathbb{1}_{\xi^{(n)}(\theta_i + A_n) = 1}$, then $\xi^{(n)}(\theta_i + (0, c_n)) \geq 2$. Hence

$$\left| \chi^{(n)}(A \times I) - \tilde{\xi}^{(n)}(I) \right| \leq \sum_{i=1}^n \mathbb{1}_{\xi^{(n)}(\theta_i + (0,c_n)) \geq 2} \leq \Xi^{(3)}(\mathcal{A}),$$

where the last inequality comes from the definition (2.3), where $\mathcal{A}$ is the set of points (θ, x_1, x_2) with $\theta \in (0, 2\pi)$ and $(x_1, x_2) \in (\theta, \theta + c_n)^2$. To prove that this positive random variable converges in law to 0, we consider its expectation

$$\int_0^{2\pi} d\theta \int_{(\theta, \theta + c_n)^2} \rho_3^{\mathrm{U}(n)}(\theta, x_1, x_2) \, dx_1 \, dx_2 = 2\pi \int_{(0, c_n)^2} \rho_3^{\mathrm{U}(n)}(0, x_1, x_2) \, dx_1 \, dx_2$$

and show it goes to 0. Thanks to the multilinearity of the determinant,

$$\rho_3^{\mathrm{U}(n)}(0, x_1, x_2)$$
$$= \begin{vmatrix} K^{\mathrm{U}(n)}(0) & K^{\mathrm{U}(n)}(x_1) - K^{\mathrm{U}(n)}(0) & K^{\mathrm{U}(n)}(x_2) - K^{\mathrm{U}(n)}(0) \\ K^{\mathrm{U}(n)}(x_1) & K^{\mathrm{U}(n)}(0) - K^{\mathrm{U}(n)}(x_1) & K^{\mathrm{U}(n)}(x_1 - x_2) - K^{\mathrm{U}(n)}(x_1) \\ K^{\mathrm{U}(n)}(x_2) & K^{\mathrm{U}(n)}(x_1 - x_2) - K^{\mathrm{U}(n)}(x_2) & K^{\mathrm{U}(n)}(0) - K^{\mathrm{U}(n)}(x_2) \end{vmatrix}.$$

As $|K^{\mathrm{U}(n)}|_\infty = O(n)$ and $|K'^{\mathrm{U}(n)}|_\infty = O(n^2)$, the first column of this determinant is $O(n)$, and the two others are $O(n^2 c_n)$. Thus

$$\rho_3^{\mathrm{U}(n)}(0, x_1, x_2) = O(n^{7/3}).$$

The integration domain is $c_n^2 = O(n^{-8/3})$, concluding the proof.   □

Let $\tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1, \ldots, \theta_k)$, $k \geq 0$, be the correlation functions of the point process $\tilde{\xi}^{(n)}$. If, for any $k \geq 1$, the convergence of the factorial moment

$$
\mathbb{E}\left(\frac{(\tilde{\xi}^{(n)}(I))!}{(\tilde{\xi}^{(n)}(I) - k)!}\right) = \int_{I^k} \tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1, \ldots, \theta_k)\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_k
$$

(2.5)

$$
\xrightarrow[n\to\infty]{} \left(\frac{1}{24\pi}\int_A u^2\,\mathrm{d}u\right)^k \left(\int_I \frac{\mathrm{d}u}{2\pi}\right)^k
$$

can be shown, then Theorem 1.1 will be proved, thanks to the above Lemma 2.4. The way to show (2.5) relies on three steps, to apply a simple dominated convergence argument:

- if all θ_k's are distinct, $\tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1, \ldots, \theta_k)$ converges to $(\frac{1}{48\pi^2}\int_A u^2\,\mathrm{d}u)^k$ as $n \to \infty$ (Lemma 2.5);
- in the set

(2.6) $$\Omega^{\mathrm{U}(n)} = \{(\theta_1, \ldots, \theta_k) \in I^k : \theta_i \notin \theta_j + A_n, 1 \leq i, j \leq k\},$$

  $\tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1, \ldots, \theta_k)$ is uniformly bounded (Lemma 2.6);
- even if $\tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1, \ldots, \theta_k)$ is not uniformly bounded in the complement of $\Omega^{\mathrm{U}(n)}$ in I^k ($\overline{\Omega}^{\mathrm{U}(n)}$), the contribution to the integral is negligible because the volume of $\overline{\Omega}^{\mathrm{U}(n)}$ decreases sufficiently fast (Lemma 2.7).

LEMMA 2.5 (Simple convergence).   *Let $\theta_1, \ldots, \theta_k$ be distinct elements in I^k. Then*

$$
\tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1, \ldots, \theta_k) \xrightarrow[n\to\infty]{} \left(\frac{1}{48\pi^2}\int_A u^2\,\mathrm{d}u\right)^k.
$$

PROOF.   First note that, as all the θ_k's are distinct, for sufficiently large n, the point $(\theta_1, \ldots, \theta_k)$ is in $\Omega^{\mathrm{U}(n)}$; see (2.6). This means that if $\theta_1, \ldots, \theta_k$ are points of $\tilde{\xi}^{(n)}$, the point in each of the $\theta_i + A_n$ is not another one of the θ_j's. This makes the combinatorics easy: the correlation functions of $\tilde{\xi}^{(n)}$ can be explicitly given in terms of those of $\xi^{(n)}$, as noted in [30], by an inclusion–exclusion argument: for sufficiently large n,

$$
\tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1, \ldots, \theta_k)
$$

(2.7)

$$
= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{\theta_1+A_n} \mathrm{d}x_1 \cdots \int_{\theta_k+A_n} \mathrm{d}x_k
$$

$$
\times \int_{((\theta_1+A_n)\sqcup\cdots\sqcup(\theta_k+A_n))^m} \rho_{2k+m}^{\mathrm{U}(n)}(\theta_1, x_1, \ldots, \theta_k, x_k,
$$

$$
y_1, \ldots, y_m)\,\mathrm{d}y_1 \cdots \mathrm{d}y_m.
$$

Note that there is no convergence issue here as $\rho_{2k+m}^{\mathrm{U}(n)} \equiv 0$ if $2k + m > n$. We first show that the term corresponding to $m = 0$ in the above sum gives the expected asymptotics. The determinantal aspect of the process makes things easy. $\rho_{2k}^{\mathrm{U}(n)}(\theta_1, x_1, \ldots, \theta_k, x_k)$ is a $2k \times 2k$ determinant, and only the terms in the 2×2 diagonal blocks make a significant contribution (which leads to the idea of asymptotic independence). More precisely, we write formally

$$\rho_{2k}^{\mathrm{U}(n)}(\theta_1, x_1, \ldots, \theta_k, x_k) = \det_{1 \leq i, j \leq k} \begin{pmatrix} K^{\mathrm{U}(n)}(\theta_i - \theta_j) & K^{\mathrm{U}(n)}(\theta_i - x_j) \\ K^{\mathrm{U}(n)}(x_i - \theta_j) & K^{\mathrm{U}(n)}(x_i - x_j) \end{pmatrix}.$$

As $|x_i - \theta_i| = O(n^{-4/3})$ and $K^{\mathrm{U}(n)}(x) = \sin(nx/2)/\sin(x/2)$ if $i \neq j$ all terms of the corresponding 2×2 above matrix are $O(1)$. Moreover, the above determinant is unchanged by subtracting an odd column to the following even column, and then by subtracting an odd line to the following even line. In this way, the diagonal 2×2 matrices becomes

$$\begin{pmatrix} K^{\mathrm{U}(n)}(0) & K^{\mathrm{U}(n)}(\theta_i - x_i) - K^{\mathrm{U}(n)}(0) \\ K^{\mathrm{U}(n)}(x_i - \theta_i) - K^{\mathrm{U}(n)}(0) & 2K^{\mathrm{U}(n)}(0) - K^{\mathrm{U}(n)}(x_i - \theta_i) - K^{\mathrm{U}(n)}(\theta_i - x_i) \end{pmatrix}$$
$$= \begin{pmatrix} O(n) & O(n^{2/3}) \\ O(n^{2/3}) & O(n^{1/3}) \end{pmatrix},$$

where the last equality relies on $|K^{\mathrm{U}(n)}|_\infty = O(n)$, $|K'^{\mathrm{U}(n)}|_\infty = O(n^2)$ and $|K''^{\mathrm{U}(n)}|_\infty = O(n^3)$. As a consequence, in the expansion of the determinant over all permutations of $\mathcal{S}_{2k}$, the terms corresponding to entries only in the diagonal 2×2 block matrices have order at most $n^{(4/3)k}$, while all other terms have a strictly lower order (at most $n^{(4/3)k-(2/3)}$). As the integration domain of $\rho_{2k}^{\mathrm{U}(n)}$ is $O(n^{-(4/3)k})$, the only permutations hopefully giving a nonzero limit need to come from the block diagonal 2×2 matrices. Indeed they give a nontrivial limit: their contribution is exactly

$$\prod_{i=1}^{k} \int_{\theta_i + A_n} \rho_2^{\mathrm{U}(n)}(\theta_i, x)\, dx = \left(\frac{1}{(2\pi)^2} \int_{A_n} n^2 \left(1 - \left(\frac{\sin(nx/2)}{n\sin(x/2)} \right)^2 \right) dx \right)^k.$$

A simple change of variable $x = n^{4/3}u$ allows us to conclude, thanks to the easy limit, uniform on compacts,

$$\frac{1}{(2\pi)^2} n^{2/3} \left(1 - \left(\frac{\sin(n^{-1/3}u/2)}{n\sin(n^{-4/3}u/2)} \right)^2 \right) \xrightarrow[n\to\infty]{} \frac{1}{48\pi^2} u^2.$$

Our last task is to show that in the limit (2.7) is equivalent to its $m = 0$ term. By iterations of the Hadamard–Fisher inequality, Lemma 2.3,

$$\rho_{2k+m}^{\mathrm{U}(n)}(\theta_1, x_1, \ldots, \theta_k, x_k, y_1, \ldots, y_m) \leq \rho_{2k}^{\mathrm{U}(n)}(\theta_1, x_1, \ldots, \theta_k, x_k) \prod_{i=1}^{m} \rho_1^{\mathrm{U}(n)}(y_i).$$

The contribution of all terms with $m \geq 1$ in (2.7) is therefore bounded by

$$\left(\int_{\theta_1 + A_n} dx_1 \cdots \int_{\theta_k + A_n} dx_k \, \rho_{2k}^{U(n)}(\theta_1, x_1, \ldots, \theta_k, x_k) \right) \sum_{m \geq 1} \frac{1}{m!} \left(\int_{a_n} \rho_1^{U(n)}(y) \right)^m,$$

where the integration domain $a_n = (\theta_1 + A_n) \sqcup \cdots \sqcup (\theta_k + A_n)$ has size $O(n^{-4/3})$ and $\rho_1^{U(n)}(y) = n$. The first term of the above product converges, as previously proved (it corresponds to $m = 0$), so the whole term goes to 0 as $n \to \infty$, concluding the proof. $\square$

LEMMA 2.6 (Uniform boundness). *There is a constant c depending only on A such that, for any $n \geq 1$ and $(\theta_1, \ldots, \theta_k) \in \Omega^{U(n)}$ [see (2.6)],*

$$\tilde{\rho}_k^{U(n)}(\theta_1, \ldots, \theta_k) < c.$$

PROOF. As previously mentioned, formula (2.7) is true whenever, for all distinct i and j, $\theta_j \notin \theta_i + A_n$, that is, $(\theta_1, \ldots, \theta_n)$ is in $\Omega^{U(n)}$. Using the Hadamard–Fisher inequality as in the proof of the previous lemma, $\tilde{\rho}_k^{U(n)}(\theta_1, \ldots, \theta_k)$ is therefore bounded by

$$\left(\int_{\theta_1 + A_n} dx_1 \cdots \int_{\theta_k + A_n} dx_k \, \rho_{2k}^{U(n)}(\theta_1, x_1, \ldots, \theta_k, x_k) \right) \sum_{m \geq 0} \frac{1}{m!} \left(\int_{a_n} \rho_1^{U(n)}(y) \right)^m$$

with $a_n = (\theta_1 + A_n) \sqcup \cdots \sqcup (\theta_k + A_n)$. Once again, the Hadamard–Fisher inequality gives

$$\rho_{2k}^{U(n)}(\theta_1, x_1, \ldots, \theta_k, x_k) \leq \prod_{i=1}^{k} \rho_2^{U(n)}(\theta_i, x_i).$$

This gives the upper bound, uniform in $(\theta_1, \ldots, \theta_n) \in \Omega^{U(n)}$,

$$\left(\int_{A_n} \rho_2^{U(n)}(0, x) \, dx \right)^k \sum_{m \geq 0} \frac{1}{m!} \left(\int_{a_n} \rho_1^{U(n)}(y) \right)^m$$

converging to

$$\left(\frac{1}{48\pi^2} \int_A u^2 \, du \right)^k$$

as previously seen. $\square$

REMARK. A better upper bound in the previous proof can be obtained as follows. By a direct ensembles argument, $\tilde{\rho}_k^{U(n)}(\theta_1, \ldots, \theta_k)$ is bounded by

$$\int_{\theta_1 + A_n} dx_1 \cdots \int_{\theta_k + A_n} dx_k \, \rho_{2k}^{U(n)}(\theta_1, x_1, \ldots, \theta_k, x_k).$$
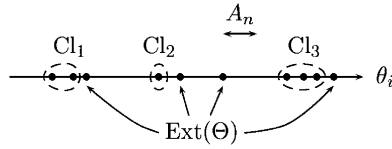
FIG. 3.   *Maximal clusters.*

This also comes from the fact that the inclusion–exclusion series (2.7) is alternate. We know by Fisher–Hadamard that this upper bound is lower than $(\int_{A_n} \rho_2^{\mathrm{U}(n)}(0,x)\,\mathrm{d}x)^k$, which is interpreted as follows: $\tilde{\rho}_k^{\mathrm{U}(n)}$ converges to its limit from below, which is a sign of repulsion before the asymptotic independence.

LEMMA 2.7 (Negligible set).   *Let $\overline{\Omega}^{\mathrm{U}(n)}$ be the complement of $\Omega^{\mathrm{U}(n)}$ in I^k; see* (2.6). *Then*

$$\int_{\overline{\Omega}^{\mathrm{U}(n)}} \tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1,\ldots,\theta_k)\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_k \underset{n\to\infty}{\longrightarrow} 0.$$

PROOF.   Let $(\theta_1,\ldots,\theta_k)\in\overline{\Omega}^{\mathrm{U}(n)}$, and note $\Theta=\{\theta_1,\ldots,\theta_k\}$ the set of these points. For notational convenience, one can suppose $\theta_1<\cdots<\theta_k$. A set of points $\theta_s<\cdots<\theta_t$ is said to be a cluster of Θ if, for all $s\leq k\leq t$, $\theta_{k+1}\in\theta_k+A_n$. Let $\mathrm{Ext}(\Theta)$ be the set of points which cannot be included in a maximal cluster (see Figure 3)

$$\mathrm{Ext}(\Theta)=\{\theta_i:1\leq i\leq k,(\theta_i+A_n)\cap\Theta=\varnothing\}.$$

This way we get a partition

$$\Theta=\mathrm{Ext}(\Theta)\bigsqcup_{i=1}^{\ell}\mathrm{Cl}_i,$$

where there are ℓ maximal clusters $\mathrm{Cl}_1,\ldots,\mathrm{Cl}_\ell$. Suppose that $\mathrm{Ext}(\Theta)=\{\theta_{i_1}<\cdots<\theta_{i_p}\}$ where $p=|\mathrm{Ext}(\Theta)|$. Then the following obvious bound holds:

$$\tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1,\ldots,\theta_k)\leq\int_{\theta_{i_1}+A_n}\mathrm{d}x_1\cdots\int_{\theta_{i_p}+A_n}\mathrm{d}x_p\,\rho_{k+p}^{\mathrm{U}(n)}(\theta,\ldots,\theta_k,x_1,\ldots,x_p)$$

$$\leq\prod_{j=1}^{\ell}\rho_{|\mathrm{Cl}_i|}^{\mathrm{U}(n)}(\mathrm{Cl}_i)\prod_{j=1}^{p}\int_{\theta_{i_j}+A_n}\rho_2^{\mathrm{U}(n)}(\theta_{i_j},x_j)\,\mathrm{d}x_j,$$

where we used the Hadamard–Fisher inequality. This last product of p elements is bounded, uniformly in $(\theta_1,\ldots,\theta_k)$, as it is equal to the pth power of

$$\int_{A_n}\rho_2^{\mathrm{U}(n)}(0,x)\,\mathrm{d}x,$$

which converges. Concerning the first product, we analyze each term in the following way. For any $1 \leq t \leq k - 1$, the correlation $\rho_t^{\mathrm{U}(n)}$ is a $t \times t$ determinant of $K^{\mathrm{U}(n)}$. Suppose that all arguments of $\rho_t^{\mathrm{U}(n)}$ are included in an interval of size c_n. By subtracting the first row from the $t - 1$ others, one obtains that the first row is $O(n)$, and the others are $O(n^2 c_n)$ because $|K'^{\mathrm{U}(n)}|_\infty = O(n^2)$. Hence $\rho_t^{\mathrm{U}(n)} = O(n^{2t-1} c_n^{t-1})$. As a consequence, as all points of a cluster Cl_i are within distance $O(n^{-4/3})$,

$$\rho_{|\mathrm{Cl}_i|}^{\mathrm{U}(n)}(\mathrm{Cl}_i) = O\big(n^{(2/3)|\mathrm{Cl}_i|+1/3}\big).$$

This leads to the upper bound

$$\tilde{\rho}_k^{\mathrm{U}(n)}(\theta_1, \ldots, \theta_k) = O\big(n^{(2/3)(k-p)+\ell/3}\big).$$

But the size of the set of points $(\theta_1, \ldots, \theta_k) \in \overline{\Omega}^{\mathrm{U}(n)}$ with such a clusters configuration is $O(n^{-(4/3)(k-p)})$, because there needs to be $k - p$ small gaps [i.e., of size $O(n^{-4/3})$] between successive θ_k's. Hence the total contribution of such clusters to the integral over $\overline{\Omega}^{\mathrm{U}(n)}$ is

$$O\big(n^{\ell/3-(2/3)(k-p)}\big),$$

which goes top 0 because $\ell \leq k - p$ (with equality if all clusters have only one point) and $k - p < 2(k - p)$ [$k = p$ is not possible: this would mean that $(\theta_1, \ldots, \theta_k) \notin \overline{\Omega}^{\mathrm{U}(n)}$]. □

As previously explained, the three above lemmas complete the proof of Theorem 1.1.

PROOF OF COROLLARY 1.2.   Note that the events $\{x_\ell < n^{4/3} \tau_\ell^{(n)} < y_\ell, 1 \leq \ell \leq k\}$ and

$$\big\{\chi^{(n)}((72\pi)^{1/3}(x_k, y_k), (0, 2\pi)) \geq 1,$$
$$\chi^{(n)}((72\pi)^{1/3}(x_\ell, y_\ell), (0, 2\pi)) = 1, 1 \leq \ell \leq k-1,$$
$$\chi^{(n)}((72\pi)^{1/3}(y_{\ell-1}, x_\ell), (0, 2\pi)) = 0, 1 \leq \ell \leq k\big\}$$

are almost surely the same ($y_0 = 0$). The independence property of the limit Poisson point process χ in disjoint subsets therefore yields

$$\mathbb{P}\big(x_\ell < n^{4/3} \tau_\ell^{(n)} < y_\ell, 1 \leq \ell \leq k\big)$$

$$\xrightarrow[n \to \infty]{} (1 - e^{-(y_k^3 - x_k^3)}) \prod_{\ell=1}^{k-1} (y_\ell^3 - x_\ell^3) e^{-(y_\ell^3 - x_\ell^3)} \prod_{\ell=1}^{k} e^{-(x_\ell^3 - y_{\ell-1}^3)},$$

where we noted that, for any interval (a, b), $\chi^{(n)}((72\pi)^{1/3}(a, b), (0, 2\pi))$ is a Poisson random variable with parameter $(b^3 - a^3)$. A straightforward simplification of the above products gives the expected result. Concerning the limiting

density of $n^{4/3}\tau_k^{(n)}$, we proceed in two steps. First, from formula (1.1), the joint density of $n^{4/3}(\tau_1^{(n)},\ldots,\tau_k^{(n)})$ is, in the limit and on the simplex $0 < u_1 < \cdots < u_k$, proportional to

$$u_1^2 u_2^2 \cdots u_k^2 e^{-u_k^3}.$$

Consequently,

$$
\begin{aligned}
\mathbb{P}(\tau_k < x) &= c_k \int_0^x u_k^2 e^{-u_k^3} \int_{0<u_1<\cdots<u_{k-1}<u_k} u_1^2 u_2^2 \cdots u_{k-1}^2 \, \mathrm{d}u_1 \cdots \mathrm{d}u_{k-1} \\
&= c_k \int_0^x u_k^{2+3(k-1)} e^{-u_k^3} \int_{0<v_1<\cdots<v_{k-1}<1} v_1^2 v_2^2 \cdots v_{k-1}^2 \, \mathrm{d}v_1 \cdots \mathrm{d}v_{k-1},
\end{aligned}
$$

so the density of τ_k is proportional to $x^{3k-1} e^{-x^3}$. $\quad\square$

2.4. *The Gaussian unitary ensemble.* The small gaps asymptotics for the GUE are obtained exactly in the same way as for the unitary group. The only difference is that the determinantal kernel is not translation invariant anymore, leading to some complications. More precisely, let (h_n) be the Hermite polynomials, more precisely the successive monic orthogonal polynomials with respect to the Gaussian weight $e^{-x^2/2} \, \mathrm{d}x$. Following [1], where the following results on the determinantal aspect of the GUE can be found, we introduce the functions

$$\psi_k(x) = \frac{e^{-x^2/4}}{\sqrt{\sqrt{2\pi}k!}} h_k(x).$$

Then the set of points $\{\lambda_1,\ldots,\lambda_n\}$ with law (1.2) is a determinantal point process with kernel (with respect to the Lebesgue measure on $\mathbb{R}$) given by

$$
\begin{aligned}
K^{\mathrm{GUE}(n)}(x,y) &= n \frac{\psi_n(x\sqrt{n})\psi_{n-1}(y\sqrt{n}) - \psi_{n-1}(x\sqrt{n})\psi_n(y\sqrt{n})}{x - y} \\
&= n^{3/2}\left(\psi_{n-1}(y\sqrt{n}) \int_0^1 \psi_n'(tx\sqrt{n} + (1-t)y\sqrt{n}) \, \mathrm{d}t \right. \\
&\quad \left. - \psi_n(y\sqrt{n}) \int_0^1 \psi_{n-1}'(tx\sqrt{n} + (1-t)y\sqrt{n}) \, \mathrm{d}t \right).
\end{aligned}
$$

(2.8)

We will now discuss why Lemma 2.4 through Lemma 2.7 still hold in the case of the GUE ensemble, restricted to $(-2 + \varepsilon_0, 2 - \varepsilon_0)$ for a given $\varepsilon_0 > 0$. Note that the Hadamard–Fischer inequality, an important tool for the proof of all the following lemmas, still holds for the Gaussian unitary ensemble because the set of its eigenvalues is a determinantal process.

- We now note

$$\xi^{(n)} = \sum_{i=1}^n \delta_{\lambda_i} \mathbb{1}_{|\lambda_i|<2-\varepsilon_0}$$

and its thinning $\tilde{\xi}^{(n)}$ obtained from $\xi^{(n)}$ by only keeping the eigenvalues λ_k for which $\xi^{(n)}(\lambda_k + n^{-4/3}A = 1)$. The absence of successive small gaps, Lemma 2.4, only requires in its proof that $|K^{\text{GUE}(n)}|_\infty = \text{O}(n)$, $|\partial_x K^{\text{GUE}(n)}|_\infty = \text{O}(n^2)$. This is proved in Lemma 2.8.

- The analog of the simple convergence of the correlation function associated to the thinned point process, Lemma 2.5, is now, for any distinct points $\lambda_1, \ldots, \lambda_k$ in $(-2 + \varepsilon_0, 2 - \varepsilon_0)$,

$$\tilde{\rho}_k^{\text{GUE}(n)}(\lambda_1, \ldots, \lambda_k) \xrightarrow[n \to \infty]{} \left(\frac{1}{48\pi^2} \int_A u^2 \, \mathrm{d}u \right)^k \prod_{i=1}^k (4 - \lambda_i^2)^2.$$

The proof, in the same way as the unitary case, requires uniform bounds on the partial derivatives of the kernel, given in Lemma 2.8, and the asymptotics

$$\int_{\lambda_i + A_n} \rho_2^{\text{GUE}(n)}(\lambda_i, x) \, \mathrm{d}x \xrightarrow[n \to \infty]{} \left(\frac{1}{48\pi^2} \int_A u^2 \, \mathrm{d}u \right) (4 - \lambda_i^2)^2,$$

which is a direct consequence of Lemma 2.9.

- The uniform boundedness result, Lemma 2.6, only requires the Hadamard–Fischer inequality, Lemma 2.3, which applies to any determinantal point process.

- Finally, we give the analog of Lemma 2.7 in the following way. Let

$$\Omega^{\text{GUE}(n)} = \{(\lambda_1, \ldots, \lambda_k) \in I^k : \lambda_i \notin \lambda_j + A_n, 1 \le i, j \le k\},$$

and $\overline{\Omega}^{\text{GUE}(n)}$ be its complement in I^k, where I is any Borel set in $(-2 + \varepsilon_0, 2 - \varepsilon_0)$.

$$\int_{\overline{\Omega}^{\text{GUE}(n)}} \tilde{\rho}_k^{\text{GUE}(n)}(\lambda_1, \ldots, \lambda_k) \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_k \xrightarrow[n \to \infty]{} 0.$$

The only estimate necessary for this result is that the first-order partial derivative of the kernel is uniformly $\text{O}(n)$, which is one of the estimates of the following Lemma 2.8.

LEMMA 2.8. *Let $\varepsilon_0 > 0$. Uniformly for $x, y \in (-2 + \varepsilon_0, 2 - \varepsilon_0)$, $K^{\text{GUE}(n)}(x, y)$ is $\text{O}(n)$, the first-order partial derivatives of $K^{\text{GUE}(n)}$ are $\text{O}(n^2)$ and the second-order ones are $\text{O}(n^3)$.*

Moreover, under the additional condition $|x - y| > \delta > 0$, $K^{\text{GUE}(n)}(x, y)$ is uniformly bounded, by a constant depending only on ε_0 and δ.

PROOF. From the Plancherel–Rotach asymptotics for the Hermite polynomials (Theorem 8.22.9 in [32]), for any nonnegative integer k, $\psi_{n-k}(\sqrt{n}x)$ is $\text{O}(1/\sqrt{n})$, uniformly in $x \in (-2 + \varepsilon_0, 2 - \varepsilon_0)$. Consequently, if $|x - y| > \delta$, from the first line of (2.8), $K^{\text{GUE}(n)}$ is uniformly $\text{O}(1)$.

To prove the first assertion of the lemma, we use the stability property of the functions (ψ_k) by derivation (see [1], Lemma 3.2.7),

$$(2.9) \qquad \psi_n'(x) = -\frac{x}{2}\psi_n(x) + \sqrt{n}\psi_{n-1}(x).$$

Injecting this expression of ψ_n' in the last two lines of (2.8) and using $\psi_n = O(1/\sqrt{n})$ yields $K^{\text{GUE}(n)}(x, y) = O(n)$ uniformly for $x, y \in (-2 + \varepsilon_0, 2 - \varepsilon_0)$. Iterating this procedure, any partial derivative of $K^{\text{GUE}(n)}$ of order k are n^{k+2} times a linear combination of factors of type $\psi_{n-i} \int_0^1 \psi_{n-j}$, which shows that any partial derivative of order k is $O(n^{k+1})$. $\square$

The Plancherel–Rotach asymptotics for Hermite polynomials also yield a precise evaluation of the correlation functions evaluated at close points, in particular the following result.

LEMMA 2.9. *Let $\varepsilon_0 > 0$, $c > 0$. Then as $n \to \infty$, uniformly in $x \in (-2 + \varepsilon_0, 2 - \varepsilon_0)$, $|u| < cn^{-4/3}$,*

$$\rho_2^{\text{GUE}(n)}(x, x + u) = \frac{1}{48\pi^2} n^4 (4 - x^2)^2 u^2 + O(n).$$

PROOF. The intuition for this result is as follows. From the well-known convergence to the sine kernel,

$$\frac{1}{n\rho_{\text{sc}}(x)} K^{\text{GUE}(n)}\left(x, x + \frac{v}{n\rho_{\text{sc}}(x)}\right) \underset{n \to \infty}{\longrightarrow} \frac{\sin(\pi v)}{\pi v}.$$

If this convergence is sufficiently uniform, one expects for small u a Taylor expansion

$$K^{\text{GUE}(n)}(x, x + u) \approx \frac{\sin(\pi n\rho_{\text{sc}}(x)u)}{\pi u} \approx n\rho_{\text{sc}}(x) - \frac{\pi^2}{6}\rho_{\text{sc}}(x)^3 n^3 u^2.$$

Injecting this expansion in

$$\rho_2^{\text{GUE}(n)}(x, x + u) = K^{\text{GUE}(n)}(x, x) K^{\text{GUE}(n)}(x + u, x + u) - K^{\text{GUE}(n)}(x, x + u)^2$$

gives the expected result. To justify the accuracy of the sine kernel approximation for u close to 0, we rely on Corollary 1 in [10] (more general asymptotics in the full complex plane for, in particular, Hermite polynomials were obtained in [8]): using the Plancherel–Rotach asymptotics of Hermite polynomials the authors show that[5]

$$K^{\text{GUE}(n)}(x, y) = \frac{1}{2\pi\sqrt{\sin\omega\sin\theta}} \frac{\sin(n(a_\theta - a_\omega))}{\sin(\omega - \theta)} + O(1)$$

uniformly for x, y in $(-2 + \varepsilon_0, 2 - \varepsilon_0)$ where $x = 2\cos\omega$, $y = 2\cos\theta$, $0 \le \theta, \omega \le \pi$, $a_\omega = \sin(2\omega) - 2\omega$, $a_\theta = \sin(2\theta) - 2\theta$. As $a_\theta - a_\omega = 2(\omega - \theta)\sin^2\theta +$

---

[5]Note that the normalization in [10] is different, as their semicircle law is supported on $(-\sqrt{n}, \sqrt{n})$.

$O((\omega - \theta)^2)$ and $\omega - \theta = (y - x)/\sqrt{4 - x^2} + O((x - y)^2)$, a simple expansion yields

$$(2.10) \qquad \begin{aligned} &\frac{1}{\rho_{sc}(x)} K^{GUE(n)}(x, y) \\ &\qquad = \frac{1}{2\pi^2 \rho_{sc}(x)\sqrt{\rho_{sc}(x)\rho_{sc}(y)}} \\ &\qquad\quad \times \frac{\sin(2n(\omega - \theta)\sin^2\theta + O(n(\omega - \theta)^2))}{\sin((x - y)/\sqrt{4 - y^2} + O((x - y)^2))} + O(1) \\ &\qquad = \frac{\sin(n\pi(x - y)\rho_{sc}(x) + O(n(x - y)^2))}{\pi(x - y)\rho_{sc}(x) + O((x - y)^2)} + O(1). \end{aligned}$$

For $x - y = O(n^{-4/3})$, this implies

$$\frac{1}{\rho_{sc}(x)} K^{GUE(n)}(x, y) = n - \frac{1}{6} n^3 \pi^2 (x - y)^2 \rho_{sc}(x)^3 + O(1)$$

and the expected result for $\rho_2^{GUE(n)}(x, x + u)$. $\square$

The above discussion completes the proof of Theorem 1.4. Its Corollary 1.5 follows exactly in the same way as Corollary 1.2 from Theorem 1.1.

Corollary 1.6 is a straightforward consequence of the scission of the limiting Lévy measure: if $\tilde{\chi} = \{(a_j, b_j)\}$ is a Poisson point process with measure $\mathbb{E}\tilde{\chi}(A \times B) = \mu_1(A)\mu_2(B)$, conditionally to the b_k's the a_k's are distributed independently of each other, independently of the b_k's, as a Poisson point process with intensity proportional to μ_1. In particular, in our situation, the abscissa associated to the minimal ordinate is distributed with density proportional to

$$(4 - x^2)^2.$$

3. Large gaps.

3.1. *The unitary group*: *Asymptotics of Toeplitz determinants*.   To evaluate the extreme gaps, we first investigate the queuing distribution of one given spacing. A large part of the literature concerns the probability of having no eigenvalues in a given interval (e.g., [12, 35]). A rigorous derivation of the queuing distribution for the large gaps was only given recently, thanks to the steepest descent method for Riemann–Hilbert problems [7, 23] or by operator theory tools for Toeplitz determinants [13].

The link with the nearest neighbor distribution is given by the following lemma, explained in [25], Appendix A.8.

LEMMA 3.1.   *Let $u \sim \mu_{\mathrm{U}(n)}$, with eigenangles $0 \leq \theta_1 < \cdots < \theta_n < 2\pi$, and $\xi^{(n)} = \sum_{i=1}^{n} \delta_{\theta_i}$. Then*

$$n\mathbb{P}(\theta_2 - \theta_1 > x) = -\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\xi^{(n)}(0, x) = 0).$$

Moreover, the probability of having no eigenvalues in an arc of size 2α is equal to the Toeplitz determinant (this is a consequence of Heine's formula)

$$D_n(\alpha) = \det_{1 \leq j,k \leq n}\left(\frac{1}{2\pi}\int_{\alpha}^{2\pi-\alpha} e^{\mathrm{i}(j-k)\theta}\,\mathrm{d}\theta\right).$$

All the asymptotics we need in the following are direct consequences of the precise analysis of $D_n(\alpha)$ given by Deift et al. [7] and Krasovsky [23]. More precisely they prove that for some sufficiently large s_0 and any $\varepsilon > 0$, uniformly in $s_0/n < \alpha < \pi - \varepsilon$,

$$(3.1)\qquad \log D_n(\alpha) = n^2 \log \cos\frac{\alpha}{2} - \frac{1}{4}\log\left(n\sin\frac{\alpha}{2}\right) + c_0 + \mathrm{O}\left(\frac{1}{n\sin(\alpha/2)}\right),$$

$$(3.2)\quad \frac{\mathrm{d}}{\mathrm{d}\alpha}\log D_n(\alpha) = -\frac{n^2}{2}\tan\frac{\alpha}{2} - \frac{1}{8}\cot\frac{\alpha}{2} + \mathrm{O}\left(\frac{1}{n\sin^2(\alpha/2)}\right)$$

for an explicit constant c_0, which remained conjectural for 20 years. From this, we can give upper bounds on $n\mathbb{P}(\theta_2 - \theta_1 > u)$, the expectation of the number of gaps greater than u.

LEMMA 3.2.   *Given any $a > 0$ and $\varepsilon > 0$, uniformly in $u \in (a\sqrt{\log n}/n, 2\pi - \varepsilon)$, as $n \to \infty$*

$$n\mathbb{P}(\theta_2 - \theta_1 > u) \leq un^{2+\mathrm{o}(1)}e^{-n^2u^2/32}.$$

PROOF.   From Lemma 3.1,

$$n\mathbb{P}(\theta_2 - \theta_1 > u) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}D_n(\alpha) = -\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\log D_n(\alpha)\right)D_n(\alpha)$$

evaluated for $u = 2\alpha$. The first term is evaluated thanks to (3.2): it is $\mathrm{O}(un^{2+\mathrm{o}(1)})$ uniformly in $(a\sqrt{\log n}/n, \pi - \varepsilon)$. Concerning the second term, (3.1) and the inequality $\log\cos x \leq -x^2/2$ on $[0, \pi/2)$ imply that it is $e^{-n^2\alpha^2/8}n^{\mathrm{o}(1)}$, completing the proof. $\square$

Note that substituting $u = 2\alpha = \frac{\sqrt{\lambda\log n}}{n}$ in the asymptotics (3.1) and (3.2) yields

$$(3.3)\qquad\qquad n\mathbb{P}(\theta_2 - \theta_1 > u) = n^{1-\lambda/32+\mathrm{o}(1)},$$

where the o(1) terms are uniform in $\lambda \in [a, b]$, for any given positive a and b. Hence, if $\lambda > 32$, the expected number of gaps greater than $\frac{\sqrt{\lambda\log n}}{n}$ goes to 0. If

$\lambda < 32$, this goes to ∞. The transition in the number of large gaps at $\lambda = 32$ is a strong clue for Theorem 1.3, but it is not sufficient: concluding the proof requires additional knowledge about the correlations between gaps, Lemma 3.8, assumed for the moment. Take $p > 0$. We denote

$$X_n = \frac{n}{\sqrt{32 \log n}} \mathcal{T}_{\ell_n}^{(n)},$$

where $\ell_n = n^{o(1)}$. To prove that $X_n - 1$ converges to 0 in L^p, we pick some arbitrarily small $\varepsilon > 0$, bound $\mathbb{E}(|X_n - 1|^p \mathbb{1}_{1-\varepsilon < X_n < 1+\varepsilon})$ by ε^p, and we need to prove that both of the following terms converge to 0:

$$\mathbb{E}(|X_n - 1|^p \mathbb{1}_{X_n < 1-\varepsilon}) \leq \mathbb{P}(X_n < 1 - \varepsilon),$$

(3.4) $$\mathbb{E}(|X_n - 1|^p \mathbb{1}_{X_n > 1+\varepsilon}) \leq \mathbb{E}(|X_n - 1|^p \mathbb{1}_{X_n > 1+\varepsilon, \mathcal{T}_1^{(n)} < \pi/2})$$

$$+ \left(\frac{2\pi n}{\sqrt{32 \log n}}\right)^p \mathbb{P}(\mathcal{T}_1^{(n)} > \pi/2)$$

(here the value $\pi/2$ is arbitrary, any angle strictly smaller than π would be appropriate for this proof). First, decompose the unit circle into 8 fixed angles of size $\pi/4$. If $\mathcal{T}_1^{(n)} > \pi/2$, one of these arcs is free of eigenvalues. From (3.1), the probability of this event decreases exponentially, so (3.4) converges to 0.

Moreover, integrating by parts,

$$\mathbb{E}(|X_n - 1|^p \mathbb{1}_{X_n > 1+\varepsilon, \mathcal{T}_1^{(n)} < \pi/2})$$

$$= \int_{1+\varepsilon}^{\infty} p(u-1)^{p-1} \mathbb{P}(X_n > u, \mathcal{T}_1^{(n)} < \pi/2) \, du$$

$$+ p\varepsilon^{p-1} \mathbb{P}(X_n > 1 + \varepsilon, \mathcal{T}_1^{(n)} < \pi/2)$$

$$\leq \int_{1+\varepsilon}^{(n/\sqrt{32 \log n})(\pi/2)} p(u-1)^{p-1} \mathbb{P}(X_n > u) \, du + p\varepsilon^{p-1} \mathbb{P}(X_n > 1 + \varepsilon),$$

because X_n needs to be shorter than $\frac{n}{\sqrt{32 \log n}} \mathcal{T}_1^{(n)}$. The probability that X_n is greater than u is obviously shorter than $n\mathbb{P}(\theta_2 - \theta_1 > u\sqrt{32 \log n}/n)$, the expectation of the number of gaps greater than $u\sqrt{32 \log n}/n$. Hence the above quantity goes to 0 thanks to the uniform estimate of Lemma 3.2.

Finally, showing that $\mathbb{P}(X_n < 1 - \varepsilon) \to 0$ requires an additional argument, the negative correlation property for empty sets events, Lemma 3.8. In particular, this negative correlation implies the following result.

LEMMA 3.3. *Consider a set of disjoint arcs on the unit circle. Let M_n be the number of such intervals free of eigenangles, that is, those I_k's such that $\xi^{(n)}(I_k) = 0$. Then $\mathrm{Var}(M_n) \leq \mathbb{E}(M_n)$.*

PROOF.   This is a straightforward consequence of Lemma 3.8: as $\xi^{(n)}$ is a determinantal point process, for disjoints I and J, $\mathbb{P}(\xi^{(n)}(I \cup J) = 0) \leq \mathbb{P}(\xi^{(n)}(I_j) = 0)\mathbb{P}(\xi^{(n)}(I_k) = 0)$. Hence, noting m_n the number of initial arcs,

$$
\begin{aligned}
\mathbb{E}(M_n^2) &= \sum_{1 \leq j,k \leq m_n} \mathbb{P}\big(\xi^{(n)}(I_j \cup I_k) = 0\big) \\
&\leq 2 \sum_{1 \leq j < k \leq m_n} \mathbb{P}\big(\xi^{(n)}(I_j) = 0\big)\mathbb{P}\big(\xi^{(n)}(I_k) = 0\big) + \sum_{1 \leq j \leq m_n} \mathbb{P}\big(\xi^{(n)}(I_j) = 0\big) \\
&\leq \mathbb{E}(M_n)^2 + \mathbb{E}(M_n)
\end{aligned}
$$

as expected. $\square$

Consider now a number m_n of disjoint intervals $I_1, \ldots, I_{m_n}$ of length $(1 - \varepsilon)\frac{\sqrt{32 \log n}}{n}$ in $(0, 2\pi)$. We can find $m_n = \lfloor 2\pi n / \sqrt{32 \log n} \rfloor$ of them. Let M_n be the number of such intervals free of eigenangles, that is, those I_k's such that $\xi^{(n)}(I_k) = 0$. If there are less than ℓ_n gaps larger than $(1 - \varepsilon)\frac{\sqrt{32 \log n}}{n}$, either there are less than ℓ_n intervals I_k's free of eigenangles, or there is a gap between successive eigenangles containing two intervals,

$$
\mathbb{P}(X_n < 1 - \varepsilon) \leq \mathbb{P}(M_n < \ell_n) + \mathbb{P}\bigg(\mathcal{T}_1^{(n)} \geq 2(1 - \varepsilon)\frac{\sqrt{32 \log n}}{n}\bigg).
$$

This last term is bounded by the expectation of the number of gaps greater than $2(1 - \varepsilon)\frac{\sqrt{32 \log n}}{n}$; from (3.3) this goes to 0 if $2(1 - \varepsilon) > 1$ (true for ε sufficiently small).

Concerning $\mathbb{P}(M_n < \ell_n)$, first note that by the estimate (3.3), $\mathbb{E}(M_n) = n^{1-(1-\varepsilon)^2+o(1)}$, so $\ell_n = o(\mathbb{E}(M_n))$. This allows to use Chebyshev's inequality, for sufficiently large n.

$$
\begin{aligned}
\mathbb{P}(M_n < \ell_n) &\leq \mathbb{P}\big(|M_n - \mathbb{E}(M_n)| > \mathbb{E}(M_n) - \ell_n\big) \\
&\leq \frac{\mathrm{Var}(M_n)}{(\mathbb{E}(M_n) - \ell_n)^2} \leq \frac{\mathbb{E}(M_n)}{(\mathbb{E}(M_n) - \ell_n)^2},
\end{aligned}
$$

where we used Lemma 3.3 in the last inequality. This last term is equivalent to $1/\mathbb{E}(M_n)$, thus going to 0, which completes the proof.

3.2. *The GUE*: *Comparison of Fredholm determinants.*   For the proof of Theorem 1.7, precise asymptotics like (3.1) and (3.2) related to unitary groups are not available in the GUE context. This difficulty can be overcome, our main observation being Lemma 3.5: the probability that an interval is free of eigenvalues is equivalent in the GUE(n) and U(n) cases, up to a normalization, if the interval size is shorter than the expected extreme gap size. The proof relies on a comparison of the Fredholm determinants associated to $K^{\mathrm{GUE}(n)}$ and $K^{\mathrm{U}(n)}$.

The rest of the proof is similar to the one concerning $U(n)$. Indeed, consider an $\varepsilon > 0$, $p > 0$, $\tilde{T}_{\ell_n}^{(n)}$ the ℓ_nth largest gap in I, and

$$\tilde{X}_n = \frac{\tilde{T}_{\ell_n}^{(n)}}{t_n}, \qquad t_n = \frac{\sqrt{32 \log n}}{n \inf_I \sqrt{4 - x^2}}.$$

We then decompose

$$\mathbb{E}(|\tilde{X}_n - 1|^p) \leq \varepsilon^p + \mathbb{P}(\tilde{X}_n < 1 - \varepsilon) + \mathbb{E}(|\tilde{X}_n - 1|^p \mathbb{1}_{\tilde{X}_n > 1 + \varepsilon}).$$

This last expectation is, by integration by parts,

$$\int_\varepsilon^\infty p v^{p-1} \mathbb{P}(\tilde{X}_n > 1 + v)\, dv + p \varepsilon^{p-1} \mathbb{P}(\tilde{X}_n > 1 + \varepsilon).$$

The probability $\mathbb{P}(\tilde{X}_n > 1 + v)$ is lower than the expectation of the number of gaps greater than $(1 + v)t_n$. Lemma 3.6 therefore yields $\mathbb{E}(|\tilde{X}_n - 1|^p \mathbb{1}_{\tilde{X}_n > 1 + \varepsilon}) \to 0$ as $n \to \infty$.

Concerning $\mathbb{P}(\tilde{X}_n < 1 - \varepsilon)$, we proceed as for the unitary group: for $I = (a, b)$ with $a < b$, consider $\tilde{m}_n = \lfloor (b - a)/((1 - \varepsilon)t_n) \rfloor$ disjoint intervals of length $(1 - \varepsilon)t_n$ included in I. Let $\tilde{M}_n$ be the number of these intervals free of eigenvalues. Then

$$\mathbb{P}(\tilde{X}_n < 1 - \varepsilon) \leq \mathbb{P}(\tilde{M}_n < \ell_n) + \mathbb{P}\big(\tilde{T}_1^{(n)} > 2(1 - \varepsilon)t_n\big).$$

From Lemma 3.6 this last probability goes to 0 as $n \to \infty$ if $2(1 - \varepsilon) > 1$, $\varepsilon < 1/2$. Moreover, from Lemma 3.7,

$$(3.5) \qquad\qquad \mathbb{E}(\tilde{M}_n) \gg n^\delta$$

for some $\delta > 0$ depending only on ε and I. Moreover, from the negative correlation property Lemma 3.8 and the same reasoning as Lemma 3.3,

$$(3.6) \qquad\qquad \text{Var}(\tilde{M}_n) \leq \mathbb{E}(\tilde{M}_n).$$

As $\ell_n = n^{o(1)}$, from (3.5) $\mathbb{E}(\tilde{M}_n) - \ell_n > 0$ for sufficiently large n, which allows us to use Chebyshev's inequality,

$$\mathbb{P}(\tilde{M}_n < \ell_n) \leq \mathbb{P}\big(|\tilde{M}_n - \mathbb{E}(\tilde{M}_n)| > \mathbb{E}(\tilde{M}_n) - \ell_n\big)$$

$$\leq \frac{\text{Var}(\tilde{M}_n)}{(\mathbb{E}(\tilde{M}_n) - \ell_n)^2} \leq \frac{\mathbb{E}(\tilde{M}_n)}{(\mathbb{E}(\tilde{M}_n) - \ell_n)^2},$$

the last inequality being (3.6). From (3.5) this last term is equivalent to $1/\mathbb{E}(\tilde{M}_n)$ and going to 0, as $n \to \infty$, completing the proof.

LEMMA 3.4. *Let $\delta_n = o(1)$. The following asymptotics hold for the unitary group and* GUE *kernels*:

(1) *Uniformly for x, y in $(0, 2\pi)$ and $|x - y| = \mathrm{O}(\delta_n)$,*

$$\frac{2\pi}{n} K^{\mathrm{U}(n)}(x, y) - \frac{\sin(n(x - y)/2)}{n(x - y)/2} = \mathrm{O}\left(\frac{\delta_n}{n}\right).$$

(2) *Let $\varepsilon_0 > 0$. Uniformly for x, y in $(-2 + \varepsilon_0, 2 - \varepsilon_0)$ and $|x - y| = \mathrm{O}(\delta_n)$,*

$$\frac{1}{n\rho_{\mathrm{sc}}(x)} K^{\mathrm{GUE}(n)}(x, y) - \frac{\sin(n\pi\rho_{\mathrm{sc}}(x)(x - y))}{n\pi\rho_{\mathrm{sc}}(x)(x - y)} = \mathrm{O}\left(\frac{1}{n}\right) + \mathrm{O}(\delta_n) + \mathrm{O}(n\delta_n^2).$$

PROOF.   For the unitary group, the kernel is explicit so

$$\frac{2\pi}{n} K^{\mathrm{U}(n)}(x, y) - \frac{\sin(n(x - y)/2)}{n(x - y)/2}$$

$$= \sin\left(n\frac{x - y}{2}\right) \frac{1}{n(x - y)/2} \left(\frac{((x - y)/2)}{\sin((x - y)/2)} - 1\right).$$

As $|x - y| = \mathrm{O}(\delta_n) \to 0$, by expansion of sin at third order $(\frac{x-y}{2})/\sin(\frac{x-y}{2}) - 1 = \mathrm{O}((x - y)^2)$, which completes the proof.

Concerning the Gaussian unitary ensemble, the same type of asymptotics hold, being a direct consequence of formula (2.10). Note that when taking $\delta_n = \mathrm{O}(1/n)$, the speed of convergence to the sine kernel is $1/n^2$ for the unitary group, much better that $1/n$ for the GUE, whose correlation kernel is not translation invariant. □

LEMMA 3.5.   *Let $\delta_n = \mathrm{O}(\sqrt{\log n}/n)$, $\varepsilon_0 > 0$. Then uniformly for x in $(-2 + \varepsilon_0, 2 - \varepsilon_0)$,*

$$\left| \mathbb{P}^{\mathrm{GUE}(n)}\left(\lambda_i \notin \left[x, x + \frac{\delta_n}{\rho_{\mathrm{sc}}(x)}\right], 1 \leq i \leq n\right) \right.$$

$$\left. - \mathbb{P}^{\mathrm{U}(n)}(\theta_i \notin [0, 2\pi\delta_n], 1 \leq i \leq n) \right| \leq n^{\mathrm{o}(1)-1}.$$

PROOF.   By inclusion–exclusion, the probability that a determinantal point process with kernel K has no points in a measurable subset A is the Fredholm determinant (see, e.g., Lemma 3.2.4 in [1])

$$(3.7) \qquad \det(\mathrm{Id} - K_A) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{A^k} \det_{k \times k} (K(x_i, x_j)) \, \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

To compare empty sets probabilities, we therefore need to compare Fredholm determinants. A classical inequality (see, e.g., [16], Chapter IV, (5.14)) is

$$(3.8) \qquad |\det(\mathrm{Id} + A) - \det(\mathrm{Id} + B)| \leq |A - B|_1 e^{1 + |A|_1 + |B|_1},$$

where $|T|_1$ is the trace norm of a nuclear operator T. However, for positive (like A or B) operators $T(f)(x) = \int K(x, y) f(y) \, \mathrm{d}y$ the trace norm is $\int |K(x, x)| \, \mathrm{d}x$,

but for nonpositive operators, like $A - B$, the trace norm is difficult to evaluate, even for a compactly supported continuous kernel K. However, in such a case, the Hilbert–Schmidt norm is computable.

$$(3.9) \qquad |T|_2^2 = \iint |K(x, y)|^2 \, dx \, dy.$$

This is the reason why, instead of the Fredhom determinant and the inequality (3.8), we will use the modified Carleman–Fredholm determinant

$$\det_2(\mathrm{Id} + T) = \det(\mathrm{Id} + T) e^{-\operatorname{Tr} T}$$

and the inequality

$$(3.10) \qquad |\det_2(\mathrm{Id} + A) - \det_2(\mathrm{Id} + B)| \le |A - B|_2 e^{(|A|_2 + |B|_2 + 1)^2/2},$$

which can be found in [16], Chapter IV, (7.11). For our purpose, note that from (3.7), after a simple change of variables, the probability that there are no eigenvalues in $[x, x + \delta_n/(\rho_{sc}(x))]$ is equal to

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{(n\delta_n)^k} \det_{k \times k} \left(\frac{1}{n\rho_{sc}} K^{\mathrm{GUE}(n)} \left(x + \frac{y_i}{n\rho_{sc}(x)}, x + \frac{y_j}{n\rho_{sc}(x)} \right) \right) dy_1 \cdots dy_k.$$

In the same way, the probability that there are no eigenvalues in $[0, 2\pi\delta_n]$ is

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{(n\delta_n)^k} \det_{k \times k} \left(\frac{2\pi}{n} K^{\mathrm{U}(n)} \left(\frac{2\pi}{n} y_i, \frac{2\pi}{n} y_j \right) \right) dy_1 \cdots dy_k.$$

Hence inequality (3.10) will be applied with A and B integral operators with respective kernel

$$A(u, v) = -\frac{1}{n\rho_{sc}} K^{\mathrm{GUE}(n)}_{(0, n\delta_n)} \left(x + \frac{u}{n\rho_{sc}(x)}, x + \frac{v}{n\rho_{sc}(x)} \right)$$

and

$$B(u, v) = -\frac{2\pi}{n} K^{\mathrm{U}(n)}_{(0, n\delta_n)} \left(\frac{2\pi}{n} u, \frac{2\pi}{n} v \right).$$

From Lemma 3.4, the infinite norm between the two kernels above is $O(n\delta_n^2) = O(\log n / n)$, so by (3.9), integrating on a domain of area $(n\delta_n)^2 = O(\log n)$,

$$|A - B|_2 = \left(O((\log n)^2/n^2) O(\log n) \right)^{1/2} = O((\log n)^{3/2}/n).$$

Moreover, consider a parameter $\alpha_n > 0$, and decompose

$$|A|_2^2 = \iint |A(x, y)|^2 \mathbb{1}_{|x-y| < \alpha_n} \, dx \, dy + \iint |A(x, y)|^2 \mathbb{1}_{|x-y| > \alpha_n} \, dx \, dy.$$

From Lemma 3.4, if $|x - y| > \alpha_n$, then $|A(x, y)|$ is smaller than $1/(\pi\alpha_n) + O(\log n / n)$, and when $|x - y| < \alpha_n$, it is bounded by $1 + O(\log n / n)$. Consequently,

$$|A|_2^2 = O(\alpha_n \sqrt{\log n}) + O(\log n / \alpha_n^2) = O((\log n)^{2/3})$$

by choosing $\alpha_n = (\log n)^{1/6}$. In the same way, $|B|_2^2 = O((\log n)^{2/3})$. Hence

$$(3.11) \quad |\det_2(\mathrm{Id} + A) - \det_2(\mathrm{Id} + B)| = O\left(\frac{(\log n)^{3/2}}{n}\right)e^{O((\log n)^{2/3})} \leq n^{o(1)-1}.$$

Finally, using once again Lemma 3.4, here on the diagonal,

$$(3.12) \qquad \mathrm{Tr}\, A = -\int A(x, x)\, dx = -n\delta_n + O((\log n)^{3/2}/n),$$

and the same for $\mathrm{Tr}\, B$. Finally, write

$$|\det(\mathrm{Id} + A) - \det(\mathrm{Id} + B)|$$
$$\leq e^{\mathrm{Tr}\, A}|\det_2(\mathrm{Id} + A) - \det_2(\mathrm{Id} + B)| + |e^{\mathrm{Tr}\, A - \mathrm{Tr}\, B} - 1||\det(\mathrm{Id} + B)|.$$

Formulas (3.11) and (3.12) show that the first term is bounded by $n^{o(1)-1}$. Moreover, $\det(\mathrm{Id} + B)$ is a probability, so bounded by 1, and the estimates of $\mathrm{Tr}\, A$ and $\mathrm{Tr}\, B$, formula (3.12), show that $e^{\mathrm{Tr}\, A - \mathrm{Tr}\, B} - 1$ is $O(\log n/n)$, completing the proof. $\square$

LEMMA 3.6. *Let* $s(I) = \inf_I \sqrt{4 - x^2}$, $\varepsilon > 0$ *and* $\alpha_n = \sqrt{32 \log n}/n$. *There are some constants* $c_1, c_2 > 0$ *depending only on* ε *and* I, *such that for any* $v > \varepsilon$ *and* $n \geq 1$,

$$\mathbb{E}|\{i : \lambda_i \in I, \lambda_{i+1} - \lambda_i > (1 + v)\alpha_n/s(I)\}| \leq c_1 n^{-c_2 v}.$$

PROOF. The first step allows reasoning on fixed intervals instead of gaps. More precisely, note $I = [a, b]$ with $a < b$, and consider the intervals of length $(1 + \frac{v}{2})\alpha_n/s(I)$ by successive slips of size $s_n = \frac{v}{2}\alpha_n/s(I)$.

$$J_k = \left[a + ks_n, a + ks_n + \left(1 + \frac{v}{2}\right)\frac{\alpha_n}{s(I)}\right], \qquad 0 \leq k \leq p_n = \lfloor (b - a)/s_n \rfloor.$$

There an injective map associating to any eigenvalues gap of size at least $(1 + v)\alpha_n/s(I)$ an interval J_k included in this gap, for example, the one with lower index. Consequently,

$$\mathbb{E}|\{i : \lambda_i \in I, \lambda_{i+1} - \lambda_i > (1 + v)\alpha_n/s(I)\}| \leq \sum_{k=0}^{p_n} \mathbb{P}(J_k = 0),$$

where we use the abbreviation $\mathbb{P}(J_k = 0) = \mathbb{P}(\lambda_i \notin J_k, 1 \leq i \leq n)$. The second step consists in obtaining uniform upper bounds for these empty intervals probabilities. For this purpose, the negative correlation property, Lemma 3.8, is used by partitioning the interval J_k:

$$J_k = J_k^{(1)} \sqcup J_k^{(2)} \bigsqcup_{j=1}^{q_v} L_k^{(j)} \sqcup M_k,$$

where:

- $J_k^{(1)}$ has length $(1 - \varepsilon')\alpha_n/s(I)$, with ε' to be chosen positive but sufficiently small compared to ε;
- $J_k^{(2)}$ has length $(\frac{\varepsilon}{2} + \varepsilon')\alpha_n/s(I)$;
- the intervals $L_k^{(j)}$ all have length $\alpha_n/(2s(I))$, and their number is $q_v = \lfloor v - \varepsilon \rfloor$;
- M_k is a residual interval.

The negative correlation property yields

$$\mathbb{P}^{\mathrm{GUE}(n)}(J_k = 0) \leq \mathbb{P}^{\mathrm{GUE}(n)}(J_k^{(1)} = 0)\mathbb{P}^{\mathrm{GUE}(n)}(J_k^{(2)} = 0) \prod_{j=1}^{q_v} \mathbb{P}^{\mathrm{GUE}(n)}(L_k^{(j)} = 0).$$

This now can be upper-bounded using Lemma 3.5 because all intervals are shorter than $\alpha_n/s(I)$: for example, noting x one extremity of $J_k^{(1)}$,

$$\mathbb{P}^{\mathrm{GUE}(n)}(J_k^{(1)} = 0) = \mathbb{P}^{\mathrm{U}(n)}([0, (1 - \varepsilon')\alpha_n\sqrt{4 - x^2}/s(I)] = 0) + n^{\mathrm{o}(1)-1}$$

$$\leq \mathbb{P}^{\mathrm{U}(n)}([0, (1 - \varepsilon')\alpha_n] = 0) + n^{\mathrm{o}(1)-1} = n^{-(1-\varepsilon')^2+\mathrm{o}(1)}$$

with o(1) not depending on the index k, and where the last estimate relies on (3.1). In the same way,

$$\mathbb{P}^{\mathrm{GUE}(n)}(J_k^{(2)} = 0) \leq n^{-(\varepsilon/2+\varepsilon')^2+\mathrm{o}(1)}, \qquad \mathbb{P}^{\mathrm{GUE}(n)}(L_k^{(j)} = 0) \leq n^{-1/4+\mathrm{o}(1)}.$$

Gathering all these results,

$$\mathbb{E}|\{i : \lambda_i \in I, \lambda_{i+1} - \lambda_i > (1 + v)\alpha_n/s(I)\}| \leq n^{1-(1-\varepsilon')^2-(\varepsilon/2+\varepsilon')^2-(1/4+\mathrm{o}(1))q_v}.$$

We can chose ε' sufficiently small such that $1 - (1 - \varepsilon')^2 - (\frac{\varepsilon}{2} + \varepsilon')^2 < 0$ (e.g., $\varepsilon' = \varepsilon^2/8$). For such a choice, for sufficiently large n not depending on v, the above exponent is smaller than $-c - \frac{1}{8}\lfloor v - \varepsilon \rfloor$ for some $c > 0$. Such a function is smaller than $-c_2 v$ on $v > \varepsilon$, for some $c_2 > 0$. $\square$

LEMMA 3.7.   *Let $I = [a, b]$ with $a < b$, $s(I) = \inf_I \sqrt{4 - x^2}$, $\varepsilon \in (0, 1)$ and $\alpha_n = \sqrt{32 \log n}/n$. Consider a maximal number*

$$\tilde{m}_n = \lfloor (b - a)/((1 - \varepsilon)\alpha_n/s(I)) \rfloor$$

of disjoint intervals $I_1, \ldots, I_{m_n}$ of length $(1 - \varepsilon)\alpha_n/s(I)$ included in I. Let

$$\tilde{M}_n = |\{1 \leq j \leq m_n : \lambda_i \notin I_j, 1 \leq i \leq n\}|$$

be the number of those intervals containing no eigenvalues. Then

$$\mathbb{E}(\tilde{M}_n) \gg n^\delta$$

for some $\delta > 0$ depending only on ε and I.

PROOF. We first take a restricted interval to avoid the fluctuations in the spectral measure: there is $I' = (a', b') \subset I$ with $a' < b'$ such that

$$(3.13) \qquad (1 - \varepsilon) \frac{\sup_{I'} \sqrt{4 - x^2}}{s(I)} \leq 1 - \frac{\varepsilon}{2}.$$

Then, for a given interval $I_k = [x, x + (1 - \varepsilon)\alpha_n / s(I)]$ included in I', from Lemma 3.5,

$$\mathbb{P}^{\mathrm{GUE}(n)}(\lambda_i \notin I_k, 1 \leq i \leq n)$$
$$= \mathbb{P}^{\mathrm{U}(n)}(\theta_i \notin [0, (1 - \varepsilon)\sqrt{4 - x^2}/s(I)], 1 \leq i \leq n) + n^{\mathrm{o}(1)-1}$$

with o(1) uniform in x. From (3.13) this is greater than

$$\mathbb{P}^{\mathrm{U}(n)}\left(\theta_i \notin \left[0, \left(1 - \frac{\varepsilon}{2}\right)\right], 1 \leq i \leq n\right) + n^{\mathrm{o}(1)-1} = n^{-(1-\varepsilon/2)^2 + \mathrm{o}(1)}$$

from (3.1). There are $n^{1+\mathrm{o}(1)}$ intervals I_k's included in I', so as $n \to \infty$

$$\mathbb{E}(\tilde{M}_n) \geq n^{1-(1-\varepsilon/2)^2 + \mathrm{o}(1)} \gg n^\delta$$

with $\delta = (1 - (1 - \varepsilon/2)^2)/2$, for example. □

3.3. *The negative association property.* As previously noted, to deduce the asymptotics of the largest gaps, the correlation between distinct gaps is required.

In the context of a point process on a finite set $\mathcal{E}$, let Λ and Λ' be distinct disjoint subsets of $\mathcal{E}$, and write 0^Λ for the event that the elements of Λ are free of particles. Shirai and Takahashi [28] showed that for determinantal point processes, the empty sets events are negatively correlated.

$$(3.14) \qquad \mathbb{P}(0^{\Lambda \cup \Lambda'}) \leq \mathbb{P}(0^\Lambda)\mathbb{P}(0^{\Lambda'}).$$

This negative association property has received considerable attention in the past few years, in the context of ASEP, for example. Still, for discrete determinantal point processes, formula (3.14) was generalized to all increasing events [24], and general criteria for the negative association property were given in [4].

The following continuous analog of (3.14) holds. It can be shown by a simple discretization, relying on results from [4, 24, 28]. We give another justification, which relies on a work of Georgii and Yoo [15].

LEMMA 3.8 (Negative correlation of the vacuum events). *Let $\xi^{(n)}$ be the point process associated to the eigenvalues of Haar distributed unitary matrix (resp., an element of the* GUE). *Let I_1 and I_2 be compact disjoint subsets of $[0, 2\pi)$ (resp., $\mathbb{R}$). Then*

$$(3.15) \qquad \mathbb{P}\big(\xi^{(n)}(I_1 \cup I_2) = 0\big) \leq \mathbb{P}\big(\xi^{(n)}(I_1) = 0\big)\mathbb{P}\big(\xi^{(n)}(I_2) = 0\big).$$

PROOF.   The general negative correlation result, Corollary 3.3 in [15], requires a locally trace class operator K with a restriction on its spectrum. More precisely, consider the unitary case, the GUE proof, being similar. For an operator K acting on $L^2((0, 2\pi), \mu)$, if $\mathrm{spec}(K) \subset [0, 1]$, there exist a unique determinantal point process ξ with kernel K; see [30]; under the additional hypothesis $\mathrm{spec}(K) \subset [0, 1)$, Georgii and Yoo proved that for any compact and disjoint Borel sets $\Lambda \subset \Delta \subset [0, 2\pi)$,

$$\mathbb{P}_\mu\big(\xi(\Lambda) = 0 | \xi(\Delta/\Lambda) = 0\big) \leq \mathbb{P}_\mu\big(\xi(\Lambda) = 0\big).$$

In the case of Haar distributed unitary matrices, $K = K^{\mathrm{U}(n)}$ is a nuclear operator with kernel $K^{\mathrm{U}(n)}(x, y) = \frac{1}{2\pi} \frac{\sin(n(x-y)/2)}{\sin((x-y)/2)}$, and defines a projection: 1 is in its spectrum, and the general statement does not directly apply. To care for this minor problem, look at the restriction $K_\Lambda^{\mathrm{U}(n)}$ of $K^{\mathrm{U}(n)}$ to a compact subset Λ of $(0, 2\pi)$ ($K_\Lambda^{\mathrm{U}(n)} = P_\Lambda K^{\mathrm{U}(n)} P_\Lambda$, P_Λ being the projection on Λ). Suppose that the set $(0, 2\pi)/\Lambda$ has a nonempty interior. As a projection of $K^{\mathrm{U}(n)}$, $K_\Lambda^{\mathrm{U}(n)}$ is still nonnegative and trace class. As for any determinantal point process,

$$(3.16) \qquad \mathbb{E}\big(z^{\xi^{(n)}(\Lambda)}\big) = \det\big(\mathrm{Id} + (z - 1)K_\Lambda^{\mathrm{U}(n)}\big)$$

in the sense of Fredholm determinants of a trace class operator. Suppose that $K_\Lambda^{\mathrm{U}(n)}$ has an eigenvalue $\lambda \geq 1$. Then by choosing $z = 1 - 1/\lambda \geq 0$, (3.16) yields

$$\mathbb{E}\big(z^{\xi^{(n)}(I)}\big) = 0 \qquad \text{if } \lambda > 1, \qquad \mathbb{P}\big(\xi^{(n)}(I) = 0\big) = 0 \qquad \text{if } \lambda = 1.$$

In each case, this is absurd because the joint law of the eigenvalues is absolutely continuous with respect to the Lebesgue measure on $(0, 2\pi)^n$, and $(0, 2\pi)/\Lambda$ has a nonempty interior: both quantities need to be strictly positive. Hence $K_{I_1 \cup I_2}^{\mathrm{U}(n)}$ is a trace class operator with spectrum in $[0, 1)$, and the result from [15] applies. $\square$

REMARK.   Another way to prove Lemma 3.8 is as follows: the inequality can be stated for the determinantal point process with kernel $\alpha K^{\mathrm{U}(n)}$ with $0 < \alpha < 1$, and then the inequality remains true for $K^{\mathrm{U}(n)}$ by continuity of the application $K \mapsto \det(\mathrm{Id} - K)$ in the set of trace class operators, by (3.8).

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COURANT INSTITUTE
  OF MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY
251 MERCER STREET
NEW YORK, NEW YORK 10012
USA
E-MAIL: benarous@cims.nyu.edu

DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
ONE OXFORD STREET
CAMBRIDGE, MASSACHUSETTS 02138
USA
E-MAIL: bourgade@math.harvard.edu