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CHAOS OF A MARKOV OPERATOR AND THE FOURTH MOMENT CONDITION

BY M. LEDOUX

Université de Toulouse and Institut Universitaire de France

We analyze from the viewpoint of an abstract Markov operator recent results by Nualart and Peccati, and Nourdin and Peccati, on the fourth moment as a condition on a Wiener chaos to have a distribution close to Gaussian. In particular, we are led to introduce a notion of chaos associated to a Markov operator through its iterated gradients and present conditions on the (pure) point spectrum for a sequence of chaos eigenfunctions to converge to a Gaussian distribution. Convergence to gamma distributions may be examined similarly.

1. Introduction. In a striking contribution [20], Nualart and Peccati discovered a few years ago that the fourth moment of homogeneous polynomial chaos on Wiener space characterizes convergence toward the Gaussian distribution. Specifically, and in a simplified (finite dimensional) setting, let $F:\mathbb{R}^N \to \mathbb{R}$, $1 \le k \le N$, be defined by

$$F = F(x) = \sum_{i_1,\dots,i_k=1}^{N} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}, \qquad x = (x_1,\dots,x_N) \in \mathbb{R}^N, \tag{1}$$

where $a_{i_1,\dots,i_k}$ are real numbers vanishing on diagonals and symmetric in the indices. Assume by homogeneity that $\int_{\mathbb{R}^N} F^2 \, d\gamma_N = 1$ where

$$d\gamma_N(x) = (2\pi)^{-N/2} \mathrm{e}^{-|x|^2/2} \, dx$$

is the standard Gaussian measure on $\mathbb{R}^N$. Such a function F will be called homogeneous of degree k. Let now F_n on $\mathbb{R}^{N_n}$, $n \in \mathbb{N}$, $N_n \to \infty$, be a sequence of such homogeneous polynomials of fixed degree k. The main theorem of Nualart and Peccati [20] expresses that the sequence of distributions of the F_n's converges toward the standard Gaussian distribution γ_1 on the real line if and only if

$$\int_{\mathbb{R}^{N_n}} F_n^4 \, d\gamma_{N_n} \to 3 \tag{2}$$

(3 being the fourth moment of the standard normal). The result actually holds for homogeneous chaos on the infinite dimensional Wiener space, and the equivalence

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is further described in terms of convergence of contractions. The proof of [20] relies on multiplication formulas for homogeneous chaos and the use of stochastic calculus.

Since [20] was published, numerous improvements and developments on this theme have been considered; cf., for example, [13–15, 17, 19, 23], ... An introduction to some of these developments (with emphasis on multiplication formulas) is the recent monograph [22] by Peccati and Taqqu. In particular, the work by Nualart and Ortiz-Latorre [19] introduces a technological breakthrough with a new proof only based on Malliavin calculus and the use of integration by parts on Wiener space. In this work, the convergence of $(F_n)_{n\in\mathbb{N}}$ to a Gaussian distribution [and thus also (2)] is also shown to be equivalent to the fact that

$$\mathrm{Var}_{\gamma_{N_n}}(|\nabla F_n|^2) \to 0, \tag{3}$$

where $\mathrm{Var}_{\gamma_{N_n}}$ is the variance with respect to γ_{N_n}. Based upon this observation, recent work by Nourdin and Peccati [13, 14] develops the tool of the so-called Stein method (cf., e.g., [5, 6, 26, 27]) in order to quantify the convergence toward the Gaussian distribution. Relying also on multiplication formulas and the use of integration by parts on Wiener space, one key step in the investigation [13] is expressed by the following inequality: for a given homogeneous function F of degree k on $\mathbb{R}^N$ normalized in $\mathrm{L}^2(\gamma_N)$,

$$\mathrm{Var}_{\gamma_N}(|\nabla F|^2) \le C_k\left(\int_{\mathbb{R}^N} F^4\, d\gamma_N - 3\right), \tag{4}$$

where $C_k > 0$ only depends on k. In particular, the proximity of $\int_{\mathbb{R}^N} F^4\, d\gamma_N$ to 3 controls the variance of $|\nabla F|^2$. Now, Stein's method for homogeneous chaos on Wiener space as developed in [13] expresses that

$$d(\nu, \gamma_1) \le C\, \mathrm{Var}_{\gamma_N}(|\nabla F|^2)^{1/2}, \tag{5}$$

where $d(\nu, \gamma_1)$ stands for some appropriate distance between the law ν of F and γ_1, so that $|\nabla F|^2$ being close to a constant forces the distribution of F to be close to a Gaussian distribution. The conjunction of (4) and (5) thus describes how the fourth moment condition controls convergence to a Gaussian.

The primary motivation of this work is to understand what structure of a functional F allows for the preceding results, in particular thus the control by the fourth moment of the distance to the Gaussian distribution. In the process of this investigation, we will revisit the preceding results and conclusions in the setting of a symmetric Markov operator, including, as a particular example, the Ornstein–Uhlenbeck operator $\mathrm{L} = \Delta - x \cdot \nabla$, corresponding to the Wiener space setting. In order to achieve this goal, observe that the homogeneous polynomial F of (1) is an eigenfunction with eigenvalue k of the Ornstein–Uhlenbeck operator, that is, $-\mathrm{L}F = kF$. We shall therefore try to understand what is necessary for an eigenfunction F of a Markov operator in order to satisfy an inequality such as (4). This investigation leads us to define a notion of chaos eigenfunction with respect to such a Markov operator with pure point spectrum consisting of a countable sequence of eigenvalues, the homogeneous polynomial F of (1) being one example with respect to the Ornstein–Uhlenbeck operator. The main achievement of this work is then the formulation of an explicit condition on the sequence of eigenvalues under which a chaos eigenfunction satisfies an inequality such as (4).

The basic data will thus be a Markov operator L on some state space $(E, \mathcal{F})$ with invariant and reversible probability measure μ and symmetric bilinear carré du champ operator

$$\Gamma(f, g) = \tfrac{1}{2}[\mathrm{L}(fg) - f\mathrm{L}g - g\mathrm{L}f],$$

acting on functions f, g in a suitable domain $\mathcal{A}$. For simplicity, we often write $\Gamma(f) = \Gamma(f, f)$ which is always nonnegative. By invariance and symmetry of μ with respect to L, the definition of the carré du champ operator Γ yields the integration by parts formula

$$\int_E f(-\mathrm{L}g)\, d\mu = \int_E g(-\mathrm{L}f)\, d\mu = \int_E \Gamma(f, g)\, d\mu.$$

In particular $\int_E \mathrm{L}f\, d\mu = 0$ since $\mathrm{L}1 = 0$ by the Markov property. The operator L is said, in addition, to be a diffusion operator if, for every smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, and every $f \in \mathcal{A}$,

$$\mathrm{L}\varphi(f) = \varphi'(f)\mathrm{L}f + \varphi''(f)\Gamma(f).$$

Alternatively, Γ is a derivation in the sense that $\Gamma(\varphi(f), g) = \varphi'(f)\Gamma(f, g)$.

We refer to the lecture notes [1], Chapter 2, by Bakry for an introduction to this abstract framework of Markov and carré du champ operators and a discussion of some of the examples emphasized below. Additional general references include [7] for further probabilistic interpretations and [4, 8] for constructions in terms of Dirichlet forms; see also [12] and the forthcoming [3]. One prototype example of a Markov diffusion operator is the Ornstein–Uhlenbeck operator acting on say the algebra $\mathcal{A}$ of polynomial functions f on $E = \mathbb{R}^N$ as $\mathrm{L}f(x) = \Delta f(x) - x \cdot \nabla f(x)$, with invariant and reversible probability measure the Gaussian distribution $\mu = \gamma_N$ and carré du champ $\Gamma(f) = |\nabla f|^2$. One could consider its infinite dimensional extension on Wiener space (cf. [4] and [18], Chapter 1), but for simplicity in the exposition we stick here on the finite dimensional case as a reference example. The preceding general setting also includes discrete examples, such as the two-point space and its products. Namely, on $E = \{-1, +1\}^N$, let $\mathrm{L}f = \frac{1}{2}\sum_{i=1}^N D_i f$ where $D_i f(x) = f(\tau_i(x)) - f(x)$, $x = (x_1, \dots, x_i, \dots, x_N)$, $\tau_i(x) = (x_1, \dots, -x_i, \dots, x_N)$. L is invariant and symmetric with respect to the uniform measure μ on $\{-1, +1\}^N$ with carré du champ $\Gamma(f) = \frac{1}{4}\sum_{i=1}^N (D_i f)^2$, but is not a diffusion operator.

These two examples actually entail a crucial chaos structure in the sense that the generators L may be diagonalized in a sequence of orthogonal polynomials (Hermite polynomials in the Gaussian case, Walsh polynomials in the cube example);

see, for example, [1], Chapter 1, [18], Chapter 1, [10], Chapter 2, [22], Chapter 5. More precisely, setting for $\underline{k} = (k_1, \dots, k_N) \in \mathbb{N}^N$, $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, $H_{\underline{k}}(x) = h_{k_1}(x_1) \cdots h_{k_N}(x_N)$, with $(h_k)_{k \in \mathbb{N}}$ the sequence of orthonormal Hermite polynomials on the real line, any function $f : \mathbb{R}^N \to \mathbb{R}$ in $\mathrm{L}^2(\gamma_N)$ may be written as

$$f = \sum_{k \in \mathbb{N}} \sum_{|\underline{k}|=k} \langle f, H_{\underline{k}} \rangle H_{\underline{k}},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathrm{L}^2(\gamma_N)$ and where the second sum runs over all $\underline{k} \in \mathbb{N}^N$ with $|\underline{k}| = k_1 + \cdots + k_N = k$. An element $H = H_{\underline{k}}$ with $|\underline{k}| = k$ is an eigenfunction of the Ornstein–Uhlenbeck operator with $-\mathrm{L}H = kH$ and the spectrum of the operator $-\mathrm{L}$ thus consists of the sequence of the nonnegative integers. For fixed $k \in \mathbb{N}$, linear combinations

$$F = \sum_{|\underline{k}|=k} a_{\underline{k}} H_{\underline{k}} \tag{6}$$

define generic eigenfunctions (chaos) of $-\mathrm{L}$ with eigenvalue k, the homogeneous function F of (1) being one example.

Similarly, if $f : \{-1, +1\}^N \to \mathbb{R}$,

$$f = \sum_{k=0}^{N} \sum_{|A|=k} \langle f, W_A \rangle W_A,$$

where the second sum runs over all subsets A of $\{1, \dots, N\}$ with k elements and

$$W_A(x) = \prod_{i \in A} x_i, \qquad x = (x_1, \dots, x_N) \in \{-1, +1\}^N, A \subset \{1, \dots, N\},$$

are the so-called Walsh polynomials. For the discrete operator $\mathrm{L}f = \frac{1}{2} \sum_{i=1}^N D_i f$, $-\mathrm{L}W_A = kW_A$ if $|A| = k$. The spectrum of $-\mathrm{L}$ is thus equal to $\mathbb{N}$, and linear combinations

$$F = \sum_{|A|=k} a_A W_A \tag{7}$$

describe the family of eigenfunctions (chaos) of $-\mathrm{L}$ with eigenvalue k.

A further example is Poisson space. In dimension one, let μ be the Poisson law on $\mathbb{N}$ with parameter $\theta > 0$. For a function $f : \mathbb{N} \to \mathbb{R}$ with finite support say, let $Df(j) = f(j) - f(j-1)$ for every $j \in \mathbb{N}$ $[f(-1) = 0]$. The Poisson operator may then be defined as $\mathrm{L}f(j) = \theta Df(j+1) - jDf(j)$, $j \in \mathbb{N}$. It is not a diffusion. The associated carré du champ operator is given by $2\Gamma(f)(j) = \theta Df(j+1)^2 + jDf(j)^2$, $j \in \mathbb{N}$. The operator $-\mathrm{L}$ has a spectrum given by the sequence of the integers and is diagonalized along the Charlier orthogonal polynomials. Multi-dimensional Poisson models are similar.

Laplacians $\mathrm{L} = \Delta$ on (compact) Riemannian manifolds, and acting on families of smooth functions, also enter this framework. These Laplacians are diffusion operators and, in the compact case, have again a spectrum consisting of a countable sequence of eigenvalues; cf., for example, [9].

This work will analyze properties of eigenfunctions of such Markov operators L, that is, functions $F : E \to \mathbb{R}$ (in the domain of L) such that $-\mathrm{L}F = \lambda F$ for some $\lambda > 0$. (We emphasize that F and λ are thus rather eigenfunction and eigenvalue of $-\mathrm{L}$ which is nonnegative.) The ultimate goal of this work is to find conditions on such an eigenfunction F of a diffusion operator L in order that the analog of (4) holds, and that the fourth moment condition then ensures the proximity with the Gaussian distribution. We outline here the various steps of the investigation. The first step will be to show (following [13] in the Ornstein–Uhlenbeck setting) that Stein's method applied to an eigenfunction F indicates that it has a Gaussian distribution if (and only if) its carré du champ $\Gamma(F)$ is constant; see Proposition 1 below. More precisely, in accordance with (5), for suitable families of functions $\varphi : \mathbb{R} \to \mathbb{R}$, and whenever $\int_E F^2 \, d\mu = 1$,

$$\left| \int_{\mathbb{R}} \varphi(F) \, d\mu - \int_{\mathbb{R}} \varphi \, d\gamma_1 \right| \le C_\varphi \operatorname{Var}_\mu(\Gamma(F))^{1/2}, \tag{8}$$

where Var_μ is the variance with respect to μ.

On the basis of this result, the fourth moment condition appears quite naturally by the integration by parts formula since (assuming the necessary domain and integrability conditions)

$$\lambda \int_E F^4 \, d\mu = \int_E F^3(-\mathrm{L}F) \, d\mu = 3 \int_E F^2 \Gamma(F) \, d\mu.$$

Moreover, $\int_E \Gamma(F) \, d\mu = \int_E F(-\mathrm{L}F) \, d\mu = \lambda \int_E F^2 \, d\mu$, so that, still assuming by homogeneity that $\int_E F^2 \, d\mu = 1$,

$$\lambda \left(\frac{1}{3} \int_E F^4 \, d\mu - 1 \right) = \int_E F^2 \big(\Gamma(F) - \lambda \big) \, d\mu. \tag{9}$$

This identity is the first indication that the proximity of $\int_E F^4 \, d\mu$ with 3 actually amounts to the proximity of $\Gamma(F)$ with its constant mean value λ.

The next step in the investigation, the main result of this note, describes a chaos structure of an eigenfunction F of a Markov operator L (not necessarily diffusive) with spectrum consisting in a sequence $S = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ of eigenvalues in order that whenever F is such a chaos with eigenvalue λ_k normalized in $\mathrm{L}^2(\mu)$,

$$\operatorname{Var}_\mu(\Gamma(F)) \le C_k \int_E F^2 \big(\Gamma(F) - \lambda_k \big) \, d\mu \tag{10}$$

for some finite constant C_k only depending on S. The relations (8), (9) and (10) together therefore describe how the fourth moment condition $\int_E F^4 \, d\mu \sim 3$ ensures that $\Gamma(F)$ is close to constant and thus that the distribution of F is close to

Gaussian. This family of inequalities may then be used to describe convergence to a Gaussian distribution of a sequence of such chaos eigenfunctions. The abstract chaos structure underlying these results is defined by means of the iterated gradients of the Markov operator L and is shown to easily cover the examples of Wiener, Walsh or Poisson chaos. For example, the chaos structure of the homogeneous polynomial F of (6) actually amounts to the fact that $\nabla^{k+1} F = 0$. The proof of (10) will proceed by a standard and direct algebraic Γ-calculus on eigenfunctions involving the iterated gradients of the operator L and avoiding any type of multiplication formulas for chaos.

Turning to the content of this note, Section 2 briefly presents Stein's method applied to an eigenfunction of a Markov diffusion operator. The next section discusses the iterated gradients and the associated Γ-calculus on eigenfunctions, of fundamental use in the investigation. Section 4 introduces the notion of chaos of a Markov operator with pure point spectrum and presents the aforementioned main result (10), proved in Section 6. The last section briefly describes analogous conclusions for convergence to gamma distributions covering recent results of [14].

It should be carefully emphasized that the present exposition develops more the algebraic and spectral descriptions of the problem under investigation [and concentrates on a proof of (10)] rather than the analytic issues on domains and classes of functions involved in the analysis. In particular, we work with families of functions in the domain of the Markov operator and its carré du champ and with eigenfunctions assumed to satisfy all the necessary domain and integrability conditions required to develop integration by parts and the associated Γ-calculus. These properties are classically and easily satisfied for the main examples in mind, the Gaussian case, the discrete cube or the setting of the Laplace operator on a compact Riemannian manifold. Note, however, that the extension from the finite dimensional Gaussian setting to the infinite dimensional one requires basic analysis on Wiener space as presented, for example, in the first chapter of [18] (see also [22]) in order to fully justify the domain issues and the various conclusions. These aspects, carefully developed in the aforementioned references, are not discussed here. Further conditions ensuring the validity of the results presented here might be developed in broader contexts.

2. Stein's method for eigenfunctions. We start our investigation with a brief exposition of Stein's lemma applied to eigenfunctions of a diffusion operator. We refer to [5, 6, 26, 27] and the references therein for general introductions on Stein's method. The results below are mere adaptations of the investigation [13] by Nourdin and Peccati in Wiener space to which we refer for further details. Throughout this section, L is thus a diffusion operator with invariant and reversible measure μ and carré du champ Γ as described in the Introduction. All the necessary domain and integrability conditions on the eigenfunctions under investigation are implicitly assumed, and are satisfied for the main Ornstein–Uhlenbeck example; cf. [13].

We first illustrate, at a qualitative level, Stein's method in this abstract context. Given a measurable map $F : E \to \mathbb{R}$, say that L commutes to F if there exists a Markov operator $\mathcal{L}$ on the real line such that for every $\varphi : \mathbb{R} \to \mathbb{R}$ (in the domain of $\mathcal{L}$ and such that $\varphi \circ F$ is in the domain of L)

$$\mathrm{L}(\varphi \circ F) = (\mathcal{L}\varphi)(F).$$

In this case, the image measure μ_F of μ by F is the invariant measure of $\mathcal{L}$.

One model factorization operator $\mathcal{L}$ on $\mathbb{R}$ is the Ornstein–Uhlenbeck operator $\mathcal{L}\psi = \psi'' - x\psi'$ with invariant measure the standard Gaussian distribution $d\gamma_1(x) = \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$. Let then F be an eigenfunction of $-\mathrm{L}$ with eigenvalue $\lambda > 0$. The observation here, at the root of Stein's argument, is that whenever $\Gamma = \Gamma(F)$ is (μ-almost everywhere) constant, then L commutes to F through the Ornstein–Uhlenbeck operator $\mathcal{L}$, and thus the distribution μ_F of F is Gaussian. Namely, note first that by integration by parts, $\int_E \Gamma \, d\mu = \int_E F(-\mathrm{L}F) \, d\mu = \lambda \int_E F^2 \, d\mu$ so that if Γ is constant and F is normalized in $\mathrm{L}^2(\mu)$, then $\Gamma = \lambda$. Then, for $\varphi : \mathbb{R} \to \mathbb{R}$ smooth enough, by the chain rule formula for the diffusion operator L,

$$\mathrm{L}(\varphi \circ F) = \varphi'(F)\mathrm{L}F + \varphi''(F)\Gamma = -\lambda F\varphi'(F) + \varphi''(F)\Gamma.$$

Hence, if $\Gamma = \lambda$,

$$\mathrm{L}(\varphi \circ F) = \lambda(\mathcal{L}\varphi)(F)$$

so that L commutes to F, and thus μ_F is the invariant measure of the Ornstein–Uhlenbeck operator $\mathcal{L}$ characterized as the Gaussian distribution γ_1.

For an eigenfunction F, $\Gamma = \Gamma(F)$ constant thus forces the distribution of F to be Gaussian. Now, as such, this observation is not of much use and to describe convergence to normal as for sequences of homogeneous polynomials in the Introduction, it should be suitably quantified in the form of inequality (8) in order to express that the proximity of Γ with a constant value forces the distribution of F to be close to Gaussian. This is the content of the classical Stein lemma as described in the next statement.

PROPOSITION 1. *Let F be an eigenfunction of $-\mathrm{L}$ with eigenvalue $\lambda > 0$ and set $\Gamma = \Gamma(F)$. Denote by μ_F the distribution of F. Given $\varphi : \mathbb{R} \to \mathbb{R}$ integrable with respect to μ_F and γ_1, let ψ be a smooth solution of the associated Stein equation $\varphi - \int_{\mathbb{R}} \varphi \, d\gamma_1 = \psi' - x\psi$. Then,*

$$\left| \int_{\mathbb{R}} \varphi \, d\mu_F - \int_{\mathbb{R}} \varphi \, d\gamma_1 \right| \le \frac{C_\varphi}{\lambda} \left(\int_E (\Gamma - \lambda)^2 \, d\mu \right)^{1/2}, \tag{11}$$

where $C_\varphi = \|\psi'\|_\infty^2$. In particular, if $\int_E F^2 \, d\mu = 1$,

$$\left| \int_{\mathbb{R}} \varphi \, d\mu_F - \int_{\mathbb{R}} \varphi \, d\gamma_1 \right| \le \frac{C_\varphi}{\lambda} \mathrm{Var}_\mu(\Gamma)^{1/2}.$$

PROOF. Since μ_F is the distribution of F under μ, and by the Stein equation,

$$\int_{\mathbb{R}} \varphi\, d\mu_F - \int_{\mathbb{R}} \varphi\, d\gamma_1 = \int_E \varphi(F)\, d\mu - \int_{\mathbb{R}} \varphi\, d\gamma_1 = \int_E [\psi'(F) - F\psi(F)]\, d\mu.$$

Now $-\mathrm{L}F = \lambda F$ so that

$$\psi'(F) - F\psi(F) = \psi'(F) + \lambda^{-1}\mathrm{L}F\psi(F)$$

and hence, after integration by parts with respect to the operator L and the use of the diffusion property,

$$\int_{\mathbb{R}} \varphi\, d\mu_F - \int_{\mathbb{R}} \varphi\, d\gamma_1 = \int_E \psi'(F)[1 - \lambda^{-1}\Gamma]\, d\mu.$$

Together with the Cauchy–Schwarz inequality,

$$\left|\int_{\mathbb{R}} \varphi\, d\mu_F - \int_{\mathbb{R}} \varphi\, d\gamma_1\right| \le \left(\int_E \psi'(F)^2\, d\mu\right)^{1/2} \left(\int_E [1 - \lambda^{-1}\Gamma]^2\, d\mu\right)^{1/2},$$

which amounts to (11). If $\int_E F^2\, d\mu = 1$, then $\int_E \Gamma\, d\mu = \int_E F(-\mathrm{L}F)\, d\mu = \lambda$ and thus $\int_E (\Gamma - \lambda)^2\, d\mu = \mathrm{Var}_\mu(\Gamma)$. The proof of Proposition 1 is complete. □

Proposition 1 is thus investigated in [13] for Wiener chaos. As is discussed there (Lemma 1.2 and Theorem 3.1), the constant C_φ in (11) of Proposition 1 can be uniformly bounded inside specific classes of functions. For instance, $C_\varphi \le 2$ when φ is the characteristic function of a Borel set (corresponding to the total variation distance) and $C_\varphi \le 1$ when φ is the characteristic function of a half-line (corresponding to the Kolmogorov distance).

For the further purposes, observe, as is classical (cf. [26, 27]), that Stein's strategy may be developed similarly for the Laguerre operator on the positive half-line $\mathcal{L}_p\psi = x\psi'' + (p - x)\psi'$, $p > 0$, with invariant measure the gamma distribution $dg_p(x) = \Gamma(p)^{-1}x^{p-1}\mathrm{e}^{-x}\, dx$. Let F be an eigenfunction of $-\mathrm{L}$ with eigenvalue $\lambda > 0$ and $\Gamma = \Gamma(F)$. As above, for every $\varphi : \mathbb{R} \to \mathbb{R}$ smooth enough, setting $G = F + p$,

$$\begin{aligned} \mathrm{L}(\varphi \circ G) &= \varphi'(G)\mathrm{L}F + \varphi''(G)\Gamma \\ &= -\lambda F\varphi'(G) + \varphi''(G)\Gamma \\ &= \lambda\left((p - G)\varphi'(G) + \frac{1}{\lambda}\Gamma\varphi''(G)\right). \end{aligned}$$

In this case, if $\Gamma = \lambda G$,

$$\mathrm{L}(\varphi \circ G) = \lambda(\mathcal{L}_p\varphi)(G)$$

so that μ_G is the invariant measure of $\mathcal{L}_p$ characterized as the gamma distribution g_p.

For this example of the Laguerre operator, the criterion for an eigenfunction F to have a gamma distribution is thus that $\Gamma = \lambda(F + p)$. On the basis of this qualitative description of Stein's method for the Laguerre operator, the next statement illustrates the analog of Proposition 1 for this model.

PROPOSITION 2. *Let F be an eigenfunction of $-\mathrm{L}$ with eigenvalue $\lambda > 0$, and set $\Gamma = \Gamma(F)$. Let $p > 0$ and denote by μ_{F+p} the distribution of $F + p$. Given $\varphi : \mathbb{R} \to \mathbb{R}$ integrable with respect to μ_{F+p} and g_p, let ψ be a smooth solution of the associated Stein equation $\varphi - \int_{\mathbb{R}} \varphi \, dg_p = x\psi' + (p - x)\psi$. Then,*

$$\left| \int_{\mathbb{R}} \varphi \, d\mu_{F+p} - \int_{\mathbb{R}} \varphi \, dg_p \right| \le \frac{C_\varphi}{\lambda} \left(\int_E \left(\Gamma - \lambda(F + p)\right)^2 d\mu \right)^{1/2}, \tag{12}$$

where $C_\varphi = \|\psi'\|_\infty^2$. In particular, if $\int_E F^2 \, d\mu = p$,

$$\left| \int_{\mathbb{R}} \varphi \, d\mu_{F+p} - \int_{\mathbb{R}} \varphi \, dg_p \right| \le \frac{C_\varphi}{\lambda} \mathrm{Var}_\mu(\Gamma - \lambda F)^{1/2}.$$

PROOF. Set again $G = F + p$. Start as in the proof of Proposition 1, namely

$$\begin{aligned} \int_{\mathbb{R}} \varphi \, d\mu_G - \int_{\mathbb{R}} \varphi \, dg_p &= \int_E \varphi(G) \, d\mu - \int_{\mathbb{R}} \varphi \, dg_p \\ &= \int_E [G\psi'(G) + (p - G)\psi(G)] \, d\mu. \end{aligned}$$

Since $-\mathrm{L}F = \lambda F$, and thus $\mathrm{L}G = \lambda(p - G)$,

$$G\psi'(G) + (p - G)\psi(G) = G\psi'(G) + \lambda^{-1}\mathrm{L}G\psi(G).$$

After integration by parts with respect to the operator L and the use of the diffusion property,

$$\int_{\mathbb{R}} \varphi \, d\mu_G - \int_{\mathbb{R}} \varphi \, dg_p = \int_E \psi'(G)[G - \lambda^{-1}\Gamma] \, d\mu.$$

The conclusion follows similarly from the Cauchy–Schwarz inequality. □

Proposition 2 is similarly investigated in [13] in the context of Stein's method on Wiener space. Again the the constant C_φ in (12) may be bounded only in terms of p inside specific classes of functions; cf. [13], Lemma 1.3 and Theorem 3.11. Analogs of Stein's lemma in the context of the preceding statements have been investigated on discrete Poisson or Bernoulli spaces in [16, 21, 24]. In those examples, the control of the variance of Γ is not enough to ensure proximity to a Gaussian distribution and has to be supplemented by various additional conditions.

3. Iterated gradients. This section presents the family of the iterated gradients of a Markov operator and the basic (algebraic) Γ-calculus on eigenfunctions at the root of the investigation. Given a symmetric Markov operator L as above (not necessarily a diffusion operator), recall following [1, 11], the iterated gradients Γ_m, $m \ge 2$, associated to L defined according to the rule defining $\Gamma = \Gamma_1$ as

$$\Gamma_m(f, g) = \tfrac{1}{2}[\mathrm{L}\Gamma_{m-1}(f, g) - \Gamma_{m-1}(f, \mathrm{L}g) - \Gamma_{m-1}(g, \mathrm{L}f)]$$

for functions f, g in a suitable class $\mathcal{A}$. By extension, $\Gamma_0(f, g) = fg$. For simplicity, set $\Gamma_m(f) = \Gamma_m(f, f)$. Note that in general $\Gamma_m(f)$ for $m \geq 2$ is not necessarily nonnegative. The Γ_2 operator has been introduced first by Bakry and Émery [2] to describe curvature properties of Markov operators and to provide a simple criterion to ensure spectral gap and functional inequalities; cf. [1], Chapter 6, [12] and [3]. This criterion will be used in Proposition 4 below. The iterated gradients Γ_m have been exploited in [11] toward variance and entropy expansions.

The following elementary lemma will be of constant use throughout this note and concentrates on the significant properties of the iterated gradients of a given eigenfunction. Recall that we assume the necessary domain and integrability conditions to justify the relevant identities.

LEMMA 3. *Let F be an eigenfunction of $-\mathrm{L}$ with eigenvalue λ. Set $\Gamma_m = \Gamma_m(F)$, $m \geq 1$. Then, for every $m \geq 1$,*

$$\Gamma_m = \tfrac{1}{2}\mathrm{L}\Gamma_{m-1} + \lambda\Gamma_{m-1} = \left(\tfrac{1}{2}\mathrm{L} + \lambda\,\mathrm{Id}\right)^{m-1}\Gamma. \tag{13}$$

Furthermore, for every $m, n \geq 1$,

$$\int_E \Gamma_n \Gamma_m \, d\mu = \int_E \Gamma_{n-1}\Gamma_{m+1}\, d\mu. \tag{14}$$

In particular, by selecting $n = 1$, for every $m \geq 1$,

$$\int_E \Gamma\Gamma_m \, d\mu = \int_E F^2 \Gamma_{m+1}\, d\mu. \tag{15}$$

PROOF. Equality (13) is an immediate consequence of the definition of Γ_m and the eigenfunction property

$$\Gamma_m(F) = \tfrac{1}{2}\mathrm{L}\Gamma_{m-1}(F) - \Gamma_{m-1}(F, \mathrm{L}F) = \tfrac{1}{2}\mathrm{L}\Gamma_{m-1}(F) + \lambda\Gamma_{m-1}(F).$$

The conclusion follows by iteration.

Recalling the notation $\Gamma_m = \Gamma_m(F)$, multiply the preceding identity by Γ_n and integrate with respect to μ to get, by symmetry,

$$2\int_E \Gamma_n\Gamma_m\, d\mu = \int_E \Gamma_{m-1}\mathrm{L}\Gamma_n\, d\mu + 2\lambda\int_E \Gamma_n\Gamma_{m-1}\, d\mu.$$

Changing the role of n and $m-1$, by symmetry again,

$$2\int_E \Gamma_{m-1}\Gamma_{n+1}\, d\mu = \int_E \Gamma_{m-1}\mathrm{L}\Gamma_n\, d\mu + 2\lambda\int_E \Gamma_{m-1}\Gamma_n\, d\mu$$

and the identity (14) follows. The proof of the lemma is complete. □

The following statement is a first illustration of the method developed next. It expresses a kind of rigidity result under the geometric Γ_2 curvature condition mentioned previously.

PROPOSITION 4. *Assume that the operator* L *is of curvature* $\rho > 0$ *in the sense of Bakry–Émery* [2] ([1], *Chapter* 6), *that is*, $\Gamma_2(f) \geq \rho\Gamma(f)$ *for every* $f \in \mathcal{A}$. *If* F *is an eigenfunction of* $-\mathrm{L}$ *with eigenvalue* ρ, *then* $\Gamma(F)$ *is* (μ-*almost everywhere*) *constant. In case* L *is a diffusion operator, the distribution of* F *is Gaussian.*

It might be useful to recall ([1], Chapter 6, [3, 12]) that under the curvature condition of the statement, $\lambda \geq \rho$ for every nonzero eigenvalue λ of $-\mathrm{L}$. In particular, L is ergodic in the sense that if $\Gamma(f) = 0$, then f is constant (μ-almost everywhere). It is also worthwhile mentioning that for the model space consisting of the Ornstein–Uhlenbeck diffusion operator $\mathrm{L} = \Delta - x \cdot \nabla$ with invariant measure γ_N, $\rho = 1$ and the eigenfunctions with eigenvalue 1 are the linear functions

$$F(x) = \sum_{i=1}^{N} a_i x_i, \qquad x = (x_1, \dots, x_N) \in \mathbb{R}^N,$$

whose distributions are of course Gaussian. Since Gaussian Wiener chaos of order larger than or equal to 2 do not contain any nonzero Gaussian variable [10] and [20], Proposition 4 thus expresses a kind of rigidity property in the sense that if F is a nonzero eigenfunction of the Ornstein–Uhlenbeck operator L with eigenvalue λ, then F is Gaussian if and only if $\Gamma(F)$ is constant, and if and only if $\lambda = \rho = 1$.

The proof of Proposition 4 is rather straigthforward. Write as before $\Gamma_m(F) = \Gamma_m$, $m \geq 1$. By Lemma 3 [formula (13)], $\Gamma_2 = \frac{1}{2}\mathrm{L}\Gamma + \rho\Gamma$. Therefore, under the curvature condition $\Gamma_2(f) \geq \rho\Gamma(f)$, $\mathrm{L}\Gamma \geq 0$. But then

$$0 \leq \int_E \Gamma \mathrm{L}\Gamma \, d\mu = -\int_E \Gamma(\Gamma) \, d\mu \leq 0,$$

so that $\Gamma = \Gamma(F)$ is (μ-almost everywhere) constant. The final assertion of the statement then follows from Stein's lemma (Proposition 1).

4. Chaos of a Markov operator. This section is devoted to the main conclusions of this work. We are thus given, on a state space E, a Markov operator L with symmetric and invariant probability measure μ and carré du champ Γ (acting on a suitable algebra of functions $\mathcal{A}$). Assume in addition that L has a pure point spectrum consisting of a countable sequence of eigenvalues $S = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ (more precisely, S is the spectrum of $-\mathrm{L}$) (cf. [3, 25, 28]). Since $\lambda_1 > 0$, L is ergodic [in the sense that if $\Gamma(f) = 0$, then f is constant].

Given the spectrum $S = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$, define for every $k \in \mathbb{N}$ the polynomial of degree k in the real variable X,

$$Q_k(X) = \prod_{i=0}^{k-1} (X - \lambda_i) = \sum_{i=1}^{k} \frac{1}{i!} Q_k^{(i)}(0) X^i$$

($Q_0 \equiv 1$). Define then the bilinear form (acting on $\mathcal{A} \times \mathcal{A}$)

$$Q_k(\Gamma) = \sum_{i=1}^{k} \frac{1}{i!} Q_k^{(i)}(0)\Gamma_i.$$

The following main definition introduces the notion of chaos associated to L and its spectrum S.

DEFINITION 5. An eigenfunction F of $-\mathrm{L}$ with eigenvalue λ_k ($-\mathrm{L}F = \lambda_k F$) is said to be a chaos of degree $k \geq 1$ relative to S if $Q_{k+1}(\Gamma)(F) = 0$ (μ-almost everywhere). We call F a chaos eigenfunction (with eigenvalue λ_k).

Motivation for the preceding definition is provided by the Ornstein–Uhlenbeck operator with spectrum $S = \mathbb{N}$. Namely, it is easily shown in this case (see [11], Section 2) that $Q_k(\Gamma)(F) = |\nabla^k F|^2$. Any eigenfunction F as in (6) is such that $\nabla^k F$ is constant and $\nabla^{k+1} F = 0$ leading thus to Definition 5. In the infinite dimensional setting of an abstract Wiener space (E, H, μ) with separable Hilbert space H, referring to [18], Chapter 1, for notation and terminology, the Ornstein–Uhlenbeck operator L has domain $\mathbb{D}^{2,2}$ and $Q_k(\Gamma)(F) = \|D^k F\|^2_{H^{\otimes k}}$ for any $F \in \mathbb{D}^{k,2}$ where D is the derivative operator (use as in the finite dimensional case the commutation $[\mathrm{L}, D] = D$ and the chain rule formula [18], Proposition 1.4.5). Now, if $J_k F$ denotes the projection of F (in $\mathbb{D}^{k,2}$) on the kth Wiener chaos, $\mathrm{L}J_k F = -kJ_k F$ and $D^k(J_k F) = J_0 D^k F = \mathbb{E}(D^k F)$ so that $J_k F$ thus defines a k-chaos in the sense of Definition 5. For example, in case $H = \mathrm{L}^2(T, \mathcal{B}, \nu)$ where ν is a σ-finite atomless measure on a measurable space $(T, \mathcal{B})$, the elements $J_k F$ may be represented as multiple stochastic integrals

$$I_k(f_k) = \int_T \cdots \int_T f_k(t_1, \dots, t_k) W(dt_1) \cdots W(dt_k)$$

of symmetric functions f_k on $\mathrm{L}^2(T^k)$ with respect to the white noise W and

$$D^k I_k(f_k) = \{f_k(t_1, \dots, t_k); t_1, \dots, t_k \in T\}.$$

The discrete operator $\mathrm{L}f = \frac{1}{2}\sum_{i=1}^N D_i f$ on the cube $\{-1, +1\}^N$ and the Poisson operator are further instances entering this definition with again $S = \mathbb{N}$ (see [11], Section 2). On the cube $\{-1, +1\}^N$, for example,

$$Q_k(\Gamma)(F) = \frac{1}{2^{2k}} \sum (D_{i_1} \cdots D_{i_k} F)^2,$$

where the sum is over distinct $i_1, \dots, i_k \in \{1, \dots, N\}$ and thus any F of the form (7) is a k-chaos ($k < N$).

There are of course examples of eigenfunctions which are not chaos. For instance, the Laguerre operator on the positive half-line $\mathcal{L}_p \psi = x\psi'' + (p - x)\psi'$, $p > 0$, has spectrum equal to $\mathbb{N}$ (with eigenvectors the Laguerre orthogonal polynomials with respect to the gamma distribution g_p), but the eigenfunction $F = x - p$ with eigenvalue 1 is not a 1-chaos as $Q_2(\Gamma)(F) = -\frac{1}{2}F$.

According to the preceding examples, another possible definition of k-chaos would have been that $Q_k(\Gamma)(F)$ is constant. (If F is normalized in $\mathrm{L}^2(\mu)$, then ([11], page 443),

$$\int_E Q_k(\Gamma)(F)\,d\mu = \int_E F Q_k(-\mathrm{L})F\,d\mu = Q_k(\lambda_k),$$

hence $Q_k(\Gamma)(F) = Q_k(\lambda_k)$.) Now, it is easily checked [using (13) of Lemma 3] that if F is an eigenfunction of $-\mathrm{L}$ with eigenvalue λ_k, then $\mathrm{L}Q_k(\Gamma)(F) = 2Q_{k+1}(\Gamma)(F)$. In particular therefore, if $Q_k(\Gamma)(F)$ is constant, then $Q_{k+1}(\Gamma)(F) = 0$. Conversely, if $Q_{k+1}(\Gamma)(F) = 0$, by ergodicity, $Q_k(\Gamma)(F)$ is constant. It will turn out more simple in the proofs of the main results to use the first definition of chaos [as $Q_{k+1}(\Gamma)(F) = 0$].

The following statements are the main results of this work. Recall the polynomials $Q_k(X)$ and set, for $k \geq 1$, $X \in \mathbb{R}$,

$$R_{k+1}(X) = \frac{1}{X^2}\left[Q_{k+1}(X) - Q_{k+1}^{(1)}(0)X\right] = \sum_{i=2}^{k+1} \frac{1}{i!} Q_{k+1}^{(i)}(0) X^{i-2}$$

and

$$T_{k+1}(X) = R_{k+1}(X + \lambda_k) - R_{k+1}(\lambda_k).$$

Thus, for example, $Q_2(X) = X^2 - \lambda_1 X$, $R_2 \equiv 1$ and $T_2 \equiv 0$, $Q_3(X) = X^3 - (\lambda_1 + \lambda_2)X^2 + \lambda_1\lambda_2 X$, $R_3(X) = X - (\lambda_1 + \lambda_2)$ and $T_3(X) = X$. Set furthermore

$$\pi_k = \lambda_1 \cdots \lambda_k, \qquad k \geq 1 \qquad (\pi_0 = 1).$$

The following theorem puts forward the fundamental identity at the root of this work.

THEOREM 6. *In the preceding setting, let F be a k-chaos eigenfunction with eigenvalue λ_k, $k \geq 1$. Set $\Gamma = \Gamma(F)$. Then*

$$\pi_{k-1} \int_E \Gamma^2\,d\mu = \pi_k \int_E F^2 \Gamma\,d\mu + (-1)^k \int_E \Gamma T_{k+1}\left(\frac{\mathrm{L}}{2}\right)\Gamma\,d\mu. \tag{16}$$

COROLLARY 7. *In the preceding setting, let F be a k-chaos eigenfunction with eigenvalue λ_k, $k \geq 1$. Set $\Gamma = \Gamma(F)$. If*

$$(-1)^k T_{k+1}\left(-\frac{\lambda_n}{2}\right) \leq 0 \qquad \textit{for every } n \in \mathbb{N}, \tag{17}$$

then

$$\int_E \Gamma^2\,d\mu \leq \lambda_k \int_E F^2 \Gamma\,d\mu. \tag{18}$$

In particular, if F is normalized in $\mathrm{L}^2(\mu)$, *then* $\int_E \Gamma \, d\mu = \int_E F(-\mathrm{L}F) \, d\mu = \lambda_k$ *and thus*

$$\mathrm{Var}_\mu(\Gamma) \le \lambda_k \left(\int_E F^2 \Gamma \, d\mu - \lambda_k \right). \tag{19}$$

Under the additional diffusion hypothesis on L, according to (9), inequality (19) of Corollary 7 may be expressed equivalently as

$$\mathrm{Var}_\mu(\Gamma) \le \lambda_k^2 \left(\frac{1}{3} \int_E F^4 \, d\mu - 1 \right). \tag{20}$$

In particular, if $\int_E F^4 \, d\mu = 3$, then $\Gamma = \Gamma(F)$ is constant and by Stein's lemma (Proposition 1), the distribution of F is Gaussian.

The next statement describes a fundamental instance for which the spectral condition (17) in Corollary 7 is fulfilled.

THEOREM 8. *The spectral condition* (17) *in Corollary* 7,

$$(-1)^k T_{k+1}\left(-\frac{\lambda_n}{2}\right) \le 0 \qquad \textit{for every } n \in \mathbb{N}$$

is satisfied when $S = (\lambda_n)_{n \in \mathbb{N}} = \mathbb{N}$.

As a consequence of this result, the conclusions of Corollary 7 apply to the examples of the Ornstein–Uhlenbeck, Bernoulli and Poisson operators. As such, some of the main conclusions of [13] are covered by the preceding general statement, and in particular the initial result of [20], namely that if $(F_n)_{n \in \mathbb{N}}$ is a sequence of homogeneous Gaussian chaos, normalized in $\mathrm{L}^2(\gamma_{N_n})$, $N_n \to \infty$, then $(F_n)_{n \in \mathbb{N}}$ converges to a Gaussian distribution as soon as $\int_E F_n^4 \, d\mu \to 3$.

For discrete models as the cube or the Poisson space, the picture is less satisfactory. For instance on the cube $E = \{-1, +1\}^{N_n}$, $N_n \to \infty$, if $F_n = \sum_{|A|=k} a_A^n W_A$, $n \in \mathbb{N}$, is a sequence of Walsh chaos of degree k normalized in $\mathrm{L}^2(\mu)$ for the uniform measure μ, and if $\int_E F_n^2 \Gamma(F_n) \, d\mu \to k$, then as an application of Corollary 7, $\Gamma(F_n) \to k$ in $\mathrm{L}^2(\mu)$. Now $\Gamma(F)$ being constant in this case is not always discriminative [as shown by the example of $F(x) = x_1 \cdots x_k$] and further conditions have to be imposed on the sequence $(F_n)_{n \in \mathbb{N}}$ to ensure convergence toward a Gaussian distribution. This analysis has been recently achieved in [16]. Similar additional conditions have been studied on Poisson spaces in [21, 24]. The input of Corollary 7 on convergence of chaos in these discrete examples is that it reduces the convergence $\Gamma(F_n) \to \lambda_k$ in $\mathrm{L}^2(\mu)$ by the weaker condition $\int_E F_n^2 \Gamma(F_n) \, d\mu \to \lambda_k$.

5. Chaos of order 1 and 2. Before turning to the general proofs of Theorem 6 and Corollary 7, and to get a better feeling about these statements, we discuss in this section the particular values $k = 1$ and $k = 2$. Recall that we write for simplicity $\Gamma_m = \Gamma_m(F)$, $m \geq 1$, for an eigenfunction F.

When $k = 1$, that is, $Q_2(\Gamma) = \Gamma_2 - \lambda_1 \Gamma = 0$, multiplying this identity by F^2 and integrating with respect to μ, it follows thanks to Lemma 3 [formula (15)] that

$$\int_E \Gamma^2 \, d\mu = \lambda_1 \int_E F^2 \Gamma \, d\mu.$$

Now here $R_2 \equiv 1$, and thus $T_2 \equiv 0$, so that both the fundamental identity (16) and the spectral condition (17) are automatically satisfied.

When $k = 2$, start from $Q_3(\Gamma) = \Gamma_3 - (\lambda_1 + \lambda_2)\Gamma_2 - \lambda_1\lambda_2\Gamma = 0$. Multiplying by F^2 and integrating, it follows similarly thanks to Lemma 3 [formula (15)] that

$$\int_E \Gamma\Gamma_2 \, d\mu - (\lambda_1 + \lambda_2) \int_E \Gamma^2 \, d\mu + \lambda_1\lambda_2 \int_E F^2 \Gamma \, d\mu = 0.$$

By (13) of Lemma 3, $\Gamma_2 = \frac{1}{2}\mathrm{L}\Gamma + \lambda_2\Gamma$ so that

$$\frac{1}{2} \int_E \Gamma \mathrm{L}\Gamma \, d\mu - \lambda_1 \int_E \Gamma^2 \, d\mu + \lambda_1\lambda_2 \int_E F^2 \Gamma \, d\mu = 0.$$

Here $R_3(X) = X - (\lambda_1 + \lambda_2)$ and $T_3(X) = X$ so that the fundamental identity (16) holds, and the spectral condition (17) amounts to $\lambda_n \geq 0$ for every $n \in \mathbb{N}$.

One observation on which we will come back in the next section is that, in the case $k = 2$, only the inequality $Q_3(\Gamma) \geq 0$ is used in order to reach the conclusions of Corollary 7. A further observation is that for chaos of order 1 or 2, the spectral condition (17) is fulfilled for any sequence of eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. This is clearly not the case when $k \geq 3$.

6. Proofs of Theorems 6 and 8. In this section, we establish Theorem 6, Corollary 7 and Theorem 8. Let thus F be a k-chaos with eigenvalue λ_k. (If necessary, we may assume that $k \geq 3$ according to the preceding section.) Write as usual Γ_m for $\Gamma_m(F)$, $m \geq 1$.

As in the preceding section for chaos of order 1 or 2, start as a first step from the chaos hypothesis $Q_{k+1}(\Gamma) = 0$. Multiply this identity by F^2 and integrate with respect to μ. By definition of Q_{k+1} and (15) of Lemma 3,

$$\begin{aligned}
0 &= \int_E F^2 Q_{k+1}(\Gamma) \, d\mu \\
&= \sum_{i=1}^{k+1} \frac{1}{i!} Q_{k+1}^{(i)}(0) \int_E F^2 \Gamma_i \, d\mu \qquad (21) \\
&= Q_{k+1}^{(1)}(0) \int_E F^2 \Gamma \, d\mu + \sum_{i=2}^{k+1} \frac{1}{i!} Q_{k+1}^{(i)}(0) \int_E \Gamma\Gamma_{i-1} \, d\mu.
\end{aligned}$$

Now, by (13) of Lemma 3,

$$\begin{aligned}
&\sum_{i=2}^{k+1} \frac{1}{i!} Q_{k+1}^{(i)}(0) \int_E \Gamma \Gamma_{i-1}\, d\mu \\
&\quad = \sum_{i=2}^{k+1} \frac{1}{i!} Q_{k+1}^{(i)}(0) \int_E \Gamma \left(\frac{1}{2}\mathrm{L} + \lambda_k \,\mathrm{Id}\right)^{i-2} \Gamma\, d\mu \\
&\quad = \sum_{i=2}^{k+1} \frac{1}{i!} Q_{k+1}^{(i)}(0) \sum_{\ell=0}^{i-2} \binom{i-2}{\ell} \frac{1}{2^\ell} \lambda_k^{i-2-\ell} \int_E \Gamma \mathrm{L}^\ell \Gamma\, d\mu \\
&\quad = \sum_{\ell=0}^{k-1} \sum_{i=\ell+2}^{k+1} \binom{i-2}{\ell} \frac{1}{i!} Q_{k+1}^{(i)}(0) \lambda_k^{i-2-\ell} \frac{1}{2^\ell} \int_E \Gamma \mathrm{L}^\ell \Gamma\, d\mu.
\end{aligned}$$

Recalling the definition of the polynomial R_{k+1}, note that

$$\sum_{i=\ell+2}^{k+1} \binom{i-2}{\ell} \frac{1}{i!} Q_{k+1}^{(i)}(0) \lambda_k^{i-2-\ell} = \frac{1}{\ell!} R_{k+1}^{(\ell)}(\lambda_k).$$

Hence

$$\sum_{i=2}^{k+1} \frac{1}{i!} Q_{k+1}^{(i)}(0) \int_E \Gamma \Gamma_{i-1}\, d\mu = \sum_{\ell=0}^{k-1} \frac{1}{\ell!} R_{k+1}^{(\ell)}(\lambda_k) \frac{1}{2^\ell} \int_E \Gamma \mathrm{L}^\ell \Gamma\, d\mu.$$

Now

$$\sum_{\ell=0}^{k-1} \frac{1}{\ell!} R_{k+1}^{(\ell)}(\lambda_k) X^\ell = R_{k+1}(X + \lambda_k) = T_{k+1}(X) + R_{k+1}(\lambda_k)$$

so that

$$\sum_{i=2}^{k+1} \frac{1}{i!} Q_{k+1}^{(i)}(0) \int_E \Gamma \Gamma_{i-1}\, d\mu = \int_E \Gamma T_{k+1}\left(\frac{\mathrm{L}}{2}\right) \Gamma\, d\mu + R_{k+1}(\lambda_k) \int_E \Gamma^2\, d\mu.$$

The fundamental identity (16) of Theorem 6 then follows from (21) together with the fact that

$$Q_{k+1}^{(1)}(0) = (-1)^k \lambda_1 \cdots \lambda_k = (-1)^k \pi_k$$

and

$$R_{k+1}(\lambda_k) = (-1)^{k+1} \lambda_1 \cdots \lambda_{k-1} = (-1)^{k+1} \pi_{k-1}.$$

The proof is complete.

Corollary 7 is deduced from Theorem 6 through the following classical and elementary property, consequence of the point spectrum hypothesis.

LEMMA 9. *If P is a polynomial, $\int_E uP(\mathrm{L})u\,d\mu \geq 0$ for every u [in the $\mathrm{L}^2(\mu)$-domain of $P(\mathrm{L})$] if (and only if) $P(-\lambda_n) \geq 0$ for every $n \in \mathbb{N}$.*

PROOF. For each $n \in \mathbb{N}$, denote by E_n the eigenspace associated to the eigenvalue λ_n so that $\mathrm{L}^2(\mu) = \bigoplus_{n\in\mathbb{N}} E_n$ since $S = (\lambda_n)_{n\in\mathbb{N}}$ is the spectrum of L. Decompose then u in $\mathrm{L}^2(\mu)$ as $u = \sum_{n\in\mathbb{N}} u_n$ with $u_n \in E_n$, $n \in \mathbb{N}$, so that $P(\mathrm{L})u = \sum_{n\in\mathbb{N}} P(-\lambda_n)u_n$ and

$$\int_E uP(\mathrm{L})u\,d\mu = \sum_{n\in\mathbb{N}} P(-\lambda_n)\int_E u_n^2\,d\mu$$

from which conclusion follows. $\square$

As mentioned for chaos of order 2, when k is even, only the inequality $Q_{k+1}(\Gamma) \geq 0$ is used in order to reach the conclusions of Corollary 7.

We next turn to the proof of Theorem 8, checking the spectral condition (17) $(-1)^k T_{k+1}(-\frac{\lambda_n}{2}) \leq 0$, $n \in \mathbb{N}$, for $S = (\lambda_n)_{n\in\mathbb{N}} = \mathbb{N}$. Since in this case $T_{k+1}(X) = R_{k+1}(X+k) - (-1)^{k+1}(k-1)!$, we have to show that

$$\left(\frac{n}{2}-k\right)^{-2}\left[\prod_{i=0}^{k}\left(\frac{n}{2}-i\right) - k!\left(\frac{n}{2}-k\right)\right] \geq (k-1)!.$$

When $\frac{n}{2} = k$, the expression on the left-hand side is equal to $k!\sum_{i=1}^{k}\frac{1}{i}$ so that the conclusion holds in this case. When $\frac{n}{2} \neq k$, we need to show that

$$\left(\frac{n}{2}-k\right)^{-1}\left[\prod_{i=0}^{k-1}\left(\frac{n}{2}-i\right) - k!\right] \geq (k-1)!.$$

Assume first that $n \geq 2k+1$. Then

$$\begin{aligned}\prod_{i=0}^{k-1}\left(\frac{n}{2}-i\right) &= \left(\frac{n}{2}-k+1\right)\prod_{i=0}^{k-2}\left(\frac{n}{2}-i\right)\\ &\geq \left(\frac{n}{2}-k+1\right)\prod_{i=2}^{k}\left(i+\frac{1}{2}\right) \geq \left(\frac{n}{2}-k+1\right)k!.\end{aligned}$$

Hence

$$\left(\frac{n}{2}-k\right)^{-1}\left[\prod_{i=0}^{k-1}\left(\frac{n}{2}-i\right) - k!\right] \geq k!,$$

which answers this case. We turn to the case where $n \leq 2k-1$ for which it is necessary to check that

$$\prod_{i=0}^{k-1}\left(\frac{n}{2}-i\right) \leq \frac{n}{2}(k-1)!.$$

It is enough to assume that n is odd, $n = 2p - 1$, $1 \le p \le k$. Then

$$\prod_{i=0}^{k-1}\left(\frac{n}{2} - i\right) = \prod_{i=0}^{p-1}\left(\frac{n}{2} - i\right)\prod_{i=p}^{k-1}\left(\frac{n}{2} - i\right) \le \prod_{i=1}^{p}\left(i - \frac{1}{2}\right)\prod_{i=1}^{k-p}\left(i - \frac{1}{2}\right).$$

Therefore, the inequality to establish amounts to

$$\prod_{i=1}^{p-1}\left(i - \frac{1}{2}\right)\prod_{i=1}^{k-p}\left(i - \frac{1}{2}\right) \le (p-1)!(k-p)! \le (k-1)!,$$

which is trivially satisfied. The claims thus holds in this case too. Theorem 8 is therefore established.

7. Convergence to gamma distributions. In this last section, we briefly address the analogs of Theorem 6 and Corollary 7 in the context of convergence to gamma distributions on the basis of the corresponding Stein characterization of Proposition 2. The main conclusion is obtained by a simple variation on the fundamental identity (16) of Theorem 6. In particular, the analysis covers the recent results of [14] (see also [13]) in the context of Wiener chaos.

The framework is the one of the preceding sections, with a Markov operator L with spectrum $S = (\lambda_n)_{n\in\mathbb{N}}$ and invariant and reversible probability measure μ and carré du champ Γ. Recall $\pi_k = \lambda_1 \cdots \lambda_k$, $k \ge 1$, and the polynomials R_{k+1} and T_{k+1} of Theorem 6.

The following theorem addresses approximation of a k-chaos F by a gamma distribution via the control of $\mathrm{Var}_\mu(\Gamma - \lambda_k F)$ as emphasized in Proposition 2. As announced, the proof is an easy modification on the fundamental identity (16) of Theorem 6.

THEOREM 10. *Let F be a k-chaos with eigenvalue λ_k, $k \ge 1$, such that $\int_E F^2\,d\mu = p > 0$. Set $\Gamma = \Gamma(F)$. Under the spectral condition* (17) $(-1)^k T_{k+1}(-\frac{\lambda_n}{2}) \le 0$ *for every $n \in \mathbb{N}$, it holds*

$$\mathrm{Var}_\mu(\Gamma - \lambda_k F) \le \lambda_k \int_E F^2\Gamma\,d\mu + A_k \int_E F\Gamma\,d\mu - pB_k - p^2\lambda_k^2,$$

where

$$A_k = \frac{2(-1)^k\lambda_k}{\pi_{k-1}} R_{k+1}\left(\frac{\lambda_k}{2}\right) \quad \textit{and} \quad B_k = \frac{(-1)^k\lambda_k^2}{\pi_{k-1}} R_{k+1}\left(\frac{\lambda_k}{2}\right).$$

In the diffusion case,

$$\lambda_k \int_E F^4\,d\mu = 3\int_E F^2\Gamma\,d\mu \quad \text{and} \quad \lambda_k \int_E F^3\,d\mu = 2\int_E F\Gamma\,d\mu$$

so that the conclusion of the theorem reads

$$\mathrm{Var}_\mu(\Gamma - \lambda_k F) \le \frac{\lambda_k^2}{3}\int_E F^4\,d\mu + \frac{A_k\lambda_k}{2}\int_E F^3\,d\mu - pB_k - p^2\lambda_k^2.$$

Consider now the example where $S = \mathbb{N}$ for which we know from Theorem 8 that the spectral condition (17) holds. The inequality of Theorem 10 takes a nicer form when $k \ge 2$ is even. Indeed in this case $(-1)^k\lambda_k R_{k+1}(\frac{\lambda_k}{2}) = -2k!$ so that

$$\frac{1}{k}\,\mathrm{Var}_\mu(\Gamma - \lambda_k F) \le \int_E F^2\Gamma\,d\mu - 4\int_E F\Gamma\,d\mu + 2pk - p^2k.$$

In particular in the diffusion case,

$$\frac{3}{k^2}\,\mathrm{Var}_\mu(\Gamma - \lambda_k F) \le \int_E F^4\,d\mu - 6\int_E F^3\,d\mu + 6p - 3p^2. \tag{22}$$

This inequality (22) then ensures, through Stein's lemma (Proposition 2), that if $(F_n)_{n\in\mathbb{N}}$ is a sequence of k-chaos such that $\int_E F_n^2\,d\mu = p$ for every n and

$$\int_E F_n^4\,d\mu - 6\int_E F_n^3\,d\mu + 6p - 3p^2 \to 0,$$

then $(F_n + p)_{n\in\mathbb{N}}$ converges in distribution to the gamma distribution with parameter p, that is the main result of [14].

PROOF OF THEOREM 10. Let thus F be a k-chaos with $\int_E F^2\,d\mu = p$, hence $\int_E \Gamma\,d\mu = p\lambda_k$. Set $U = \Gamma - \lambda_k F$ (so $\int_E U\,d\mu = p\lambda_k$). It is immediately checked that

$$\begin{aligned}\int_E \Gamma^2\,d\mu &= \int_E U^2\,d\mu + 2\lambda_k\int_E F\Gamma\,d\mu - p\lambda_k^2\\ &= \mathrm{Var}_\mu(U) + 2\lambda_k\int_E F\Gamma\,d\mu - p(1-p)\lambda_k^2\end{aligned}$$

and, for every $\ell \ge 1$,

$$\int_E \Gamma \mathrm{L}^\ell\Gamma\,d\mu = \int_E U\mathrm{L}^\ell U\,d\mu + 2(-1)^\ell\lambda_k^{\ell+1}\int_E F\Gamma\,d\mu - p(-1)^\ell\lambda_k^{\ell+2}.$$

Therefore, the fundamental identity (16) of Theorem 6 takes the form, after a little algebra,

$$\begin{aligned}(-1)^k&\int_E UT_{k+1}\Big(\frac{\mathrm{L}}{2}\Big)U\,d\mu - \pi_{k-1}\,\mathrm{Var}_\mu(U)\\ &+ \pi_k\int_E F^2\Gamma\,d\mu + 2(-1)^k\lambda_k R_{k+1}\Big(\frac{\lambda_k}{2}\Big)\int_E F\Gamma\,d\mu\\ &- p(-1)^k\lambda_k^2 R_{k+1}\Big(\frac{\lambda_k}{2}\Big) - p^2\lambda_k^2\pi_{k-1} = 0.\end{aligned}$$

Under the spectral condition (17) $(-1)^k T_{k+1}(-\frac{\lambda_n}{2}) \leq 0$ for every $n \in \mathbb{N}$,

$$\pi_{k-1} \operatorname{Var}_\mu(U) \leq \pi_k \int_E F^2 \Gamma\, d\mu + 2(-1)^k \lambda_k R_{k+1}\left(\frac{\lambda_k}{2}\right) \int_E F\Gamma\, d\mu$$
$$- p(-1)^k \lambda_k^2 R_{k+1}\left(\frac{\lambda_k}{2}\right) - p^2 \lambda_k^2 \pi_{k-1},$$

which amounts to the statement of the theorem. The proof is complete. □

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REFERENCES

[1] BAKRY, D. (1994). L'hypercontractivité et son utilisation en théorie des semigroupes. In *Lectures on Probability Theory* (*Saint-Flour*, 1992). *Lecture Notes in Math.* **1581** 1–114. Springer, Berlin. MR1307413

[2] BAKRY, D. and ÉMERY, M. (1985). Diffusions hypercontractives. In *Séminaire de Probabilités*, *XIX*, 1983/84. *Lecture Notes in Math.* **1123** 177–206. Springer, Berlin. MR0889476

[3] BAKRY, D., GENTIL, I. and LEDOUX, M. (2012). Forthcoming monograph.

[4] BOULEAU, N. and HIRSCH, F. (1991). *Dirichlet Forms and Analysis on Wiener Space*. *de Gruyter Studies in Mathematics* **14**. de Gruyter, Berlin. MR1133391

[5] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q. M. (2011). *Normal Approximation by Stein's Method*. Springer, Heidelberg.

[6] CHEN, L. H. Y. and SHAO, Q.-M. (2005). Stein's method for normal approximation. In *An Introduction to Stein's Method*. *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.* **4** 1–59. Singapore Univ. Press, Singapore. MR2235448

[7] DELLACHERIE, C., MAISONNEUVE, B. and MEYER, P.-A. (1992). Probabilités et potentiel. Chapitres XVII à XXIV: Processus de Markov (fin), Compléments de calcul stochastique. Herman, Paris.

[8] FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (1994). *Dirichlet Forms and Symmetric Markov Processes*. *de Gruyter Studies in Mathematics* **19**. de Gruyter, Berlin. MR1303354

[9] GALLOT, S., HULIN, D. and LAFONTAINE, J. (1990). *Riemannian Geometry*, 2nd ed. Springer, Berlin. MR1083149

[10] JANSON, S. (1997). *Gaussian Hilbert Spaces*. *Cambridge Tracts in Mathematics* **129**. Cambridge Univ. Press, Cambridge. MR1474726

[11] LEDOUX, M. (1995). L'algèbre de Lie des gradients itérés d'un générateur markovien—développements de moyennes et entropies. *Ann. Sci. École Norm. Sup.* (4) **28** 435–460. MR1334608

[12] LEDOUX, M. (2000). The geometry of Markov diffusion generators: Probability theory. *Ann. Fac. Sci. Toulouse Math.* (6) **9** 305–366. MR1813804

[13] NOURDIN, I. and PECCATI, G. (2009). Stein's method on Wiener chaos. *Probab. Theory Related Fields* **145** 75–118. MR2520122

[14] NOURDIN, I. and PECCATI, G. (2009). Noncentral convergence of multiple integrals. *Ann. Probab.* **37** 1412–1426. MR2546749

[15] NOURDIN, I., PECCATI, G. and REINERT, G. (2010). Invariance principles for homogeneous sums: Universality of Gaussian Wiener chaos. *Ann. Probab.* **38** 1947–1985. MR2722791

[16] NOURDIN, I., PECCATI, G. and REINERT, G. (2010). Stein's method and stochastic analysis of Rademacher functionals. *Electron. J. Probab.* **15** 1703–1742. MR2735379

[17] NOURDIN, I., PECCATI, G. and RÉVEILLAC, A. (2010). Multivariate normal approximation using Stein's method and Malliavin calculus. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 45–58. MR2641769

[18] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. Springer, Berlin. MR2200233

[19] NUALART, D. and ORTIZ-LATORRE, S. (2008). Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stochastic Process. Appl.* **118** 614–628. MR2394845

[20] NUALART, D. and PECCATI, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** 177–193. MR2118863

[21] PECCATI, G., SOLÉ, J. L., TAQQU, M. S. and UTZET, F. (2010). Stein's method and normal approximation of Poisson functionals. *Ann. Probab.* **38** 443–478. MR2642882

[22] PECCATI, G. and TAQQU, M. (2011). *Wiener Chaos: Moments, Cumulants and Diagrams*. Springer, Berlin.

[23] PECCATI, G. and TUDOR, C. A. (2005). Gaussian limits for vector-valued multiple stochastic integrals. In *Séminaire de Probabilités XXXVIII. Lecture Notes in Math.* **1857** 247–262. Springer, Berlin. MR2126978

[24] PECCATI, G. and ZHENG, C. (2010). Multi-dimensional Gaussian fluctuations on the Poisson space. *Electron. J. Probab.* **15** 1487–1527. MR2727319

[25] REED, M. and SIMON, B. (1975). *Methods of Modern Mathematical Physics* **2**. Academic Press, New York.

[26] REINERT, G. (2005). Three general approaches to Stein's method. In *An Introduction to Stein's Method. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.* **4** 183–221. Singapore Univ. Press, Singapore. MR2235451

[27] STEIN, C. (1986). *Approximate Computation of Expectations. Institute of Mathematical Statistics Lecture Notes—Monograph Series* **7**. IMS, Hayward, CA. MR0882007

[28] YOSIDA, K. (1995). *Functional Analysis*, 6th ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **123**. Springer, Berlin. MR0617913

INSTITUT DE MATHÉMATIQUES DE TOULOUSE
UNIVERSITÉ DE TOULOUSE
F-31062 TOULOUSE
FRANCE
E-MAIL: ledoux@math.univ-toulouse.fr