The Annals of Probability
2012, Vol. 40, No. 4, 1715–1758
DOI: 10.1214/11-AOP660

BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH ROUGH DRIVERS

BY JOSCHA DIEHL AND PETER FRIZ

Technical University Berlin and Weierstrass Institute for Applied Analysis and Stochastics

Backward stochastic differential equations (BSDEs) in the sense of Pardoux–Peng [*Lecture Notes in Control and Inform. Sci.* **176** (1992) 200–217] provide a non-Markovian extension to certain classes of nonlinear partial differential equations; the nonlinearity is expressed in the so-called driver of the BSDE. Our aim is to deal with drivers which have very little regularity in time. To this end, we establish continuity of BSDE solutions with respect to rough path metrics in the sense of Lyons [*Rev. Mat. Iberoam.* **14** (1998) 215–310] and so obtain a notion of "BSDE with rough driver." Existence, uniqueness and a version of Lyons' limit theorem in this context are established. Our main tool, aside from rough path analysis, is the stability theory for quadratic BSDEs due to Kobylanski [*Ann. Probab.* **28** (2000) 558–602].

1. Introduction. We recall that *backward stochastic differential equations* (BSDEs) are stochastic equations of the type

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\,dr - \int_t^T Z_r\,dW_r. \tag{1}$$

Here, W is an m-dimensional Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. The *terminal data* ξ is assumed to be $\mathcal{F}_T$-measurable, the *driver* $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$ is a predictable random field; a solution to this equation is a $(1+m)$-dimensional adapted solution process of the form $(Y_t, Z_t)_{0\le t\le T}$; subject to some integrability properties depending on the framework imposed by the type of assumptions on f. Equation (1) can also be written in differential form:

$$-dY_t = f(t, Y_t, Z_t)\,dt - Z_t\,dW_t.$$

The aim of this paper, partially motivated from the recent progress on partial differential equations driven by rough path [4, 5, 8, 14, 23], is to consider

$$-dY_t = f(t, Y_t, Z_t)\,dt + H(Y_t)\,d\zeta_t - Z_t\,dW_t,$$

Received August 2010; revised March 2011.

[1]Supported by an IRTG (Berlin-Zurich) Ph.D.-scholarship.

[2]Supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC Grant agreement nr. 258237.

MSC2010 subject classifications. 60H10, 60H15.

Key words and phrases. Backward stochastic differential equations, backward doubly stochastic differential equations, rough path analysis, stochastic partial differential equations, viscosity theory.

where ζ is (at first) a smooth d-dimensional driving signal—accordingly $H = (H_1, \dots, H_d)$—followed by a discussion in which we establish *rough path stability* of the solution process (Y, Z) as a function of ζ. Note that we do *not* establish any sort of rough path stability in W. Indeed when $f \equiv 0$ in (1), BSDE theory reduces to martingale representation, an intrinsically stochastic result which does not seem amenable to a rough pathwise approach.[3] We are able to carry out our analysis in a framework in which the ω-dependence of the terms driven by ζ factorizes through an Itô process. That is, we consider, for fixed $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$,

$$\begin{aligned} dX_t &= b(\omega; t)\,dt + \sigma(\omega; t)\,dW_t, \qquad t_0 \le t \le T;\, X_{t_0} = x_0 \in \mathbb{R}^n, \\ -dY_t &= f(\omega; t, Y_t, Z_t)\,dt + H(X_t, Y_t)\,d\zeta - Z_t\,dW, \\ &\qquad\qquad\qquad\qquad\qquad\qquad t_0 \le t \le T;\, Y_T = \xi \in L^\infty(\mathcal{F}_T). \end{aligned}$$

Our main-result is, under suitable conditions on f and $H = (H_1, \dots, H_d)$, that any sequence (ζ^n) which is Cauchy in rough path metric gives rise to a solution (Y, Z) of the *BSDE with rough driver*

$$-dY_t = f(\omega; t, Y_t, Z_t)\,dt + H(X_t, Y_t)\,d\boldsymbol{\zeta} - Z_t\,dW_t, \tag{2}$$

where $\boldsymbol{\zeta}$ denotes the (rough path) limit of (ζ^n) and where indeed (Y, Z) depends only on $\boldsymbol{\zeta}$ and not on the particular approximating sequence. An interesting feature of this result, which somehow encodes the particular structure of the above equation, is that one does *not* need to construct or understand the iterated integrals of ζ and W; but only those of ζ which is tantamout to speak of the rough path $\boldsymbol{\zeta}$. This is in strict contrast to the usual theory of rough differential equations in which both $d\zeta$ and dW figure as driving differentials, for example, in equations of the form $dy = V_1(y)\,d\zeta + V_2(y)\,dW$.

If we specialize to a fully Markovian setting, say $\xi = g(X_T)$, $\sigma(\omega; t) = \sigma(t, X_t(\omega))$, $b(\omega; t) = b(t, X_t(\omega))$, $f(\omega; t, y, z) = f(t, X_t(\omega), y, z)$, $H = H(X_t, Y_t)$, we find that the solution to (2), evaluated at $t = t_0$, yields a solution to the (terminal value problem of the) *rough partial differential equation*

$$-du = (\mathcal{L}u)\,dt + f(t, x, u, Du\,\sigma(t, x))\,dt + H(x, u)\,d\boldsymbol{\zeta}, \qquad u_T(x) = g(x),$$

where $\mathcal{L}$ denotes the generator of X. If one is interested in the Cauchy problem, $\tilde{u}(t, x) = u(T - t, x)$ satisfies,

$$d\tilde{u} = (\mathcal{L}\tilde{u})\,dt + f(x, \tilde{u}, D\tilde{u}\,\sigma(t, x))\,dt + H(x, \tilde{u})\,d\tilde{\boldsymbol{\zeta}}, \tilde{u}_0(x) = g(x), \tag{3}$$

where $\tilde{\boldsymbol{\zeta}} = \zeta(T - \cdot)$.

To the best of our knowledge, (2) is the first attempt to introduce rough path methods [13, 18–20] in the field of backward stochastic differential equations [10,

[3] See however the recent work of Liang et al. [16] in which martingale representation is replaced by an abstract transformation.

15, 21]. Of course, there are many hints in the literature toward the possibility of doing so: we mention in particular the Pardoux–Peng [22] theory of *backward doubly stochastic differential equations* (BDSDEs) which amounts to replacing $d\boldsymbol{\zeta}$ in (2) by another set of Brownian differentials, say dB, independent of W. This theory was then employed by Buckdahn and Ma [3] to construct (stochastic viscosity) solutions to (3) with $d\boldsymbol{\zeta}$ replaced by a Brownian differential and the assumption that the vector fields $H_1(x,\cdot),\dots,H_d(x,\cdot)$ commute.

This paper is structured as follows. In Section 2, we state and prove our main result concerning the existence and uniqueness of BSDEs with rough drivers. Section 3 specializes the setting to a purely Markovian one. In this context, BSDEs with rough drivers are connected to rough partial differential equations, which we analyze in their own right. In Section 4, we establish the connection to BDSDEs.

2. BSDE with rough driver. We fix once and for all a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F})_t,\mathbb{P})$, which carries an m-dimensional Brownian motion W. Let $\mathcal{F}_t$ be the usual filtration of W. Denote by $H^2_{[0,T]}(\mathbb{R}^m)$ the space of predictable processes X in $\mathbb{R}^m$ such that $\|X\|^2 := \mathbb{E}[\int_0^T |X|_r^2\,dr] < \infty$. Denote by $H^\infty_{[0,T]}(\mathbb{R})$ the space of predictable processes that are almost surely bounded. We will say a sequence converges in H^∞ if it converges uniformly on $[0,T]$, $\mathbb{P}$-a.s. For a random variable ξ we denote by $\|\xi\|_\infty$ its essential supremum, for a process Y we denote by $\|Y\|_\infty$ the essential supremum of $\sup_{0\le t\le T}|Y_t|$.

For a smooth path ζ in $\mathbb{R}^d$ and $\xi \in L^\infty(\mathcal{F}_T)$, we consider the BSDE

$$Y_t = \xi + \int_t^T f(r,Y_r,Z_r)\,dr + \int_t^T H(X_r,Y_r)\,d\zeta(r) - \int_t^T Z_r\,dW_r, \qquad t\le T, \tag{4}$$

where the $\mathbb{R}^n$-valued semimartingale X has the form

$$X_t = x + \int_0^t \sigma_r\,dW_r + \int_0^t b_r\,dr.$$

Here, $H=(H_1,\dots,H_d)$ with $H_k:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$, $k=1,\dots,d$ and $\int_t^T H(X_r, Y_r)\,d\zeta(r) := \sum_{k=1}^d \int_t^T H_k(X_r,Y_r)\dot{\zeta}^k(r)\,dr$. W is an m-dimensional Brownian motion (hence Z is a row vector taking values in $\mathbb{R}^{m\times 1}$ identified with $\mathbb{R}^m$). $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$ is a predictable random function, $x\in\mathbb{R}^n$, σ is a predictable process taking values in $\mathbb{R}^{n\times m}$, b is a predictable process taking values in $\mathbb{R}^n$.

DEFINITION 1. We call equation (4) *BSDE with data* (ξ,f,H,ζ). We call (Y,Z) a solution if $Y\in H^\infty_{[0,T]}(\mathbb{R})$, $Z\in H^2_{[0,T]}(\mathbb{R}^m)$ and (4) is satisfied.

For a vector x, we denote the Euclidean norm as usual by $|x|$. For a matrix X, we denote by $|X|$, depending on the situation, either the 1-norm (operator norm), the 2-norm (Euclidean norm) or the ∞-norm (operator norm of the transpose). This slight abuse of notation will not lead to confusion, as all inequalities will be valid up to multiplicative constants.

We introduce the following assumptions:

(A1) There exists a constant $C_\sigma > 0$ such that $\mathbb{P}$-a.s. for $t \in [0, T]$

$$|\sigma_t(\omega)| \leq C_\sigma.$$

(A2) There exists a constant $C_b > 0$ such that $\mathbb{P}$-a.s. for $t \in [0, T]$

$$|b_t(\omega)| \leq C_b.$$

(F1) There exists a constant $C_{1,f} > 0$ such that $\mathbb{P}$-a.s. for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$[4]

$$|f(\omega; t, y, z)| \leq C_{1,f} + C_{1,f}|z|^2,$$
$$|\partial_z f(\omega; t, y, z)| \leq C_{1,f} + C_{1,f}|z|.$$

(F2) There exists a constant $C_{2,f} > 0$ such that $\mathbb{P}$-a.s. for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$

$$\partial_y f(\omega; t, y, z) \leq C_{2,f}.$$

For given real numbers $\gamma > p \geq 1$, we have the following assumption:

$(H_{p,\gamma})$ Let $H(x, \cdot) = (H_1(x, \cdot), \dots, H_d(x, \cdot))$ be a collection of vector fields on $\mathbb{R}$, parameterized by $x \in \mathbb{R}^n$. Assume that for some $C_H > 0$, we have joint regularity of the form

$$\sup_{i=1,\dots,d} |H_i|_{\mathrm{Lip}^{\gamma+2}(\mathbb{R}^{n+1})} \leq C_H.$$

As a consequence of Theorems 2.3 and 2.6 in [15], we get the following lemma.

LEMMA 2. *Assume* (A1), (A2), (F1), (F2) *and let H be Lipschitz on $\mathbb{R}^n \times \mathbb{R}$. Let $\xi \in L^\infty(\mathcal{F}_T)$ and a smooth path ζ be given. Then there exists a unique solution to the BSDE with data (ξ, f, H, ζ).*

We want to give meaning to equation (4), where the smooth path ζ is replaced by a general geometric rough path $\boldsymbol{\zeta} \in C^{0,p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$, where $G^{[p]}(\mathbb{R}^d)$ is the free step-$[p]$ nilpotent group over $\mathbb{R}^d$, realized as subset of $\mathbb{R} \oplus \mathbb{R}^d \oplus \cdots \oplus$

[4]When we use partial derivatives, we assume implicitly that the function in question is continuously differentiable in the respective variable. In fact, throughout, it would suffice to assume (local) Lipschitzness and bound the Lipschitz constant analogously.

$(\mathbb{R}^d)^{[p]}$, equipped with Carnot–Caratheodory metric.[5] We give our main result, the proof of which we present at the end of the section.

THEOREM 3. *Let $p \geq 1$, $\gamma > p$ and $\zeta^n, n = 1, 2, \dots,$ be smooth paths in $\mathbb{R}^d$. Assume $\zeta^n \to \boldsymbol{\zeta}$ in p-variation, for a path $\boldsymbol{\zeta} \in C^{0,p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$. Let $\xi \in L^\infty(\mathcal{F}_T)$. Let f be a random function satisfying* (F1) *and* (F2). *Moreover, assume* (A1), (A2) *and* $(H_{p,\gamma})$. *For $n \geq 1$, denote by (Y^n, Z^n) the solutions to the BSDE with data (ξ, f, H, ζ^n).*

Then there exists a process $(Y, Z) \in H^\infty_{[0,T]} \times H^2_{[0,T]}$ such that

$$\begin{aligned} Y^n &\to Y \qquad \text{uniformly on } [0,T]\ \mathbb{P}\text{-a.s.}, \\ Z^n &\to Z \qquad \text{in } H^2_{[0,T]}. \end{aligned}$$

The process is unique in the sense, that it only depends on the limiting rough path $\boldsymbol{\zeta}$ and not on the approximating sequence. We write (formally [6]*)*

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\,dr + \int_t^T H(X_r, Y_r)\,d\boldsymbol{\zeta}(r) - \int_t^T Z_r\,dW_r. \tag{5}$$

Moreover, the solution mapping

$$\begin{aligned} C^{0,p\text{-var}}\big([0,T], G^{[p]}(\mathbb{R}^d)\big) \times L^\infty(\mathcal{F}_T) &\to H^\infty_{[0,T]} \times H^2_{[0,T]}, \\ (\boldsymbol{\zeta}, \xi) &\mapsto (Y, Z) \end{aligned}$$

is continuous.

The problem in showing convergence of the processes (Y^n, Z^n) in the statement of the theorem lies in the fact, that in general the Lipschitz constants for the corresponding BSDEs will tend to infinity as $n \to \infty$. It does not seem possible then, to directly control the solutions via a priori bounds, a standard tool in the theory of BSDEs (see, e.g., [10]). We will take another approach and transform the BSDEs corresponding to the smooth paths ζ^n into BSDEs which are easier to analyze.

---

[5] In a Brownian context, one can take $2 < p < 3$ and $G^{[p]}(\mathbb{R}^d) \cong \mathbb{R}^d \oplus so(d)$ is the state space for d-dimensional Brownian motion and it's Lévy area. More generally, $G^{[p]}(\mathbb{R}^d)$ is the "correct" state space for a geometric p-rough path; the space of such paths subject to p-variation regularity (in rough path sense) yields a complete metric space under p-variation rough path metric. Technical details of geometric rough path spaces (as found, e.g., in Section 9 of [13]) will not be necessary for the understanding of the present paper.

[6] The "integral" $\int H(X,Y)\,d\boldsymbol{\zeta}$ is *not* a rough integral defined in the usual rough path theory (e.g., [19] or [13]); regularity issues aside one misses the iterated integrals of X (and thus W) against those of ζ. For what it's worth, in the present context (5) can be taken as an implicit definition of $\int H(X,Y)\,d\boldsymbol{\zeta}$. (Somewhat similar in spirit: Föllmer's Itô integral which appears in his Itô formula *sans probabilité*.) More pragmatically, notation (5) is justified a posteriori through our uniqueness result; in addition it is consistent with standard BSDE notation when $\boldsymbol{\zeta}$ happens to be a smooth path.

We start by defining the flow (parametrized by x)

$$\phi(t,x,y) = y + \int_t^T \sum_{k=1}^d H_k(x,\phi(r,x,y))\,d\zeta^k(r). \tag{6}$$

Let ϕ^{-1} be the y-inverse of ϕ, then

$$\phi^{-1}(t,x,y) = y - \int_t^T \sum_{k=1}^d \partial_y\phi^{-1}(r,x,y) H_k(x,y)\,d\zeta^k(r).$$

We have the following lemma.

LEMMA 4. *Assume* (A1), (A2), (F1), (F2) *and let H be Lipschitz on $\mathbb{R}^n \times \mathbb{R}$. Let $\xi \in L^\infty(\mathcal{F}_T)$ and a smooth path ζ be given and let ϕ be the corresponding flow defined in* (6). *Let (Y,Z) be the unique solution to the BSDE with data* (ξ, f, H, ζ).

Then the process $(\tilde{Y},\tilde{Z})$ defined as

$$\tilde{Y}_t := \phi^{-1}(t,X_t,Y_t), \qquad \tilde{Z}_t := -\frac{\partial_x\phi(t,X_t,\tilde{Y}_t)}{\partial_y\phi(t,X_t,\tilde{Y}_t)}\sigma_t + \frac{1}{\partial_y\phi(t,X_t,\tilde{Y}_t)}Z_t,$$

satisfies the BSDE

$$\tilde{Y}_t = \xi + \int_t^T \tilde{f}(r,X_r,\tilde{Y}_r,\tilde{Z}_r)\,dr - \int_t^T \tilde{Z}_r\,dW_r, \tag{7}$$

where [*throughout, ϕ and all its derivatives will always be evaluated at* $(t,x,\tilde{y})$]

$$\tilde{f}(t,x,\tilde{y},\tilde{z}) := \frac{1}{\partial_y\phi}\biggl\{f(t,\phi,\partial_y\phi\tilde{z} + \partial_x\phi\sigma_t) + \langle\partial_x\phi, b_t\rangle + \frac{1}{2}\operatorname{Tr}[\partial_{xx}\phi\sigma_t\sigma_t^T] \\ + \langle\tilde{z}, (\partial_{xy}\phi\sigma_t)^T\rangle + \frac{1}{2}\partial_{yy}\phi|\tilde{z}|^2\biggr\}.$$

REMARK 5. This ("Doss-Sussman") transformation is well known and has been recently applied to BDSDEs [3] and rough partial differential equations [12]. We include details for the reader's convenience.

PROOF OF LEMMA 4. Denoting $\psi := \phi^{-1}$ and $\theta_r := (r, X_r, Y_r)$, we have by Itô's formula

$$\psi(t,X_t,Y_t) = \xi - \int_t^T \sum_{k=1}^d \partial_y\psi(\theta_r)H_k(X_r,Y_r)\dot{\zeta}^k(r)\,dr - \int_t^T \langle\partial_x\psi(\theta_r), b_r\rangle\,dr \\ - \int_t^T \langle\partial_x\psi(\theta_r), \sigma_r\,dW_r\rangle + \int_t^T \partial_y\psi(\theta_r) f(r,Y_r,Z_r)\,dr \\ + \int_t^T \sum_{k=1}^d \partial_y\psi(\theta_r)H_k(X_r,Y_r)\dot{\zeta}^k(r)\,dr - \int_t^T \partial_y\psi(\theta_r)Z_r\,dW_r$$

$$
\begin{aligned}
&- \frac{1}{2}\int_t^T \mathrm{Tr}[\partial_{xx}\psi(\theta_r)\sigma_r\sigma_r^T]\,dr - \frac{1}{2}\int_t^T \partial_{yy}\psi(\theta_r)|Z_r|^2\,dr \\
&- \int_t^T \langle \partial_{xy}\psi(\theta_r), \sigma_r Z_r^T\rangle\,dr \\
= \xi + &\int_t^T \bigg[\partial_y\psi(\theta_r) f(r, Y_r, Z_r) - \langle \partial_x\psi(\theta_r), b_r\rangle \\
&\qquad - \frac{1}{2}\mathrm{Tr}[\partial_{xx}\psi(\theta_r)\sigma_r\sigma_r^T] - \frac{1}{2}\partial_{yy}\psi(\theta_r)|Z_r|^2 \\
&\qquad\qquad - \langle \partial_{xy}\psi(\theta_r), \sigma_r Z_r^T\rangle\bigg]\,dr \\
&- \int_t^T \langle \partial_x\psi(\theta_r)\sigma_r + \partial_y\psi(\theta_r)Z_r, dW_r\rangle.
\end{aligned}
$$

Now, by deriving the identity $\psi(t, x, \phi(t, x, \tilde{y})) = \tilde{y}$ we get

$$
\begin{aligned}
0 &= \partial_x\psi + \partial_y\psi\partial_x\phi, \\
0 &= \partial_{xx}\psi + \partial_{yx}\psi \otimes \partial_x\phi + [\partial_{xy}\psi + \partial_{yy}\psi\partial_x\phi] \otimes \partial_x\phi + \partial_y\psi\partial_{xx}\phi \\
&= \partial_{xx}\psi + 2\partial_{xy}\psi \otimes \partial_x\phi + \partial_{yy}\psi\partial_x\phi \otimes \partial_x\phi + \partial_y\psi\partial_{xx}\phi, \\
1 &= \partial_y\psi\partial_y\phi, \\
0 &= \partial_{xy}\psi\partial_y\phi + \partial_{yy}\psi\partial_x\phi\partial_y\phi + \partial_y\psi\partial_{xy}\phi, \\
0 &= \partial_{yy}\psi(\partial_y\phi)^2 + \partial_y\psi\partial_{yy}\phi.
\end{aligned}
$$

And hence,

$$
\begin{aligned}
\partial_{yy}\psi &= -\frac{\partial_{yy}\phi}{(\partial_y\phi)^3}, \qquad \partial_x\psi = -\frac{\partial_x\phi}{\partial_y\phi}, \\
\partial_{xy}\psi &= \frac{\partial_{yy}\phi}{(\partial_y\phi)^3}\partial_x\phi - \frac{\partial_{xy}\phi}{(\partial_y\phi)^2}, \\
\partial_{xx}\psi &= 2\bigg[\frac{\partial_{yy}\phi}{(\partial_y\phi)^3}\partial_x\phi - \frac{\partial_{xy}\phi}{(\partial_y\phi)^2}\bigg] \otimes \partial_x\phi \\
&\qquad + \frac{\partial_{xx}\phi}{(\partial_y\phi)^3}\partial_x\phi \otimes \partial_x\phi - \frac{1}{\partial_y\phi}\partial_{xx}\phi.
\end{aligned}
$$

If we define

$$
\begin{aligned}
\tilde{Y}_t &:= \psi(t, X_t, Y_t) = \phi^{-1}(t, X_t, Y_t), \\
\tilde{Z}_t &:= \partial_x\psi(t, X_t, Y_t)\sigma_t + \partial_y\psi(t, X_t, Y_t)Z_t \\
&= -\frac{\partial_x\phi(t, X_t, \tilde{Y}_t)}{\partial_y\phi(t, X_t, \tilde{Y}_t)}\sigma_t + \frac{1}{\partial_y\phi(t, X_t, \tilde{Y}_t)}Z_t,
\end{aligned}
$$

and [ψ and its derivatives are always evaluated at $(t, x, \phi(t, x, \tilde{y}))$, ϕ and its derivatives are evaluated at $(t, x, \tilde{y})$]

$$\begin{aligned}
\tilde{f}(t, x, \tilde{y}, \tilde{z}) &:= \partial_y \psi f\left(t, \phi, \partial_y \phi\left(\tilde{z} + \frac{\partial_x \phi \sigma_t}{\partial_y \phi}\right)\right) - \langle \partial_x \psi, b_t \rangle - \frac{1}{2} \mathrm{Tr}[\partial_{xx} \psi \sigma_t \sigma_t^T] \\
&\qquad - \frac{1}{2} \partial_{yy} \psi \left| \frac{\tilde{z} - \partial_x \psi \sigma_t}{\partial_y \psi} \right|^2 - \left\langle \partial_{xy} \psi, \sigma_t \left(\frac{\tilde{z} - \partial_x \psi \sigma_t}{\partial_y \psi} \right)^T \right\rangle \\
&= \frac{1}{\partial_y \phi} \biggl\{ f(t, \phi, \partial_y \phi \tilde{z} + \partial_x \phi \sigma_t) + \langle \partial_x \phi, b_t \rangle + \frac{1}{2} \mathrm{Tr}[\partial_{xx} \phi \sigma_t \sigma_t^T] \\
&\qquad + \langle \tilde{z}, (\partial_{xy} \phi \sigma_t)^T \rangle + \frac{1}{2} \partial_{yy} \phi |\tilde{z}|^2 \biggr\},
\end{aligned}$$

we therefore obtain

$$\tilde{Y}_t = \xi + \int_t^T \tilde{f}(r, x, \tilde{Y}_r, \tilde{Z}_r)\, dr - \int_t^T \tilde{Z}_r\, dW_r. \qquad \square$$

DEFINITION 6. We call equation (7) *BSDE with data* $(\xi, \tilde{f}, 0, 0)$.

The BSDE (4) only makes sense for a smooth path ζ. On the other hand, equation (6) yields a flow of diffeomorphisms for a general geometric rough path $\boldsymbol{\zeta} \in C^{0,p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$, $p \geq 1$. Hence, we can, also in this case, consider the function $\tilde{f}$ from the previous lemma. We now record important properties for this induced function.

LEMMA 7. *Let* $p \geq 1$, $\boldsymbol{\zeta} \in C^{0,p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$ *and* $\gamma > p$. *Assume* (A1), (A2), (F1), (F2) *and* $(H_{p,\gamma})$. *Let* ϕ *be the flow corresponding to equation* (6) (*now solved as a rough differential equation*). *Then the function*

$$\begin{aligned}
\tilde{f}(t, x, \tilde{y}, \tilde{z}) &:= \frac{1}{\partial_y \phi} \biggl\{ f(t, \phi, \partial_y \phi \tilde{z} + \partial_x \phi \sigma_t) + \langle \partial_x \phi, b_t \rangle \\
&\qquad + \frac{1}{2} \mathrm{Tr}[\partial_{xx} \phi \sigma_t \sigma_t^T] + \langle \tilde{z}, (\partial_{xy} \phi \sigma_t)^T \rangle + \frac{1}{2} \partial_{yy} \phi |\tilde{z}|^2 \biggr\}
\end{aligned} \tag{8}$$

satisfies the following properties:

- *There exists a constant* $\tilde{C}_{1,f} > 0$ *depending only on* C_σ, C_b, $C_{1,f}$, C_H *and* $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ *such that*
$$|\tilde{f}(t, x, \tilde{y}, \tilde{z})| \leq \tilde{C}_{1,f} + \tilde{C}_{1,f} |\tilde{z}|^2, \qquad |\partial_{\tilde{z}} \tilde{f}(t, x, \tilde{y}, \tilde{z})| \leq \tilde{C}_{1,f} + \tilde{C}_{1,f} |\tilde{z}|.$$
- *There exists a constant* $\tilde{C}_{2,f} > 0$ *that only depends on* C_σ, C_b, $C_{2,f}$, C_H *and* $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ *such that for every* ε *there exists an* $h_\varepsilon > 0$ *that only depends on* C_σ, C_b, C_H *and* $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ *such that on* $[T - h_\varepsilon, T]$ *we have*
$$\partial_{\tilde{y}} \tilde{f}(t, x, \tilde{y}, \tilde{z}) \leq \tilde{C}_{2,f} + \varepsilon |\tilde{z}|^2.$$

PROOF. (i) Note that

$$\begin{aligned}
|\tilde{f}(t,x,\tilde{y},\tilde{z})| &\le \left|\frac{1}{\partial_y\phi}\right|\Big(|f(t,\phi,\partial_y\phi\tilde{z}+\partial_x\phi\sigma_t)| + |\langle\partial_x\phi, b_t\rangle| \\
&\qquad + \left|\frac{1}{2}\operatorname{Tr}[\partial_{xx}\phi\sigma_t\sigma_t^T]\right| + |\langle\tilde{z},(\partial_{xy}\phi\sigma_t)^T\rangle| + \left|\frac{1}{2}\partial_{yy}\phi\right||\tilde{z}|^2\Big) \\
&\le \left|\frac{1}{\partial_y\phi}\right|\Big(C_{1,f} + C_{1,f}|\partial_y\phi\tilde{z}+\partial_x\phi\sigma_t|^2 + |\partial_x\phi||b_t| \\
&\qquad + \frac{1}{2}|\partial_{xx}\phi||\sigma_t\sigma_t^T| + |\tilde{z}||\partial_{xy}\phi\sigma_t| + \frac{1}{2}|\partial_{yy}\phi||\tilde{z}|^2\Big) \\
&\le \left|\frac{1}{\partial_y\phi}\right|\Big(C_{1,f} + C_{1,f}2(|\partial_y\phi|^2|\tilde{z}| + |\partial_x\phi||\sigma_t^T|) + |\partial_x\phi||b_t| \\
&\qquad + \frac{1}{2}|\partial_{xx}\phi||\sigma_t|^2 + |\tilde{z}||\partial_{xy}\phi||\sigma_t^T| + \frac{1}{2}|\partial_{yy}\phi||\tilde{z}|^2\Big) \\
&\le \tilde{C}_{1,f} + \tilde{C}_{1,f}|\tilde{z}|^2.
\end{aligned}$$

Here we have used (A1), (A2) and (F1). For the boundedness of the flow and its derivatives, we have used Lemma B.1. Note that $\tilde{C}_{1,f}$ hence only depends on C_σ, C_b, $C_{1,f}$, C_H and $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$.

(ii) Note that

$$\begin{aligned}
&|\partial_{\tilde{z}}\tilde{f}(t,x,\tilde{y},\tilde{z})| \\
&\quad = \left|\partial_z f(t,\phi,\partial_y\phi\tilde{z}+\partial_x\phi\sigma_t) + \frac{1}{\partial_y\phi}(\partial_{xy}\phi\sigma_t + \partial_{yy}\phi\tilde{z})\right| \\
&\quad \le C_{1,f} + C_{1,f}(|\partial_y\phi||\tilde{z}| + |\partial_x\phi||\sigma_t|) + \left|\frac{\partial_{xy}\phi}{\partial_y\phi}\right||\sigma_t| + \left|\frac{\partial_{yy}\phi}{\partial_y\phi}\right||\tilde{z}| \\
&\quad \le \tilde{C}_{1,f} + \tilde{C}_{1,f}|\tilde{z}|.
\end{aligned}$$

Here we have used (A1), (A2) and (F1). For the boundedness of the flow and its derivatives, we have used Lemma B.1. Note that again, $\tilde{C}_{1,f}$ hence only depends on C_σ, C_b, $C_{1,f}$, C_H and $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$. Without loss of generality, we can choose it to be the same constant as in the estimate for (i).

(iii) Note that

$$\begin{aligned}
&\partial_{\tilde{y}}\tilde{f}(t,x,\tilde{y},\tilde{z}) \\
&\quad = -\frac{\partial_{yy}\phi}{(\partial_y\phi)^2}\Big\{f(t,\phi,\partial_y\phi\tilde{z}+\partial_x\phi\sigma_t) + \langle\partial_x\phi, b_t\rangle \\
&\qquad\qquad + \frac{1}{2}\operatorname{Tr}[\partial_{xx}\phi\sigma_t\sigma_t^T] + \langle\tilde{z},(\partial_{xy}\phi\sigma_t)^T\rangle + \frac{1}{2}\partial_{yy}\phi|\tilde{z}|^2\Big\}
\end{aligned}$$

$$+\frac{1}{\partial_y \phi}\Big\{\partial_y \phi \partial_y f(t,\phi,\partial_y\phi\tilde{z}+\partial_x\phi\sigma_t)+\langle\partial_{yx}\phi,b_t\rangle$$
$$+\frac{1}{2}\operatorname{Tr}[\partial_{yxx}\phi\sigma_t\sigma_t^T]+\langle\tilde{z},(\partial_{yxy}\phi\sigma_t)^T\rangle+\frac{1}{2}\partial_{yyy}\phi|\tilde{z}|^2\Big\}.$$

Hence, using our assumptions on f, we get

$$\begin{aligned}
\partial_{\tilde{y}}\tilde{f}(t,x,\tilde{y},\tilde{z}) \le{}& \left|\frac{\partial_{yy}\phi}{(\partial_y\phi)^2}\right|\Big\{C_{2,f}+C_{2,f}|\partial_y\phi\tilde{z}+\partial_x\phi\sigma_t|^2+|\partial_x\phi||b_t| \\
&\qquad+\frac{1}{2}|\partial_{xx}\phi||\sigma_t|^2+|\tilde{z}||\partial_{xy}\phi||\sigma_t|+\frac{1}{2}|\partial_{yy}\phi||\tilde{z}|^2\Big\} \\
&+\partial_y f(t,\phi,\partial_y\phi\tilde{z}+\partial_x\phi\sigma_t) \\
&+\frac{1}{\partial_y\phi}\Big\{|\partial_{yx}\phi||b_t|+\frac{1}{2}|\partial_{yxx}\phi||\sigma_t| \\
&\qquad+(1+|\tilde{z}|^2)|\partial_{yxy}\phi||\sigma_t|_{op}+\frac{1}{2}\partial_{yyy}\phi|\tilde{z}|^2\Big\} \\
\le{}& \left|\frac{\partial_{yy}\phi}{(\partial_y\phi)^2}\right|\Big\{C_{2,f}+C_{2,f}2|\partial_x\phi|^2|\sigma_t|^2+|\partial_x\phi||b_t| \\
&\qquad+\frac{1}{2}|\partial_{xx}\phi||\sigma_t|^2+|\partial_{xy}\phi||\sigma_t|\Big\} \\
&+\partial_y f(t,\phi,\partial_y\phi\tilde{z}+\partial_x\phi\sigma_t) \\
&+\frac{1}{\partial_y\phi}\Big\{|\partial_{yx}\phi||b_t|+\frac{1}{2}|\partial_{yxx}\phi||\sigma_t|+|\partial_{yxy}\phi||\sigma_t|\Big\} \\
&+\Big\{\left|\frac{\partial_{yy}\phi}{(\partial_y\phi)^2}\right|C_{2,f}2|\partial_y\phi|^2+\left|\frac{\partial_{yy}\phi}{(\partial_y\phi)^2}\right||\partial_{xy}\phi||\sigma_t| \\
&\qquad+\left|\frac{\partial_{yy}\phi}{(\partial_y\phi)^2}\right|\frac{1}{2}|\partial_{yy}\phi|+\frac{1}{\partial_y\phi}|\partial_{yxy}\phi||\sigma_t^T|+\frac{1}{\partial_y\phi}\frac{1}{2}\partial_{yyy}\phi\Big\}|\tilde{z}|^2 \\
\le{}& \tilde{C}_{2,f}+\Big\{\left|\frac{\partial_{yy}\phi}{(\partial_y\phi)^2}\right|C_{2,f}2|\partial_y\phi|^2+\left|\frac{\partial_{yy}\phi}{(\partial_y\phi)^2}\right||\partial_{xy}\phi||\sigma_t| \\
&\qquad+\left|\frac{\partial_{yy}\phi}{(\partial_y\phi)^2}\right|\frac{1}{2}|\partial_{yy}\phi|+\frac{1}{\partial_y\phi}|\partial_{yxy}\phi||\sigma_t^T|+\frac{1}{\partial_y\phi}\frac{1}{2}\partial_{yyy}\phi\Big\} \\
&\qquad\times|\tilde{z}|^2,
\end{aligned}$$

where $\tilde{C}_{2,f}$ only depends on C_σ, C_b, C_H and $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ (here we have used Lemma B.1 to bound the flow and its derivatives).

By (A1), (A2) σ and b are bounded. Then, by the properties of the flow, the term in front of $|\tilde{z}|^2$ goes uniformly to zero as t approaches T. To be specific:

using $(H_{p,\gamma})$ we obtain, again by Lemma B.1, that for every $\varepsilon > 0$ there exists an $h_\varepsilon > 0$, depending on C_σ, C_b, C_H and $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ such that on $[T - h_\varepsilon, T]$ we have

$$\partial_{\tilde{y}} \tilde{f}(t, x, \tilde{y}, \tilde{z}) \le \tilde{C}_{2,f} + \varepsilon|\tilde{z}|^2. \qquad \square$$

We are now ready to prove Theorem 3.

PROOF OF THEOREM 3. For the sake of unified notation, we write (Y^0, Z^0) for the (rough BSDE) solution (Y, Z); similarly, we write $\boldsymbol{\zeta}^0$ for the rough path $\boldsymbol{\zeta}$.

1. *Existence*

Let $\phi^n, n \ge 0$ be the (ODE, for $n \ge 1$ and RDE, when $n = 0$) solution flow (parametrized by x)

$$\phi^n(t, x, y) = y + \int_t^T H(x, \phi^n(r, x, y))\, d\zeta^n(r).$$

By Lemma B.1, we have for all $n \ge 0$, $x \in \mathbb{R}^n$, that $\phi^n(t, x, \cdot)$ is a flow of C^3-diffeomorphisms. Let $\psi^n(t, x, \cdot)$ be its y-inverse. We have that $\phi^n(t, \cdot, \cdot)$ and its derivatives up to order three are bounded (Lemma B.1). The same holds true for $\psi^n(t, \cdot, \cdot)$ and its derivatives up to order three.

Moreover, by Lemma B.2 we have that locally uniformly on $[0, T] \times \mathbb{R}^n \times \mathbb{R}$

$$\begin{aligned}
&\left(\phi^n, \frac{1}{\partial_y \phi^n}, \partial_y \phi^n, \partial_{yy}\phi^n, \partial_x \phi^n, \partial_{xx}\phi^n, \partial_{yx}\phi^n\right) \\
&\qquad \to \left(\phi^0, \frac{1}{\partial_y \phi^0}, \partial_y \phi^0, \partial_{yy}\phi^0, \partial_x \phi^0, \partial_{xx}\phi^0, \partial_{yx}\phi^0\right).
\end{aligned} \tag{9}$$

Denote for $n \ge 0$ the function

$$\begin{aligned}
\tilde{f}^n(t, x, \tilde{y}, \tilde{z}) := \frac{1}{\partial_y \phi^n} \Big\{ & f(t, \phi^n, \partial_y \phi^n \tilde{z} + \partial_x \phi^n \sigma_t) + \langle \partial_x \phi^n, b_t\rangle \\
& + \frac{1}{2}\operatorname{Tr}[\partial_{xx}\phi^n \sigma_t \sigma_t^T] + \langle \tilde{z}, (\partial_{xy}\phi^n \sigma_t)^T\rangle + \frac{1}{2}\partial_{yy}\phi^n |\tilde{z}|^2 \Big\}.
\end{aligned}$$

Now, we have seen above that for $n \ge 1$, the process

$$\begin{aligned}
(\tilde{Y}^n, \tilde{Z}^n) &:= L^n(Y^n, Z^n) \\
&:= \bigg((\phi^n)^{-1}(\cdot, X_\cdot, Y^n_\cdot), \\
&\qquad - \frac{\partial_x \phi^n(\cdot, X_\cdot, (\phi^n)^{-1}(\cdot, X_\cdot, Y^n_\cdot))}{\partial_y \phi^n(\cdot, X_\cdot, (\phi^n)^{-1}(\cdot, X_\cdot, Y^n_\cdot))}\sigma_\cdot \\
&\qquad + \frac{1}{\partial_y \phi^n(\cdot, X_\cdot, (\phi^n)^{-1}(\cdot, X_\cdot, Y^n_\cdot))} Z^n_\cdot \bigg)
\end{aligned}$$

solves the BSDE with data $(\xi, \tilde{f}^n, 0, 0)$.

Note that although $(\xi, \tilde{f}^n, 0, 0)$ is a quadratic BSDE, existence and uniqueness of a solution are guaranteed for $n \geq 1$ by the fact that the mapping L^n is one to one and by the existence of a unique solution to the untransformed BSDE (Lemma 2).

For $n = 0$, using the good properties of $\tilde{f}^0$ demonstrated in Lemma 7, there exists a solution $(\tilde{Y}^0, \tilde{Z}^0) \in H^\infty_{[0,T]} \times H^2_{[0,T]}$ to the BSDE with data $(\xi, \tilde{f}^0, 0, 0)$ by Theorem 2.3 in [15]. Note that it is a priori not unique, but we will show that it is at least unique on a small time interval up to T.

We now construct the process (Y^0, Z^0) of the statement on subintervals of $[0, T]$. First of all, notice that we can choose the constant $\tilde{C}_{1,f}$ appearing in Lemma 7 uniformly for all $n \geq 0$. Let $M := \|\xi\|_\infty + T\tilde{C}_{1,f}$. By Corollary 2.2 in [15], we have

$$\|\tilde{Y}^n\|_\infty \leq M, \qquad n \geq 0. \tag{10}$$

Now by Lemma 7:

- There exists $\tilde{C}_{1,f} > 0$ that only depends on C_σ, C_b, $C_{1,f}$, C_H and $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ such that
$$|\tilde{f}^0(t, x, y, z)| \leq \tilde{C}_{1,f} + \tilde{C}_{1,f}|z|^2,$$
$$|\partial_z \tilde{f}^0(t, x, y, z)| \leq \tilde{C}_{1,f} + \tilde{C}_{1,f}|z|.$$
- There exists a constant $\tilde{C}_{2,f} > 0$ that only depends on C_σ, C_b, $C_{2,f}$, C_H and $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ such that for every ε there exists an $h_\varepsilon > 0$ that only depends on C_σ, C_b, C_H and $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ such that on $[T - h_\varepsilon, T]$ we have
$$\partial_y \tilde{f}^0(t, x, y, z) \leq \tilde{C}_{2,f} + \varepsilon|z|^2.$$

Hence, we can choose $h = h_{\delta(\tilde{C}_{1,f}, M)}$, such that for $t \in [T - h, T]$ we have

$$\partial_y \tilde{f}(t, x, y, z) \leq \tilde{C}_{2,f} + \delta(\tilde{C}_{1,f}, M)|z|^2.$$

Here δ is the universal function given in the statement of Theorem A.2. We can then apply Theorem A.2 to get uniqueness of our solution $(\tilde{Y}^0, \tilde{Z}^0)$ on $[T - h, T]$. Now, as a consequence of (9) we have

$$\tilde{f}^n \to \tilde{f}^0 \qquad \text{uniformly on compacta.}$$

Hence, by the argument of Theorem 2.8 in [15] we have that

$$\begin{aligned} &\tilde{Y}^n \to \tilde{Y}^0 \qquad \text{uniformly on } [T - h, T]\ \mathbb{P}\text{-a.s.}, \\ &\tilde{Z}^n \to \tilde{Z}^0 \qquad \text{in } H^2_{[T-h,T]}. \end{aligned} \tag{11}$$

Moreover, if we define

$$Y^0_t := \phi^0(t, X_t, \tilde{Y}^0_t), \qquad t \in [T - h, T],$$

$$Z^0_t := \partial_y \phi^0(t, X_t, \tilde{Y}^0_t)\left[\tilde{Z}^0_t + \frac{\partial_x \phi^0(t, X_t, \tilde{Y}^0_t)}{\partial_y \phi^0(t, X_t, \tilde{Y}^0_t)}\sigma_t\right], \qquad t \in [T - h, T],$$

and remembering that by construction

$$Y_t^n = \phi^n(t, X_t, \tilde{Y}_t^n),$$
$$Z_t^n = \partial_y \phi^n(t, X_t, \tilde{Y}_t^n)\Big[\tilde{Z}_t^n + \frac{\partial_x \phi^n(t, X_t, \tilde{Y}_t^n)}{\partial_y \phi^n(t, X_t, \tilde{Y}_t^n)} \sigma_t\Big],$$

and using (9) we get

$$\begin{aligned} &Y^n \to Y^0 \qquad \text{uniformly on } [T-h, T]\ \mathbb{P}\text{-a.s.}, \\ &Z^n \to Z^0 \qquad \text{in } H^2_{[T-h,T]}. \end{aligned} \tag{12}$$

Let us proceed to the next subinterval. To make the rough path disappear in the BSDE, we will use a similar transformation via a flow as above. As before we need to control the driver of the transformed BSDE, this time near $T - h$. For this reason, we have to start the flow anew. First, we rewrite the BSDEs for $n \geq 1$ as

$$Y_t^n = Y_{T-h}^n + \int_t^T f(r, Y_r^n, Z_r^n)\, dr - \int_t^{T-h} H(X_r, Y_r^n)\, d\zeta_r^n - \int_t^{T-h} Z_r^n\, dW_r.$$

Then define the flow $\phi^{n,T-h}$ started at time $T - h$, that is,

$$\phi^{n,T-h}(t, x, y) = y + \int_t^{T-h} H(x, \phi^{n,T-h}(r, x, y))\, d\zeta^n(r), \qquad t \leq T - h.$$

On $[0, T - h]$ define

$$\begin{aligned} (\tilde{Y}_\cdot^{n,T-h}, \tilde{Z}_\cdot^{n,T-h}) := \Big(&(\phi^{n,T-h})^{-1}(\cdot, X_\cdot, Y_\cdot^n), \\ &- \frac{\partial_x \phi^{n,T-h}(\cdot, X_\cdot, (\phi^{n,T-h})^{-1}(\cdot, X_\cdot, Y_\cdot^n))}{\partial_y \phi^{n,T-h}(\cdot, X_\cdot, (\phi^{n,T-h})^{-1}(\cdot, X_\cdot, Y_\cdot^n))} \sigma_\cdot \\ &+ \frac{1}{\partial_y \phi^{n,T-h}(\cdot, X_\cdot, (\phi^{n,T-h})^{-1}(\cdot, X_\cdot, Y_\cdot^n))} Z_\cdot^n \Big). \end{aligned}$$

Then

$$\tilde{Y}_t^{n,T-h} = Y_{T-h}^n + \int_t^{T-h} \tilde{f}^{n,T-h}(r, X_r, \tilde{Y}_r^{n,T-h}, \tilde{Z}_r^{n,T-h})\, dr - \int_t^{T-h} \tilde{Z}_r^{n,T-h}\, dW_r,$$

where

$$\begin{aligned} \tilde{f}^{n,T-h}(t, x, \tilde{y}, \tilde{z}) := \frac{1}{\partial_y \phi^{n,T-h}} \Big\{ &f(t, \phi^{n,T-h}, \partial_y \phi^{n,T-h} \tilde{z} + \partial_x \phi^{n,T-h} \sigma_t) \\ &+ \langle \partial_x \phi^{n,T-h}, b_t \rangle + \frac{1}{2} \operatorname{Tr}[\partial_{xx} \phi^{n,T-h} \sigma_t \sigma_t^T] \\ &+ \langle \tilde{z}, (\partial_{xy} \phi^{n,T-h} \sigma_t)^T \rangle + \frac{1}{2} \partial_{yy} \phi^{n,T-h} |\tilde{z}|^2 \Big\}. \end{aligned}$$

This BSDE is also defined for $n = 0$ and as before we get via Lemma 7 for the same h and the same $\tilde{C}_{1,f}$ and $\tilde{C}_{2,f}$ as before (here the explicit dependence of these constants is crucial), that on $[T - 2h, T - h]$ we have

$$\partial_y \tilde{f}^{0,T-h}(t, x, y, z) \le \tilde{C}_{2,f} + \delta(\tilde{C}_{1,f}, M)|z|^2.$$

Hence, we can apply Comparison Theorem A.2 to get uniqueness of our solution $(\tilde{Y}^{0,T-h}, \tilde{Z}^{0,T-h})$ on $[T - 2h, T - h]$. Now, also note that for the terminal value we have from (12) and (10)

$$\begin{aligned} Y^n_{T-h} &\to Y^0_{T-h} \qquad \mathbb{P}\text{-a.s.}, \\ |Y^n_{T-h}| &\le M, \qquad n \ge 1. \end{aligned}$$

Hence, again by the argument of Theorem 2.8 in [15][7]

$$\begin{aligned} &\tilde{Y}^{n,T-h} \to \tilde{Y}^{0,T-h} && \text{uniformly on } [T-2h, T-h]\ \mathbb{P}\text{-a.s.}, \\ &\tilde{Z}^{n,T-h} \to \tilde{Z}^{0,T-h} && \text{in } H^2_{[T-2h,T-h]}. \end{aligned}$$

Finally, reversing the transformation, we get as above

$$\begin{aligned} &Y^n \to Y^0 && \text{uniformly on } [T-2h, T-h]\ \mathbb{P}\text{-a.s.}, \\ &Z^n \to Z^0 && \text{in } H^2_{[T-2h,T-h]}. \end{aligned}$$

Then, we can iterate this procedure on suberintervals of length h up to time 0. Without loss of generality, we can assume that $T = Nh$ for an $N \in \mathbb{N}$. Then patching the results together we get

$$\sup_{t \le T} |Y^n_t - Y^0_t| \le \sum_{k=1}^{N} \sup_{(k-1)h \le t \le kh} |Y^n_t - Y^0_t| \to 0 \qquad \mathbb{P}\text{-a.s.}$$

and

$$\mathbb{E}\left[\int_0^T |Z^n_r - Z^0_r|^2\, dr\right] = \sum_{k=1}^{N} \mathbb{E}\left[\int_{(k-1)h}^{kh} |Z^n_r - Z^0_r|^2\, dr\right] \to 0.$$

[7] Note that Theorem 2.8 in [15] demands convergence in L^∞ of the terminal value. A closer look at the proof though, reveals that $\mathbb{P}$-a.s. convergence combined with a uniform deterministic bound (M in our case) is enough. To be specific: the convergence of the terminal value is only used at two instances for Theorem 2.8 and this is in the proof of Proposition 2.4 (which is the main ingredient for Theorem 2.8). First, it is used on page 568, right before Step 2 where it reads "By Lebesgue's dominated …". Second, it is used on page 570, before the end of the proof where it reads "from which we deduce that …". In both cases, the above stated requirement is enough.

2. *Uniqueness*

Let $\bar{\zeta}^n, n \geq 1$ be another sequence of smooth paths that converges to $\boldsymbol{\zeta}$ in p-variation. Let $(\bar{Y}^n, \bar{Z}^n)$ be the solutions to BSDEs with data $(\xi, f, H, \bar{\zeta}^n)$. Then as above

$$\begin{aligned}\tilde{\bar{Y}}^n &\to \tilde{Y}^0 && \text{uniformly on } [T-h, T]\ \mathbb{P}\text{-a.s.},\\ \tilde{\bar{Z}}^n &\to \tilde{Z}^0 && \text{in } H^2_{[T-h,T]}.\end{aligned}$$

And hence,

$$\begin{aligned}\bar{Y}^n &\to Y^0 && \text{uniformly on } [T-h, T]\ \mathbb{P}\text{-a.s.},\\ \bar{Z}^n &\to Z^0 && \text{in } H^2_{[T-h,T]}.\end{aligned}$$

Note that the choice of h in the proof of existence only depended on properties of the limiting function $\tilde{f}^0$, so we can use the same value here. One can now iterate this argument up to time 0 to get

$$\begin{aligned}\bar{Y}^n &\to Y^0 && \text{uniformly on } [0, T]\ \mathbb{P}\text{-a.s.},\\ \bar{Z}^n &\to Z^0 && \text{in } H^2_{[0,T]},\end{aligned}$$

as desired.

3. *Continuity of the solution map*

We note that for a given $B > 0$, all terminal values ξ such that $|\xi| \leq B$ and all geometric p-rough paths with $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]} \leq B$ we can choose an $h = h(B) > 0$ such that the above constructed unique solution (Y^0, Z^0) to the BSDE (5) is given by

$$Y_t^0 = \begin{cases} \phi^{0,T}(t, X_t, \tilde{Y}_t^T), & t \in [T-h, T],\\ \phi^{0,T-h}(t, X_t, \tilde{Y}_t^{T-h}), & t \in [T-2h, T-h],\\ \dots & \\ \phi^{0,h}(t, X_t, \tilde{Y}_t^h), & t \in [0, h],\end{cases}$$

$$Z_t^0 = \begin{cases} \partial_y \phi^{0,T}(t, X_t, \tilde{Y}_t^{0,T})\Big[\tilde{Z}_t^{0,T} + \dfrac{\partial_x \phi^{0,T}(t, X_t, \tilde{Y}_t^{0,T})}{\partial_y \phi^0(t, X_t, \tilde{Y}_t^{0,T})}\sigma_t\Big], \\ \qquad t \in [T-h, T],\\ \partial_y \phi^{0,T-h}(t, X_t, \tilde{Y}_t^{0,T-h})\Big[\tilde{Z}_t^{0,T-h} + \dfrac{\partial_x \phi^{0,T-h}(t, X_t, \tilde{Y}_t^{0,T-h})}{\partial_y \phi^{0,T-h}(t, X_t, \tilde{Y}_t^{0,T-h})}\sigma_t\Big], \\ \qquad t \in [T-2h, T-h],\\ \dots \\ \partial_y \phi^{0,h}(t, X_t, \tilde{Y}_t^{0,h})\Big[\tilde{Z}_t^{0,h} + \dfrac{\partial_x \phi^{0,h}(t, X_t, \tilde{Y}_t^{0,h})}{\partial_y \phi^{0,h}(t, X_t, \tilde{Y}_t^{0,h})}\sigma_t\Big], \\ \qquad t \in [0, h],\end{cases}$$

where we used the unique solutions to the following BSDEs:

$$
\begin{aligned}
\tilde{Y}_t^{0,T} &= \xi + \int_t^T \tilde{f}^{0,T}(r, X_r, \tilde{Y}_r^{0,T}, \tilde{Z}_r^{0,T})\, dr - \int_t^T \tilde{Z}_r^{0,T}\, dW_r, \\
\tilde{Y}_t^{0,T-h} &= \phi^{0,T}(T-h, X_{T-h}, \tilde{Y}_{T-h}^{0,T}) + \int_t^{T-h} \tilde{f}^{0,T-h}(r, X_r, \tilde{Y}_r^{0,T-h}, \tilde{Z}_r^{0,T-h})\, dr \\
&\quad - \int_t^{T-h} \tilde{Z}_r^{0,T-h}\, dW_r, \\
&\dots \\
\tilde{Y}_t^{0,h} &= \phi^{0,2h}(h, X_h, \tilde{Y}_h^{0,2h}) + \int_t^h \tilde{f}^{0,h}(r, X_r, \tilde{Y}_r^{0,h}, \tilde{Z}_r^{0,h})\, dr - \int_t^h \tilde{Z}_r^{0,h}\, dW_r.
\end{aligned}
$$

From this representation and stability results on BSDEs (Theorem 2.8 in [15]), it easily follows that the solution map

$$
C^{0,p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d)) \times L^\infty(\mathcal{F}_T) \to H^\infty_{[0,T]} \times H^2_{[0,T]}
$$

is continuous in balls of radius B. Since this is true for every $B > 0$ we get the desired result. □

3. The Markovian setting—connection to rough PDEs. We now specialize to a Markovian model. We are interested in solving the following forward backward stochastic differential equation for $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$:

$$
\begin{aligned}
X_t^{t_0,x_0} &= x_0 + \int_{t_0}^t \sigma(r, X_r^{t_0,x_0})\, dW_r + \int_{t_0}^t b(r, X_r^{t_0,x_0})\, dr, \qquad t \in [t_0, T], \\
Y_t^{t_0,x_0} &= g(X_T^{t_0,x_0}) + \int_t^T f(r, X_r^{t_0,x_0}, Y_r^{t_0,x_0}, Z_r^{t_0,x_0})\, dr \\
&\quad + \int_t^T H(X_r^{t_0,x_0}, Y_r^{t_0,x_0})\, d\boldsymbol{\zeta}_r - \int_t^T Z_r^{t_0,x_0}\, dW_r, \qquad t \in [t_0, T].
\end{aligned}
\tag{13}
$$

Here $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $f : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}$, $H = (H_1, \dots, H_d) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^d$, $\boldsymbol{\zeta} : [0,T] \to \mathbb{R}^d$ are continuous mappings, on which more assumptions will be presented later.

Assume for the moment that $\boldsymbol{\zeta}$ is actually a smooth path. Then (13) is connected to the PDE

$$
\begin{aligned}
&\partial_t u(t,x) + \tfrac{1}{2} \operatorname{Tr}[\sigma(t,x)\sigma(t,x)^T D^2 u(t,x)] + \langle b(t,x), Du(t,x)\rangle \\
&\qquad + f(t,x,u(t,x), Du(t,x)\sigma(t,x)) + H(x, u(t,x))\dot{\zeta}_t = 0, \\
&\qquad\qquad\qquad t \in [0,T), x \in \mathbb{R}^n, \\
&u(T,x) = g(x), \qquad x \in \mathbb{R}^n.
\end{aligned}
\tag{14}
$$

We will make this connection explicit after introducing the following adaption (and strengthening) of previous assumptions:

(MA1) There exists a constant $C_\sigma > 0$ such that for $(t, x) \in [0, T] \times \mathbb{R}^n$

$$|\sigma(t, x)| \le C_\sigma,$$
$$|\partial_{x_i} \sigma(t, x)| \le C_\sigma, \qquad i = 1, \dots, n.$$

(MA2) There exists a constant $C_b > 0$ such that for $(t, x) \in [0, T] \times \mathbb{R}^n$

$$|b(t, x)| \le C_b,$$
$$|\partial_x b(t, x)| \le C_b.$$

(MF1) There exists a constant $C_{1,f} > 0$ such that for $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$

$$|f(t, x, y, z)| \le C_{1,f},$$
$$|\partial_z f(t, x, y, z)| \le C_{1,f}.$$

(MF2) There exists a constant $C_{2,f} > 0$ such that such that for $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$

$$\partial_y f(t, x, y, z) \le C_{2,f}.$$

(MF3) There exists a constant $C_{3,f} > 0$ such that such that for $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$

$$\partial_x f(t, x, y, z) \le C_{3,f} + C_{3,f}|z|^2,$$

and f is uniformly continuous in x, uniformly in (t, y, z).

(MG1) g is bounded and uniformly continuous.

We again consider for a smooth (or rough) path ζ the flow (parametrized by x)

$$\phi(t, x, y) = y + \int_t^T \sum_{k=1}^d H_k(x, \phi(r, x, y))\, d\zeta^k(r). \tag{15}$$

In what follows, $BC([0, T] \times \mathbb{R}^n)$ [resp., $BC(\mathbb{R}^n)$] denotes the space of bounded continuous functions on $[0, T] \times \mathbb{R}^n$ (resp., $\mathbb{R}^n$) with the topology of uniform convergence on compacta. Similarly, $BUC([0, T] \times \mathbb{R}^n)$ [resp., $BUC(\mathbb{R}^n)$] denotes the space of bounded uniformly continuous functions on $[0, T] \times \mathbb{R}^n$ (resp., $\mathbb{R}^n$) with the topology of uniform convergence on compacta.

PROPOSITION 8. *Assume* (MA1), (MA2), (MF1), (MF2), (MF3), (MG1) *and let H be Lipschitz on $\mathbb{R}^n \times \mathbb{R}$. Let ζ be a smooth path. Then there exists a unique viscosity solution*[8] *to* (14) *in $BUC([0, T], \mathbb{R}^n)$.*

It is given by

$$u(t, x) := Y_t^{t,x},$$

where for every $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ the process $(Y^{t_0,x_0}, Z^{t_0,x_0})$ is the solution to (13).

[8]For an introduction to the theory of viscosity solutions, we refer the reader to [7].

PROOF. The fact that u is a bounded, uniformly continuous viscosity solution follows from Proposition 2.5 and Theorem 3.4 in [1]. Uniqueness of a bounded viscosity solution to (14) follows from Theorem C.1. Note, that since $\partial y f$ is bounded, we can choose $h_\varepsilon \equiv 0$ in the statement of the theorem and hence get uniqueness on the entire interval $(0, T]$. Every BUC function on $(0, T] \times \mathbb{R}^n$ has a unique extension to $[0, T] \times \mathbb{R}^n$. Hence, u is unique in $BUC([0, T], \mathbb{R}^n)$. $\square$

REMARK 9. It is also possible to show existence of a (unique) solution to (14) by purely deterministic methods; see, for example, Theorem 2 in [9].

Let now $\zeta^n, n = 1, 2, \ldots,$ be smooth paths in $\mathbb{R}^d$. Let $\gamma > p \geq 1$ and assume $\zeta^n \to \boldsymbol{\zeta}^0$ in p-variation, for a $\boldsymbol{\zeta}^0 \in C^{0,p\text{-var}}([0, T], G^{[p]}(\mathbb{R}^d))$. Assume (MA1), (MA2), (MF1), (MF2), (MF3), (MG1) and $(H_{p,\gamma})$, so that especially Theorem 3 holds true. It follows that the corresponding u^n (as given in Proposition 8) converge *pointwise* to some function u^0, that is,

$$u^n(t, x) \to u^0(t, x), \qquad t \in [0, T], x \in \mathbb{R}^n.$$

Again, the limiting function u^0 does not depend on the approximating sequence, but only on the limiting rough path $\boldsymbol{\zeta}^0$. We could hence define this u^0 to be the solution solution to (14). But it is not straightforward, via this approach, to show uniform convergence on compacta as well as continuity of the solution map. We hence work directly on the PDEs, as in [5] and [12]. First, we get the respective versions of Lemma 4 and Lemma 7.

LEMMA 10. *Assume* (MA1), (MA2), (MF1), (MF2), (MG1) *and let* $H(x, \cdot) = (H_1(x, \cdot), \ldots, H_d(x, \cdot))$ *be a collection of Lipschitz vector fields on* $\mathbb{R}$. *Let a smooth path* ζ *be given. Let* u *be the unique viscosity solution to* (14).

Then $v(t, x) := \phi^{-1}(t, x, u(t, x))$ *is a viscosity solution to*

$$\begin{aligned} &\partial_t v(t, x) + \tfrac{1}{2} \operatorname{Tr}[\sigma(t, x)\sigma(t, x)^T D^2 v(t, x)] + \langle b(t, x), Dv(t, x)\rangle \\ &\qquad + \tilde{f}(t, x, v(t, x), Dv(t, x)\sigma(t, x)) = 0, \qquad t \in [0, T), x \in \mathbb{R}^n, \\ &v(T, x) = g(x), \qquad x \in \mathbb{R}^n, \end{aligned}$$

where [*in what follows the* ϕ *will always be evaluated at* $(t, x, \tilde{y})$]

$$\begin{aligned} \tilde{f}(t, x, \tilde{y}, \tilde{z}) = \frac{1}{\partial_y \phi} \Big\{ & f\big(t, x, \phi, \partial_y \phi \tilde{z} + \partial_x \phi \sigma(t, x)\big) \\ & + \langle \partial_x \phi, b(t, x)\rangle + \frac{1}{2} \operatorname{Tr}[\partial_{xx} \phi \sigma(t, x)\sigma(t, x)^T] \\ & + \langle \tilde{z}, (\partial_{xy} \phi \sigma(t, x))^T\rangle + \frac{1}{2} \partial_{yy} \phi |\tilde{z}|^2 \Big\}. \end{aligned}$$

PROOF. This is an application of Lemma 5 in [12]. □

LEMMA 11. *Let* $p \geq 1$, $\boldsymbol{\zeta} \in C^{0,p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$ *and* $\gamma > p$. *Assume* (MA1), (MA2), (MF1), (MF2), (MF3), (MG1) *and* $(H_{p,\gamma})$. *Let* ϕ *be the flow corresponding to equation* (15) (*solved as a rough differential equation*). *Then*

$$\tilde{f}(t,x,\tilde{y},\tilde{z}) = \frac{1}{\partial_y \phi}\bigg\{ f\big(t,x,\phi,\partial_y\phi\tilde{z} + \partial_x\phi\sigma(t,x)\big) + \langle \partial_x\phi, b(t,x)\rangle + \frac{1}{2}\operatorname{Tr}[\partial_{xx}\phi\sigma(t,x)\sigma(t,x)^T] + \langle \tilde{z}, (\partial_{xy}\phi\sigma(t,x))^T\rangle + \frac{1}{2}\partial_{yy}\phi|\tilde{z}|^2\bigg\}$$

satisfies:

- *There exists a constant* $\tilde{C}_{1,f} > 0$ *depending only on* C_σ, C_b, $C_{1,f}$, C_H *and* $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ *such that for* $(t,x,\tilde{y},\tilde{z}) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m$
$$|\tilde{f}(t,x,\tilde{y},\tilde{z})| \leq \tilde{C}_{1,f} + \tilde{C}_{1,f}|\tilde{z}|^2,$$
$$|\partial_{\tilde{z}}\tilde{f}(t,x,\tilde{y},\tilde{z})| \leq \tilde{C}_{1,f} + \tilde{C}_{1,f}|\tilde{z}|.$$
- *There exists a constant* $\tilde{C}_{2,f} > 0$ *that only depends on* C_σ, C_b, $C_{2,f}$, C_H *and* $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ *such that for every* $\varepsilon > 0$ *there exists an* $h_\varepsilon > 0$ *that only depends on* C_σ, C_b, C_H *and* $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ *such that for* $(t,x,\tilde{y},\tilde{z}) \in [T-h_\varepsilon, T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m$
$$\partial_{\tilde{y}}\tilde{f}(t,x,\tilde{y},\tilde{z}) \leq \tilde{C}_{2,f} + \varepsilon|\tilde{z}|^2.$$
- *There exists a* $\tilde{C}_{3,f} > 0$ *that only depends on* C_σ, C_b, $C_{2,f}$, $C_{3,f}$, C_H *and* $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$ *such that for* $(t,x,\tilde{y},\tilde{z}) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m$
$$\partial_x\tilde{f}(t,x,\tilde{y},\tilde{z}) \leq \tilde{C}_{3,f} + \tilde{C}_{3,f}|\tilde{z}|^2.$$

PROOF. The first three inequalities follow as in Lemma 7. Now for $i \leq n$, we have

$$\begin{aligned}
&\partial_{x_i}\tilde{f}(t,x,\tilde{y},\tilde{z})\\
&\quad= -\partial_{x_i y}\phi\frac{1}{\partial_y\phi}\tilde{f}(t,x,\tilde{y},\tilde{z})\\
&\qquad+\frac{1}{\partial_y\phi}\Big[\partial_y f\big(t,x,\phi,\partial_y\phi\tilde{z}+\partial_x\phi\sigma(t,x)\big)\partial_{x_i}\phi\\
&\qquad\qquad+\partial_z f\big(t,x,\phi,\partial_y\phi\tilde{z}+\partial_x\phi\sigma(t,x)\big)\\
&\qquad\qquad\quad\times\big(\partial_{x_i y}\phi\tilde{z}+\partial_{x_i x}\phi\sigma(t,x)+\partial_x\phi\partial_{x_i}\sigma(t,x)\big)^T
\end{aligned}$$

$$
\begin{aligned}
&+ \langle \partial_{x_i x}\phi, b(t,x)\rangle + \langle \partial_x \phi, \partial_{x_i} b(t,x)\rangle \\
&+ \frac{1}{2}\operatorname{Tr}[\partial_{x_i xx}\phi\sigma(t,x)\sigma(t,x)^T] + \frac{1}{2}\operatorname{Tr}[\partial_{xx}\phi\partial_{x_i}\sigma(t,x)\sigma(t,x)^T] \\
&+ \frac{1}{2}\operatorname{Tr}[\partial_{xx}\phi\sigma(t,x)\partial_{x_i}\sigma(t,x)^T] + \langle \tilde{z}, (\partial_{x_i xy}\phi\sigma(t,x))^T\rangle \\
&\qquad\qquad + \langle \tilde{z}, (\partial_{xy}\phi\partial_{x_i}\sigma(t,x))^T\rangle + \frac{1}{2}\partial_{x_i yy}\phi|\tilde{z}|^2\Big].
\end{aligned}
$$

So

$$
\begin{aligned}
&|\partial_{x_i}\tilde{f}(t,x,\tilde{y},\tilde{z})| \\
&\quad \le |\partial_{x_i y}\phi|\left|\frac{1}{\partial_y\phi}\right||\tilde{f}(t,x,\tilde{y},\tilde{z})| \\
&\qquad + \left|\frac{1}{\partial_y\phi}\right|\Big[\big|\partial_y f\big(t,x,\phi,\partial_y\phi\tilde{z} + \partial_x\phi\sigma(t,x)\big)\big||\partial_{x_i}\phi| \\
&\qquad\qquad + \big|\partial_z f\big(t,x,\phi,\partial_y\phi\tilde{z} + \partial_x\phi\sigma(t,x)\big)\big| \\
&\qquad\qquad\quad \times \big(|\partial_{x_i y}\phi||\tilde{z}| + |\partial_{x_i x}\phi||\sigma(t,x)| + |\partial_x\phi||\partial_{x_i}\sigma(t,x)|\big) \\
&\qquad\qquad + |\langle\partial_{x_i x}\phi||b(t,x)| + |\partial_x\phi||\partial_{x_i} b(t,x)| \\
&\qquad\qquad + \frac{1}{2}|\partial_{x_i xx}\phi||\sigma(t,x)|^2 + |\partial_{xx}\phi||\partial_{x_i}\sigma(t,x)||\sigma(t,x)| \\
&\qquad\qquad + |\tilde{z}||\partial_{x_i xy}\phi||\sigma(t,x)| + |\tilde{z}||\partial_{xy}\phi||\partial_{x_i}\sigma(t,x)| + \frac{1}{2}|\partial_{x_i yy}\phi||\tilde{z}|^2\Big] \\
&\quad \le \tilde{C}_{3,f} + \tilde{C}_{3,f}|\tilde{z}|^2
\end{aligned}
$$

with a constant $\tilde{C}_{3,f}$ only depending on C_σ, C_b, $C_{1,f}$, $C_{2,f}$, $C_{3,f}$, C_H and $\|\boldsymbol{\zeta}\|_{p\text{-var};[0,T]}$. Here, we have used the first inequality of the statement to bound $\tilde{f}$, (MF1), (MF2), (MF3) to bound f and its y and z derivatives and Lemma B.1 to bound the flow and its derivatives.

Summing over i then yields the desired result. □

THEOREM 12. *Let $\gamma > p \ge 1$ and let $\zeta^n, n = 1, 2, \ldots$ be smooth paths in $\mathbb{R}^d$. Assume*

$$\zeta^n \to \boldsymbol{\zeta}$$

in p-variation, for a rough path $\boldsymbol{\zeta} \in C^{0,p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$. Assume (MA1), (MA2), (MF1), (MF2), (MF3), (MG1) *and* ($H_{p,\gamma}$). *Let $u^n \in BUC([0,T]\times\mathbb{R}^n)$ be the unique solution to* (14) *with driving path ζ^n* (*Proposition* 8). *Then there exists $u \in BC([0,T]\times\mathbb{R}^n)$, only dependent on $\boldsymbol{\zeta}$ but not on the approximating sequence ζ^n, such that*

$$u^n \to u \qquad \textit{locally uniformly}.$$

We write (formally)

$$
\begin{aligned}
(16)\qquad & du + \bigl[\tfrac{1}{2}\operatorname{Tr}[\sigma(t,x)\sigma(t,x)^T D^2u(t,x)] + \langle b(t,x), Du(t,x)\rangle \\
&\qquad\qquad + f(t,x,u(t,x), Du(t,x)\sigma(t,x))\bigr]\,dt \\
&\qquad + H(x,u(t,x))\,d\boldsymbol{\zeta}(t) = 0, \qquad t\in[0,T), x\in\mathbb{R}^n, \\
& u(T,x) = g(x), \qquad x\in\mathbb{R}^n.
\end{aligned}
$$

Furthermore, the solution map

$$
\begin{aligned}
C^{0,p\text{-var}}\bigl([0,T], G^{[p]}(\mathbb{R}^d)\bigr) \times BUC(\mathbb{R}^n) &\to BC([0,T]\times\mathbb{R}^n), \\
(\boldsymbol{\zeta}, g) &\mapsto u
\end{aligned}
$$

is continuous.

At last we have the stochastic representation

$$u(t,x) = Y_t^{t,x},$$

where $Y^{t,x}$ is (the Y-component of) the solution to the BSDE (13).

REMARK 13. Equations like (16) have been considered in [12]. The setting there is more general in the sense that the vector field H in front of the rough path is allowed to also depend on the gradient. On the other hand, their f is independent of the gradient and H is linear.

For the proof, we apply the same ideas as in the proof of Theorem 1 in [5]. We however mimic our analysis of the BSDE case (Theorem 3) and proceed on small intervals; a similar approach was carried out in Lions–Souganidis [17].

REMARK 14. We suspect the solution to actually lie in $BUC([0,T]\times\mathbb{R}^n)$. Showing this would involve adapting the comparison theorem C.1 to directly yield a modulus of continuity for solutions, as it has been done in [9] under different assumptions on the coefficients.

PROOF. For the sake of unified notation, we write u^0 for the (rough PDE) solution u; similarly, we write $\boldsymbol{\zeta}^0$ for the rough path $\boldsymbol{\zeta}$.

1. *Existence*

Let $\phi^n, n\ge 0$ be the (ODE, for $n\ge 1$ and RDE, when $n=0$) solution flow (parametrized by x)

$$\phi^n(t,x,y) = y + \int_t^T H(x,\phi^n(r,x,y))\,d\zeta^n(r).$$

Then, by Lemma 10, for $n\ge 1$, u^n is a solution to (14) if and only if $v^n(t,x) := (\phi^n)^{-1}(t,x,u^n(t,x))$ is a solution to

$$
\begin{aligned}
(17)\qquad & \partial_t v^n(t,x) + \tfrac{1}{2}\operatorname{Tr}[\sigma(t,x)\sigma(t,x)^T D^2 v^n(t,x)] + \langle b(t,x), Dv^n(t,x)\rangle \\
&\qquad + \tilde{f}^n(t,x,v^n(t,x), Dv^n(t,x)\sigma(t,x)) = 0, \qquad t\in[0,T), x\in\mathbb{R}^n, \\
& v^n(T,x) = g(x), \qquad x\in\mathbb{R}^n,
\end{aligned}
$$

where

$$\tilde{f}^n(t,x,\tilde{y},\tilde{z}) = \frac{1}{\partial_y \phi^n}\bigg\{ f(t,x,\phi^n, \partial_y \phi^n \tilde{z} + \partial_x \phi^n \sigma(t,x)) + \langle \partial_x \phi^n, b(t,x)\rangle \\ + \frac{1}{2}\operatorname{Tr}[\partial_{xx}\phi^n \sigma(t,x)\sigma(t,x)^T] \\ + \langle \tilde{z}, (\partial_{xy}\phi^n \sigma(t,x))^T\rangle + \frac{1}{2}\partial_{yy}\phi^n |\tilde{z}|^2 \bigg\}.$$

In the proof of Theorem 3, we have already seen that $\tilde{f}^n \to \tilde{f}^0$, locally uniformly. From the method of semi-relaxed limits (Lemma 6.1, Remarks 6.2–6.4 in [7]), the pointwise (relaxed) limits

$$\bar{v}^0 := \limsup{}^* v^n, \qquad \underline{v}^0 := \liminf{}_* v^n,$$

are viscosity (sub resp. super) solutions to the (transformed) PDE (17) for $n = 0$. Here, we have used the fact, that $\bar{v}^0$ and $\underline{v}^0$ are indeed finite, say bounded in norm by $M > 0$. This follows from the Feyman–Kac representation (Proposition 8) for each u^n, in combination with bounds [uniform in (t_0, x_0) and n] on the corresponding BSDEs (Corollary 2.2 in [15]). [Although not completely obvious, such uniform bounds can also be obtained without BSDE arguments; one would need to exploit comparison for (14), and then (17), clearly valid when $n \geq 1$, with rough path estimates for RDE solutions which will serve as sub- and supersolutions without spatial structure.] Note also, that the semi-relaxed limiting procedure preserves the terminal value (see, e.g., Proposition 5.1 in [11] or Section 10 in [5]).

By Lemma 11, the function $\tilde{f}^0$ satisfies the conditions of Theorem C.1. Hence, the PDE (17) for $n = 0$ satisfies comparison on $[T-h, T]$ for h sufficiently small (as long as $T - h > 0$), and h only depends on M and the constants $\tilde{C}_{2,f}$, $\tilde{C}_{1,f}$ and $\tilde{C}_{2,f}$ for $\tilde{f}^0$ given by Lemma 11. So $v^0(t,x) := \bar{v}^0(t,x) = \underline{v}^0(t,x)$, $t \in [T-h, T]$ is the unique (and continuous, since $\bar{v}$, $\underline{v}$ are respectively upper resp. lower semicontinuous) solution to (17) with $n = 0$ on $[T-h, T]$. Moreover, using a Dini-type argument (Remark 6.4 in [7]), one sees that this limit must be uniform on compact sets. Undoing the transformation, we see that $u^n \to u^0$ locally uniformly on $[T-h, T]$, where $u^0(t,x) := \phi^0(t,x,v^0(t,x))$, $t \in [T-h, T]$.

We proceed to the next subinterval. We use the same argument as above, we just work with a different transformation. For $n \geq 0$, let $\phi^{n,T-h}$ be the solution flow started at time $T-h$, that is,

$$\phi^{n,T-h}(t,x,y) = y + \int_t^{T-h} H(x, \phi^{n,T-h}(r,x,y))\, d\zeta^n(r).$$

Then, for $n \geq 1$, $u^n|_{[0,T-h]}$ is a solution to

$$\partial_t u^n(t,x) + \frac{1}{2}\operatorname{Tr}[\sigma(t,x)\sigma(t,x)^T D^2 u^n(t,x)] + \langle b(t,x), Du^n(t,x)\rangle$$
$$+ f(t,x,u^n(t,x), Du^n(t,x)\sigma(t,x)) + H(x,u^n(t,x))\dot{\zeta}_r = 0,$$
$$t \in [0,T-h], x \in \mathbb{R}^n,$$
$$u(T-h,x) = \phi^n\big(T-h,x,v^n(T-h,x)\big), \qquad x \in \mathbb{R}^n.$$

if and only if $v^{n,T-h}(t,x) := (\phi^{n,T-h})^{-1}(t,x,u^n(t,x))$ is a solution to

$$\partial_t v^{n,T-h}(t,x) + \frac{1}{2}\operatorname{Tr}[\sigma(t,x)\sigma(t,x)^T D^2 v^{n,T-h}(t,x)] + \langle b(t,x), Dv^{n,T-h}(t,x)\rangle$$
$$+ \tilde{f}^{n,T-h}(t,x,v^{n,T-h}(t,x), \sigma(t,x)Dv^{n,T-h}(t,x)) = 0,$$
$$t \in (0,T-h), x \in \mathbb{R}^n,$$
$$v^{n,T-h}(T,x) = \phi^n\big(T-h,x,v^n(T-h,x)\big), \qquad x \in \mathbb{R}^n,$$

where of course $\tilde{f}^{n,T-h}$ is defined as $\tilde{f}^n$ was, with ϕ^n replaced by $\phi^{n,T-h}$.

Now we have already shown that the terminal values of these PDEs converge, for example,

$$\phi^n\big(T-h,\cdot,v^n(T-h,\cdot)\big) \to \phi\big(T-h,\cdot,v(T-h,\cdot)\big) \qquad \text{locally uniformly.}$$

As before, one also shows that $\tilde{f}^{n,T-h} \to \tilde{f}^{0,T-h}$, locally uniformly. By Theorem C.1, we again get comparison, now on $[T-2h, T-h]$, and hence again via the method of semi-relaxed limits we arrive at[9]

$$v^{n,T-h} \to v^{0,T-h} \qquad \text{locally uniformly on } [T-2h,T-h] \times \mathbb{R}^n.$$

Hence, $u^n \to u^0$ locally uniformly on $[T-2h, T-h]$, where $u^0(t,x) = \phi^{0,T-h}(t,x,v^{0,T-h}(t,x))$. Iterating this argument up to time 0, we get

$$u^n \to u^0 \qquad \text{locally uniformly on } [0,T] \times \mathbb{R}^n,$$

where u^0 is defined on intervals of length h as above.[10]

2. *Uniqueness, continuity of solution map*

Uniqueness of the limit and continuity of the solution map now follow by the same arguments as in the proof of Theorem 3, adapted to the PDE setting.

[9] Remark 6.3 in [7] does not take into account converging terminal values. But the result is immediate: the relaxed limit is a sub resp. super solution by Lemma 6.3 and by Proposition 2 in [5] their terminal value is exactly the limit of the given converging terminal values.

[10] The attentive reader will observe that convergence at $t = 0$ is not immediate, since Theorem C.1 was not formulated to give comparison at $t = 0$. But we can argue by extending the coefficients as well as the (rough) paths ζ^n for $t \in [-1, 0]$ as

$$\sigma(t,x) := \sigma(0,x), \qquad b(t,x) := b(0,x), \qquad f(t,x,y,z) := f(0,x,y,z), \qquad \zeta^n_t := \zeta^n_0$$

and considering the PDEs on the interval $[-1, T]$.

3. *Stochastic representation*

Let $\zeta^n \to \zeta^0$ as above. Denote by u^n the solution to the corresponding PDE (rough PDE for $n = 0$). Denote by $Y^{n,t,x}$ the solution to the BSDE (13) (BSDE with rough driver for $n = 0$) corresponding to the path ζ^n.

Then, using the result from step 1, the stochastic representation in the case of a smooth path from Proposition 8 and the convergence of the BSDEs from Theorem 3, we get

$$u^0(t,x) = \lim_{n\to\infty} u^n(t,x) = \lim_{n\to\infty} Y_t^{n,t,x} = Y_t^{0,t,x}. \qquad \square$$

4. Connection to BDSDEs. Let $\Omega^1 = C([0,T],\mathbb{R}^d)$, $\Omega^2 = C([0,T],\mathbb{R}^m)$, with the respective Wiener measures $\mathbb{P}^1$, $\mathbb{P}^2$ on them. Let $\Omega = \Omega^1 \times \Omega^2$, with the product measure $\mathbb{P} := \mathbb{P}^1 \otimes \mathbb{P}^2$. For $(\omega^1,\omega^2) \in \Omega$ let $B(\omega^1,\omega^2) = \omega^1$ be the coordinate mapping with respect to the first component. Analogously, $W(\omega^1,\omega^2) = \omega^2$ is the coordinate mapping with respect to the second component. In particular, B is a d-dimensional Brownian motion and W is an independent m-dimensional Brownian motion.

Define $\mathcal{F}_t := \mathcal{F}^B_{t,T} \vee \mathcal{F}^W_{0,t}$, where $\mathcal{F}^B_{t,T} := \sigma(B_r : r \in [t,T])$, $\mathcal{F}^W_{0,t} := \sigma(W_r : r \in [0,t])$. Note that $\mathcal{F}$ is not a filtration, since it is neither increasing nor decreasing. In this setting, Pardoux and Peng [22] considered backward doubly stochastic differential equations (BDSDEs). An $\mathcal{F}$-adapted process (Y,Z) is called a solution to the BDSDE

$$Y_t = \xi + \int_t^T f(r,Y_r,Z_r)\,dr + \int_t^T H(X_r,Y_r)\circ dB_r - \int_t^T Z_r\,dW_r, \tag{18}$$

if $\mathbb{E}[\sup_{t\le T}|Y_t|^2] < \infty$, $\mathbb{E}[\int_0^T |Z_r|^2\,dr] < \infty$ and (Y,Z) satisfies $\mathbb{P}$-a.s. (18) for $t \le T$. Here X is again the semimartingale

$$X_t = x + \int_0^t \sigma_r\,dW_r + \int_0^t b_r\,dr.$$

Under appropriate (essentially Lipschitz) conditions on f and H they were able to show existence and uniqueness of a solution.[11] The connection to BSDEs with rough driver is given by the following theorem.

THEOREM 15. *Let $p \in (2,3)$, $\gamma > p$. Let $\xi \in L^\infty(\mathcal{F}_T)$. Let f be a random function satisfying* (F1) *and* (F2). *Moreover, assume* (A1), (A2), (F1), (F2) *and* $(H_{p,\gamma})$.

Then by Theorem 1.1 *in* [22] *there exists a unique solution (Y,Z) to the BDSDE*

$$Y_t = \xi + \int_t^T f(r,Y_r,Z_r)\,dr + \int_t^T H(X_r,Y_r)\circ dB_r - \int_t^T Z_r\,dW_r.$$

[11] Pardoux and Peng considered equations where the Stratonovich integral was actually a backward integral. But if H is smooth enough, the formulations are equivalent. See also Section 4 in [3].

Let $\mathbf{B}_t = \exp(B_t + A_t)$ be the Enhanced Brownian motion (over B)[12], *especially $\mathbf{B} \in C_0^{0,p\text{-var}}([0,T], G^2(\mathbb{R}^d))$ $\mathbb{P}^1$ a.s. By setting $\mathbf{B} = 0$ on a null set, we get $\mathbf{B} \in C_0^{0,p\text{-var}}([0,T], G^2(\mathbb{R}^d))$. By Theorem 3 we can, for every $\omega^1 \in \Omega^1$, construct the solution to the BSDE with rough driver*

$$\begin{aligned} Y^{rp}(\omega^1,\cdot)_t = \xi(\cdot) + \int_t^T f(r, Y_r^{rp}, Z_r^{rp})\,dr + \int_t^T H(X_r, Y^{rp}(\omega^1,\cdot))\,d\mathbf{B}_r(\omega^1) \\ - \int_t^T Z^{rp}(\omega^1,\cdot)\,dW_r(\cdot), \qquad t \in [0,T]. \end{aligned}$$

We then have for $\mathbb{P}^1$-a.e. ω^1 that $\mathbb{P}^2$-a.s.

$$Y_t(\omega^1,\cdot) = Y_t^{rp}(\omega^1,\cdot), \qquad t \le T$$

and

$$Z_t(\omega^1,\cdot) = Z_t^{rp}(\omega^1,\cdot), \qquad dt \otimes \mathbb{P}^2\text{-a.s.}$$

PROOF. As in the proof of Theorem 3, in the BDSDE setting, one can transform the integral belonging to the Brownian motion B away. In [3] it was shown that if we let ϕ be the stochastic (Stratonovich) flow

$$\phi(\omega^1; t, x, y) = y + \int_t^T H(x, \phi(\omega^1; r, x, y)) \circ dB_r(\omega^1),$$

then with $\tilde{Y}_t := \phi^{-1}(t, X_t, Y_t)$, $\tilde{Z}_t := \frac{1}{\partial_y \phi(t, X_t, Y_t)} Z_t$ we have $\mathbb{P}$-a.s.

$$\begin{aligned} \tilde{Y}_t(\omega^1,\omega^2) = \xi(\omega^2) + \int_t^T \tilde{f}(\omega^1,\omega^2; r, X_r, \tilde{Y}_r(\omega^1,\omega^2), \tilde{Z}_r(\omega^1,\omega^2))\,dr \\ - \int_t^T \tilde{Z}_r(\omega^1,\omega^2)\,dW_r(\omega^2), \qquad t \le T. \end{aligned} \tag{19}$$

Here

$$\begin{aligned} \tilde{f}(\omega^1,\omega^2; t, x, \tilde{y}, \tilde{z}) := \frac{1}{\partial_y \phi}\bigg\{ f(\omega^2; t, \phi, \partial_y \phi \tilde{z} + \partial_x \phi \sigma_t) + \langle \partial_x \phi, b_t \rangle \\ + \frac{1}{2}\operatorname{Tr}[\partial_{xx}\phi \sigma_t \sigma_t^T] + \langle \tilde{z}, (\partial_{xy}\phi \sigma_t)^T \rangle + \frac{1}{2}\partial_{yy}\phi |\tilde{z}|^2 \bigg\}, \end{aligned}$$

where ϕ and its derivatives are always evaluated at $(\omega^1; x, \tilde{y})$. Especially by a Fubini type theorem (e.g., Theorem 3.4.1 in [2]), there exists Ω_0^1 with $\mathbb{P}^1(\Omega_0^1) = 1$ such that for $\omega^1 \in \Omega_0^1$ equation (19) holds true $\mathbb{P}^2$ a.s.

[12] $\mathbf{B}$ is precisely d-dimensional Brownian motion enhanced with its iterated integrals in Stratonovich sense; it is in $1-1$ correspondence with Brownian motion enhanced with Lévy's area; exp denotes the exponential map from the Lie algebra $\mathbb{R}^d \oplus so(d)$ to the group, realized inside the truncated tensor algebra. See, for example, Section 13 in [13] for more details.

On the other hand, we can construct ω^1-wise the rough flow

$$\phi^{rp}(\omega^1; t, x, y) = y + \int_t^T H(x, \phi^{rp}(\omega^1; r, x, y))\, d\mathbf{B}_r(\omega^1).$$

Assume for the moment that we have global comparison, so that we can solve the transformed BSDE uniquely, that is, for every $\omega^1 \in \Omega^1$, we have $\mathbb{P}^2$ a.s.

$$\begin{aligned}\tilde{Y}_t^{rp}(\omega^1, \omega^2) = \xi(\omega^2) &+ \int_t^T \tilde{f}^{rp}(\omega^1; r, \tilde{Y}_r^{rp}(\omega^1, \omega^2), \tilde{Z}_r^{rp}(\omega^1, \omega^2))\, dr \\ &- \int_t^T \tilde{Z}_r^{rp}(\omega^1, \omega^2)\, dW_r(\omega^2), \qquad t \le T,\end{aligned}$$

where

$$\begin{aligned}&\tilde{f}^{rp}(\omega^1, \omega^2; t, x, \tilde{y}, \tilde{z}) \\ &\quad := \frac{1}{\partial_y \phi^{rp}} \biggl\{ f(\omega^2; t, \phi^{rp}, \partial_y \phi^{rp} \tilde{z} + \partial_x \phi^{rp} \sigma_t) + \langle \partial_x \phi^{rp}, b_t \rangle \\ &\qquad\qquad + \frac{1}{2} \operatorname{Tr}[\partial_{xx} \phi^{rp} \sigma_t \sigma_t^T] + \langle \tilde{z}, (\partial_{xy} \phi^{rp} \sigma_t)^T \rangle + \frac{1}{2} \partial_{yy} \phi^{rp} |\tilde{z}|^2 \biggr\},\end{aligned}$$

where ϕ and its derivatives are always evaluated at $(\omega^1; x, \tilde{y})$. It is a classical rough path result that there exists Ω_1^1 with $\mathbb{P}^1(\Omega_1^1) = 1$ such that for $\omega^1 \in \Omega_1^1$ we have

$$\phi^{rp}(\omega^1; \cdot, \cdot, \cdot) = \phi(\omega^1; \cdot, \cdot, \cdot).$$

Combining above results, we have for $\omega^1 \in \Omega_0^1 \cap \Omega_1^1$ that $(\tilde{Y}_t(\omega^1, \cdot), \tilde{Z}_t(\omega^1, \cdot))$ and $(\tilde{Y}_t^{rp}(\omega^1, \cdot), \tilde{Z}_t^{rp}(\omega^1, \cdot))$ satisfy the same BSDE. Hence, we have by uniqueness

$$\tilde{Y}_t(\omega^1, \cdot) = \tilde{Y}_t^{rp}(\omega^1, \cdot), \qquad t \le T, \mathbb{P}^2\text{-a.s.}$$

and

$$\tilde{Z}_t(\omega^1, \cdot) = \tilde{Z}_t^{rp}(\omega^1, \cdot), \qquad dt \otimes \mathbb{P}^2\text{-a.s.}$$

By reversing the transformation, we get the desired result for Y and Z.

Now, since comparison does *not* necessarily hold globally, we must argue differently. Define $A^k := \{\omega^1 \in \Omega^1 : \|\mathbf{B}(\omega^1)\|_{p\text{-var}} \le k\}$. Then on A^k we have for an $h = h(k) > 0$ comparison on $[T - h, T]$, and we argue on subsequent intervals as above. Now, since $\mathbb{P}(\bigcup_k A^k) = 1$, we get the desired result. $\square$

APPENDIX A: COMPARISON FOR BSDES

DEFINITION A.1. Let $\xi \in L^\infty(\mathcal{F}_T)$, W an m-dimensional Brownian motion and f a predictable function on $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^m$.

We call an adapted process (Y, Z, C) a *supersolution to the BSDE with data* (ξ, f) if $Y \in H^\infty_{[0,T]}$, $Z \in H^2_{[0,T]}$, C is an adapted right continuous increasing process and

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\,dr - \int_t^T Z_r\,dW_r + \int_t^T dC_r, \qquad t \le T.$$

We call (Y, Z, C) a *subsolution to the BSDE with data* (ξ, f) if $(Y, Z, -C)$ is a supersolution.

The following statement, as well as its proof, are based on Theorem 2.6 in [15].

THEOREM A.2. *There exists a* (*universal*) *strictly positive function* $\delta : \mathbb{R}^2_+ \to (0, \infty)$ *such that the following statement is true.*

Let $(Y^{(1)}, Z^{(1)}, C^{(1)})$ *be a supersolution to the BSDE with data* $(\xi^{(1)}, f^{(1)})$. *Let* $(Y^{(2)}, Z^{(2)}, C^{(2)})$ *be a subsolution to the BSDE with data* $(\xi^{(2)}, f^{(2)})$. *Let* $M \in \mathbb{R}_+$ *be a bound for* $Y^{(1)}$ *and* $Y^{(2)}$, *that is,*

$$\|Y^{(1)}\|_\infty, \qquad \|Y^{(2)}\|_\infty \le M.$$

Assume that $\mathbb{P}$-*a.s.*

$$f^{(1)}\big(Y^{(1)}_t, Z^{(1)}_t\big) \le f^{(2)}\big(t, Y^{(1)}_t, Z^{(1)}_t\big) \qquad \forall t \in [0, T],$$
$$\xi^{(1)} \le \xi^{(2)}.$$

Assume that there exist constants $C > 0, L > 0, K > 0$ *such that for* $(t, y, z) \in [0, T] \times [-M, M] \times \mathbb{R}^m$

$$\big|f^{(2)}(t, y, z)\big| \le L + C|z|^2 \qquad \mathbb{P}\text{-a.s.},$$
$$\big|\partial_z f^{(2)}(t, y, z)\big| \le K + C|z| \qquad \mathbb{P}\text{-a.s.}$$

Assume that there exists a constant $N > 0$ *such that for* $(t, y, z) \in [0, T] \times [-M, M] \times \mathbb{R}^m$

$$\partial_y f^{(2)}(t, y, z) \le N + \delta(C, M)|z|^2 \qquad \mathbb{P}\text{-a.s.} \tag{20}$$

Then $\mathbb{P}$-*a.s.*

$$Y^{(1)}_t \le Y^{(2)}_t, \qquad 0 \le t \le T. \tag{21}$$

REMARK A.3. We note that, as in Theorem 2.6 of [15], the assumptions could be weakened by replacing the constants L, K, N with deterministic functions $l_t \in L^1(0, T)$, $k_t \in L^2(0, T)$ and $n_t \in L^1(0, T)$.

In our application of Theorem A.2 in the proof of Theorem 3, condition (20) is not satisfied on $[0, T]$. But we are able to choose $h > 0$ small enough, such that it is satisfied on $[T - h, T]$. Comparison (21) then holds on $[T - h, T]$.

PROOF. Let $\lambda > 0$, $B > 1$ be constants, to be specified later on. We begin by constructing several functions, on whose properties we will rely later in the proof. Define

$$\gamma(\tilde{y}) := \gamma_{\lambda,B}(\tilde{y}) := \frac{1}{\lambda}\log\left(\frac{e^{\lambda B\tilde{y}}+1}{B}\right) - M, \qquad \tilde{y} \in \mathbb{R}.$$

Then

$$\gamma^{-1}(y) = \frac{1}{\lambda B}\log\bigl(Be^{\lambda(y+M)} - 1\bigr), \qquad \gamma'(\tilde{y}) = B\frac{1}{1+e^{-\lambda B\tilde{y}}}.$$

Denote $g(y) := e^{-\lambda(y+M)}$, then $0 < g \le 1$, on $[-M, M]$. Define

$$w(y) := \gamma'(\gamma^{-1}(y)) = B - g(y).$$

Then

$$w'(y) = \lambda g(y), \qquad w''(y) = -\lambda^2 g(y),$$
$$\frac{w'(y)}{w(y)} = \frac{\lambda g(y)}{B - g(y)}, \qquad \frac{w''(y)}{w(y)} = \frac{-\lambda^2 g(y)}{B - g(y)}.$$

In particular, $w > 0$ on $[-M, M]$.

Define $\alpha(y) := \gamma^{-1}(y)$. Then, since $(Y^{(1)}, Z^{(1)}, C^{(1)})$ is a supersolution to the BSDE with data $(\xi^{(1)}, f^{(1)})$, Itô's formula gives

$$\alpha(Y_t^{(1)}) = \alpha(Y_0^{(1)}) - \int_0^t \alpha'\bigl(Y_r^{(1)}\bigr) f^{(1)}\bigl(r, Y_r^{(1)}, Z_r^{(1)}\bigr)\,dr + \int_0^t \alpha'\bigl(Y_r^{(1)}\bigr) Z_r^{(1)}\,dW_r$$
$$- \int_0^t \alpha'\bigl(Y_r^{(1)}\bigr)\,dC_r + \int_0^t \alpha''\bigl(Y_r^{(1)}\bigr)\bigl|Z_r^{(1)}\bigr|^2\,dr.$$

Define

$$\tilde{Y^{(1)}} := \alpha\bigl(Y^{(1)}\bigr), \qquad \tilde{Z^{(1)}} := \frac{Z^{(1)}}{\gamma'(\tilde{Y}^{(1)})} = \frac{Z^{(1)}}{w(Y^{(1)})}.$$

and

$$F^{(1)}(t, \tilde{y}, \tilde{z}) := \frac{1}{\gamma'(\tilde{y})}\left[f^{(1)}(t, \gamma(\tilde{y}), \gamma'(\tilde{y})\tilde{z}) + \frac{1}{2}\gamma''(\tilde{y})|\tilde{z}|^2\right].$$

Since $\alpha' > 0$ we have that $(\tilde{Y}^{(1)}, \tilde{Z}^{(1)}, \int_0^\cdot \alpha'(Y_r^{(1)})\,dC_r^{(1)})$ is a supersolution to the BSDE with data $(\alpha(\xi^{(1)}), F^{(1)})$. Analogously, we have that $(\tilde{Y}^{(2)}, \tilde{Z}^{(2)}, \int_0^\cdot \alpha'(Y_r^{(2)})\,dC_r^{(2)})$ is a subsolution to the BSDE with data $(\alpha(\xi^{(2)}), F^{(2)})$. Since α is increasing, it is now enough to verify that $\tilde{Y}^{(1)} \le \tilde{Y}^{(2)}$.

We will verify that $F^{(2)}$ satisfies the conditions of Proposition 2.9 in [15]. Especially we will show that there exist constants $A, G > 0$ such that

$$\partial_y f^{(2)}(t, y, z) + A\bigl|\partial_z f^{(2)}(t, y, z)\bigr|^2 \le G \qquad \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m. \tag{22}$$

For simplicity, denote $F := F^{(2)}$, $f := f^{(2)}$. Denote $y = \gamma(\tilde{y})$, $z = \gamma'(\tilde{y})\tilde{z} = w(y)\tilde{z}$. For convenience, w and its derivatives will always be evaluated at y. Then

$$\partial_{\tilde{z}} F(t,\tilde{y},\tilde{z}) = \partial_z f(t,y,z) + z\frac{w'}{w},$$

$$\partial_{\tilde{y}} F(t,\tilde{y},\tilde{z}) = \frac{1}{w}\left[\frac{1}{2}w''|z|^2 + w'\big(\partial_z f(t,y,z)z - f(t,y,z)\big)\right] + \partial_y f(t,y,z).$$

Hence,

$$\partial_{\tilde{y}} F(t,\tilde{y},\tilde{z}) \le \frac{1}{w}\left[\frac{1}{2}w''|z|^2 + w'(|z|[K + C|z|] + L + C|z|^2)\right] + \partial_y f(t,y,z)$$

and

$$|\partial_{\tilde{z}} F(t,\tilde{y},\tilde{z})|^2 \le \left[K + C|z| + \frac{w'}{w}|z|\right]^2.$$

So, for $A > 0$,

$$\begin{aligned}(\partial_{\tilde{y}} F + A|\partial_{\tilde{z}} F|^2)(t,\tilde{y},\tilde{z}) \le{}& |z|^2\left[\frac{1}{2}\frac{w''}{w} + \frac{w'}{w}2C + A\left(C + \frac{w'}{w}\right)^2\right] \\ &+ K|z|\left[\frac{w'}{w} + 2A\left(C + \frac{w'}{w}\right)\right] \\ &+ \frac{w'}{w}L + \partial_y f(t,y,z) + AK^2.\end{aligned}$$

Note, that for the second term we have

$$\begin{aligned}K|z|\left[\frac{w'}{w} + 2A\left(C + \frac{w'}{w}\right)\right] &\le K|z|\left[(1+2A)\left(C + \frac{w'}{w}\right)\right] \\ &\le A\left(C + \frac{w'}{w}\right)^2|z|^2 + \frac{(1+2A)^2}{A}K^2.\end{aligned}$$

Hence,

$$\begin{aligned}(\partial_{\tilde{y}} F + A|\partial_{\tilde{z}} F|^2)(t,\tilde{y},\tilde{z}) \le{}& |z|^2\left[\frac{1}{2}\frac{w''}{w} + \frac{w'}{w}2C + 2A\left(C + \frac{w'}{w}\right)^2\right] \\ &+ \frac{w'}{w}L + \partial_y f(t,y,z) + \left(A + \frac{(1+2A)^2}{A}\right)K^2.\end{aligned}$$

Now

$$\begin{aligned}&\frac{1}{2}\frac{w''}{w} + \frac{w'}{w}2C + 2A\left(C + \frac{w'}{w}\right)^2 \\ &\quad= \frac{1}{2}\frac{w''}{w} + \frac{w'}{w}2C + 2AC^2 + 4AC\frac{w'}{w} + 2A\left(\frac{w'}{w}\right)^2\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\lambda^2}{2}\frac{g(y)}{B-g(y)} + 2C(1+2A)\frac{\lambda g(y)}{B-g(y)} + 2AC^2 + 2A\frac{\lambda^2 g(y)^2}{(B-g(y))^2} \\
&= \frac{g(y)}{(B-g(y))^2}\left[-\frac{\lambda^2}{2}\big(B-g(y)\big) + 2C(1+2A)\lambda\big(B-g(y)\big) + 2A\lambda^2 g(y)\right] \\
&\quad + 2AC^2 \\
&= \frac{g(y)}{(B-g(y))^2}\left[\frac{\lambda^2}{2}\big((1+4A)g(y)-B\big) + 2C(1+2A)\lambda\big(B-g(y)\big)\right] \\
&\quad + 2AC^2.
\end{aligned}
$$

For all $A < 1$, we hence have

$$
\begin{aligned}
&\frac{1}{2}\frac{w''}{w} + \frac{w'}{w}2C + 2A\left(C + \frac{w'}{w}\right)^2 \\
&\qquad \le \frac{g(y)}{(B-g(y))^2}\left[\frac{\lambda^2}{2}\big(5g(y)-B\big) + 2C3\lambda\big(B-g(y)\big)\right] + 2AC^2.
\end{aligned}
$$

Now, choose $B = 6$. Hence, $5g(y) - B \le -1$, $y \in [-M, M]$. Then choose $\lambda = \lambda(C)$ sufficiently large such that the term in square brackets is strictly negative, say smaller then -1 for all $y \in [-M, M]$. This is possible since it is a polynomial in λ and the leading power has a negative coefficient. Then for $y \in [-M, M]$

$$
\begin{aligned}
\frac{g(y)}{(6-g(y))^2}\left[\frac{\lambda^2}{2}\big(5g(y)-6\big) + 2C3\lambda\big(6-g(y)\big)\right] &\le -\frac{g(y)}{(6-g(y))^2} \\
&\le -\frac{1}{36}e^{-\lambda 2M} =: -2\delta,
\end{aligned}
$$

where δ depends only M and λ and hence only on M and C, that is,

$$
\delta = \delta(C, M) = \tfrac{1}{72}e^{-\lambda(C)2M}.
$$

Now choose $A \in (0, 1)$ small enough such that $2AC^2 < \delta$. If for some $N > 0$ we have

$$
\partial_y f(t, y, z) \le N + \delta(C, M)|z|^2,
$$

it follows that

$$
\begin{aligned}
(\partial_{\tilde{y}} F + A|\partial_{\tilde{z}} F|^2)(t, \tilde{y}, \tilde{z}) &\le \frac{w'}{w}L + N + \left(A + \frac{(1+2A)^2}{A}\right)K^2 \\
&\le \frac{\lambda}{B-1}L + N + \left(A + \frac{(1+2A)^2}{A}\right)K^2 \\
&=: G.
\end{aligned}
$$

So we have shown (22) and comparison then follows from Proposition 2.9 in [15]. $\square$

APPENDIX B: FLOW PROPERTIES

Consider the solution flow ϕ to

$$\phi(t,x,y) = y + \int_t^T H(x,\phi(r,x,y))\,d\zeta_r, \tag{23}$$

where H and ζ will be specified in a moment. We need to control

$$\partial_y\phi - 1, \partial_x\phi, \partial_{xx}\phi, \partial_{xy}\phi, \partial_{yy}\phi, \partial_{yyy}\phi, \partial_{xyy}\phi, \partial_{xxy}\phi$$

over a small interval $[T-h,T]$. Note that each of the above expressions is 0 when evaluated at $t=T$.

LEMMA B.1. *Let $p \geq 1$, $\boldsymbol{\zeta} \in C^{0,p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$ and $\gamma > p$. Assume that $H_i = H_i(x,y)$ has joint regularity of the form*

$$\sup_{i=1,\dots,d} |H_i(\cdot,\cdot)|_{\mathrm{Lip}^{\gamma+2}(R^{n+1})} \leq c_1$$

and

$$\|\zeta\|_{p\text{-var};[0,T]} \leq c_2.$$

Then, the solution to (23) *induces a flow of C^3 diffeomorphisms, parametrized by $x \in \mathbb{R}^n$, and there exists a positive $L = L(c_1,c_2,T)$ so that, uniformly over $x \in R^n$, $y \in R$ and $t \in [0,T]$*

$$\max\left\{\partial_x\phi, \partial_y\phi, \frac{1}{\partial_y\phi}, \partial_{xx}\phi, \partial_{xy}\phi, \partial_{yy}\phi, \partial_{yyy}\phi, \partial_{xyy}\phi, \partial_{xxy}\phi\right\} < L.$$

Moreover, for every $\varepsilon > 0$ there exists a positive $\delta = \delta(\varepsilon, c_1, c_2)$ so that, uniformly over $x \in R^n$, $y \in R$ and $t \in [T-\delta, T]$

$$\max\{\partial_x\phi, \partial_y\phi - 1, \partial_{xx}\phi, \partial_{xy}\phi, \partial_{yy}\phi, \partial_{yyy}\phi, \partial_{xyy}\phi, \partial_{xxy}\phi\} < \varepsilon.$$

PROOF. Consider the extended RDE

$$d\xi = 0,$$
$$-d\phi = H(\xi,\phi)\,d\zeta$$

with terminal data $(\xi_T, \phi_T) = (x,y)$. The assumption on (H_i) implies that (ξ,ϕ) evolves according to a rough differential equation with $\mathrm{Lip}^{\gamma+2}$-vector fields. In this case, the ensemble

$$\hat{\phi} = (\xi, \phi, \partial_x\phi, \partial_y\phi, \partial_{xx}\phi, \partial_{xy}\phi, \partial_{yy}\phi, \partial_{yyy}\phi, \partial_{xyy}\phi, \partial_{xxy}\phi)$$

can be seen to be the (unique[13], nonexplosive) solution to an RDE along $\mathrm{Lip}^{\gamma-1}_{\mathrm{loc}}$ vector fields. Thanks to nonexplosivity we can, for fixed terminal data

$$\hat{\phi}_T = (x, y, 0, 1, 0, 0, 0, 0, 0, 0),$$

localize the problem and assume without loss of generality that the above ensemble is driven along $\mathrm{Lip}^{\gamma-1}$ vector fields. Since we want estimates that are *uniform* in x, y we make another key observations: there is no loss of generality in taking $(x, y) = (0, 0)$ provided H is replaced by $H_{x,y} = H(x + \cdot, y + \cdot)$. This also shifts the derivatives [evaluated at some (x, y)] to derivatives evaluated at $(0, 0)$. As announced, we can now safely localize, and assume that the vector fields required for $\hat{\phi}$, obtain by taking formal (x, y) derivatives in

$$\begin{aligned} d\xi &= 0, \\ -d\phi &= H(\xi, \phi)\, d\zeta, \end{aligned}$$

are globally $\mathrm{Lip}^{\gamma-1}$. A basic estimate (Theorem 10.14 in [13]) for RDE solutions implies that for some $C = C(p, \gamma)$

$$|\hat{\phi}_t - \hat{\phi}_T| \le |\hat{\phi}|_{p\text{-var};[t,T]} = C \times \varphi_p(|H_{x,y}|_{\mathrm{Lip}^{\gamma+2}} \|\zeta\|_{p\text{-var};[T-h,T]}),$$

where $\varphi_p(x) = \max(x, x^p)$. At last, we note that $|H_{x,y}|_{\mathrm{Lip}^{\gamma+2}} = |H|_{\mathrm{Lip}^{\gamma+2}}$ thanks to invariance of such Lip norms under translation. The proof is then easily finished. □

LEMMA B.2. *Assume the setting of the previous lemma. Assume that ζ^n, $n \ge 1$ is a sequence of p rough paths that converge to a rough path ζ^0 in p-variation.*

Then locally uniformly on $[0, T] \times \mathbb{R}^n \times \mathbb{R}$

$$\begin{aligned} &\left(\phi^n, \frac{1}{\partial_y \phi^n}, \partial_y \phi^n, \partial_{yy}\phi^n, \partial_x \phi^n, \partial_{xx}\phi^n, \partial_{yx}\phi^n\right) \\ &\qquad \to \left(\phi^0, \frac{1}{\partial_y \phi^0}, \partial_y \phi^0, \partial_{yy}\phi^0, \partial_x \phi^0, \partial_{xx}\phi^0, \partial_{yx}\phi^0\right). \end{aligned}$$

PROOF. Using enlargement of the state space as in the proof of Lemma B.1 we can apply the same reasoning as in Theorems 11.14 and 11.15 in [13] to get the desired result. □

---

[13]This is actual a subtle point since uniqueness in general requires $\mathrm{Lip}^{\gamma}_{\mathrm{loc}}$-regularity. The point is that the RDEs obtain by differentiating the flow have a special structure so that for the final level of derivatives only rough integration is need; as is well known, for this it suffices to have $\mathrm{Lip}^{\gamma-1}_{\mathrm{loc}}$ regularity. Chapter 11 in [13] contains a detailed discussion of this.

APPENDIX C: COMPARISON FOR PDES

We consider the equation

$$
\begin{aligned}
&-\partial_t u - \tfrac{1}{2}\operatorname{Tr}[\sigma(t,x)\sigma(t,x)^T D^2 u] - \langle b(t,x), Du\rangle \\
&\qquad - f(t,x,u,Du\sigma(t,x)) = 0, \qquad (t,x) \in (0,T) \times \mathbb{R}^n,
\end{aligned} \tag{24}
$$

where $f : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, are a continuous functions.

The following statement as well as its proof are a modification of Theorem 3.2 in [15]. (The statement is not in its most general form, but adjusted to what we need in the main text.)

THEOREM C.1. *Assume that there exists a constant $C_b > 0$ such that for $(t,x), (t,y) \in [0,T] \times \mathbb{R}^n$*

$$
\begin{aligned}
|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(y)| &\le C_b|x-y|, \\
|b(t,x)| &\le C_b.
\end{aligned}
$$

Assume that there exists a constant $C_\sigma > 0$ such that for $(t,x), (t,y) \in [0,T] \times \mathbb{R}^n$

$$
\begin{aligned}
|\sigma(t,x) - \sigma(y)| &\le C_\sigma|x-y|, \\
|\sigma(t,x)| &\le C_\sigma.
\end{aligned}
$$

Assume that there exists a constant $C_{1,f} > 0$ such that for $(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$

$$
\begin{aligned}
|f(t,x,y,z)| &\le C_{1,f}(1+|z|^2), \\
|\partial_z f(t,x,y,z)| &\le C_{1,f}(1+|z|).
\end{aligned}
$$

Assume that there exists a constant $C_{2,f}$ such that for every $\varepsilon > 0$ there exists an $h_\varepsilon \in (0,T]$ such that for $(t,x,y,z) \in [T-h_\varepsilon, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ we have

$$
\partial_y f(t,x,y,z) \le C_{2,f} + \varepsilon|z|^2. \tag{25}
$$

Assume that there exists a constant $C_{3,f} > 0$ such that for $(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$

$$
|\partial_x f(t,x,y,z)| \le C_{3,f}(1+|z|^2).
$$

Let u, v be a bounded semicontinuous sub-(resp., super-)solution to (24) on $(0,T) \times \mathbb{R}^n$, with $u(T,\cdot) \le v(T,\cdot)$. Then there exists an $\varepsilon^* = \varepsilon^*(\|u\|_\infty \vee \|v\|_\infty, C, C_{2,f}) > 0$ such that for $(t,x) \in (T-h_{\varepsilon^*}, T] \times \mathbb{R}^n$ we have

$$
u(t,x) \le v(t,x).
$$

PROOF. 1. Reduction

We will first transform the PDE into a PDE with coefficient that satisfies a certain structure condition (equation (24) in [15]). Set $M := \max\{\|u\|_\infty, \|v\|_\infty\} + 1$.

Let $\lambda > 0$, $A > 1$, $K > 0$ be constants to be chosen later. We begin by constructing several functions, whose properties we will rely on later in the proof. Define

$$\varphi(\tilde{y}) := \frac{1}{\lambda} \ln\left(\frac{e^{\lambda A \tilde{y}} + 1}{A}\right) : \mathbb{R} \to \left(-\frac{\ln(A)}{\lambda}, \infty\right).$$

We will have to choose $A \geq e^{\lambda 2 M e^{Kt}}$, since we shall need later on that $\{e^{Kt}(y - M) : y \in [-M, M]\}$ is contained in the range of φ. Then

$$\varphi'(\tilde{y}) = A \frac{1}{1 + e^{-\lambda A \tilde{y}}}, \qquad \varphi^{-1}(y) = \frac{1}{\lambda A} \ln(A e^{\lambda y} - 1).$$

Define $r(y) := \varphi^{-1}(e^{Kt}(y - M))$, its inverse $s(\tilde{y}) := \varphi(\tilde{y}) e^{-Kt} + M$ and $g(y) := e^{-\lambda e^{Kt}(y-M)} : [-M, M] \to [1, e^{\lambda 2 M e^{Kt}}]$. Then $g'(y) = -\lambda e^{Kt} g(y)$. Define

$$w(y) := e^{-Kt} \varphi'(r(y)) = \partial_{\tilde{y}} s|_{\tilde{y} = r(y)} = e^{-Kt}\big[A - e^{-\lambda e^{Kt}(y-M)}\big] = e^{-Kt}[A - g(y)],$$

which is nonnegative for $A \geq e^{\lambda 2 M e^{Kt}}$. Then

$$w'(y) = \lambda g(y), \qquad w''(y) = -e^{Kt} \lambda^2 g(y).$$

Let now $u(t, x)$ be a solution to (24). Let $\tilde{u}(t, x) := r(u(t, x))$. Then $u(t, x) = s(\tilde{u}(t, x))$, and hence

$$\partial_{x_i} u(t, x) = \varphi'(\tilde{u}(t, x)) e^{-Kt} \partial_{x_i} \tilde{u}(t, x),$$

$$\partial_{x_j x_i} u(t, x) = \varphi''(\tilde{u}(t, x)) e^{-Kt} \partial_{x_j} \tilde{u}(t, x) \partial_{x_i} \tilde{u}(t, x) + \varphi'(\tilde{u}(t, x)) e^{-Kt} \partial_{x_j x_i} \tilde{u}(t, x),$$

that is,

$$Du(t, x) = \varphi'(\tilde{u}(t, x)) e^{-Kt} D\tilde{u}(t, x),$$

$$D^2 u(t, x) = \varphi''(\tilde{u}(t, x)) e^{-Kt} D\tilde{u}(t, x) \otimes D\tilde{u}(t, x) + \varphi'(\tilde{u}(t, x)) e^{-Kt} D^2 \tilde{u}(t, x).$$

Hence,

$$\begin{aligned}
\partial_t \tilde{u}(t, x)
&= \frac{1}{\varphi'(\tilde{u}(t, x))}\big[K e^{Kt}\big(u(t, x) - M\big) + e^{Kt} \partial_t u(t, x)\big] \\
&= \frac{1}{\varphi'(\tilde{u}(t, x))} K e^{Kt}\big(u(t, x) - M\big) \\
&\quad - \frac{1}{\varphi'(\tilde{u}(t, x))} e^{Kt}\left[\frac{1}{2} \operatorname{Tr}[\sigma(t, x)\sigma(t, x)^T D^2 u(t, x)] + \langle b(t, x), Du(t, x)\rangle\right] \\
&\quad - \frac{1}{\varphi'(\tilde{u}(t, x))} e^{Kt} f(t, x, u(t, x), Du(t, x)\sigma(t, x))
\end{aligned}$$

$$
\begin{aligned}
&= K\frac{\varphi(\tilde{u}(t,x))}{\varphi'(\tilde{u}(t,x))} - \frac{1}{2}\operatorname{Tr}[\sigma(t,x)\sigma(t,x)^T D^2\tilde{u}(t,x)] \\
&\quad - \frac{\varphi''(\tilde{u}(t,x))}{\varphi'(\tilde{u}(t,x))}\frac{1}{2}\operatorname{Tr}[\sigma(t,x)\sigma(t,x)^T D\tilde{u}(t,x)\otimes D\tilde{u}(t,x)] \\
&\quad - \langle b(t,x), D\tilde{u}(t,x)\rangle \\
&\quad - \frac{1}{\varphi'(\tilde{u}(t,x))}e^{Kt} f(t,x,s(\tilde{u}(t,x)),\varphi'(\tilde{u}(t,x))e^{-Kt}D\tilde{u}(t,x)\sigma(t,x)).
\end{aligned}
$$

Analogously, by resorting to test functions, one shows, that if u (resp., v) is a viscosity sub-(resp., super-) solution to (24), then $\tilde{u}(t,x) := r(u(t,x))$ [resp., $\tilde{v}(t,x) := r(v(t,x))$] is a viscosity sub-(resp., super-) solution to

$$
\begin{aligned}
&-\partial_t\tilde{u}(t,x) - \tfrac{1}{2}\operatorname{Tr}[\sigma(t,x)\sigma(t,x)^T D^2\tilde{u}(t,x)] - \langle(b(t,x), D\tilde{u}(t,x)\rangle \\
&\qquad - \tilde{f}(t,x,\tilde{u}(t,x), D\tilde{u}(t,x)\sigma(t,x)) = 0, \qquad t\in(0,T), x\in\mathbb{R}^n,
\end{aligned}
\tag{26}
$$

where, denoting from now on $y = s(\tilde{y})$, $z = w(y)\tilde{z}$,

$$
\begin{aligned}
\tilde{f}(t,x,\tilde{y},\tilde{z}) &= -K\frac{\varphi(\tilde{y})}{\varphi'(\tilde{y})} + \frac{\varphi''(\tilde{y})}{\varphi'(\tilde{y})}\frac{1}{2}|\tilde{z}|^2 \\
&\qquad + \frac{1}{\varphi'(\tilde{y})}e^{Kt} f(t,x,s(\tilde{y}),\varphi'(\tilde{y})e^{-Kt}\tilde{z}) \\
&= -K\frac{y-M}{w(y)} + w'(y)\frac{1}{2}|\tilde{z}|^2 + \frac{1}{w(y)} f(t,x,y,w(y)\tilde{z}).
\end{aligned}
$$

We also obviously have $\tilde{u}(T,\cdot) \le \tilde{v}(T,\cdot)$.

We will bound the $\tilde{y}$-derivative of $\tilde{f}$, while at the same time choosing the constants K,λ,A. First,

$$
\begin{aligned}
\partial_{\tilde{y}}\tilde{f}(t,x,\tilde{y},\tilde{z}) &= -K\left(1-(y-M)\frac{w'(y)}{w(y)}\right) + \frac{1}{2}\frac{w''(y)}{w(y)}|z|^2 \\
&\qquad - \frac{w'(y)}{w(y)} f(t,x,y,z) + \partial_y f(t,x,y,z) \\
&\qquad + \frac{w'(y)}{w(y)}\partial_z f(t,x,y,z)z \\
&\le -K\left(1-(y-M)\frac{w'(y)}{w(y)}\right) + \frac{1}{2}\frac{w''(y)}{w(y)}|z|^2 \\
&\qquad + \frac{w'(y)}{w(y)}C_{1,f}(1+|z|^2) + \partial_y f(t,x,y,z) \\
&\qquad + \frac{w'(y)}{w(y)}C_{1,f}(1+|z|)|z|
\end{aligned}
$$

$$\le \frac{|z|^2}{w(y)}\left(\frac{1}{2}w''(y) + C_{1,f}w'(y) + C_{1,f}w'(y)\right)$$
$$- K\left(1 - (y-M)\frac{w'(y)}{w(y)}\right) + \partial_y f(t,x,y,z)$$
$$+ C_{1,f}\frac{w'(y)}{w(y)} + C_{1,f}\frac{w'(y)}{w(y)}|z|.$$

Now using

$$C_{1,f}\frac{w'(y)}{w(y)}|z| \le \frac{|z|^2}{w(y)}w'(y) + \frac{w'(y)}{w(y)}C_{1,f}^2,$$

we get

$$\begin{aligned}\partial_{\tilde{y}}\tilde{f}(t,x,\tilde{y},\tilde{z}) \le \frac{|z|^2}{w(y)}\left(\frac{1}{2}w''(y) + (2C_{1,f}+1)w'(y)\right)\\ - K + \partial_y f(t,x,y,z) + \frac{w'(y)}{w(y)}\left(C_{1,f} + K(y-M) + C_{1,f}^2\right).\end{aligned} \tag{27}$$

Note that

$$C_{1,f} + K(y-M) + C_{1,f}^2 \le C_{1,f} - K + C_{1,f}^2, \qquad y \in [-(M-1), M-1].$$

Hence, we can choose $K_0 = K_0(C_{1,f})$ sufficiently large, such that

$$C_{1,f} + K_0(y-M) + C_{1,f}^2 \le -1, \qquad y \in [-(M-1), M-1].$$

Then we have that for all choices of $K_0 > K$, and all choices $\lambda > 0$ that the last term in (27),

$$\frac{w'(y)}{w(y)}\left(C_{1,f} + K(y-M) + C_{1,f}^2\right)$$
$$= \frac{e^{-\lambda e^{Kt}(y-M)}}{A - e^{-\lambda e^{Kt}(y-M)}}\lambda e^{Kt}\left(C_{1,f} + K(y-M) + C_{1,f}^2\right),$$

is negative for $y \in [-(M-1), M-1]$ as long as $A > e^{\lambda 2Me^{Kt}}$. We now fix $K = K(C_{1,f}, C_{2,f}) = \max\{K_0(C_{1,f}), C_{2,f}\} + 1$. Then

$$\begin{aligned}\tfrac{1}{2}w''(y) + (2C_{1,f}+1)w'(y) &= -\tfrac{1}{2}e^{Kt}\lambda^2 g(y) + \lambda(2C_{1,f}+1)g(y)\\ &\le -\tfrac{1}{2}\lambda^2 g(y) + \lambda(2C_{1,f}+1)g(y)\\ &= g(y)\lambda\left[(2C_{1,f}+1) - \tfrac{1}{2}\lambda\right].\end{aligned}$$

So, if we choose $\lambda = \lambda(C_{1,f}) = 4C_{1,f} + 4$, we have

$$\tfrac{1}{2}w''(y) + (2C_{1,f}+1)w'(y) \le g(y)(4C_{1,f}+4)(-1) \le -(4C_{1,f}+4) \le -1.$$

We now fix $A = A(\lambda(C_{1,f}), M, K(C_{1,f}, C_{2,f})) = A(M, C_{1,f}, C_{2,f}) = e^{\lambda 2Me^{KT}} + 1$. Then for the first term in (27)

$$
\begin{aligned}
&\frac{|z|^2}{w(y)}\left(\frac{1}{2}w''(y) + (2C_{1,f}+1)w'(y)\right)\\
&\qquad = \frac{|z|^2}{e^{\lambda 2Me^{KT}} + 1 - e^{-\lambda e^{Kt}(y-M)}} e^{Kt}\left(\frac{1}{2}w''(y) + (2C_{1,f}+1)w'(y)\right)\\
&\qquad \le -\frac{|z|^2}{e^{\lambda 2Me^{KT}} + 1 - e^{-\lambda e^{Kt}(y-M)}} e^{Kt}\\
&\qquad \le -\frac{|z|^2}{e^{\lambda 2Me^{KT}}} e^{Kt}\\
&\qquad < -\delta|z|^2 < 0,
\end{aligned}
$$

with

$$
\delta = \delta(\lambda(C_{1,f}), K(C_{1,f}, C_{2,f}), M) = \delta(M, C_{1,f}, C_{2,f}) = \frac{e^{Kt}}{e^{\lambda 2Me^{KT}} + 1} > 0.
$$

We now set $\varepsilon^* = \varepsilon^*(M, C_{2,f}, C_{2,f}) := \frac{\delta}{2}$. Then on $[T - h_{\varepsilon^*}, T]$ we have

$$
\partial_y f(t, x, y, z) \le C_{2,f} + \frac{\delta}{2}|z|^2,
$$

and hence we get that on $[T - h_{\varepsilon^*}, T]$ (remember that $K \ge C_{2,f} + 1$)

$$
\begin{aligned}
\partial_{\tilde{y}} \tilde{f}(t, x, \tilde{y}, \tilde{z}) &\le \frac{|z|^2}{w(y)}\left(\frac{1}{2}w''(y) + (2C_{1,f}+1)w'(y)\right)\\
&\qquad - K + \partial_y f(t, x, y, z) + \frac{w'(y)}{w(y)}\left(C_{1,f} + K(y-M) + C_{1,f}^2\right)\\
&\le -\delta|z|^2 - K + C_{2,f} + \frac{\delta}{2}|z|^2\\
&\le -\delta|z|^2 + \frac{\delta}{2}|z|^2 - 1\\
&= -\frac{\delta}{2}|z|^2 - 1\\
&= -\frac{\delta}{2}|w(y)|^2|\tilde{z}|^2 - 1\\
&\le -\tilde{K}(1 + |\tilde{z}|^2) \qquad \text{for } y \in [-(M-1), M-1].
\end{aligned}
$$

for some $\tilde{K} > 0$.

Moreover by the definition of $\tilde{f}$ and the assumptions on f it is straightforward to bound the other partial derivatives of $\tilde{f}$. So in total we get $\tilde{K}, \tilde{C} > 0$ such that for $t \in [T - h_{\varepsilon^*}, T]$, $y \in [-(M-1), M-1]$, $\tilde{z} \in \mathbb{R}^m$

$$\begin{aligned} &\partial_{\tilde{y}} \tilde{f}(t, x, \tilde{y}, \tilde{z}) \leq -\tilde{K}(1 + |\tilde{z}|^2), \\ &|\partial_x \tilde{f}(t, x, \tilde{y}, \tilde{z})| \leq \tilde{C}(1 + |\tilde{z}|^2), \\ &|\partial_{\tilde{z}} \tilde{f}(t, x, \tilde{y}, \tilde{z})| \leq \tilde{C}(1 + |\tilde{z}|). \end{aligned} \tag{28}$$

Let $\underline{M} := r(-(M-1))$, $\bar{M} := r(M-1)$. Then, since u, v take values in $[-(M-1), M-1]$, $\tilde{u}, \tilde{v}$ take values in $[\underline{M}, \bar{M}]$. We can then define

$$\tilde{\tilde{f}}(t, x, \tilde{y}, \tilde{z}) := \begin{cases} \tilde{f}(t, x, \underline{M}, \tilde{z}) - \tilde{K}(1 + |\tilde{z}|^2)(\tilde{y} - \underline{M}), & \tilde{y} < \underline{M}, \\ \tilde{f}(t, x, \tilde{y}, \tilde{z}), & \tilde{y} \in [\underline{M}, \bar{M}], \\ \tilde{f}(t, x, \bar{M}, \tilde{z}) - \tilde{K}(1 + |\tilde{z}|^2)(\tilde{y} - \bar{M}), & \bar{M} < \tilde{y}. \end{cases}$$

This function $\tilde{\tilde{f}}$ then satisfies[14] for some $\tilde{K}, \tilde{C} > 0$ and for all $t \in [T - h_{\varepsilon^*}, T]$, $\tilde{y} \in \mathbb{R}$, $\tilde{z} \in \mathbb{R}^m$

$$\begin{aligned} &\partial_{\tilde{y}} \tilde{\tilde{f}}(t, x, \tilde{y}, \tilde{z}) \leq -\tilde{K}(1 + |\tilde{z}|^2), \\ &|\partial_x \tilde{\tilde{f}}(t, x, \tilde{y}, \tilde{z})| \leq \tilde{C}(1 + |\tilde{z}|^2), \\ &|\partial_{\tilde{z}} \tilde{\tilde{f}}(t, x, \tilde{y}, \tilde{z})| \leq \tilde{C}(1 + |\tilde{z}| + |\tilde{y}||\tilde{z}|) \end{aligned} \tag{29}$$

and $\tilde{u}, \tilde{v}$ are also sub-(resp., super-) solution to (26) with $\tilde{f}$ replaced by $\tilde{\tilde{f}}$.[15] We can hence assume the validity of (29) for $\tilde{f}$.

2. Comparison under structure condition

Let $\tilde{u}$, $\tilde{v}$ be a semicontinuous sub-(resp., super-)solution to

$$\begin{aligned} -\partial_t \tilde{u}(t, x) - \tfrac{1}{2} \operatorname{Tr}[\sigma(t, x)\sigma(t, x)^T D^2 \tilde{u}(t, x)] - \langle (b(t, x), D\tilde{u}(t, x) \rangle \\ - \tilde{f}(t, x, \tilde{u}(t, x), D\tilde{u}(t, x)\sigma(t, x)) = 0, \qquad t \in (0, T), x \in \mathbb{R}^n, \end{aligned}$$

where $\tilde{f}$ satisfies (29) for $t \in [0, T]$, $\tilde{y} \in \mathbb{R}$, $\tilde{z} \in \mathbb{R}^m$. Let $\tilde{u}$ be bounded above and $\tilde{v}$ be bounded.

[14]Note that $\tilde{\tilde{f}}$ is not necessarily continuously differentiable in $\tilde{y}$ anymore, but, as was noted on page 1718, we can directly work with functions that are only (locally) Lipschitz and bound the corresponding Lipschitz constants.

[15]The reason one wants bounds globally in y is that is that the proof involves $\tilde{u}_\gamma = \tilde{u} - \gamma/t$ which is unbounded.

In fact, it is possible to carry out the comparison proof without penalizing $t = 0$. It suffices to use a slightly more general version of the parabolic theorem of sums such as established in [9]. Following this approach would also lead to comparison at $t = 0$, if we take into consideration the remarks in [6] on the accessibility of a subsolution.

Assume $\tilde{u}(T,\cdot) \le \tilde{v}(T,\cdot)$. We show $\tilde{u} \le \tilde{v}$ on $(0,T] \times \mathbb{R}^n$.

First of all, note that $\tilde{u}_\gamma(t,x) := \tilde{u}(t,x) - \frac{\gamma}{t}$ is also a subsolution. Since $\tilde{u} \le \tilde{v}$ follows from $\tilde{u}_\gamma \le \tilde{v}$ in the limit $\gamma \to 0$, it suffices to prove comparison under the additional assumption

$$\lim_{t\to 0} \tilde{u}(t,x) = -\infty \qquad \text{uniformly on } \mathbb{R}^n.$$

Define

$$L := \sup_{x\in\mathbb{R}^n, t\in(0,T]} [\tilde{u}(t,x) - \tilde{v}(t,x)]$$

and also

$$L(h) := \sup_{|x-x'|\le h, t\in(0,T]} [\tilde{u}(t,x) - \tilde{v}(t,x')],$$
$$L' := \lim_{h\to 0} L(h).$$

One has of course $L \le L'$. We will show $L' \le 0$. Consider

$$\psi_{\varepsilon,\eta}(t,x,x') := \tilde{u}(t,x) - \tilde{v}(t,x') - \frac{|x-x'|^2}{\varepsilon^2} - \eta(|x|^2 + |x'|^2).$$

Let $L_{\varepsilon,\eta}$ be the maximum of $\psi_{\varepsilon,\eta}$ and $(\hat{t},\hat{x},\hat{x}') = (\hat{t}_{\varepsilon,\eta}, \hat{x}_{\varepsilon,\eta}, \hat{x}'_{\varepsilon,\eta}) \in (0,T] \times \mathbb{R}^n$ a maximizing point, which exists by the assumptions on $\tilde{u}$ and $\tilde{v}$.

We argue by contradiction. Hence, assume $\tilde{u}(s,z) - \tilde{v}(s,z) > \delta$ for some (s,z). Then also $L' > \delta$. We first argue, that for small enough values of ε, η the optimizing time parameter $\hat{t}$ cannot be T. Indeed, assuming $\hat{t} = T$ we can estimate

$$\begin{aligned} \delta - 2\eta|z|^2 &= \psi_{\varepsilon,\eta}(s,z,z) \\ &\le \psi_{\varepsilon,\eta}(T,\hat{x},\hat{x}') \\ &= \sup_{x,x'}\left[u(T,x) - v(T,x') - \frac{|x-x'|^2}{\varepsilon^2} - \eta(|x|^2 + |x'|^2)\right]. \end{aligned}$$

Now by Theorem 3.1 in [7], applied to $u(T,x) - \eta|x|^2$ and $v(T,x) + \eta|x'|^2$, we get

$$\begin{aligned} \lim_{\varepsilon\to 0} \psi_{\varepsilon,\eta}(T,\hat{x},\hat{x}') &= \sup_x [u(T,x) - v(T,x) - 2\eta|x|^2] \\ &\le \sup_x [u(T,x) - v(T,x)] \le 0. \end{aligned}$$

It follows that for ε, η small enough, $\hat{t} \ne T$. Also, by assumption $\tilde{u}(s,z) - \tilde{v}(s,z) > \delta$; hence, we have for η small enough, that $\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}') \ge L_{\varepsilon,\eta} \ge \delta > 0$. We assume to be in this scenario from now on.

By applying the parabolic theorem on sums (e.g., Theorem 8.3 in [7]), we get

$$(b, \hat{p}, \hat{X}) \in \bar{\mathcal{P}}^{2,+}\tilde{u}(\hat{t}, \hat{x}),$$
$$(b', \hat{p}', \hat{X}') \in \bar{\mathcal{P}}^{2,+}\tilde{v}(\hat{t}, \hat{x}'),$$

such that $b - b' = 0$, $\hat{p} = 2\frac{\hat{x}-\hat{x}'}{\varepsilon^2} + 2\eta\hat{x}$, $\hat{p}' = 2\frac{\hat{x}-\hat{x}'}{\varepsilon^2} - 2\eta\hat{x}'$ and

$$\begin{pmatrix} \hat{X} & 0 \\ 0 & -\hat{X}' \end{pmatrix} \leq \frac{4}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 4\eta\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{30}$$

Indeed, defining $\varphi(t, x, x') := \frac{|x-x'|^2}{\varepsilon^2} + \eta(|x|^2 + |x'|^2)$ we have

$$A := D^2\varphi(t, x, x') = \frac{2}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\eta\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then

$$A^2 = \left(\frac{8}{\varepsilon^4} + 8\frac{\eta}{\varepsilon^2}\right)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 4\eta^2\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

By the Theorem on Sums, for every $a > 0$ there exist said elements of the jets such that

$$\begin{pmatrix} \hat{X} & 0 \\ 0 & -\hat{X}' \end{pmatrix} \leq A + aA^2.$$

Hence, we can choose a so small such that (30) holds.

By the viscosity property,

$$\begin{aligned} 0 &\leq \mathrm{Tr}[\sigma\sigma^T(\hat{t}, \hat{x})\hat{X}] - \mathrm{Tr}[\sigma\sigma^T(\hat{t}, \hat{x}')\hat{X}'] + \langle b(\hat{t}, \hat{x}), \hat{p}\rangle - \langle b(\hat{t}, \hat{x}'), \hat{p}'\rangle \\ &\quad + \tilde{f}(\hat{t}, \hat{x}, \tilde{u}(\hat{t}, \hat{x}), \hat{p}\sigma(\hat{t}, \hat{x})) - \tilde{f}(\hat{t}, \hat{x}', \tilde{v}(\hat{t}, \hat{x}'), \hat{p}'\sigma(\hat{t}, \hat{x}')) \\ &= (\mathrm{i}) + (\mathrm{ii}) + (\mathrm{iii}). \end{aligned}$$

Where

$$\begin{aligned} (\mathrm{i}) &:= \mathrm{Tr}[\sigma\sigma^T(\hat{t}, \hat{x})\hat{X}] - \mathrm{Tr}[\sigma\sigma^T(\hat{t}, \hat{x}')\hat{X}'], \\ (\mathrm{ii}) &:= \langle b(\hat{t}, \hat{x}), \hat{p}\rangle - \langle b(\hat{t}, \hat{x}'), \hat{p}'\rangle, \\ (\mathrm{iii}) &:= \tilde{f}(\hat{t}, \hat{x}, \tilde{u}(\hat{t}, \hat{x}), \hat{p}\sigma(\hat{t}, \hat{x})) - \tilde{f}(\hat{t}, \hat{x}', \tilde{v}(\hat{t}, \hat{x}'), \hat{p}'\sigma(\hat{t}, \hat{x}')). \end{aligned}$$

Multiplying (30) with $\binom{\sigma(\hat{t},\hat{x})}{\sigma(\hat{t},\hat{x}')}$ from the right side, with $\binom{\sigma(\hat{t},\hat{x})}{\sigma(\hat{t},\hat{x}')}^T$ from the left and then taking the trace, we get

$$\begin{aligned} (\mathrm{i}) &= \mathrm{Tr}[\sigma\sigma^T(\hat{t}, \hat{x})\hat{X}] - \mathrm{Tr}[\sigma\sigma^T(\hat{t}, \hat{x}')\hat{X}'] \\ &\leq \frac{4}{\varepsilon^2}\|\sigma(\hat{t}, \hat{x}) - \sigma(\hat{t}, \hat{x}')\|_2^2 + 4\eta\big(\|\sigma(\hat{t}, \hat{x})\|_2^2 + \|\sigma(\hat{t}, \hat{x}')\|_2^2\big) \\ &\leq C_\sigma\frac{4}{\varepsilon^2}|\hat{x} - \hat{x}'|^2 + 8\eta C_\sigma^2. \end{aligned}$$

Moreover,

$$\begin{aligned}
\text{(ii)} &= \langle b(\hat{t},\hat{x}'),\hat{p}'\rangle - \langle b(\hat{t},\hat{x}),\hat{p}\rangle \\
&\le |b(\hat{t},\hat{x}') - b(\hat{t},\hat{x})|\left|2\frac{\hat{x}-\hat{x}'}{\varepsilon^2}\right| + |b(\hat{t},\hat{x})||2\eta\hat{x}' + 2\eta\hat{x}| \\
&\le C_b|\hat{x}-\hat{x}'|\left|2\frac{\hat{x}-\hat{x}'}{\varepsilon^2}\right| + C_b 2\eta(|\hat{x}'| + |\hat{x}|) \\
&= C_b 2\frac{|\hat{x}-\hat{x}'|^2}{\varepsilon^2} + C_b 2\eta(|\hat{x}'| + |\hat{x}|).
\end{aligned}$$

We have

$$\begin{aligned}
\text{(iii)} &= \tilde{f}(\hat{t},\hat{x},\tilde{u}(\hat{t},\hat{x}),\hat{p}\sigma(\hat{t},\hat{x})) - \tilde{f}(\hat{t},\hat{x}',\tilde{v}(\hat{t},\hat{x}'),\hat{p}'\sigma(\hat{t},\hat{x}')) \\
&= \int_0^1 \bigl[\partial_x \tilde{f}((*))(\hat{x}-\hat{x}') + \partial_{\tilde{y}}\tilde{f}((*))\bigl(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')\bigr) \\
&\qquad + \partial_{\tilde{z}}\tilde{f}((*))\bigl(\hat{p}\sigma(\hat{t},\hat{x}) - \hat{p}'\sigma(\hat{t},\hat{x}')\bigr)\bigr]\,d\lambda,
\end{aligned}$$

where

$$(*) := \bigl(\hat{t}, \lambda\hat{x} + (1-\lambda)\hat{x}', \lambda\tilde{u}(\hat{t},\hat{x}) + (1-\lambda)\tilde{v}(\hat{t},\hat{x}'), \lambda\hat{p}\sigma(\hat{t},\hat{x}) + (1-\lambda)\hat{p}'\sigma(\hat{t},\hat{x}')\bigr).$$

We know $|\tilde{v}(\hat{t},\hat{x}')| \le \|\tilde{v}\|_\infty < \infty$ and by the upper boundedness of $\tilde{u}$ and by the definition of the maximizier we get $\infty > C \ge \tilde{u}(\hat{t},\hat{x}) \ge \tilde{u}(T,0) - \|\tilde{v}\|_\infty > \infty$. Hence, $\lambda\tilde{u}(\hat{t},\hat{x}) + (1-\lambda)\tilde{v}(\hat{t},\hat{x}')$ is always bounded and we can assume that actually

$$|\partial_{\tilde{z}}\tilde{f}(t,x,\tilde{y},\tilde{z})| \le \tilde{C}(1 + |\tilde{z}|).$$

Remember moreover that we assume η small enough, such that $\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}') \ge L_{\varepsilon,\eta} \ge \delta > 0$. Especially we have $|\hat{x}-\hat{x}'| \le \varepsilon\sqrt{\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')}$. Hence, we can estimate [let $(**) := \lambda\hat{p}\sigma(\hat{t},\hat{x}) + (1-\lambda)\hat{p}'\sigma(\hat{t},\hat{x}')$]

$$\begin{aligned}
\text{(iii)} &\le \int_0^1 \bigl[\tilde{C}\bigl(1 + |(**)|^2\bigr)|\hat{x}-\hat{x}'| - \tilde{K}\bigl(1 + |(**)|^2\bigr)\bigl(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')\bigr) \\
&\qquad + \tilde{C}\bigl(1 + |(**)|\bigr)|\hat{p}\sigma(\hat{t},\hat{x}) - \hat{p}'\sigma(\hat{t},\hat{x}')|\bigr]\,d\lambda \\
&\le \int_0^1 \Bigl[\tilde{C}\bigl(1 + |(**)|^2\bigr)\varepsilon\sqrt{\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')} \\
&\qquad - \tilde{K}\bigl(1 + |(**)|^2\bigr)\bigl(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')\bigr) \\
&\qquad + \tilde{C}\vartheta\bigl(1 + |(**)|\bigr)^2\bigl(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')\bigr) \\
&\qquad + \frac{1}{\vartheta(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}'))}|\hat{p}\sigma(\hat{t},\hat{x}) - \hat{p}'\sigma(\hat{t},\hat{x}')|^2\Bigr]\,d\lambda
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \Big[\tilde{C}\big(1 + |(**)|^2\big)\varepsilon\sqrt{\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')} \\
&\qquad - \tilde{K}\big(1 + |(**)|^2\big)\big(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')\big) \\
&\qquad + \tilde{C}\vartheta 2\big(1 + |(**)|^2\big)\big(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')\big) \\
&\qquad + \frac{1}{\vartheta(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}'))}|\hat{p}\sigma(\hat{t},\hat{x}) - \hat{p}'\sigma(\hat{t},\hat{x}')|^2 \Big]\, d\lambda.
\end{aligned}
$$

Choose $\vartheta = \frac{\tilde{C}}{6\tilde{K}}$. We then have for $\varepsilon^2 < \frac{\tilde{C}\sqrt{\delta}}{3\tilde{K}}$

$$
\begin{aligned}
\text{(iii)} &\le -\frac{\tilde{K}}{3}\big(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')\big) + |\hat{p}\sigma(\hat{t},\hat{x}) - \hat{p}'\sigma(\hat{t},\hat{x}')|^2 \frac{1}{\vartheta(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}'))} \\
&\le -\frac{\tilde{K}}{3}\big(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')\big) \\
&\qquad + (|\hat{p}|C_\sigma|\hat{x} - \hat{x}'| + |\hat{p} - \hat{p}'|C_\sigma)^2 \frac{1}{\vartheta(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}'))} \\
&\le -\frac{\tilde{K}}{3}\big(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')\big) \\
&\qquad + \left(2\frac{|\hat{x} - \hat{x}'|^2}{\varepsilon^2} + 2C_\sigma\eta|\hat{x}||\hat{x} - \hat{x}'| + C_\sigma|2\eta\hat{x} + 2\eta\hat{x}'|\right)^2 \\
&\qquad \times \frac{1}{\vartheta(\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}'))}.
\end{aligned}
$$

Now, Lemma 3.5 in [15] (see also Lemma 2 in [5]) yields

$$
\liminf_{\varepsilon\to 0}\liminf_{\eta\to 0}[\tilde{u}(\hat{t},\hat{x}) - \tilde{v}(\hat{t},\hat{x}')] = L',
$$

$$
\limsup_{\varepsilon\to 0}\limsup_{\eta\to 0}\frac{|\hat{x} - \hat{x}'|}{\varepsilon} = 0,
$$

$$
\limsup_{\varepsilon\to 0}\limsup_{\eta\to 0}\eta(|\hat{x}|^2 + |\hat{x}'|^2) = 0.
$$

Hence, we get $\limsup_{\varepsilon\to 0}\limsup_{\eta\to 0}\text{(i)} \le 0$, $\limsup_{\varepsilon\to 0}\limsup_{\eta\to 0}\text{(ii)} \le 0$, and $\limsup_{\varepsilon\to 0}\limsup_{\eta\to 0}\text{(iii)} \le -\frac{\tilde{K}}{3}L'$. Combining, we arrive at

$$
0 \le -\frac{\tilde{K}}{3}L',
$$

which is the desired contradiction. $\square$

Acknowledgment. Part of this work was carried out during the 2010 SPDE programme at INI, Cambridge.

REFERENCES

[1] BARLES, G., BUCKDAHN, R. and PARDOUX, E. (1997). Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep*. **60** 57–83. MR1436432

[2] BOGACHEV, V. I. (2007). *Measure Theory. Vol. I*. Springer, Berlin. MR2267655

[3] BUCKDAHN, R. and MA, J. (2001). Stochastic viscosity solutions for nonlinear stochastic partial differential equations. I. *Stochastic Process. Appl*. **93** 181–204. MR1828772

[4] CARUANA, M. and FRIZ, P. (2009). Partial differential equations driven by rough paths. *J. Differential Equations* **247** 140–173. MR2510132

[5] CARUANA, M., FRIZ, P. K. and OBERHAUSER, H. (2011). A (rough) pathwise approach to a class of non-linear stochastic partial differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** 27–46. MR2765508

[6] CHEN, Y. G., GIGA, Y. and GOTO, S. (1991). Remarks on viscosity solutions for evolution equations. *Proc. Japan Acad. Ser. A Math. Sci*. **67** 323–328. MR1151347

[7] CRANDALL, M. G., ISHII, H. and LIONS, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc*. (*N.S.*) **27** 1–67. MR1118699

[8] DEYA, A., GUBINELLI, M. and TINDEL, S. (2009). Non-linear rough heat equations. Preprint. Available at arXiv:0911.0618.

[9] DIEHL, J., FRIZ, P. K. and OBERHAUSER, H. (2011). Parabolic comparison revisited and applications. Preprint. Available at arXiv:1102.5774.

[10] EL KAROUI, N., PENG, S. and QUENEZ, M. C. (1997). Backward stochastic differential equations in finance. *Math. Finance* **7** 1–71. MR1434407

[11] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[12] FRIZ, P. K. and OBERHAUSER, H. (2010). Rough path stability of SPDEs arising in non-linear filtering. Preprint. Available at arXiv:1005.1781.

[13] FRIZ, P. K. and VICTOIR, N. B. (2010). *Multidimensional Stochastic Processes as Rough Paths*: *Theory and Applications*. *Cambridge Studies in Advanced Mathematics* **120**. Cambridge Univ. Press, Cambridge. MR2604669

[14] GUBINELLI, M. and TINDEL, S. (2010). Rough evolution equations. *Ann. Probab.* **38** 1–75. MR2599193

[15] KOBYLANSKI, M. (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.* **28** 558–602. MR1782267

[16] LIANG, G., LYONS, T. J. and QIAN, Z. (2009). Backward stochastic dynamics on a filtered probability space. Preprint. Available at arXiv:0904.0377.

[17] LIONS, P.-L. and SOUGANIDIS, P. E. (2000). Fully nonlinear stochastic pde with semilinear stochastic dependence. *C. R. Acad. Sci. Paris Sér. I Math*. **331** 617–624. MR1799099

[18] LYONS, T. (1994). Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young. *Math. Res. Lett*. **1** 451–464. MR1302388

[19] LYONS, T. and QIAN, Z. (2002). *System Control and Rough Paths*. *Oxford Mathematical Monographs*. Oxford Univ. Press, Oxford. MR2036784

[20] LYONS, T. J., CARUANA, M. and LÉVY, T. (2007). *Differential Equations Driven by Rough Paths*. *Lecture Notes in Math*. **1908**. Springer, Berlin. MR2314753

[21] PARDOUX, É. and PENG, S. G. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett*. **14** 55–61. MR1037747

[22] PARDOUX, É. and PENG, S. G. (1994). Backward doubly stochastic differential equations and systems of quasilinear SPDEs. *Probab. Theory Related Fields* **98** 209–227. MR1258986

[23] TEICHMANN, J. (2009). Another approach to some rough and stochastic partial differential esquations. Preprint. Available at arXiv:0908.2814v1.

INSTITUT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT BERLIN
STRASSE DES 17, JUNI 136
10623 BERLIN
GERMANY
E-MAIL: joscha.diehl@gmail.com

INSTITUT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT BERLIN
STRASSE DES 17, JUNI 136
10623 BERLIN
GERMANY
AND
WEIERSTRASS—INSTITUT FÜR ANGEWANDTE ANALYSIS UND STOCHASTIK
MOHRENSTRASS E 39
10117 BERLIN
GERMANY
E-MAIL: friz@math.tu-berlin.de
friz@wias-berlin.de